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A resolvent trace formula of Jacquet-Zagier type for Hilbert Maass forms

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Abstract

In this paper, we give an explicit trace formula of Jacquat-Zagier type for Hilbert Maass forms of parallel weight 0 by means of the resolvent kernel function of the Laplacian on the complex upper half-plane. This is an analogy of Sugiyama-Tsuzuki's trace formula.

Keywords: Trace formula, Hilbert Maass form

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Contents

1 Introduction

1.1 Background

Zagier found a generalized Eichler-Selberg trace formula involving symmetric square L-functions by means of the Rankin-Selberg method in computing the trace formula of Hecke operators of elliptic cusp forms ([21]). An analogy of his work for Maass forms has been obtained by himself ([22]), and a generalization of Zagier's formula for Hilbert modular forms has been done by Mizumoto and Takase ([13], [17]) under the assumption that the narrow class number of the base field is one.

In ([10]), Jacquet and Zagier constructed a similar result for general cusp forms on the adelization $\mathrm{GL}_2(\mathbb{A}_F)$ for arbitrary number field F under quite a general setting. Their formula can be regarded as deformation by a complex variable z of the usual Arthur-Selberg trace formula for $\mathrm{GL}(2)$ because the latter one can be recovered from the former one by computing the residue at $z = 1$ ([20]). However, their formula has an abstract form, and many terms remain uncomputed. One of the motivations for researches in this field is to calculate explicit trace formulas of Jacquet-Zagier type for certain test functions and to get generalizations of classical results such as ([13], [17], [21], [22]) in adelic setting.

Throughout the paper, we let F denote a totally real number field. Sugiyama and Tsuzuki constructed a generalized Zagier's formula for Hilbert modular forms with square-free levels and proved a non-vanishing property of symmetric square L-functions $L(z, \pi; \mathrm{Ad})$ for Hilbert modular forms ([16]) without the assumption on the class number of F. Their only assumption for F is that the prime 2 splits completely in $F/\mathbb{Q}$.

In this paper, we give an analogy of Sugiyama-Tsuzuki's trace formula for Hilbert Maass forms which recovers Zagier's formula for Maass forms ([22]) by means of a suitable test function constructed from the resolvent kernel function of the Laplace operator on the complex upper half-plane at infinite places. For a technical reason, we adopt the same assumptions for F as theirs.

The remarkable feature of our main results is that the terms which can be written by using the integration involving the Hecke L-functions appear in the spectral side from the theory of Eisenstein series on $\mathrm{GL}_2(\mathbb{A}_F)$. Besides, it is interesting that the hyperbolic terms and the elliptic terms in the geometric side can be written by using the certain hypergeometric series in two variables being a generalization of the Appell series $F_3(\alpha, \alpha', \beta, \beta', \gamma, x, y)$ ([1]).

The trace formula proved in this article is highly expected to obtain a non-vanishing property of the symmetric square L-function $L(z, \pi; \mathrm{Ad})$ of Hilbert Maass forms and to have applications for many arithmetic problems.

1.2 Notations

Let F be a totally real number field with degree n_F, $\mathfrak{o}$ be the integer ring of F, and $\mathbb{A}$ be the adele ring of F. We denote the set of finite places of F and the set of infinite places of F by Σ_{fin} and Σ_∞ respectively and set $\Sigma_F = \Sigma_{\mathrm{fin}} \cup \Sigma_\infty$. The completion of F at $v \in \Sigma_F$ is denoted by F_v and the modulus of F_v is by $|\ |_v$. Then, the idele norm of $\mathbb{A}^\times$ is defined by $|\ |_{\mathbb{A}} = \prod_{v \in \Sigma_F} |\ |_v$.

Choose an embedding $\mathbb{R}_+ \to \mathbb{A}^\times$ by $x \mapsto \left(1_{\mathrm{fin}}, x^{\frac{1}{n_F}}, \ldots, x^{\frac{1}{n_F}}\right)$. Occasionally, we regard $\mathbb{R}_+$ as the subgroup of $\mathbb{A}^\times$ by this embedding. Then we have a splitting $\mathbb{A}^\times = \mathbb{A}^1 \times \mathbb{R}_+$ where $\mathbb{A}^1$ is the kernel of the idele norm map. Let Ξ be the set of all characters on $\mathbb{A}^\times$ which are trivial on $F^\times \mathbb{R}_+$.

For $v \in \Sigma_{\mathrm{fin}}$, let $\mathfrak{o}_v$ be the integer ring of F_v, $\mathfrak{p}_v$ be the prime ideal of $\mathfrak{o}_v$, and q_v be the order of the residue field $\mathfrak{o}_v/\mathfrak{p}_v$. For each $v \in \Sigma_{\mathrm{fin}}$, we fix a prime element $\varpi_v \in \mathfrak{p}_v$.

For a non-zero ideal $\mathfrak{a} \subset \mathfrak{o}$, we set $\mathrm{N}(\mathfrak{a}) = \prod_{v\in\Sigma_{\mathrm{fin}}} q_v^{\mathrm{ord}_v(\mathfrak{a})}$ and denote the set of $v \in \Sigma_{\mathrm{fin}}$ dividing $\mathfrak{a}$ by $S(\mathfrak{a})$. Let $\psi = \prod_{v\in\Sigma_F} \psi_v$ be the additive character of $\mathbb{A}/F$ such that $\psi_v(x) = e^{2\pi\sqrt{-1}x}$ for $v \in \Sigma_\infty$ and $x \in \mathbb{R}$. Let $U(\mathfrak{a})$ be the compact subgroup of $\mathbb{A}^\times$ defined by

$$U(\mathfrak{a}) = \left\{ a = (a_v)_{v\in\Sigma_F} \in \mathbb{A}^\times \;\middle|\; \begin{array}{ll} a_v \in \mathfrak{o}_v^\times & (v \in \Sigma_{\mathrm{fin}} - S(\mathfrak{a})) \\ a_v \in 1 + \mathfrak{p}_v^{\mathrm{ord}_v(\mathfrak{a})} & (v \in S(\mathfrak{a})) \\ a_v = 1 & (v \in \Sigma_\infty) \end{array} \right\}$$

and $\Xi(\mathfrak{a})$ be the set of all $\chi \in \Xi$ which are trivial on $U(\mathfrak{a})$.

Let B, H, and N be F-subgroups of $G = \mathrm{GL}(2)$ defined symbolically as $B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$, $H = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\}$, and $N = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$. Let Z be the center of G. For $v \in \Sigma_F$, we define $\mathbf{K}_v$ as $G(\mathfrak{o}_v)$ if $v \in \Sigma_{\mathrm{fin}}$ and $O(2)$ if $v \in \Sigma_\infty$. Let $G_\mathbb{A} = \prod'_{v\in\Sigma_F} G_v$, $\mathbf{K} = \prod_{v\in\Sigma_F} \mathbf{K}_v$ be the standard maximal compact subgroup of $G_\mathbb{A}$, and $\mathbf{K}_{\mathrm{fin}} = \prod_{v\in\Sigma_{\mathrm{fin}}} \mathbf{K}_v$. Given a non-zero ideal $\mathfrak{n} \subset \mathfrak{o}$, we set $\mathbf{K}_0(\mathfrak{n}\mathfrak{o}_v) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{K}_v \;\middle|\; c \in \mathfrak{n}\mathfrak{o}_v \right\}$ for $v \in \Sigma_{\mathrm{fin}}$ and $\mathbf{K}_0(\mathfrak{n}) = \prod_{v\in\Sigma_{\mathrm{fin}}} \mathbf{K}_0(\mathfrak{n}\mathfrak{o}_v)$. Let $\Pi_{\mathrm{cus}}(\mathfrak{n})$ be the set of all irreducible cuspidal representations $\pi \cong \otimes'_{v\in\Sigma_F}\pi_v$ of $Z_\mathbb{A}\backslash G_\mathbb{A}$ such that π_v has a non-zero $\mathbf{K}_0(\mathfrak{n}\mathfrak{o}_v)$-invariant vector for $v \in \Sigma_{\mathrm{fin}}$ and has a non-zero $SO(2)$-invariant vector for $v \in \Sigma_\infty$.

For any complex-valued function f and g on a set X, we write $f(x) \ll g(x)$ if there exists a constant $C > 0$ such that $|f(x)| \le C|g(x)|$ for all $x \in X$. We write $f(x) \asymp g(x)$ when both $f(x) \ll g(x)$ and $g(x) \ll f(x)$ hold. If we emphasize the dependence of the implied constant C on some parameters $a, b, c, \ldots$, we write $f(x) \ll_{a,b,c,\ldots} g(x)$.

Let $\chi = \prod_{v\in\Sigma_F} \chi_v \in \Xi$. For $v \in \Sigma_\infty$, we set $\epsilon(\chi_v) \in \{0, 1\}$ and $a(\chi_v) \in \mathbb{R}$ so that

$$\chi_v(x) = \mathrm{sgn}(x)^{\epsilon(\chi_v)}|x|^{\sqrt{-1}a(\chi_v)}, \quad x \in \mathbb{R}.$$

The Hecke L-function $L_F(s, \chi)$ treated in this article has the Euler product on $\mathrm{Re}(s) > 1$ of the form

$$L_F(s, \chi) = \prod_{v\in\Sigma_F} L_v(s, \chi_v)$$

where

$$L_v(s, \chi_v) = \begin{cases} (1 - \chi_v(\varpi_v)q_v^{-s})^{-1} & (v \in \Sigma_{\mathrm{fin}},\ \chi_v : \text{unramified}) \\ 1 & (v \in \Sigma_{\mathrm{fin}},\ \chi_v : \text{ramified}) \\ \Gamma_\mathbb{R}\left(s + \sqrt{-1}a(\chi_v) + \epsilon(\chi_v)\right) & (v \in \Sigma_\infty) \end{cases}$$

with $\Gamma_\mathbb{R}(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$. Let $\zeta_F(s) = L(s, 1)$ be the completed Dedekind zeta function of F and $\zeta_v(s) = L_v(s, 1)$ be the Euler factor of $\zeta_F(s)$ at $v \in \Sigma_F$. The function $L(s, \chi)$ has a meromorphic continuation to $\mathbb{C}$ and is entire or holomorphic except for simple poles at $s = 0, 1$ if $\chi \neq 1$ or $\chi = 1$ respectively.

The Euler product of $L_v(s, \chi_v)$ over $v \in \Sigma_{\mathrm{fin}}$ is denoted by $L_{\mathrm{fin}}(s, \chi)$. For $\chi \in \Xi(\mathfrak{o})$, it is known that $L_{\mathrm{fin}}(s, \chi)$ has no zero on the vertical line $\mathrm{Re}(s) = 1$. From the proof of [18, Theorem 3.11], we obtain the estimate

$$\left|L_{\mathrm{fin}}(1 + \sqrt{-1}u, \chi)^{-1}\right| \ll \log\left\{ \prod_{v\in\Sigma_F} \left(3 + |\tfrac{u}{2} + a(\chi_v)|\right) \right\}, \quad u \in \mathbb{R} \tag{1.1}$$

with the implied constant independent of $\chi \in \Xi(\mathfrak{o})$. Let $[\sigma_1, \sigma_2]$ be a closed interval contained in open interval $(0,1)$. Then, by means of the Phragmén-Lindelöf principle, there exists a constant N_0 depending only on the field F such that

$$\left|L_{\rm fin}(\sigma+\sqrt{-1}u,\chi)\right| \ll_{\sigma_1,\sigma_2} \prod_{v\in\Sigma_\infty} \left(1+|\tfrac{u}{2}+a(\chi_v)|\right)^{N_0}, \quad \sigma\in[\sigma_1,\sigma_2], u\in\mathbb{R} \qquad (1.2)$$

with the implied constant independent of $\chi \in \Xi(\mathfrak{o})$ (see [9, Lemma 5.2]). Moreover, when $\chi \neq 1$, the above estimate holds uniformly for $\sigma \in [0,1]$.

1.3 Explanation of the main results

Let $v \in \Sigma_F$ and $I(\chi)$ be the normalized induced representation $\mathrm{Ind}_{B_v}^{G_v}(\chi \boxtimes \chi^{-1})$ for a quasi-character χ on $F_v^\times$.

Let $\pi \cong \otimes'_{v\in\Sigma_F}\pi_v \in \Pi_{\rm cus}(\mathfrak{n})$ and $\mathfrak{f}_\pi \subset \mathfrak{o}$ be the conductor of π. Then, for $v \in \Sigma_F - S(\mathfrak{f}_\pi)$, it is known that

$$\pi_v \cong \begin{cases} I\left(|\cdot|_v^{\frac{\nu_v(\pi_v)}{2}}\right) & (v\in\Sigma_{\rm fin}-S(\mathfrak{f}_\pi)) \\ I\left(\mathrm{sgn}(\cdot)^{\epsilon(\pi_v)}|\cdot|_v^{\frac{\nu_v(\pi_v)}{2}}\right) & (v\in\Sigma_\infty)\end{cases} \qquad (1.3)$$

where $\nu_v(\pi_v) \in \mathfrak{X}_v := \begin{cases}\mathbb{C}/4\pi\sqrt{-1}(\log q_v)^{-1}\mathbb{Z} & (v\in\Sigma_{\rm fin}) \\ \mathbb{C} & (v\in\Sigma_\infty)\end{cases}$ and $\epsilon(\pi_v)\in\{0,1\}$ for $v\in\Sigma_\infty$. We may choose the parameter $\nu_v(\pi_v)$ for each $v \in \Sigma_F - S(\mathfrak{f}_\pi)$ so that

$$\nu_v(\pi_v) \in \begin{cases}\sqrt{-1}[0,2\pi(\log q_v)^{-1}] \cup (\{0,2\pi\sqrt{-1}(\log q_v)^{-1}\}+(0,1)) & (v\in\Sigma_{\rm fin}-S(\mathfrak{f}_\pi)) \\ \sqrt{-1}\mathbb{R}_+\cup(0,1) & (v\in\Sigma_\infty).\end{cases} \qquad (1.4)$$

For $v\in\Sigma_{\rm fin}$ and a $\mathbf{K}_v$-spherical irreducible representation $\pi_v \cong I(|\cdot|_v^s)$, we define

$$Q(\pi_v) = \frac{x_v(\pi_v)}{q_v^{\frac12}+q_v^{-\frac12}}, \qquad x_v = q_v^s + q_v^{-s}$$

where (q_v^s, q_v^{-s}) is the Satake parameter of π_v.

In the following, we define many functions to describe our main results. For $v\in\Sigma_{\rm fin}$ and $\delta \in F_v$, let ε_δ be the character of $F_v^\times$ corresponding to the extension $F_v(\sqrt{\delta})/F_v$ by local class field theory. For $z, s\in\mathbb{C}$, we define

$$\begin{aligned}\mathcal{O}_{v,\epsilon}^{\delta,(z)}(a) =& \frac{\zeta_v(-z)}{L_v(\frac{-z+1}{2},\varepsilon_\delta)}\left(\frac{1+q_v^{\frac{z+1}{2}}}{1+q_v}\right)^{\epsilon}|a|_v^{\frac{-z+1}{4}} \\ &+ \frac{\zeta_v(z)}{L_v(\frac{z+1}{2},\varepsilon_\delta)}\left(\frac{1+q_v^{\frac{-z+1}{2}}}{1+q_v}\right)^{\epsilon}|a|_v^{\frac{z+1}{4}} \quad \left(\epsilon\in\{0,1\}, \quad a\in F_v^\times\right)\end{aligned}$$

and

$$\mathcal{S}_v^{\delta,(z)}(s;a) = \begin{cases} -q_v^{-\frac{s+1}{2}}\frac{\zeta_v(s+\frac{z+1}{2})\zeta_v(s+\frac{-z+1}{2})}{L_v(s+1,\varepsilon_\delta)}|a|_v^{\frac{s+1}{2}} & (|a|_v\le 1) \\ -q_v^{-\frac{s+1}{2}}\left\{\frac{\zeta_v(-z)\zeta_v(s+\frac{z+1}{2})}{L_v(\frac{-z+1}{2},\varepsilon_\delta)}|a|_v^{\frac{-z+1}{4}} + \frac{\zeta_v(z)\zeta_v(s+\frac{-z+1}{2})}{L_v(\frac{z+1}{2},\varepsilon_\delta)}|a|_v^{\frac{z+1}{4}}\right\} & (|a|_v>1)\end{cases}.$$

We remark that

$$\mathcal{O}_{v,0}^{\delta,(z)}(a) = 1, \qquad \mathcal{O}_{v,1}^{\delta,(z)}(a) = \begin{cases} 2 & (v \text{ splits in } F(\sqrt{\delta})/F) \\ 1 & (v \text{ ramifies in } F(\sqrt{\delta})/F) \\ 0 & (v \text{ remains prime in } F(\sqrt{\delta})/F) \end{cases} \qquad \text{for all } a \in \mathfrak{o}_v^\times.$$

by a simple calculation. For a finite subset $S \subset \Sigma_{\text{fin}}$, a non-zero square-free ideal $\mathfrak{n} \subset \mathfrak{o}$ such that $S \cap S(\mathfrak{n}) = \emptyset$, an element $\Delta \in F^\times$, and a non-zero fractional ideal $\mathfrak{a} \subset F$, we define the well-defined product

$$\mathbf{B}_{\mathfrak{n}}^{(z)}(\mathbf{s}_S|\Delta;\mathfrak{a}) = \prod_{v \in \Sigma_{\text{fin}} - (S \cup S(\mathfrak{n}))} \mathcal{O}_{v,0}^{\Delta,(z)}(a_v) \prod_{v \in S(\mathfrak{n})} \mathcal{O}_{v,1}^{\Delta,(z)}(a_v) \prod_{v \in S} \mathcal{S}_v^{\Delta,(z)}(s_v;a_v)$$

where $(a_v) \in \mathbb{A}_{\text{fin}}^\times$ is an idele corresponding to $\mathfrak{a}$ and $\mathbf{s}_S = (s_v)_{v \in S} \in \mathfrak{X}_S := \prod_{v \in S} \mathfrak{X}_v$.

For any condition P, we put $\delta(P) = 1$ if P is true and $\delta(P) = 0$ if P is false respectively. For $v \in \Sigma_\infty$ and $z, s \in \mathbb{C}$, we define

$$\mathcal{O}_v^{+,(z)}(s;a) = \frac{\pi^{-\frac{1}{2}}}{4}|a|_v|1-a^{-2}|_v^{\frac{s+1}{2}} \frac{\Gamma(\frac{s}{2}+\frac{z+1}{4})\Gamma(\frac{s}{2}-\frac{z-1}{4})}{\Gamma(s+1)}\delta(a>1)$$
$$\times F_3^{(1,0)}\left(\begin{matrix} \frac{s}{2}+\frac{z+1}{4}, \frac{s}{2}-\frac{z-1}{4}, \frac{s+1}{2}; \frac{z+1}{4}, \frac{-z+1}{4} \\ s+1; \frac{s+1}{2} \end{matrix}; 1-a^{-2}, 1-a^2\right) \qquad (a \in F_v^\times)$$

and

$$\mathcal{O}_v^{-,(z)}(s;a) = (a^2+1)\frac{\Gamma(\frac{s+1}{2})\Gamma(\frac{s}{2}+\frac{z+1}{4})\Gamma(\frac{s}{2}-\frac{z-1}{4})}{\Gamma(s+1)\Gamma(\frac{s}{2}+1)}$$
$$\times F_3^{(1,0)}\left(\begin{matrix} \frac{s}{2}+\frac{z+1}{4}, \frac{s}{2}-\frac{z-1}{4}, \frac{s+1}{2}; \frac{z+3}{4}, \frac{-z+3}{4} \\ s+1; \frac{s}{2}+1 \end{matrix}; 1, -a^2\right) \qquad (a \in F_v^\times)$$

where $F_3^{(1,0)}$ is a certain hypergeometric function in two variables. The details of this function are described in §5.

We assume that $\mathfrak{n} \subset \mathfrak{o}$ is a non-zero square-free ideal from now on. For $\pi \cong \otimes'_{v \in \Sigma_F} \pi_v \in \Pi_{\text{cus}}(\mathfrak{n})$, the symmetric square L-function $L(s,\pi;\text{Ad})$ of π on $\text{Re}(s) > 1$ is defined by the Euler product

$$L(s,\pi;\text{Ad}) = \prod_{v \in \Sigma_F} L_v(s,\pi_v,\text{Ad})$$

where

$$L_v(s,\pi_v,\text{Ad}) = \begin{cases} (1-q_v^{-s+\nu_v(\pi_v)})^{-1}(1-q_v^{-s-\nu_v(\pi_v)})^{-1}(1-q_v^{-s})^{-1} & (v \in \Sigma_{\text{fin}} - S(\mathfrak{f}_\pi)) \\ (1-q_v^{-s+1})^{-1} & (v \in S(\mathfrak{f}_\pi)) \\ \Gamma_{\mathbb{R}}(s+\nu_v(\pi_v))\Gamma_{\mathbb{R}}(s-\nu_v(\pi_v))\Gamma_{\mathbb{R}}(s) & (v \in \Sigma_\infty) \end{cases}$$

where $\nu_v(\pi_v)$ is the spectral parameter of π_v defined by (1.3). It is known that $L(s,\pi;\text{Ad})$ is continued to an entire function ([6]).

For a sufficiently large positive integer N and fixed parameters $\mathbf{s}_S = (s_v)_{v \in S} \in \mathfrak{X}_S$ with $\text{Re}(s_v) > 1$ and $\mathfrak{s}_\infty = (\mathfrak{s}_v = (s_{i,v})_{1 \le i \le N})_{v \in \Sigma_\infty} \in (\mathbb{C}^N)^{\Sigma_\infty}$ with $\text{Re}(s_{i,v}) > 1$ and $\text{Re}(s_{i,v}) \neq \text{Re}(s_{j,v})$ if $i \neq j$, let $I_{\text{cus}}^0(\mathbf{s}_S,\mathfrak{s}_\infty,z)$, $I_{\text{Eis}}^0(\mathbf{s}_S,\mathfrak{s}_\infty,z)$, and $I_{\text{res}}^0(\mathbf{s}_S,\mathfrak{s}_\infty,z)$ be meromorphic functions on the region $-1 < \text{Re}(z) < 1$ defined as follows.

The first term is given by

$$I_{\text{cus}}^0(\mathbf{s}_S,\mathfrak{s}_\infty,z) = \sum_{\pi \in \Pi_{\text{cus}}(\mathfrak{n})} C(\mathfrak{n},\pi)\mathbb{P}_{E_*}(\mathfrak{n},\pi) \tag{1.5}$$

where

$$C(\mathfrak{n},\pi) = D_F^{-1}\,[\mathbf{K}_{\mathrm{fin}}:\mathbf{K}_0(\mathfrak{n})]^{-1}\prod_{v\in\Sigma_\infty}\left[\frac{1}{2}\prod_{i=1}^{N}\left\{\frac{s_{i,v}^2-\nu_v(\pi_v)^2}{4}\right\}^{-1}\right]$$
$$\times\prod_{v\in S}\left\{\left(q_v^{\frac{1+\nu_v(\pi_v)}{2}}+q_v^{\frac{1-\nu_v(\pi_v)}{2}}\right)-\left(q_v^{\frac{s_v+1}{2}}+q_v^{\frac{-s_v+1}{2}}\right)\right\}^{-1}$$

and

$$\mathbb{P}_{E_*}(\mathfrak{n},\pi)=\frac{1}{2}\mathrm{N}(\mathfrak{f}_\pi)^{\frac{z-1}{2}}D_F^{z-\frac{1}{2}}\left\{\prod_{v\in S(\mathfrak{n}\mathfrak{f}_\pi^{-1})}\left(1+\frac{Q(I(|\ |_v^{\frac{z}{2}}))-Q(\pi_v)^2}{1-Q(\pi_v)^2}\right)\right\}\frac{L(\frac{z+1}{2},\pi;\mathrm{Ad})}{L(1,\pi;\mathrm{Ad})}.$$

Here, D_F is the absolute value of the discriminant of $F/\mathbb{Q}$.

The second term is given by

$$I^0_{\mathrm{Eis}}(\mathbf{s}_S,\mathfrak{s}_\infty,z)=\frac{\mathrm{vol}(\mathbb{A}^1/F^\times)^{-1}}{8\pi\sqrt{-1}}\sum_{\chi\in\Xi(\mathfrak{o})}\int_{\sqrt{-1}\mathbb{R}}I^0_{\mathrm{Eis},\chi}(u,z)du \tag{1.6}$$

where

$$\begin{aligned}
&I^0_{\mathrm{Eis},\chi}(u,z)\\
=&D_F^{z-2}\,[\mathbf{K}_{\mathrm{fin}}:\mathbf{K}_0(\mathfrak{n})]^{-\frac{3}{2}}\,\frac{\zeta_F(\frac{z+1}{2})L_F(\frac{z+1}{2}+u,\chi^2)L_F(\frac{z+1}{2}-u,\chi^{-2})}{L_F(1+u,\chi^2)L_F(1-u,\chi^{-2})}\\
&\times\prod_{v\in S(\mathfrak{n})}\left[(q_v^{\frac{1}{2}}-q_v^{-\frac{1}{2}})(1-q_v^{-\frac{3}{2}})+(q_v^{-\frac{1}{2}}+q_v^{-1})\left\{q_v^{\frac{z}{2}}+q_v^{-\frac{z}{2}}-q_v^{-u}\chi_v(\varpi_v)^2-q_v^{u}\chi_v(\varpi_v)^{-2}\right\}\right]\\
&\qquad\qquad\qquad\qquad\qquad\times L_v(1+u,\chi_v^2)L_v(1-u,\chi_v^{-2})\\
&\times\prod_{v\in S}\left(-q_v^{-\frac{s_v+1}{2}}\right)L_v\left(\tfrac{s_v+u}{2},\chi_v\right)L_v\left(\tfrac{s_v-u}{2},\chi_v^{-1}\right)\\
&\times\prod_{v\in\Sigma_\infty}\left[\prod_{i=1}^{N}\left\{\tfrac{s_{i,v}^2}{4}+\left(\tfrac{u}{2\sqrt{-1}}+a(\chi_v)\right)^2\right\}^{-1}\right].
\end{aligned}\tag{1.7}$$

The third term is given by

$$I^0_{\mathrm{res}}(\mathbf{s}_S,\mathfrak{s}_\infty,z)=\frac{1}{2}D_F^{\frac{z}{4}}\sum_{\substack{\chi\in\Xi(\mathfrak{o})\\ \chi^2=1}}\left\{I^0_{\mathrm{res},\chi}(z)+I^0_{\mathrm{res},\chi}(-z)\right\} \tag{1.8}$$

where

$$I^0_{\mathrm{res},\chi}(z)=D_F^{\frac{z}{4}-1}\frac{\zeta_F(z+1)}{\zeta_F(\frac{z+3}{2})}\prod_{v\in S(\mathfrak{n})}\frac{(1-q_v^{-1})(1+q_v^{-\frac{z+1}{2}})}{q_v+1}\zeta_v(\tfrac{z+3}{2})$$
$$\times\prod_{v\in S}\frac{\zeta_v(\frac{z+3}{2})}{\zeta_v(\frac{z+1}{2})}\mathcal{S}_{\mathrm{res},v,\chi_v}(s_v,z)\prod_{v\in\Sigma_\infty}\left[\prod_{i=1}^{N}\left\{\tfrac{s_{i,v}^2}{4}-\tfrac{(z+1)^2}{16}\right\}^{-1}\right]$$

and

$$\mathcal{S}_{\mathrm{res},v,\chi_v}(s_v,z)$$
$$=\frac{-q_v^{-\frac{s_v+1}{2}}}{1-q_v^{-s_v}}\times\Big[L_v(\tfrac{s_v}{2}-\tfrac{z+1}{4},\chi_v)L_v(\tfrac{s_v}{2}+\tfrac{z+1}{4},\chi_v)\left\{1+q^{-s}-2q_v^{-s-1}-\chi_v(\varpi_v)q_v^{-\frac{s}{2}-\frac{z+1}{4}}(1-q_v^{-1})\right\}$$
$$+(1-q_v^{-1})\zeta_v(\tfrac{z+1}{2})\zeta_v(s_v+\tfrac{z+1}{2})\left\{q_v^{-\frac{z+1}{2}}(1+2q_v^{-s})-q_v^{-s-z-1}+2\chi_v(\varpi_v)q_v^{-\frac{s}{2}-\frac{z+1}{4}}\right\}\Big]. \quad (1.9)$$

For $\Delta\in F^\times-(F^\times)^2$, let ε_Δ be the quadratic character corresponding to the quadratic extension $F(\sqrt{\Delta})/F$, $\mathfrak{d}_\Delta$ be the relative discriminant of $F(\sqrt{\Delta})/F$, and $\mathfrak{f}_\Delta$ be the fractional ideal such that $(\Delta)=\mathfrak{d}_\Delta\mathfrak{f}_\Delta^2$.

Theorem 1.1. *Let N be a sufficiently large positive integer, $\mathbf{s}_S=(s_v)_{v\in S}\in\mathfrak{X}_S$, $\mathfrak{s}_\infty=(\mathfrak{s}_v=(s_{i,v})_{1\le i\le N})_{v\in\Sigma_\infty}\in(\mathbb{C}^N)^{\Sigma_\infty}$ with $\min\{\mathrm{Re}(s_v)|v\in S\}>2\max\{\mathrm{Re}(s_{i,v})|^{v\in\Sigma_\infty;}_{1\le i\le N}\}+1$, $\mathrm{Re}(s_{i,v})>1$ and $\mathrm{Re}(s_{i,v})\neq\mathrm{Re}(s_{j,v})$ if $i\neq j$, $\mathfrak{n}\subset\mathfrak{o}$ be a non-zero square-free ideal, and $S\subset\Sigma_{\mathrm{fin}}$ be a finite subset such that $S\cap S(\mathfrak{n})=\emptyset$. Suppose that the prime 2 splits completely in $F/\mathbb{Q}$ and $|2|_v=1$ for all $v\in S\cup S(\mathfrak{n})$. Then, for $z\in\mathbb{C}$ such that $-1<\mathrm{Re}(z)<1$, we have the identity*

$$I^0_{\mathrm{cus}}(\mathbf{s}_S,\mathfrak{s}_\infty,z)+I^0_{\mathrm{Eis}}(\mathbf{s}_S,\mathfrak{s}_\infty,z)+I^0_{\mathrm{res}}(\mathbf{s}_S,\mathfrak{s}_\infty,z)$$
$$=J^0_{\mathrm{uni}}(\mathbf{s}_S,\mathfrak{s}_\infty,z)+J^0_{\mathrm{hyp}}(\mathbf{s}_S,\mathfrak{s}_\infty,z)+J^0_{\mathrm{ell}}(\mathbf{s}_S,\mathfrak{s}_\infty,z)$$

where the three terms on the right-hand side are described explicitly as follows.

The first term is given by

$$J^0_{\mathrm{uni}}(\mathbf{s}_S,\mathfrak{s}_\infty,z)=D_F^{\frac{z}{4}}\left\{\Lambda_F(-z)\hat{J}^1_{\mathrm{uni}}(\mathbf{s}_S,\mathfrak{s}_\infty,z)+\Lambda_F(z)\hat{J}^1_{\mathrm{uni}}(\mathbf{s}_S,\mathfrak{s}_\infty,-z)\right\}$$

where we set $\Lambda_F(z)=D_F^{\frac{z}{2}}\zeta_F(z)$ and

$$\hat{J}^1_{\mathrm{uni}}(\mathbf{s}_S,\mathfrak{s}_\infty,z)=D_F^{\frac{z-2}{4}}\prod_{v\in S}\frac{-q_v^{-\frac{s_v+1}{2}}}{1-q_v^{-s_v-\frac{z+1}{2}}}\prod_{v\in S(\mathfrak{n})}\frac{1+q_v^{\frac{z+1}{2}}}{1+q_v}$$
$$\times\prod_{v\in\Sigma_\infty}2^{-\frac{z+3}{2}}\pi^{-\frac{z+3}{4}}\Gamma\left(\tfrac{-z+1}{4}\right)\left\{\sum_{i=1}^N\mu_v^{(i)}(\mathfrak{s}_\infty)\frac{\Gamma(\frac{s_{i,v}}{2}+\frac{z+1}{4})}{\Gamma(\frac{s_{i,v}}{2}-\frac{z-3}{4})}\right\}$$

with

$$\mu_v^{(i)}(\mathfrak{s}_\infty)=\prod_{\substack{1\le j\le N\\ j\neq i}}\left(\frac{s_{j,v}^2-s_{i,v}^2}{4}\right)^{-1}.$$

The second term is given by the absolutely convergent sum

$$J^0_{\mathrm{hyp}}(\mathbf{s}_S,\mathfrak{s}_\infty,z)=\frac{1}{2}D_F^{\frac{-1}{2}}\zeta_F\left(\frac{-z+1}{2}\right)\sum_{a\in\mathfrak{o}(S)^\times_+\setminus\{1\}}\mathbf{B}^{(z)}_{\mathfrak{n}}\left(\mathbf{s}_S|1;\frac{a}{(a-1)^2}\mathfrak{o}\right)$$
$$\times\prod_{v\in\Sigma_\infty}\left\{\sum_{i=1}^N\mu_v^{(i)}(\mathfrak{s}_\infty)\mathcal{O}_v^{+,(z)}\left(s_{i,v};\frac{a+1}{a-1}\right)\right\},$$

where $\mathfrak{o}(S)^\times_+$ denotes the set of all totally positive S-units.

The third term is given by the absolutely convergent sum

$$J^0_{\mathrm{ell}}(\mathbf{s}_S,\mathfrak{s}_\infty,z)=\frac{1}{2}D_F^{\frac{z-1}{2}}\sum_{(t:n)_F}\mathrm{N}(\mathfrak{d}_F)^{\frac{z+1}{4}}L\left(\frac{z+1}{2},\varepsilon_\Delta\right)\mathbf{B}^{(z)}_{\mathfrak{n}}\left(\mathbf{s}_S|\Delta;n\mathfrak{f}_\Delta^{-2}\right)$$
$$\times\prod_{v\in\Sigma_\infty}\left\{\sum_{i=1}^N\mu_v^{(i)}(\mathfrak{s}_\infty)\mathcal{O}_v^{\mathrm{sgn}(\Delta^{(v)}),(z)}\left(s_{i,v};t|\Delta|_v^{\frac{-1}{2}}\right)\right\},$$

where $(t:n)_F$ runs over cosets $\{(ct,c^2n)\in F^2\mid c\in F^\times\}$ such that $\Delta=t^2-4n\in F^\times-(F^\times)^2$, $(t,n)\in\{(c_vt_v,c_v^2n_v)|c_v\in F_v^\times,t\in\mathfrak{o}_v,n_v\in\mathfrak{o}_v^\times\}$ for all $v\in\Sigma_{\mathrm{fin}}-S$, and $\mathrm{ord}_v(n\mathfrak{f}_\Delta^{-2})<0$ for all $v\in S(\mathfrak{n})$ with $\varepsilon_{\Delta,v}$ being unramified and non-trivial.

