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Scale-invariant Controller Synthesis in Mass Chains

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Abstract: This paper studies the disturbance suppression problem for a homogeneous mass chain, i.e., a chain of identical point masses interconnected by identical mechanical impedances. The particular focus is placed on whether the disturbance attenuation level of the chain of arbitrary length can be uniformly bounded with a size-independent controller. We explicitly represent the scalar transfer functions from the disturbances to a given intermass displacement as a function of the number of masses. This is an extension of the previous work by the author that established a boundedness of the H-infinity norm when one end of the chain is perturbed. We propose a new method that drastically simplifies its derivation process and provide the complete forms of all the transfer functions of our interest.

Keywords: Disturbance rejection, admittance, linear systems, decentralized systems, large-scale systems

1. INTRODUCTION

We study the disturbance propagation in the mass chain of Fig. 1. More specifically, we investigate how the length of the mass chain affects the transfer functions from the disturbances to a given intermass displacement.

With the prevalence of large-scale networked systems, it has become of great interest to consider the systems that involve very large numbers of locally interacting subsystems. Examples of such systems include the platooning of vehicles (Levine and Athans (1966)), frequency and voltage regulation problems in electrical power systems (Machowski et al. (1997); Simpson-Porco et al. (2016)), and flocking and consensus phenomena (Bamieh et al. (2012)). In such applications, the numbers of subsystems is often subject to change. Therefore, one problem is ensuring the ability to control a system behaviour independent of the size of the systems.

In practice, however, such criteria are rarely satisfied. This is due to the fact, at least in part, that several key performance measures relating to global behaviours of large-scale networks simply do not scale. Notable examples include the string instability (e.g., Seiler et al. (2004); Feng et al. (2019)) or network incoherence phenomena (e.g., Bamieh et al. (2012)). Also bidirectional control, which our problem formulation corresponds to in the context of vehicle platooning, is often subject to the inevitable string instability (Barooah and Hespanha (2005); Herman et al. (2015)) except for few favourable situations (Farnam and Sarlette (2017)). Nevertheless, it appears that average or local performance measures, for example those in (Carli et al. (2009); Pates (2015); Bamieh et al. (2012); Yamamoto and Smith (2016); Pates and Yamamoto (2018)), can be guaranteed independent of the size of the network.

For a simple mass chain depicted in Fig. 1, the author has shown that the H^∞-norm of the transfer function from the movable point displacement to a given intermass displacement is uniformly bounded with respect to N for certain choices of interconnection admittances (Yamamoto and Smith (2016)). The conditions have been given with respect to a dimensionless parameter h depending on the admittance and mass. This has been achieved by deriving the form of complex iterative maps to evaluate how the transfer function changes with N. A graphical means to design a suitable interconnection admittances was provided so that the supremum of the H^∞-norm over N is no greater than a prescribed value. This can be thought of as a scale-invariant H^∞ control design for an infinite family of plants in which the interconnection admittance is the controller.

One drawback of the method that Yamamoto and Smith (2016) provided is that the derivation process for the complex iterative maps was rather complicated and hard to generalize. In the present paper we provide a much simpler way to evaluate the system dynamics as a function of N. The closed-form expressions are given not only for the transfer function from the movable point displacement but also from the disturbances on each mass.

2. PROBLEM FORMULATION

2.1 General notation

The set of natural, real and complex numbers is denoted by $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$, respectively. $\mathbb{R}^{m\times n}$ is the set of m-by-n real matrices. $\mathbb{C}_+$ is the closed right-half plane. H^∞ is the standard Hardy space on the right-half plane and $\|\cdot\|_\infty$ represents the H^∞-norm. The (i,j) entry of a matrix A is denoted by $[A]_{i,j}$.

