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A Renewed Look at Zeros of Sampled-Data Systems—From the Lifting Viewpoint

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Abstract: This paper studies the properties of the zeros of sampled-data systems. Since the basic study by Åström et al. that showed there can exist unstable zeros that have no relevance to the original continuous-time plant, there have been various studies that circumvent this effect by introducing various techniques. The present paper studies zeros of sampled-data systems from the standpoint of lifting. We will show the following: a) such unstable zeros are an artifact induced by the combination of the zero-order hold and the synchronous sampling; b) such zeros disappear when considered in the lifted plant; c) further, such an effect can be bypassed by a multirate technique. We will also review relationships with the existing techniques such as two-delay control. Analysis of some examples is given.

Keywords: Sampled-data systems, zeros, multirate systems, two-delay control

1. INTRODUCTION

In their notable paper Åström et al. (1984) have shown the striking result that a standard sample-hold approximation of a continuous-time plant, in other words, the pulse transfer function, can have unstable zeros that have seemingly no relevance to the zeros of the original continuous-time system. Unlike the case for poles, there seem to be no straightforward relationships for correspondence of zeros. Under certain conditions, this means that a sampled-held continuous-time plant can be non-minimum phase even though the original continuous-time plant is in fact minimum phase.

This naturally raised some concerns and doubts about the utility of sampled-data systems, particularly in relation to robust control and adaptive control, where unstable zeros crucially either degrade their performances or make them inapplicable.

Since then, there are a number of research activities either to analyze this effect, or to circumvent the defect induced by the sample-hold process. Blachuta (1999) gave some detailed analysis of zeros as the sampling time approaches zero. See also Méndez-Barrios et al. (2015) for a detailed perturbation analysis of zeros. Fu and Dumont (1989) explored a condition under which the sample/held pulse transfer function remains to be minimum phase if the original plant is of minimum phase. Bai and Wu (2002) studied the distribution of limiting zeros of pulse transfer functions. Hara et al. (1989) have studied the relationship between real poles and real zeros of pulse transfer functions.

In addition, there are multirate approaches introducing an extra sampling in the intersampling periods, or else, employing multirate generalized hold. Hagiwara and Araki (1988) has shown that any state feedback controller can be otained by an equivalent multirate controller. Along the same line, Mita et al. (1990) introduced what they call two-delay control which guarantees the minimum phase property for the obtained sampled system. They have shown that for almost every choice of an extra intersampling point, the resulting augmented system remains minimum phase.

However, their analysis relies heavily on matrix manipulations, and is not easy to penetrate. Particularly, it is difficult to keep track of the relationship with the continuous-time behavior of the original plant. Hence it is difficult to see why the the zero-free property is lost by sampling, because there is no explicit reference to the intersample behavior.

This paper aims at clarifying these issues from the viewpoint of modern sampled-data control theory, particularly lifting; see, e.g., Chen and Francis (1995); Bamieh et al. (1991); Yamamoto (1994). The idea is to express a given continuous-time plant $P(s)$ via lifting, and convert it to an infinite-dimensional discrete-time system. A crucial observation here is that while the zero-order hold only restricts the class of inputs, sampling of the output seriously limits the richness of the observed outputs, thereby presenting a limited capability of the continuous-time system.

Hence we first opt for proving that by converting the system to the discrete-time domain via lifting, any extra zeros that are not derived from the original continuous-time plant disappear aside from some few exceptions. Hence this shows that the extra zeros appearing by sample and hold are an artifact due to a matched sampling/hold timing. While this fact in itself seems obvious, there is a subtlety due to the infinite-dimensionality introduced by lifting, and it is indeed nontrivial to prove this fact. We will see this in the next section. We will also see that by multirate sampling/hold, one can generally bypass such a defect due to a matched sampling and hold.

