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Tracking of Signals Beyond the Nyquist Frequency

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***Abstract*— This paper studies the problem of tracking or disturbance rejection for sampled-data control systems, where the tracking signal can have frequency components higher than the Nyquist frequency. In view of the well-known sampling theorem, one recognizes that any high-frequency components may be detected only as an alias in the low base band, and hence it is impossible to recover or detect such frequency components. This paper examines the basic underlying assumption, and shows that this assumption depends crucially on the underlying analog model. We show that it is indeed possible to recover such high-frequency signals, and also that, by introducing multirate signal processing techniques, it is possible to track or reject such frequency components. Detailed analysis of multirate closed-loop systems and zeros and poles are given. It is shown via examples that tracking of high-frequency signals beyond the Nyquist frequency can be achieved with satisfactory accuracy.**

I. Introduction

This note raises and studies the following question: Suppose we are given a sampled-data control system with the objective of tracking a reference signal or rejecting a disturbance having frequency components higher than the Nyquist frequency, i.e., half the sampling frequency.

Can we track such a high-frequency tracking signal by suitably designing a digital controller?

In view of the well-known sampling theorem, this may appear to be asking something impossible. The sampling theorem (e.g, [11], [18]) tells us that if the target signal is beyond the so-called Nyquist frequency, it may appear only as an aliased signal in the base band that is below the Nyquist frequency, and cannot be recovered from such knowledge. The formula given by the sampling theorem recovers only those residing in the base band that is below the Nyquist frequency.

On the other hand, such a problem is not at all an artificial one, given various practical constraints in constructing a control system. Consider, for example, frequency regulation in electric power supply. Today many power supply systems (for example, battery backup systems) go through an inverter, and the produced power needs to be regulated to a standard power frequency, say 60 [Hz]. Due to various limitations, we cannot necessarily take advantage of the capacity of microprocessors or sensors, so that the sampling rate in sensing the output power signals may be limited. It would be very desirable to accomplish such a tracking objective without demanding much on sensing and sampling. Such an objective is also quite common in position control of hard disk drives. Due to physical limitations, the sampling rate cannot be taken fast enough to cover resonance disturbances. In such a case, it is also desired to reject such disturbances based on a low sampling rate; see, e.g., [2], [19] for details.

A close examination of the sampling theorem reveals that the signal recovery limitation below the Nyquist frequency arises from the very basic assumption on the analog signal model. Namely, in the sampling theorem, one assumes that the original analog signal is perfectly band limited below the Nyquist frequency. It does not say anything about the situation where this basic assumption does not hold.

In fact, the authors have initiated a new signal processing approach based on H^∞ sampled-data control, assuming an analog signal generator model that is not fully band limited below the Nyquist frequency [16]. The method has been successful in sound processing and other applications [8], [9]; in particular, sound-processing chips produced by the SANYO corporation has proven to be successful to the extent of producing over 60 million chips [14], [15].

This success suggests that there is some room for improvement in controlling signals beyond the Nyquist frequency which was considered to be an absolute limit of signal processing in the past. The same idea was applied to repetitive control to allow for high frequency tracking [10]. However, these are basically open-loop type signal shaping problem, and not directly applicable to feedback control.

The objective of the present paper is to show that it is indeed possible to achieve the above tracking objective provided that

- we have proper a priori knowledge about the plant and the tracking signal, and
- we can produce intersample input signals via upsampling.

Let us briefly review pertinent facts on sampled-data control. Since the introduction of lifting [4], [12], what modern sampled-data control theory has established is that one can control and optimize the intersample behavior with a discrete-time controller; details may be found, for example, in [3], [5], [13]. Such developments usually assume that the sampling occurs at the same timing both at sensing and control. On the other hand, in the signal processing literature, multirate processing, utilizing up- and down-samplers, are

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known to be quite effective [17]. In particular, it allows more elaborate signal manipulation in the intersampling periods. The combination of this multirate processing and H^∞ control is fully used in [16]. The advantage of introducing multirate processing, particularly upsampling, is that it gives more freedom in handling and reconstructing intersample signals.