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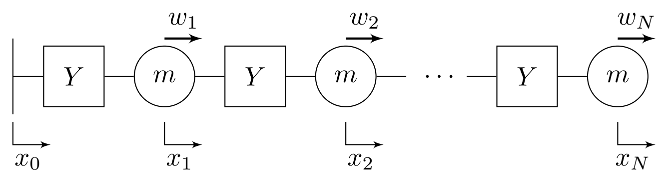

Fig. 1. Chain of N masses m connected by a mechanical admittance Y and connected to a movable point x_0.

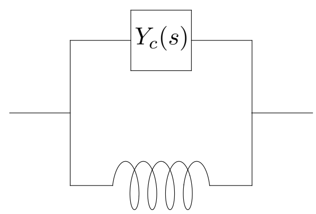

Fig. 2. Interconnection configuration. $Y(s) = k/s + Y_c(s)$.

2.2 Chain model

We consider a chain of N identical point masses m connected by identical mechanical networks (Fig. 1). Each mechanical network provides an equal and opposite force on each mass and is assumed here to have negligible mass. It is also assumed that each mechanical network consists of a spring component in parallel with other components (Fig. 2). Here we assume that the spring coefficient k is a given parameter and our task is to find a controller represented by an admittance $Y_c(s)$ that effectively supresses the disturbance in the chain.

The system is excited by a movable point $x_0(t)$ and external force acting on the ith mass, $w_i(t)$, $i \in \{1, 2, \ldots, N\}$. The displacement of the ith mass is then denoted by $x_i(t)$. We assume that the initial conditions of the movable point, external forces, and the mass displacements are all zero.

The equations of motion in the Laplace transformed domain are then

$$ms^2\hat{x}_i = sY(s)(\hat{x}_{i-1} - \hat{x}_i) + sY(s)(\hat{x}_{i+1} - \hat{x}_i) + \hat{w}_i \quad \text{for } i = 1, \ldots, N-1,$$

$$ms^2\hat{x}_N = sY(s)(\hat{x}_{N-1} - \hat{x}_N) + \hat{w}_N$$

where $\hat{}$ denotes the Laplace transform. Letting $h(s) := sZ(s)m$ where Z is a mechanical impedance defined by $Z = Y^{-1}$, we obtain

$$\hat{x} = (h(s)I - L_N)^{-1}\phi_1\hat{x}_0 + \frac{1}{sY(s)}(h(s)I - L_N)^{-1}\hat{w} \quad (1)$$

where I is the identity matrix,

$$\hat{x} = [\hat{x}_1, \ldots, \hat{x}_N]^\top, \quad \hat{w} = [\hat{w}_1, \ldots, \hat{w}_N]^\top,$$
$$\phi_1 = [1,\ 0, \ldots,\ 0]^\top \in \mathbb{R}^N,$$
$$L_N := \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & -1 \end{bmatrix} \in \mathbb{R}^{N\times N}. \quad (2)$$

Then the dynamics of the intermass displacement $e_i := x_{i-1} - x_i$ is given by

$$\hat{e} = G_{ex_0}(s)\hat{x}_0 + G_{ew}(s)\hat{w} \quad (3)$$

where $\hat{e} = [\hat{e}_1, \ldots, \hat{e}_N]^\top$ and

$$G_{ex_0}(s) = \left(I + M(h(s)I - L_N)^{-1}\right)\phi_1,$$
$$=: [G_{e_1x_0}(s), \ldots, G_{e_Nx_0}(s)]^\top$$
$$G_{ew}(s) = \frac{1}{sY(s)}M(h(s)I - L_N)^{-1},$$
$$M = \begin{bmatrix} -1 & 0 & \cdots & \cdots & 0 \\ 1 & -1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & -1 \end{bmatrix} \in \mathbb{R}^{N\times N}. \quad (4)$$

To obtain an explicit inverse of the tridiagonal matrix $hI - L_N$, let us introduce the characteristic polynomials of $L_i \in \mathbb{R}^{i\times i}$ in the variable h:

$$d_i := \det(hI - L_i), \quad (5)$$

and also the characteristic polynomials of $\overline{L}_i \in \mathbb{R}^{i\times i}$ in h:

$$\overline{d}_i := \det(hI - \overline{L}_i) \quad (6)$$

where

$$\overline{L}_i := \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{bmatrix}. \quad (7)$$

