Mass Chains with Passive Interconnection: Complex Iterative Maps and Scalability

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Kaoru Yamamoto and Malcolm C. Smith

Abstract— This paper introduces the problem of passive control of a chain of N identical masses in which there is an identical passive connection between neighbouring masses and a similar connection to a movable point. The problem arises in the design of multi-storey buildings which are subjected to earthquake disturbances, but applies in other situations, for example vehicle platoons. The paper will study the scalar transfer functions from the disturbance to a given intermass displacement. It will be shown that these transfer functions can be conveniently represented in the form of complex iterative maps and that these maps provide a method to establish boundedness in N of the $\mathcal{H}_\infty$-norm of these transfer functions for certain choices of interconnection impedance.

I. INTRODUCTION

This paper studies disturbance propagation in a chain of N identical masses with passive interconnection. We assume that neighbouring masses are connected by identical two-terminal passive mechanical impedances. The first mass is also connected by the same impedance to a movable point (Fig. 1). The problem is motivated by the problem of vibration suppression in multi-storey buildings subjected to earthquake disturbances, but applies also to other situations, for example vehicle platoons. We will study frequency response properties of the transmission from the movable point to individual intermass displacements (inter-storey drifts in the building application). In particular we will study the maximum modulus ($\mathcal{H}_\infty$-norm) of such frequency responses as a function of interconnection impedances and the dependence of the norm on N. We will investigate the issue of "scalability", namely the possibility that the $\mathcal{H}_\infty$-norm of such frequency responses remains bounded as a function of N.

Scalability has been an active area of research in automatic control of vehicles, where a related concept is that of "string stability"; see for example [1]–[4]. Using the terminology in this field, our problem formulation corresponds to "symmetric bidirectional control", i.e., control laws in which the control action for each vehicle is equally dependent on the spacing errors with the predecessor and the follower. In [2] a general result has been shown that, using symmetric bidirectional control, the infinity norm of the transfer function vector from lead vehicle trajectory to spacing error grows without bound as the number of vehicles increases, if the combined vehicle-controller dynamics contains a double integrator. This corresponds to a positive static (spring) stiffness in the case of a mass chain with passive interconnection, which is the usual case. However, it may be noted that
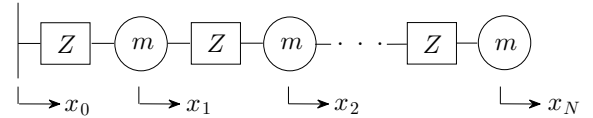
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Fig. 1. Chain of N masses m connected by a passive mechanical impedance $Z(s)$ (admittance $Y(s) = Z(s)^{-1}$), and connected to a movable point x_0.

well-regulated individual intermass displacements may be a satisfactory performance objective for buildings.

In this paper we will study the scalar transfer functions from the disturbance to a given intermass displacement. It will be shown that these transfer functions can be expressed recursively in N and take the form of complex iterative maps parameterised by a dimensionless quantity derived from the impedance and mass. This form gives a convenient method to accurately compute these transfer functions. The fixed points of these recursions will be shown to provide a lower bound on the peak amplification over all N. It will also be shown that the $\mathcal{H}_\infty$-norm of these transfer functions is bounded above independently of the length of the mass chain for suitable choice of the interconnection impedance. A graphical method will be developed to design impedance functions which achieve a given supremal $\mathcal{H}_\infty$-norm (over all N).

II. BACKGROUND ON PASSIVE MECHANICAL NETWORKS

A mechanical one-port network with force-velocity pair (F, v) is *passive* if for all square integrable pairs $F(t)$ and $v(t)$ on $(-\infty, T]$, $\int_{-\infty}^{T} F(t)v(t)dt \geq 0$ [5]. For a linear time-invariant network the impedance $Z(s)$ is defined by the ratio $\hat{v}(s)/\hat{F}(s)$ where ˆ denotes Laplace transform, and $Y(s) = Z(s)^{-1}$ is called the admittance. Such a network can be shown to be passive if and only if $Z(s)$ or $Y(s)$ is positive real [6], [7]. A real-rational function $G(s)$ is *positive real* if (i) $G(s)$ is analytic for $\text{Re}(s) > 0$ and (ii) $\text{Re}(G(s)) > 0$ for $\text{Re}(s) > 0$.