The paper is organized as follows: Section 2 introduces the continuous-time lifting and formulates the problem. Section 3 gives a characterization of invariant zeros and proves that the invariant zeros of a lifted continuous-time system are derived from the zeros of the original continuous-time plant in the form $e^{\mu h}$ where μ is such a zero and h is the sampling period. Section 4 shows that by introducing multirate sampling, one can bypass the difficulty of unstable zeros, as well as discussing the relationship with other multirate techniques such as the two-delay control initiated by Mita et al. (1990). Section 5 discusses some examples and illustrates the results.

Notation and convention

Throughout the paper, we use the standard notation, e.g., $\mathbb{R}$ for the set of real numbers and $\mathbb{Z}_+$ for the set of nonnegative integers. s denotes the Laplace variable and $\hat{f}(s)$ denote the Laplace transform of a function f. The function value of f at $t \in \mathbb{R}$ is denoted as $f(t)$ with parentheses, while $g[k]$ denotes the value of g at an integer k. (This is a convenient, albeit a little sloppy, notation that allows us to distinguish the functions defined on real variables or on integers.) We say that a plant is *minimum phase* if there exist no unstable zeros; otherwise, it is said to be *non-minimum phase*.

2. PROBLEM FORMULATION

Consider a linear, time-invariant, continuous-time plant $P(s)$ described by the following state space equations:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $u \in \mathbb{R}^m$. Throughout the paper, we assume that (C, A) is observable.

Suppose that the input $u(t)$ is sampled with sampling period h. If the zero-order hold is used as in the setting in Åström et al. (1984), the input to the plant for $kh \le t < (k+1)h$ is the constant $u[k]$ and we obtain the following description at sampling instants $t = kh$, $k \in \mathbb{Z}_+$,

$$\begin{aligned} x[k+1] &= e^{Ah}x[k] + \int_0^h e^{A(h-\tau)}Bu[k]\mathrm{d}\tau \\ y[k] &= Cx[k] + Du[k]. \end{aligned} \tag{2}$$

Åström et al. (1984) have shown that, for single-input-single-output systems, i.e., $m = p = 1$, the transfer function of the discrete-time system (2) always has unstable zeros for sufficiently small sampling period when the relative degree of $P(s)$ by (1) is greater than 2.

However, the above formulation considers only the behavior of $P(s)$ at the sampling instants. In order to take the intersample behavior into account, we introduce the continuous-time lifting $\mathcal{L}$ (see, e.g., Chen and Francis (1995); Bamieh et al. (1991); Yamamoto (1994)):

$$\begin{aligned} \mathcal{L}:\ & L^2_{loc} \to \ell^2(L^2[0,h)) : x(\cdot) \mapsto \{x[k](\cdot)\}_{k=0}^{\infty}, \\ & x[k](\theta) := x(kh+\theta). \end{aligned}$$

Lifting the continuous-time plant $P(s)$ with period h, we obtain:

$$\begin{aligned} P_{Lh} : x[k+1] &= e^{Ah}x[k] + \int_0^h e^{A(h-\tau)}Bu[k](\tau)\ \mathrm{d}\tau \\ y[k](\theta) &= Ce^{A\theta}x[k] + Du[k](\theta) \\ &\quad + \int_0^\theta Ce^{A(\theta-\tau)}Bu[k](\tau)\ \mathrm{d}\tau. \end{aligned} \tag{3}$$

In the following section, we characterize the zeros in the lifted system P_{Lh} and show that there is a simple relationship to the corresponding zeros in the continuous-time plant $P(s)$.

3. ZEROS AND THE ASSOCIATED ZERO DIRECTIONS

In this section, we will show that the extra zeros of (2) introduced by the combination of sample and hold actions can disappear once we lift such a plant. At first, this might seem somewhat trivial, for one may note that the lifted system virtually represents the original continuous-time system. However, there is a subtle point, due to the infinite-dimensionality introduced by lifting. We will discuss this later.

Consider the linear discrete-time system Σ:

$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k] \\ y[k] &= Cx[k] + Du[k] \end{aligned} \tag{4}$$

where the state x, input u and output y belong, respectively, to Banach spaces X, U and Y. We do not assume finite-dimensionality of these spaces.