Multirate sampled-data control has had some history in the control literature; see, e.g., [1], [6], [7]. However, they use full information obtained by multirate sampling, and the focus is on extending the capability of control. In contrast, here once output is sampled, we do not perform further so the basic sampling period is fixed. Upsampling is performed only on the side of computing the control signals.

In this framework, we propose a new design method for tracking sinusoids by allowing upsampled input signals while maintaining the advantage of analog signal generating model.

II. Problem Formulation

Consider the sampled-data system depicted in Fig. 1.

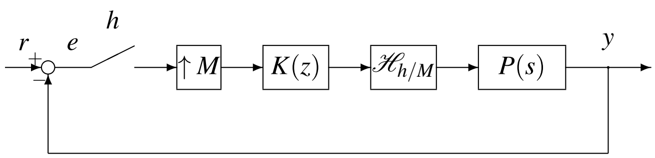

Fig. 1. Sampled feedback system

$P(s)$ is a linear, time-invariant, continuous-time plant, and $K(z)$ is a linear, time-invariant, discrete-time controller. The error e is sampled with sampling period h, and after sampled, it is upsampled by factor M to allow for a faster control processing. The action of $\uparrow M$ is given as follows:

$$(\uparrow M)(e)[kh+\ell] = \begin{cases} e[kh] & \text{if } \ell = 0 \\ 0 & \ell = h/M, \dots (M-1)h/M. \end{cases} \quad (1)$$

$\mathscr{H}_{h/M}$ is the zero-order hold that holds the output as constant for the period of h/M.

We consider the following problem:

Problem 1: *In the block diagram Fig. 1, consider the reference input* $\sin\omega t$ *where* ω *is greater than the Nyquist frequency* π/h. *Find a discrete-time controller* $K(z)$ *such that the output* $y(t)$ *nearly tracks the reference* $r(t) = \sin\omega t$ *or its delayed signal* $r(t-L) = \sin\omega(t-L)$.

We attack this problem according to the following scenario:

- Lift the system Fig. 1 as a state space model [5], [3], [12].
- Describe transmission/invariant zeros in the obtained state space model.
- Characterize the transmission zero directions, which governs the intersample behavior of the real tracking performance.

We then convert this problem to a H^∞ sampled-data control problem, and show that our objective is indeed achieved.

A. State Space Description of the Lifted Multirate System

It is necessary to describe the system in Fig. 1 as a time-invariant discrete-time system with a single sampling period h. A complication arises due to the mixture of continuous-time plant $P(s)$ and also upsampler $\uparrow M$. In order to deal with these, we need to introduce both continuous-time lifting and discrete-time lifting (blocking) for the upsampler.

Let $P(s)$ and $K(z)$ be described by the following state space equations:

$$P(s): \begin{cases} \frac{d}{dt}x_c(t) & = A_c x_c(t) + B_c u(t) \\ y(t) & = C_c x_c(t) \end{cases} \quad (2)$$

$$K(z): \begin{cases} x_d[k+1] & = A_d x_d[k] + B_d w_d[k] \\ y_d[k] & = C_d x_d[k] + D_d w_d[k]. \end{cases} \quad (3)$$

Here, and in what follows, we employ the convention that function values are specified as $f(t)$ with parentheses when t is a continuous variable, and as $g[k]$ when k takes on integer values.

In order to give a unified description of the equations above as a single discrete-time system, we introduce the continuous-time lifting [5], [4], [3], [12]:

$$\begin{aligned} \mathscr{L}: \quad & L^2_{loc}[0,\infty) \to \ell^2(L^2[0,h)) : x(\cdot) \mapsto \{x[k](\cdot)\}_{k=0}^\infty, \quad (4) \\ & x[k](\theta) := x(kh+\theta). \end{aligned}$$

Lifting the continuous-time plant $P(s)$ in the period h, we obtain

$$\tilde{\Sigma}_P: \begin{cases} x_c[k+1] & = e^{A_c h}x_c[k] + \int_0^h e^{A_c(h-\tau)}B_c u[k](\tau)d\tau \\ y_c[k](\theta) & = C_c e^{A_c\theta}x_c[k] + \int_0^\theta C_c e^{A_c(\theta-\tau)}B_c u[k](\tau)d\tau. \end{cases} \quad (5)$$

We also need to perform discrete-time lifting (i.e., blocking) for the discrete-time controller combined with upsampler (1), because upsampler makes it time-varying in the activating timing. To remedy this, we must lift it with period h.