The passive components considered are springs, dampers and inerters. The *inerter* is a mechanical two-terminal, one-port device with the property that the applied force at the terminals is proportional to the relative acceleration between the terminals, i.e., $F = b(\dot{v}_2 - \dot{v}_1)$ where b is the constant of proportionality called the *inertance* which has units of kilograms [8] and v_1, v_2 are the terminal velocities. The inerter completes a standard analogy between mechanical

and electrical networks which allows classical results from electrical network synthesis to be translated over exactly to mechanical systems. In particular, any real-rational positive-real function can be realised as the impedance or admittance of a network with springs, inerters, and dampers only [8].

III. PROBLEM FORMULATION

A. Notation

The set of natural, real and complex numbers is denoted by $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$, respectively. $\mathbb{R}^{m \times n}$ is the set of m by n real matrices. $\mathbb{R}_+$ is the set of non-negative numbers and $\mathbb{C}_+$ is the closed right-half plane. $\mathcal{H}_\infty$ is the standard Hardy space on the right-half plane and $\|\cdot\|_\infty$ represents the $\mathcal{H}_\infty$-norm.

B. Chain model

We consider a chain of N identical masses m connected by identical passive mechanical networks (Fig. 1). Each passive mechanical network provides an equal and opposite force on each mass and is assumed here to have negligible mass. The system is excited by a movable point $x_0(t)$ and the displacement of the ith mass is denoted by $x_i(t)$, $i \in \{1, 2, \ldots, N\}$. We assume that the initial conditions of the movable point and the mass displacements are all zero.

The equations of motion in the Laplace transformed domain are

$$ms^2 \hat{x}_i = sY(s)(\hat{x}_{i-1} - \hat{x}_i) + sY(s)(\hat{x}_{i+1} - \hat{x}_i)$$
$$\text{for } i = 1, \ldots, N-1,$$
$$ms^2 \hat{x}_N = sY(s)(\hat{x}_{N-1} - \hat{x}_N).$$

In matrix form this can be written as

$$ms^2 \hat{x} = sY(s)H\hat{x} + sY(s)\phi_1 \hat{x}_0$$

whence

$$\hat{x} = (h(s)I - H)^{-1}\phi_1 \hat{x}_0 \tag{1}$$

where I is the $N \times N$ identity matrix,

$$h(s) = sZ(s)m, \quad Z = Y^{-1},$$
$$\hat{x} = [\hat{x}_1, \ldots, \hat{x}_N]^{\mathrm{T}},$$
$$\phi_1 \in \mathbb{R}^N, \quad \phi_1 = [1, 0, \ldots, 0]^{\mathrm{T}},$$
$$H \in \mathbb{R}^{N \times N},$$
$$H = \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & -2 & 1 \\ 0 & \cdots & 0 & 1 & -1 \end{bmatrix}.$$

Let us consider the characteristic polynomials d_i of $H \in \mathbb{R}^{i \times i}$ in the variable h given by

$$d_i = \det(hI - H)$$
$$= \begin{vmatrix} h+2 & -1 & 0 & \cdots & 0 \\ -1 & h+2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & h+2 & -1 \\ 0 & \cdots & 0 & -1 & h+1 \end{vmatrix} \tag{2}$$
$$\text{for } i = 1, \ldots, N.$$

Then $d_1 = h + 1$. Suppose also $d_{-1} = 1$ and $d_0 = 1$. From (2) we find that

$$d_i(h) = (h+2)d_{i-1}(h) - d_{i-2}(h) \quad \text{for } i = 1, \ldots, N. \tag{3}$$

Equation (1) can be written using d_i as

$$\hat{x} = \frac{\text{adj}(h(s)I - H)}{\det(h(s)I - H)}\phi_1 \hat{x}_0$$
$$= \frac{1}{d_N} \begin{pmatrix} d_{N-1} & * & \cdots \\ \vdots & \vdots & \\ d_0 & * & \cdots \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \hat{x}_0$$
$$= \begin{pmatrix} d_{N-1}/d_N \\ \vdots \\ d_0/d_N \end{pmatrix} \hat{x}_0. \tag{4}$$

Then the intermass displacement of the ith mass defined by $\delta_i = x_i - x_{i-1}$ in the Laplace domain is given by

$$\hat{\delta}_i = ((d_{N-i} - d_{N-i+1})/d_N) \hat{x}_0 =: T_{\hat{x}_0 \to \hat{\delta}_i} \hat{x}_0$$
$$\text{for } i = 1, \ldots, N. \tag{5}$$

IV. STABILITY OF PASSIVE INTERCONNECTION

We first establish some properties of the sequence $d_i(h)$, namely that they are Hurwitz with real distinct roots in the interval $(-4, 0)$ for $i = 1, 2, \ldots$ and thus form a Sturm sequence [9].