We have a corollary of Theorem 1.1 which recovers Zagier's formula for Maass forms ([22]). Let $\mathfrak{B}$ be the space of all entire functions such that $\alpha(s)=\alpha(-s)$ and satisfying the estimate

$$|\alpha(s)|\ll_{a,b,l}(1+|\mathrm{Im}(s)|)^{-l},\qquad \mathrm{Re}(s)\in[a,b]$$

for any closed interval $[a,b]\subset\mathbb{R}$ and $l>0$. For $v\in S$, let $\mathcal{A}_v$ be the space of all holomorphic functions on $\mathfrak{X}_v$ such that $\alpha(s)=\alpha(-s)$. For $v\in\Sigma_\infty$, set $\mathcal{A}_v=\mathfrak{B}$. We set the holomorphic 1-form $d\mu_v(s)$ on $\mathfrak{X}_v$ by

$$d\mu_v(s)=\begin{cases}2^{-1}(\log q_v)(q_v^{\frac{s+1}{2}}-q_v^{\frac{-s+1}{2}})ds & (v\in S)\\ s\ ds & (v\in\Sigma_\infty)\end{cases}.$$

Let $c>1$ and $L_v(c)$ be the contour on $\mathfrak{X}_v$ defined by

$$L_v(c)=\begin{cases}\{c+\sqrt{-1}y|\ -2\pi(\log q_v)^{-1}\le y\le 2\pi(\log q_v)^{-1}\} & (v\in S)\\ \{c+\sqrt{-1}y|\ y\in\mathbb{R}\} & (v\in\Sigma_\infty)\end{cases},$$

For $v\in S$, $\alpha\in\mathcal{A}_v$, $z\in\mathbb{C}$, and $a\in F_v^\times$, set

$$\hat{\mathcal{S}}_v^{\delta,(z)}(\alpha;a)=\frac{1}{2\pi\sqrt{-1}}\int_{L_v(c)}\mathcal{S}_v^{\delta,(z)}(s;a)\alpha(s)d\mu_v(s).$$

Let $v\in\Sigma_\infty$, $\alpha\in\mathcal{A}_v$, and $a\in F_v^\times$, we define

$$\hat{\mathcal{O}}_v^{\pm,(z)}(\alpha;a)=\frac{1}{2\pi\sqrt{-1}}\int_{L_v(c)}\mathcal{O}_v^{\pm,(z)}(s;a)\alpha(s)d\mu_v(s).$$

Let $\mathcal{A}_{S\cup\Sigma_\infty}=\bigotimes_{v\in S\cup\Sigma_\infty}\mathcal{A}_v$. For a pure tensor $\alpha=\otimes_{v\in S\cup\Sigma_\infty}\alpha_v$ of $\mathcal{A}_{S\cup\Sigma_\infty}$, a non-zero square-free integral ideal $\mathfrak{n}$ such that $S\cap S(\mathfrak{n})=\emptyset$, an element of $\Delta\in F^\times$, a non-zero fractional ideal of F, $\chi\in\Xi(\mathfrak{o})$, and $z,u\in\mathbb{C}$, we set

$$\mathbf{B}^{(z)}_{\mathfrak{n}}(\alpha|\Delta;\mathfrak{a})=\prod_{v\in\Sigma_{\mathrm{fin}}-(S\cup S(\mathfrak{n}))}\mathcal{O}^{\Delta,(z)}_{v,0}(a_v)\prod_{v\in S(\mathfrak{n})}\mathcal{O}^{\Delta,(z)}_{v,1}(a_v)\prod_{v\in S}\hat{\mathcal{S}}_v^{\Delta,(z)}(\alpha_v;a_v),$$

$$\Upsilon^{(z)}(\alpha)=\prod_{v\in S}\frac{1}{2\pi\sqrt{-1}}\int_{L_v(c)}\frac{-q_v^{-\frac{s+1}{2}}}{1-q_v^{-s-\frac{z+1}{2}}}\alpha_v(s)d\mu_v(s)$$
$$\times\prod_{v\in\Sigma_\infty}\frac{1}{2\pi\sqrt{-1}}\int_{L_v(c)}\frac{\Gamma(\frac{s}{2}+\frac{z+1}{4})}{\Gamma(\frac{s}{2}-\frac{z-3}{4})}\alpha_v(s)d\mu_v(s),$$

$$\Theta_{\mathrm{Eis},\chi}(\alpha;u)=2^{n_F}\prod_{v\in S}\frac{1}{2\pi\sqrt{-1}}\int_{L_v(c)}(-q_v^{-\frac{s+1}{2}})L_v(\tfrac{s+u}{2},\chi_v)L_v(\tfrac{s-u}{2},\chi_v^{-1})\alpha_v(s)d\mu_v(s)$$
$$\times\prod_{v\in\Sigma_\infty}\alpha_v\left(u+2\sqrt{-1}a(\chi_v)\right), \tag{1.10}$$

and

$$\Theta_{\mathrm{res},\chi}(\alpha;z)=2^{n_F}\prod_{v\in S}\frac{1}{2\pi\sqrt{-1}}\frac{\zeta_v(\frac{z+1}{2})}{\zeta_v(\frac{z+3}{2})}\int_{L_v(c)}\mathcal{S}_{\mathrm{res},v,\chi_v}(s,z)\alpha_v(s)d\mu_v(s)\prod_{v\in\Sigma_\infty}\alpha_v(\tfrac{z+1}{2}).$$

Moreover, we define

$$\mathbb{I}^0_{\mathrm{cusp}}(\alpha|\mathfrak{n};z)=\frac{(-1)^{\#S}}{2}D_F^{z-\frac{3}{2}}[\mathbf{K}_{\mathrm{fin}}:\mathbf{K}_0(\mathfrak{n})]^{-1}$$
$$\times\sum_{\pi\in\Pi_{\mathrm{cus}}(\mathfrak{n})}\mathrm{N}(\mathfrak{f}_\pi)^{\frac{z-1}{2}}\prod_{v\in S(\mathfrak{n}\mathfrak{f}_\pi^{-1})}\left(1+\frac{Q(I(|\ |_v^{\frac{z}{2}}))-Q(\pi_v)^2}{1-Q(\pi_v)^2}\right)$$
$$\times\frac{L(\frac{z+1}{2},\pi;\mathrm{Ad})}{L(1,\pi;\mathrm{Ad})}\prod_{v\in S\cup\Sigma_\infty}\alpha_v(\nu_v(\pi_v)),$$
$$\mathbb{I}^0_{\mathrm{Eis}}(\alpha|\mathfrak{n};z)=\frac{\mathrm{vol}(\mathbb{A}^1/F^\times)^{-1}}{8\pi\sqrt{-1}}\sum_{\chi\in\Xi(\mathfrak{o})}\int_{\sqrt{-1}\mathbb{R}}\mathbb{I}^0_{\mathrm{Eis},\chi}(\alpha;u,z)du,$$
$$\mathbb{I}^0_{\mathrm{res}}(\alpha|\mathfrak{n};z)=\frac{1}{2}D_F^{\frac{z}{4}}\sum_{\substack{\chi\in\Xi(\mathfrak{o})\\ \chi^2=1}}\left\{\mathbb{I}^0_{\mathrm{res},\chi}(\alpha;z)+\mathbb{I}^0_{\mathrm{res},\chi}(\alpha;-z)\right\}$$

where

$$\mathbb{I}^0_{\mathrm{Eis},\chi}(\alpha;u,z)$$
$$=D_F^{z-2}\left[\mathbf{K}_{\mathrm{fin}}:\mathbf{K}_0(\mathfrak{n})\right]^{-\frac{3}{2}}\frac{\zeta_F(\frac{z+1}{2})L_F(\frac{z+1}{2}+u,\chi^2)L_F(\frac{z+1}{2}-u,\chi^{-2})}{L_F(1+u,\chi^2)L_F(1-u,\chi^{-2})}\Theta_{\mathrm{Eis},\chi}(\alpha;u)$$
$$\times\prod_{v\in S(\mathfrak{n})}\left[(q_v^{\frac{1}{2}}-q_v^{-\frac{1}{2}})(1-q_v^{-\frac{3}{2}})+(q_v^{-\frac{1}{2}}+q_v^{-1})\left\{q_v^{\frac{z}{2}}+q_v^{-\frac{z}{2}}-q_v^{-u}\chi_v(\varpi_v)^2-q_v^{u}\chi_v(\varpi_v)^{-2}\right\}\right]$$
$$\times L_v(1+u,\chi_v^2)L_v(1-u,\chi_v^{-2})$$

and

$$\mathbb{I}^0_{\mathrm{res},\chi}(\alpha;z)=D_F^{\frac{z}{4}-1}\frac{\zeta_F(z+1)}{\zeta_F(\frac{z+3}{2})}\Theta_{\mathrm{res},\chi}(\alpha;z)$$
$$\times\prod_{v\in S(\mathfrak{n})}\frac{(1-q_v^{-1})(1+q_v^{-\frac{z+1}{2}})}{q_v+1}\zeta_v(\tfrac{z+3}{2}).$$

Corollary 1.2. *Let $\mathfrak{n}$ and S be as in Theorem 1.1. Suppose that the prime 2 splits completely in $F/\mathbb{Q}$ and $|2|_v=1$ for all $v\in S\cup S(\mathfrak{n})$. Then, for $\alpha=\otimes_{v\in S\cup\Sigma_\infty}\alpha_v\in\mathcal{A}_{S\cup\Sigma_\infty}$ and $z\in\mathbb{C}$ such that $-1<\mathrm{Re}(z)<1$, we have the identity*

$$\mathbb{I}^0_{\mathrm{cusp}}(\alpha|\mathfrak{n};z)+\mathbb{I}^0_{\mathrm{Eis}}(\alpha|\mathfrak{n};z)+\mathbb{I}^0_{\mathrm{res}}(\alpha|\mathfrak{n};z)=\mathbb{J}^0_{\mathrm{uni}}(\alpha|\mathfrak{n};z)+\mathbb{J}^0_{\mathrm{hyp}}(\alpha|\mathfrak{n};z)+\mathbb{J}^0_{\mathrm{ell}}(\alpha|\mathfrak{n};z)$$

where the three terms on the right-hand side are described explicitly as follows.

The first term is given by

$$\mathbb{J}^0_{\text{uni}}(\alpha|\mathfrak{n};z) = D_F^{\frac{z}{4}}\left\{\Lambda_F(-z)\hat{\mathbb{J}}^1_{\text{uni}}(\alpha|\mathfrak{n};z)+\Lambda_F(z)\hat{\mathbb{J}}^1_{\text{uni}}(\alpha|\mathfrak{n};-z)\right\}$$

with

$$\hat{\mathbb{J}}^1_{\text{uni}}(\alpha|\mathfrak{n};z) = \left\{2^{-\frac{z+3}{2}}\pi^{-\frac{z+3}{4}}\Gamma\left(\tfrac{-z+1}{4}\right)\right\}^{n_F} D_F^{\frac{z-2}{4}}\Upsilon^{(z)}(\alpha)\prod_{v\in S(\mathfrak{n})}\frac{1+q_v^{\frac{z+1}{2}}}{1+q_v}.$$

The second term is given by the absolutely convergent sum

$$\mathbb{J}^0_{\text{hyp}}(\alpha|\mathfrak{n};z) = \frac{1}{2}D_F^{\frac{-1}{2}}\zeta_F\left(\frac{-z+1}{2}\right)\sum_{a\in\mathfrak{o}(S)^\times_+\setminus\{1\}}\mathbf{B}^{(z)}_{\mathfrak{n}}\left(\alpha|1;\frac{a}{(a-1)^2}\mathfrak{o}\right)\prod_{v\in\Sigma_\infty}\hat{\mathcal{O}}_v^{+,(z)}\left(\alpha_v;\frac{a+1}{a-1}\right).$$

The third term is given by the absolutely convergent sum

$$\mathbb{J}^0_{\text{ell}}(\alpha|\mathfrak{n};z) = \frac{1}{2}D_F^{\frac{z-1}{2}}\sum_{(t:n)_F}\mathrm{N}(\mathfrak{d}_F)^{\frac{z+1}{4}}L\left(\frac{z+1}{2},\varepsilon_\Delta\right)\mathbf{B}^{(z)}_{\mathfrak{n}}\left(\alpha|\Delta;n\mathfrak{f}_\Delta^{-2}\right)\times\prod_{v\in\Sigma_\infty}\hat{\mathcal{O}}_v^{\mathrm{sgn}(\Delta^{(v)}),(z)}\left(\alpha_v;t|\Delta|_v^{\frac{-1}{2}}\right),$$

where $(t:n)_F$ runs over the same cosets as in Theorem 1.1.

1.4 The organization

Let us explain the content of this paper. In §2, we construct the kernel function. In §3, we recall the theory of Eisenstein series and the spectral decomposition for $\mathcal{L}^2(Z_\mathbb{A}G_F\backslash G_\mathbb{A})$ and define the smoothed Eisenstein series investigated in ([16]). In §4, we calculate the spectral side. In §5, we observe the geometric side and compute the identity term and the unipotent term. We state a brief explanation of the hypergeometric series in two variables dealt with in this article. The computations of the hyperbolic term and the elliptic term are done in §6 and §7. In §8, we prove our main results.

2 Construction of the kernel function

In this section, we construct the test function treated in this article. The functions at finite places are the same as the ones in ([16]). For infinite places, the matrix coefficient of the discrete series representation of $G_\mathbb{R}/Z_\mathbb{R}$ is dealt with in ([16]). In this article, we replace it by the resolvent kernel function of the Laplacian on the upper half-plane.

We set the Haar measures as the following way. For $v\in\Sigma_{\text{fin}}$, let dx_v be the Haar measure on F_v such that $\int_{\mathfrak{o}_v}dx_v = q^{-\frac{\mathrm{d}_v}{2}}$ where d_v is the local exponent of the different of $F/\mathbb{Q}$. We fix the Haar measure $d^\times x_v$ on $F_v^\times$ by $d^\times x_v = \zeta_v(1)\frac{dx_v}{|x|_v}$. For $v\in\Sigma_\infty$, let dx_v and be the standard Lebesgue measure on $\mathbb{R}$. Set the Haar measure $d^\times x_v$ on $\mathbb{R}^\times$ by $d^\times x_v = \frac{dx_v}{|x|_v}$. We set the Haar measures on $\mathbb{A}$ and $\mathbb{A}^\times$ by $dx=\prod_{v\in\Sigma_F}dx_v$ and $d^\times x = \prod_{v\in\Sigma_F}d^\times x_v$ respectively. We also define the Haar measures on H_v, N_v, and Z_v by the group isomorphisms $H_v\cong(F_v^\times)^2$, $N_v\cong F_v$, and $Z_v\cong F_v^\times$ accordingly, i.e. $dh_v = d^\times t_{1,v}d^\times t_{2,v}$ if $h_v=\begin{pmatrix}t_{1,v}&0\\0&t_{2,v}\end{pmatrix}$, $dn_v=dx_v$

if $n_v = \begin{pmatrix} 1 & x_v \\ 0 & 1 \end{pmatrix}$, and $dz_v = da_v$ if $z_v = \begin{pmatrix} a_v & 0 \\ 0 & a_v \end{pmatrix}$. Let dk_v be the Haar measure dk_v on $\mathbf{K}_v$ such that $\mathrm{vol}(\mathbf{K}_v; dk_v) = \int_{\mathbf{K}_v} dk_v = 1$. Then, our Haar measure dg_v on G_v is defined as $dg_v = dh_v dn_v dk_v$ by the Iwasawa decomposition $G_v = H_v N_v K_v$. On $\mathbf{K}$, $H_{\mathbb{A}}$, $N_{\mathbb{A}}$, $Z_{\mathbb{A}}$, and $G_{\mathbb{A}}$, we always use the product measures of their factors. More precisely,

$$\begin{aligned} \int_{G_{\mathbb{A}}} f(g)dg &= \int_{\mathbf{K}} \int_{\mathbb{A}^\times} \int_{\mathbb{A}^\times} \int_{\mathbb{A}} f\left(\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} k\right) d^\times a_1 d^\times a_2 dx dk \\ &= \int_{\mathbf{K}} \int_{\mathbb{A}^\times} \int_{\mathbb{A}^\times} \int_{\mathbb{A}} f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} k\right) \left|\frac{a_2}{a_1}\right|_{\mathbb{A}} d^\times a_1 d^\times a_2 dx dk \end{aligned}$$

holds for all $f \in \mathcal{L}^1(G_{\mathbb{A}})$.

2.1 Convergence lemmas

Recall that a Siegel domain $\mathfrak{S}_1$ in $G(\mathbb{A})^1 = \{g \in G_{\mathbb{A}} \mid |\det g|_{\mathbb{A}} = 1\}$ is a subset of $G(\mathbb{A})^1$ of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} k$$

where $k \in \mathbf{K}$, x is in a compact subset of $\mathbb{A}$, a and d lie in a compact subset of $\mathbb{A}^1$, and $t_1, t_2 \in \mathbb{A}^\times$ such that $t_{1,v} = t_{2,v} = 1$ for all $v \in \Sigma_{\mathrm{fin}}$ and $\left|\frac{t_{1,v}}{t_{2,v}}\right|_v > c$ for all $v \in \Sigma_\infty$ and some fixed positive number $c > 0$. It is known that if $\mathfrak{S}_1$ is sufficiently large, then $G(\mathbb{A})^1 = G(F)\mathfrak{S}_1$. We fix such a domain $\mathfrak{S}_1$.

We introduce convergence estimates investigated in ([16]) by means of a certain norm on $G_{\mathbb{A}}$. For $v \in \Sigma_F$ and g_v, let $(\mathrm{Ad}(g_v)_{ij})_{1 \le i,j \le 3}$ be the representing matrix of the endomorphism on $\mathfrak{sl}_2(F_v)$ given by $X \mapsto gXg^{-1}$ with respect to the F_v-basis $\left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\}$ of $\mathfrak{sl}_2(F_v)$. Set

$$||g_v||_v = \begin{cases} \max\{|\mathrm{Ad}(g_v)_{ij}|_v \mid 1 \le i,j \le 3\} & (v \in \Sigma_{\mathrm{fin}}) \\ \left(\sum_{1 \le i,j \le 3} |\mathrm{Ad}(g_v)_{ij}|^2\right)^{\frac{1}{2}} & (v \in \Sigma_\infty) \end{cases}$$

for $g_v \in G_v$ and

$$||g||_{\mathbb{A}} = \prod_{v \in \Sigma_F} ||g_v||_v$$

for $g = (g_v) \in G_{\mathbb{A}}$. We remark that the norm $||\cdot||_{\mathbb{A}}$ is $Z_{\mathbb{A}}$-invariant and bi-$\mathbf{K}$-invariant, take values in $[1, +\infty)$, and satisfies $||gh||_{\mathbb{A}} \le ||g||_{\mathbb{A}} ||h||_{\mathbb{A}}$ for all $g, h \in G_{\mathbb{A}}$. We define the function $y : G_{\mathbb{A}} \to \mathbb{R}_+$ as

$$y\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} k\right) = \left|\frac{a}{d}\right|_{\mathbb{A}}, \quad \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in B_{\mathbb{A}}, \quad k \in \mathbf{K}$$

by the Iwasawa decomposition of $G_{\mathbb{A}}$. Then we have $||g||_{\mathbb{A}} \asymp y(g)$, $g \in \mathfrak{S}_1$.

Proposition 2.1. ([16, Proposition 2.2]) *Let $\varphi : G_{\mathbb{A}} \to \mathbb{C}$ be a function such that $\varphi(zg) = \varphi(g)$ for all $z \in Z_{\mathbb{A}}, g \in G_{\mathbb{A}}$ and satisfies $|\varphi(g)| \ll ||g||_{\mathbb{A}}^{-m}$ on $G_{\mathbb{A}}$ with $m > 1$. Then, the series*

$$K_\varphi(g,h) = \sum_{\gamma \in Z_F \backslash G_F} \varphi(g^{-1}\gamma h)$$

converges absolutely and locally uniformly on $g, h \in G_{\mathbb{A}}$. Moreover, the following holds.

(i) $\sup_{h\in G_{\mathbb{A}}}|K_\varphi(g,h)| \ll ||g||_{\mathbb{A}}^m$, $g\in G_{\mathbb{A}}$ *with the implied constant independent of* g.
(ii) *For any* $g\in G_{\mathbb{A}}$, *the function* $h\mapsto K_\varphi(g,h)$ *belongs to* $\mathcal{L}^q(Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}})$ *for any* $q>0$.

Proof. See ([16, pp. 2988–2989]). □

2.2 The resolvent kernel function of the Laplacian

We introduce basic properties of the resolvent kernel function of the Laplacian from ([8]). For $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1$, we define $\mathrm{G}_s:\mathbb{R}_+\to\mathbb{C}$ as

$$\mathrm{G}_s(u)=\frac{\Gamma(\frac{s+1}{2})^2}{4\pi\Gamma(s+1)}u^{-\frac{s+1}{2}}{}_2F_1\left(\tfrac{s+1}{2},\tfrac{s+1}{2};s+1;-\frac{1}{u}\right)$$

where ${}_2F_1$ is the Gaussian hypergeometric function. From ([8, Lemma 1.7]), G_s satisfies the conditions

$$\mathrm{G}_s(u)=\frac{1}{4\pi}\log\frac{1}{u}+O(1),\qquad (u\to+0), \tag{2.1}$$

$$\mathrm{G}'_s(u)=-(4\pi u)^{-1}+O(1),\qquad (u\to+0), \tag{2.2}$$

$$\mathrm{G}_s(u)\ll u^{-\frac{\mathrm{Re}(s)+1}{2}},\qquad (u\to+\infty). \tag{2.3}$$

It is remarkable that the growths of $\mathrm{G}_s(u)$ and $\mathrm{G}'_s(u)$ when u tends to $+0$ are independent of s. For distinct $s_1,s_2\in\mathbb{C}$ with $\mathrm{Re}(s_1)>1$ and $\mathrm{Re}(s_2)>1$, $\lim_{u\to+0}(\mathrm{G}_{s_1}(u)-\mathrm{G}_{s_2}(u))$ exists because $\mathrm{G}_{s_1}(u)-\mathrm{G}_{s_2}(u)$ and $\mathrm{G}'_{s_1}(u)-\mathrm{G}'_{s_2}(u)$ are bounded for u near to 0 from (2.1) and (2.2). Moreover, the limit can be written explicitly as

$$\lim_{u\to+0}(\mathrm{G}_{s_1}(u)-\mathrm{G}_{s_2}(u))=-\frac{1}{4\pi}\left(\frac{\Gamma'(\frac{s_1+1}{2})}{\Gamma(\frac{s_1+1}{2})}-\frac{\Gamma'(\frac{s_2+1}{2})}{\Gamma(\frac{s_2+1}{2})}\right) \tag{2.4}$$

by ([8, (10.24)]).

Let $\mathbb{H}=\{x+\sqrt{-1}y\in\mathbb{C}\mid x\in\mathbb{R},y>0\}$ be the complex upper half-plane. The hyperbolic distance between $z\in\mathbb{H}$ and $w\in\mathbb{H}$ is given by

$$\rho(z,w)=\log\frac{|z-\overline{w}|+|z-w|}{|z-\overline{w}|-|z-w|}.$$

Then, we have

$$\cosh\rho(z,w)=1+2u(z,w)$$

where

$$u(z,w)=\frac{|z-w|^2}{4\mathrm{Im}(z)\mathrm{Im}(w)}.$$

Let $\Delta_0=-y^2(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})$ be the Laplace operator on $\mathbb{H}$. Then, for fixed $w\in\mathbb{H}$, the function $z\to\mathrm{G}_s(u(z,w))$ $(z\in\mathbb{H}-\{w\})$ is an eigenfunction of Δ_0 with eigenvalue $\frac14(1-s^2)$, i.e.

$$\left(\Delta_0-\frac{1-s^2}{4}\right)\mathrm{G}_s(u(\cdot,w))=0. \tag{2.5}$$

Let R_s be the integral operator on $\mathbb{H}$ whose kernel function is given by $(z,w)\mapsto\mathrm{G}_s(u(z,w))$, namely

$$(\mathrm{R}_sf)(z)=\int_{\mathbb{H}}\mathrm{G}_s(u(z,w))f(w)d\mu w \tag{2.6}$$

where $d\mu z = \frac{dxdy}{y^2}$ is the standard $SL_2(\mathbb{R})$-invariant measure on $\mathbb{H}$ and $f : \mathbb{H} \to \mathbb{C}$ be a measurable function on $\mathbb{H}$ such that the integral (2.6) converges absolutely. Then, R_s is the right inverse function of $\Delta_0 - \frac{1-s^2}{4}$.

Proposition 2.2. [8, Theorem 1.17] *Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. If f is smooth and bounded on $\mathbb{H}$, then*

$$\left(\Delta_0 - \frac{1-s^2}{4}\right)\mathrm{R}_s f = f.$$

Remark 2.3. We note that Δ and R_s in [8] coincide with $-\Delta_0$ and $-\mathrm{R}_{2s-1}$ respectively in our notations.

Let N be a positive integer and $\mathfrak{s}_\infty = (\mathfrak{s}_v = (s_{i,v})_{1\le i\le N})_{v\in\Sigma_\infty} \in (\mathbb{C}^N)^{\Sigma_\infty}$ with $\mathrm{Re}(s_{i,v}) > 1$ and $\mathrm{Re}(s_{i,v}) \neq \mathrm{Re}(s_{j,v})$ if $i \neq j$ for all $v \in \Sigma_\infty$. We define

$$\mu_v^{(i)}(\mathfrak{s}_\infty) = \prod_{\substack{1\le j\le N\\ j\neq i}} \left(\frac{s_{j,v}^2 - s_{i,v}^2}{4}\right)^{-1}$$

and $\varphi_{\mathfrak{s}_v} : \mathbb{R}_+ \to \mathbb{C}$ for $v \in \Sigma_\infty$ by

$$\varphi_{\mathfrak{s}_v}(u) = \sum_{i=1}^{N} \mu_v^{(i)}(\mathfrak{s}_\infty)\mathrm{G}_{s_{i,v}}(u).$$

Lemma 2.4. *For $v \in \Sigma_\infty$ and $\lambda \in \mathbb{C}$, we have*

$$\sum_{i=1}^{N} \mu_v^{(i)}(\mathfrak{s}_\infty)\left(\lambda - \frac{1-s_{i,v}^2}{4}\right)^{-1} = \prod_{i=1}^{N}\left(\lambda - \frac{1-s_{i,v}^2}{4}\right)^{-1}. \tag{2.7}$$

Proof. As meromorphic functions on $\lambda \in \mathbb{C}$, the possible singularities of both sides of (2.7) occur at $\lambda = \frac{1-s_{i,v}^2}{4}$ $(1 \le j \le N)$ as simple poles. By a simple calculation, their residues at $\lambda = \frac{1-s_{i,v}^2}{4}$ coincide. Hence the difference of the left-hand side and the right-hand side is entire and tends to 0 when $|\lambda| \to +\infty$. By Liouville's theorem, (2.7) holds. □

Corollary 2.5. *If $N \ge 2$, we have*

$$\sum_{i=1}^{N} \mu_v^{(i)}(\mathfrak{s}_\infty) = 0.$$

Proof. By multiplying λ in (2.7) and taking $|\lambda| \to +\infty$, the claim follows. □

Lemma 2.6. *If $N \ge 2$, we have*

$$\lim_{u\to+0} \varphi_{\mathfrak{s}_v}(u) = -\frac{1}{4\pi}\sum_{i=1}^{N} \mu_v^{(i)}(\mathfrak{s}_\infty)\frac{\Gamma'(\frac{s_{i,v}+1}{2})}{\Gamma(\frac{s_{i,v}+1}{2})}. \tag{2.8}$$

Proof. By Corollary 2.5, and (2.4), we have

$$\begin{aligned}
\lim_{u\to+0}\varphi_{\mathfrak{s}_v}(u) &= \lim_{u\to+0}\sum_{i=1}^{N}\mu_v^{(i)}(\mathfrak{s}_\infty)\mathrm{G}_{s_{i,v}}(u)\\
&= \lim_{u\to+0}\sum_{i=1}^{N-1}\mu_v^{(i)}(\mathfrak{s}_\infty)\left(\mathrm{G}_{s_{i,v}}(u)-\mathrm{G}_{s_{N,v}}(u)\right)\\
&= -\frac{1}{4\pi}\sum_{i=1}^{N-1}\mu_v^{(i)}(\mathfrak{s}_\infty)\left(\frac{\Gamma'(\frac{s_{i,v}+1}{2})}{\Gamma(\frac{s_{i,v}+1}{2})}-\frac{\Gamma'(\frac{s_{N,v}+1}{2})}{\Gamma(\frac{s_{N,v}+1}{2})}\right)\\
&= -\frac{1}{4\pi}\sum_{i=1}^{N}\mu_v^{(i)}(\mathfrak{s}_\infty)\frac{\Gamma'(\frac{s_{i,v}+1}{2})}{\Gamma(\frac{s_{i,v}+1}{2})}.
\end{aligned}$$

□

From now on, we regard $\varphi_{\mathfrak{s}_v}$ as a continuous function on $[0,\infty)$ by Lemma 2.6.

For $v\in\Sigma_\infty$, let $\mathfrak{s}_\infty=(\mathfrak{s}_v=(s_{i,v})_{1\le i\le N})_{v\in\Sigma_\infty}\in(\mathbb{C}^N)^{\Sigma_\infty}$ be as above. We define the function $\Phi_{\mathfrak{s}_v}:G_v(=G_\mathbb{R})\to\mathbb{C}$ by

$$\Phi_{\mathfrak{s}_v}(g)=\begin{cases}\varphi_{\mathfrak{s}_v}\left(\sinh^2 t_g\right) & (\det g>0)\\ 0 & (\det g<0)\end{cases} \tag{2.9}$$

where $t_g\in\mathbb{R}_{\ge0}$ such that $g\in Z_v\mathbf{K}_v\begin{pmatrix}e^{t_g} & 0\\ 0 & e^{-t_g}\end{pmatrix}\mathbf{K}_v$. By the second equation of ([12, p. 47, (2.4.1)]) we can rewrite $\Phi_{\mathfrak{s}_v}(g)$ for $g\in G_v$ with $\det g>0$ and $t_g>0$ as

$$\begin{aligned}
\Phi_{\mathfrak{s}_v}(g) &= \frac{1}{4\pi}\sum_{i=1}^{N}\mu_v^{(i)}(\mathfrak{s}_\infty)\frac{\Gamma(\frac{s_{i,v}+1}{2})^2}{\Gamma(s_{i,v}+1)}(\cosh^2 t_g)^{-\frac{s_{i,v}+1}{2}}{}_2F_1(\tfrac{s_{i,v}+1}{2},\tfrac{s_{i,v}+1}{2};s_{i,v}+1;\cosh^{-2}t_g)\\
&= \frac{1}{4\pi}\sum_{i=1}^{N}\mu_v^{(i)}(\mathfrak{s}_\infty)\frac{\Gamma(\frac{s_{i,v}+1}{2})^2}{\Gamma(s_{i,v}+1)}\left\{\tfrac{4(ad-bc)}{a^2+b^2+c^2+d^2+2(ad-bc)}\right\}^{\frac{s_{i,v}+1}{2}}\\
&\qquad\times{}_2F_1\left(\tfrac{s_{i,v}+1}{2},\tfrac{s_{i,v}+1}{2};s_{i,v}+1;\tfrac{4(ad-bc)}{a^2+b^2+c^2+d^2+2(ad-bc)}\right)
\end{aligned} \tag{2.10}$$

if $g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ because $\cosh^{-2}t_g=\frac{4(ad-bc)}{a^2+b^2+c^2+d^2+2(ad-bc)}$.

From (2.3), we have the growth estimate

$$|\Phi_{\mathfrak{s}_v}(g)|_v\ll||g||_v^{-m}\quad(g\in G_v) \tag{2.11}$$

with $m=\min\left\{\frac{\mathrm{Re}(s_{i,v})+1}{2}\mid 1\le i\le N\right\}$.

Theorem 2.7. *Let $\phi:G_v\to\mathbb{C}$ be a bounded C^∞-function such that*

(a) *$\phi(zgk)=\tau_\ell(k)\phi(g)$ for all $z\in Z_v$, $k\in SO(2)$, and $g\in G_v$.*

(b) *$R(\Delta_v)\phi=\lambda\phi$ for some $\lambda\in\mathbb{C}$ with $\lambda\ne\frac{1-s_{i,v}^2}{4}$, $(1\le i\le N)$.*

where τ_ℓ be the character of $SO(2)$ given by $\tau_\ell(\left(\begin{smallmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{smallmatrix}\right))=e^{\sqrt{-1}\ell\theta}$ for $\ell\in\mathbb{Z}$, R is the right regular representation of G_v on $C^\infty(G_v)$, and Δ_v is the Casimir element in the universal enveloping algebra of G_v. Then we have

$$\int_{Z_v\backslash G_v}\Phi_{\mathfrak{s}_v}(g)\phi(g)dg=\begin{cases}\frac{1}{2}\left\{\prod_{i=1}^N\left(\lambda-\frac{1-s_{i,v}^2}{4}\right)^{-1}\right\}\phi(1) & (\ell=0)\\ 0 & (\ell\ne0)\end{cases}.$$

Proof.

$$\begin{aligned}
\int_{Z_v\backslash G_v} \Phi_{\mathfrak{s}_v}(g)\phi(g)dg &= \int_{SL_2(\mathbb{R})} \Phi_{\mathfrak{s}_v}(g)\phi(g)dg + \int_{SL_2(\mathbb{R})} \Phi_{\mathfrak{s}_v}(\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right) g)\phi(\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right) g)dg \\
&= \int_{SL_2(\mathbb{R})} \Phi_{\mathfrak{s}_v}(g)\phi(g)dg \\
&= \int_{\mathbb{H}}\int_{SO(2)} \Phi_{\mathfrak{s}_v}\left(\begin{pmatrix} y^{\frac12} & xy^{-\frac12} \\ 0 & y^{-\frac12}\end{pmatrix} k\right)\phi\left(\begin{pmatrix} y^{\frac12} & xy^{-\frac12} \\ 0 & y^{-\frac12}\end{pmatrix} k\right)\frac{dxdy}{y^2}dk \\
&= \int_{\mathbb{H}} \Phi_{\mathfrak{s}_v}\left(\begin{pmatrix} y^{\frac12} & xy^{-\frac12} \\ 0 & y^{-\frac12}\end{pmatrix}\right)\phi\left(\begin{pmatrix} y^{\frac12} & xy^{-\frac12} \\ 0 & y^{-\frac12}\end{pmatrix}\right)\frac{dxdy}{y^2}\int_{SO(2)}\tau_\ell(k)dk \\
&= \int_{\mathbb{H}} \varphi_{\mathfrak{s}_v}\left(u(\sqrt{-1},z)\right)\tilde{\phi}(z)d\mu z\int_{SO(2)}\tau_\ell(k)dk. \qquad (2.12)
\end{aligned}$$

Here, we write $\phi\left(\begin{pmatrix} y^{\frac12} & xy^{-\frac12} \\ 0 & y^{-\frac12}\end{pmatrix}\right) = \tilde{\phi}(z)$ for $z = x + \sqrt{-1}y$. We note that $R(\Delta_v)\phi$ and $\Delta_0\tilde{\phi}$ coincide. The latter integral of (2.12) is equal to $\frac12$ or 0 if $\ell = 0$ or $\ell \neq 0$ respectively. When $\ell = 0$, (2.12) is equal to

$$\begin{aligned}
\int_{\mathbb{H}} \varphi_{\mathfrak{s}_v}\left(u(\sqrt{-1},z)\right)\tilde{\phi}(z)d\mu z &= \frac12\sum_{i=1}^N \mu_v^{(i)}(\mathfrak{s}_\infty)\int_{\mathbb{H}} \mathrm{G}_{s_{i,v}}\left(u(\sqrt{-1},z)\right)\tilde{\phi}(z)d\mu z \\
&= \frac12\sum_{i=1}^N \mu_v^{(i)}(\mathfrak{s}_\infty)\mathrm{R}_{s_{i,v}}\tilde{\phi}(\sqrt{-1}) \\
&= \frac12\sum_{i=1}^N \mu_v^{(i)}(\mathfrak{s}_\infty)\left(\lambda - \tfrac{1-s_{i,v}^2}{4}\right)^{-1}\mathrm{R}_{s_{i,v}}\left\{\left(\Delta_0 - \tfrac{1-s_{i,v}^2}{4}\right)\tilde{\phi}\right\}(\sqrt{-1}).
\end{aligned}$$

By ([8, Lemma 1.18]), Proposition 2.2, and Lemma 2.4, we obtain

$$\begin{aligned}
\int_{\mathbb{H}} \varphi_{\mathfrak{s}_v}\left(u(i,z)\right)\tilde{\phi}(z)d\mu z &= \frac12\sum_{i=1}^N \mu_v^{(i)}(\mathfrak{s}_\infty)\left(\lambda - \tfrac{1-s_{i,v}^2}{4}\right)^{-1}\left\{\left(\Delta_0 - \tfrac{1-s_{i,v}^2}{4}\right)\mathrm{R}_{s_{i,v}}\right\}\tilde{\phi}(\sqrt{-1}) \\
&= \frac12\sum_{i=1}^N \mu_v^{(i)}(\mathfrak{s}_\infty)\left(\lambda - \tfrac{1-s_{i,v}^2}{4}\right)^{-1}\tilde{\phi}(\sqrt{-1}) \\
&= \frac12\left\{\prod_{i=1}^N\left(\lambda - \tfrac{1-s_{i,v}^2}{4}\right)^{-1}\right\}\phi(1).
\end{aligned}$$

□

2.3 Green functions on $G(F_v)$ for a non-archimedean place v

Let $v \in \Sigma_{\mathrm{fin}}$. For $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > 1$, we define $\Phi_{v,s} : G_v \to \mathbb{C}$ as

$$\Phi_{v,s}(g) = \left(q_v^{-\frac{s+1}{2}} - q_v^{\frac{s+1}{2}}\right)^{-1} q_v^{-\frac{(s+1)}{2}(n_1-n_2)}$$

where n_1 and n_2 are integers such that $g \in \mathbf{K}_v\begin{pmatrix}\varpi_v^{n_1} & 0 \\ 0 & \varpi_v^{n_2}\end{pmatrix}\mathbf{K}_v$ and $n_1 \geq n_2$. We also have

$$\Phi_{v,s}(g) = \left(q_v^{-\frac{s+1}{2}} - q_v^{\frac{s+1}{2}}\right)^{-1}\left\{|\det g|_v^{-1}\max(|a|_v, |b|_v, |c|_v, |d|_v)^2\right\}^{-\frac{s+1}{2}} \qquad (2.13)$$

for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. It is easy to check that $\Phi_{v,s}$ is a unique function on G_v with the properties:

(a) $\Phi_{v,s}(zkgk') = \Phi_{v,s}(g)$ for all $z \in Z_v$, $k, k' \in \mathbf{K}_v$, and $g \in G_v$,

(b) $R\left(\mathbb{T}_v - (q_v^{\frac{1+s}{2}} + q_v^{\frac{1-s}{2}})q_v^{\frac{3d_v}{2}}\mathbf{1}_{\mathbf{K}_v}\right)\Phi_{v,s} = \mathbf{1}_{Z_v\mathbf{K}_v}$,

(c) $\Phi_{v,s}(g) = O(1)$, $g \in G_v$,

where $\mathbb{T}_v = q_v^{\frac{3d_v}{2}}\mathbf{1}_{\mathbf{K}_v\left(\begin{smallmatrix} \varpi_v & 0 \\ 0 & 1 \end{smallmatrix}\right)\mathbf{K}_v}$. For a compact subset U of the half-plane $\mathrm{Re}(s) > 1$, from (2.13), we obtain the growth estimate

$$|\Phi_{v,s}(g)| \ll_U ||g||_v^{-\frac{\mathrm{Re}(s)+1}{2}} \quad (g \in G_v, s \in U). \tag{2.14}$$

Lemma 2.8. ([16, Lemma 2.4]) *Let $\varphi : G_v \to \mathbb{C}$ be a smooth function such that $\varphi(zgk) = \varphi(g)$ for all $z \in Z_v$, $k \in \mathbf{K}_v$, and $g \in G_v$. Then we have*

$$\int_{Z_v\backslash G_v} \Phi_{v,s}(g_v)\left[R\left(\mathbb{T}_v - (q_v^{\frac{1+s}{2}} + q_v^{\frac{1-s}{2}})q_v^{\frac{3d_v}{2}}\mathbf{1}_{\mathbf{K}_v}\right)\varphi\right](g_v)dg_v = q_v^{-\mathrm{d}_v}\varphi(1)$$

as long as the integral of the left-hand side is absolutely convergent.