Proposition 2.1: When lifted with period h, the discrete-time controller $K(z)$ is expressible as

$$\begin{aligned} \tilde{\Sigma}_K : x_d[k+1] &:= x_d(kh+h) = A_d^M x_d[k] + A_d^{M-1}B_d e[k](0) \\ &=: \overline{A_d}x_d[k] + \overline{B_d}e[k](0) \\ y_d[k] &:= \begin{bmatrix} y_d(kh) \\ y_d(kh+h/M) \\ \vdots \\ y_d(kh+(M-1)h/M) \end{bmatrix} \\ &= \begin{bmatrix} C_d \\ C_d A_d \\ \vdots \\ C_d A_d^{M-1} \end{bmatrix} x_d[k] + \begin{bmatrix} D_d \\ C_d B_d \\ \vdots \\ C_d A_d^{M-2} B_d \end{bmatrix} e[k](0) \\ &=: \overline{C_d}x_d[k] + \overline{D_d}e[k](0). \end{aligned}$$

Define a generalized hold function $H(\theta)$ by

$$H(\theta) := [\chi_{[0,h/M)}(\theta), \chi_{[h/M,2h/M)}(\theta), \dots, \chi_{[(M-1)h/M,h)}(\theta)], \quad (6)$$

where $\chi_{[ih/M,(i+1)h/M)}(\theta)$, $i=0,\ldots,M-1$ denotes the characteristic function of the interval $[ih/M,(i+1)h/M)$. Then the lifted input $u[k](\theta)$ for P can be written simply as

$$u[k](\theta) = H(\theta)y_d[k]. \tag{7}$$

Proof Observe first that, between time kh and $(k+1)h$, the following formulas hold inductively, in light of the definition (1) of the upsampler $\uparrow M$:

$$\begin{cases} x_d(kh+h/M) &= A_d x_d(kh) + B_d e_d(kh) \\ x_d(kh+2h/M) &= A_d x_d(kh+h/M) = A_d^2 x_d(kh) \\ & \qquad +A_d B_d e_d(kh) \\ & \vdots \\ x_d(kh+h) &= A_d x_d(kh+(M-1)h/M) \\ & \quad = A_d^M x_d(kh) + A_d^{M-1} B_d e_d(kh). \end{cases}$$

$$\begin{cases} y_d(kh) &= C_d x_d(kh) + D_d e_d(kh) \\ y_d(kh+2h/M) &= C_d x_d(kh+h/M) \\ & \quad = C_d A_d x_d(kh) + C_d B_d e_d(kh) \\ & \vdots \\ y_d(kh+(M-1)h/M) &= C_d x_d(kh+(M-1)h/M) \\ & = C_d A_d^M x_d(kh) + C_d A_d^{M-1} B_d e_d(kh). \end{cases}$$

Stacking them up together yields the formula for $y_d[k]$, and then $u[k]$. □

Now define

$$B(\theta) := \int_0^\theta e^{A_c(\theta-\tau)} B_c H(\tau)\, \mathrm{d}\tau. \tag{8}$$

Then the lifted $\tilde{\Sigma}_K$ and $\tilde{\Sigma}_P$ are represented as

$$\begin{aligned} \tilde{\Sigma}_K : x_d[k+1] &=: \overline{A_d} x_d[k] + \overline{B_d} e[k](0) \\ y_d[k] &=: \overline{C_d} x_d[k] + \overline{D_d} e[k](0) \\ u[k](\theta) &=: H(\theta) y_d[k] \end{aligned} \tag{9}$$

$$\begin{aligned} \tilde{\Sigma}_P : x_c[k+1] &= e^{A_c h} x_c[k] + B(h) y_d[k] \\ y[k](\theta) &= C_c e^{A_c\theta} x_c[k] + C_c B(\theta) y_d[k]. \end{aligned} \tag{10}$$