Then $d_1 = h + 1$ and $\overline{d}_1 = h + 2$. Using the Laplace expansion, we find that

$$d_i(h) = (h+2)d_{i-1}(h) - d_{i-2}(h), \quad (8)$$
$$\overline{d}_i(h) = (h+2)\overline{d}_{i-1}(h) - \overline{d}_{i-2}(h), \quad \text{for } i = 1, \ldots, N \quad (9)$$

with initial conditions

$$d_{-1} = 1, d_0 = 1, \overline{d}_{-1} = 0, \overline{d}_0 = 1. \quad (10)$$

Using d_i and $\overline{d}_i$, the inverse of $hI - L_N$ can be written explicitly as

$$\left[(hI - L_N)^{-1}\right]_{i,j} = \overline{d}_{j-1}(h)d_{N-i}(h) \quad \text{for } i \geq j, \quad (11)$$

which can be easily deduced from the theorem provided by Usmani (1994). Note that $hI - L_N$ is symmetric and hence $\left[(hI - L_N)^{-1}\right]_{i,j} = \left[(hI - L_N)^{-1}\right]_{j,i}$.

The transfer functions $G_{e_ix_0}(s)$ and $G_{ew}(s)$ in (3) are then written as (suppressing the dependence on $h(s)$ in d_i and $\overline{d}_i$)

$$G_{e_ix_0}(s) = \frac{d_{N-i+1} - d_{N-i}}{d_N}, \quad (12)$$

$$G_{e_iw_j}(s) = \begin{cases} \dfrac{1}{sY(s)}\dfrac{1}{d_N}\overline{d}_{j-1}(d_{N-i+1} - d_{N-i}) & \text{for } i > j, \\ \dfrac{1}{sY(s)}\dfrac{1}{d_N}d_{N-j}(\overline{d}_{i-2} - \overline{d}_{i-1}) & \text{for } i \leq j, \end{cases} \quad (13)$$

where $G_{e_iw_j}(s)$ is the (i, j)-entry of $G_{ew}(s)$.

3. STABILITY OF INTERCONNECTION

Treating h as the independent variable, we first state the following properties of the sequence $d_i(h)$ given in Yamamoto and Smith (2016).

Proposition 1. (Yamamoto and Smith (2016)). $d_i(h)$ has negative real distinct roots which interlace the roots of $d_{i+1}(h)$ for $i = 1, 2, \ldots$. The roots of $d_i(h)$ lie in the interval $(-4, 0)$ for $i = 1, 2, \ldots$.

Proof. See (Yamamoto and Smith, 2016, Theorem 1).

We will say that the mass chain of Fig. 1 is stable if all poles in the transfer functions $G_{ex_0}(s)$ and $G_{ew}(s)$ have negative real parts. We may observe that poles in $G_{ex_0}(s)$ can only occur at an s for which $d_N(h(s)) = 0$. Hence, from Proposition 1, we see that all poles in $G_{ex_0}(s)$ have negative real parts (in the s-domain) if and only if $h(s) \in \mathbb{C} \setminus (-4, 0)$ for all $s \in \mathbb{C}_+$. This test can be conducted with standard tools such as the Routh-Hurwitz stability criterion.

Proposition 2. Let $n(s)$ and $d(s)$ be coprime polynomials such that $h(s) = n(s)/d(s)$. If the polynomial $n(s)+Kd(s)$ is Hurwitz for any $K \in (0, 4)$, $G_{ex_0}(s)$ is stable.

