Sampled-data Filters with Compactly Supported Acquisition Prefilters

Yamamoto, Yutaka
Kyoto University : Professor Emeritus

Yamamoto, Kaoru
Department of Electrical Engineering, Kyushu University

Nagahara, Masaaki
Institute of Environmental Science and Technology, The University of Kitakyushu

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Sampled-data Filters with Compactly Supported Acquisition Prefilters

Yutaka Yamamoto[1], Kaoru Yamamoto[2], and Masaaki Nagahara[3],

Abstract— **This paper studies the problem of reconstructing continuous-time signals from discrete-time uniformly sampled data. This signal reconstruction problem has been studied by the authors in various contexts, and led to a new signal processing paradigm. The key idea there is to employ a physically realizable signal generator model, and design an (sub)optimal filter via $H^\infty(\mathbb{C}_+)$ optimal sampled-data control theory. The present paper aims at extending this framework to a more general setting where observed data are acquired through an acquisition device (prefilter) that has compact support. In this way, the framework can capture the properties of processing signals with a localized acquisition filter. We give a general setup as well as approximate solution methods along with their convergence results. A simulation is presented to illustrate some properties of the result.**

I. Introduction

Modern sampled-data control theory finds yet another area of application, i.e., digital signal processing [12]. As discussed there, one of the central problems in digital signal processing is the reconstruction of the original analog signal from its sampled data. Shannon [5] first considered this problem in the context of uniform and ideal sampling. He effectively used the sampling theorem [13] as the central guiding methodology, and proved that if the original signal is perfectly band-limited below the so-called Nyquist frequency, one could fully recover the original analog signal from its sampled data. Since then, this Shannon paradigm has dominated digital signal processing to date.

Recently, we have developed and proposed a completely new methodology based on H^∞ sampled-data control theory: [12], [3], [4]. The present paper intends to extend this framework to a general context with compactly supported acquisition prefilters.

Making use of the fact that modern sampled-data control theory can optimize intersample behavior, these papers successfully derived new design formulas for digital filters that can be used to recover the original analog signals.

A crucial element there is that we assume a natural model that describes the signal class to be processed. Instead of assuming the artificial perfect band-limiting hypothesis, we assume that the target signals are generated by a linear, time-invariant, stable, finite-dimensional system. Such a system gives a stable filter that controls the decay curve toward high-frequency. Unlike the case for the sampling theorem, this model gives a decay curve beyond the Nyquist frequency, yet it gives rise to enough information that allows us to recover optimal high frequency behavior via sampled-data control theory, which is prohibited by the Shannon paradigm. The new theory has proven to be successful, and in some part it is incorporated in actual devices.

In this new theory, however, the signal generator (or signal acquisition device) is assumed to be a linear, time-invariant, stable and finite-dimensional system. This is a natural setting in the following sense: In many cases, we consider (analog) signals generated by a physical system, be it natural or artificial (man-made), which obeys the underlying physical laws. In this sense, it is very natural to consider signals generated by such systems, and this leads to the basic setting considered in our previous work, e.g., [12], [3], [4].

On the other hand, there are situations that do not readily satisfy the above hypotheses. Suppose we are encountered with an unknown class of signals, for which there is no generating model available. We receive these signals through some device, and we obtain their sampled values. We may want to identify some characteristics of such signals through those observed data.

The theory of wavelets is built on such assumptions. To this end, one may assume an acquisition device, and obtain the sampled-data once filtered after this acquisition device. In such a context, we often assume that such an acquisition device is described by a prefilter that has compact support. This allows for a finer resolution in the time domain, and time-local properties are better preserved. For example, when there is a singularity in the target signal (e.g., discontinuity), it is easier to detect it in such a framework. This is the core of the time-frequency analysis enabled by wavelet theory.

To make sampled-data signal processing theory compatible with such a time-local approach, we need to extend our theory to the new context where the signal processing prefilter is derived from an acquisition device that has compact support. This assumption does not satisfy the basic hypotheses in [12], and we need to generalize our framework to this context. The theory requires an extension of the tools and settings of those given in [12], and it is the target of the present paper.

II. Problem formulation

As in our previous work [12], [8], we consider the sampled-data system depicted in Fig. 1.