Definition 1. Let Σ be as above given by (4). Assume that Σ is observable. A complex number λ is an *invariant zero* if there exist $x \in X$ and $u \in U$ such that

$$\begin{bmatrix} \lambda I - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = 0 \tag{5}$$

for some $[x^T\ u^T]^T \neq 0$.

Remark 2. If either $D = 0$ or if U is finite-dimensional and B is of full column rank, an eigenvalue λ of A cannot be a zero according to this definition. If it were, the corresponding mode would become unobservable, contradicting the observability assumption. To see this, suppose λ is a zero with associated zero direction $[\tilde{x}^T, \tilde{u}^T]^T$ where $\tilde{x}$ is a nonzero eigenvector of A corresponding to λ. Then we have $B\tilde{u} = 0$, and by the linear independence of the columns of B, $\tilde{u}$ must be zero. This implies that $C\tilde{x} = 0$ and by observability this is impossible. (Note that this argument does not work when U is infinite-dimensional, e.g., the plant derived from lifting.) The case for $D = 0$ is similar.

Let us start with the following abstract characterization of zeros.

Theorem 3. Let Σ be a discrete-time system described by (4). Suppose that the spectrum $\sigma(A)$ of A consists only of eigenvalues, i.e., $\lambda \notin \sigma(A)$ means that λ belongs to the *resolvent set. Then $\lambda \notin \sigma(A)$ is an invariant zero of Σ if and only if there exists a nonzero pair $(x_0, u_0) \in X \times U$ such that the output $y[k] \equiv 0$ for every $k \in \mathbb{Z}_+$ with initial state x_0 and input $u[k] = \lambda^k u_0$.*

Proof. First, suppose λ is an invariant zero of Σ. This means

$$\begin{aligned}(\lambda I - A)\, x_0 - Bu_0 &= 0 \\ Cx_0 + Du_0 &= 0\end{aligned} \tag{6}$$

for some $x_0 \in X$ and $u_0 \in U$, not simultaneously zero. Observe that, from (4) and (6), we have

$$\begin{aligned}x[1] &= Ax_0 + Bu_0 = \lambda x_0 \\ x[2] &= Ax[1] + Bu[1] = A\lambda x_0 + B\lambda u_0 = \lambda^2 x_0\end{aligned}$$

and inductively

$$x[k] = \lambda^k x_0, \quad \forall k \in \mathbb{Z}_+.$$

This yields

$$y[k] = Cx[k] + Du[k] = \lambda^k (Cx_0 + Du_0) = 0, \quad \forall k \in \mathbb{Z}_+.$$

Conversely, suppose $y[k] \equiv 0$ with $x_0 \in X$ and $u[k] = \lambda^k u_0$. Then

$$y[0] = Cx_0 + Du_0 = 0. \tag{7}$$

Now $y \equiv 0$ means

$$C(zI - A)^{-1} x_0 + \frac{Du_0}{z - \lambda} + C(zI - A)^{-1} B \frac{u_0}{z - \lambda} \equiv 0. \tag{8}$$

Put $w := (\lambda I - A)x_0$. Since $\lambda \notin \sigma(A)$, the resolvent $(\lambda I - A)^{-1}$ exists and is continuous. Then $x_0 = (\lambda I - A)^{-1} w$. This implies

$$\begin{aligned}C(zI - A)^{-1} x_0 &= C(zI - A)^{-1} (\lambda I - A)^{-1} w \\ &= \frac{C}{\lambda - z} \left\{ (zI - A)^{-1} - (\lambda I - A)^{-1} \right\} w\end{aligned}$$

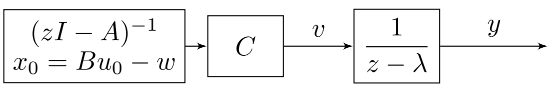
by the resolvent identity:
$(\lambda I - A)^{-1} - (\mu I - A)^{-1} = (\mu - \lambda)(\lambda I - A)^{-1} (\mu I - A)^{-1}$ (Yosida (1978)). Hence (8) becomes

$$\begin{aligned}&\frac{C}{z - \lambda} (\lambda I - A)^{-1} w + \frac{Du_0}{z - \lambda} \\ &- \frac{C}{z - \lambda} (zI - A)^{-1} w + \frac{C}{z - \lambda} (zI - A)^{-1} Bu_0 \equiv 0.\end{aligned}$$