Proof. Since $d_N(h(s)) = 0$ only if $h(s) \in (-4, 0)$, the condition of the proposition implies that $h(s) \in \mathbb{C} \setminus (-4, 0)$ for all $s \in \mathbb{C}_+$, and consequently that $G_{ex_0}(s)$ is stable. □

For $G_{ew}(s)$, we further need to check if $sY(s) \neq 0$ for all $s \in \mathbb{C}_+$. Putting these conditions together, we have the following proposition.

Proposition 3. The mass chain of Fig. 1 is stable if $h(s) \in \mathbb{C} \setminus (-4, 0)$ and $sY(s) \neq 0$ for all $s \in \mathbb{C}_+$.

Example. Consider a parallel spring–damper combination with admittance $Y(s) = k/s + c$ for the interconnection. Then

$$h(s) = sZ(s)m = \frac{ms^2}{cs + k}. \tag{14}$$

Since m, k, c are positive constants, the polynomial $ms^2 + K(cs + k)$ is Hurwitz for all $K > 0$. From Proposition 2, $h(s) \in \mathbb{C} \setminus (-4, 0)$ and hence $d_N(h(s)) \neq 0$ for all $s \in \mathbb{C}_+$. Furthermore, $sY(s) = cs + k \neq 0$ for all $s \in \mathbb{C}_+$. Hence the mass chain is stable.

Remark 1. We note that Hara et al. (2014) have investigated the stability of systems described by "generalized frequency variables," which is $h(s)$ in our case. Here we provide an explicit condition for stability specific to our model.

Remark 2. If we only consider passive interconnections, the stability of $G_{ex_0}(s)$ can be tested by only looking at the imaginary axis. That is, for a positive real impedance $Z(s) \not\equiv 0$, $G_{ex_0}(s)$ is stable if $h(j\omega)$ does not take values in the interval $(-4, 0)$ (Yamamoto and Smith, 2016, Theorem 2).

4. INTERMASS DISPLACEMENTS

We now define the following functions:

$$F_N^{(i,j)}(h) := \begin{cases} \dfrac{1}{d_N} \overline{d}_{j-1}(d_{N-i+1} - d_{N-i}) & \text{for } i > j, \\ \dfrac{1}{d_N} d_{N-j}(\overline{d}_{i-2} - \overline{d}_{i-1}) & \text{for } i \leq j. \end{cases} \tag{15}$$

Then, for $h(s) = sZ(s)m$,

$$\begin{aligned} G_{e_1x_0}(s) &= 1 + F_N^{(1,1)}(h(s)), \\ G_{e_ix_0}(s) &= F_N^{(i,1)}(h(s)) \text{ for } i > 1, \\ G_{e_iw_j}(s) &= \frac{1}{sY(s)} F_N^{(i,j)}(h(s)). \end{aligned} \tag{16}$$

Yamamoto and Smith (2016) have shown that, for any $i \leq N$, $\|G_{e_ix_0}(s)\|_\infty$ is uniformly bounded for all $N \in \mathbb{N}$ with a suitable choice of interconnection by deriving a form of an iterated Möbius transformation to represent $F_N^{(i,1)}(h)$. Closed-form solutions of $F_N^{(i,1)}(h)$ have been given by making use of a conjugacy transformation to establish the formal upper bounds. However, the derivation process of these iterated Möbius transformations was rather complicated and hard to generalize.

The following theorem gives these closed-form expressions in much simpler way and also gives the expressions of $F_N^{(i,j)}(h)$ for all (i, j).

Theorem 4. Let $\zeta \in \mathbb{C}$ be the root of

$$z^2 - (h + 2)z + 1 = 0 \tag{17}$$

satisfying $|\zeta| \leq 1$. For any $i, j \in \mathbb{N}$,

$$F_N^{(i,j)} = \begin{cases} \dfrac{\zeta^{i-j}(1 - \zeta^{2j})\left(1 - \zeta^{2(N-i+1)}\right)}{(1 + \zeta)(1 + \zeta^{2N+1})} & \text{for } i > j, \\ -\dfrac{\zeta^{j-i+1}(1 + \zeta^{2i-1})\left(1 + \zeta^{2(N-j)+1}\right)}{(1 + \zeta)(1 + \zeta^{2N+1})} & \text{for } i \leq j. \end{cases} \tag{18}$$

suppressing the dependence on h in $F_N^{(i,j)}$ and ζ.