Theorem 1:

1. $d_i(h)$ has negative real distinct roots which interlace the roots of $d_{i+1}(h)$ for $i = 1, 2, \ldots$.
2. The roots of $d_i(h)$ lie in the interval $(-4, 0)$ for $i = 1, 2, \ldots$.

   Proof:

1. This can be proved, using induction, by considering the sign of $d_i(h)$ at the roots of $d_{i-1}(h)$ and at $h = 0$ and as $h \to -\infty$.
2. Let $\mathbb{P} = \cup_N \sigma(H) : N = 1, 2, \ldots$ and denote its closure by $\bar{\mathbb{P}}$. Note that a Gershgorin disc bound on the eigenvalues of H [2], [3], [10] (this holds for all N) implies $\bar{\mathbb{P}} \in [-4, 0]$. It is straightforward to check that $d_i(0) = 1$ and $d_i(-4) = (-1)^i(2i + 1)$. Hence, the roots of $d_i(h)$ lie in the interval $(-4, 0)$. ∎

We will say that the system of Fig. 1 is stable if all poles in the transfer functions $T_{\hat{x}_0 \to \hat{\delta}_i}$ have negative real parts.

Theorem 2: For $Z(s)$ positive real, the system of Fig. 1 is stable if $sZ(s)m$ does not take values in the interval $(-4, 0)$ for any s with $\text{Re}(s) = 0$.

Proof: From (4), poles in $T_{\hat{x}_0 \to \hat{\delta}_i}$ can only occur at an s for which $d_N(h(s)) = 0$. The result now follow from Theorem 1 on noting that $\text{Re}(Z(s)) > 0$ for $\text{Re}(s) > 0$. ∎

V. FIRST INTERMASS DISPLACEMENT

It is shown in this section that the intermass displacement of the first mass in a chain of N masses is given by a recursion. Dividing both sides of (3) by d_{i-1}, we obtain

$$(d_i/d_{i-1})(h) = h + 2 - (d_{i-2}/d_{i-1})(h) \quad \text{for } i = 1, \dots, N. \tag{6}$$

Let $F_N(h) = 1 - d_{N-1}(h)/d_N(h)$ $(= -T_{\hat{x}_0 \to \hat{\delta}_1}$ in a chain of N masses) and substituting this into (6) for $i = N$ gives the recursion:

$$F_N = \frac{F_{N-1} + h}{F_{N-1} + h + 1}. \tag{7}$$

If we set $F_0 = 0$ then the above recursion is true for all $N \geq 1$. In the language of complex iterative maps, the sequence F_N is the orbit of 0 for the recursion (7).

A. Convergence to Fixed Points

For the recursion

$$z_i = f(z_{i-1}) \quad \text{for } i = 1, 2, \dots,$$

a complex number z_f is called a fixed point (e.g., [11]) when $z_f = f(z_f)$. The fixed point z_f is called an attractive fixed point when the sequence z_i converges to z_f for z_0 sufficiently close to z_f. The convergence is guaranteed if $|f'(z_f)| < 1$.

For (7) the fixed points satisfy

$$z = \frac{z + h}{z + h + 1}.$$

This gives two fixed points:

$$\mu_\pm := \frac{-h \pm h\sqrt{1 + 4/h}}{2}.$$

Note that we take the principal value [12] of $\sqrt{1 + 4/h}$, i.e., $\sqrt{|1 + 4/h|}e^{j\varphi/2}$ where $\varphi = \angle(1 + 4/h)$, $-\pi < \varphi \leq \pi$. We find that

$$|f'(\mu_+)| = \left| \frac{\mu_+}{h + \mu_+} \right|^2 = \left| \frac{1 - \sqrt{1 + 4/h}}{1 + \sqrt{1 + 4/h}} \right|^2,$$

$$|f'(\mu_-)| = \left| \frac{\mu_-}{h + \mu_-} \right|^2 = \left| \frac{1 + \sqrt{1 + 4/h}}{1 - \sqrt{1 + 4/h}} \right|^2.$$

Since $\text{Re}\left(\sqrt{1 + 4/h}\right) \geq 0$ we see that

$$|f'(\mu_+)| = |f'(\mu_-)|^{-1} \leq 1.$$

Thus μ_+ is an attractor and μ_- is a repeller except when $|f'(\mu_+)| = 1 = |f'(\mu_-)|$, which occurs when $h \in [-4, 0]$. Local convergence to μ_+ follows immediately for $h \notin [-4, 0]$. However, a stronger result can be shown.