This yields

$$\frac{C}{z - \lambda} (zI - A)^{-1} (Bu_0 - w) \equiv 0. \tag{9}$$

Consider the block diagram with 0 initial state in the block $1/(z - \lambda)$ as follows:



Identity (9) implies $y \equiv 0$, and hence $v \equiv 0$. Since (C, A) is observable, this implies

$$Bu_0 - w = 0,$$

which implies

$$(\lambda I - A)x_0 - Bu_0 = 0.$$

This completes the proof. □

In the theorem above, we have excluded eigenvalues from our consideration. What happens if $\lambda \in \sigma(A)$? We can in fact show that (5) is still satisfied for some x_0 and u_0, but λ is not necessarily a zero, i.e., the pair x_0 and u_0 may be simultaneously zero.

Lemma 4. Assume the same hypothesis on the system (A, B, C, D) in Theorem 3, and let λ be an eigenvalue of A with eigenvector x_0. If λ is an invariant zero of (A, B, C, D) such that (5) holds for (x_0, u_0), then the corresponding output for x_0 and $u[k] = \lambda^k u_0$ is identically zero. Conversely, if the output for the initial state x_0 and input $u[k] = \lambda^k u_0$ with $u_0 \neq 0$ is identically zero, then λ is an invariant zero.

Proof. The proof for the first half is the same as that for Theorem 3.

Conversely, suppose that the output $y[k]$ against the input $u[k] = \lambda^k u_0$ and initial state x_0 is identically zero. We first note from (7) that $y[0] = Cx_0 + Du_0 = 0$. It follows that

$$y[n] = CA^n x_0 + \lambda^n Du_0 + \sum_{k=0}^{n} CA^{n-k} B \lambda^k u_0 = 0, \quad \forall n.$$

Note here that λ cannot be zero. For if it were, $CA^n x_0 = 0$ for all $n \geq 0$. By the observability hypothesis, this would imply $x_0 = 0$, contradicting that x_0 be an eignevector. Since

$$CA^k x_0 + \lambda^k Du_0 = \lambda^k (Cx_0 + Du_0) = 0, \quad k = 0, 1, 2, \ldots,$$

we have

$$\sum_{k=0}^{n} CA^{n-k} B \lambda^k u_0 = 0, \quad \forall n.$$

Since $\lambda \neq 0$, it easily follows that

$$CA^n Bu_0 = 0, \quad \forall n.$$

By the observability of (C, A) we have $Bu_0 = 0$. Hence (5) is satisfied with nonzero u_0. □

Remark 5. The condition $u_0 \neq 0$ cannot be dispensed with. If u_0 were zero, it would imply $Du_0 = 0$, and hence $Cx_0 = 0$ as well. This would readily imply that $CA^k x_0 = 0$ for every $k \geq 0$, yielding $x_0 = 0$ by observability. But this contradicts x_0 being an eigenvector. In this case λ is not an invariant zero.

Remark 6. We note that the same proof applies to the continuous-time case with straightforward modifications.

We now prove the main assertion of this paper: *Let P be a continuous-time plant, and P_{Lh} its lifted system. The zeros of P_{Lh} are derived from zeros μ of P as $\lambda = e^{\mu h}$ except* $\lambda = 0$. This might seem straightforward in view of Theorem 3, but there is a subtle point. According to Definition 1, λ is an invariant zero if and only if (5) holds for some $x_0 \in X$ and $u_0 \in U$, not simultaneously zero. In the present case, $U = L^2[0, h)$ and u_0 can be any function in $L^2[0, h)$. On the other hand, for a zero in the continuous-time case, U is finite-dimensional, and the corresponding input must be of the form $e^{\mu t} v_0$ for some finite-dimensional vector v_0. However, as noted above, for the lifted case the initial input vector $u_0(\theta)$ is not necessarily confined to functions of form $e^{\mu t} v_0$. This means that for an invariant zero of the lifted plant, we must show that the initial function $u_0(\cdot) \in L^2[0, h)$ must be of the form $u_0(\theta) = e^{\mu \theta} v_0$ for some $v_0 \in \mathbb{R}^m$ and $\mu \in \mathbb{C}$.