[1]Professor Emeritus, Kyoto University, Kyoto 606-8510, Japan yy@i.kyoto-u.ac.jp. This work was supported in part by the Japan Society for the Promotion of Science under Grants-in-Aid for Scientific Research No. 15H04021 and 24360163. The author wishes to thank DIGITEO and Laboratoire des Signaux et Systemes (L2S, UMR CNRS), CNRS-CentraleSupelec-University Paris-Sud and Inria Saclay for their financial support while part of this research was conducted.

[2]Department of Electrical Engineering, Kyushu University, Fukuoka 819-0395, Japan k.yamamoto@ieee.org. This work was supported in part by the Swedish Research Council through the LCCC Linnaeus Center.

[3]Institute of Environmental Science and Technology, The University of Kitakyushu, Fukuoka 808-0135, Japan. nagahara@ieee.org. This work was supported in part by JSPS KAKENHI grant numbers 15H02668 and 16KK0134, and SCOPE grant number 172310003.

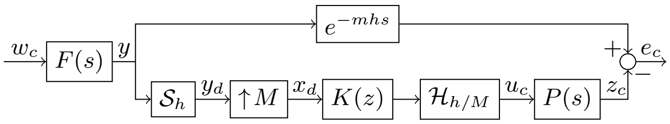


Fig. 1: Signal reconstruction error system

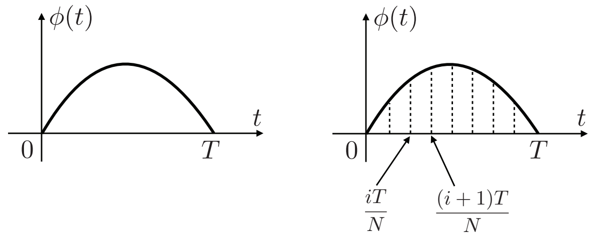


Fig. 2: Sampling kernel $\phi(t)$ (left) and fast discretization of $\phi(t)$ (right)

Here the exogenous signal w_c goes through a filter $F(s)$ and becomes the actual target analog signal y. Contrary to our previous work [12], [3], [4], [8], etc., this $F(s)$ is not necessarily derived from a linear, finite-dimensional, time-invariant system. We assume that $F(s)$ is given as the Laplace transform of a function $\phi(\cdot)$ that has compact support on $[0,T] \subset [0,\infty)$. This $F(s)$ may not necessarily represent the physical characteristic of a signal generating system, but rather represents a characteristic of our signal acquisition device. The totality of such y constitutes the signal class to be reconstructed. The resulting analog signal y is then sampled with a sampling period h, and then becomes the digital signal that must be processed. The objective here is to reconstruct the analog signal y from given sampled data $y(kh)$, $k = 0,1,2,\cdots$. Note that the prefilter $F(s)$ is stable because ϕ has compact support, and the signal y is obtained as the convolution $\phi * w_c$ in the time domain.

Fig. 2 (left) shows an example of the sampling kernel $\phi(t)$.

The discrete-time signal y_d is first upsampled by factor M by the upsampler $\uparrow M$

$$\uparrow M : y_d \mapsto x_d : x_d[k] = \begin{cases} y_d[l], & k = Ml,\ l = 0,1,\ldots \\ 0, & \text{otherwise} \end{cases},$$

and becomes another discrete-time signal x_d with sampling period h/M. The discrete-time signal x_d is then processed by a digital filter $K(z)$ to be designed, and becomes a continuous-time signal u_c by going through the zero-order hold $\mathcal{H}_{h/M}$ (which works in sampling period h/M), and then becomes the final signal z_c by passing through an analog buffer filter $P(s)$. Here $P(s)$ can be assumed to be 1 for simplicity. An advantage here is that one can use the fast hold device $\mathcal{H}_{h/M}$ thereby making possible more precise signal restoration. The objective here is to design a digital filter $K(z)$ for given $F(s)$, M and $P(s)$, to optimally reconstruct the filtered signal y. To allow for a processing time, we introduce some delay L for reconstructing y, i.e., $y(t-L)$ instead of $y(t)$; for convenience we take L to be an integer multiple of the sampling period h, i.e., $L = mh$ for some positive integer m.