Theorem 3: Let F_0 be arbitrary. The sequence $\{F_N\}$ defined by (7) converges to μ_+ when $h \notin (-4, 0)$ (except
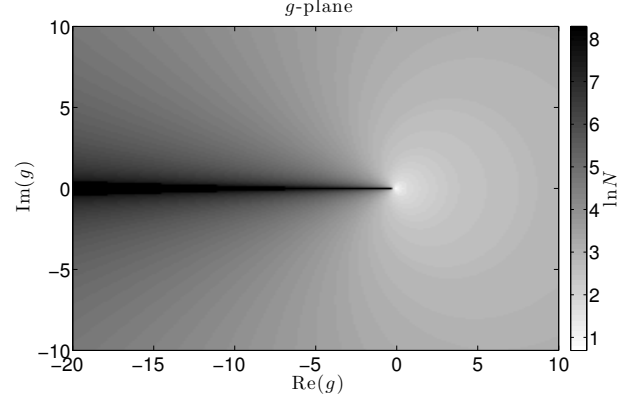

Fig. 2. Speed of convergence of the sequence $\{F_N\}$ to the fixed point μ_+. The shading at a specific point g denotes the smallest N such that $|F_N(h) - \mu_+(h)| < 10^{-5}$.

when $F_0 = \mu_-$). $\{F_N\}$ does not converge to either μ_+ or μ_- when $h \in (-4, 0)$ (except when $F_0 = \mu_+$ or μ_-).

Proof: It may be observed that (7) takes the form of a Möbius transformation, which is (i) parabolic when $h = 0$ or -4, (ii) elliptic when $h \in (-4, 0)$, and (iii) loxodromic when $h \notin [-4, 0]$. The following facts can be shown by the use of a conjugacy transformation (see [13]). When (7) is parabolic there is a unique fixed point and $\{F_N\}$ converges pointwise for any initial condition. When (7) is loxodromic there are two fixed points, an attractor and a repeller, in this case μ_+ and μ_- respectively, and $\{F_N\}$ converges pointwise to μ_+ for any initial condition other than μ_-. When (7) is elliptic $\{F_N\}$ fails to converge for any initial condition other than the fixed points. ∎

Here we are concerned with the orbit of 0 for the recursion (7). In this case Theorem 3 specialises to: $\{F_N\}$ converges to μ_+ when $h \notin (-4, 0)$ but fails to converge otherwise. For the purpose of graphical representations we now introduce the inverse of h:

$$g(s) = h^{-1}(s) = Y(s)/(sm). \tag{8}$$

Fig. 2 shows the speed of convergence in the g-plane of $\{F_N(h)\}$ to μ_+. The shading at a specific point g in the complex plane denotes the smallest N such that $|F_N(h) - \mu_+(h)| < 10^{-5}$; black (white) indicates large (small) values of N. We see that the speed of the convergence is slower when g is closer to the real axis between $(-\infty, -1/4)$ (corresponding to $h \in (-4, 0)$). Hence, if $h(s) \notin (-4, 0)$ for all $s \in \mathbb{C}_+$, $\lim_{N \to \infty} \|F_N\|_\infty = \sup_\omega |\mu_+(h(j\omega))|$ for $F_0 = 0$. Furthermore, $\sup_\omega |\mu_+(h(j\omega))| \leq \sup_N \|F_N\|_\infty$. A contour map of the magnitude of the fixed point μ_+ in the g-plane is shown in Fig. 3. The outermost boundary represents $\ln|\mu_+| = -1.5$ and the spacing of the boundaries is 0.1 where $\ln|\mu_+|$ takes the value $-1.5, -0.5, 0.5 \dots$.