The next theorem indeed guarantees this. For simplicity of the argument, we confine ourselves to the single-input-single-output case, i.e., $m = p = 1$. The general case is similar, with suitable assumptions on non-degeneracy of input and output spaces.

Theorem 7. Let P be the plant (1)*, and P_{Lh} its lifting* (3)*. A complex number $\lambda \neq 0$ is an invariant zero of the lifted system P_{Lh} if and only if $\lambda = e^{\mu h}$ for some invariant zero μ of the continuous-time plant P.*

Proof. Let μ be an invariant zero of P. Then,

$$(\mu I - A)\,x_0 - Bu_0 = 0,$$
$$Cx_0 + Du_0 = 0.$$

With x_0 and $u(t) = e^{\mu t}u_0$, the output $y(t) \equiv 0$, $t \geq 0$ by Theorem 3 (applied to the continuous-time case; Remark 10). This also implies for P_{Lh} with x_0 and

$$u[k](\theta) := \exp(\mu(kh+\theta)) = e^{k\mu h}e^{\mu\theta}u_0,$$

the output $y[k](\theta) \equiv 0$. Again by Theorem 3, $e^{\mu h}$ is a zero of P_{Lh}.

Conversely, let $\lambda \neq 0$ be a zero of the lifted system P_{Lh}. Then there exists μ such that $\lambda = e^{\mu h}$. Also, suppose

$$\begin{cases} (e^{\mu h}I - e^{Ah})x_0 - \displaystyle\int_0^h e^{A(h-\tau)}Bu_0(\tau)\,\mathrm{d}\tau & = 0 \\ Ce^{At}x_0 + Du_0(t) + \displaystyle\int_0^t Ce^{A(t-\tau)}Bu_o(\tau)\,\mathrm{d}\tau & = 0. \end{cases}$$

Define

$$u[k](\theta) := e^{\mu kh}u_0(\theta).$$

Then by the necessity part of the previous theorem,

$$y[k](\theta) \equiv 0 \quad \forall k, \theta.$$

This means that $y(t) \equiv 0$ with initial state x_0 and input $u[k](\theta) = e^{\mu kh}u_0(\theta)$.

To complete the proof, we need to show $u_0(\theta) = e^{\mu\theta}v_0$ so that $u[k](\theta) = e^{\mu(kh+\theta)}v_0$, i.e., $e^{\mu t}v_0$ for $t = kh + \theta$. Now note that

$$y(t) = Du(t) + Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau.$$

Suppose, for the moment, $D = 0$. Then

$$Ce^{At}x_0 + (g * u)(t) \equiv 0$$

where

$$g(t) = Ce^{At}B$$

This implies that $g*u$ must cancel the exponential $Ce^{At}x_0$. Hence $\hat{u}(s) = \hat{g}^{-1}(s)C(sI-A)^{-1}x_0$ with rational transfer function $\hat{g}(s)$. Hence $\hat{u}(s)$ itself must be rational. Hence $u(t)$ must be an exponential polynomial, i.e., of the form $\sum p_i(t)e^{\mu_i t}$. This $u(t)$ must also satisfy $u(t+kh) = e^{\mu kh}u(t)$ for every $k \geq 0$. Hence $u(t)$ must be of form $u(t) = e^{\mu t}v_0$. When $D \neq 0$, take $\tilde{g} := \delta D + g$. Then $\hat{\tilde{g}}(s)$ is still rational, and the conclusion remains valid. □

Remark 8. Mirkin et al. (1999) have shown the result for the special case where the plant (1) or its lifting (3) is left-invertible. This amounts to requiring that D be left invertible, i.e., relative degree zero for the SISO case, and it does not apply to the general situation.