Fig. 1 shows the signal reconstruction error system block diagram. The delay in the upper portion of the diagram corresponds to the fact that we allow a certain amount of time delay for signal reconstruction. Let T_{ew} denotes the input/output operator from w_c to $e_c(t) := y(t-mh) - z_c(t)$. Our design objective is as follows:

Problem 1: Given stable $F(s)$ and $P(s)$ and an attenuation level $\gamma > 0$, find a digital filter $K(z)$ such that

$$\|T_{ew}\|_\infty = \sup_{w_c \in L^2[0,\infty)} \frac{\|T_{ew} w_c\|_2}{\|w_c\|_2} < \gamma.$$

Remark 2.1: The above L^2-induced norm $\|T_{ew}\|_\infty$ is indeed the H^∞-norm of the operator T_{ew} [11].

In the sequel, we may denote the sampling and hold operations with associated upsamplers, etc., simply by $\mathcal{S}$ and $\mathcal{H}$, respectively, to make the notation simpler.

III. Examples of Acquisition Filters

Several examples can be listed as an acquisition filter characteristic function ϕ. Most of them are commonly utilized in wavelet expansion. The simplest is perhaps the Haar scaling function ϕ_0:

$$\phi_0(t) := \begin{cases} 1, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

where T can vary depending on our choice.

A more elaborate choice would be that of the second-order B-spline function

$$B_2(t) := \begin{cases} t, & 0 \le t \le T/2 \\ 2-t, & T/2 \le t \le T \\ 0, & \text{elsewhere.} \end{cases} \tag{2}$$

A. Cardinal Exponential Splines

Unser and Blu [7], [6] introduced the notion of *cardinal exponential splines*. The first-order exponential splines are obtained by truncating usual exponential functions to a bounded interval $[0,T]$ for some positive T:

$$\beta_\alpha(t) := \begin{cases} e^{\alpha t}, 0 \le t \le T, \\ 0, \quad \text{elsewhere.} \end{cases} \tag{3}$$

Typically, T is normalized to 1, but we leave it as a free parameter here.

What is interesting to us is the following lemma that asserts that the usual sampled-data filter $K[z]$ designed for the anti-aliasing prefilter $F(s) = 1/(s-\alpha)$, i.e., the Laplace transform of $e^{\alpha t}$, works also for the present case with $\beta_\alpha(t)$ whose Laplace transform is

$$\frac{1 - e^{-T(s-\alpha)}}{s-\alpha}.$$

Here α is assumed to be negative to guarantee stability.

Lemma 3.1: Suppose $\alpha < 0$. Let $F_0(s) = 1/(s-\alpha)$, $F_T := (1 - e^{-T(s-\alpha)})/(s-\alpha)$ and let $K[z]$ be a filter designed for Fig. 1 with $F = F_0$ such that

$$\|T_{ew}\| = \left\|(e^{-mhs} - \mathcal{H}K\mathcal{S})F_0\right\| < \epsilon. \tag{4}$$

Then K also satisfies

$$\|T_{ew}\| = \left\|(e^{-mhs} - \mathcal{H}K\mathcal{S})F_T\right\| < 2\epsilon. \tag{5}$$

Proof Observe that (4) implies

$$\left\|(e^{-mhs} - \mathcal{H}K\mathcal{S})e^{\alpha T}e^{-Ts}F_0\right\| = e^{\alpha T}\left\|(e^{-mhs} - \mathcal{H}K\mathcal{S})F_0\right\| < \epsilon$$

because $\alpha < 0$. It follows that

$$\begin{aligned}\left\|(e^{-mhs} - \mathcal{H}K\mathcal{S})F_T\right\| &= \left\|(e^{-mhs} - \mathcal{H}K\mathcal{S})(F_0 - e^{-T(s-\alpha)}F_0)\right\| \\ &\le \left\|(e^{-mhs} - \mathcal{H}K\mathcal{S})F_0\right\| \\ &\quad + \left\|(e^{-mhs} - \mathcal{H}K\mathcal{S})e^{-T(s-\alpha)}F_0\right\| \\ &< 2\epsilon. \end{aligned}$$

□

This means that for the first-order cardinal exponential splines, the (sub)optimal filter $K[z]$ can be designed without really involving a special configuration. Note, however, that although the filter $K[z]$ can be taken to be the same, the processing result can be quite different since the obtained acquired signal y and its sampled values for F_T are different from those obtained through F_0. The one for F_T are more localised due to the compact support nature of F_T.

IV. Design for the General Case

The input/output relation with compactly supported impulse response function ϕ is not realizable as a linear, time-invariant, finite-dimensional system. However, it can be well realized as a linear, time-invariant, *infinite-dimensional* system as we see now.