2.4 The kernel function

Let $N \geq 2$ be a sufficiently large integer, $\mathbf{s}_S = (s_v)_{v\in S} \in \mathfrak{X}_S$ with $\mathrm{Re}(s_v) > 1$, and $\mathfrak{s}_\infty = (\mathfrak{s}_v = (s_{i,v})_{1\leq i\leq N})_{v\in\Sigma_\infty} \in (\mathbb{C}^N)^{\Sigma_\infty}$ with $\mathrm{Re}(s_{i,v}) > 1$ and $\mathrm{Re}(s_{i,v}) \neq \mathrm{Re}(s_{j,v})$ if $i \neq j$. We fix a finite subset $S \subset \Sigma_{\mathrm{fin}}$ such that $S \cap S(\mathfrak{n}) = \emptyset$ and $|2|_v = 1$ for all $v \in S$. Set

$$\Phi(g) = \Phi(\mathfrak{n}|\mathbf{s}_S, \mathfrak{s}_\infty; g) = \prod_{v\in\Sigma_F} \Phi_v(g_v) \tag{2.15}$$

where

$$\Phi_v(g_v) = \begin{cases} \mathbf{1}_{Z_v\mathbf{K}_0(\mathfrak{n}\mathfrak{o}_v)}(g_v) & (v \in \Sigma_{\mathrm{fin}} - S) \\ \Phi_{v,s_v}(g_v) & (v \in S) \\ \Phi_{\mathfrak{s}_v}(g_v) & (v \in \Sigma_\infty) \end{cases}$$

for $g = (g_v)_{v\in\Sigma_F} \in G_{\mathbb{A}}$. We define a function on $G_{\mathbb{A}} \times G_{\mathbb{A}}$ by

$$\mathbf{\Phi}(g, h) = \mathbf{\Phi}(\mathfrak{n}|\mathbf{s}_S, \mathfrak{s}_\infty; g, h) = \sum_{\gamma\in Z_F\backslash G_F} \Phi(\mathfrak{n}|\mathbf{s}_S, \mathfrak{s}_\infty; g^{-1}\gamma h), \quad g, h \in G_{\mathbb{A}}. \tag{2.16}$$

Lemma 2.9. *Let $\mathcal{U}$ be a compact set of $\{\mathbf{s} = (s_v)_{v\in S} \in \mathfrak{X}_S \mid \mathrm{Re}(s_v) > 1\}$. Then,*

$$|\Phi(g)| \ll_{\mathcal{U}} ||g||_v^{-m}, \quad g \in G_{\mathbb{A}}$$

and the series (2.16) *converges absolutely and satisfies*

$$\sum_{\gamma\in Z_F\backslash G_F} |\Phi(g^{-1}\gamma h)| \ll_{\mathcal{U}} y(g)^m, \quad g \in \mathfrak{S}_1, h \in G_{\mathbb{A}}$$

with $m = \min\{\frac{\mathrm{Re}(s_v)+1}{2} \mid v \in S\} \cup \{\frac{\mathrm{Re}(s_{i,v})+1}{2} \mid v \in \Sigma_\infty, 1 \leq i \leq N\}$.

Proof. This follows from (2.11), (2.14), and Proposition 2.1. □

3 The spectral decomposition of the kernel function

3.1 The theory of Eisenstein series

We introduce a brief explanation of the theory of Eisenstein series from ([5]). For $\chi \in \Xi$ and $z \in \mathbb{C}$, let $H(\chi, z)$ be the space of all smooth functions on $G_{\mathbb{A}}$ such that

$$f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) = \chi\left(\frac{a}{d}\right)\left|\frac{a}{d}\right|_{\mathbb{A}}^{\frac{z+1}{2}} f(g) \qquad \text{for all } \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in B_{\mathbb{A}}, g \in G_{\mathbb{A}}.$$

The group $G_{\mathbb{A}}$ acts on $H(\chi, z)$ by the right translation. The functions in $H(\chi, z)$ are uniquely determined by their restriction to $\mathbf{K}$. Set $H(\chi) = H(\chi, 0)$. Then, the hermitian pairing

$$\langle f | f' \rangle = \int_{\mathbf{K}} f(k)\overline{f'(k)} dk, \quad f, f' \in H(\chi)$$

becomes a $G_{\mathbb{A}}$-invariant inner product on $H(\chi)$. For $z \in \mathbb{C}$, we set the representation π_z of $G_{\mathbb{A}}$ on $H(\chi)$ by

$$(\pi_z(x)f)(g) = y(gx)^{\frac{z}{2}} y(g)^{-\frac{z}{2}} f(gx), \quad x, g \in G_{\mathbb{A}}$$

and the map $S_z : H(\chi) \to H(\chi, z)$ by

$$S_z f(g) = y(g)^{\frac{z}{2}} f(g), \quad f \in H(\chi), g \in G_{\mathbb{A}}.$$

Then, S_z is a bijective $G_{\mathbb{A}}$-intertwining operator between $(\pi_z, H(\chi))$ and $(R, H(\chi, z))$ where R denotes the right regular representation, i.e. it holds that

$$S_z(\pi_z(x)f)(g) = S_z f(gx), \quad x, g \in G_{\mathbb{A}}. \tag{3.1}$$

For $f \in H(\chi)$ and $g \in G_{\mathbb{A}}$, the corresponding Eisenstein series is defined by

$$Ef(g, z) = \sum_{\gamma \in B_F \backslash G_F} S_z f(\gamma g), \qquad z \in \mathbb{C}, \ \mathrm{Re}(z) > 1$$

which converges locally uniformly.

Let $\mathscr{S}(\mathbb{A})$ be the set of all Schwartz-Bruhat functions and $S(\mathbb{A})$ be the subspace of $\mathscr{S}(\mathbb{A})$ spanned by

$$\left\{ \Psi = \prod_{v \in \Sigma_F} \Psi_v \in \mathscr{S}(\mathbb{A}) \;\middle|\; \begin{array}{l} (a)\ \Psi_v = \mathbf{1}_{\mathbf{K}_v} \text{ for almost all } v \in \Sigma_{\mathrm{fin}} \\ (b)\ \Psi_v \text{ is locally constant and has compact support for } v \in \Sigma_{\mathrm{fin}} \\ (c)\ \Psi_v(x) = e^{-\pi x^2} \times (\text{polynomial in } x) \text{ for } v \in \Sigma_\infty \end{array} \right\}.$$

The spaces $\mathscr{S}(\mathbb{A}^2)$ and $S(\mathbb{A}^2)$ are defined analogously.

For $\Psi \in \mathscr{S}(\mathbb{A}^2)$, we set $\widehat{\Psi}, \dot{\widehat{\Psi}}, \ddot{\widehat{\Psi}} \in \mathscr{S}(\mathbb{A}^2)$ as follows:

$$\widehat{\Psi}[(x, y)] = \int_{\mathbb{A}} \int_{\mathbb{A}} \Psi[(u, v)] \psi(-xu - yv) du dv, \quad \text{(the Fourier transform)},$$

$$\dot{\widehat{\Psi}}[(x, y)] = \int_{\mathbb{A}} \Psi[(u, y)] \psi(-xu) du, \quad \text{(the Fourier transform w.r.t. the first variable)},$$

$$\ddot{\widehat{\Psi}}[(x, y)] = \int_{\mathbb{A}} \Psi[(x, v)] \psi(-yv) dv, \quad \text{(the Fourier transform w.r.t. the second variable)}.$$

For $\Psi \in S(\mathbb{A}^2)$, the function

$$f(g,\Psi,\chi,z) = \chi(\det g)|\det g|_{\mathbb{A}}^{\frac{z+1}{2}} \int_{\mathbb{A}^\times} \Psi\left[(0,t)g\right] \chi^2(t)|t|_{\mathbb{A}}^{z+1} d^\times t$$

belongs to $H(\chi,z)$. The corresponding Eisenstein series

$$E(g,\Psi,\chi,z) = \sum_{\gamma \in B_F \backslash G_F} f(\gamma g,\Psi,\chi,z)$$

again converges locally uniformly for $\mathrm{Re}(z) > 1$. Moreover, $E(g,\Psi,\chi,z)$ has a meromorphic continuation to $\mathbb{C}$ and satisfies the functional equation

$$E(g,\Psi,\chi,z) = E({}^t g^{-1},\widehat{\Psi},\chi^{-1},-z). \tag{3.2}$$

The possible poles of $E(g,\Psi,\chi,z)$ occur only at $z = -1, 1$ if $\chi^2 = 1$. When it has poles at $z = -1, 1$, these are simple poles with residues being given by

$$\operatorname*{Res}_{z=1} E(g,\Psi,\chi,z) = \mathrm{vol}(\mathbb{A}^1/F^\times)\widehat{\Psi}(0)\chi(\det g), \tag{3.3}$$

$$\operatorname*{Res}_{z=-1} E(g,\Psi,\chi,z) = -\mathrm{vol}(\mathbb{A}^1/F^\times)\Psi(0)\chi(\det g). \tag{3.4}$$

We denote the subspace of $H(\chi)$ consisting of all right $\mathbf{K}_0(\mathfrak{n}) \times SO(2)^{\Sigma_\infty}$-invariant functions by $H(\chi)_{\mathfrak{n}}$.

Lemma 3.1. *For $\chi \in \Xi - \Xi(\mathfrak{o})$, it holds that $H(\chi)_{\mathfrak{n}} = \{0\}$.*

Proof. Because $\chi \notin \Xi(\mathfrak{o})$, there exists $\varepsilon = (\varepsilon_v)_{v\in\Sigma_F} \in U(\mathfrak{o})$ such that $\chi(\varepsilon) \neq 1$. For any $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\begin{pmatrix} a_v & b_v \\ c_v & d_v \end{pmatrix}\right)_{v\in\Sigma_F} \in \mathbf{K}$ and $x = (x_v)_{v\in\Sigma_F} \in \mathbb{A}$, by an easy calculation,

$$k^{-1}\begin{pmatrix} \varepsilon & x \\ 0 & 1 \end{pmatrix} k = (\det k)^{-1}\begin{pmatrix} * & * \\ -c^2x + ac(1-\varepsilon) & * \end{pmatrix}. \tag{3.5}$$

We can check that the the lower left component of (3.5) at $v \in S(\mathfrak{n})$ belongs to $\mathfrak{p}_v$ if $c_v \in \mathfrak{p}_v$ or $c \notin \mathfrak{p}_v$ for suitable $x_v \in \mathfrak{o}_v$. We fix such a x_v for $v \in S(\mathfrak{n})$ and set $x_v = 0$ for $v \in \Sigma_F - S(\mathfrak{n})$. Then, k and $\begin{pmatrix} \varepsilon & x \\ 0 & 1 \end{pmatrix} k$ belong to the same right coset of $\mathbf{K}_0(\mathfrak{n}) \times SO(2)^{\Sigma_\infty}$ in $\mathbf{K}$. Hence, for any $f \in H(\chi)_{\mathfrak{n}}$, we have

$$f(k) = f\left(\begin{pmatrix} \varepsilon & x \\ 0 & 1 \end{pmatrix} k\right) = \chi(\varepsilon)f(k).$$

Therefore, we obtain $f(k) = 0$ since $\chi(\varepsilon) \neq 1$. □

When $\chi \in \Xi(\mathfrak{o})$, we remark that $H(\chi)$ is finite dimensional and its orthonormal basis is constructed as follows.

Let $v \in S(\mathfrak{n}) \cup \Sigma_\infty$. Set functions on $\mathbf{K}_v$ as

$$f_{v,0} = (q_v+1)^{\frac{1}{2}}\mathbf{1}_{\mathbf{K}_0(\mathfrak{p}_v)}, \quad f_{v,1} = (1+q_v^{-1})^{\frac{1}{2}}\mathbf{1}_{\mathbf{K}_v - \mathbf{K}_0(\mathfrak{p}_v)}.$$

for $v \in S(\mathfrak{n})$ and

$$f_{v,0}(k) = (\det k)^{\epsilon(\chi_v)}, \quad k \in O(2)$$

for $v \in \Sigma_\infty$ where $\epsilon(\chi_v) \in \{0,1\}$ such that $\chi(-1) = (-1)^{\epsilon(\chi_v)}$. We put the index set $A(\chi)_{\mathfrak{n}} = \{\iota = (\iota_v)_{v \in S(\mathfrak{n})} \in \{0,1\}^{S(\mathfrak{n})}\}$. For $\iota \in A(\chi)_{\mathfrak{n}}$, let $f_\iota^\chi \in H(\chi)$ be an element such that its restriction to $\mathbf{K}$ is given by

$$\prod_{v \in \Sigma_{\mathrm{fin}} - S(\mathfrak{n})} \mathbf{1}_{\mathbf{K}_v} \prod_{v \in S(\mathfrak{n})} f_{v,\iota_v} \prod_{v \in \Sigma_\infty} f_{v,0}.$$

Then,

$$B(\chi)_{\mathfrak{n}} := \{f_\iota^\chi \mid \iota \in A(\chi)_{\mathfrak{n}}\} \tag{3.6}$$

forms an orthonormal basis of $H(\chi)_{\mathfrak{n}}$. We also fix an orthonormal basis $B(\chi)$ of $H(\chi)$ so that it contains $H(\chi)_{\mathfrak{n}}$.

Lemma 3.2. *Let $\chi \in \Xi(\mathfrak{o})$ and $f_\iota^\chi \in B(\chi)_{\mathfrak{n}}$ ($\iota \in A(\chi)_{\mathfrak{n}}$). Then, $S_z f_\iota^\chi(g)$ and the corresponding Eisenstein series $Ef_\iota^\chi(g,z)$ can be written as*

$$S_z f_\iota^\chi(g) = D_F^{\frac{1}{2}} L_F(z+1,\chi^2)^{-1} \left(\prod_{v \in S(\mathfrak{n})} L_v(z+1,\chi_v^2)\right) f(g,\Psi_\iota,\chi,z)$$

and

$$Ef_\iota^\chi(g,z) = D_F^{\frac{1}{2}} L_F(z+1,\chi^2)^{-1} \left(\prod_{v \in S(\mathfrak{n})} L_v(z+1,\chi_v^2)\right) E(g,\Psi_\iota,\chi,z)$$

respectively. Here, $\Psi_\iota \in S(\mathbb{A}^2)$ is defined by

$$\Psi_\iota[(x,y)] = \prod_{v \in \Sigma_F} \Psi_{\iota,v}[(x_v,y_v)] \tag{3.7}$$

where

$$\Psi_{\iota,v}[(x_v,y_v)] = \begin{cases} \mathbf{1}_{\mathfrak{o}_v^2}[(x_v,y_v)] & (v \in \Sigma_{\mathrm{fin}} - S(\mathfrak{n})) \\ (q_v+1)^{\frac{1}{2}} \mathbf{1}_{\mathfrak{p}_v \times \mathfrak{o}_v^\times}[(x_v,y_v)] & (v \in S(\mathfrak{n}), \iota_v = 0) \\ (1+q_v^{-1})^{\frac{1}{2}} \mathbf{1}_{\mathfrak{o}_v^\times \times \mathfrak{o}_v}[(x_v,y_v)] & (v \in S(\mathfrak{n}), \iota_v = 1) \\ e^{-\pi(x_v^2+y_v^2)} & (v \in \Sigma_\infty) \end{cases}. \tag{3.8}$$

Proof. By analytic continuation, we may assume that $\mathrm{Re}(z) > 0$. We consider the local integrals

$$f_v(g_v,\Psi_{\iota,v},\chi_v,z) = \chi_v(\det g_v)|\det g_v|_v^{\frac{z+1}{2}} \int_{F_v^\times} \Psi_{\iota,v}[(0,t_v)g_v]\chi_v^2(t_v)|t_v|_v^{z+1} d^\times t_v.$$

Set $g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} k = \left(\begin{pmatrix} a_v & b_v \\ 0 & d_v \end{pmatrix} k_v\right)_{v \in \Sigma_F} \in G_{\mathbb{A}}$ by the Iwasawa decomposition. Then we have the following calculations.

(i) When $v \in \Sigma_{\mathrm{fin}} - S(\mathfrak{n})$, because $\Psi_{\iota,v}$ is right $\mathbf{K}_v$-invariant, we have

$$\begin{aligned}
f_v(g_v, \Psi_{\iota,v}, \chi_v, z) &= \chi_v(a_v d_v)|a_v d_v|_v^{\frac{z+1}{2}} \int_{F_v^\times} \mathbf{1}_{\mathfrak{o}_v^2}[(0, d_v t_v)]\chi_v^2(t_v)|t_v|_v^{z+1} d^\times t_v \\
&= \chi_v\left(\frac{a_v}{d_v}\right)\left|\frac{a_v}{d_v}\right|_v^{\frac{z+1}{2}} \int_{F_v^\times} \mathbf{1}_{\mathfrak{o}_v^2}[(0, t_v)]\chi_v^2(t_v)|t_v|_v^{z+1} d^\times t_v \\
&= \chi_v\left(\frac{a_v}{d_v}\right)\left|\frac{a_v}{d_v}\right|_v^{\frac{z+1}{2}} \int_{\mathfrak{o}_v - \{0\}} \chi_v^2(t_v)|t_v|_v^{z+1} d^\times t_v \\
&= \chi_v\left(\frac{a_v}{d_v}\right)\left|\frac{a_v}{d_v}\right|_v^{\frac{z+1}{2}} \sum_{n=0}^{\infty} \int_{\varpi_v^n \mathfrak{o}_v^\times} \chi_v^2(t_v)|t_v|_v^{z+1} d^\times t_v \\
&= \chi_v\left(\frac{a_v}{d_v}\right)\left|\frac{a_v}{d_v}\right|_v^{\frac{z+1}{2}} q_v^{-\frac{\mathrm{d}_v}{2}} \sum_{n=0}^{\infty} \chi_v^2(\varpi_v) q_v^{-n(z+1)} \\
&= q_v^{-\frac{\mathrm{d}_v}{2}} L_v(z+1, \chi_v^2)\chi_v\left(\frac{a_v}{d_v}\right)\left|\frac{a_v}{d_v}\right|_v^{\frac{z+1}{2}}.
\end{aligned}$$

(ii) When $v \in S(\mathfrak{n})$ and $\iota_v = 0$, we have

$$\begin{aligned}
f_v(g_v, \Psi_{\iota,v}, \chi_v, z) &= (q_v+1)^{\frac{1}{2}} \chi_v(a_v d_v)|a_v d_v|_v^{\frac{z+1}{2}} \\
&\qquad \times \int_{F_v^\times} \mathbf{1}_{\mathfrak{p}_v \times \mathfrak{o}_v^\times}[(0, d_v t_v)k_v]\chi_v^2(t_v)|t_v|_v^{z+1} d^\times t_v \\
&= (q_v+1)^{\frac{1}{2}} \chi_v\left(\frac{a_v}{d_v}\right)\left|\frac{a_v}{d_v}\right|_v^{\frac{z+1}{2}} \int_{F_v^\times} \mathbf{1}_{\mathfrak{p}_v \times \mathfrak{o}_v^\times}[(0, t_v)k_v]\chi_v^2(t_v)|t_v|_v^{z+1} d^\times t_v.
\end{aligned}$$

It can be easily checked that the integral in the last line is equal to $q_v^{-\frac{\mathrm{d}_v}{2}}$ or 0 according as $k_v \in \mathbf{K}_0(\mathfrak{n}\mathfrak{o}_v)$ or $k_v \notin \mathbf{K}_0(\mathfrak{n}\mathfrak{o}_v)$. Hence we get

$$f_v(g_v, \Psi_{\iota,v}, \chi_v, z) = q_v^{-\frac{\mathrm{d}_v}{2}} \chi_v\left(\frac{a_v}{d_v}\right)\left|\frac{a_v}{d_v}\right|_v^{\frac{z+1}{2}} f_{v,0}(k_v).$$

(iii) When $v \in S(\mathfrak{n})$ and $\iota_v = 1$, by a similar calculation in (ii), we have

$$f_v(g_v, \Psi_{\iota,v}, \chi_v, z) = q_v^{-\frac{\mathrm{d}_v}{2}} \chi_v\left(\frac{a_v}{d_v}\right)\left|\frac{a_v}{d_v}\right|_v^{\frac{z+1}{2}} f_{v,1}(k_v).$$

(iv) When $v \in \Sigma_\infty$, because $\Psi_{\iota,v}$ is right $\mathbf{K}_v$-invariant, we have

$$\begin{aligned}
f_v(g_v, \Psi_{\iota,v}, \chi_v, z) &= \chi_v(a_v d_v)|a_v d_v|_v^{\frac{z+1}{2}} \int_{F_v^\times} e^{-\pi d_v^2 t_v^2}\chi_v^2(t_v)|t_v|_v^{z+1} d^\times t_v \\
&= \chi_v\left(\frac{a_v}{d_v}\right)\left|\frac{a_v}{d_v}\right|_v^{\frac{z+1}{2}} \cdot 2\int_0^\infty e^{-\pi t^2}\chi_v^2(t) t^{z+1} d^\times t \\
&= \Gamma_{\mathbb{R}}\left(z+1+\sqrt{-1}a(\chi_v^2)\right)\chi_v\left(\frac{a_v}{d_v}\right)\left|\frac{a_v}{d_v}\right|_v^{\frac{z+1}{2}}.
\end{aligned}$$

By the above calculations, we obtain

$$\begin{aligned} f(g,\Psi_\iota,\chi,z) &= \prod_{v\in\Sigma_F} f_v(g_v,\Psi_{\iota,v},\chi_v,z) \\ &= D_F^{-\frac{1}{2}}\left(\prod_{v\in\Sigma_F-S(\mathfrak{n})} L_v(z+1,\chi_v^2)\right)\chi\left(\frac{a}{d}\right)\left|\frac{a}{d}\right|_{\mathbb{A}}^{\frac{z+1}{2}} f_\iota^\chi(k) \\ &= D_F^{-\frac{1}{2}}\left(\prod_{v\in\Sigma_F-S(\mathfrak{n})} L_v(z+1,\chi_v^2)\right) S_z f_\iota^\chi(g). \end{aligned}$$

By multiplying $D_F^{\frac{1}{2}}$ and L-factors, the proof is completed. □

For later use, we calculate $\widehat{\dot{\Psi}_\iota}$ as the following lemma at the end of this section.

Lemma 3.3. *For $\iota\in A(\chi)_\mathfrak{n}$, let $\Psi_\iota\in S(\mathbb{A}^2)$ be as in (3.7) and (3.8). Then*

$$\widehat{\dot{\Psi}_\iota}[(x,y)] = \prod_{v\in\Sigma_F}\widehat{\dot{\Psi}_{\iota,v}}[(x_v,y_v)]$$

where

$$\widehat{\dot{\Psi}_{\iota,v}}[(x_v,y_v)] = \begin{cases} q_v^{-\frac{\mathrm{d}_v}{2}}\mathbf{1}_{\mathfrak{p}_v^{-\mathrm{d}_v}\times\mathfrak{o}_v}[(x_v,y_v)] & (v\in\Sigma_{\mathrm{fin}}-S(\mathfrak{n})) \\ q_v^{-\frac{\mathrm{d}_v}{2}}(q_v+1)^{\frac{1}{2}}q_v^{-1}\mathbf{1}_{\mathfrak{p}_v^{-\mathrm{d}_v-1}\times\mathfrak{o}_v^\times}[(x_v,y_v)] & (v\in S(\mathfrak{n}),\iota_v=0) \\ q_v^{-\frac{\mathrm{d}_v}{2}}(1+q_v^{-1})^{\frac{1}{2}}\left(\mathbf{1}_{\mathfrak{p}_v^{-\mathrm{d}_v}\times\mathfrak{o}_v}-q_v^{-1}\mathbf{1}_{\mathfrak{p}_v^{-\mathrm{d}_v-1}\times\mathfrak{o}_v}\right)[(x_v,y_v)] & (v\in S(\mathfrak{n}),\iota_v=1) \\ e^{-\pi(x_v^2+y_v^2)} & (v\in\Sigma_\infty) \end{cases}.$$

Proof. The claim follows from the following formulas:

$$\begin{aligned} &\int_{F_v}\mathbf{1}_{\mathfrak{p}_v^n}(u)\psi_v(-yu)du = q_v^{-\frac{\mathrm{d}_v}{2}}q_v^{-n}\mathbf{1}_{\mathfrak{p}_v^{-\mathrm{d}_v-n}}(y), \quad n\in\mathbb{Z},\ v\in\Sigma_{\mathrm{fin}}, \\ &\int_{F_v}\mathbf{1}_{\mathfrak{o}_v^\times}(u)\psi_v(-yu)du = q_v^{-\frac{\mathrm{d}_v}{2}}\left(\mathbf{1}_{\mathfrak{p}_v^{-\mathrm{d}_v}}(y)-q_v^{-1}\mathbf{1}_{\mathfrak{p}_v^{-\mathrm{d}_v-1}}(y)\right), \quad v\in\Sigma_{\mathrm{fin}}, \\ &\int_{\mathbb{R}} e^{-\pi u^2}e^{-2\pi\sqrt{-1}yu}du = e^{-\pi y^2}. \end{aligned}$$

□

3.2 The spectral decomposition of $\mathbf{\Phi}(g,h)$

It is known that the Hilbert space $\mathcal{L}^2=\mathcal{L}^2(Z_\mathbb{A}G_F\backslash G_\mathbb{A})$ decomposes as a direct sum

$$\mathcal{L}^2=\mathcal{L}^2_{\mathrm{cus}}\oplus\mathcal{L}^2_{\mathrm{Eis}}\oplus\mathcal{L}^2_{\mathrm{res}}$$

where $\mathcal{L}^2_{\mathrm{cus}}$ is the space of cusp forms, $\mathcal{L}^2_{\mathrm{res}}$ is the space spanned by the functions $\chi\circ\det$ with $\chi\in\Xi$ and $\chi^2=1$, and $\mathcal{L}^2_{\mathrm{Eis}}=(\mathcal{L}^2_{\mathrm{cus}}\oplus\mathcal{L}^2_{\mathrm{res}})^\perp$ is the space which can be written by using Eisenstein series. Let us recall the function $\mathbf{\Phi}(g,h)$ on $G_\mathbb{A}\times G_\mathbb{A}$ defined as in (2.16). For fixed $g\in G_\mathbb{A}$, the function $h\mapsto\mathbf{\Phi}(g,h)$ belongs to $\mathcal{L}^2$. For $\flat\in\{\mathrm{cus},\mathrm{Eis},\mathrm{res}\}$, we denote the projection of the function $h\mapsto\mathbf{\Phi}(g,h)$ to $\mathcal{L}^2_\flat$ by $\mathbf{\Phi}_\flat(g,h)$.

From the construction of $\mathbf{\Phi}$, the function $\mathbf{\Phi}(g,\bullet)$ is right $\mathbf{K}_0(\mathfrak{n})\times SO(2)^{\Sigma_\infty}$-invariant. In particular, $\mathbf{\Phi}_{\rm cus}(g,\bullet)$ belongs to

$$(\mathcal{L}^2_{\rm cus})^{\mathbf{K}_0(\mathfrak{n})\times SO(2)^{\Sigma_\infty}} = \bigoplus_{\pi\in\Pi_{\rm cus}(\mathfrak{n})} \pi^{\mathbf{K}_0(\mathfrak{n})\times SO(2)^{\Sigma_\infty}}. \tag{3.9}$$

It is known that the space $\pi^{\mathbf{K}_0(\mathfrak{n})\times SO(2)^{\Sigma_\infty}}$ is finite dimensional for each $\pi\in\Pi_{\rm cus}(\mathfrak{n})$.

Remark 3.4. In the above discussion, we note that the equation

$$\mathbf{\Phi}(g,h) = \mathbf{\Phi}_{\rm cus}(g,h) + \mathbf{\Phi}_{\rm Eis}(g,h) + \mathbf{\Phi}_{\rm res}(g,h) \tag{3.10}$$

holds as elements in $\mathcal{L}^2(Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}})$ for fixed $g\in G_{\mathbb{A}}$. We state that (3.10) actually holds for all $g,h\in G_{\mathbb{A}}$. In fact, each function in (3.10) is continuous. The continuities of $\mathbf{\Phi}_{\rm cus}$, $\mathbf{\Phi}_{\rm Eis}$, and $\mathbf{\Phi}_{\rm res}$ are proved in Propositions 3.6, 4.9, and 4.11 respectively.

For $\pi\cong\otimes'_{v\in\Sigma_F}\pi_v\in\Pi_{\rm cus}(\mathfrak{n})$, let $\mathcal{B}_{\rm cus}(\pi,\mathfrak{n})\subset C^\infty(Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}})^{\mathbf{K}_0(\mathfrak{n})\times SO(2)^{\Sigma_\infty}}$ be an orthonormal basis of $\pi^{\mathbf{K}_0(\mathfrak{n})\times SO(2)^{\Sigma_\infty}}$. By ([19, §2.5]), we have

$$\begin{aligned} R(\mathbb{T}_v)\varphi &= \left(q_v^{\frac{1+\nu_v(\pi_v)}{2}} + q_v^{\frac{1-\nu_v(\pi_v)}{2}}\right)\varphi, \qquad (v\in S) \\ R(\Delta_v)\varphi &= \frac{1-\nu_v(\pi_v)^2}{4}\varphi, \qquad (v\in\Sigma_\infty) \end{aligned} \tag{3.11}$$

for all $\varphi\in\mathcal{B}_{\rm cus}(\pi,\mathfrak{n})$.

Lemma 3.5. ([19, Proposition 15.2]) *For any $m\in\mathbb{N}$, there exists $M_m\in\mathbb{N}$ such that the estimate*

$$|\varphi(g)| \ll \left\{\prod_{v\in\Sigma_\infty}(1+|\nu_v(\pi_v)|)^{M_m}\right\} y(g)^{-m}, \quad \pi\in\Pi_{\rm cus}(\mathfrak{n}),\ \varphi\in\mathcal{B}_{\rm cus}(\pi,\mathfrak{n}),\ g\in\mathfrak{S}_1$$

holds.

Proof. See ([19, pp. 123–124]). □

Proposition 3.6. *We set*

$$\begin{aligned} C(\mathfrak{n},\pi) = D_F^{-1}\,[\mathbf{K}_{\rm fin}:\mathbf{K}_0(\mathfrak{n})]^{-1} \prod_{v\in\Sigma_\infty}\left[\frac{1}{2}\prod_{i=1}^{N}\left\{\frac{s_{i,v}^2-\nu_v(\pi_v)^2}{4}\right\}^{-1}\right] \\ \times\prod_{v\in S}\left\{\left(q_v^{\frac{1+\nu_v(\pi_v)}{2}}+q_v^{\frac{1-\nu_v(\pi_v)}{2}}\right)-\left(q_v^{\frac{s_v+1}{2}}+q_v^{\frac{-s_v+1}{2}}\right)\right\}^{-1}, \end{aligned} \tag{3.12}$$

for $\pi\in\Pi_{\rm cus}(\mathfrak{n})$.
(a) For any $m\in\mathbb{N}$, there exist $N_m\in\mathbb{N}$ such that if $N\geq N_m$, the estimate

$$\sum_{\pi\in\Pi_{\rm cus}(\mathfrak{n})}\sum_{\varphi\in\mathcal{B}_{\rm cus}(\pi,\mathfrak{n})}|C(\mathfrak{n},\pi)||\varphi(g)||\varphi(h)| \ll y(g)^{-m}y(h)^{-m}, \quad g,h\in\mathfrak{S}_1 \tag{3.13}$$

holds.

(b) If $N \ge N_m$ for some $m \in \mathbb{N}$, we have

$$\mathbf{\Phi}_{\rm cus}(g,h) = \sum_{\pi \in \Pi_{\rm cus}(\mathfrak{n})} \sum_{\varphi \in \mathcal{B}_{\rm cus}(\pi,\mathfrak{n})} C(\mathfrak{n},\pi)\overline{\varphi(g)}\varphi(h) \tag{3.14}$$

for all $g, h \in G_{\mathbb{A}}$. Moreover, $\mathbf{\Phi}_{\rm cus}(g,h)$ is continuous.

Proof. (a) From (1.4) and the assumptions of $\mathbf{s}_S$ and $\mathfrak{s}_\infty$, the product (3.12) has an upper bound independent of $\pi \in \Pi_{\rm cus}(\mathfrak{n})$. Hence we can easily check that

$$|C(\mathfrak{n},\pi)| \ll \prod_{v \in \Sigma_\infty} (1+|\nu_v(\pi_v)|)^{-2N}, \quad \pi \in \Pi_{\rm cus}(\mathfrak{n}).$$

For any $m \in \mathbb{N}$, we have

$$|C(\mathfrak{n},\pi)||\varphi(g)||\varphi(h)| \ll \left\{\prod_{v \in \Sigma_\infty} (1+|\nu_v(\pi_v)|)^{-2N+2M_m}\right\} y(g)^{-m} y(h)^{-m}, \quad \pi \in \Pi_{\rm cus}(\mathfrak{n}),\ g,h \in \mathfrak{S}_1$$

for some $M_m \in \mathbb{N}$ by Lemma (3.5). We note that $|\mathfrak{B}_{\rm cus}(\pi,\mathfrak{n})| \le 2^{|S(\mathfrak{n})|}$. Therefore, (3.13) holds by Weyl's low.