4. MULTIRATE SAMPLING/HOLD

We have seen that extra zeros of (2) that are not derived as $e^{\mu h}$ for zeros μ of (1) are artifact of matching sample and hold actions. This artifact disappears when we consider those zeros for lifted systems. This is due to the observation of the continuous-time output in the lifted system. However, in the sampled-data context, it is not possible to invoke the continuous-time observation. To remedy this, various multirate techniques have been studied, e.g., Mita et al. (1990). We will examine such techniques from the present viewpoint, and show that one can make the extra zeros in the sampled system (2) disappear by multirate sampling/hold. More specifically, when (2) has a zero λ, the next theorem shows that the output is nonzero for almost every t in the intersampling periods against the input $u[k] = \lambda^k u_0$. This explains the existing technique such as the two-delay output control introduced in Mita et al. (1990) where another sampling point is taken between the original sampling points to discard the zeros.

Theorem 9. Suppose that system (1) does not have an invariant zero at 0*. Let λ be an invariant zero of the system* (2)*. Then for the input $u[k] = \lambda^k u_0$, $k = 0, 1, 2, \ldots$, $u_0 \in U$, for almost every θ the intersample output $y[k](\theta)$, $0 \leq \theta \leq h$ takes a nonzero value for all k.*

Proof. Let us first describe the output taking the intersample behavior into account:

$$y[k](\theta) = Ce^{A\theta}x[k] + \lambda^k(D + \int_0^\theta Ce^{A\tau}B\mathrm{d}\tau)u_0, \quad (10)$$

where

$$x[k] = e^{Akh}x_0 + \sum_{i=0}^{k-1} e^{A(k-i-1)h}\mathbf{B}\lambda^i u_0,$$

$$\mathbf{B}u := \int_0^h e^{A\tau}Bd\tau$$

Then $y[k](\theta)$ is clearly an analytic function in θ for $\theta \in [0, h]$ for every $k \in \mathbb{Z}_+$. It cannot be identically zero. For if it were, (1) would have an invariant zero μ with $\lambda = e^{\mu h}$, by Theorem 7. Since $u[k]$ is constant on $[0, h]$, this means that $\mu = 0$ which contradicts the hypothesis. Then it can have at most finitely many zeros between $[0, h]$. Since this holds for every $k \in \mathbb{Z}_+$, there exist at most countably many θ's where $y[k](\theta)$ vanish for all k. Then for almost every fixed θ, $y[k](\theta)$ must be nonzero for all k. □

Mita et al. (1990) proposed what they call two-delay output control using the intersample output $y(kh + ih)$ by taking a suitable $0 < i < 1$ in addition to the original output $y(kh)$, $k \in \mathbb{Z}_+$. They have shown that for almost every i, this augmented system no longer has zeros which appear in the original sample/hold system, by computing the rank of the augmented system matrix. The theorem above explains the idea why this works and these zeros disappear.

Yamamoto et al. (2016) also studied a multirate technique from a different angle. It gave some characterization of the zeros of the closed-loop system consisting of a continuous-time plant with an upsampled discrete-time controller. The controller does not utilize an intersample extra sampling, but can produce a multirate output which becomes a control input to the plant. It is shown that by suitably invoking a proper H^∞ sampled-data control, the closed-loop system can have the capability of tracking high frequency components beyond the Nyquist frequency in spite of the limited observation below that frequency. A proper use of upsampling and robust control design makes this possible. The precise relationship with the results here is a theme for future study.

We also note here that while Freudenberg et al. (1997) studied robustness issues of shifting zeros via a generalized hold, it is mainly for the techniques attempting to remove/shift zeros of the continuous-time plant, and not for the removal of extra discrete-time zeros arising as an artifact of sample and hold, as discussed here or in two-delay control.

Remark 10. According to Zeng and Allgower (2016), a more elaborate analysis for the maximum number of zeros in the sampling interval $[0, h)$, which yields more precise estimate on the number of upsampling needed to make the resulting system free from the extra zeros.