It is important to note that by introducing the generalized hold $H(\cdot)$, and $\overline{A_d}, \overline{B_d}, \overline{C_d}, \overline{D_d}$, the multirate system with upsampler $\uparrow M$ can be placed in the same form as the single-rate system. This is a great advantage in deriving the closed-loop system equation in the next section.

III. Closed-loop equation

Substituting (9) and (10) into the block diagram Fig. 1, we readily obtain the following closed-loop equations (with e taken as output):

$$\begin{bmatrix} x_d[k+1] \\ x_c[k+1] \end{bmatrix} = \begin{bmatrix} \overline{A_d} & -\overline{B_d} C_c \\ B(h)\overline{C_d} & e^{A_c h} - B(h)\overline{D_d} C_c \end{bmatrix} \begin{bmatrix} x_d[k] \\ x_c[k] \end{bmatrix} + \begin{bmatrix} \overline{B_d}\delta_0 \\ B(h)\overline{D_d}\delta_0 \end{bmatrix} r[k](\theta) \tag{11}$$

where δ_0 is Dirac's delta, acting on $r[k](\theta)$ as $\delta_0 r[k](\theta) := r[k](0)$. That is, it represents the sampler at each time k.

We then have the following formula for $e[k](\theta)$:

$$\begin{aligned} e[k](\theta) &= r[k](\theta) - y[k](\theta) \\ &= r[k](\theta) - C_c e^{A_c\theta} x_c[k] - C_c B(\theta) v[k] \\ &= r[k](\theta) - C_c e^{A_c\theta} x_c[k] \\ & \quad - C_c B(\theta)(\overline{C_d} x_d[k] + \overline{D_d} e[k](0)) \\ &= r[k](\theta) - C_c e^{A_c\theta} x_c[k] \\ & \quad - C_c B(\theta)\left(\overline{C_d} x_d[k] + \overline{D_d}(r[k](0) - C_c x_c[k])\right) \\ &= \begin{bmatrix} -C_c B(\theta)\overline{C_d} & -C_c e^{A_c\theta} + C_c B(\theta)\overline{D_d} C_c \end{bmatrix} \begin{bmatrix} x_d[k] \\ x_c[k] \end{bmatrix} \\ & \quad + (I - C_c B(\theta)\overline{D_d}\delta_0) r[k](\theta). \end{aligned} \tag{12}$$

IV. Zeros and tracking

According to [12], the tracking performance of the closed-loop system (11), (12) (i.e., Fig. 1) is determined by

1) transmission zero, and
2) the corresponding zero direction, which is the initial intersample function of the tracking signal.

Hence we here characterize such zeros and corresponding directions. According to the equations (11), (12), a complex λ is an invariant zero, with a corresponding initial function $v(\cdot)$ if and only if

$$\left[\begin{array}{cc|c} \lambda I - \overline{A_d} & \overline{B_d} C_c & \overline{B_d}\delta_0 \\ -B(h)\overline{C_d} & \lambda I - e^{A_c h} + B(h)\overline{D_d} C_c & B(h)\overline{D_d}\delta_0 \\ \hline C_c B(\theta)\overline{C_d} & C_c e^{A_c\theta} - C_c B(\theta)\overline{D_d} C_c & I - C_c B(\theta)\overline{D_d}\delta_0 \end{array}\right] \begin{bmatrix} x_d \\ x_c \\ v(\theta) \end{bmatrix} = 0. \tag{13}$$

The third row yields

$$\begin{aligned} v(\theta) - C_c B(\theta)\overline{D_d} v(0) &= -C_c B(\theta)\overline{C_d} x_d - C_c e^{A_c\theta} x_c \\ & \quad + C_c B(\theta)\overline{D_d} C_c x_c. \end{aligned} \tag{14}$$