B. Bounds on Iterative Map

Making use of a conjugacy transformation, we can write

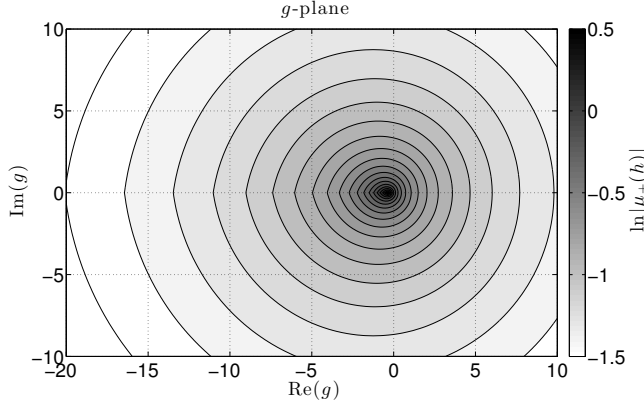$$f(z) = \frac{z + h}{z + h + 1} = \varphi \lambda \varphi^{-1}$$

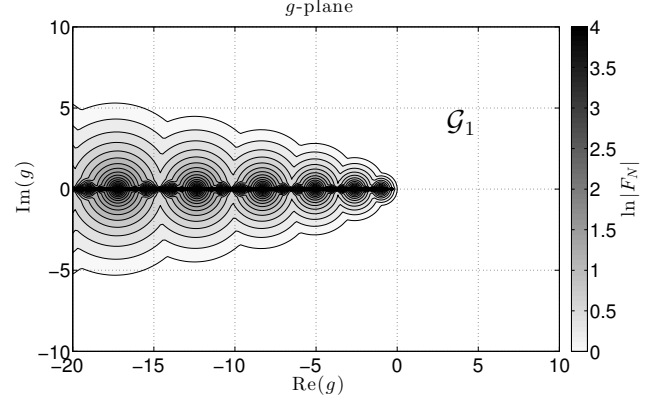Fig. 3. Contour map of the magnitude of the fixed point μ_+ in the g-plane.

Fig. 4. Region of the complex values of g for which $|F_N| \leq \gamma \in \mathbb{R}_+$ for all N.

where

$$\varphi(z) = \frac{z\mu_- - \mu_+}{z - 1},$$

$$\lambda(z) = \frac{1 - \mu_+}{1 - \mu_-} z =: \kappa z$$

are Möbius transformations. Then

$$F_N = f^N(0) = \varphi\lambda^N\varphi^{-1}(0)$$

$$= \mu_+ \frac{1 - k^N}{1 - \frac{\mu_+}{\mu_-} k^N}$$

$$= \mu_+ \frac{1 - (1 - \mu_+)^{2N}}{1 + (1 - \mu_+)^{2N+1}} \qquad (9)$$

since $\kappa = (1 - \mu_+)^2$ and $\mu_+/\mu_- = -(1 - \mu_+)$. We can show that

$$|1 - \mu_+| < 1 \quad \text{when} \quad h \notin [-4, 0]$$

and $|1 - \mu_+| \to 1$ as $h \to [-4, 0]$. We now make use of (9) to establish upper bounds on $|F_N|$ for suitable choices of $h(s)$.

Theorem 4: Suppose $Z(s) = (k/s + Y_1(s))^{-1}$ where k is a positive constant and Y_1 is a positive-real admittance for which $Re\,(Y_1(j\omega))$ is bounded below and $Re\,(j\omega Y_1(j\omega))$ is bounded above. Then

$$\sup_\omega \sup_N |F_N\,(h(j\omega))|$$

is finite where $h(j\omega) = mj\omega Z(j\omega)$.

Sketch of proof: The derivation of the upper bound makes use of (9). A typical $h(j\omega)$ takes the form $-c_1\omega^2 + jc_2\omega^3 + \ldots$ for ω close to 0, and hence approaches the interval $[-4, 0]$ where the complex iterative map is unbounded. However, the iteration is bounded at 0, and the manner in which $h(j\omega) \to 0$ is critical. The estimate of the denominator in (9) requires a worst-case estimate of the exponent N for complex values of $1 - \mu_+$. ∎

We remark that the conditions on $Y_1(s)$ in Theorem 4 are satisfied by typical passive networks of interest. For example, for a damper and inerter in parallel, (which means

that $Z(s)$ is the impedance of a parallel spring-damper-inerter network) $Y_1(j\omega) = c + bj\omega$, here $Re\,(Y_1(j\omega)) \geq c$ and $Re\,(j\omega Y_1(j\omega)) \leq 0$. We also note that $\sup_N |F_N| = \sup_{N \geq 1} |F_N|$ since $F_0 = 0$.