Suppose that the least upper bound of $\operatorname{supp}\phi$ is $T > 0$. Then we easily see that

$$\phi = \delta_T * (\delta_{-T} * \phi) = (\delta_{-T})^{-1} * (\delta_{-T} * \phi),$$

where δ_a is the Dirac delta distribution and $(\delta_a)^{-1}$ denotes its inverse with respect to convolution. This means that the impulse response ϕ is *pseudorational* in the sense of [9]. Hence it can be realized as follows:

Let $q := \delta_{-T}$ and $p := \delta_{-T} * \phi$. We first compute the state space X^q associated to the "denominator" q. According to [9], we have

$$\begin{aligned} X^q &= \{x \in L^2_{loc}[0,\infty) : (\delta_{-1} * x)|_{[0,\infty)} = 0\} \\ &\equiv L^2[0,1], \end{aligned}$$

where $L^2_{loc}[0,\infty)$ denotes the space of functions that are *locally* L^2. Denote by $x_t(\cdot)$ the state at time t belonging to the space $L^2[0,1]$, our realization takes the form

$$\frac{d}{dt}x_t(\theta) = \frac{\partial}{\partial\theta}x_t(\theta) + \phi(\theta)u(t) \tag{6}$$

$$=: Ax_t + Bu(t) \tag{7}$$

$$y(t) = x_t(0). \tag{8}$$

We must also specify the domain of A. According to [9] again, we must have

$$\begin{aligned} D(A) &:= \{x \in X^q : \frac{dx}{d\theta} \in X^q\} \\ &= \{x : dx/d\theta \in L^2_{loc}[0,\infty), \operatorname{supp}(dx/d\theta) \subset [0,T]\} \\ &= \{x : dx/d\theta \in L^2[0,T] \text{ and } x(T) = 0\}. \end{aligned} \tag{9}$$

But this realization is not very convenient for deriving a formula for the solution of our problem. Following [2], we can give an approximation of (6), (8) as follows:

The state $x \in L^2[0,T]$ and the "B" element above can be approximated by piecewise constant step functions as

$$x = \sum_{i=1}^{N} x_i \chi_{[(i-1)\tau, i\tau)}(\theta)$$

where x_i denotes the averaging value $x((i-1)\tau)$ with $\tau = T/N$, and $\chi_{[(i-1)\tau,i\tau)}$ is the characteristic function of the interval $[(i-1)\tau, i\tau)$. Similarly for ϕ. Taking the forward difference approximation for $A = \partial/\partial\theta$, we obtain the following approximation for (6) and (8):

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \frac{1}{\tau}\begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & & 0 \\ & & \ddots & & 0 \\ & & \ddots & 1 & -1 \\ 0 & 0 & \cdots & & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} + \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{bmatrix} u(t) \tag{10}$$

$$y(t) = x_1. \tag{11}$$

Note that in (10) the last row should correspond to the difference $(x((N-1)\tau) - x(T))/\tau$, which is equal to x_N/τ by $x(T) = 0$. The condition $x(T) = 0$ follows from (9).

We can invoke this approximation in the usual sampled-data design filter method as developed in, e.g., [12]. However, the resulting formula may not be so convenient for actual computation when N is large. For this purpose, it is more straightforward to resort directly to fast-sample/fast-hold approximation of the convolution operator via ϕ. We will show this formula in the next section.

V. Solution Method via Fast-sample/fast-hold approximation

The acquisition kernel ϕ is generally not realizable by a finite-dimensional LTI system, so that the standard H^∞ sampled-data theory is not directly applicable. Instead, we employ the fast-sample/fast-hold (FSFH) approximation method. This method approximates continuous-time inputs and outputs via a sampler and hold that operate in the period $h/\overline{N}$ for some positive integer $\overline{N}$. Here we let $\overline{N} = Ml_1$ where l_1 is a positive integer.