(b) For any $\varphi \in \mathcal{B}_{\rm cus}(\pi,\mathfrak{n})$, we have

$$\begin{aligned}
\langle \mathbf{\Phi}_{\rm cus}(g,\bullet)|\varphi\rangle &= \langle \mathbf{\Phi}(g,\bullet)|\varphi\rangle \\
&= \int_{Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}}} \mathbf{\Phi}(g,h)\overline{\varphi(h)}dh \\
&= \int_{Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}}} \left(\sum_{\gamma \in Z_F\backslash G_F} \Phi(g^{-1}\gamma h)\right)\overline{\varphi(h)}dh \\
&= \int_{Z_{\mathbb{A}}\backslash G_{\mathbb{A}}} \Phi(g^{-1}h)\overline{\varphi(h)}dh \\
&= \int_{Z_{\mathbb{A}}\backslash G_{\mathbb{A}}} \Phi(h)\overline{\varphi(gh)}dh.
\end{aligned} \tag{3.15}$$

The right-hand side of (3.15) is equal to

$$\int_{Z^{S,\Sigma_\infty}\backslash G^{S,\Sigma_\infty}} \int_{Z_S\backslash G_S} \int_{Z_{\Sigma_\infty}\backslash G_{\Sigma_\infty}} \Phi(h_{\Sigma_\infty}h_S h^{S,\Sigma_\infty})\overline{\varphi(g_{\Sigma_\infty}h_{\Sigma_\infty}g_S h_S g^{S,\Sigma_\infty}h^{S,\Sigma_\infty})}dh_{\Sigma_\infty}dh_S dh^{S,\Sigma_\infty}$$

where $Z^{S,\Sigma_\infty}\backslash G^{S,\infty} = \prod'_{v \in \Sigma_{\rm fin}-S} Z_v\backslash G_v$, $Z_S\backslash G_S = \prod_{v \in S} Z_v\backslash G_v$, $Z_{\Sigma_\infty}\backslash G_{\Sigma_\infty} = \prod_{v \in \Sigma_\infty} Z_v\backslash G_v$, and the Haar measures are defined so that the product formula holds.

Since φ is right $\mathbf{K}_0(\mathfrak{n})$-invariant and $\Phi(h_{\Sigma_\infty}h_S h^{S,\Sigma_\infty}) = 0$ unless $h^{S,\Sigma_\infty} \in \prod'_{v \in \Sigma_{\rm fin}-S} Z_v\backslash Z_v\mathbf{K}_0(\mathfrak{n}\mathfrak{o}_v)$, the h^{S,Σ_∞}-integral yields the volume factor

$$\begin{aligned}
\prod_{v \in \Sigma_{\rm fin}-S} {\rm vol}(Z_v\backslash Z_v\mathbf{K}_0(\mathfrak{n}\mathfrak{o}_v);dh_v) &= \prod_{v \in S(\mathfrak{n})}[\mathbf{K}_v:\mathbf{K}_0(\mathfrak{n}\mathfrak{o}_v)]^{-1} \prod_{v \in \Sigma_{\rm fin}-S} {\rm vol}(Z_v\backslash Z_v\mathbf{K}_v;dh_v) \\
&= [\mathbf{K}_{\rm fin}:\mathbf{K}_0(\mathfrak{n})]^{-1} \prod_{v \in \Sigma_{\rm fin}-S} q_v^{-{\rm d}_v}.
\end{aligned}$$

The h_S-integral and h_{Σ_∞}-integral can be computed by Theorem 2.7 and Lemma 2.8 respectively. Hence, we have

$$\langle \mathbf{\Phi}_{\text{cus}}(g, \bullet)|\varphi\rangle = C(\mathfrak{n}, \pi)\overline{\varphi(g)}$$

Therefore, (3.14) holds as $\mathcal{L}^2$-functions on $h \in G_{\mathbb{A}}$ for fixed $g \in G_{\mathbb{A}}$. If $N \geq N_m$ for some $m \in \mathbb{N}$, (3.14) is a uniformly convergent sum by (a) and hence continuous. □

3.3 The smoothed Eisenstein series

Let

$$E(z; g) = E\mathbf{1}_{\mathbf{K}}(g, z) = D_F^{\frac{1}{2}}\zeta_F(z+1)^{-1}E(g, \Psi_0, 1, z)$$

be the standard Eisenstein series on $G_{\mathbb{A}}$ where

$$\Psi_0[(x, y)] = \prod_{v \in \Sigma_{\text{fin}}} \mathbf{1}_{\mathfrak{o}_v^2}[(x_v, y_v)] \prod_{v \in \Sigma_\infty} e^{-\pi(x_v^2 + y_v^2)}, \tag{3.16}$$

for $(x, y) = ((x_v, y_v))_{v \in \Sigma_F} \in \mathbb{A}^2$. We also have

$$E(z; g) = \sum_{\gamma \in B_F \backslash G_F} y(\gamma g)^{\frac{z+1}{2}}, \quad \text{Re}(z) > 1, g \in G_{\mathbb{A}}.$$

The Fourier transform of Ψ_0 is given by

$$\widehat{\Psi_0}[(x, y)] = D_F^{-1} \prod_{v \in \Sigma_{\text{fin}}} \mathbf{1}_{(\varpi_v^{-\mathrm{d}_v}\mathfrak{o}_v)^2}[(x_v, y_v)] \prod_{v \in \Sigma_\infty} e^{-\pi(x_v^2 + y_v^2)}$$

for $(x, y) = ((x_v, y_v))_{v \in \Sigma_F} \in \mathbb{A}^2$. Hence, by easy calculation and (3.2), we have the functional equation

$$E^*(z; g) = E^*(-z; g) \tag{3.17}$$

where $E^*(z; g) = \Lambda_F(z)E(z; g) = D_F^{\frac{z+1}{2}} E(g, \Psi_0, 1, z)$ and $\Lambda_F(z) = D_F^{\frac{z}{2}}\zeta_F(z)$. Here, we note that $E(z; g) = E(z; {}^t g^{-1})$ because ${}^t g^{-1} = (\det g)^{-1} w_0 g w_0^{-1}$ with $w_0 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $E(z; g)$ is left $Z_{\mathbb{A}}G_F$-invariant and is right $\mathbf{K}$-invariant. Then $E^*(z; g)$ has a meromorphic continuation to $\mathbb{C}$ and the simple poles occur only at $z = 0, 1$ with residues are given by

$$\operatorname*{Res}_{z=1} E^*(z; g) = \text{vol}(\mathbb{A}^\times/F^\times), \quad \operatorname*{Res}_{z=-1} E^*(z; g) = -\text{vol}(\mathbb{A}^\times/F^\times)$$

from (3.3) and (3.4). Recall the functional equation $\Lambda_F(z) = \Lambda_F(1-z)$ and the Fourier expansion

$$E^*(z; g) = \Lambda_F(-z)y(g)^{\frac{z+1}{2}} + \Lambda_F(z)y(g)^{\frac{-z+1}{2}} + \Lambda_F(-z) \sum_{a \in F^\times} W_\psi\left(z; \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right) \tag{3.18}$$

where $W_\psi(z; g)$ is the global Whittaker function defined as

$$W_\psi(z; g) = \int_{\mathbb{A}} y\left(w_0 \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) g\right)^{\frac{z+1}{2}} \psi(-x)dx, \qquad g \in G_{\mathbb{A}}.$$

Then, we have $W_\psi(z; g) = \prod_{v \in \Sigma_F} W_v(z; g_v)$ where $W_v(z; g_v)$ is the $\mathbf{K}_v$-invariant Whittaker function on G_v determined by

$$W_v\left(z; \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{cases} \zeta_v(z+1)^{-1}\delta(\varpi_v^{\mathrm{d}_v} t \in \mathfrak{o}_v)|\varpi_v^{\mathrm{d}_v} t|_v^{\frac{1}{2}} \frac{|\varpi_v^{\mathrm{d}_v+1}t|_v^{\frac{z}{2}} - |\varpi_v^{\mathrm{d}_v+1}t|_v^{-\frac{z}{2}}}{q_v^{-\frac{z}{2}} - q_v^{\frac{z}{2}}}, & (v \in \Sigma_{\text{fin}}) \\ \zeta_v(z+1)^{-1}2|t|_v^{\frac{1}{2}} K_{\frac{z}{2}}(2\pi|t|_v), & (v \in \Sigma_\infty) \end{cases} \tag{3.19}$$

for $t \in F_v^\times$ where $K_\alpha(s)$ is the modified Bessel function of the second kind.

Lemma 3.7. [16, Lemma 3.1] *For any* $0 < \sigma_1 < \sigma_2$*, there exists* $m > 0$ *such that, for any element* D *in the universal enveloping algebra of* $(\prod_{v\in\Sigma_\infty} G_v) \cap G^1(\mathbb{A})$*, it holds that*

$$|(z-1)R(D)E^*(z;g)| \ll_D y(g)^m, \quad \mathrm{Re}(z) \in [\sigma_1, \sigma_2], \; g \in \mathfrak{S}_1.$$

Proof. See ([16, p. 2993]). □

Now we define the smoothed Eisenstein series. Let $\mathcal{B}_1$ be the space of all entire functions β such that $\beta(0) = \beta(\pm 1) = \beta'(\pm 1) = 0$ and

$$|\beta(z)| \ll_{\sigma_1,\sigma_2} (1 + |\mathrm{Im}(z)|)^{-M}, \quad \mathrm{Re}(z) \in [\sigma_1, \sigma_2] \tag{3.20}$$

for any $M > 0$ and interval $[\sigma_1, \sigma_2] \subset \mathbb{R}$.

For $\beta \in \mathcal{B}_1$ and $\sigma > 0$, we set

$$\mathcal{E}^*_\beta(g) = \int_{L_\sigma} \beta(z)E^*(z;g)dz, \quad g \in G_{\mathbb{A}}$$

where the contour L_σ is the oriented vertical line from $\sigma - \sqrt{-1}\infty$ to $\sigma + \sqrt{-1}\infty$.

Proposition 3.8. ([16, Proposition 3.2]) *The contour integral* $\mathcal{E}^*_\beta(g)$ *converges absolutely and is independent of* $\sigma > 0$*. For any* $M > 0$*,*

$$|\mathcal{E}^*_\beta(g)| \ll y(g)^{-M}, \quad g \in \mathfrak{S}_1.$$

Proof. See [16, p. 2994]. □

Let $I \subset \mathbb{R}$ be an open interval. A meromorphic function f on the vertical strip $\mathrm{Re}(z) \in I$ which is holomorphic away from the possible poles on real axis is said to be vertically of moderate growth if for any interval $[\sigma_1, \sigma_2] \subset I$, there exists $N_1 > 0$ such that $|f(z)| \ll (1 + |\mathrm{Im}(z)|)^{N_1}$ for any $z \in \mathcal{T}_\delta = \{z \in \mathbb{C} | \mathrm{Re}(z) \in [\sigma_1, \sigma_2], |\mathrm{Im}(z)| \geq \delta\}$.

We need the following proposition to obtain our main results.

Proposition 3.9. ([16, Lemma 4.2]) *Let* F *be a meromorphic function on the vertical strip* $\mathrm{Re}(z) \in (\sigma_1, \sigma_2)$*, which is vertically of moderate growth and is holomorphic away from possible simple poles at* $z = 0, \pm 1$*. Assume that there exists* $\sigma \in (\sigma_1, \sigma_2)$*,*$\sigma \neq 0, \pm 1$ *such that* $\int_{L_\sigma} \beta(z)F(z)dz = 0$ *for all* $\beta \in \mathcal{B}_1$*. Then we have* $F(z) = 0$ *identically on the vertical strip* $\mathrm{Re}(z) \in (\sigma_1, \sigma_2)$*.*

Proof. See ([16, p. 2999]). □

4 The smoothed convolution

Let $\Phi(g) = \Phi(\mathfrak{n}|\mathbf{s}_S, \mathfrak{s}_\infty; g)$ and $\mathbf{\Phi}(g,h) = \mathbf{\Phi}(\mathfrak{n}|\mathbf{s}_S, \mathfrak{s}_\infty; g, h)$ be as in (2.15) and (2.16) respectively. For $\beta \in \mathcal{B}_1$, set

$$\mathbb{I}(\mathfrak{n}|\mathbf{s}_S, \mathfrak{s}_\infty; \beta) = \int_{Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}}} \mathbf{\Phi}(g,g)\mathcal{E}^*_\beta(g)dg \tag{4.1}$$

and

$$\mathbb{I}_\flat(\mathfrak{n}|\mathbf{s}_S, \mathfrak{s}_\infty; \beta) = \int_{Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}}} \mathbf{\Phi}_\flat(g,g)\mathcal{E}^*_\beta(g)dg, \quad \flat \in \{\mathrm{cus}, \mathrm{Eis}, \mathrm{res}\}. \tag{4.2}$$

The aim of this section is to prove the following theorem.

Theorem 4.1. *For each* $\flat \in \{\mathrm{cus}, \mathrm{Eis}, \mathrm{res}\}$, *there exists a holomorphic function* $\hat{I}^0_\flat(\mathbf{s}_S, \mathfrak{s}_\infty, z)$ *on the region* $-1 < \mathrm{Re}(z) < 1$ *which is vertically of moderate growth and satisfies*

$$\mathbb{I}(\mathfrak{n}|\mathbf{s}_S, \mathfrak{s}_\infty; \beta) = \int_{L_\sigma} \beta(z) \left(\hat{I}^0_{\mathrm{cus}}(\mathbf{s}_S, \mathfrak{s}_\infty, z) + \hat{I}^0_{\mathrm{Eis}}(\mathbf{s}_S, \mathfrak{s}_\infty, z) + \hat{I}^0_{\mathrm{res}}(\mathbf{s}_S, \mathfrak{s}_\infty, z) \right) dz$$

for $-1 < \sigma < 1$.

4.1 The calculation of $\mathbb{I}_{\mathrm{cus}}(\mathfrak{n}|\mathbf{s}_S, \mathfrak{s}_\infty; \beta)$

If $-1 < \sigma < 1$ and N is sufficiently large, from Propositions 3.6 and 3.8, we have the following termwise integration

$$\begin{aligned}
\mathbb{I}_{\mathrm{cus}}(\mathfrak{n}|\mathbf{s}_S, \mathfrak{s}_\infty; \beta) &= \sum_{\pi \in \Pi_{\mathrm{cus}}(\mathfrak{n})} \sum_{\varphi \in \mathcal{B}_{\mathrm{cus}}(\pi, \mathfrak{n})} C(\mathfrak{n}, \pi) \int_{Z_\mathbb{A} G_F \backslash G_\mathbb{A}} |\varphi(g)|^2 \mathcal{E}^*_\beta(g) dg \\
&= \sum_{\pi \in \Pi_{\mathrm{cus}}(\mathfrak{n})} C(\mathfrak{n}, \pi) \int_{L_\sigma} \beta(z) \mathbb{P}_{E^*(z)}(\mathfrak{n}, \pi) dz \\
&= \int_{L_\sigma} \beta(z) \left(\sum_{\pi \in \Pi_{\mathrm{cus}}(\mathfrak{n})} C(\mathfrak{n}, \pi) \mathbb{P}_{E^*(z)}(\mathfrak{n}, \pi) \right) dz \qquad (4.3)
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{P}_{E^*(z)}(\mathfrak{n}, \pi) &= \sum_{\varphi \in \mathcal{B}_{\mathrm{cus}}(\pi, \mathfrak{n})} \int_{Z_\mathbb{A} G_F \backslash G_\mathbb{A}} |\varphi(g)|^2 E^*(z; g) dg \\
&= \Lambda_F(z+1) \sum_{\varphi \in \mathcal{B}_{\mathrm{cus}}(\pi, \mathfrak{n})} \int_{Z_\mathbb{A} G_F \backslash G_\mathbb{A}} |\varphi(g)|^2 E(z; g) dg.
\end{aligned}$$

In the above calculation, we have applied Fubini's theorem. In fact,

$$\int_{Z_\mathbb{A} G_F \backslash G_\mathbb{A}} \int_{L_\sigma} |\beta(z) \Lambda_F(z+1) E(z, g)|\, |dz| |\varphi(g)|^2 dg < +\infty, \quad \varphi \in C^\infty(Z_\mathbb{A} G_F \backslash G_\mathbb{A}) \cap \mathcal{L}^2_{\mathrm{cus}}$$

holds from the assumption of $\beta \in \mathcal{B}_1$, the property of moderate growth of Λ_F, the property of rapid decrease of φ, and Lemma 3.7.

By the locally uniform convergence of the sum in (4.3) on $-1 < \mathrm{Re}(z) < 1$, the function

$$\hat{I}^0_{\mathrm{cus}}(\mathbf{s}_S, \mathfrak{s}_\infty, z) = \sum_{\pi \in \Pi_{\mathrm{cus}}(\mathfrak{n})} C(\mathfrak{n}, \pi) \mathbb{P}_{E^*(z)}(\mathfrak{n}, \pi) \qquad (4.4)$$

is holomorphic on the region $-1 < \mathrm{Re}(z) < 1$.

To calculate (4.4), we recall the Fourier expansion of $\varphi \in \mathcal{B}_{\mathrm{cus}}(\pi, \mathfrak{n})$ as

$$\varphi\left(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) g\right) = \sum_{a \in F^\times} \phi_\varphi\left(\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) g\right) \psi(x), \quad x \in \mathbb{A},\ g \in G_\mathbb{A} \qquad (4.5)$$

where ϕ_φ is the ψ-Whittaker function given by

$$\phi_\varphi(g) = \int_{\mathbb{A}/F} \varphi\left(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) g\right) \psi(-x) dx.$$

If $\mathrm{Re}(z) > 1$, by the unfolding procedure, we have

$$\begin{aligned}
\int_{Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}}}|\varphi(g)|^2E(z;g)dg &= \int_{Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}}}\sum_{\gamma\in B_F\backslash G_F}y(\gamma g)^{\frac{z+1}{2}}|\varphi(g)|^2dg \\
&= \int_{Z_{\mathbb{A}}B_F\backslash G_{\mathbb{A}}}y(g)^{\frac{z+1}{2}}|\varphi(g)|^2dg \\
&= \int_{Z_{\mathbb{A}}N_{\mathbb{A}}H_F\backslash G_{\mathbb{A}}}y(g)^{\frac{z+1}{2}}\left(\int_{N_F\backslash N_{\mathbb{A}}}|\varphi(ng)|^2dn\right)dg \\
&= \int_{Z_{\mathbb{A}}N_{\mathbb{A}}H_F\backslash G_{\mathbb{A}}}y(g)^{\frac{z+1}{2}}\left(\int_{\mathbb{A}/F}\left|\varphi\left(\left(\begin{smallmatrix}1&x\\0&1\end{smallmatrix}\right)g\right)\right|^2dx\right)dg \\
&= \int_{Z_{\mathbb{A}}N_{\mathbb{A}}H_F\backslash G_{\mathbb{A}}}y(g)^{\frac{z+1}{2}}\sum_{a\in F^\times}\left|\phi_\varphi\left(\left(\begin{smallmatrix}a&0\\0&1\end{smallmatrix}\right)g\right)\right|^2dg \\
&= \int_{\mathbb{A}^\times}\int_{\mathbf{K}}|t|_{\mathbb{A}}^{\frac{z-1}{2}}\left|\phi_\varphi\left(\left(\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right)k\right)\right|^2d^\times tdk. \qquad (4.6)
\end{aligned}$$

For $\pi \cong \otimes'_{v\in\Sigma_F}\pi_v \in \Pi_{\mathrm{cus}}(\mathfrak{n})$, let V_{π_v} be the ψ_v-Whittaker model of π_v. Let $\phi_{0,v} \in V_{\pi_v}$ be an element such that its explicit form is given as in ([19, p. 22]) and ([15, pp. 379–380]) for $v \in \Sigma_\infty$ and $v \in \Sigma_{\mathrm{fin}} - S(\mathfrak{n}\mathfrak{f}_\pi^{-1})$ respectively. Then, by the theory of local new forms explained in ([19]), we have

$$V_{\pi_v}^{\mathbf{K}_0(\mathfrak{n}\mathfrak{o}_v)} = \begin{cases}\mathbb{C}\phi_{0,v} & (v\in\Sigma_{\mathrm{fin}}-S(\mathfrak{n}\mathfrak{f}_\pi^{-1}))\\ \mathbb{C}\phi_{0,v}\oplus\mathbb{C}\phi_{1,v} & (v\in S(\mathfrak{n}\mathfrak{f}_\pi^{-1}))\end{cases}$$

where $\phi_{1,v} = \pi_v\left(\left(\begin{smallmatrix}\varpi_v^{-1}&0\\0&1\end{smallmatrix}\right)\right)\phi_{0,v} - Q(\pi_v)\phi_{0,v}$.

For a non-zero ideal $\mathfrak{c}\subset\mathfrak{o}$ with $\mathfrak{n}\mathfrak{f}_\pi^{-1}\subset\mathfrak{c}$, let $\varphi_{\pi,\mathfrak{c}} \in \pi^{\mathbf{K}_0(\mathfrak{n})\times SO(2)^{\Sigma_\infty}}$ be an element such that

$$\phi_{\varphi_{\pi,\mathfrak{c}}}(g) = \left\{\prod_{v\notin S(\mathfrak{c})}\phi_{0,v}(g_v)\right\}\left\{\prod_{v\in S(\mathfrak{c})}\phi_{1,v}(g_v)\right\}, \quad g\in G_{\mathbb{A}}.$$

Then $\{\varphi_{\pi,\mathfrak{c}}\}_{\mathfrak{n}\mathfrak{f}_\pi^{-1}\subset\mathfrak{c}\subset\mathfrak{o}}$ forms an orthogonal basis of $\varphi_{\pi,\mathfrak{c}} \in \pi^{\mathbf{K}_0(\mathfrak{n})\times SO(2)^{\Sigma_\infty}}$ and satisfies

$$||\varphi_{\pi,\mathfrak{o}}||^2 = 2\mathrm{N}(\mathfrak{f}_\pi)\left\{\prod_{v\in S(\mathfrak{f}_\pi)}(q_v+1)^{-1}\right\}L(1,\pi,\mathrm{Ad}), \qquad (4.7)$$

$$||\varphi_{\pi,\mathfrak{c}}||^2 = ||\varphi_{\pi,\mathfrak{o}}||^2\prod_{v\in S(\mathfrak{c})}\{1-Q(\pi_v)^2\} \qquad (4.8)$$

by ([19, Corollary 2.15]) and ([19, Lemma 2.4]).

To compute $\mathbb{P}_{E^*(z)}(\mathfrak{n},\pi)$, we take $\mathcal{B}_{\mathrm{cus}}(\pi,\mathfrak{n}) = \left\{||\varphi_{\pi,\mathfrak{c}}||^{-1}\varphi_{\pi,\mathfrak{c}} \mid \mathfrak{n}\mathfrak{f}_\pi^{-1}\subset\mathfrak{c}\subset\mathfrak{o}\right\}$. From (4.6), we have

$$\begin{aligned}
&\mathbb{P}_{E^*(z)}(\mathfrak{n},\pi)\\
&= \Lambda_F(z+1)\sum_{\substack{\mathfrak{c}\\ \mathfrak{n}\mathfrak{f}_\pi^{-1}\subset\mathfrak{c}\subset\mathfrak{o}}}||\varphi_{\pi,\mathfrak{c}}||^{-2}\left\{\prod_{v\notin S(\mathfrak{c})}Z_v(z,\phi_{0,v},\phi_{0,v})\right\}\left\{\prod_{v\in S(\mathfrak{c})}Z_v(z,\phi_{1,v},\phi_{1,v})\right\} \qquad (4.9)
\end{aligned}$$

where

$$Z_v(z,\phi,\phi')=\int_{F_v^\times}\int_{\mathbf{K}_v}|t|_v^{\frac{z-1}{2}}\phi\left(\left(\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right)k\right)\overline{\phi'\left(\left(\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right)k\right)}dkd^\times t,\quad \phi,\phi'\in V_{\pi_v}.$$

Lemma 4.2.
(i) For any $v\in\Sigma_{\rm fin}-S(\mathfrak{f}_\pi)$, we have

$$Z_v(z,\phi_{0,v},\phi_{0,v})=q_v^{{\rm d}_v(\frac{z}{2}-1)}\frac{\zeta_v(\frac{z+1}{2})}{\zeta_v(z+1)}L_v(\tfrac{z+1}{2},\pi_v,{\rm Ad}).$$

(ii) For any $v\in S(\mathfrak{n}\mathfrak{f}_\pi^{-1})$

$$Z_v(z,\phi_{1,v},\phi_{1,v})=\left\{\frac{q_v^{\frac{z}{2}}+q_v^{-\frac{z}{2}}}{q_v^{\frac{1}{2}}+q_v^{-\frac{1}{2}}}-Q(\pi_v)^2\right\}Z_v(z,\phi_{0,v},\phi_{0,v}).$$

(iii) For any $v\in S(\mathfrak{f}_\pi)$, we have

$$Z_v(z,\phi_{0,v},\phi_{0,v})=(q_v+1)^{-1}q_v^{\frac{z+1}{2}}\times q_v^{{\rm d}_v(\frac{z}{2}-1)}\frac{\zeta_v(\frac{z+1}{2})}{\zeta_v(z+1)}L_v(\tfrac{z+1}{2},\pi_v,{\rm Ad}).$$

(iv) For any $v\in\Sigma_\infty$, we have

$$Z_v(z,\phi_{0,v},\phi_{0,v})=\frac{\Gamma_{\mathbb{R}}(\frac{z+1}{2})}{\Gamma_{\mathbb{R}}(z+1)}L_v(\tfrac{z+1}{2},\pi_v,{\rm Ad}).$$

Proof. For (i), we can refer to the proof of ([19, Lemma 2.14]). (ii) and (iii) follow from ([19, p. 22]). We can prove (iv) by the same calculation as in the proof of ([16, Lemma 3.3]). □

By substituting (4.7), (4.8), and the results of Lemma 4.2 into (4.9), we obtain

$$\begin{aligned}\mathbb{P}_{E^*(z)}(\mathfrak{n},\pi)&=\frac{1}{2}{\rm N}(\mathfrak{f}_\pi)^{\frac{z-1}{2}}D_F^{z-\frac{1}{2}}\zeta_F(\tfrac{z+1}{2})\sum_{\substack{\mathfrak{c}\\ \mathfrak{n}\mathfrak{f}_\pi^{-1}\subset\mathfrak{c}\subset\mathfrak{o}}}\left\{\prod_{v\in S(\mathfrak{c})}\frac{Q(I(|\ |_v^{\frac{z}{2}}))-Q(\pi_v)^2}{1-Q(\pi_v)^2}\right\}\\&\qquad\times\frac{L(\frac{z+1}{2},\pi,{\rm Ad})}{L(1,\pi,{\rm Ad})}\\&=\frac{1}{2}{\rm N}(\mathfrak{f}_\pi)^{\frac{z-1}{2}}D_F^{z-\frac{1}{2}}\zeta_F(\tfrac{z+1}{2})\left\{\prod_{v\in S(\mathfrak{n}\mathfrak{f}_\pi^{-1})}\left(1+\frac{Q(I(|\ |_v^{\frac{z}{2}}))-Q(\pi_v)^2}{1-Q(\pi_v)^2}\right)\right\}\\&\qquad\times\frac{L(\frac{z+1}{2},\pi,{\rm Ad})}{L(1,\pi,{\rm Ad})}.\end{aligned}$$

From (4.3), The function $\hat{I}^0_{\rm cus}(\mathbf{s}_S,\mathfrak{s}_\infty,z)$ satisfies

$$\mathbb{I}_{\rm cus}(\mathfrak{n}|\mathbf{s}_S,\mathfrak{s}_\infty;\beta)=\int_{L_\sigma}\beta(z)\hat{I}^0_{\rm cus}(\mathbf{s}_S,\mathfrak{s}_\infty,z)dz,\qquad(-1<\sigma<1)$$

and is vertically of moderate growth by Lemma 3.7.

4.2 Explicit formulas of $\mathbf{\Phi}_{\mathrm{Eis}}(g,h)$ and $\mathbf{\Phi}_{\mathrm{res}}(g,h)$

By the theory of Eisenstein series in [7] and noting that the function $h \mapsto \mathbf{\Phi}(g,h)$ belongs to $\mathcal{L}^q(Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}})$ for any $q>0$ (cf. Proposition 2.1 (ii)), explicit formulas for $\mathbf{\Phi}_{\mathrm{Eis}}(g,h)$ and $\mathbf{\Phi}_{\mathrm{res}}(g,h)$ are given as follows:

$$\begin{aligned}\mathbf{\Phi}_{\mathrm{Eis}}(g,h) =& \frac{\mathrm{vol}(\mathbb{A}^1/F^\times)^{-1}}{8\pi}\\ &\times \sum_{\chi\in\Xi}\sum_{f,f'\in B(\chi)}\int_{\mathbb{R}}\langle \pi_{\sqrt{-1}u}(\Phi)f|f'\rangle Ef(g,\sqrt{-1}u)\overline{Ef'(h,\sqrt{-1}u)}du, \end{aligned}\tag{4.10}$$

$$\mathbf{\Phi}_{\mathrm{res}}(g,h) = \mathrm{vol}\left(Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}}\right)^{-1}\sum_{\substack{\chi\in\Xi\\ \chi^2=1}}\overline{\chi(\det g)}\chi(\det h)\int_{Z_{\mathbb{A}}\backslash G_{\mathbb{A}}}\Phi(x)\chi(\det x)dx. \tag{4.11}$$

Here, the matrix coefficient $\langle \pi_{\sqrt{-1}u}(\Phi)f|f'\rangle$ occuring in the left-hand side of (4.10) is given by

$$\begin{aligned}\langle \pi_{\sqrt{-1}u}(\Phi)f|f'\rangle &= \int_{Z_{\mathbb{A}}\backslash G_{\mathbb{A}}}\langle \pi_{\sqrt{-1}u}(g)f|f'\rangle\Phi(g)dg\\ &= \int_{Z_{\mathbb{A}}\backslash G_{\mathbb{A}}}\left(\int_{\mathbf{K}}\left(\pi_{\sqrt{-1}u}(g)f\right)(k)\overline{f'(k)}dk\right)\Phi(g)dg\\ &= \int_{Z_{\mathbb{A}}\backslash G_{\mathbb{A}}}\left(\int_{\mathbf{K}}y(kg)^{\frac{\sqrt{-1}u}{2}}y(k)^{-\frac{\sqrt{-1}u}{2}}f(kg)\overline{f'(k)}dk\right)\Phi(g)dg\\ &= \int_{Z_{\mathbb{A}}\backslash G_{\mathbb{A}}}\int_{\mathbf{K}}y(g)^{\frac{\sqrt{-1}u}{2}}f(g)\overline{f'(k)}\Phi(k^{-1}g)dkdg\\ &= \int_{\mathbb{A}^\times}\int_{\mathbb{A}}\int_{\mathbf{K}}\int_{\mathbf{K}}f(k')\overline{f'(k)}\Phi\left(k^{-1}\left(\begin{smallmatrix}a&0\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&x\\0&1\end{smallmatrix}\right)k'\right)\\ &\qquad\qquad\times\chi(a)|a|_{\mathbb{A}}^{\frac{1+\sqrt{-1}u}{2}}dkdk'dxd^\times a. \end{aligned}\tag{4.12}$$

4.2.1 The matrix coefficient

We recall the space $H(\chi)_{\mathfrak{n}}$ for $\chi\in\Xi$, the index set $A(\chi)_{\mathfrak{n}} = \{\iota = (\iota_v)_{v\in S(\mathfrak{n})}\in\{0,1\}^{S(\mathfrak{n})}\}$, and the orthonormal basis $B(\chi)_{\mathfrak{n}}$ for $\chi\in\Xi(\mathfrak{o})$ constructed as in (3.6).

Lemma 4.3. *For $\chi\in\Xi$, if $f\in H(\chi)_{\mathfrak{n}}^{\perp}$ or $f'\in H(\chi)_{\mathfrak{n}}^{\perp}$, then*

$$\langle \pi_{\sqrt{-1}u}(\Phi)f|f'\rangle = 0.$$

Proof. Let $H(\chi)_{\mathfrak{n}}|_{\mathbf{K}}$ be the space of all functions on $\mathbf{K}$ which come from the restriction of some function in $H(\chi)_{\mathfrak{n}}$ to $\mathbf{K}$. More precisely, $H(\chi)_{\mathfrak{n}}|_{\mathbf{K}}$ consists of all right $\mathbf{K}_0(\mathfrak{n})\times SO(2)^{\Sigma_\infty}$-invariant functions on $\mathbf{K}$ such that

$$f\left(\begin{pmatrix}a&b\\0&d\end{pmatrix}k\right) = \chi\left(\frac{a}{d}\right)f(k)\qquad\text{for all }\begin{pmatrix}a&b\\0&d\end{pmatrix}\in B_{\mathbb{A}}\cap\mathbf{K}, k\in\mathbf{K}.$$

Suppose $f'\in H(\chi)_{\mathfrak{n}}^{\perp}$. It is easy to check that the function on $\mathbf{K}$ defined as

$$k\mapsto\int_{\mathbb{A}^\times}\int_{\mathbb{A}}\int_{\mathbf{K}}f(k')\Phi\left(k^{-1}\left(\begin{smallmatrix}a&0\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&x\\0&1\end{smallmatrix}\right)k'\right)\chi(a)|a|_{\mathbb{A}}^{\frac{1+\sqrt{-1}u}{2}}dk'dxd^\times a$$

belongs to $H(\chi)_{\mathfrak{n}}|_{\mathbf{K}}$ since Φ is bi-$\mathbf{K}_0(\mathfrak{n}) \times SO(2)^{\Sigma_\infty}$-invariant. Therefore, the matrix coefficient is given by the inner product of some function in $H(\chi)_{\mathfrak{n}}$ and f' since the inner product is determined by their restrictions to $\mathbf{K}$, hence is equal to 0. In the case of $f \in H(\chi)_{\mathfrak{n}}^{\perp}$, the proof is similar. □

Lemma 4.4. *For $v \in \Sigma_{\rm fin}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, we define the function $\mathcal{F}_s : F_v^\times \to \mathbb{C}$ by*

$$\mathcal{F}_s(a) = \int_{F_v} \max\{|a|_v, |ax|_v, 1\}^{-s-1}\, dx, \qquad (a \in F_v^\times). \tag{4.13}$$

Then, we have

$$\mathcal{F}_s(a) = q_v^{-\frac{\mathrm{d}_v}{2}} \left(\frac{1-q_v^{-s-1}}{1-q_v^{-s}}\right) |a|_v^{-1} \max(|a|_v, 1)^{-s}.$$

Proof. If $|a|_v \le 1$, by separating the integral into $|x| \le |a|_v^{-1}$ and $|x| > |a|_v^{-1}$, we have

$$\begin{aligned}
\mathcal{F}_s(a) &= \left(\int_{|x|_v \le |a|_v^{-1}} + \int_{|x|_v > |a|_v^{-1}}\right) \max(|a|_v, |ax|_v, 1)^{-s-1} dx \\
&= \int_{|x|_v \le |a|_v^{-1}} dx + \int_{|x|_v > |a|_v^{-1}} |ax|_v^{-s-1} dx \\
&= q^{-\frac{\mathrm{d}_v}{2}} |a|_v^{-1} + |a|_v^{-1} \int_{|x|_v > 1} |x|_v^{-s-1} dx \\
&= q^{-\frac{\mathrm{d}_v}{2}} \left\{1 + \frac{(1-q_v^{-1})q_v^{-s}}{1-q_v^{-s}}\right\} |a|_v^{-1} \\
&= q^{-\frac{\mathrm{d}_v}{2}} \left(\frac{1-q_v^{-s-1}}{1-q_v^{-s}}\right) |a|_v^{-1}.
\end{aligned}$$

If $|a|_v > 1$, by separating the integral into $|x| \le 1$ and $|x| > 1$, we have

$$\begin{aligned}
\mathcal{F}_s(a) &= \left(\int_{|x|_v \le 1} + \int_{|x|_v > 1}\right) \max(|a|_v, |ax|_v, 1)^{-s-1} dx \\
&= \int_{|x|_v \le 1} |a|_v^{-s-1} dx + \int_{|x|_v > 1} |ax|_v^{-s-1} dx \\
&= \left(q_v^{-\frac{\mathrm{d}_v}{2}} + \int_{|x|_v > 1} |x|_v^{-s-1} dx\right) |a|_v^{-s-1} \\
&= q_v^{-\frac{\mathrm{d}_v}{2}} \left\{1 + \frac{(1-q_v^{-1})q_v^{-s}}{1-q_v^{-s}}\right\} |a|_v^{-s-1} \\
&= q_v^{-\frac{\mathrm{d}_v}{2}} \left(\frac{1-q_v^{-s-1}}{1-q_v^{-s}}\right) |a|_v^{-s-1}.
\end{aligned}$$

Therefore, the lemma follows. □

The following lemma gives an explicit formula of the matrix coefficient $\langle \pi_{\sqrt{-1}u}(\Phi) f_\iota^\chi | f_\kappa^\chi \rangle$.