Substituting $v(0) = -C_c x_c$ into (14) gives

$$\begin{aligned} v(\theta) + C_c B(\theta)\overline{D_d} C_c x_c &= -C_c B(\theta)\overline{C_d} x_d - C_c e^{A_c\theta} x_c \\ & \quad + C_c B(\theta)\overline{D_d} C_c x_c. \end{aligned} \tag{15}$$

Hence

$$v(\theta) = -C_c B(\theta)\overline{C_d} x_d - C_c e^{A_c\theta} x_c. \tag{16}$$

Now let us make the following assumption:

Assumption A: There is no pole-zero cancellation between the lifted discrete-time controller and the continuous-time plant.

Under this assumption, and also closed-loop stability, we readily see that an unstable λ (i.e., $|\lambda| \geq 1$) is an invariant zero with associated zero direction $v(\theta)$ (i.e., satisfying (13) and (16)) if and only if

$$G_{er}(\lambda)[v] = 0,$$

where $G_{er}(z)$ denotes the transfer operator from r to e. That is, the value λ is an invariant zero if and only if it is a transmission zero from r to e, with associated initial direction $v(\theta)$.

Now from (13) and $v(0) = -C_c x$, we obtain

$$(\lambda I - \overline{A_d})x_d + \overline{B_d}C_c x_c - \overline{B_d}C_c x_c = (\lambda I - \overline{A_d})x_d = 0, \quad (17)$$

$$-B(h)\overline{C_d}x_d + (\lambda I - e^{A_c h} + B(h)\overline{D_d}C_c)x_c = -B(h)\overline{C_d}x_d + (\lambda I - e^{A_c h})x_c = 0. \quad (18)$$

Existence of a nonzero solution to (17) and (18) is a necessary and sufficient condition for λ to be a transmission zero of $G_{er}(z)$ with $v(\theta) = -C_c B(\theta)\overline{C_d}x_d - C_c e^{A_c\theta}x_c$ as an associated zero direction function. Now note that (17) and (18) can admit a solution in the following ways:

1) there exists a solution $[x_d^T, x_c^T]^T$ with nonzero x_d, or
2) a solution of type $[0, x_c^T]^T$ with nonzero x_c exists.

In the first case, x_d is an eigenvector of the discrete-time controller and λ is a pole of $K(z)$. In the second case, λ is a pole of the lifted continuous-time plant and x_c is a corresponding eigenvector. This means that, under the above Assumption A, the poles of K or lifted P always yields a transmission zero. Since we have started with the characterization of invariant zeros as above, we can reverse the argument as well, and thus have the following theorem:

Theorem 4.1: Under the assumption A, and the assumption of the closed-loop stability, the unstable poles of lifted K and P induce a transmission zero of the closed-loop transfer operator $G_{er}(z)$ and vice versa.

As a corollary, consider the case $\lambda = e^{j\omega h}$ being an eigenvalue of $\overline{A_d}$. Let x_d be the corresponding eigenvector, and then $\overline{C_d}x_d \neq 0$ (otherwise, the controller is not observable). It follows that by taking $H(\theta)$ to be $e^{j\omega\theta}$ ($0 \leq \theta \leq h$), the output of the discrete-time controller becomes $e^{jk\omega h}e^{j\omega\theta}y_d = e^{j\omega t}y_d$, where $y_d = \overline{C_d}x_d$. That is, the discrete-time controller can work as an internal model for $1/(s - j\omega)$. Taking a combination with the complex conjugate, this can work as an internal model for $\sin\omega t$, with this suitable choice of $H(\theta)$. When $\mathscr{H}_{h/M}$ is a zero-order hold, it cannot exactly produce this sinusoidal hold function, but it can still approximate such a hold function.