The multirate system given by Fig. 1 can be cast into a single-rate sampled-data system via lifting [10], [1], and the H^∞ control problem can be solved for the generalized plant Fig. 3, where the filter $\tilde{K}(z)$ is a linear and time-invariant, single-input/M-output system. Once the optimal filter $\tilde{K}(z)$ is obtained, one can obtain the interpolation filter $K(z)$ by

$$K(z) = \begin{bmatrix} 1 & z^{-1} & \ldots & z^{-M+1} \end{bmatrix} \tilde{K}(z^M).$$

The design procedure of this problem by the FSFH approximation for a linear and time-invariant $F(s)$ is given in

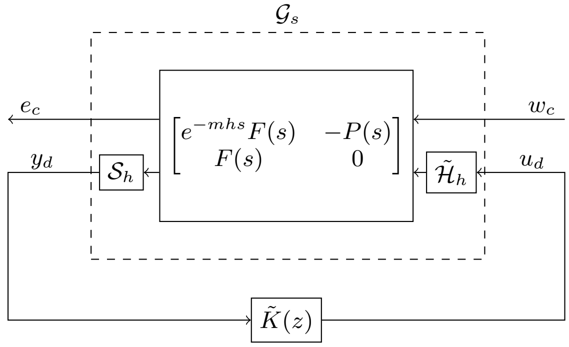


Fig. 3: Sampled-data control system

[12] and we employ the similar approach here. The FSFH approximation of the filter F in the current setting is given as follows. Let $\overline{\phi}_N$ be the averaging approximant of the impulse response ϕ. This is given by the step function approximation with step size $\tau = T/N$. For simplicity, we assume that τ is an integer multiple of $h/\overline{N}$. More details follow in the next section where we discuss the convergence of this approximation. Then,

$$F_{\overline{N}}(z) := \sum_{i=0}^{l_2-1} \overline{\phi}_N(ih/\overline{N})z^{-i}$$

where $l_2 = T\overline{N}/h$.

As in [12], the FSFH approximation of $P(s)$ is given by

$$P_{\overline{N}}(z) = \left[\begin{array}{c|cccc} A_P^{\overline{N}} & A_P^{\overline{N}-1}B_P & A_P^{\overline{N}-2}B_P & \dots & B_P \\ \hline C_P & D_P & 0 & \dots & 0 \\ C_P A_P & C_P B_P & D_P & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ C_P A_P^{\overline{N}-1} & C_P A_P^{\overline{N}-2}B_P & C_P A_P^{\overline{N}-3}B_P & \dots & D_P \end{array}\right]$$

where $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is the packed notation for continuous-time transfer function $D + C(sI - A)^{-1}B$, and

$$P(s) = \left[\begin{array}{c|c} A_{P_c} & B_{P_c} \\ \hline C_{P_c} & D_{P_c} \end{array}\right],$$

$$A_P = e^{A_{P_c}h/\overline{N}},\ B_P = \int_0^{h/\overline{N}} e^{A_{P_c}t}B_{P_c}\ dt.$$

The sampled-data error system T_{ew} can then be approximated by the discrete-time, linear, time-invariant system $T_{\overline{N}}(z)$ as follows:

$$T_{\overline{N}}(z) = z^{-m}F_{\overline{N}}(z) - P_{\overline{N}}(z)H\widetilde{K}(z)SF_{\overline{N}}(z)$$

where

$$H := \operatorname{diag}\{I_l\} \in \mathbb{R}^{\overline{N}\times M},\ I_l := [1, 1, \dots, 1]^T \in \mathbb{R}^{l_1},$$
$$S := [1, 0, \dots, 0]^T \in \mathbb{R}^{1\times\overline{N}}.$$

Our design problem (Problem 1) can be approximated by $\|T_{\overline{N}}\|_\infty < \gamma$. The convergence of this approximate design problem is discussed in the next section.

VI. Convergence of the Approximate Design

Be it the averaging approximation or the FSFH one, we need to guarantee that the approximate design converges to the solution to the original problem.

For simplicity, we assume $\phi(\cdot)$ is a continuous function. Let $\overline{\phi}_N$ be the averaging approximant of ϕ, that is, we take the step function

$$\sum_{k=0}^{N-1} \phi(k\tau)\chi_{[k\tau,(k+1)\tau)}$$

where $\tau = T/N$ and $\chi_{[a,b)}$ is the characteristic function of the interval $[a, b)$. (For the general case where ϕ is not continuous, we need only take the average of ϕ on each interval $[k\tau, (k+1)\tau)$.

Let $F_N(s)$ denote the Laplace transform of $\overline{\phi}_N$, i.e., $F_N := \mathcal{L}[\overline{\phi}_N]$. We start with the following lemma:

Lemma 6.1: Under the assumption, for each fixed $u \in L^2$ $\overline{\phi}_N * u$ converges uniformly to $\phi * u$ on $[0, \infty)$ as $N \to \infty$. Furthermore, this convergence is uniform in u in the unit ball of L^2.