5. EXAMPLES

Example 1: Consider the first-order plant:

$$P(s) = \frac{s-1}{s+1}.$$

This has a minimal realization

$$\begin{cases} \dot{x}(t) &= -x(t) + 2u(t) \\ y(t) &= -x(t) + u(t). \end{cases}$$

Solving (5) for this system, we find that P has a zero at 1 with an associated zero directional vector $[x_0\ u_0]^T = [1\ 1]^T$. Indeed, with initial state $x_0 = 1$ and $u(t) = e^t$, we can immediately see that $x(t) = e^t$ and $y(t) \equiv 0$ for $t \geq 0$.

Now let us consider the lifted system P_{Lh}:

$$\begin{cases} x[k+1] = e^{-h}x[k] + \displaystyle\int_0^h 2e^{-(h-\tau)}u[k](\tau)\mathrm{d}\tau \\ y[k](\theta) = -e^{-\theta}x[k] + u[k](\theta) - \displaystyle\int_0^\theta 2e^{-(\theta-\tau)}u[k](\tau)\mathrm{d}\tau. \end{cases} \tag{11}$$

If P_{Lh} has a zero, then there exist some initial state $x_0 \in X$ and initial input function $u_0(\cdot) \in L^2[0, h)$ such that $y[k](\theta) \equiv 0$ for all k, θ, or equivalently

$$y(t) = -e^{-t}x_0 + u(t) - \int_0^t 2e^{-(t-\tau)}u(\tau)\mathrm{d}\tau \equiv 0, \quad t \geq 0$$

where $u(kh + \theta) = u[k](\theta)$. Taking the Laplace transform of the above equation yields

$$\begin{aligned} &-\frac{1}{s+1}x_0 + \left(1 - \frac{2}{s+1}\right)\hat{u}(s) \equiv 0 \\ \iff\ &\hat{u}(s) = \frac{1}{s-1}x_0 \\ \iff\ &u(t) = e^t x_0. \end{aligned}$$

This implies that P_{Lh} has a zero only at e^h and $y[k](\theta) \equiv 0$ for all k, θ with initial state $x_0 = 1$ and $u[k](\theta) = e^{kh}e^{\theta}$, i.e., the lifted input of $u(t) = e^t$.

Indeed, from (11), we see

$$\begin{aligned} x[k+1] &= e^{-h}x[k] + \int_0^h 2e^{-(h-\tau)}e^{\tau}\ \mathrm{d}\tau \\ &= e^{-h}x[k] + (e^{(k+1)h} - e^{(k-1)h}). \end{aligned}$$

Then

$$\begin{aligned} x[1] &= e^{-h} + (e^h - e^{-h}) = e^h, \\ x[2] &= e^{-h}e^h + e^{2h} - 1 = e^{2h}, \end{aligned}$$

and inductively, $x[k] = e^{kh}$. Hence,

$$\begin{aligned} y[k](\theta) &= -e^{-\theta}e^{kh} + e^{kh}e^{\theta} - 2\int_0^\theta e^{-(\theta-\tau)}e^{kh}e^{\tau}\ \mathrm{d}\tau \\ &= -e^{-\theta}e^{kh} + e^{kh}e^{\theta} - e^{-\theta}e^{kh}(e^{2\theta} - 1) \\ &\equiv 0 \quad \forall k, \theta. \end{aligned}$$

as expected.

Example 2: Take the following third-order plant with relative degree 3 considered also by Mita et al. (1990):

$$P(s) = \frac{1}{(s+1)(s+2)(s+3)}.$$

Discretizing $P(s)$ using the zero-order hold gives the following pulse transfer function

$$H(z) = \frac{p(z)}{6(z - e^{-h})(z - e^{-2h})(z - e^{-3h})} \tag{12}$$

where $p(z) = (1 - 3e^{-h} + 3e^{-2h} - e^{-3h})z^2 + (2e^{-h} - 4e^{-2h} + 4e^{-4h} - 2e^{-5h})z + (e^{-3h} - 3e^{-4h} + 3e^{-5h} - e^{-6h})$.

One of the two zeros of $H(z)$ becomes unstable at $h \approx 0.96$ and approaches -3.732 as h tends to 0 (Fig. 1) as claimed by Åström et al. (1984).