Proof. We first solve the recurrence relations (8) and (9). Observe that the other root of (17) is ζ^{-1} and

$$\zeta + \zeta^{-1} = h + 2. \tag{19}$$

Substituting this into (8) gives

$$d_i = \left(\zeta + \zeta^{-1}\right) d_{i-1} - d_{i-2} \tag{20}$$

which implies

$$\begin{aligned} d_i - \zeta d_{i-1} &= \zeta^{-1}(d_{i-1} - \zeta d_{i-2}) = \zeta^{-i}(d_0 - \zeta d_{-1}) \\ &= \zeta^{-i}(1 - \zeta) \end{aligned} \tag{21}$$

and

$$\begin{aligned} d_i - \zeta^{-1} d_{i-1} &= \zeta(d_{i-1} - \zeta^{-1} d_{i-2}) = \zeta^i(d_0 - \zeta^{-1} d_{-1}) \\ &= \zeta^i(1 - \zeta^{-1}). \end{aligned} \tag{22}$$

Subtracting (21) from (22) and rearranging give

$$d_i = \frac{1 + \zeta^{2i+1}}{\zeta^i(1 + \zeta)}. \tag{23}$$

Similarly, from the following equations

$$\overline{d}_i - \zeta \overline{d}_{i-1} = \zeta^{-1}(\overline{d}_{i-1} - \zeta \overline{d}_{i-2}) = \zeta^{-i}(\overline{d}_0 - \zeta \overline{d}_{-1}) = \zeta^{-i} \tag{24}$$

and

$$\overline{d}_i - \zeta^{-1} \overline{d}_{i-1} = \zeta(\overline{d}_{i-1} - \zeta^{-1} \overline{d}_{i-2}) = \zeta^i(\overline{d}_0 - \zeta \overline{d}_{-1}) = \zeta^i, \tag{25}$$

we obtain

$$\bar{d}_i = \frac{1+\zeta^{2i+2}}{\zeta^i(1-\zeta^2)}. \tag{26}$$

Substituting (23) and (26) into (15) gives the closed-form expression (18). □

4.1 Limits of the Sequences

It may be observed that the sequence $\left(F_N^{(i,j)}\right)$ in (18) is convergent for a fixed $\zeta \in \mathbb{C}$ with $|\zeta| < 1$ or $\zeta = \pm 1$, and divergent otherwise. The following theorem provides the condition for the convergence on h and its limit.

Theorem 5. The sequence $\left(F_N^{(i,j)}\right)$ converges pointwise to a limit $\mu^{(i,j)}$ for each $h \in \mathbb{C} \setminus (-4, 0)$ but fails to converge otherwise. In particular,

(1) For $h \in \mathbb{C} \setminus [-4, 0)$,

$$\mu^{(i,j)}(h) = \begin{cases} \dfrac{\zeta^{i-j}(1-\zeta^{2j})}{(1+\zeta)} & \text{for } i > j, \\ -\dfrac{\zeta^{j-i+1}(1+\zeta^{2i-1})}{(1+\zeta)} & \text{for } i \le j. \end{cases} \tag{27}$$

(2) For $h = -4$,

$$\mu^{(i,j)}(h) = \begin{cases} (-1)^{j-1}2j & \text{for } i > j, \\ (-1)^{i-j-2}(2i-1) & \text{for } i \le j. \end{cases} \tag{28}$$

Proof. Recall that ζ is the root of (17) satisfying $|\zeta| \le 1$. Since $\zeta + \zeta^{-1} = h + 2$,

$$|\zeta| = 1 \iff h = 2(\cos(\angle\zeta) - 1) \in [-4, 0]. \tag{29}$$

For $h = 0$, $\zeta = 1$ and from (18),

$$F_N^{(i,j)}(0) = \begin{cases} 1 & \text{for } i > j, \\ -1 & \text{for } i \le j. \end{cases} \tag{30}$$