Proof Let $u \in B_1$, the unit ball of L^2. Since ϕ and $\overline{\phi}_N$ have compact support, we have

$$\begin{aligned} &\left|\int_0^t (\overline{\phi}_N(t-\tau) - \phi(t-\tau))u(\tau)d\tau\right| \\ &\quad\le \int |\overline{\phi}_N(t-\tau) - \phi(t-\tau)| \cdot |u(\tau)|d\tau \\ &\quad\le \left\|\overline{\phi}_N - \phi\right\|_2 \|u\|_2 = \left\|\overline{\phi}_N - \phi\right\|_2. \end{aligned} \tag{12}$$

Since the estimate on the right-hand side of (12) is independent of t, the uniform convergence follows. □

We also need the following lemma which asserts that F_N converges to F in H^∞ norm.

Lemma 6.2: Under the same hypotheses, F_N converges to F in H^∞ norm as $N \to \infty$.

Proof We have

$$\begin{aligned} &\left|\int_0^T (\overline{\phi}_N - \phi(t))e^{-j\omega t}dt\right| \\ &\le \int_0^T |\overline{\phi}_N - \phi(t))|dt \\ &\le \left\{\int_0^T |\overline{\phi}_N - \phi(t))|^2 dt\right\}^{1/2} \left\{\int_0^T 1dt\right\}^{1/2} \\ &= \sqrt{T}\left\|\overline{\phi}_N - \phi\right\|_2. \end{aligned} \tag{13}$$

The last term clearly goes to zero as $N \to \infty$, independently of ω. Hence $\|F_N - F\|_\infty \to 0$. □

Theorem 6.3: Let $\overline{\phi}_N$ be the approximant of ϕ as above. For a given $\delta > 0$, take N such that

$$\sup_{0\le t<\infty} |(\overline{\phi}_N * u - \phi * u)(t)| < \delta, \tag{14}$$

and

$$\|F_N - F\|_\infty < \delta. \tag{15}$$

Let K be a filter designed for F_N such that

$$\left\|(e^{-mhs} - \mathcal{H}K\mathcal{S})F_N\right\| < \epsilon. \tag{16}$$

Then we have

$$\left\|(e^{-mhs} - \mathcal{H}K\mathcal{S})F\right\| < \epsilon + \delta + C\delta, \quad (17)$$

for some constant C depending on K.

Proof Observe that

$$\begin{aligned}
&\left\|(e^{-mhs} - \mathcal{H}K\mathcal{S})F - (e^{-mhs} - \mathcal{H}K\mathcal{S})F_N\right\|_\infty \\
&\quad = \left\|e^{-mhs}(F - F_N) - \mathcal{H}K\mathcal{S}(F - F_N)\right\|_\infty \\
&\quad \le \left\|e^{-mhs}(F - F_N)\right\|_\infty + \left\|\mathcal{H}K\mathcal{S}(F - F_N)\right\|_\infty . \quad (18)
\end{aligned}$$

Now note that $\mathcal{H}K\mathcal{S}$ gives a continuous operator from $C[0,\infty)$ (with uniform convergence topology) to L^2. Let C denote its operator norm (gain). Then it follows that

$$\begin{aligned}
\left\|(e^{-mhs} - \mathcal{H}K\mathcal{S})F\right\|_\infty &\le \left\|(e^{-mhs} - \mathcal{H}K\mathcal{S})F_N\right\|_\infty \\
&\quad + \left\|(e^{-mhs} - \mathcal{H}K\mathcal{S})F - (e^{-mhs} - \mathcal{H}K\mathcal{S})F_N\right\|_\infty \\
&\le \left\|(e^{-mhs} - \mathcal{H}K\mathcal{S})F_N\right\|_\infty \\
&\quad + \left\|(e^{-mhs} - \mathcal{H}K\mathcal{S})F - (e^{-mhs} - \mathcal{H}K\mathcal{S})F_N\right\|_\infty \\
&\qquad \le \epsilon + \delta + C\delta \quad (19)
\end{aligned}$$

by (18). Hence (17) follows. □

This theorem guarantees that a filter K designed for F_N with sufficiently large N can also work for F. This guarantees the validity of the present approximate design method.