Lemma 4.5. *Let* $\chi = \prod_{v \in \Sigma_F} \chi_v \in \Xi(\mathfrak{o})$ *and* $\iota, \kappa \in A(\chi)_{\mathfrak{n}}$. *Then we have*

$$\begin{aligned}\langle \pi_{\sqrt{-1}u}(\Phi) f_\iota^\chi | f_\kappa^\chi \rangle = {} & \delta(\iota = \kappa) D_F^{-1} \left[\mathbf{K}_{\mathrm{fin}} : \mathbf{K}_0(\mathfrak{n})\right]^{-1} \\ & \times \prod_{v \in S} \left(-q_v^{-\frac{s_v+1}{2}}\right) L_v\left(\tfrac{s_v+\sqrt{-1}u}{2}, \chi_v\right) L_v\left(\tfrac{s_v-\sqrt{-1}u}{2}, \overline{\chi_v}\right) \\ & \times \prod_{v \in \Sigma_\infty} \left[\prod_{i=1}^N \left\{\tfrac{s_{i,v}^2}{4} + \left(\tfrac{u}{2} + a(\chi_v)\right)^2\right\}^{-1}\right]. \end{aligned} \tag{4.14}$$

Proof. By the product formula, the matrix coefficient is equal to the product of local integrals, i.e.

$$\begin{aligned}\langle \pi_{\sqrt{-1}u}(\Phi) f_\iota^\chi | f_\kappa^\chi \rangle = \prod_{v \in \Sigma_F} \int_{F_v^\times} \int_{F_v} \int_{\mathbf{K}_v} \int_{\mathbf{K}_v} & f_{v,\iota_v}(k'_v) \overline{f_{v,\kappa_v}(k_v)} \Phi_v\left(k_v^{-1} \left(\begin{smallmatrix} a_v & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & x_v \\ 0 & 1 \end{smallmatrix}\right) k'_v\right) \\ & \times \chi_v(a_v)|a_v|_v^{\frac{1+\sqrt{-1}u}{2}} dk_v dk'_v dx_v d^\times a_v \end{aligned} \tag{4.15}$$

where Φ_v is the v-component of the test function Φ. We calculate separately for the seven cases. In the following computation, we use the decomposition

$$\mathbf{K}_v = \mathbf{K}_0(\mathfrak{p}_v) \bigsqcup \left(\bigsqcup_{\xi \in \mathfrak{o}_v/\mathfrak{p}_v} \begin{pmatrix} \xi & -1 \\ 1 & 0 \end{pmatrix} \mathbf{K}_0(\mathfrak{p}_v) \right), \quad v \in \Sigma_{\mathrm{fin}}. \tag{4.16}$$

(i) When $v \in S(\mathfrak{n})$, $\iota_v = 0$, and $\kappa_v = 0$, the v-component of (4.15) is equal to

$$\begin{aligned} & (q_v+1) \int_{F_v^\times} \int_{F_v} \int_{\mathbf{K}_0(\mathfrak{p}_v)} \int_{\mathbf{K}_0(\mathfrak{p}_v)} \mathbf{1}_{Z_v \mathbf{K}_0(\mathfrak{p}_v)} \left(k_v^{-1} \left(\begin{smallmatrix} a_v & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & x_v \\ 0 & 1 \end{smallmatrix}\right) k'_v\right) \\ & \qquad \times \chi_v(a_v)|a_v|_v^{\frac{1+\sqrt{-1}u}{2}} dk_v dk'_v dx_v d^\times a_v \\ & = (q_v+1)^{-1} \int_{F_v^\times} \int_{F_v} \mathbf{1}_{Z_v \mathbf{K}_0(\mathfrak{p}_v)} \left(\left(\begin{smallmatrix} a_v & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & x_v \\ 0 & 1 \end{smallmatrix}\right)\right) \chi_v(a_v)|a_v|_v^{\frac{1+\sqrt{-1}u}{2}} dx_v d^\times a_v. \end{aligned} \tag{4.17}$$

It is obvious that $\left(\begin{smallmatrix} a_v & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & x_v \\ 0 & 1 \end{smallmatrix}\right)$ belongs to $Z_v \mathbf{K}_0(\mathfrak{p}_v)$ if and only if $a_v \in \mathfrak{o}_v^\times$ and $x \in \mathfrak{o}_v$. Hence (4.17) is equal to

$$(q_v+1)^{-1} \int_{\mathfrak{o}_v^\times} \int_{\mathfrak{o}_v} \chi_v(a_v)|a_v|_v^{\frac{1+\sqrt{-1}u}{2}} dx_v d^\times a_v = q_v^{-\mathrm{d}_v}(q_v+1)^{-1}.$$

(ii) When $v \in S(\mathfrak{n})$, $\iota_v = 0$, and $\kappa_v = 1$, the v-component of (4.15) is equal to

$$\begin{aligned} & q_v^{-\frac{1}{2}}(q_v+1) \int_{F_v^\times} \int_{F_v} \int_{\mathbf{K}_0(\mathfrak{p}_v)} \int_{\mathbf{K}_v - \mathbf{K}_0(\mathfrak{p}_v)} \mathbf{1}_{Z_v \mathbf{K}_0(\mathfrak{p}_v)} \left(k_v^{-1} \left(\begin{smallmatrix} a_v & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & x_v \\ 0 & 1 \end{smallmatrix}\right) k'_v\right) \\ & \qquad \times \chi_v(a_v)|a_v|_v^{\frac{1+\sqrt{-1}u}{2}} dk_v dk'_v dx_v d^\times a_v \\ & = q_v^{-\frac{1}{2}} \int_{F_v^\times} \int_{F_v} \sum_{\xi \in \mathfrak{o}_v/\mathfrak{p}_v} \int_{\mathbf{K}_0(\mathfrak{p}_v)} \mathbf{1}_{Z_v \mathbf{K}_0(\mathfrak{p}_v)} \left(k_v^{-1} \left(\begin{smallmatrix} 0 & 1 \\ -1 & \xi \end{smallmatrix}\right) \left(\begin{smallmatrix} a_v & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & x_v \\ 0 & 1 \end{smallmatrix}\right)\right) \\ & \qquad \times \chi_v(a_v)|a_v|_v^{\frac{1+\sqrt{-1}u}{2}} dk_v dx_v d^\times a_v \\ & = q_v^{-\frac{1}{2}}(1+q_v)^{-1} \sum_{\xi \in \mathfrak{o}_v/\mathfrak{p}_v} \int_{F_v^\times} \int_{F_v} \mathbf{1}_{Z_v \mathbf{K}_0(\mathfrak{p}_v)} \left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & \xi \end{smallmatrix}\right) \left(\begin{smallmatrix} a_v & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & x_v \\ 0 & 1 \end{smallmatrix}\right)\right) \\ & \qquad \times \chi_v(a_v)|a_v|_v^{\frac{1+\sqrt{-1}u}{2}} dx_v d^\times a_v. \end{aligned} \tag{4.18}$$

It can be easily checked that $\left(\begin{smallmatrix} 0 & 1 \\ -1 & \xi \end{smallmatrix}\right)\left(\begin{smallmatrix} a_v & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & x_v \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ -a_v & \xi - a_v x_v \end{smallmatrix}\right)$ never belongs to $Z_v\mathbf{K}_0(\mathfrak{p}_v)$. Hence (4.18) is equal to 0.

(iii) When $v \in S(\mathfrak{n})$, $\iota_v = 1$, and $\kappa_v = 0$, the v-component of (4.15) is equal to 0 by a similar calculation in (ii).

(iv) When $v \in S(\mathfrak{n})$, $\iota_v = 1$, and $\kappa_v = 1$, the v-component of (4.15) is equal to

$$\begin{aligned}
&(1+q_v^{-1})\int_{F_v^\times}\int_{F_v}\int_{\mathbf{K}_v-\mathbf{K}_0(\mathfrak{p}_v)}\int_{\mathbf{K}_v-\mathbf{K}_0(\mathfrak{p}_v)} \mathbf{1}_{Z_v\mathbf{K}_0(\mathfrak{p}_v)}\left(k_v^{-1}\left(\begin{smallmatrix} a_v & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & x_v \\ 0 & 1 \end{smallmatrix}\right)k_v'\right) \\
&\qquad\qquad\qquad\qquad \times \chi_v(a_v)|a_v|_v^{\frac{1+\sqrt{-1}u}{2}}\,dk_v dk_v' dx_v d^\times a_v. \\
&= (1+q_v^{-1})\int_{F_v^\times}\int_{F_v}\sum_{\xi,\xi'\in\mathfrak{o}_v/\mathfrak{p}_v}\int_{\mathbf{K}_0(\mathfrak{p}_v)}\int_{\mathbf{K}_0(\mathfrak{p}_v)} \mathbf{1}_{Z_v\mathbf{K}_0(\mathfrak{p}_v)}\left(k_v^{-1}\left(\begin{smallmatrix} 0 & 1 \\ -1 & \xi \end{smallmatrix}\right)\left(\begin{smallmatrix} a_v & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & x_v \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \xi' & -1 \\ 1 & 0 \end{smallmatrix}\right)k_v'\right) \\
&\qquad\qquad\qquad\qquad \times \chi_v(a_v)|a_v|_v^{\frac{1+\sqrt{-1}u}{2}}\,dk_v dk_v' dx_v d^\times a_v \\
&= q_v^{-1}(1+q_v)^{-1}\sum_{\xi,\xi'\in\mathfrak{o}_v/\mathfrak{p}_v}\int_{F_v^\times}\int_{F_v} \mathbf{1}_{Z_v\mathbf{K}_0(\mathfrak{p}_v)}\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & \xi \end{smallmatrix}\right)\left(\begin{smallmatrix} a_v & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & x_v \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \xi' & -1 \\ 1 & 0 \end{smallmatrix}\right)\right) \\
&\qquad\qquad\qquad\qquad \times \chi_v(a_v)|a_v|_v^{\frac{1+\sqrt{-1}u}{2}}\,dx_v d^\times a_v. \qquad (4.19)
\end{aligned}$$

For any $\xi, \xi' \in \mathfrak{o}_v/\mathfrak{p}_v$, we can check that $\left(\begin{smallmatrix} 0 & 1 \\ -1 & \xi \end{smallmatrix}\right)\left(\begin{smallmatrix} a_v & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & x_v \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \xi' & -1 \\ 1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ -a_v x_v - a_v\xi' + \xi & a_v \end{smallmatrix}\right)$ belongs to $Z_v\mathbf{K}_0(\mathfrak{p}_v)$ if and only if $a_v \in \mathfrak{o}_v^\times$ and $x_v \in (a_v^{-1}\xi - \xi') + \mathfrak{p}_v$. Hence (4.19) is equal to

$$\begin{aligned}
&q_v^{-1}(q_v+1)^{-1}\sum_{\xi,\xi'\in\mathfrak{o}_v/\mathfrak{p}_v}\int_{\mathfrak{o}_v^\times}\int_{(a^{-1}\xi-\xi')+\mathfrak{p}_v} \chi_v(a_v)|a_v|_v^{\frac{1+\sqrt{-1}u}{2}}\,dx_v d^\times a_v \\
&= q_v^{-1}(q_v+1)^{-1}\sum_{\xi,\xi'\in\mathfrak{o}_v/\mathfrak{p}_v} q_v^{-\mathrm{d}_v}\cdot q_v^{-1} \\
&= q_v^{-\mathrm{d}_v}(q_v+1)^{-1}.
\end{aligned}$$

(v) When $v \in S$, recall (2.13). Since $\Phi_v = \Phi_{v,s_v}$ is bi-$\mathbf{K}_v$- invariant, the v-component of (4.15) is equal to

$$\begin{aligned}
&\int_{F_v^\times}\int_{F_v} \Phi_{v,s_v}\left(\left(\begin{smallmatrix} a_v & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & x_v \\ 0 & 1 \end{smallmatrix}\right)\right)\chi_v(a_v)|a_v|_v^{\frac{1+\sqrt{-1}u}{2}}\,dx_v d^\times a_v \\
&= \left(q_v^{-\frac{s_v+1}{2}} - q_v^{\frac{s_v+1}{2}}\right)^{-1}\int_{F_v^\times}\int_{F_v} |a_v|_v^{\frac{s_v+1}{2}}\max(|a_v|_v, |a_v x_v|_v, 1)^{-(s_v+1)} \\
&\qquad\qquad\qquad\qquad \times \chi_v(a_v)|a_v|_v^{\frac{1+\sqrt{-1}u}{2}}\,dx_v d^\times a_v \\
&= \left(q_v^{-\frac{s_v+1}{2}} - q_v^{\frac{s_v+1}{2}}\right)^{-1}\int_{F_v^\times} \mathcal{F}_{s_v}(a_v)\chi_v(a_v)|a_v|_v^{\frac{s_v+\sqrt{-1}u}{2}+1}\,d^\times a_v. \qquad (4.20)
\end{aligned}$$

where $\mathcal{F}_{s_v}$ is the complex-valued function on $F_v^\times$ defined as in (4.13). Hence by Lemma 4.4, the integral (4.20) is equal to

$$
\begin{aligned}
&q_v^{-\frac{\mathrm{d}_v}{2}} \cdot \frac{-q_v^{-\frac{s_v+1}{2}}}{1-q_v^{-s_v}} \int_{F_v^\times} \max(|a_v|_v, 1)^{-s_v} \chi_v(a_v)|a_v|_v^{\frac{s_v+\sqrt{-1}u}{2}} d^\times a_v \\
&= q_v^{-\frac{\mathrm{d}_v}{2}} \cdot \frac{-q_v^{-\frac{s_v+1}{2}}}{1-q_v^{-s_v}} \left(\int_{|a_v|_v \le 1} \chi_v(a_v)|a_v|_v^{\frac{s_v+\sqrt{-1}u}{2}} d^\times a_v + \int_{|a_v|_v > 1} \chi_v(a_v)|a_v|_v^{\frac{-s_v+\sqrt{-1}u}{2}} d^\times a_v \right) \\
&= q_v^{-\mathrm{d}_v} \cdot \frac{-q_v^{-\frac{s_v+1}{2}}}{1-q_v^{-s_v}} \left(\frac{1}{1-\chi_v(\varpi_v) q_v^{-\frac{s_v+\sqrt{-1}u}{2}}} + \frac{\overline{\chi_v(\varpi_v)} q_v^{-\frac{s_v-\sqrt{-1}u}{2}}}{1-\overline{\chi_v(\varpi_v)} q_v^{-\frac{s_v-\sqrt{-1}u}{2}}} \right) \\
&= q_v^{-\mathrm{d}_v} \cdot \left(-q_v^{-\frac{s_v+1}{2}}\right) \left(1-\chi_v(\varpi_v) q_v^{-\frac{s_v+\sqrt{-1}u}{2}}\right)^{-1} \left(1-\overline{\chi_v(\varpi_v)} q_v^{-\frac{s_v-\sqrt{-1}u}{2}}\right)^{-1} \\
&= q_v^{-\mathrm{d}_v} \cdot \left(-q_v^{-\frac{s_v+1}{2}}\right) L_v\left(\tfrac{s_v+\sqrt{-1}u}{2}, \chi_v\right) L_v\left(\tfrac{s_v-\sqrt{-1}u}{2}, \overline{\chi_v}\right).
\end{aligned}
$$

(vi) When $v \in \Sigma_{\mathrm{fin}} - (S \cup S(\mathfrak{n}))$, the v-component of (4.15) is equal to

$$
\begin{aligned}
&\int_{F_v^\times} \int_{F_v} \int_{\mathbf{K}_v} \int_{\mathbf{K}_v} \mathbf{1}_{Z_v \mathbf{K}_v} \left(k_v^{-1} \left(\begin{smallmatrix} a_v & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & x_v \\ 0 & 1 \end{smallmatrix}\right) k_v'\right) \chi_v(a_v)|a_v|_v^{\frac{1+\sqrt{-1}u}{2}} dk_v dk_v' dx_v d^\times a_v \\
&= \int_{F_v^\times} \int_{F_v} \mathbf{1}_{Z_v \mathbf{K}_v} \left(\left(\begin{smallmatrix} a_v & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & x_v \\ 0 & 1 \end{smallmatrix}\right) \right) \chi_v(a_v)|a_v|_v^{\frac{1+\sqrt{-1}u}{2}} dx_v d^\times a_v \\
&= \int_{\mathfrak{o}_v^\times} \int_{\mathfrak{o}_v} \chi_v(a_v)|a_v|_v^{\frac{1+\sqrt{-1}u}{2}} dx_v d^\times a_v \\
&= q_v^{-\mathrm{d}_v}.
\end{aligned}
$$

(vii) When $v \in \Sigma_\infty$, the v-component of (4.15) can be rewritten as

$$
\begin{aligned}
&\int_{\mathbb{R}^\times} \int_{\mathbb{R}} \int_{O(2)} \int_{O(2)} (\det k^{-1}k')^{\epsilon(\chi_v)} \Phi_{\mathfrak{s}_v} \left(k^{-1} \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) k'\right) \\
&\qquad \times (\mathrm{sgn}(a))^{\epsilon(\chi_v)} |a|_v^{\frac{1}{2}+\sqrt{-1}(\frac{u}{2}+a(\chi_v))} dk dk' dx d^\times a. \qquad (4.21)
\end{aligned}
$$

We note that $\Phi_{\mathfrak{s}_v}(g)$ vanishes unless $\det g > 0$ and bi-$SO(2)$-invariant. By separating the integral with respect to k and k' into four parts, $\{k \in SO(2), k' \in SO(2)\}$, $\{k \in SO(2), k' \notin SO(2)\}$, $\{k \notin SO(2), k' \in SO(2)\}$, and $\{k \notin SO(2), k' \notin SO(2)\}$ and using the change of variables, (4.21) is equal to

$$
\begin{aligned}
&4\int_0^\infty \int_{\mathbb{R}} \int_{SO(2)} \int_{SO(2)} \Phi_{\mathfrak{s}_v}\left(k^{-1}\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)k'\right) a^{\frac{1}{2}+\sqrt{-1}(\frac{u}{2}+a(\chi_v))} dk dk' dx d^\times a \\
&= \int_0^\infty \int_{\mathbb{R}} \Phi_{\mathfrak{s}_v}\left(\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\right) a^{\frac{1}{2}+\sqrt{-1}(\frac{u}{2}+a(\chi_v))} dx d^\times a \\
&= \int_0^\infty \int_{\mathbb{R}} \Phi_{\mathfrak{s}_v}\left(\left(\begin{smallmatrix} a & x \\ 0 & 1 \end{smallmatrix}\right)\right) a^{-\frac{1}{2}+\sqrt{-1}(\frac{u}{2}+a(\chi_v))} dx d^\times a \\
&= \frac{1}{4\pi}\sum_{i=1}^N \mu_v^{(i)}(\mathfrak{s}_\infty)\frac{\Gamma(\frac{s_{i,v}+1}{2})^2}{\Gamma(s_{i,v}+1)} \\
&\times \int_0^\infty \int_{\mathbb{R}} \left\{\tfrac{4a}{(a+1)^2+x^2}\right\}^{\frac{s_{i,v}+1}{2}} {}_2F_1\left(\tfrac{s_{i,v}+1}{2}, \tfrac{s_{i,v}+1}{2}; s_{i,v}+1; \tfrac{4a}{(a+1)^2+x^2}\right) a^{-\frac{1}{2}+\sqrt{-1}(\frac{u}{2}+a(\chi_v))} dx d^\times a
\end{aligned}
\tag{4.22}
$$

by (2.10). For $s, w \in \mathbb{C}$ with $\mathrm{Re}(s) > |\mathrm{Re}(w)|$, we set

$$
I(s,w) = \frac{1}{4\pi}\frac{\Gamma(\frac{s+1}{2})^2}{\Gamma(s+1)}\int_0^\infty \int_{\mathbb{R}} \left\{\tfrac{4a}{(a+1)^2+x^2}\right\}^{\frac{s+1}{2}} {}_2F_1\left(\tfrac{s+1}{2}, \tfrac{s+1}{2}; s+1; \tfrac{4a}{(a+1)^2+x^2}\right) a^{\frac{w-1}{2}} dx d^\times a. \tag{4.23}
$$

A formal computation and the change of variable $x \to (a+1)x$ give

$$
\begin{aligned}
I(s,w) &= \frac{1}{4\pi}\sum_{n=0}^\infty \frac{\Gamma(\frac{s+1}{2}+n)^2}{n!\Gamma(s+1+n)}\int_0^\infty \int_{\mathbb{R}} \left\{\tfrac{4a}{(a+1)^2+x^2}\right\}^{\frac{s+1}{2}+n} a^{\frac{w-1}{2}} dx d^\times a \\
&= \frac{1}{4\pi}\sum_{n=0}^\infty \frac{\Gamma(\frac{s+1}{2}+n)^2}{n!\Gamma(s+1+n)} 4^{\frac{s+1}{2}+n}\int_{\mathbb{R}}(1+x^2)^{-\frac{s+1}{2}-n}dx \int_0^\infty a^{\frac{s+w}{2}+n}(a+1)^{-s-2n}d^\times a \\
&= \frac{1}{4\pi}\sum_{n=0}^\infty \frac{\Gamma(\frac{s+1}{2}+n)^2}{n!\Gamma(s+1+n)} 4^{\frac{s+1}{2}+n}\cdot \pi^{\frac{1}{2}}\frac{\Gamma(\frac{s}{2}+n)}{\Gamma(\frac{s+1}{2}+n)}\cdot\frac{\Gamma(\frac{s+w}{2}+n)\Gamma(\frac{s-w}{2}+n)}{\Gamma(s+2n)} \\
&= \sum_{n=0}^\infty \frac{\Gamma(\frac{s+w}{2}+n)\Gamma(\frac{s-w}{2}+n)}{n!\Gamma(s+1+n)} \\
&= \frac{\Gamma(\frac{s+w}{2})\Gamma(\frac{s-w}{2})}{\Gamma(s+1)} {}_2F_1(\tfrac{s+w}{2}; \tfrac{s-w}{2}; s+1; 1) \\
&= \frac{\Gamma(\frac{s+w}{2})\Gamma(\frac{s-w}{2})}{\Gamma(s+1)}\cdot\frac{\Gamma(s+1)\Gamma(1)}{\Gamma(\frac{s-w}{2}+1)\Gamma(\frac{s+w}{2}+1)} \\
&= (\tfrac{s+w}{2})^{-1}(\tfrac{s-w}{2})^{-1} \\
&= \left(\tfrac{s^2-w^2}{4}\right)^{-1} \\
&= \left(\tfrac{1-w^2}{4} - \tfrac{1-s^2}{4}\right)^{-1}.
\end{aligned}
$$

In the above computation, we use the well-known formulas

$$
\int_{\mathbb{R}}(1+x^2)^{-\nu}dx = \pi^{\frac{1}{2}}\frac{\Gamma(\nu-\frac{1}{2})}{\Gamma(\nu)}, \quad \mathrm{Re}(\nu) > \frac{1}{2}, \tag{4.24}
$$

$$
\int_0^\infty a^s(a+1)^{-\nu}d^\times a = \frac{\Gamma(s)\Gamma(\nu-s)}{\Gamma(\nu)}, \quad \mathrm{Re}(\nu) > \mathrm{Re}(s) > 0,
$$

$$\Gamma(2s) = \pi^{-\frac{1}{2}} 2^{2s-1} \Gamma(s)\Gamma(s+\tfrac{1}{2}),$$

and

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re}(c-a-b) > 0.$$

Because $\Gamma(s+n) \asymp \Gamma(\mathrm{Re}(s)+n)$ $(n \in \mathbb{Z}_{\geq 0})$ holds for $s \in \mathbb{C}$ with $\mathrm{Re}(s) \neq 0, -1, -2, \ldots$ by Stirling's formula, we can obtain a dominated series by replacing s and w by $\mathrm{Re}(s)$ and $\mathrm{Re}(w)$ respectively. Hence the above calculation is justified.

By putting $s = s_{i,v}$ and $w = 2\sqrt{-1}(\frac{u}{2} + a(\chi_v))$, (4.22) is equal to

$$\begin{aligned}
&\sum_{i=1}^{N} \mu_v^{(i)}(\mathfrak{s}_\infty) I\left(s_{i,v}, 2\sqrt{-1}(\tfrac{u}{2} + a(\chi_v))\right) \\
&= \sum_{i=1}^{N} \mu_v^{(i)}(\mathfrak{s}_\infty) \left\{ \tfrac{1+4\left(\frac{u}{2}+a(\chi_v)\right)^2}{4} - \tfrac{1-s_{i,v}^2}{4} \right\}^{-1} \\
&= \prod_{i=1}^{N} \left\{ \tfrac{1+4\left(\frac{u}{2}+a(\chi_v)\right)^2}{4} - \tfrac{1-s_{i,v}^2}{4} \right\}^{-1} \\
&= \prod_{i=1}^{N} \left\{ \tfrac{s_{i,v}^2}{4} + \left(\tfrac{u}{2} + a(\chi_v)\right)^2 \right\}^{-1}.
\end{aligned}$$

□

Remark 4.6. The matrix coefficient $\langle \pi_{\sqrt{-1}u}(\Phi) f_\iota^\chi | f_\iota^\chi \rangle$ is independent of $\iota \in A(\chi)_\mathfrak{n}$.

4.2.2 The continuities of $\mathbf{\Phi}_{\mathrm{Eis}}$ and $\mathbf{\Phi}_{\mathrm{res}}$

By Lemmas 3.1, 4.3, and 4.5, we can rewrite (4.10) as

$$\begin{aligned}
\mathbf{\Phi}_{\mathrm{Eis}}(g,h) =& \frac{\mathrm{vol}(\mathbb{A}^1/F^\times)^{-1}}{8\pi} \\
&\times \sum_{\chi \in \Xi(\mathfrak{o})} \sum_{\iota \in A(\chi)_\mathfrak{n}} \int_{\mathbb{R}} D'_\chi(\mathfrak{n}|\mathbf{s}_S, \mathfrak{s}_\infty, \sqrt{-1}u) E f_\iota^\chi(g, \sqrt{-1}u) \overline{E f_\iota^\chi(h, \sqrt{-1}u)} du
\end{aligned} \tag{4.25}$$

where

$$\begin{aligned}
D'_\chi(\mathfrak{n}|\mathbf{s}_S, \mathfrak{s}_\infty, z) =& D_F^{-1} \left[\mathbf{K}_{\mathrm{fin}} : \mathbf{K}_0(\mathfrak{n})\right]^{-1} \\
&\times \prod_{v \in S} \left(-q_v^{-\frac{s_v+1}{2}}\right) L_v\left(\tfrac{s_v+z}{2}, \chi_v\right) L_v\left(\tfrac{s_v-z}{2}, \overline{\chi_v}\right) \\
&\times \prod_{v \in \Sigma_\infty} \left[\prod_{i=1}^{N} \left\{ \tfrac{s_{i,v}^2}{4} + \left(\tfrac{z}{2\sqrt{-1}} + a(\chi_v)\right)^2 \right\}^{-1} \right].
\end{aligned} \tag{4.26}$$

We remark that the estimate

$$\left| D'_\chi(\mathfrak{n}|\mathbf{s}_S, \mathfrak{s}_\infty, \sqrt{-1}u) \right| \ll \prod_{v \in \Sigma_F} \left(1 + |\tfrac{u}{2} + a(\chi_v)|\right)^{-2N}, \quad \chi \in \Xi(\mathfrak{o}), u \in \mathbb{R} \tag{4.27}$$

holds.

Lemma 4.7. *Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n_F-1} \in \mathfrak{o}^\times$ be a fundamental system of units of F. For any $\chi = \prod_{v \in \Sigma_F} \chi_v \in \Xi(\mathfrak{o})$, we have*

$$\sum_{v \in \Sigma_\infty} a(\chi_v) = 0 \tag{4.28}$$

and

$$\sum_{v \in \Sigma_\infty} (\log |\varepsilon_j|_v) a(\chi_v) \in \pi\mathbb{Z}, \quad j \in \{1, 2, \ldots, n_F - 1\}. \tag{4.29}$$

Proof. Recall that χ is trivial on $F^\times$ and $\mathbb{R}_+$. Hence, for any $x \in \mathbb{R}_+$, and $i \in \{1, 2, \ldots, n_F - 1\}$, we have

$$\chi((1_{\mathrm{fin}}, x^{\frac{1}{n_F}}, \ldots, x^{\frac{1}{n_F}})) = \prod_{v \in \Sigma_\infty} \chi_v(x^{\frac{1}{n_F}}) = \prod_{v \in \Sigma_\infty} x^{\frac{\sqrt{-1}}{n_F} a(\chi_v)} = e^{\frac{\sqrt{-1}}{n_F} (\log x)(\sum_{v \in \Sigma_F} a(\chi_v))} = 1$$

and

$$\chi(\varepsilon_j^2) = \prod_{v \in \Sigma_F} \chi_v(\varepsilon_j^2) = \prod_{v \in \Sigma_\infty} \chi_v(\varepsilon_j^2) = e^{2\sqrt{-1} \sum_{v \in \Sigma_\infty} (\log |\varepsilon_j|_v) a(\chi_v)} = 1.$$

Therefore, we get (4.28) and (4.29). $\square$

Let $\{v_1, v_2, \ldots, v_{n_F}\} = \Sigma_\infty$. It is well-known that the $n_F \times n_F$-matrix

$$B = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \log |\varepsilon_1|_{v_1} & \log |\varepsilon_1|_{v_2} & \cdots & \log |\varepsilon_1|_{v_{n_F}} \\ \vdots & \vdots & \ddots & \vdots \\ \log |\varepsilon_{n_F-1}|_{v_1} & \log |\varepsilon_{n_F-1}|_{v_2} & \cdots & \log |\varepsilon_{n_F-1}|_{v_{n_F}} \end{pmatrix}$$

has non-zero determinant. By Lemma 4.7, the set

$$L(\Xi(\mathfrak{o})) = \left\{ \begin{pmatrix} a(\chi_{v_1}) \\ a(\chi_{v_2}) \\ \vdots \\ a(\chi_{v_{n_F}}) \end{pmatrix} + \pi n \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^{n_F} \;\middle|\; \chi \in \Xi(\mathfrak{o}), n \in \mathbb{Z} \right\}$$

is contained in the lattice $\pi B^{-1} \mathbb{Z}^{n_F}$.

Remark 4.8. If $\chi, \chi' \in \Xi(\mathfrak{o})$ satisfy

$$a(\chi_v) = a(\chi'_v) \text{ for any } v \in \Sigma_\infty, \tag{4.30}$$

then we have $(\chi^{-1}\chi')^2 = 1$. Hence, for fixed χ, the number of χ' satisfying (4.30) is at most $\#\{\chi \in \Xi(\mathfrak{o}) \mid \chi^2 = 1\} \le h_F^+$. Here, h_F^+ is the narrow class number of F.

Proposition 4.9. *The sum* (4.25) *converges locally uniformly absolutely on $g, h \in G_\mathbb{A}$. In particular, there exists a constant $N_1 > 0$ such that the estimate*

$$\sum_{\chi \in \Xi(\mathfrak{o})} \sum_{\iota \in A(\chi)_\mathfrak{n}} \int_\mathbb{R} \left| D'_\chi(\mathfrak{n} | \mathbf{s}_S, \mathfrak{s}_\infty, \sqrt{-1}u) E f_\iota^\chi(g, \sqrt{-1}u) \overline{E f_\iota^\chi(h, \sqrt{-1}u)} \right| du$$
$$\ll y(g)^{N_1} y(h)^{N_1}, \quad g, h \in \mathfrak{S}_1$$

holds if N is large enough. In particular, $\mathbf{\Phi}_{\mathrm{Eis}}(g, h)$ is continuous.

Proof. By the estimate in ([19, Proposition 15.1]), there exists a constant $N_1 > 0$ such that

$$Ef_\iota^\chi(g, \sqrt{-1}u) \ll \prod_{v\in\Sigma_\infty} \left(1 + |\tfrac{u}{2} + a(\chi_v)|\right)^{N_1} y(g)^{N_1}, \quad u \in \mathbb{R},\ g \in \mathfrak{S}_1 \tag{4.31}$$

with the implied constant independent of $\chi \in \Xi(\mathfrak{o})$ and $\iota \in A(\chi)_\mathfrak{n}$. From (4.27) and (4.31), it suffices to show that

$$\sum_{\chi\in\Xi(\mathfrak{o})} \int_\mathbb{R} \prod_{v\in\Sigma_\infty} \left(1 + |\tfrac{u}{2} + a(\chi_v)|\right)^{-r} du < +\infty \tag{4.32}$$

for large $r > 0$. We set the function $f : \mathbb{R}^{n_F} \to \mathbb{R}$ as

$$f(x_1, x_2, \dots, x_{n_F}) = \left\{\prod_{j=1}^{n_F-1} \left(1 + \left|\frac{x_{n_F}}{2} + x_j\right|\right)^{-r}\right\} \left(1 + \left|\frac{x_{n_F}}{2} - x_1 - \cdots - x_{n_F-1}\right|\right)^{-r}.$$

Then it holds that

$$f(x_1, x_2, \dots, x_{n_F}) \asymp \prod_{j=1}^{n_F} (1 + |x_j|)^{-r} \tag{4.33}$$

and (4.32) is equal to

$$\sum_{\chi\in\Xi(\mathfrak{o})} \int_\mathbb{R} f\left(a(\chi_{v_1}), \dots, a(\chi_{v_{n_F-1}}), u\right) du.$$

From (4.33), this series converges if and only if

$$\sum_{\chi\in\Xi(\mathfrak{o})} \sum_{n\in\mathbb{Z}} f\left(a(\chi_{v_1}), \dots, a(\chi_{v_{n_F-1}}), \pi n\right) du < +\infty. \tag{4.34}$$

The left-hand side of (4.34) is bounded above by

$$h_F^+ \sum_{\boldsymbol{a}\in L(\Xi(\mathfrak{o}))} f(\boldsymbol{a}) \le h_F^+ \sum_{\boldsymbol{a}\in\pi B^{-1}\mathbb{Z}^{n_F}} f(\boldsymbol{a}). \tag{4.35}$$

Again from (4.33), the right-hand side of (4.35) converges for $r > 1$ since

$$\begin{aligned}&\int_\mathbb{R} \cdots \int_\mathbb{R} \prod_{j=1}^{n_F} (1 + |x_j|)^{-r} dx_1 \dots dx_{n_F} \\ &= \left(\int_\mathbb{R} (1 + |x|)^{-r} dx\right)^{n_F} < +\infty.\end{aligned}$$

The proof is completed. □

Next, we prove the continuity of $\mathbf{\Phi}_{\mathrm{res}}$.

Lemma 4.10. *For $\chi \in \Xi - \Xi(\mathfrak{o})$, we have*

$$\int_{Z_\mathbb{A}\backslash G_\mathbb{A}} \Phi(g)\chi(\det g) dg = 0.$$

Proof. Let $a = (a_v)_{v\in\Sigma_F} \in \mathbb{A}^\times$ be an idele such that $a_v \in \mathfrak{o}_v^\times$ for $v \in \Sigma_{\mathrm{fin}}$, $a_v = 1$ for $v \in \Sigma_\infty$, and $\chi(a) \neq 1$. Then we can prove the statement by the change of variable $g \to \left(\begin{smallmatrix} a & 0 \\ 0 & 1\end{smallmatrix}\right) g$. □

From Lemma 4.10, we can rewrite (4.11) as

$$\mathbf{\Phi}_{\rm res}(g,h) = \mathrm{vol}\,(Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}})^{-1} \sum_{\substack{\chi\in\Xi(\mathfrak{o})\\ \chi^2=1}} \overline{\chi(\det g)}\chi(\det h) \int_{Z_{\mathbb{A}}\backslash G_{\mathbb{A}}} \Phi(x)\chi(\det x)dx. \tag{4.36}$$

Proposition 4.11. *The function $\mathbf{\Phi}_{\rm res}(g,h)$ is continuous and bounded.*

Proof. This is obvious because (4.36) is a finite sum. □

4.3 The calculation of $\mathbb{I}_{\rm res}(\mathfrak{n}|\mathbf{s}_S,\mathfrak{s}_\infty;\beta)$

Note that the function $\mathbf{\Phi}_{\rm res}(g,g)$ is a constant function on $g\in G_{\mathbb{A}}$ from (4.36). By the calculation in ([16, 5.1]), we have the following lemma.