For $h = 4$, $\zeta = -1$. Since it is tricky to compute $F_N^{(i,j)}(-4)$ from (18), we use the expression (15). It is straightforward to obtain that, from (8) and (9),

$$d_i(-4) = (-1)^i(2i+1), \quad \bar{d}_i(-4) = (-1)^i(i+1). \tag{31}$$

Substituting these into (15) gives

$$F_N^{(i,j)}(-4) = \begin{cases} (-1)^{j-1}2j\dfrac{2(N-i+1)}{2N+1} & \text{for } i > j, \\ (-1)^{i-j-2}(2i-1)\dfrac{2(N-j+1)}{2N+1} & \text{for } i \le j \end{cases} \tag{32}$$

and $\lim_{N\to\infty} F_N^{(i,j)}(-4) = \mu^{(i,j)}(-4)$ given in the theorem. For $h \in (-4, 0)$, ζ is imaginary with $|\zeta| = 1$ and hence the sequence $\left(F_N^{(i,j)}\right)$ does not converge as observed from (18).

For $h \in \mathbb{C} \setminus [-4, 0]$, $|\zeta| < 1$ and evaluating limits of (18) gives $\mu^{(i,j)}(h)$. □

Hence, if $h(s) \in \mathbb{C} \setminus (-4, 0)$ for all $s \in \mathbb{C}_+$,

$$\sup_\omega \lim_{N\to\infty} |F_N^{(i,j)}(h(j\omega))| = \sup_\omega |\mu^{(i,j)}(h(j\omega))|. \tag{33}$$

Furthermore,

$$\sup_N \|F_N^{(i,j)}(h(s))\|_\infty \ge \sup_\omega |\mu^{(i,j)}(h(j\omega))|. \tag{34}$$

That is, $\sup_\omega |\mu^{(i,j)}(h(j\omega))|$ gives a lower bound of the supremum of H^∞-norm of $F_N^{(i,j)}(h(s))$ over N. A contour

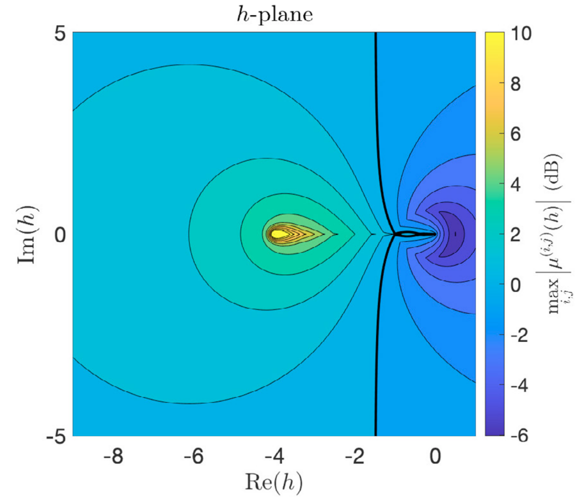

Fig. 3. Contour plot of $\max_{i,j} |\mu^{(i,j)}(h)|$ with $1 \le i, j \le 100$. The thick curve represents a contour of level 0 dB.