We now illustrate graphically the boundedness result of Theorem 4. Fig. 4 shows the region of the complex values of $g\,(= h^{-1})$ for which $\max_N |F_N(h)| \leq \gamma$ with $1 \leq N \leq 200$ for a positive constant γ. The spacing of the boundaries is 0.2 where $\ln(\gamma)$ takes the value $0, 0.2, 0.4, \ldots$. The outermost boundary represents $\gamma = 1$ and $\mathcal{G}_1$ denotes the set $\{g \in \mathbb{C} : \max_N |F_N| \leq 1\}$. This means $\max_N \|F_N\|_\infty \leq 1$ if and only if $g \in \mathcal{G}_1$ for $s \in \mathbb{C}_+$. Note that from Fig. 2 the sequence $\{F_N\}$ converges to the fixed point μ_+ quickly when $g \in \mathcal{G}_1$ so the choice of $N = 200$ is large enough to accurately determine the shape of the boundary in the figure.

VI. HIGHER INTERMASS DISPLACEMENTS

In the previous section we saw that the first intermass displacement is given by the recursion (7). This section shows that the other intermass displacements are also given recursively.

Theorem 5: For any $i = 1, 2, \ldots$, the intermass displacements $F_N^{(i)} = -T_{\hat{x}_0 \to \hat{\delta}_i}$ (see (5)) satisfy the recursion:

$$F_N^{(i)} = \frac{d_{i-2} F_{N-1}^{(i)} + h}{F_{N-1}^{(i)} + d_i} \qquad (10)$$

for $N = i, i+1, \ldots$, where $F_{i-1}^{(i)} = 0$.

Proof: Define

$$f(N, i) = \overbrace{(d_{N-i+1} - d_{N-i})(d_{N-i} - d_{N-i-1}} + d_{N-1}d_i)$$
$$- d_N d_{i-2}(d_{N-i} - d_{N-i-1}) - h d_{N-1}d_N. \qquad (11)$$

From (5) we see that the theorem is equivalent to $f(N, i) = 0$ for all $i \in \mathbb{N}$ and $i \leq N \in \mathbb{N}$. The proof will follow by induction after establishing the following facts:

1. $f(N, 1) = 0$ for all $N \geq 1$.
2. $f(N, 2) = 0$ for all $N \geq 2$.
3. $f(N, i) = f(N, i-1) + f(N-1, i-1) - f(N-1, i-2)$ for any $i \geq 3, N \geq i$.

We now establish these facts in turn.

1. $f(N, 1) = (d_{N-1} - d_{N-2})(d_N - d_{N-1} - d_N d_{-1})$
$$+ d_{N-1}(d_1(d_N - d_{N-1}) - h d_N)$$
$$= d_{N-1}(d_N - (h+2)d_{N-1} + d_{N-2})$$
$$= 0.$$

   Note that $d_N = (h+2)d_{N-1} - d_{N-2}$ given from (3).

2. $f(N, 2) = (d_{N-2} - d_{N-3})(d_{N-1} - d_{N-2} - d_N d_0)$
$$+ d_{N-1}d_2(d_{N-1} - d_{N-2} - h d_{N-1} d_N)$$
$$= (d_{N-1} - (h+1)d_{N-2})(h+1)d_{N-1}$$
$$+ d_{N-1}(h^2 + 3h + 1)(d_{N-1} - d_{N-2})$$
$$- h d_{N-1} d_N$$
$$= -h d_{N-1}(d_N - (h+2)d_{N-1} + d_{N-2})$$
$$= 0.$$

3. Consider the expression

$$X(N, i) = f(N, i) - f(N, i-1)$$
$$- f(N-1, i-1) + f(N-1, i-2). \quad (12)$$

It may be observed that four terms in (12) corresponding to the overbrace in (11) cancel pairwise. Also the four terms in (12) of the form $-h d_{N-1} d_N$ cancel pairwise. Thus

$$X(N, i) = (d_{N-i} - d_{N-i-1})(d_{N-1}d_{i-3} - d_N d_{i-2})$$
$$+ (d_{N-i+1} - d_{N-i})(d_{N-1}d_i + d_N d_{i-3}$$
$$- d_{N-2}d_{i-1} - d_{N-1}d_{i-4})$$
$$+ (d_{N-i+2} - d_{N-i+1})(d_{N-2}d_{i-2} - d_{N-1}d_{i-1}).$$
$$(13)$$