Remark 4.2: In fact, if $e^{j\omega h}$ is an eigenvalue of $\overline{A_d} = A_d^M$, $e^{j\omega h/M}$ is an eigenvalue of A_d. Then, by taking $\mathscr{H}_{h/M}$ to be $e^{j\omega\theta}$ for $0 \leq \theta \leq h/M$, it is seen that the output of the controller produces $e^{j\omega t}$, because at each step the output is kept multiplied by $e^{j\omega h/M}$. Since the difference between the zero-order hold and the sinusoidal hold $e^{j\omega\theta}$ is small for $0 \leq \theta \leq h/M$, the controller output can produce an approximation of $e^{j\omega t}$. This indeed occurs in the subsequent Fig. 5 in Example 6.1.

V. Design method

So far, we have only characterized the zeros and tracking conditions, and have not shown how we can handle high-frequency tracking beyond the Nyquist frequency.

Observe first that our system Fig. 1 cannot be used as it is for a design block diagram for H^∞ sampled-data control. Sampling is not a bounded operator on L^2, and hence system Fig. 1 as it is cannot be used as a design model. To remedy this, we place a strictly proper anti-alias filter $F(s)$ in front of the adding point of the error. In other words, the reference signal is pre-filtered by $F(s)$. This is advantageous in that we can control frequency weighting in the input reference signals. We here emphasize that *unlike the usual case of $F(s)$ where we put more emphasis on the low-frequency range, we attempt to place more emphasis on the frequency that we wish to track.* This is a rather non-standard idea different from usual sampled-data control, and the objective here is to show that *this does indeed work for the tracking purpose of this paper*.

Another attempt we devise here is that we allow some delays in tracking. That is, instead of taking the error $e(t) = r(t) - y(t)$, we try to minimize the delayed error $\tilde{e}(t) := r(t - L) - y(t)$ for some positive L as stated in Problem 1. This is under analogy from the case of delayed signal construction in [16] where we can achieve better performance by allowing certain delays in signal reconstruction. While its advantage in performance is yet to be investigated in the future, this will give us at least more freedom in controller design.

Incorporating these changes into Fig. 1, we obtain the following generalized plant Fig. 2 for design. Here L is a design parameter; we usually take L to be an integer multiple of h, with some small number such as 4 – 10.

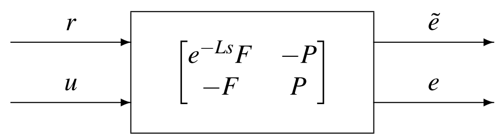

Fig. 2. Generalized plant

VI. Example, design, and simulation

To show the effectiveness of the present framework, particularly its tracking capability beyond the Nyquist frequency, consider the following simple example:

Example 6.1: Consider the plant

$$P(s) := \frac{1}{s^2 + 2s + 1} \quad (19)$$

with (normalized) sampling period $h = 1$ in Fig. 1. The Nyquist frequency is then π [rad/sec] which is just equal to 0.5 [Hz]. Suppose that we are given the tracking signal $r = \sin\omega t$, where $\omega = 3\pi/2$ [rad/sec], which is equal to 0.75 [Hz]. This is clearly above the Nyquist frequency, and a normal signal-processing intuition or a digital control thinking may tell us that it is impossible to track.

The basic idea is that we place more weight on this high frequency signal rather than the low frequency range below the Nyquist frequency. In fact, we take the weighting function

$$F(s) := \frac{s}{s^2 + 0.1s + (3\pi/2)^2}. \quad (20)$$

which has a clear peak at $3\pi/2$ [rad/sec] and also deemphasizes low-frequency.

The response against the sinusoid $r(t) = \sin(3\pi/2)t$ is shown in Fig. 3, and its (delayed) error is shown in Fig. 4 with $M = 8$ and $L = 4h$. These figures clearly show that