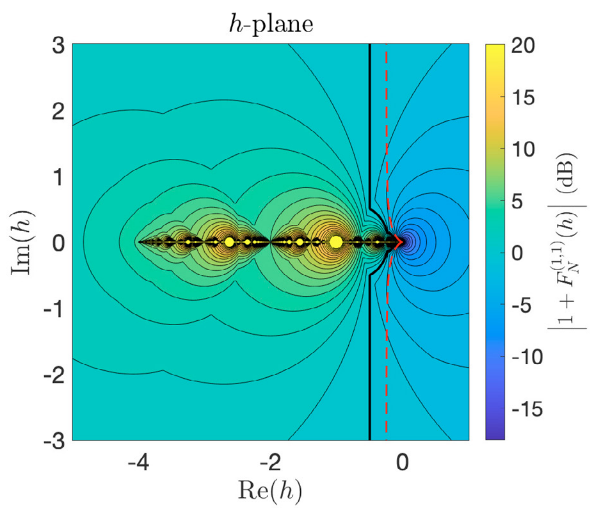

Fig. 4. Nyquist diagram of $h(s) = s^2/(2s+1)$ (red, dashed) and contour plot of $\max_{1\le N\le 200} |1+F_N^{(1,1)}|$. The thick black curve represents a contour of level 0 dB.

plot of the maximum magnitude of $\mu^{(i,j)}(h)$ over (i, j) with $0 \le, i, j \le 100$ in the h-plane is shown in Fig. 3. The thick black curve represents $\max_{i,j} |\mu^{(i,j)}(h)| = 0$ (dB). The figure shows that the asymptotic value of $F_N^{(i,j)}(h(j\omega))$ as $N \to \infty$ is directly related to the proximity of $h(j\omega)$ to the point -4. From Theorem 5 we see that the magnitude of $|\mu^{(i,j)}(-4)|$ grows as we increase the indices and we must avoid this region.

4.2 Uniform Boundedness

When the interconnection is restricted to be passive, Yamamoto and Smith (2016) have derived a sufficient condition such that $\|G_{e_i x_0}(s)\|_\infty$ is uniformly bounded for any $N \in \mathbb{N}$. We demonstrate the boundedness result in Fig. 4. The figure shows a contour plot of $\max_{1\le N\le 200} |1 + F_N^{(1,1)}(h)|$ with the thick black curve representing a contour of level 0 dB. (We may note that $N = 200$ is large enough to accurately determine the

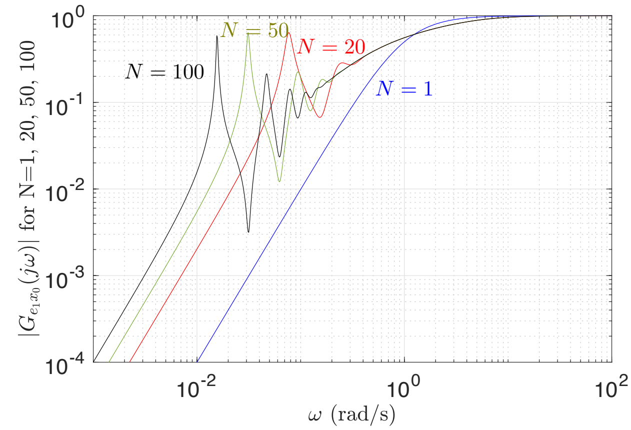

Fig. 5. The magnitude plot of $G_{e_1x_0}(j\omega)$ with $h(s) = s^2/(2s+1)$ for $N = 1, 20, 50, 100$.

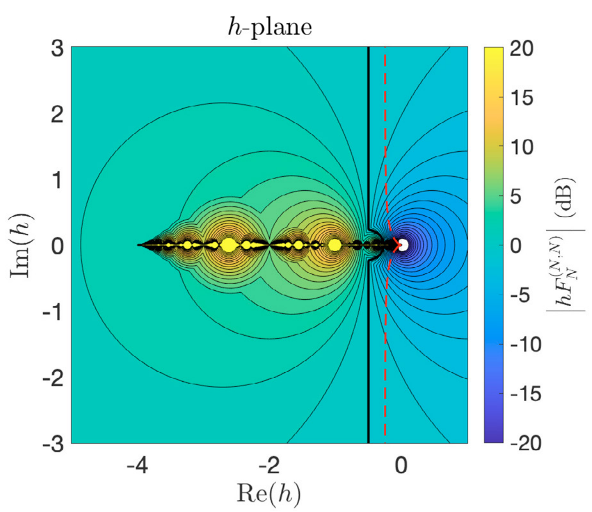

Fig. 6. Nyquist diagram of $h(s) = s^2/(2s+1)$ (red, dashed) and contour plot of $\max_{1\le N\le 200} |hF_N^{(1,N)}|$. The thick black curve represents a contour of level 0 dB.