Lemma 4.12. *For any $\beta\in\mathcal{B}_1$,*

$$\int_{Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}}} \mathcal{E}^*_\beta(g)dg = 0.$$

Proof. See ([16, pp. 2999–3000]). □

Proposition 4.13. *For any $\beta\in\mathcal{B}_1$,*

$$\mathbb{I}_{\rm res}(\mathfrak{n}|\mathbf{s}_S,\mathfrak{s}_\infty;\beta)=0.$$

Proof. This is immediate from Lemma 4.12. □

4.4 The calculation of $\mathbb{I}_{\rm Eis}(\mathfrak{n}|\mathbf{s}_S,\mathfrak{s}_\infty;\beta)$

The calculations occurring in this section are based on ([10]). We set

$$\mathscr{H}(g,h) = \mathbf{\Phi}_{\rm Eis}(g,h) + \mathbf{\Phi}_{\rm res}(g,h), \quad g,h\in G_{\mathbb{A}}.$$

Note that the estimate

$$\mathbf{\Phi}_{\rm Eis}(g,g) \ll y(g)^{2N_1}, \quad g\in\mathfrak{S}_1 \tag{4.37}$$

holds for some $N_1>0$ by Propositions 4.9 and 4.11. Thus, the integral $\mathbb{I}_{\rm Eis}(\mathfrak{n}|\mathbf{s}_S,\mathfrak{s}_\infty;\beta)$ converges absolutely from Proposition 3.8 and (4.37).

Let $\sigma>1$, by means of the uniform absolute convergence of $E(z;g)=\sum_{\gamma\in B_F\backslash G_F} y(g)^{\frac{z+1}{2}}$ for $\mathrm{Re}(z)=\sigma$, the estimate $\int_{L_\sigma}|\beta(z)\Lambda_F(z+1)||dz|<+\infty$ from (3.20), and Proposition 4.13, we have

$$
\begin{aligned}
\mathbb{I}_{\mathrm{Eis}}(\mathfrak{n}|\mathbf{s}_S,\mathfrak{s}_\infty;\beta) &= \mathbb{I}_{\mathrm{Eis}}(\mathfrak{n}|\mathbf{s}_S,\mathfrak{s}_\infty;\beta) + \mathbb{I}_{\mathrm{res}}(\mathfrak{n}|\mathbf{s}_S,\mathfrak{s}_\infty;\beta)\\
&= \int_{Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}}} \mathscr{H}(g,g)\left(\int_{L_\sigma}\beta(z)E^*(z;g)dz\right)dg\\
&= \int_{Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}}} \mathscr{H}(g,g)\left(\int_{L_\sigma}\beta(z)\Lambda_F(z+1)E(z;g)dz\right)dg\\
&= \int_{Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}}} \mathscr{H}(g,g)\left(\sum_{\gamma\in B_F\backslash G_F}\int_{L_\sigma}\beta(z)\Lambda_F(z+1)y(\gamma g)^{\frac{z+1}{2}}dz\right)dg\\
&= \int_{Z_{\mathbb{A}}B_F\backslash G_{\mathbb{A}}} \mathscr{H}(g,g)\left(\int_{L_\sigma}\beta(z)\Lambda_F(z+1)y(g)^{\frac{z+1}{2}}dz\right)dg\\
&= \int_{Z_{\mathbb{A}}N_{\mathbb{A}}H_F\backslash G_{\mathbb{A}}}\left(\int_{N_F\backslash N_{\mathbb{A}}}\mathscr{H}(ng,ng)dn\right)\\
&\qquad\qquad\times\left(\int_{L_\sigma}\beta(z)\Lambda_F(z+1)y(g)^{\frac{z+1}{2}}dz\right)dg. \qquad (4.38)
\end{aligned}
$$

For $g\in G_{\mathbb{A}}$, the function $(x_1,x_2)\mapsto \mathscr{H}\left(\left(\begin{smallmatrix}1 & x_1\\ 0 & 1\end{smallmatrix}\right)g, \left(\begin{smallmatrix}1 & x_2\\ 0 & 1\end{smallmatrix}\right)g\right)$ on $\mathbb{A}^2$ has the Fourier expansion of the form

$$
\mathscr{H}\left(\left(\begin{smallmatrix}1 & x_1\\ 0 & 1\end{smallmatrix}\right)g, \left(\begin{smallmatrix}1 & x_2\\ 0 & 1\end{smallmatrix}\right)g\right) = \sum_{\alpha,\beta\in F} c_g(\alpha,\beta)\psi(\alpha x_1+\beta x_2)
$$

where $c_g(\alpha,\beta)$ is given by

$$
c_g(\alpha,\beta) = \int_{\mathbb{A}/F}\int_{\mathbb{A}/F}\mathscr{H}\left(\left(\begin{smallmatrix}1 & u_1\\ 0 & 1\end{smallmatrix}\right)g, \left(\begin{smallmatrix}1 & u_2\\ 0 & 1\end{smallmatrix}\right)g\right)\psi(-\alpha u_1-\beta u_2)du_1du_2.
$$

We get

$$
\begin{aligned}
\int_{N_F\backslash N_{\mathbb{A}}}\mathscr{H}(ng,ng)dn &= \sum_{\alpha\in F} c_g(\alpha,-\alpha)\\
&= \sum_{\alpha\in F}\int_{\mathbb{A}/F}\int_{\mathbb{A}/F}\mathscr{H}\left(\left(\begin{smallmatrix}1 & u_1\\ 0 & 1\end{smallmatrix}\right)g, \left(\begin{smallmatrix}1 & u_2\\ 0 & 1\end{smallmatrix}\right)g\right)\psi(\alpha(-u_1+u_2))du_1du_2\\
&= \int_{(N_F\backslash N_{\mathbb{A}})^2}\mathscr{H}(n_1g,n_2g)\,dn_1dn_2\\
&\quad+\sum_{\alpha\in F^\times}\int_{(N_F\backslash N_{\mathbb{A}})^2}\mathscr{H}\left(n_1\left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right)g, n_2\left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right)g\right)\psi(n_1^{-1}n_2)dn_1dn_2. \qquad (4.39)
\end{aligned}
$$

In the first term of (4.39), we can replace $\mathscr{H}$ by $\mathbf{\Phi}$ since $\mathscr{H}(g,h)=\mathbf{\Phi}(g,h)-\mathbf{\Phi}_{\mathrm{cus}}(g,h)$ and $\mathbf{\Phi}_{\mathrm{cus}}$ has no constant term with respect to g and h. In the second term of (4.39), we can replace $\mathscr{H}$ by $\mathbf{\Phi}_{\mathrm{Eis}}$ since $\mathbf{\Phi}_{\mathrm{res}}$ has no non-constant Fourier coefficients.

Hence, substituting (4.39) into (4.38), We obtain the decomposition

$$
\mathbb{I}_{\mathrm{Eis}}(\mathfrak{n}|\mathbf{s}_S,\mathfrak{s}_\infty;\beta) = \mathbb{I}_1(\beta)+\mathbb{I}_2(\beta) \qquad (4.40)
$$

where

$$
\mathbb{I}_j(\beta) = \int_{Z_{\mathbb{A}}N_{\mathbb{A}}H_F\backslash G_{\mathbb{A}}}\mathscr{H}_{j,N}(g,g)\left(\int_{L_\sigma}\beta(z)\Lambda_F(z+1)y(g)^{\frac{z+1}{2}}dz\right)dg \qquad (j=1,2) \qquad (4.41)
$$

with

$$\mathscr{H}_{1,N}(g,h) = \int_{(N_F\backslash N_{\mathbb{A}})^2} \mathbf{\Phi}\left(n_1 g, n_2 h\right) dn_1 dn_2, \tag{4.42}$$

$$\mathscr{H}_{2,N}(g,h) = \sum_{\alpha\in F^\times} \int_{(N_F\backslash N_{\mathbb{A}})^2} \mathbf{\Phi}_{\mathrm{Eis}}\left(n_1 \left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right) g, n_2 \left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right) h\right) \psi(n_1^{-1}n_2) dn_1 dn_2 \tag{4.43}$$

for $g, h \in G_{\mathbb{A}}$.

4.4.1 Computatoins of $\mathbb{I}_1(\beta)$

We have to show that the integral $\mathbb{I}_1(\beta)$ makes sense.

Lemma 4.14. *For any $M > 0$, $\alpha \in \mathbb{R}^\times$ and $\beta \in \mathbb{R}$, we have*

$$\left|\int_{L_\sigma} \beta(z)\Lambda_F(z+1)t^{\alpha z+\beta}dz\right| \ll_{M,\alpha,\beta} \max\left\{t, t^{-1}\right\}^{-M}, \qquad t \in \mathbb{R}_+.$$

Proof. Suppose $t \geq 1$. Due to $\beta(0) = \beta(\pm 1) = 0$ and (3.20), the function $z \mapsto \beta(z)\Lambda_F(z+1)a^{\alpha z+\beta}$ is entire and tends to 0 if $\mathrm{Re}(z)$ is bounded and $|\mathrm{Im}(z)| \to +\infty$. Hence we can shift the contour L_σ to $L_{-\frac{M}{\alpha}-\frac{\beta}{\alpha}}$ by considering the contour integral along the rectangle whose vertices are $\sigma - \sqrt{-1}T$, $\sigma + \sqrt{-1}T$, $-\frac{M}{\alpha} - \frac{\beta}{\alpha} + \sqrt{-1}T$, and $-\frac{M}{\alpha} - \frac{\beta}{\alpha} - \sqrt{-1}T$ oriented counter-clockwise and taking $T \to +\infty$. Thus,

$$\begin{aligned}\left|\int_{L_\sigma} \beta(z)\Lambda_F(z+1)t^{\alpha z+\beta}dz\right| &= \left|\int_{L_{-\frac{M}{\alpha}-\frac{\beta}{\alpha}}} \beta(z)\Lambda_F(z+1)t^{\alpha z+\beta}dz\right| \\ &\leq \int_{L_{-\frac{M}{\alpha}-\frac{\beta}{\alpha}}} |\beta(z)\Lambda_F(z+1)||dz| \times t^{-M}.\end{aligned}$$

In the case of $t < 1$, we can also prove by shifting the contour L_σ to $L_{\frac{M}{\alpha}-\frac{\beta}{\alpha}}$. □

For $c > 1$ and a finite subset $S' \subset \Sigma_{\mathrm{fin}}$, let $\mathfrak{A}^1 = \mathfrak{A}^1(c, S')$ be the set consisting of all $a = (a_v)_{v\in\Sigma_F} \in \mathbb{A}^1$ such that

$$|a_v|_v = 1 \ \text{ for all } v \in \Sigma_F - S', \qquad \max\{|a_v|_v, |a_v|_v^{-1}\} < c \ \text{ for all } v \in S'.$$

By the compactness of the space $\mathbb{A}^1/F^\times$, it holds that $\mathbb{A}^1 = \mathfrak{A}^1 F^\times$ for some c and S'. In the following, we fix such an $\mathfrak{A}^1$.

Lemma 4.15. *The estimate*

$$\left\|\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}\right\|_{\mathbb{A}} \ll \max\{|a|_{\mathbb{A}}, |a|_{\mathbb{A}}^{-1}\}, \quad a \in \mathfrak{A}^1\mathbb{R}_+$$

holds.

Proof. We write $a = a_1 x$ with $a_1 = (a_{1,v})_{v\in\Sigma_F} \in \mathfrak{A}^1$ and $x \in \mathbb{R}_+$. By an easy calculation, we have

$$\left\|\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}\right\|_v = \begin{cases} 1 & (v \in \Sigma_{\mathrm{fin}} - S') \\ \max\{|a_{1,v}|_v, |a_{1,v}|_v^{-1}\} & (v \in S') \\ \sqrt{|a_{1,v}x^{\frac{1}{n_F}}|^2 + |a_{1,v}x^{\frac{1}{n_F}}|^{-2} + 1} & (v \in \Sigma_\infty)\end{cases}$$

We can check that the factor at $v \in S'$ is bounded above by c and the factor at $v \in \Sigma_\infty$ is bounded above by $\sqrt{3}c \cdot \max\{x^{\frac{1}{n_F}}, x^{-\frac{1}{n_F}}\}$. Hence we get the following:

$$\left|\left|\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right|\right|_{\mathbb{A}} \ll \prod_{v \in \Sigma_\infty} \max\{x^{\frac{1}{n_F}}, x^{-\frac{1}{n_F}}\} = \max\{x, x^{-1}\} = \max\{|a|_{\mathbb{A}}, |a|_{\mathbb{A}}^{-1}\}.$$

□

Lemma 4.16. *The estimate*

$$\int_{Z_{\mathbb{A}} N_{\mathbb{A}} H_F \backslash G_{\mathbb{A}}} \int_{(N_F \backslash N_{\mathbb{A}})^2} \sum_{\gamma \in Z_F \backslash G_F} \left|\Phi(g^{-1} n_1^{-1} \gamma n_2 g)\right| dn_1 dn_2 \times \left| \int_{L_\sigma} \beta(z) \Lambda_F(z+1) y(g)^{\frac{z+1}{2}} dz \right| dg < +\infty \qquad (4.44)$$

holds. Hence, the integral $\mathbb{I}_1(\beta)$ *makes sense.*

Proof. By Proposition 2.1 (i) and Lemma 2.9, we have

$$\sum_{\gamma \in Z_F \backslash G_F} \left|\Phi(g^{-1} n_1^{-1} \gamma n_2 g)\right| \ll ||n_1 g||_{\mathbb{A}}^m, \quad n \in N_{\mathbb{A}}, g \in G_{\mathbb{A}}$$

for some $m > 1$. When $n_1 \in N_{\mathbb{A}}$ belongs to a compact subset, it holds that

$$||n_1 g||_{\mathbb{A}}^m \leq ||n_1||_{\mathbb{A}}^m ||g||_{\mathbb{A}}^m \ll ||g||_{\mathbb{A}}^m, \quad g \in G_{\mathbb{A}}.$$

Thus, we obtain

$$\int_{(N_F \backslash N_{\mathbb{A}})^2} \sum_{\gamma \in Z_F \backslash G_F} \left|\Phi(g^{-1} n_1^{-1} \gamma n_2 g)\right| dn_1 dn_2 \ll ||g||_{\mathbb{A}}^m, \quad g \in G_{\mathbb{A}}.$$

Let $M > m+1$, by Lemmas 4.14, 4.15, and calculating by means of the Iwasawa decomposition, the integral (4.44) is dominated by

$$\begin{aligned}
&\int_{\mathbb{A}^\times / F^\times} \max\{|a|_{\mathbb{A}}, |a|_{\mathbb{A}}^{-1}\}^{-M+m} |a|_{\mathbb{A}}^{-1} d^\times a \\
&= \mathrm{vol}(\mathbb{A}^\times / \mathrm{F}^\times) \int_0^\infty \max\{x, x^{-1}\}^{-M+m} x^{-1} d^\times x \\
&= \mathrm{vol}(\mathbb{A}^\times / \mathrm{F}^\times) \left(\int_0^1 x^{M-m-1} d^\times x + \int_1^\infty x^{-M+m-1} d^\times x \right) \\
&< +\infty.
\end{aligned}$$

We complete the proof. □

Next, by using the Bruhat decomposition of $\gamma \in G_F$:

$$G_F = B_F \bigsqcup \left(\bigsqcup_{\nu \in N_F} B_F w_0 \nu \right),$$

we can rewrite $\mathbf{\Phi}(g,h)$ as

$$\mathbf{\Phi}(g,h) = \sum_{\substack{\nu\in N_F\\ \alpha\in F^\times}} \Phi(g^{-1}\nu^{-1}\left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right)h) + \sum_{\substack{\nu_1,\nu_2\in N_F\\ \alpha\in F^\times}} \Phi(g^{-1}\nu_1^{-1}\left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right)w_0\nu_2 h).$$

Hence we get the decomposition

$$\mathscr{H}_{1,N}(g,h) = \mathscr{H}'_{1,N}(g,h) + \mathscr{H}''_{1,N}(g,h)$$

with

$$\begin{aligned}
\mathscr{H}'_{1,N}(g,h) &= \int_{N_F\backslash N_{\mathbb{A}}}\left(\int_{N_{\mathbb{A}}}\sum_{\alpha\in F^\times}\Phi(g^{-1}n_1^{-1}\left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right)n_2 h)dn_1\right)dn_2\\
&= \int_{N_{\mathbb{A}}}\sum_{\alpha\in F^\times}\Phi(g^{-1}n^{-1}\left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right)h)dn.\\
\mathscr{H}''_{1,N}(g,h) &= \int_{N_{\mathbb{A}}^2}\sum_{\alpha\in F^\times}\Phi(g^{-1}n_1^{-1}\left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right)w_0 n_2 h)dn_1 dn_2.
\end{aligned}$$

Here, the second line follows from the change of variable $n_1 \to \left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right)n_2\left(\begin{smallmatrix}\alpha^{-1} & 0\\ 0 & 1\end{smallmatrix}\right)n_1$.

We note that the functions $\mathscr{H}'_{1,N}(g,g)$ and $\mathscr{H}''_{1,N}(g,g)$ on $g\in G_{\mathbb{A}}$ are left $Z_{\mathbb{A}}N_{\mathbb{A}}H_F$-invariant. Let

$$\mathbb{I}_1(\beta) = \mathbb{I}'_1(\beta) + \mathbb{I}''_1(\beta)$$

be the corresponding decomposition of $\mathbb{I}_1(\beta)$. These integrals also make sense from Lemma 4.16.

Lemma 4.17. *For any $\beta\in\mathcal{B}_1$, we have*

$$\mathbb{I}'_1(\beta) = 0.$$

Proof. By Lemma 4.16 we have the following termwise integral:

$$\begin{aligned}
\mathbb{I}'_1(\beta) &= \sum_{\alpha\in F^\times}\int_{Z_{\mathbb{A}}N_{\mathbb{A}}H_F\backslash G_{\mathbb{A}}}\int_{N_{\mathbb{A}}}\Phi(g^{-1}n^{-1}\left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right)g)dn\left(\int_{L_\sigma}\beta(z)\Lambda_F(z+1)y(g)^{\frac{z+1}{2}}dz\right)dg\\
&= \sum_{\alpha\in F^\times}\int_{\mathbf{K}}\int_{\mathbb{A}^\times/F^\times}\int_{N_{\mathbb{A}}}\Phi(k^{-1}\left(\begin{smallmatrix}a^{-1} & 0\\ 0 & 1\end{smallmatrix}\right)n^{-1}\left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}a & 0\\ 0 & 1\end{smallmatrix}\right)k)dn\\
&\qquad\qquad\times\left(\int_{L_\sigma}\beta(z)\Lambda_F(z+1)|a|_{\mathbb{A}}^{\frac{z-1}{2}}dz\right)d^\times a dk\\
&= \sum_{\alpha\in F^\times}\int_{\mathbf{K}}\int_{N_{\mathbb{A}}}\Phi(k^{-1}n^{-1}\left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right)k)dndk\\
&\qquad\qquad\times\int_{\mathbb{A}^\times/F^\times}\left(\int_{L_\sigma}\beta(z)\Lambda_F(z+1)|a|_{\mathbb{A}}^{\frac{z+1}{2}}dz\right)d^\times a.
\end{aligned}$$

Here, we use the change of variable $n\to\left(\begin{smallmatrix}a & 0\\ 0 & 1\end{smallmatrix}\right)n\left(\begin{smallmatrix}a^{-1} & 0\\ 0 & 1\end{smallmatrix}\right)$ in the fourth line. It suffices to show that

$$\int_{\mathbb{A}^\times/F^\times}\left(\int_{L_\sigma}\beta(z)\Lambda_F(z+1)|a|_{\mathbb{A}}^{\frac{z+1}{2}}dz\right)d^\times a = 0. \tag{4.45}$$

The contour L_σ can be shifted to $L_{-\sigma}$ in the same way of the proof of Lemma 4.14. Hence, if $\sigma > 1$, we have

$$\begin{aligned}
&\int_{\mathbb{A}^\times/F^\times}\left(\int_{L_\sigma}\beta(z)\Lambda_F(z+1)|a|_{\mathbb{A}}^{\frac{z+1}{2}}dz\right)d^\times a\\
&=\mathrm{vol}(\mathbb{A}^1/\mathrm{F}^\times)\int_0^\infty\left(\int_{L_\sigma}\beta(z)\Lambda_F(z+1)x^{\frac{z+1}{2}}dz\right)d^\times x\\
&=\mathrm{vol}(\mathbb{A}^1/\mathrm{F}^\times)\left(\int_0^1+\int_1^\infty\right)\left(\int_{L_\sigma}\beta(z)\Lambda_F(z+1)x^{\frac{z+1}{2}}dz\right)d^\times x\\
&=\mathrm{vol}(\mathbb{A}^1/\mathrm{F}^\times)\int_0^1\left(\int_{L_\sigma}\beta(z)\Lambda_F(z+1)x^{\frac{z+1}{2}}dz\right)d^\times x\\
&\qquad+\mathrm{vol}(\mathbb{A}^1/\mathrm{F}^\times)\int_1^\infty\left(\int_{L_{-\sigma}}\beta(z)\Lambda_F(z+1)x^{\frac{z+1}{2}}dz\right)d^\times x.\\
&=\mathrm{vol}(\mathbb{A}^1/\mathrm{F}^\times)\int_{L_\sigma}\frac{2\beta(z)\Lambda_F(z+1)}{z+1}dz\\
&\qquad-\mathrm{vol}(\mathbb{A}^1/\mathrm{F}^\times)\int_{L_{-\sigma}}\frac{2\beta(z)\Lambda_F(z+1)}{z+1}dz.
\end{aligned}$$

Note that Fubini's theorem guarantees the above calculation.

Since $\beta(0)=\beta(\pm1)=\beta'(\pm1)=0$, the integrand is entire, which allows us to apply contour shifting again. We get (4.45). □

Next, we calculate $\mathbb{I}_1''(\beta)$.

$$\begin{aligned}
\mathbb{I}_1''(\beta)&=\int_{Z_{\mathbb{A}}N_{\mathbb{A}}H_F\backslash G_{\mathbb{A}}}\int_{N_{\mathbb{A}}^2}\sum_{\alpha\in F^\times}\Phi(g^{-1}n_1^{-1}\left(\begin{smallmatrix}\alpha&0\\0&1\end{smallmatrix}\right)w_0n_2g)dn_1dn_2\\
&\qquad\times\left(\int_{L_\sigma}\beta(z)\Lambda_F(z+1)y(g)^{\frac{z+1}{2}}dz\right)dg\\
&=\int_{\mathbf{K}}\int_{\mathbb{A}^\times/F^\times}\int_{\mathbb{A}^2}\sum_{\alpha\in F^\times}\Phi\left(k^{-1}\left(\begin{smallmatrix}a^{-1}&0\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&-u_1\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}\alpha&0\\0&1\end{smallmatrix}\right)w_0\left(\begin{smallmatrix}1&u_2\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}a&0\\0&1\end{smallmatrix}\right)k\right)du_1du_2\\
&\qquad\times\left(\int_{L_\sigma}\beta(z)\Lambda_F(z+1)|a|_{\mathbb{A}}^{\frac{z-1}{2}}dz\right)dg.
\end{aligned}\tag{4.46}$$

Substituting the identity

$$\left(\begin{smallmatrix}a^{-1}&0\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&-u_1\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}\alpha&0\\0&1\end{smallmatrix}\right)w_0\left(\begin{smallmatrix}1&u_2\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}a&0\\0&1\end{smallmatrix}\right)=\left(\begin{smallmatrix}a&0\\0&a\end{smallmatrix}\right)\begin{pmatrix}1&-a^{-1}u_1\\0&1\end{pmatrix}\left(\begin{smallmatrix}a^{-2}\alpha&0\\0&1\end{smallmatrix}\right)w_0\begin{pmatrix}1&a^{-1}u_2\\0&1\end{pmatrix}$$

into (4.46) and changing variables $u_1 \to au_1$ and $u_2 \to au_2$, we have

$$\mathbb{I}_1''(\beta) = \int_{\mathbf{K}} \int_{\mathbb{A}^\times/F^\times} \int_{\mathbb{A}^2} \sum_{\alpha \in F^\times} \Phi\left(k^{-1} \left(\begin{smallmatrix} 1 & -u_1 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} a^{-2}\alpha & 0 \\ 0 & 1 \end{smallmatrix}\right) w_0 \left(\begin{smallmatrix} 1 & u_2 \\ 0 & 1 \end{smallmatrix}\right) k\right) du_1 du_2$$
$$\times \left(\int_{L_\sigma} \beta(z)\Lambda_F(z+1)|a|_{\mathbb{A}}^{\frac{z+3}{2}} dz\right) dg$$
$$= \int_{\mathbb{A}^\times/F^\times} \sum_{\alpha \in F^\times} \xi_\beta(a^2\alpha) d^\times a$$
$$= \int_{\mathbb{A}^1/F^\times} \int_{\mathbb{R}_+} \sum_{\alpha \in F^\times} \xi_\beta(a_1^2 x^2 \alpha) d^\times x d^\times a_1 \tag{4.47}$$

where we set

$$\xi_\beta(a) = \int_{L_\sigma} \beta(z)\Lambda_F(z+1)|a|_{\mathbb{A}}^{\frac{z+3}{4}} dz \int_{\mathbf{K}} \int_{N_{\mathbb{A}}^2} \Phi\left(k^{-1} n_1^{-1} \left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) w_0 n_2 k\right) dn_1 dn_2 dk \tag{4.48}$$

for $\beta \in \mathcal{B}_1$ and $a \in \mathbb{A}^\times$.

Lemma 4.18. *The function* $\mathcal{F} : \mathbb{A}^1/F^\times \to \mathbb{C}$ *defined by*

$$\mathcal{F}(a_1) = \int_{\mathbb{R}_+} \sum_{\alpha \in F^\times} \xi_\beta(a_1 x^2 \alpha) d^\times x \in \mathbb{C} \tag{4.49}$$

is $U(\mathfrak{n})$*-invariant and continuous.*

Proof. We recall that Φ is bi-$\mathbf{K}_0(\mathfrak{n})$-invariant. Hence, the former statement is true since $\xi_\beta(a)$ is $U(\mathfrak{n})$-invariant by means of certain change of variables in the integral (4.48). It remains to prove the continuity of $\mathcal{F}$.

For $a \in \mathbb{A}^\times$, we obtain

$$\sum_{\alpha \in F^\times} \xi_\beta(a\alpha)$$
$$= \int_{L_\sigma} \beta(z)\Lambda_F(z+1)|a|_{\mathbb{A}}^{\frac{z+3}{4}} dz \int_{\mathbf{K}} \int_{N_{\mathbb{A}}^2} \sum_{\alpha \in F^\times} \Phi\left(k^{-1} n_1^{-1} \left(\begin{smallmatrix} a^{-1}\alpha & 0 \\ 0 & 1 \end{smallmatrix}\right) w_0 n_2 k\right) dn_1 dn_2 dk$$
$$= \int_{L_\sigma} \beta(z)\Lambda_F(z+1)|a|_{\mathbb{A}}^{\frac{z-1}{4}} dz$$
$$\times \int_{\mathbf{K}} \int_{(N_F\backslash N_{\mathbb{A}})^2} \sum_{\substack{\alpha \in F^\times \\ \beta_1,\beta_2 \in F}} \Phi\left(k^{-1} \left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) n_1^{-1} \left(\begin{smallmatrix} 1 & \beta_1 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1 \end{smallmatrix}\right) w_0 \left(\begin{smallmatrix} 1 & \beta_2 \\ 0 & 1 \end{smallmatrix}\right) n_2 k\right) dn_1 dn_2 dk$$
$$= \int_{L_\sigma} \beta(z)\Lambda_F(z+1)|a|_{\mathbb{A}}^{\frac{z-1}{4}} dz$$
$$\times \int_{\mathbf{K}} \int_{(N_F\backslash N_{\mathbb{A}})^2} \sum_{\gamma \in Z_F\backslash(G_F - B_F)} \Phi\left(k^{-1} \left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) n_1^{-1} \gamma n_2 k\right) dn_1 dn_2 dk. \tag{4.50}$$

By Proposition 2.1 the series (4.50) converges absolutely and locally uniformly on $a \in \mathbb{A}^\times$, and hence is continuous on $a \in \mathbb{A}^\times$.

Let $a = a_1 x^2$ with $a_1 \in \mathfrak{A}^1$ and $x \in \mathbb{R}_+$. By applying the same way as in the proof of Lemma 4.16, we get

$$\int_{\mathbf{K}} \int_{N_F \backslash N_{\mathbb{A}}} \sum_{\gamma \in Z_F \backslash (G_F - B_F)} \left| \Phi \left(k^{-1} \left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & 1 \end{smallmatrix} \right) n_1^{-1} \gamma n_2 k \right) \right| dndk \ll \left|\left| \left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & 1 \end{smallmatrix} \right) \right|\right|^m \ll \max\{x, x^{-1}\}^{2m},$$

$$a = a_1 x^2, \ a_1 \in \mathfrak{A}^1, \ x \in \mathbb{R}_+$$

for some $m > 1$. Let $M > 2m$, by Lemmas 4.14 and 4.15, it holds that

$$\left| \sum_{\alpha \in F^\times} \xi_\beta(g, a_1 x^2 \alpha) \right| \ll_M \max\left\{x, x^{-1}\right\}^{-M+2m}, \quad a \in \mathfrak{A}^1 \mathbb{R}_+.$$

Hence, there exists a constant $C > 0$ which is independent of $a_1 \in \mathfrak{A}^1$ such that

$$\left| \sum_{\alpha \in F^\times} \xi_\beta(g, a_1 x^2 \alpha) \right| \le C \max\left\{x, x^{-1}\right\}^{-M+2m}.$$

For any $\varepsilon > 0$, there exist $c > 1$ and an open neighborhood $V \subset \mathfrak{A}^1$ of a_1 such that

$$\left(\int_0^{c^{-1}} + \int_c^\infty \right) \max\left\{x, x^{-1}\right\}^{-M+2m} dx < \frac{\varepsilon}{2C}$$

and

$$\int_{c^{-1}}^{c} \left| \sum_{\alpha \in F^\times} \xi_\beta(g, a_1' x^2 \alpha) - \sum_{\alpha \in F^\times} \xi_\beta(g, a_1 x^2 \alpha) \right| dx < \frac{\varepsilon}{2} \quad \text{for any } a_1' \in V$$

by the uniform continuity on the compact subset $\{a_1 x^2 \mid a_1 \in \mathfrak{A}^1, \ c^{-1} \le x \le c\} \subset \mathfrak{A}^1 \mathbb{R}_+$. By a short computation, we have

$$|\mathcal{F}(a_1') - \mathcal{F}(a_1)| < \varepsilon \quad \text{for any } a_1' \in V.$$

Hence $\mathcal{F}$ is continuous. □

Lemma 4.19. *Let $\xi : \mathbb{A}^1/F^\times \to \mathbb{C}$ be a $U(\mathfrak{n})$-invariant continuous function. Then we have*

$$\int_{\mathbb{A}^1/F^\times} \xi(a_1^2) d^\times a_1 = \sum_{\substack{\chi \in \Xi(\mathfrak{n}) \\ \chi^2 = 1}} \xi_\chi$$

with

$$\xi_\chi = \int_{\mathbb{A}^1/F^\times} \xi(a_1) \overline{\chi(a_1)} d^\times a_1.$$

Proof. By the Stone-Weierstrass theorem, there exists a sequence $\{\xi_n\}_{n \in \mathbb{N}}$ consisting of continuous functions on $\mathbb{A}^1/F^\times$ such that

(a) $\xi_n \in \bigoplus_{\chi \in \Xi(\mathfrak{n})} \mathbb{C}\chi$ (algebraic direct sum)
(b) $\sup\left\{ |\xi(a_1) - \xi_n(a_1)| \;\middle|\; a_1 \in \mathbb{A}^1/F^\times \right\} < \frac{1}{n}$

Then we have the finite sum

$$\xi_n(a_1) = (\mathrm{vol}(\mathbb{A}^1/F^\times)^{-1} \sum_{\chi \in \Xi(\mathfrak{n})} \xi_{n,\chi}\chi(a_1), \qquad \left(\xi_{n,\chi} = \int_{\mathbb{A}^1/F^\times} \xi_n(a_1)\overline{\chi}(a_1)d^\times a_1\right).$$

By replacing a_1 by a_1^2 and integrating over $\mathbb{A}^1/F^\times$, we have

$$\begin{aligned}\int_{\mathbb{A}^1/F^\times} \xi_n(a_1^2)d^\times a_1 &= (\mathrm{vol}(\mathbb{A}^1/F^\times)^{-1} \sum_{\chi \in \Xi(\mathfrak{n})} \xi_{n,\chi} \int_{\mathbb{A}^1/F^\times} \chi(a_1^2)d^\times a_1 \\ &= \sum_{\substack{\chi \in \Xi(\mathfrak{n}) \\ \chi^2=1}} \xi_{n,\chi}.\end{aligned}$$

Since the number of $\chi \in \Xi(\mathfrak{n})$ with $\chi^2 = 1$ is finite, the statement holds by taking $n \to +\infty$. $\square$

Let $\mathcal{F}$ be as in (4.49), by (4.47) and Lemmas 4.18 and 4.19, we can rewrite $\mathbb{I}_1''(\beta)$ as

$$\begin{aligned}\mathbb{I}_1''(\beta) &= \sum_{\substack{\chi \in \Xi(\mathfrak{n}) \\ \chi^2=1}} \int_{\mathbb{A}^1/F^\times} \mathcal{F}(a_1)\overline{\chi(a_1)}d^\times a_1 \\ &= \sum_{\substack{\chi \in \Xi(\mathfrak{n}) \\ \chi^2=1}} \int_{\mathbb{A}^1/F^\times} \int_{\mathbb{R}_+} \sum_{\alpha \in F^\times} \xi_\beta(a_1x^2\alpha)d^\times x \cdot \overline{\chi(a_1)}d^\times a_1 \\ &= \frac{1}{2} \sum_{\substack{\chi \in \Xi(\mathfrak{n}) \\ \chi^2=1}} \int_{\mathbb{A}^1/F^\times} \int_{\mathbb{R}_+} \sum_{\alpha \in F^\times} \xi_\beta(a_1x\alpha)\overline{\chi(a_1)}d^\times x d^\times a_1 \qquad (x \to x^{\frac{1}{2}}) \\ &= \frac{1}{2} \sum_{\substack{\chi \in \Xi(\mathfrak{n}) \\ \chi^2=1}} \int_{\mathbb{A}^\times} \xi_\beta(a)\overline{\chi(a)}d^\times a \\ &= \frac{1}{2} \sum_{\substack{\chi \in \Xi(\mathfrak{n}) \\ \chi^2=1}} \int_{\mathbb{A}^\times} \int_{\mathbf{K}} \int_{N_{\mathbb{A}}^2} \Phi\left(k^{-1}n_1 \left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) w_0 n_2 k\right) \overline{\chi(a)} dn_1 dn_2 dk \\ &\qquad\qquad \times \left(\int_{L_\sigma} \beta(z)\Lambda_F(z+1)|a|_{\mathbb{A}}^{\frac{z+3}{4}} dz\right) d^\times a.\end{aligned}$$

Substituting $a \to a^{-1}$, we get

$$\begin{aligned}\mathbb{I}_1''(\beta) &= \frac{1}{2} \sum_{\substack{\chi \in \Xi(\mathfrak{n}) \\ \chi^2=1}} \int_{\mathbb{A}^\times} \int_{\mathbf{K}} \int_{N_{\mathbb{A}}^2} \Phi\left(k^{-1}n_1 \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) w_0 n_2 k\right) \chi(a) dn_1 dn_2 dk \\ &\qquad\qquad \times \left(\int_{L_\sigma} \beta(z)\Lambda_F(z+1)|a|_{\mathbb{A}}^{-\frac{z+3}{4}} dz\right) d^\times a \\ &= \frac{1}{2} \sum_{\substack{\chi \in \Xi(\mathfrak{n}) \\ \chi^2=1}} \int_{L_\sigma} \beta(z)\Lambda_F(z+1)\hat{I}^0_{\mathrm{res},\chi}(z)dz \qquad (4.51)\end{aligned}$$

where we set

$$\hat{I}^0_{\mathrm{res},\chi}(z) = \int_{\mathbb{A}^\times}\int_{\mathbf{K}}\int_{N_{\mathbb{A}}^2} \Phi\left(k^{-1}n_1 \left(\begin{smallmatrix} a & 0 \\ 0 & 1\end{smallmatrix}\right) w_0 n_2 k\right)\chi(a)|a|_{\mathbb{A}}^{-\frac{z+3}{4}}dn_1dn_2dkd^\times a. \tag{4.52}$$

We note that the change of the order of the integrals in (4.51) is guaranteed by the proof of Lemma 4.18. To calculate (4.52), we consider the local integrals given as

$$\begin{aligned}\hat{I}^0_{\mathrm{res},\chi,v}(z) &= \int_{F_v^\times}\int_{\mathbf{K}_v}\int_{N_v^2} \Phi_v\left(k^{-1}n_1 \left(\begin{smallmatrix} a & 0 \\ 0 & 1\end{smallmatrix}\right) w_0 n_2 k\right)\chi_v(a)|a|_v^{-\frac{z+3}{4}}dn_1dn_2dkd^\times a \\ &= \int_{F_v^\times}\int_{\mathbf{K}_v}\int_{F_v^2} \Phi_v\left(k^{-1} \left(\begin{smallmatrix} u_1 & u_1u_2-a \\ 1 & u_2\end{smallmatrix}\right) k\right)\chi_v(a)|a|_v^{-\frac{z+3}{4}}du_1du_2dkd^\times a, \qquad v\in\Sigma_F.\end{aligned}$$

We need the following two lemmas to get explicit formulas for $\hat{I}^0_{\mathrm{res},\chi,v}(z)$.