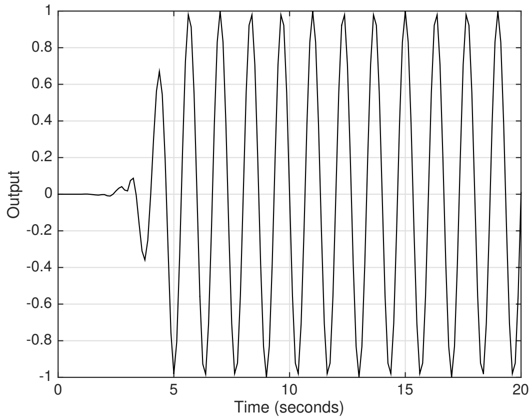

Fig. 3. System output tracking $\sin(3\pi/2)t$

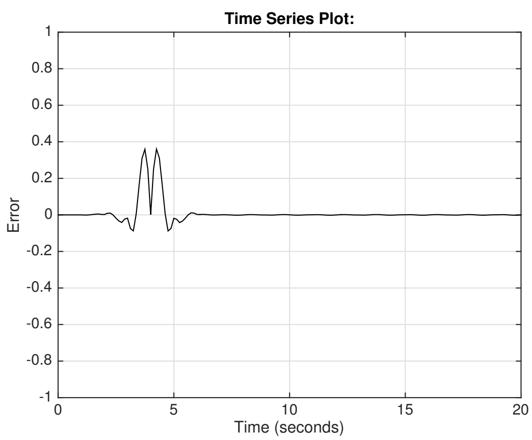

Fig. 4. Delayed error against $\sin(3\pi/2)t$

the output tracks the reference input $\sin(3\pi/2)t$, which has the natural frequency greater than the Nyquist frequency π, and the output matches the given frequency $3\pi/2$. Note also that the output shows the delay of 4 steps which is specified by the design specification. Now Fig. 5 shows the output of the discrete-time controller. This output shows that the discrete-time output indeed gives a discrete-time approximation of the sinusoid $\sin(3\pi/2)t$ (cf. Remark 4.2). This means that the discrete-time controller includes an approximate internal model that produces an approximation of the reference input. As we increase the upsampling factor M, it is expected that the designed controller produces more accurate sinusoids. Indeed, Fig. 6 show the eigenvalues of the upsampled controller, i.e., that of $\overline{A_d} = A_d^M$, and there are poles at $\pm j$, that corresponding to $\exp(3\pi j/2)t$—necessary to produce $\sin(3\pi/2)t$.

One may ask the question that this is probably due to the fact that the tracking reference frequency is not "too high."

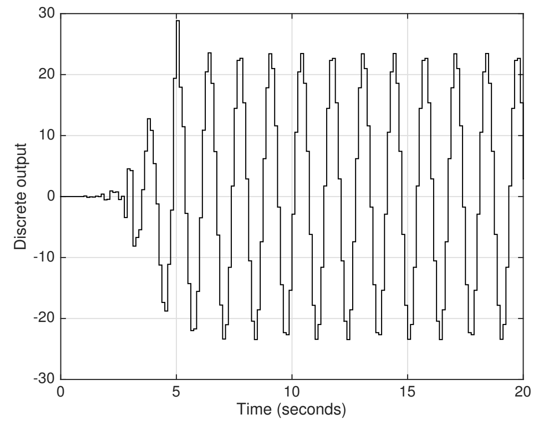

Fig. 5. Discrete-time controller output

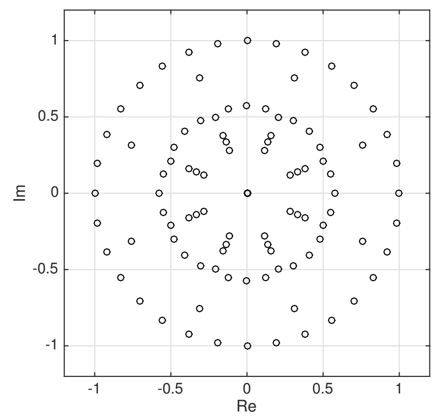

Fig. 6. Poles of the lifted controller

The following example denies this conjecture.

Example 6.2: Take the same plant $P(s) := 1/(s^2+2s+1)$, but with the objective of tracking the sinusoid $\sin(5\pi/2)t$, i.e., sinusoid at 1.25 [Hz]. We take a new weight

$$F(s) := \frac{s}{s^2+0.1s+(5\pi/2)^2},$$

which now has a peak at $5\pi/2$ [rad/sec].
The following Fig. 7, with $M = 16$ and $L = 4h$ clearly shows that the present method works as well for this even higher frequency.