VII. Numerical Example

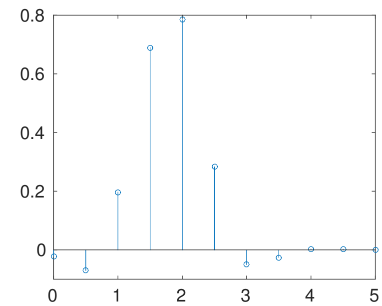
We show some numerical results. Throughout the examples, the sampling period, upsampling factor, and delay step are fixed to $h = 1$, $M = 2$, and $m = 2$, respectively.

Example 7.1: Fig. 4 shows the comparison of filter coefficients designed a) for the cardinal exponential spline $e^{-0.6t}$ ($T = 1$), and b) the usual sampled-data filter with $F(s) = 1/(s + 0.6)$. While the former exhibits more oscillatory behavior, the two filters are quite close to each other and their difference are within $0.06 \sim 0.07$ in magnitude. In view of the fact that these two filters are only suboptimal, this assures the validity of the statement at the end of Section III.

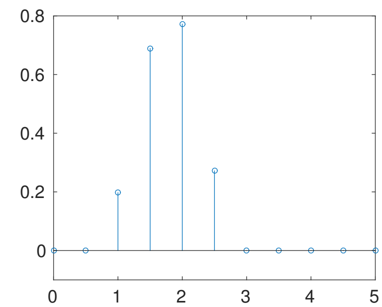
We now show some processing results for images. Consider the test image of Lena shown in Fig. 5.

This original image has a rather rough texture, and we see that by controlling the support length of the acquisition filter F, we can suitably smooth out the image. We will enlarge this image using the Haar scaling function (1) for $T = 0.5h, h, 5h, 10h$. The results are shown in Fig. 6.

For the image Fig. 6a processed with a short width ($0.5h$) of support, the resulting image shows some jaggy characters. With a wider support ($T = h$ or $T = 5h$), the results are more acceptable. They also show more smoothing skin tones. The result for the case $T = 10h$ of wider support however presents an artifact of slanted lines, which is probably due to an aliasing effect. Summarizing, in the present case, the intermediate case of $T = 5h$ shows the best compromise between resolution and the smoothing effect. What leads to the best choice is however left for future study.



(a) Filter for the exponential spline $e^{-0.6t}$



(b) Filter designed for $F(s) = 1/(s + 0.6)$

Fig. 4: Filter coefficients comparison for cardinal exponential spline with a standard sampled-data filter

Fig. 5: Test image Lena

VIII. A Relationship with Wavelet Expansion

We here give a short note on the relationship with wavelet expansion. Suppose for simplicity $\operatorname{supp}\phi \subset [0,1]$, and the sampling period $h = 1$.

Lemma 8.1: Let $\psi(t) := \phi(t-1)$. For $u \in L^2_{loc}[0,\infty)$,

$$(\phi * u)(k) = \langle \psi, \delta_{-k+1} * u \rangle, k = 1, 2, \ldots \quad (20)$$

In particular,

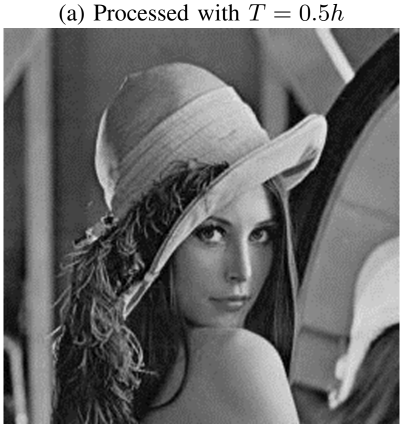
$$(\phi * u)(1) = \langle \psi, u \rangle.$$

Proof Noting $\phi(t) = 0$ outside of $[0,1]$, we have

$$\begin{aligned}
(\phi * u)(k) &= \int_0^\infty \phi(k-\tau)u(\tau)d\tau \\
&= \int_{k-1}^k \phi(k-\tau)u(\tau)d\tau \\
&= \int_{k-1}^k \psi(\tau-k+1)u(\tau)d\tau \\
&= \int_0^1 \psi(t)u(t+k-1)dt \\
&= \langle \psi, \delta_{-k+1} * u \rangle.
\end{aligned}$$