Lemma 4.20. *Let $v\in\Sigma_{\mathrm{fin}}$, $u_1,u_2\in F_v$, and $a\in F_v^\times$.*

(i) $\left(\begin{smallmatrix} u_1 & u_1u_2-a \\ 1 & u_2\end{smallmatrix}\right)\in Z_v\mathbf{K}_v$ *if and only if*

$$\left(u_1,u_2\in\mathfrak{o}_v,\ a\in\mathfrak{o}_v^\times\right)\quad\text{or}\quad\left(u_1,u_2\in\varpi_v^{-n}\mathfrak{o}_v^\times,\ a\in u_1u_2(1+\mathfrak{p}_v^n)\ \text{ for some } n\in\mathbb{N}\right).$$

(ii) $\left(\begin{smallmatrix} u_1 & u_1u_2-a \\ 1 & u_2\end{smallmatrix}\right)\in Z_v\mathbf{K}_0(\mathfrak{p}_v)$ *if and only if*

$$u_1,u_2\in\varpi_v^{-n}\mathfrak{o}_v^\times,\ a\in u_1u_2(1+\mathfrak{p}_v^n)\ \text{ for some } n\in\mathbb{N}.$$

(iii) $\left(\begin{smallmatrix} u_2 & -1 \\ -u_1u_2+a & -u_1\end{smallmatrix}\right)\in Z_v\mathbf{K}_0(\mathfrak{p}_v)$ *if and only if*

$$u_1,u_2\in\varpi_v^{-n}\mathfrak{o}_v^\times,\ a\in u_1u_2(1+\mathfrak{p}_v^{n+1})\ \text{ for some } n\in\mathbb{Z}_{\geq 0}.$$

Proof.
(i) The condition $\left(\begin{smallmatrix} u_1 & u_1u_2-a \\ 1 & u_2\end{smallmatrix}\right)\in Z_v\mathbf{K}_v$ is equivalent to

$$\begin{aligned}&\left(\begin{smallmatrix} \varpi_v^n u_1 & \varpi_v^n(u_1u_2-a) \\ \varpi_v^n & \varpi_v^n u_2\end{smallmatrix}\right)\in\mathbf{K}_v\ \text{ for some } n\in\mathbb{Z}\\ \iff\ & u_1,u_2,u_1u_2-a\in\varpi_v^{-n}\mathfrak{o}_v \text{ and } a\in\varpi^{-2n}\mathfrak{o}_v^\times \\ &\qquad\text{for some } n\in\mathbb{Z}_{\geq 0}.\end{aligned}\tag{4.53}$$

When $n=0$ it is immediate that (4.53) $\iff$ $u_1,u_2\in\mathfrak{o}_v$ and $a\in\mathfrak{o}_v^\times$.
When $n\geq 1$, by the strong triangle inequality, we have

$$q_v^{2n} = |a|_v \leq \max\{|u_1u_2-a|_v, |u_1u_2|_v\}\leq q^{2n}.$$

Since $|u_1|_v,|u_2|_v\leq q_v^n$ and $|u_1u_2-a|_v\leq q_v^n<q_v^{2n}$, it must hold that $|u_1|_v=|u_2|_v=q^n$, hence $u_1,u_2\in\varpi^{-n}\mathfrak{o}_v^\times$. Then we can write

$$u_1=\varpi_v^{-n}\varepsilon_{u_1}, u_2=\varpi_v^{-n}\varepsilon_{u_2}, a=\varpi_v^{-2n}\varepsilon_a,\quad (\varepsilon_{u_1},\varepsilon_{u_2},\varepsilon_a\in\mathfrak{o}_v^\times).$$

Since $u_1u_2-a\in\varpi_v^{-n}\mathfrak{o}_v$, we have $\varepsilon_{u_1}\varepsilon_{u_2}-\varepsilon_a\in\varpi_v^n\mathfrak{o}_v=\mathfrak{p}_v^n \iff \varepsilon_a\in\varepsilon_{u_1}\varepsilon_{u_2}(1+\mathfrak{p}_v^n)\iff$ $a\in u_1u_2(1+\mathfrak{p}_v^n)$, hence it holds that (4.53) $\iff$ $u_1,u_2\in\varpi_v^{-n}\mathfrak{o}_v^\times$ and $a\in u_1u_2(1+\mathfrak{p}_v^n)$.

(ii) By the same way as in (i) it can be easily checked that

$$\left(\begin{smallmatrix} u_1 & u_1u_2-a \\ 1 & u_2\end{smallmatrix}\right)\in Z_v\mathbf{K}_0(\mathfrak{p}_v)\iff \text{(4.53) for some } n\in\mathbb{N}.$$

Therefore, the (ii) is valid from the proof of (i).

(iii) By the same way as in (i), we also have

$$\left(\begin{smallmatrix} u_2 & -1 \\ -u_1u_2+a & -u_1 \end{smallmatrix}\right) \in Z_v\mathbf{K}_0 \iff u_1, u_2 \in \varpi_v^{-n}\mathfrak{o}_v,\ u_1u_2 - a \in \varpi_v^{-n+1}\mathfrak{o}_v,\ \text{and } a \in \varpi_v^{-2n}\mathfrak{o}_v^\times \quad \text{for some } n \in \mathbb{Z}_{\geq 0}$$
$$\iff u_1, u_2 \in \varpi_v^{-n}\mathfrak{o}_v^\times \text{ and } a \in u_1u_2(1+\mathfrak{p}_v^{n+1}) \text{ for some } n \in \mathbb{Z}_{\geq 0}.$$

□

Lemma 4.21. *For $v \in \Sigma_{\mathrm{fin}}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, we define the function $\mathscr{F}_s : F_v^\times \to \mathbb{C}$ as*

$$\mathscr{F}_s(a) = \int_{F_v^2} \max\{1, |u_1|_v, |u_2|_v, |u_1u_2 - a|_v\}^{-s-1}\, du_1du_2.$$

Then, we have the following:

(i) If $|a|_v \leq 1$,

$$\mathscr{F}_s(a) = q_v^{-\mathrm{d}_v}\left(\frac{1-q_v^{-s-1}}{1-q_v^{-s}}\right)^2.$$

(ii) If $|a|_v > 1$,

$$\mathscr{F}_s(a) = q_v^{-\mathrm{d}_v}\frac{1-q_v^{-s-1}}{(1-q_v^{-s})^2}\Bigg[(q_v^{-1}+q_v^{-s}-2q_v^{-s-1})q_v^{-sm} + (1-q_v^{-1})(1-q_v^{-s})\sum_{\substack{\ell \\ 0\leq \ell \leq m}} \max\left\{q_v^{m-\ell}, q_v^{\ell}\right\}^{-s}\Bigg]$$

with $|a|_v = q_v^m$.

Proof.
(i) If $|a|_v \leq 1$, by means of the strong triangle inequality,

$$\begin{aligned}\max\{1, |u_1|_v, |u_2|_v, |u_1u_2 - a|_v\} &= \max\{1, |u_1|_v, |u_2|_v, |u_1u_2|_v\} \\ &= \max\{1, |u_1|_v\}\max\{1, |u_2|_v\}.\end{aligned}$$

By recalling the complex-valued function $\mathcal{F}_s$ on $F_v^\times$ given by (4.13) and Lemma 4.4, we obtain

$$\begin{aligned}\mathscr{F}_s(a) &= \int_{F_v^2} \max\{1, |u_1|_v, |u_2|_v, |u_1u_2|_v\}^{-s-1}\, du_1du_2 \\ &= \int_{F_v} \max\{1, |u_1|_v\}^{-s-1}\, du_1 \int_{F_v} \max\{1, |u_2|_v\}^{-s-1}\, du_2 \\ &= \mathcal{F}_s(1)^2 \\ &= q_v^{-\mathrm{d}_v}\left(\frac{1-q_v^{-s-1}}{1-q_v^{-s}}\right)^2.\end{aligned}$$

(ii) If $|a|_v > 1$, it is easy to check that

$$\max\{1, |u_1|_v, |u_2|_v, |u_1u_2 - a|_v\} = \max\{|u_1|_v, |u_2|_v, |u_1u_2 - a|_v\}.$$

Then we have

$$\begin{aligned}
\mathscr{F}_s(a) &= \int_{F_v^2} \max\{|u_1|_v, |u_2|_v, |u_1u_2 - a|_v\}^{-s-1}\, du_1 du_2 \\
&= \left(\int_{|u_1u_2|_v > |a|_v} + \int_{|u_1u_2|_v < |a|_v} + \int_{|u_1u_2|_v = |a|_v} \right) \\
&\qquad\qquad \max\{|u_1|_v, |u_2|_v, |u_1u_2 - a|_v\}^{-s-1}\, du_1 du_2 \\
&= \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3
\end{aligned}$$

with obvious notations.

Firstly, we calculate $\mathcal{S}_1$. Putting $|a|_v = q_v^m$ $(m > 0)$, we have

$$\begin{aligned}
\mathcal{S}_1 &= \int_{|u_1u_2|_v > |a|_v} \max\{|u_1|_v, |u_2|_v, |u_1u_2 - a|_v\}^{-s-1}\, du_1 du_2 \\
&= \int_{|u_1u_2|_v > |a|_v} \max\{|u_1|_v, |u_2|_v, |u_1u_2|_v\}^{-s-1}\, du_1 du_2 \\
&= \sum_{n>m} \int_{|u_1u_2|_v = q_v^n} \max\{|u_1|_v, |u_2|_v, |u_1u_2|_v\}^{-s-1}\, du_1 du_2 \\
&= \sum_{n>m} \sum_{\ell \in \mathbb{Z}} \int_{\substack{|u_1|_v = q_v^\ell \\ |u_2|_v = q_v^{n-\ell}}} \max\left\{q_v^\ell, q_v^{n-\ell}, q_v^n\right\}^{-s-1}\, du_1 du_2 \\
&= q_v^{-\mathrm{d}_v}(1 - q_v^{-1})^2 \sum_{n>m} q_v^n \sum_{\ell \in \mathbb{Z}} \max\left\{q_v^\ell, q_v^{n-\ell}, q_v^n\right\}^{-s-1} \\
&= q_v^{-\mathrm{d}_v}(1 - q_v^{-1})^2 \sum_{n>m} q_v^n \left(\sum_{\ell<0} q_v^{-(s+1)(n-\ell)} + \sum_{0 \le \ell \le n} q_v^{-(s+1)n} + \sum_{\ell > n} q_v^{-(s+1)\ell} \right) \\
&= q_v^{-\mathrm{d}_v}(1 - q_v^{-1})^2 \sum_{n>m} q_v^n \left(\frac{q_v^{-(s+1)(n+1)}}{1 - q_v^{-s-1}} + (n+1) q_v^{-(s+1)n} + \frac{q_v^{-(s+1)(n+1)}}{1 - q_v^{-s-1}} \right) \\
&= q_v^{-\mathrm{d}_v}(1 - q_v^{-1})^2 \sum_{n>m} \left(\frac{2q_v^{-s-1}}{1 - q_v^{-s-1}} q_v^{-sn} + (n+1) q_v^{-sn} \right) \\
&= q_v^{-\mathrm{d}_v}(1 - q_v^{-1})^2 \\
&\qquad \times \left\{ \frac{2q_v^{-s-1}}{(1 - q_v^{-s-1})(1 - q_v^{-s})} q_v^{-s(m+1)} + \frac{(m+2) q_v^{-s(m+1)} - (m+1) q_v^{-s(m+2)}}{(1 - q_v^{-s})^2} \right\} \\
&= q_v^{-\mathrm{d}_v}(1 - q_v^{-1})^2 \left\{ \frac{m q_v^{-s}}{1 - q_v^{-s}} + \frac{2q_v^{-s} - q_v^{-2s} - q_v^{-3s-1}}{(1 - q_v^{-s-1})(1 - q_v^{-s})^2} \right\} q_v^{-sm}.
\end{aligned}$$

In the above computation, we use the following summation formula:

$$\sum_{n>m} (n+1) x^n = \frac{(m+2) x^{m+1} - (m+1) x^{m+2}}{(1-x)^2} \qquad (|x| < 1).$$

Secondly, we calculate $\mathcal{S}_2$ in a similar way.

$$\begin{aligned}
\mathcal{S}_2 &= \int_{|u_1u_2|_v<|a|_v} \max\{|u_1|_v, |u_2|_v, |u_1u_2-a|_v\}^{-s-1}\, du_1du_2 \\
&= \int_{|u_1u_2|_v<|a|_v} \max\{|u_1|_v, |u_2|_v, |a|_v\}^{-s-1}\, du_1du_2 \\
&= \sum_{n<m} \int_{|u_1u_2|_v=q_v^n} \max\{|u_1|_v, |u_2|_v, |a|_v\}^{-s-1}\, du_1du_2 \\
&= \sum_{n<m} \sum_{\ell\in\mathbb{Z}} \int_{\substack{|u_1|_v=q_v^\ell \\ |u_2|_v=q_v^{n-\ell}}} \max\left\{q_v^\ell, q_v^{n-\ell}, q_v^m\right\}^{-s-1}\, du_1du_2 \\
&= q_v^{-\mathrm{d}_v}(1-q_v^{-1})^2 \sum_{n<m} q_v^n \sum_{\ell\in\mathbb{Z}} \max\left\{q_v^\ell, q_v^{n-\ell}, q_v^m\right\}^{-s-1} \\
&= q_v^{-\mathrm{d}_v}(1-q_v^{-1})^2 \sum_{n<m} q_v^n \left(\sum_{\ell<n-m} q_v^{-(s+1)(n-\ell)} + \sum_{n-m\le\ell\le m} q_v^{-(s+1)m} + \sum_{\ell>m} q_v^{-(s+1)\ell} \right) \\
&= q_v^{-\mathrm{d}_v}(1-q_v^{-1})^2 \sum_{n<m} q_v^n \left(\frac{q_v^{-(s+1)(m+1)}}{1-q_v^{-s-1}} + (2m-n+1)q_v^{-(s+1)m} + \frac{q_v^{-(s+1)(m+1)}}{1-q_v^{-s-1}} \right) \\
&= q_v^{-\mathrm{d}_v}(1-q_v^{-1})^2 \left\{ \frac{2q_v^{-(s+1)(m+1)}}{1-q_v^{-s-1}} \frac{q_v^{m-1}}{1-q_v^{-1}} + \left(\frac{mq_v^{m-1}}{1-q_v^{-1}} + \frac{q_v^{m-1}(2-q_v^{-1})}{(1-q_v^{-1})^2} \right) q_v^{-(s+1)m} \right\} \\
&= q_v^{-\mathrm{d}_v} \left\{ q_v^{-1}(2-q_v^{-1}) + mq_v^{-1}(1-q_v^{-1}) + \frac{2q_v^{-s-2}(1-q_v^{-1})}{1-q_v^{-s-1}} \right\} q_v^{-sm}.
\end{aligned}$$

Finally, we calculate $\mathcal{S}_3$ separating it further.

$$\begin{aligned}
\mathcal{S}_3 &= \int_{|u_1u_2|_v=|a|_v} \max\{|u_1|_v, |u_2|_v, |u_1u_2-a|_v\}^{-s-1}\, du_1du_2 \\
&= \left(\int_{\substack{|u_1u_2|_v=q_v^m \\ |u_1|_v<1}} + \int_{\substack{|u_1u_2|_v=q_v^m \\ |u_1|_v>q_v^m}} + \int_{\substack{|u_1u_2|_v=q_v^m \\ 1\le|u_1|_v\le q_v^m}} \right) \\
&\qquad\qquad \max\{|u_1|_v, |u_2|_v, |u_1u_2-a|_v\}^{-s-1}\, du_1du_2 \\
&= \mathcal{S}_{3,1} + \mathcal{S}_{3,2} + \mathcal{S}_{3,3}
\end{aligned}$$

with obvious notations. Then $\mathcal{S}_{3,1}$ and $\mathcal{S}_{3,2}$ are calculated easily as follows.

$$\begin{aligned}
\mathcal{S}_{3,1} &= \int_{\substack{|u_1u_2|_v=q_v^m \\ |u_1|_v<1}} |u_2|_v^{-s-1} du_1du_2 \\
&= \sum_{\ell<0} q_v^{-(s+1)(m-\ell)} \int_{\substack{|u_1|_v=q_v^\ell \\ |u_2|_v=q_v^{m-\ell}}} du_1du_2 \\
&= q_v^{-\mathrm{d}_v}(1-q_v^{-1})^2 \sum_{\ell<0} q_v^{(s+1)\ell} q_v^{-sm} \\
&= q_v^{-\mathrm{d}_v}(1-q_v^{-1})^2 \frac{q_v^{-s-1}}{1-q_v^{-s-1}} q_v^{-sm}.
\end{aligned}$$

$$
\begin{aligned}
\mathcal{S}_{3,2} &= \int_{\substack{|u_1u_2|_v = q_v^m \\ |u_1|_v > q_v^m}} |u_1|_v^{-s-1} du_1 du_2 \\
&= \int_{\substack{|u_1u_2|_v = q_v^m \\ |u_2|_v < 1}} |u_1|_v^{-s-1} du_1 du_2 \\
&= \mathcal{S}_{3,1} \\
&= q_v^{-\mathrm{d}_v}(1-q_v^{-1})^2 \frac{q_v^{-s-1}}{1-q_v^{-s-1}} q_v^{-sm}.
\end{aligned}
$$

The integral $\mathcal{S}_{3,3}$ can be calculated as follows.

$$
\begin{aligned}
\mathcal{S}_{3,3} &= \int_{\substack{|u_1u_2|_v = q_v^m \\ 1 \le |u_1|_v \le q_v^m}} \max\left\{|u_1|_v, |u_2|_v, |u_1u_2 - a|_v\right\}^{-s-1} du_1 du_2 \\
&= \int_{\substack{|u_1u_2|_v = q_v^m \\ 1 \le |u_1|_v \le q_v^m}} \max\left\{|u_1|_v, |u_2|_v, |u_1u_2 - \varpi_v^{-m}|_v\right\}^{-s-1} du_1 du_2 \qquad (u_1 \to a\varpi_v^m u_1) \\
&= \sum_{\substack{\ell \\ 0 \le \ell \le m}} \int_{\substack{|u_1|_v = q_v^\ell \\ |u_2|_v = q_v^{m-\ell}}} \max\left\{q_v^\ell, q_v^{m-\ell}, |u_1u_2 - \varpi_v^{-m}|_v\right\}^{-s-1} du_1 du_2 \\
&= \sum_{\substack{\ell \\ 0 \le \ell \le m}} q_v^m \int_{|u_1|_v = |u_2|_v = 1} \max\left\{q_v^\ell, q_v^{m-\ell}, q_v^m|u_1u_2 - 1|_v\right\}^{-s-1} du_1 du_2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad (u_1 \to \varpi_v^{-\ell} u_1, u_2 \to \varpi_v^{\ell-m} u_2) \\
&= \sum_{\substack{\ell \\ 0 \le \ell \le m}} q_v^{-sm} \int_{|u_1|_v = |u_2|_v = 1} \max\left\{q_v^{\ell-m}, q_v^{-\ell}, |1-u_1|_v\right\}^{-s-1} du_1 du_2 \qquad (u_1 \to u_2^{-1} u_1) \\
&= q_v^{-\frac{\mathrm{d}_v}{2}}(1-q_v^{-1}) \sum_{\substack{\ell \\ 0 \le \ell \le m}} q_v^{-sm} \int_{u \in \mathfrak{o}_v^\times} \max\left\{q_v^{\ell-m}, q_v^{-\ell}, |1-u|_v\right\}^{-s-1} du \\
&= q_v^{-\frac{\mathrm{d}_v}{2}}(1-q_v^{-1}) \sum_{\substack{\ell \\ 0 \le \ell \le m}} q_v^{-sm} \mathcal{S}_{3,3}^{m,\ell} \qquad (4.54)
\end{aligned}
$$

where $\mathcal{S}_{3,3}^{m,\ell}$ for $0 \le \ell \le m$ is given by

$$
\begin{aligned}
\mathcal{S}_{3,3}^{m,\ell} &= \int_{u \in \mathfrak{o}_v^\times} \max\left\{q_v^{\ell-m}, q_v^{-\ell}, |1-u|_v\right\}^{-s-1} du \\
&= \left(\int_{\mathfrak{o}_v} - \int_{\mathfrak{p}_v}\right) \max\left\{q_v^{\ell-m}, q_v^{-\ell}, |1-u|_v\right\}^{-s-1} du \\
&= \int_{\mathfrak{o}_v} \max\left\{q_v^{\ell-m}, q_v^{-\ell}, |1-u|_v\right\}^{-s-1} du - \int_{\mathfrak{p}_v} \max\left\{q_v^{\ell-m}, q_v^{-\ell}, |1-u|_v\right\}^{-s-1} du
\end{aligned}
$$

Since $|1-u|_v = 1$ if $u \in \mathfrak{p}$, the second integral becomes the volume of $\mathfrak{p}_v$ and hence equals $q_v^{-\frac{\mathrm{d}_v}{2}} q_v^{-1}$. By the change of variable $u \to 1-u$ and putting $m_0 = \min\{m-\ell, \ell\}$, the first integral equals

$$
\begin{aligned}
&\int_{\mathfrak{o}_v} \max\left\{q_v^{\ell-m}, q_v^{-\ell}, |u|_v\right\}^{-s-1} du \\
=&\int_{\mathfrak{o}_v} \max\left\{q_v^{-m_0}, |u|_v\right\}^{-s-1} du \\
=&\sum_{n=0}^{\infty}\int_{\mathfrak{p}_v^n-\mathfrak{p}_v^{n+1}} \max\left\{q_v^{-m_0}, q_v^{-n}\right\}^{-s-1} du \\
=&q_v^{-\frac{\mathrm{d}_v}{2}}(1-q_v^{-1})\sum_{n=0}^{\infty} q_v^{-n}\max\left\{q_v^{-m_0}, q_v^{-n}\right\}^{-s-1} \\
=&q_v^{-\frac{\mathrm{d}_v}{2}}(1-q_v^{-1})\left\{\sum_{n=0}^{m_0} q_v^{-n}\cdot q_v^{n(s+1)} + \sum_{n=m_0+1}^{\infty} q_v^{-n}\cdot q_v^{m_0(s+1)}\right\} \\
=&q_v^{-\frac{\mathrm{d}_v}{2}}(1-q_v^{-1})\left\{\frac{1-q_v^{s}\cdot q_v^{m_0 s}}{1-q_v^{s}} + \frac{q_v^{-1}}{1-q_v^{-1}} q_v^{m_0 s}\right\} \\
=&q_v^{-\frac{\mathrm{d}_v}{2}}\left\{\frac{q_v^{-s-1}-q_v^{-s}}{1-q_v^{-s}} + \left(\frac{1-q_v^{-s-1}}{1-q_v^{-s}}\right) q_v^{m_0 s}\right\} \\
=&q_v^{-\frac{\mathrm{d}_v}{2}}\left[\frac{q_v^{-s-1}-q_v^{-s}}{1-q_v^{-s}} + \left(\frac{1-q_v^{-s-1}}{1-q_v^{-s}}\right)\max\{q_v^{\ell-m}, q_v^{-\ell}\}^{-s}\right].
\end{aligned}
$$

Hence we get

$$
\mathcal{S}_{3,3}^{m,\ell} = q_v^{-\frac{\mathrm{d}_v}{2}}\left[\frac{2q_v^{-s-1}-q_v^{-s}-q_v^{-1}}{1-q_v^{-s}} + \left(\frac{1-q_v^{-s-1}}{1-q_v^{-s}}\right)\max\left\{q_v^{\ell-m}, q_v^{-\ell}\right\}^{-s}\right]. \tag{4.55}
$$

Substituting (4.55) into (4.54), we have

$$
\begin{aligned}
\mathcal{S}_{3,3} = q_v^{-\mathrm{d}_v}(1-q_v^{-1})\Bigg[&\frac{2q_v^{-s-1}-q_v^{-s}-q_v^{-1}}{1-q_v^{-s}}(m+1)q_v^{-sm} \\
&+\left(\frac{1-q_v^{-s-1}}{1-q_v^{-s}}\right)\sum_{\substack{\ell \\ 0\le\ell\le m}} \max\left\{q_v^{m-\ell}, q_v^{\ell}\right\}^{-s}\Bigg].
\end{aligned}
$$

The above results and a computation show that

$$
\begin{aligned}
\mathscr{F}_s(a) &= \mathcal{S}_1+\mathcal{S}_2+\mathcal{S}_3 \\
&= \mathcal{S}_1+\mathcal{S}_2+\mathcal{S}_{3,1}+\mathcal{S}_{3,2}+\mathcal{S}_{3,3} \\
&= q_v^{-\mathrm{d}_v}\frac{1-q_v^{-s-1}}{(1-q_v^{-s})^2}\Bigg[(q_v^{-1}+q_v^{-s}-2q_v^{-s-1})q_v^{-sm} \\
&\qquad + (1-q_v^{-1})(1-q_v^{-s})\sum_{\substack{\ell \\ 0\le\ell\le m}} \max\left\{q_v^{m-\ell}, q_v^{\ell}\right\}^{-s}\Bigg].
\end{aligned}
$$

□

From the above two lemmas, explicit formulas for $\hat{I}^0_{\mathrm{res},\chi,v}(z)$ are given as follows.

Lemma 4.22. *For $v \in \Sigma_F$ and $z \in \mathbb{C}$, we have the following equations.*

(i) If $v \in \Sigma_{\mathrm{fin}} - (S \cup S(\mathfrak{n}))$ and $\mathrm{Re}(z) > -1$, we have

$$\hat{I}^0_{\mathrm{res},\chi,v}(z) = q_v^{-\frac{3}{2}\mathrm{d}_v} \frac{\zeta_v(\frac{z+1}{2})}{\zeta_v(\frac{z+3}{2})}.$$

(ii) If $v \in S(\mathfrak{n})$ and $\mathrm{Re}(z) > -1$, we have

$$\hat{I}^0_{\mathrm{res},\chi,v}(z) = \delta(\chi_v \text{ is unramified}) q_v^{-\frac{3}{2}\mathrm{d}_v} \frac{(1-q_v^{-1})(1+q_v^{-\frac{z+1}{2}})}{(q_v+1)} \zeta_v\left(\tfrac{z+1}{2}\right).$$

(iii) If $v \in S$ and $-1 < \mathrm{Re}(z) < 2\mathrm{Re}(s_v) - 1$, we have

$$\hat{I}^0_{\mathrm{res},\chi,v}(z) = \mathcal{S}_{\mathrm{res},v,\chi_v}(s_v, z)$$

where $\mathcal{S}_{\mathrm{res},v,\chi_v}(s_v, z)$ is defined as in (1.9).

(iv) If $v \in \Sigma_\infty$ and $-1 - 2\mathrm{Re}(s_{i,v}) < \mathrm{Re}(z) < 2\mathrm{Re}(s_{i,v}) - 1$ $(1 \le i \le N)$, we have

$$\hat{I}^0_{\mathrm{res},\chi,v}(z) = \frac{\Gamma_{\mathbb{R}}(\frac{z+1}{2})}{\Gamma_{\mathbb{R}}(\frac{z+3}{2})} \prod_{i=1}^N \left\{ \tfrac{s_{i,v}^2}{4} - \tfrac{(z+1)^2}{16} \right\}^{-1}.$$

Proof. Recall that Φ is given as in (2.15).

(i) When $v \in \Sigma_{\mathrm{fin}} - (S \cup S(\mathfrak{n}))$, the integrand is equal to

$$\mathbf{1}_{Z_v \mathbf{K}_v}\left(k^{-1} \left(\begin{smallmatrix} u_1 & u_1u_2 - a \\ 1 & u_2 \end{smallmatrix}\right) k\right) \chi_v(a)|a|_v^{-\frac{z+3}{4}} = \mathbf{1}_{Z_v \mathbf{K}_v}\left(\left(\begin{smallmatrix} u_1 & u_1u_2 - a \\ 1 & u_2 \end{smallmatrix}\right)\right) \chi_v(a)|a|_v^{-\frac{z+3}{4}}.$$

Note that χ_v is unramified and $\chi_v^2 = 1$. By Lemma 4.20 (i), we have

$$\begin{aligned}
\hat{I}^0_{\mathrm{res},\chi,v}(z) &= \int_{\mathfrak{o}_v^2} \int_{\mathfrak{o}_v^\times} \chi_v(a)|a|_v^{-\frac{z+3}{4}} du_1 du_2 d^\times a \\
&\quad + \sum_{n=1}^\infty \int_{(\varpi_v^{-n}\mathfrak{o}_v^\times)^2} \int_{a \in u_1u_2(1+\mathfrak{p}_v^n)} \chi_v(a)|a|_v^{-\frac{z+3}{4}} du_1 du_2 d^\times a \\
&= \int_{\mathfrak{o}_v^2} \int_{\mathfrak{o}_v^\times} du_1 du_2 d^\times a + \sum_{n=1}^\infty \chi_v(\varpi_v^{-2n}) q_v^{-n(\frac{z+3}{2})} \int_{(\varpi_v^{-n}\mathfrak{o}_v^\times)^2} \int_{a \in u_1u_2(1+\mathfrak{p}_v^n)} du_1 du_2 d^\times a \\
&= q_v^{-\frac{3}{2}\mathrm{d}_v} + q_v^{-\frac{3}{2}\mathrm{d}_v} \sum_{n=1}^\infty q_v^{-n(\frac{z+3}{2})} \cdot q_v^{2n}(1-q_v^{-1})^2 \cdot q_v^{-n+1}(q_v-1)^{-1} \\
&= q_v^{-\frac{3}{2}\mathrm{d}_v} + q_v^{-\frac{3}{2}\mathrm{d}_v}(1-q_v^{-1}) \sum_{n=1}^\infty q_v^{-n(\frac{z+1}{2})} \\
&= q_v^{-\frac{3}{2}\mathrm{d}_v} + q_v^{-\frac{3}{2}\mathrm{d}_v}(1-q_v^{-1}) \cdot \frac{q_v^{-\frac{z+1}{2}}}{1-q_v^{-\frac{z+1}{2}}} \\
&= q_v^{-\frac{3}{2}\mathrm{d}_v} \frac{\zeta_v(\frac{z+1}{2})}{\zeta_v(\frac{z+3}{2})}.
\end{aligned}$$

(ii) When $v \in S(\mathfrak{n})$, we use the following decomposition

$$\mathbf{K}_v = \mathbf{K}_0(\mathfrak{p}_v) \bigsqcup \left(\bigsqcup_{\xi \in \mathfrak{o}_v/\mathfrak{p}_v} \begin{pmatrix} \xi & -1 \\ 1 & 0 \end{pmatrix} \mathbf{K}_0(\mathfrak{p}_v) \right).$$

Then, for any $g \in G_v$,

$$\int_{\mathbf{K}_v} \mathbf{1}_{Z_v \mathbf{K}_0(\mathfrak{p}_v)}(k^{-1}gk)dk$$
$$= (q_v+1)^{-1} \left\{ \mathbf{1}_{Z_v \mathbf{K}_0(\mathfrak{p}_v)}(g) + \sum_{\xi \in \mathfrak{o}_v/\mathfrak{p}_v} \mathbf{1}_{Z_v \mathbf{K}_0(\mathfrak{p}_v)} \left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & \xi \end{smallmatrix}\right) g \left(\begin{smallmatrix} \xi & -1 \\ 1 & 0 \end{smallmatrix}\right) \right) \right\}.$$

Substituting $g = \left(\begin{smallmatrix} u_1 & u_1u_2-a \\ 1 & u_2 \end{smallmatrix}\right)$ and the equation $\left(\begin{smallmatrix} 0 & 1 \\ -1 & \xi \end{smallmatrix}\right)\left(\begin{smallmatrix} u_1 & u_1u_2-a \\ 1 & u_2 \end{smallmatrix}\right)\left(\begin{smallmatrix} \xi & -1 \\ 1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \xi+u_2 & -1 \\ (\xi-u_1)(\xi+u_2)+a & \xi-u_1 \end{smallmatrix}\right)$, we have

$$\begin{aligned} \hat{I}^0_{\mathrm{res},\chi,v}(z) = {} & (q_v+1)^{-1} \int_{F_v^\times} \int_{F_v^2} \mathbf{1}_{Z_v \mathbf{K}_0(\mathfrak{p}_v)} \left(\left(\begin{smallmatrix} u_1 & u_1u_2-a \\ 1 & u_2 \end{smallmatrix}\right) \right) \chi_v(a)|a|_v^{-\frac{z+3}{4}} du_1du_2d^\times a \\ & + (q_v+1)^{-1} \sum_{\xi \in \mathfrak{o}_v/\mathfrak{p}_v} \int_{F_v^\times} \int_{F_v^2} \mathbf{1}_{Z_v \mathbf{K}_0(\mathfrak{p}_v)} \left(\left(\begin{smallmatrix} \xi+u_2 & -1 \\ (\xi-u_1)(\xi+u_2)+a & \xi-u_1 \end{smallmatrix}\right) \right) \\ & \qquad \times \chi_v(a)|a|_v^{-\frac{z+3}{4}} du_1du_2d^\times a. \end{aligned}$$

By the change of variables $u_1 \to u_1+\xi$, $u_2 \to u_2-\xi$, we have

$$\begin{aligned} \hat{I}^0_{\mathrm{res},\chi,v}(z) = {} & (q_v+1)^{-1} \int_{F_v^\times} \int_{F_v^2} \mathbf{1}_{Z_v \mathbf{K}_0(\mathfrak{p}_v)} \left(\left(\begin{smallmatrix} u_1 & u_1u_2-a \\ 1 & u_2 \end{smallmatrix}\right) \right) \chi_v(a)|a|_v^{-\frac{z+3}{4}} du_1du_2d^\times a \\ & + q_v(q_v+1)^{-1} \int_{F_v^\times} \int_{F_v^2} \mathbf{1}_{Z_v \mathbf{K}_0(\mathfrak{p}_v)} \left(\left(\begin{smallmatrix} u_2 & -1 \\ -u_1u_2+a & -u_1 \end{smallmatrix}\right) \right) \chi_v(a)|a|_v^{-\frac{z+3}{4}} du_1du_2d^\times a. \end{aligned}$$