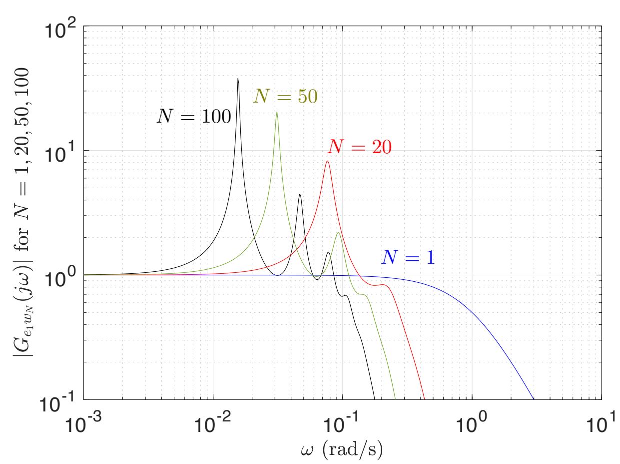

Fig. 7. The magnitude plot of $G_{e_1w_N}(j\omega)$ with $h(s) = s^2/(2s+1)$ for $N = 1, 20, 50, 100$.

shape of the boundary as discussed in (Yamamoto and Smith (2016)).) If the Nyquist diagram of $h(s)$ lies inside a contour of level γ, this indicates that $\|G_{e_ix_0}(s)\|_\infty \le \gamma$ for any $N \in \mathbb{N}$. To demonstrate this, the Nyquist diagram of $h(s) = s^2/(2s+1)$ is also plotted in Fig. 4. Since it lies inside the curve of level 0 dB, $\|G_{e_ix_0}(s)\|_\infty \le 1$ for any $N \in \mathbb{N}$, as we can confirm in Fig. 5 for $N = 1, 20, 50, 100$.

To evaluate $G_{e_iw_j}(s)$ in a similar way, we first rewrite it as

$$\begin{aligned} G_{e_iw_j}(s) &= \frac{1}{sY(s)} F_N^{(i,j)}(h(s)) \\ &= \frac{1}{ms^2} h(s) F_N^{(i,j)}(h(s)) \end{aligned} \tag{35}$$

and draw a contour plot of $\max_N |hF_N^{(i,j)}(h)|$. Figure. 6 shows the contour map for $(i,j) = (1,N)$ again with the Nyquist diagram of $h(s) = s^2/(2s+1)$. Figure 6 is very similar to Fig. 4 in appearance. However, since $1/ms^2$ is multiplied, the same interconnection $h(s) = s^2/(2s+1)$ will result in large frequency response of $G_{e_1w_N}(s)$ in the low frequency range, which is indeed the case as seen in Fig. 7. To remedy this, we need to shape the locus of $h(j\omega)$ at $\omega \approx 0$ suitably. However, because of the existence of the parallel spring, the limiting behaviour of $h(j\omega)$ as $\omega \to 0$ cannot be drastically changed using passive interconnection, i.e., positive-real $Y(s)$. Whether the use of an active controller may lead to an improvement is yet to be explored.

5. CONCLUSION

Convenient representations of transfer functions from disturbance to a given intermass displacement in a homogeneous mass chain have been presented to evaluate how the system dynamics change as the number of masses N changes. The limiting behaviour of these transfer functions as N tends to infinity has been studied. Moreover, the possibility of designing a scale-invariant controller that achieves the disturbance attenuation level independent of N has been explored. This may not be achieved by passive interconnection when the disturbance on each mass is present. The use of active controllers is considered as a future work.

6. ACKNOWLEDGMENT

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