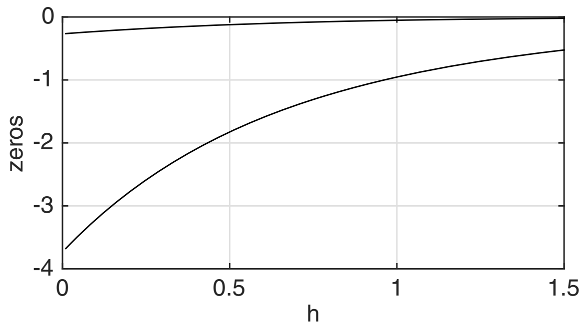


Fig. 1. Zeros of the pulse transfer function (12).

Let us now show that this unstable zero disappears when taking an intersample output as another sampling point.

The plant $P(s)$ has a realization

$$\begin{cases} \dot{x}(t) &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t) \\ &=: Ax(t) + Bu(t) \\ y(t) &= \begin{bmatrix} \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix} x(t) =: Cx(t). \end{cases}$$

Set the sampling period h to be 0.5. Then the corresponding sampled system (2) has a unstable zero $\lambda = -1.867$ with an associated zero direction:

$$\begin{bmatrix} x_0 \\ -- \\ u_0 \end{bmatrix} = \begin{bmatrix} \dfrac{1 - e^{-h}}{\lambda - e^{-h}} \\ \dfrac{1 - e^{-2h}}{2(\lambda - e^{-2h})} \\ \dfrac{1 - e^{-3h}}{3(\lambda - e^{-3h})} \\ ------ \\ 1 \end{bmatrix}.$$

Figure 2 shows the plant output

$$y[k](\theta) = Ce^{A\theta}x[k] + \int_0^\theta Ce^{A\tau}Bu[k]\mathrm{d}\tau$$

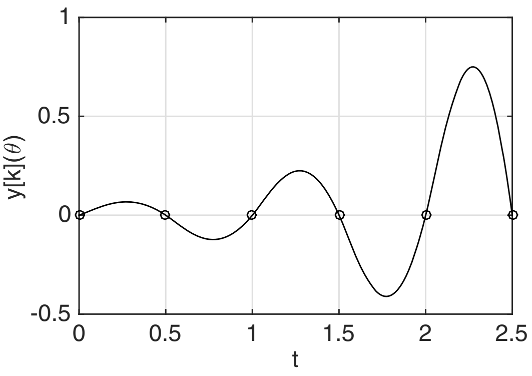


Fig. 2. Output $y[k](\theta)$ over $t = kh + \theta$.

for $0 \le \theta < h$ with the initial state x_0 and the input $u[k] = \lambda^k u_0$. We can observe that the output takes nonzero values except at sampling points $t = kh = 0.5k$. Now let us take the middle point between the original sampling points as another sampling point. Then the augmented system becomes

$$x[k+1] = e^{Ah}x[k] + \int_0^h e^{A(h-\tau)}Bu[k]\mathrm{d}\tau$$

$$y[k] = \begin{bmatrix} y(kh) \\ y(kh+h/2) \end{bmatrix} = \begin{bmatrix} C \\ Ce^{Ah/2} \end{bmatrix} x[k] + \begin{bmatrix} 0 \\ C\int_0^{h/2} e^{A\tau}B\mathrm{d}\tau \end{bmatrix} u[k].$$

Its system matrix

$$\begin{bmatrix} \lambda I - e^{Ah} & -\int_0^h e^{Ah}B\mathrm{d}\tau \\ C & 0 \\ Ce^{Ah/2} & C\int_0^{h/2} e^{A\tau}B\mathrm{d}\tau \end{bmatrix}$$

is now full rank.

6. CONCLUSION

We have shown that under some conditions the extraneous zeros introduced by sampling and hold disappear in the lifted plant. There is a clear correspondence between the zeros of the original continuous-time plant and those of the lifted discrete-time system. This also led to a clear understanding of the multirate techniques such as two-delay control.

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