VII. Discussion

We have seen that it is possible to track to a signal that contains higher frequency components than the Nyquist frequency. This is made possible even without a continuous-time internal model of the reference signal. The crux of the idea lies in the following:

- introduce a weighting function that emphasizes the desired high-frequency components,
- upsample the controller to allow for a tracking capability for high-frequency intersample signals, and

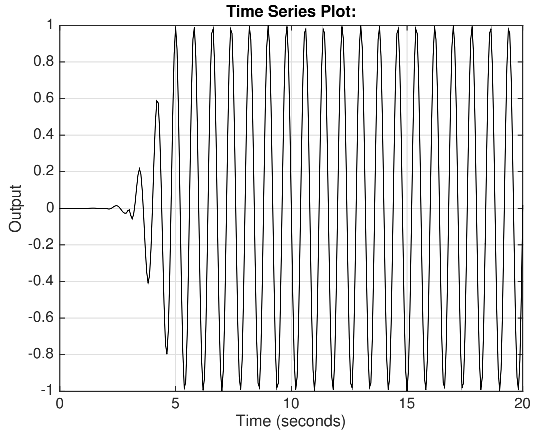

Fig. 7. System output tracking $\sin(5\pi/2)t$

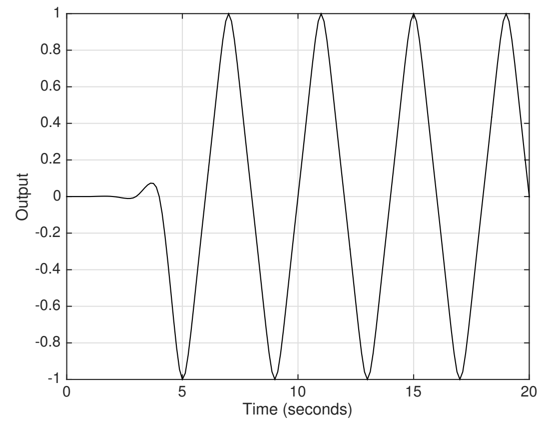

Fig. 8. System output not tracking $\sin(3\pi/2)t$ but $\sin(\pi/2)t$

- design the discrete-time controller.

While the sampler receives a discrete-time error signal that contains only low-frequency information according to the conventional sampling theorem, it can be utilized as the information carrying the tracking high-frequency reference signal. This is made possible by adjusting the weighting function that emphasizes the high-frequency component and simultaneously deemphasizes the low frequency part. This is exactly opposite to the standard thinking in which one emphasizes low frequency, but it does indeed work as we have seen in Example 6.1. In other words, the controller (and the weighting function) "fakes" the sampler, and can indeed extract the information that the received signal represents the high-frequency tracking signal, and not the low-frequency counterpart. To see this mechanism, let us show yet another example how the choice of a weight function can control such information.

Example 7.1: Take the same plant

$$P(s) := \frac{1}{s^2+2s+1}$$

as before, but with the weighting function

$$F(s) := \frac{(3\pi/4)^2}{s^2+(3\pi/2)s+(3\pi/4)^2}$$

that decays by 40 [dB/dec] with turnover frequency $3\pi/4$ [rad/sec], which is equal to 0.375 [Hz]. Designing an H^∞ controller as before with this weighting, we get the response against the sinusoid $\sin(3\pi/2)t$ as shown in Fig. 8. As can be seen from this figure, the system tracks $\sin(\pi/2)t$, not $\sin(3\pi/2)t$. This is clearly due to the fact that we have designed the controller tuned to track low-frequency inputs, and not $\sin(3\pi/2)t$, hence the controller "mistakes" the sampled values as those derived from $\sin(\pi/2)t$, which is the aliased signal from $\sin(3\pi/2)t$.

Finally, we also note that while we confined our discussions to tracking problems, it is obvious that disturbance rejection can be treated in exactly the same way.

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