Using the following substitution

$$d_i = (h+2)d_{i-1} - d_{i-2}$$
$$d_N = (h+2)d_{N-1} - d_{N-2}$$
$$d_{i-1} = (h+2)d_{i-2} - d_{i-3}$$
$$d_{i-4} = (h+2)d_{i-3} - d_{i-2}$$

in the second form of (13) and rearranging gives

$$X(N, i) = (d_{N-i} - d_{N-i-1})(d_{N-1}d_{i-3} - d_N d_{i-2})$$
$$+ (d_{N-2}d_{i-2} - d_{N-1}d_{i-1})(d_{N-i+2} - d_{N-i+1}$$
$$- (h+2)(d_{N-i+1} - d_{N-i})).$$

Now note that

$$d_{N-i+2} - d_{N-i+1} - (h+2)(d_{N-i+1} - d_{N-i})$$
$$= (h+1)d_{N-i} - d_{N-i+1}$$
$$= -(d_{N-i} - d_{N-i-1}).$$

Hence

$$X(N, i) = (d_{N-i} - d_{N-i-1})(d_{N-1}d_{i-3} - d_N d_{i-2}$$
$$- d_{N-2}d_{i-2} + d_{N-1}d_{i-1})$$
$$= (d_{N-i} - d_{N-i-1})(d_{N-1}d_{i-3} - (h+2)d_{N-1}d_{i-2}$$
$$+ d_{N-2}d_{i-2} - d_{N-2}d_{i-2} + (h+2)d_{N-1}d_{i-2}$$
$$- d_{N-1}d_{i-3})$$
$$= 0. \qquad \blacksquare$$
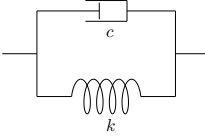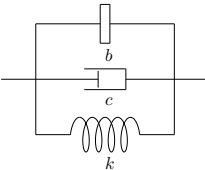
L1	L2
$Y(s) = c + \dfrac{k}{s}$	$Y(s) = bs + c + \dfrac{k}{s}$

	Layout	c [kNs/m]	b [kg]
Device 1	L1	4.0×10^3	–
Device 2	L1	6.0×10^3	–
Device 3	L2	6.0×10^3	1.0×10^5

Theorem 5 can be used to provide a graphical method to determine the maximal amplification over N in the same manner as for the first intermass displacement (Fig. 4). Fig. 5 shows the region of the complex values of g ($= h^{-1}$) for which $\max_N |F_N^{(i)}(h)| \leq \gamma \in \mathbb{R}_+$ for all $i = 1, 2, \ldots, N$ (where $1 \leq N \leq 200$) with the Nyquist diagrams of $g(s)$ of three passive vibration control devices. The layouts of these devices are shown in Table I and their structural parameters are given in Table II. We fix the parameters of the building model as $m = 1.0 \times 10^5$ kg, $k = 1.7 \times 10^5$ kN/m (based on values given in [14]). We see that devices 2 and 3 achieve $\max_N \|F_N^{(i)}\|_\infty \leq 1$ for all $i = 1, 2, \ldots, N$. It is also observed that the use of the inerters improves the high frequency performance (corresponding to the origin in the g-plane). The frequency domain plots of $\max_i |F_N^{(i)}(j\omega)|$ (Figs. 6 and 7) confirm these observations. Fig. 8 shows the boundaries which represent $\max_N |F_N^{(i)}| = 1$ where $i = 1, 2, \ldots, 5$ with $1 \leq N \leq 200$. We observe that the set $\{g \in \mathbb{C} : \max_N |F_N^{(1)}| \leq 1\}$ contains the sets $\{g \in \mathbb{C} : \max_N |F_N^{(i)}| \leq 1\}$, $i = 2, \ldots, 5$.