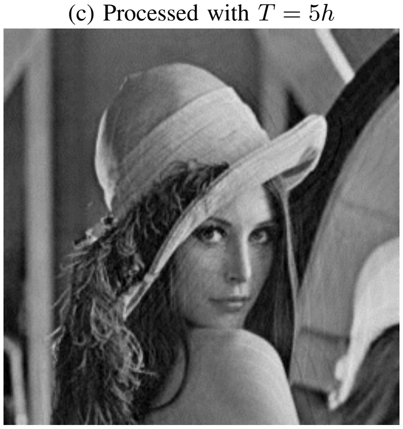
□

(a) Processed with $T = 0.5h$



(b) Processed with $T = h$

(c) Processed with $T = 5h$



(d) Processed with $T = 10h$

Fig. 6: Processing of Lena image with Haar scaling functions

This means that the sampled-values $\{(\phi * u)(k)\}_{k=1}^{\infty}$ gives the expansion coefficients of u in terms of the scaling function ψ. It thus follows that if we take

$$K(z) = I, \mathcal{H} := \psi,$$

The resulting output

$$\sum_{k=1}^{\infty}(\phi * u)(k)\psi(t-k+1) = \sum_{k=1}^{\infty}\langle\psi, \delta_{-k+1} * u\rangle\psi(t-k+1)$$

gives the expansion of u in terms of ψ.

That is, the filtered output gives the scaling function expansion by taking the mirror image of the scaling function. We can form a filterbank to go to the lower resolution expansion. This will be explored in our subsequent work.

IX. Conclusion

We have given a generalization of sampled-data signal processing theory [12], [8] to the case where the acquisition filter has compact support. This property is expected to be more adequate for signals with stronger local properties, e.g., images, rather than the usual case where more stationary nature is prevalent, for example, musical sounds. While we have seen some results in image processing, its precise advantages are yet to be seen in our future investigations.

References

[1] B. Bamieh, J. B. Pearson, B. A. Francis, and A. Tannenbaum, "A lifting technique for linear periodic systems with applications to sampled-data control," *Syst. Control Lett.*, vol. 17, no. 2, pp. 79–88, 1991.

[2] H. T. Banks and J. A. Burns, "Hereditary control problems: numerical methods based on averaging approximation," *SIAM J. Contr. Optimiz.*, vol. 16, no. 2, pp. 169–208, 1978.

[3] M. Nagahara and Y. Yamamoto, "Frequency domain min-max optimization of noise-shaping delta-sigma modulators," *IEEE Trans. Signal Processing*, vol. 60, no. 6, pp. 2828–2839, 2012.

[4] ——, "H^{∞}-optimal fractional delay filters," *IEEE Trans. Signal Processing*, vol. 61, no. 18, pp. 4473–4480, 2013.

[5] C. E. Shannon, "Communication in the presence of noise," *Proc. IRE*, vol. 38, pp. 10–21, 1949.

[6] M. Unser, "Cardinal exponential splines: Part ii—think analog, act digital," *IEEE Trans. Signal Processing*, vol. 53, no. 4, pp. 1439–1449, 2005.

[7] M. Unser and T. Blu, "Cardinal exponential splines: Part i—theory and filtering algorithms," *IEEE Trans. Signal Processing*, vol. 53, no. 4, pp. 1425–1438, 2005.

[8] K. Yamamoto, M. Nagahara, and Y. Yamamoto, "Signal reconstruction with generalized sampling," in *Proc. 56th IEEE Conf. Decision and Control*. IEEE, 2017, pp. 4003–4008.

[9] Y. Yamamoto, "Pseudo-rational input/output maps and their realizations: a fractional representation approach to infinite-dimensional systems," *SIAM J. Contr. Optimiz.*, vol. 26, pp. 1415–1430, 1988.

[10] ——, "A function space approach to sampled-data control systems and tracking problems," *IEEE Trans. Autom. Control*, vol. AC-39, no. 4, pp. 703–713, 1994.

[11] Y. Yamamoto and P. P. Khargonekar, "Frequency response of sampled-data systems," *IEEE Trans. Autom. Control*, vol. AC-41, no. 2, pp. 166–176, 1996.

[12] Y. Yamamoto, M. Nagahara, and P. P. Khargonekar, "Signal reconstruction via H^{∞} sampled-data control theory—beyond the Shannon paradigm," *IEEE Trans. Signal Processing*, vol. SP-60, no. 2, pp. 613–625, 2012.

[13] A. I. Zayed, *Advances in Shannon's Sampling Theory*. CRC Press, Boca Raton, 1993.