VII. CONCLUSIONS

The interconnection of a chain of N identical masses has been studied in which neighbouring masses are connected by identical two-terminal passive mechanical impedances, and where the first mass is also connected by the same impedance to a movable point. The problem is similar to that of symmetric bidirectional control of a vehicle string, albeit with a passivity constraint. Formulae for the transfer functions from the disturbance to a given intermass displacement have been derived in the form of complex iterative maps as a function of a dimensionless parameter depending on the impedance and mass. The maps take the form of an iterated Möbius transformation. It is shown that the fixed points of the mappings provide information on the asymptotic behaviour
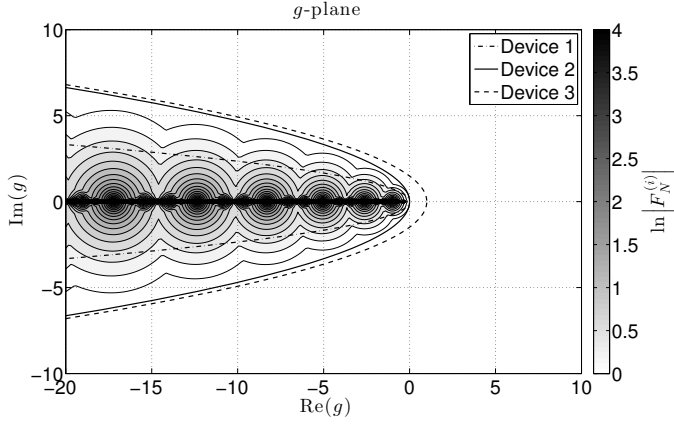
Fig. 5. Nyquist diagrams of $g(s) = Y(s)/(sm)$ for the vibration control devices in Table II and contours of $\max_{N,i}|F_N^{(i)}| = \gamma$ for $\ln(\gamma) = 0, 0.2, 0.4, \ldots$.
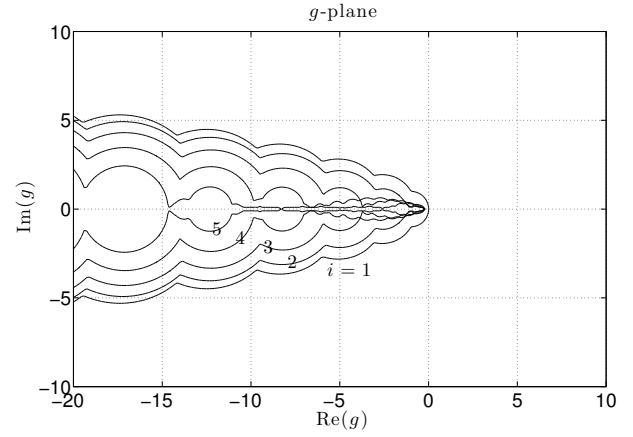

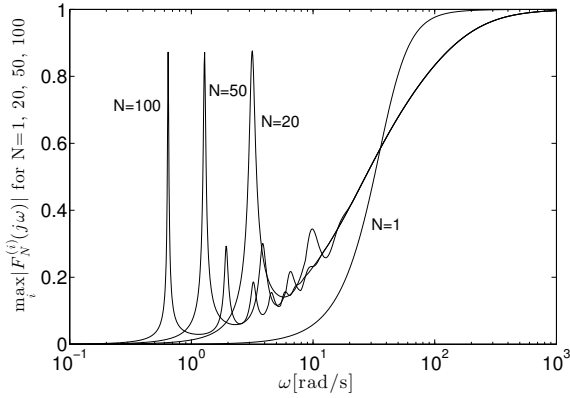Fig. 8. Boundaries of $\max_N|F_N^{(i)}| = 1$ for all N where $i = 1, 2, \ldots, 5$.

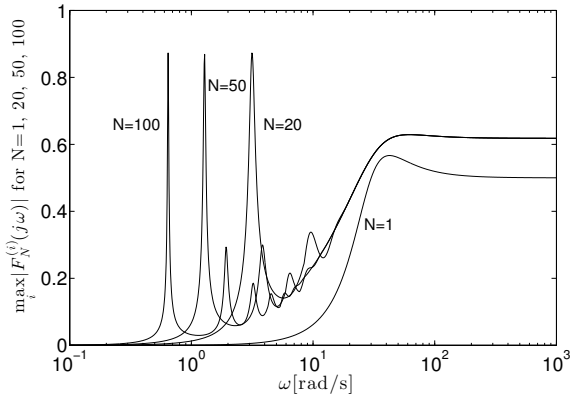Fig. 6. $\max_i|F_N^{(i)}(j\omega)|$ using Device 2 for $N = 1, 20, 50, 100$.

Fig. 7. $\max_i|F_N^{(i)}(j\omega)|$ using Device 3 for $N = 1, 20, 50, 100$.

of the disturbance transfer functions. Further, the use of a conjugacy transformation allows the iterative map to be written in a convenient form to derive formal upper bounds on $\sup_N |F_N|$. A graphical technique is introduced to provide guarantees on the disturbance amplification. The method is illustrated in the context of the design of a multi-storey building. A comparison is made between a standard spring-damper model for the lateral inter-storey suspension and the use of inerters.

VIII. ACKNOWLEDGEMENTS

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