Note that χ_v is trivial on $1+\mathfrak{p}_v$ and $\chi_v^2 = 1$. By Lemma 4.20 (ii), the first term is equal to

$$\begin{aligned} & (q_v+1)^{-1} \sum_{n=1}^{\infty} \int_{(\varpi_v^{-n}\mathfrak{o}_v^\times)^2} \chi_v(u_1u_2)|u_1u_2|_v^{-\frac{z+3}{4}} \left(\int_{u_1u_2(1+\mathfrak{p}_v^n)} d^\times a \right) du_1du_2 \\ = {} & q_v^{-\frac{\mathrm{d}_v}{2}} (q_v+1)^{-1} \sum_{n=1}^{\infty} (q_v-1)^{-1} q_v^{-n+1} \left(\int_{\varpi_v^{-n}\mathfrak{o}_v^\times} \chi_v(u)|u|_v^{-\frac{z+3}{4}} du \right)^2 \\ = {} & q_v^{-\frac{\mathrm{d}_v}{2}} q_v(q_v^2-1)^{-1} \sum_{n=1}^{\infty} q_v^{-n(\frac{z+1}{2})} \left(\chi_v(\varpi_v^{-n}) \int_{\mathfrak{o}_v^\times} \chi_v(u)du \right)^2 \qquad (u \to \varpi_v^{-n}u) \\ = {} & q_v^{-\frac{\mathrm{d}_v}{2}} q_v(q_v^2-1)^{-1} \left(\int_{\mathfrak{o}_v^\times} \chi_v(u)du \right)^2 \frac{q_v^{-\frac{z+1}{2}}}{1-q_v^{-\frac{z+1}{2}}} \\ = {} & \delta(\chi_v \text{ is unramified}) \cdot q_v^{-\frac{3}{2}\mathrm{d}_v} \frac{q_v-1}{q_v(q_v+1)} \cdot \frac{q_v^{-\frac{z+1}{2}}}{1-q_v^{-\frac{z+1}{2}}}. \end{aligned}$$

By Lemma 4.20 (iii), the second term is equal to

$$\begin{aligned}
&q_v(q_v+1)^{-1}\sum_{n=0}^{\infty}\int_{(\varpi_v^{-n}\mathfrak{o}_v^\times)^2}\chi_v(u_1u_2)|u_1u_2|_v^{-\frac{z+3}{4}}\left(\int_{u_1u_2(1+\mathfrak{p}_v^{n+1})}d^\times a\right)du_1du_2\\
=&q_v^{-\frac{\mathrm{d}_v}{2}}q_v(q_v+1)^{-1}\sum_{n=0}^{\infty}(q_v-1)^{-1}q_v^{-n}\left(\int_{\varpi_v^{-n}\mathfrak{o}_v^\times}\chi_v(u)|u|_v^{-\frac{z+3}{4}}du\right)^2\\
=&q_v^{-\frac{\mathrm{d}_v}{2}}q_v(q_v^2-1)^{-1}\sum_{n=0}^{\infty}q_v^{-n(\frac{z+1}{2})}\left(\chi_v(\varpi_v^{-n})\int_{\mathfrak{o}_v^\times}\chi_v(u)du\right)^2 \qquad (u\to\varpi_v^{-n}u)\\
=&q_v^{-\frac{\mathrm{d}_v}{2}}q_v(q_v^2-1)^{-1}\left(\int_{\mathfrak{o}_v^\times}\chi_v(u)du\right)^2\frac{1}{1-q_v^{-\frac{z+1}{2}}}\\
=&\delta(\chi_v\text{ is unramified})\cdot q_v^{-\frac{3}{2}\mathrm{d}_v}\frac{q_v-1}{q_v(q_v+1)}\cdot\frac{1}{1-q_v^{-\frac{z+1}{2}}}.
\end{aligned}$$

Hence we have

$$\hat{I}^0_{\mathrm{res},\chi,v}(z)=\delta(\chi_v\text{ is unramified})\cdot q_v^{-\frac{3}{2}\mathrm{d}_v}\frac{(1-q_v^{-1})(1+q_v^{-\frac{z+1}{2}})}{(q_v+1)}\zeta_v(\tfrac{z+1}{2})$$

(iii) When $v\in S$, we have

$$\begin{aligned}
\Phi_{v,s_v}\left(k^{-1}\left(\begin{smallmatrix}u_1 & u_1u_2-a\\ 1 & u_2\end{smallmatrix}\right)k\right)&=\Phi_{v,s_v}\left(\left(\begin{smallmatrix}u_1 & u_1u_2-a\\ 1 & u_2\end{smallmatrix}\right)\right)\\
&=\left(q_v^{-\frac{s_v+1}{2}}-q_v^{\frac{s_v+1}{2}}\right)^{-1}\left\{|a|_v^{-1}\max\{1,|u_1|_v,|u_2|_v,|u_1u_2-a|_v\}^2\right\}^{-\frac{s_v+1}{2}}\\
&=\left(q_v^{-\frac{s_v+1}{2}}-q_v^{\frac{s_v+1}{2}}\right)^{-1}|a|_v^{\frac{s_v+1}{2}}\max\{1,|u_1|_v,|u_2|_v,|u_1u_2-a|_v\}^{-(s_v+1)}.
\end{aligned}$$

From (2.13). Let $\mathscr{F}_s$ as in Lemma 4.21, then we obtain

$$\left(q_v^{-\frac{s_v+1}{2}}-q_v^{\frac{s_v+1}{2}}\right)\hat{I}^0_{\mathrm{res},\chi,v}(z)=\int_{F_v^\times}\chi_v(a)|a|_v^{\frac{s_v}{2}-\frac{z+1}{4}}\mathscr{F}_{s_v}(a)d^\times a. \tag{4.56}$$

By the formula

$$\sum_{\substack{\ell\\0\le\ell\le m}}\max\{q_v^{m-\ell},q_v^{\ell}\}^{-s}=\begin{cases}\frac{2q_v^{-s(\frac{m+1}{2})}-2q_v^{-s(m+1)}}{1-q_v^{-s}} & (m\text{ is odd})\\ q_v^{-\frac{sm}{2}}+\frac{2q_v^{-s(\frac{m}{2}+1)}-2q_v^{-s(m+1)}}{1-q_v^{-s}} & (m\text{ is even})\end{cases},\quad m>0$$

and Lemma 4.21, we can check that (4.56) is equal to the sum of four integrals as follows.

$$\left(q_v^{-\frac{s_v+1}{2}}-q_v^{\frac{s_v+1}{2}}\right)\hat{I}^0_{\mathrm{res},\chi,v}(z)=I_1+I_2+I_3+I_4$$

where

$$\begin{aligned}
I_1&=q_v^{-\mathrm{d}_v}\left(\tfrac{1-q_v^{-s_v-1}}{1-q_v^{-s_v}}\right)^2\int_{|a|_v\le1}\chi_v(a)|a|_v^{\frac{s_v}{2}-\frac{z+1}{4}}d^\times a\\
&=q_v^{-\frac{3}{2}\mathrm{d}_v}\left(\tfrac{1-q_v^{-s_v-1}}{1-q_v^{-s_v}}\right)^2L_v(\tfrac{s_v}{2}-\tfrac{z+1}{4},\chi_v)\quad(\because\chi_v\text{ is unramified}),
\end{aligned}$$

$$I_2 = q_v^{-\mathrm{d}_v} \tfrac{(1-q_v^{-s_v-1})(q_v^{-1}+q_v^{-s_v}-2q_v^{-s_v-1})}{(1-q_v^{-s_v})^2} \int_{\substack{|a|_v=q_v^m \\ m>0}} \chi_v(a) q_v^{m(\frac{s_v}{2}-\frac{z+1}{4})} \cdot q_v^{-s_v m} d^\times a$$

$$= q_v^{-\frac{3}{2}\mathrm{d}_v} \tfrac{(1-q_v^{-s_v-1})(q_v^{-1}+q_v^{-s_v}-2q_v^{-s_v-1})}{(1-q_v^{-s_v})^2} \chi_v(\varpi_v) q_v^{-\frac{s_v}{2}-\frac{z+1}{4}} L_v(\tfrac{s_v}{2}+\tfrac{z+1}{4}, \chi_v) \quad (\because \chi_v^2=1),$$

$$I_3 = q_v^{-\mathrm{d}_v} \tfrac{(1-q_v^{-s_v-1})(1-q_v^{-1})}{(1-q_v^{-s_v})^2} \int_{\substack{|a|=q_v^m \\ m>0:\mathrm{odd}}} \chi_v(a) q_v^{m(\frac{s_v}{2}-\frac{z+1}{4})} \left(2q_v^{-s_v(\frac{m+1}{2})} - 2q_v^{-s_v(m+1)}\right) d^\times a$$

$$= q_v^{-\frac{3}{2}\mathrm{d}_v} \tfrac{(1-q_v^{-s_v-1})(1-q_v^{-1})}{(1-q_v^{-s_v})^2} \sum_{\ell=1}^{\infty} \chi_v(\varpi_v^{2\ell-1}) q_v^{(2\ell-1)(\frac{s_v}{2}-\frac{z+1}{4})} \left(2q_v^{-s_v\ell} - 2q_v^{-2s_v\ell}\right)$$

$$= 2q_v^{-\frac{3}{2}\mathrm{d}_v} \tfrac{(1-q_v^{-s_v-1})(1-q_v^{-1})}{(1-q_v^{-s_v})^2} \chi_v(\varpi_v) q_v^{-\frac{s_v}{2}+\frac{z+1}{4}} \sum_{\ell=1}^{\infty} \left(q_v^{-\ell(\frac{z+1}{2})} - q_v^{-\ell(s_v+\frac{z+1}{2})}\right)$$

$$= 2q_v^{-\frac{3}{2}\mathrm{d}_v} \tfrac{(1-q_v^{-s_v-1})(1-q_v^{-1})}{(1-q_v^{-s_v})^2} \chi_v(\varpi_v) \left\{ q_v^{-\frac{s_v}{2}-\frac{z+1}{4}} \zeta_v(\tfrac{z+1}{2}) - q_v^{-\frac{3}{2}s_v-\frac{z+1}{4}} \zeta_v(s_v+\tfrac{z+1}{2}) \right\},$$

and

$$I_4 = q_v^{-\mathrm{d}_v} \tfrac{(1-q_v^{-s_v-1})(1-q_v^{-1})}{1-q_v^{-s_v}}$$
$$\times \int_{\substack{|a|=q_v^m \\ m>0:\mathrm{even}}} \chi_v(a) q_v^{m(\frac{s_v}{2}-\frac{z+1}{4})} \left(q_v^{-\frac{s_v m}{2}} + \tfrac{2q_v^{-s_v(\frac{m}{2}+1)} - 2q_v^{-s_v(m+1)}}{1-q_v^{-s}} \right) d^\times a$$

$$= q_v^{-\mathrm{d}_v} \tfrac{(1-q_v^{-s_v-1})(1-q_v^{-1})}{1-q_v^{-s_v}}$$
$$\times \sum_{\ell=1}^{\infty} \chi_v(\varpi_v^{2\ell}) q_v^{2\ell(\frac{s_v}{2}-\frac{z+1}{4})} \left\{ q_v^{-s_v\ell} + \tfrac{2(q_v^{-s_v(\ell+1)} - q_v^{-s_v(2\ell+1)})}{1-q_v^{-s_v}} \right\}$$

$$= q_v^{-\mathrm{d}_v} \tfrac{(1-q_v^{-s_v-1})(1-q_v^{-1})}{1-q_v^{-s_v}} \sum_{\ell=1}^{\infty} \left\{ \left(\tfrac{1+q_v^{-s_v}}{1-q_v^{-s_v}}\right) q_v^{-\ell(\frac{z+1}{2})} - \left(\tfrac{2q_v^{-s_v}}{1-q_v^{-s_v}}\right) q_v^{-\ell(s_v+\frac{z+1}{2})} \right\}$$

$$= q_v^{-\mathrm{d}_v} \tfrac{(1-q_v^{-s_v-1})(1-q_v^{-1})}{(1-q_v^{-s_v})^2} \left\{ (1+q_v^{-s_v}) q_v^{-\frac{z+1}{2}} \zeta_v(\tfrac{z+1}{2}) - 2q_v^{-2s_v-\frac{z+1}{2}} \zeta_v(s_v+\tfrac{z+1}{2}) \right\}.$$

Hence (iii) holds by a simple calculation.

(iv) When $v \in \Sigma_\infty$, recall that $\Phi_{\mathfrak{s}_v}$ is bi-$Z_v SO(2)$-invariant and satisfies

$$\Phi_{\mathfrak{s}_v}(\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right) g \left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)) = \Phi_{\mathfrak{s}_v}(g), \quad g \in G_v (=G_\mathbb{R})$$

by (2.10). Due to the equation

$$\begin{pmatrix} u_1 & u_1u_2 - a \\ 1 & u_2 \end{pmatrix} = \begin{pmatrix} \frac{a}{\sqrt{1+u_2^2}} & u_1\sqrt{1+u_2^2} - \frac{au_2}{\sqrt{1+u_2^2}} \\ 0 & \sqrt{1+u_2^2} \end{pmatrix} \begin{pmatrix} \frac{u_2}{\sqrt{1+u_2^2}} & \frac{-1}{\sqrt{1+u_2^2}} \\ \frac{1}{\sqrt{1+u_2^2}} & \frac{u_2}{\sqrt{1+u_2^2}} \end{pmatrix},$$

and (2.10), we get

$$\Phi_{\mathfrak{s}_v}\left(k^{-1}\begin{pmatrix} u_1 & u_1u_2 - a \\ 1 & u_2 \end{pmatrix} k\right) = \Phi_{\mathfrak{s}_v}\left(\begin{pmatrix} \frac{a}{\sqrt{1+u_2^2}} & u_1\sqrt{1+u_2^2} - \frac{au_2}{\sqrt{1+u_2^2}} \\ 0 & \sqrt{1+u_2^2} \end{pmatrix}\right),$$

for any $u_1, u_2 \in \mathbb{R}, a \in \mathbb{R}^\times, k \in O(2)$. Hence by using the change of variables $u_1 \to u_1 + \frac{au_2}{1+u_2^2}$ and $a \to (1+u_2^2)a$, we have

$$\hat{I}^0_{\mathrm{res},\chi,v}(z) = \int_{\mathbb{R}^\times}\int_{O(2)}\int_{\mathbb{R}^2}\Phi_{\mathfrak{s}_v}(k^{-1}\left(\begin{smallmatrix}u_1 & u_1u_2-a\\ 1 & u_2\end{smallmatrix}\right)k)\chi_v(a)|a|^{-\frac{z+3}{4}}du_1du_2dkd^\times a$$
$$= \int_{\mathbb{R}}(1+u_2^2)^{-\frac{z+3}{4}}du_2\int_0^\infty\int_{\mathbb{R}}\Phi_{\mathfrak{s}_v}((\begin{smallmatrix}a & u_1\\ 0 & 1\end{smallmatrix}))a^{-\frac{z+3}{4}}du_1d^\times a$$

since $\chi_v^2 = 1$. From (4.24), we have

$$\int_{\mathbb{R}}(1+u_2^2)^{-\frac{z+3}{4}}du_2 = \frac{\pi^{\frac{1}{2}}\Gamma(\frac{z+1}{4})}{\Gamma(\frac{z+3}{4})} = \frac{\Gamma_{\mathbb{R}}(\frac{z+1}{2})}{\Gamma_{\mathbb{R}}(\frac{z+3}{2})}.$$

Recall the integral $I(s,w)$ defined as in (4.23). Then, by the same way as the proof of Lemma 4.5 for $v \in \Sigma_\infty$, we have

$$\int_0^\infty\int_{\mathbb{R}}\Phi_{\mathfrak{s}_v}((\begin{smallmatrix}a & u_1\\ 0 & 1\end{smallmatrix}))a^{-\frac{z+3}{4}}du_1d^\times a = \sum_{i=1}^N \mu_v^{(i)}(\mathfrak{s}_\infty)I(s_{i,v}, -\tfrac{z+1}{2})$$
$$= \sum_{i=1}^N \mu_v^{(i)}(\mathfrak{s}_\infty)\left\{\tfrac{1-(\frac{z+1}{2})^2}{4} - \tfrac{1-s_{i,v}^2}{4}\right\}^{-1}$$
$$= \prod_{i=1}^N\left\{\tfrac{1-(\frac{z+1}{2})^2}{4} - \tfrac{1-s_{i,v}^2}{4}\right\}^{-1}$$
$$= \prod_{i=1}^N\left\{\tfrac{s_{i,v}^2}{4} - \tfrac{(z+1)^2}{16}\right\}^{-1}.$$

Here, the last line follows from Lemma 2.4.

Thus, we obtain

$$\hat{I}^0_{\mathrm{res},\chi,v}(z) = \frac{\Gamma_{\mathbb{R}}(\frac{z+1}{2})}{\Gamma_{\mathbb{R}}(\frac{z+3}{2})}\prod_{i=1}^N\left\{\tfrac{s_{i,v}^2}{4} - \tfrac{(z+1)^2}{16}\right\}^{-1}.$$

□

Suppose $-1 < \mathrm{Re}(z) < 1$, then from Lemma 4.22, we see that the integral (4.52) converges absolutely and has the formula

$$\hat{I}^0_{\mathrm{res},\chi}(z) = \delta(\chi \in \Xi(\mathfrak{o}))D_F^{-\frac{3}{2}}\frac{\zeta_F(\frac{z+1}{2})}{\zeta_F(\frac{z+3}{2})}\prod_{v\in S(\mathfrak{n})}\frac{(1-q_v^{-1})(1+q_v^{-\frac{z+1}{2}})}{q_v+1}\zeta_v(\tfrac{z+3}{2})$$
$$\times\prod_{v\in S}\frac{\zeta_v(\frac{z+3}{2})}{\zeta_v(\frac{z+1}{2})}\mathcal{S}_{\mathrm{res},v,\chi_v}(s_v,z)\prod_{v\in\Sigma_\infty}\left[\prod_{i=1}^N\left\{\tfrac{s_{i,v}^2}{4} - \tfrac{(z+1)^2}{16}\right\}^{-1}\right]. \tag{4.57}$$

Hence the function $\hat{I}^0_{\mathrm{res},\chi}(z)$ is holomorphic on the region $-1 < \mathrm{Re}(z) < 1$ and is vertically of moderate growth. From (4.51), we obtain

$$\mathbb{I}_1''(\beta) = \frac{1}{2}\int_{L_\sigma}\beta(z)\Lambda_F(z+1)\left(\sum_{\substack{\chi\in\Xi(\mathfrak{o})\\ \chi^2=1}}\hat{I}^0_{\mathrm{res},\chi}(z)\right)dz \tag{4.58}$$

for $-1 < \sigma < 1$.

4.4.2 Computations of $\mathbb{I}_2(\beta)$

We recall the Whittaker functions of Eisenstein series discussed in ([10]). Let $\chi \in \Xi$ and $f \in H(\chi)$. For $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 1$, the Bruhat decomposition gives

$$Ef(g,z) = S_z f(g) + \sum_{\nu \in N_F} S_z f(w_0 \nu g)$$

and hence

$$\int_{N_F \backslash N_{\mathbb{A}}} Ef(ng,z)\psi(-n)dn = \int_{N_{\mathbb{A}}} S_z f(w_0 ng)dn.$$

We denote this meromorphic function by $Wf(g,z)$. For $\Psi \in S(\mathbb{A}^2)$ and $\chi \in \Xi$, we apply the same construction to $E(g,\Psi,\chi,z)$. We define

$$W(g,\Psi,\chi,z) = \int_{N_F \backslash N_{\mathbb{A}}} E(ng,\Psi,\chi,z)\psi(-n)dn, \quad g \in G_{\mathbb{A}}, z \in \mathbb{C}.$$

It can also be written as

$$\begin{aligned} W(g,\Psi,\chi,z) &= \int_{N_{\mathbb{A}}} f(w_0 ng,\Psi,\chi,z)\psi(-n)dn && (4.59)\\ &= \chi(\det g)|\det g|_{\mathbb{A}}^{\frac{z+1}{2}} \int_{N_{\mathbb{A}}}\int_{\mathbb{A}^\times} \Psi[(0,t)w_0 ng]\chi^2(t)|t|_{\mathbb{A}}^{z+1}\psi(-n)d^\times t dn \\ &= \chi(\det g)|\det g|_{\mathbb{A}}^{\frac{z+1}{2}} \int_{\mathbb{A}^\times}\int_{\mathbb{A}} \Psi[(t,tx)g]\psi(-x)dx \cdot \chi^2(t)|t|_{\mathbb{A}}^{z+1}d^\times t \\ &= \chi(\det g)|\det g|_{\mathbb{A}}^{\frac{z+1}{2}} \int_{\mathbb{A}^\times} \widehat{\ddot{g\Psi}}[(t,t^{-1})] \cdot \chi^2(t)|t|_{\mathbb{A}}^{z}d^\times t && (4.60)\end{aligned}$$

where $g\Psi$ denotes the function $g\Psi[(x,y)] = \Psi[(x,y)g]$.

Lemma 4.23. ([10, Lemma 1]) *Let $\Psi \in S(\mathbb{A}^2)$ and $\chi \in \Xi$. Then, for any $\varepsilon > 0$, there exists $\Theta \in \mathscr{S}(\mathbb{A})$ such that the estimate*

$$|W(\left(\begin{smallmatrix} a & 0 \\ 0 & 1\end{smallmatrix}\right) k,\Psi,\chi,z)| \le \Theta(a)|a|_{\mathbb{A}}^{\frac{1}{2}-\frac{1}{2}|\mathrm{Re}(z)|-\varepsilon}, \quad a \in \mathbb{A}^\times, k \in \mathbf{K} \tag{4.61}$$

holds.

Proof. See ([10, pp. 8–9]). □

Now we recall the integral $\mathbb{I}_2(\beta)$ defined by (4.41). Then we can rewrite $\mathbb{I}_2(\beta)$ as

$$\begin{aligned}\mathbb{I}_2(\beta) = \int_{Z_{\mathbb{A}} N_{\mathbb{A}} \backslash G_{\mathbb{A}}} \int_{(N_F \backslash N_{\mathbb{A}})^2} &\mathbf{\Phi}_{\mathrm{Eis}}(n_1 g, n_2 g)\psi(n_1^{-1}n_2)dn_1 dn_2 \\ &\times \left(\int_{L_\sigma} \beta(z)\Lambda_F(z+1)y(g)^{\frac{z+1}{2}}dz\right)dg.\end{aligned}$$

The integral with respect to $n_1, n_2 \in N_F\backslash N_{\mathbb{A}}$ can be rewritten formally as

$$\begin{aligned}
&\int_{(N_F\backslash N_{\mathbb{A}})^2} \mathbf{\Phi}_{\mathrm{Eis}}(n_1 g, n_2 g)\psi(n_1^{-1}n_2)dn_1dn_2 \\
=&\frac{\mathrm{vol}(\mathbb{A}^1/F^\times)^{-1}}{8\pi}\int_{(N_F\backslash N_{\mathbb{A}})^2}\sum_{\chi\in\Xi(\mathfrak{o})}\sum_{\iota\in A(\chi)_{\mathfrak{n}}}\int_{\mathbb{R}} D'_\chi(\mathfrak{n}|\mathbf{s}_S, \mathfrak{s}_\infty, \sqrt{-1}u) \\
&\qquad\times Ef_\iota^\chi(n_1g, \sqrt{-1}u)\overline{Ef_\iota^\chi(n_2g, \sqrt{-1}u)}du\ \psi(n_1^{-1}n_2)dn_1dn_2 \\
=&\frac{\mathrm{vol}(\mathbb{A}^1/F^\times)^{-1}}{8\pi}\sum_{\chi\in\Xi(\mathfrak{o})}\sum_{\iota\in A(\chi)_{\mathfrak{n}}}\int_{\mathbb{R}} D'_\chi(\mathfrak{n}|\mathbf{s}_S, \mathfrak{s}_\infty, \sqrt{-1}u)Wf_\iota^\chi(g, \sqrt{-1}u)\overline{Wf_\iota^\chi(g, \sqrt{-1}u)}du
\end{aligned}$$

where $D'_\chi(\mathfrak{n}|\mathbf{s}_S, \mathfrak{s}_\infty, z)$ is defined by (4.26) The above calculation is guaranteed by the locally uniform convergence of $\mathbf{\Phi}_{\mathrm{Eis}}$, which in turn follows from Lemma 4.9 and the compactness of $N_F\backslash N_{\mathbb{A}}$.

Hence we also get formally

$$\begin{aligned}
\mathbb{I}_2(\beta) =&\frac{\mathrm{vol}(\mathbb{A}^1/F^\times)^{-1}}{8\pi} \\
&\times\int_{Z_{\mathbb{A}}N_{\mathbb{A}}\backslash G_{\mathbb{A}}}\sum_{\chi\in\Xi(\mathfrak{o})}\sum_{\iota\in A(\chi)_{\mathfrak{n}}}\int_{\mathbb{R}} D'_\chi(\mathfrak{n}|\mathbf{s}_S, \mathfrak{s}_\infty, \sqrt{-1}u)Wf_\iota^\chi(g, \sqrt{-1}u)\overline{Wf_\iota^\chi(g, \sqrt{-1}u)}du \\
&\qquad\times\left(\int_{L_\sigma}\beta(z)\Lambda_F(z+1)y(g)^{\frac{z+1}{2}}dz\right)dg \\
=&\frac{\mathrm{vol}(\mathbb{A}^1/F^\times)^{-1}}{8\pi}\sum_{\chi\in\Xi(\mathfrak{o})}\sum_{\iota\in A(\chi)_{\mathfrak{n}}}\int_{\mathbb{R}} D'_\chi(\mathfrak{n}|\mathbf{s}_S, \mathfrak{s}_\infty, \sqrt{-1}u)\mathbb{J}_\iota^\chi(\beta;\sqrt{-1}u)du \qquad (4.62)
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{J}_\iota^\chi(\beta;u) = &\int_{Z_{\mathbb{A}}N_{\mathbb{A}}\backslash G_{\mathbb{A}}} Wf_\iota^\chi(g,u)\overline{Wf_\iota^\chi(g,-\overline{u})} \\
&\times\left(\int_{L_\sigma}\beta(z)\Lambda_F(z+1)y(g)^{\frac{z+1}{2}}dz\right)dg \qquad (4.63)
\end{aligned}$$

for $u \in \mathbb{C}$ and $\beta \in \mathcal{B}_1$. The justification of the above calculation is given later by computing the integral $\mathbb{J}_\iota^\chi(\beta;u)$.

Let $\Psi_\iota \in S(\mathbb{A}^2)$ ($\iota \in A(\chi)_{\mathfrak{n}}$) be as in Lemma 3.2. Then, by Lemma 3.2, we get

$$Wf_\iota^\chi(g,u) = C_1(\chi,\mathfrak{n},u)W(g,\Psi_\iota,\chi,u) \qquad (4.64)$$

where

$$C_1(\chi,\mathfrak{n},u) = D_F^{\frac{1}{2}}L_F(u+1,\chi)^{-1}\left(\prod_{v\in S(\mathfrak{n})}L_v(u+1,\chi^2)\right).$$

Due to Lemmas 4.14 and 4.23, the integral (4.63) converges absolutely for all non-vanishing and non-singular points $u \in \mathbb{C}$ of $C_1(\chi,\mathfrak{n},u)$.

Let $\Psi_0 \in S(\mathbb{A}^2)$ defined as in (3.16), substituting the equation

$$y(g)^{\frac{z+1}{2}} = \zeta_F(z+1)^{-1}|\det g|_{\mathbb{A}}^{\frac{z+1}{2}}\int_{\mathbb{A}^\times}\Psi_0[(0,t)g]|t|_{\mathbb{A}}^{z+1}d^\times t, \quad \mathrm{Re}(z) > 0$$

into (4.63), we can write

$$\mathbb{J}_\iota^\chi(\beta;u) = \int_{N_{\mathbb{A}}\backslash G_{\mathbb{A}}} \Psi_0[(0,1)g] W f_\iota^\chi(g,u) \overline{W_\iota^\chi(g,-\overline{u})} \times \left(\int_{L_\sigma} \beta(z) D_F^{\frac{z+1}{2}} |\det g|_{\mathbb{A}}^{\frac{z+1}{2}} dz \right) dg. \qquad (4.65)$$

The following proposition is a special case of ([10, Proposition 1]), but in our case, the function $\Psi_\iota \in S(\mathbb{A}^2)$ ($\iota \in A(\chi)_{\mathfrak{n}}$) is defined explicitly, hence a more precise formula of (4.65) can be described.

Proposition 4.24. *Let $\chi \in \Xi(\mathfrak{o})$, $\iota \in A(\chi)_{\mathfrak{n}}$, and $z \in \mathbb{C}$. If $\mathrm{Re}(z) > 1$, as a function of $u \in \mathbb{C}$,*

$$J_\iota^\chi(u,z) = \int_{N_{\mathbb{A}}\backslash G_{\mathbb{A}}} \Psi_0[(0,1)g] W f_\iota^\chi(g,u) \overline{W f_\iota^\chi(g,-\overline{u})} |\det g|_{\mathbb{A}}^{\frac{z+1}{2}} dg \qquad (4.66)$$

is meromorphic on the region $|\mathrm{Re}(u)| < \frac{\mathrm{Re}(z)-1}{2}$ and is holomorphic on the imaginary axis. Moreover, an explicit form is given by

$$\begin{aligned} J_\iota^\chi(u,z) =& D_F^{\frac{z-5}{2}} [\mathbf{K}_{\mathrm{fin}} : \mathbf{K}_0(\mathfrak{n})]^{-\frac{1}{2}} C_1(\chi,\mathfrak{n},u) C_1(\chi^{-1},\mathfrak{n},-u) \\ & \times \zeta_F(\tfrac{z+1}{2})^2 L_F(u+\tfrac{z+1}{2},\chi^2) L_F(-u+\tfrac{z+1}{2},\chi^{-2}) \\ & \times \prod_{\substack{v\in S(\mathfrak{n}) \\ \iota_v = 0}} \left\{ -q_v^{-1} + q_v^{-2} + q_v^{\frac{z-1}{2}} + q_v^{-\frac{z+1}{2}} - q_v^{-u-1}\chi_v(\varpi_v)^2 - q_v^{u-1}\chi_v(\varpi_v)^{-2} \right\} \\ & \times \prod_{\substack{v\in S(\mathfrak{n}) \\ \iota_v = 1}} \left\{ q_v^{\frac{1}{2}} - q_v^{-\frac{1}{2}} + q_v^{\frac{z}{2}-1} + q_v^{-\frac{z}{2}-1} - q_v^{-u-\frac{1}{2}}\chi_v(\varpi_v)^2 - q_v^{u-\frac{1}{2}}\chi_v(\varpi_v)^{-2} \right\}. \end{aligned}$$

In preparation for proving Proposition 4.24, we observe how to proceed with the calculation of the integral (4.66). Applying the equation (4.59) to $W(g,\Psi,\chi,-\overline{u})$, we obtain

$$\begin{aligned} J_\iota^\chi(u,z) &= C_1(\chi^{-1},\mathfrak{n},-u) \int_{G_{\mathbb{A}}} \Psi_0[(0,1)g] W f_\iota^\chi(g,u) \overline{f(w_0 g, \Psi_\iota, \chi, -\overline{u})} |\det g|_{\mathbb{A}}^{\frac{z+1}{2}} dg. \\ &= C_1(\chi^{-1},\mathfrak{n},-u) \int_{G_{\mathbb{A}}} \Psi_0[(0,1)g] W f_\iota^\chi(g,u) \chi^{-1}(\det g) |\det g|_{\mathbb{A}}^{\frac{-u+z}{2}+1} \\ &\qquad \times \int_{\mathbb{A}^\times} \Psi_\iota[(t,0)g] \chi^{-2}(t) |t|_{\mathbb{A}}^{1-u} d^\times t dg. \end{aligned}$$

Here, we use the property $Wf(ng,s) = \psi(n)Wf(g,s)$, $(n \in N_{\mathbb{A}})$ in the first line. By the change of variables $t \to t^{-1}$ and $g \to \left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right) g$, we can rewrite as

$$\begin{aligned} J_\iota^\chi(u,z) &= C_1(\chi^{-1},\mathfrak{n},-u) \int_{G_{\mathbb{A}}} \Psi_0[(0,1)g] \Psi_\iota[(1,0)g] \chi^{-1}(\det g) |\det g|_{\mathbb{A}}^{\frac{-u+z}{2}+1} \\ &\qquad \times \int_{\mathbb{A}^\times} W f_\iota^\chi(\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right) g, u) \chi(t) |t|_{\mathbb{A}}^{\frac{u+z}{2}} d^\times t dg \\ &= C_1(\chi^{-1},\mathfrak{n},-u) \int_{G_{\mathbb{A}}} \Upsilon_\iota(g) \chi^{-1}(\det g) |\det g|_{\mathbb{A}}^{\frac{-u+z}{2}+1} V f_\iota^\chi(g;u,z) dg \qquad (4.67) \end{aligned}$$

where we set

$$\Upsilon_\iota(g) = \Psi_0[(0,1)g] \Psi_\iota[(1,0)g], \qquad g \in G_{\mathbb{A}}$$

and

$$Vf(g;u,z) = \int_{\mathbb{A}^\times} Wf(\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) g, u)\chi(a)|a|_{\mathbb{A}}^{\frac{u+z}{2}} d^\times a, \qquad g \in G_{\mathbb{A}}, f \in H(\chi), u, z \in \mathbb{C}$$

as in ([10, Lemma 2]). We need to prove the following two lemmas.

Lemma 4.25. *The integral $Vf_\iota^\chi(g;u,z)$ converges absolutely for $\mathrm{Re}(z) > 1$, $\mathrm{Re}(u) > \frac{1}{2} - \frac{1}{2}\mathrm{Re}(z)$ and has a meromorphic continuation in u, z. In particular, the value at $g = w_0$ is given by*

$$\begin{aligned} Vf_\iota^\chi(w_0;u,z) =& D_F^{\frac{z}{2}-1} \left[\mathbf{K}_{\mathrm{fin}} : \mathbf{K}_0(\mathfrak{n})\right]^{\frac{1}{2}} C_1(\chi, \mathfrak{n}, u) L_F(u + \tfrac{z+1}{2}, \chi^2)\zeta_F(\tfrac{z+1}{2}) \\ &\times \prod_{\substack{v \in S(\mathfrak{n}) \\ \iota_v = 0}} q_v^{\frac{z-1}{2}} L_v(u + \tfrac{z+1}{2}, \chi_v^2)^{-1} \prod_{\substack{v \in S(\mathfrak{n}) \\ \iota_v = 1}} q_v^{-\frac{1}{2}}(1 - q_v^{\frac{z-1}{2}}). \end{aligned} \tag{4.68}$$

Proof. The proof of the first claim is referred to [10, Lemma 2]. To understand details, we describe again. From (4.60), we get

$$\begin{aligned} Vf_\iota^\chi(g;u,z) &= \chi(\det g)|\det g|_{\mathbb{A}}^{\frac{u+1}{2}} C_1(\chi, \mathfrak{n}, u) \\ &\quad \times \int_{\mathbb{A}^\times}\int_{\mathbb{A}^\times} \widehat{g\ddot{\Psi}_\iota}[(at, t^{-1})]\chi^2(a)|a|_{\mathbb{A}}^{u+\frac{z+1}{2}}\chi^2(t)|t|_{\mathbb{A}}^u d^\times t d^\times a \\ &= \chi(\det g)|\det g|_{\mathbb{A}}^{\frac{u+1}{2}} C_1(\chi, \mathfrak{n}, u) \\ &\quad \times \int_{\mathbb{A}^\times}\int_{\mathbb{A}^\times} \widehat{g\ddot{\Psi}_\iota}[(a, t)]\chi^2(a)|a|_{\mathbb{A}}^{u+\frac{z+1}{2}}|t|_{\mathbb{A}}^{\frac{z+1}{2}} d^\times t d^\times a \qquad (4.69) \\ &\qquad\qquad (\text{substituting } a \to at^{-1}, t \to t^{-1}) \end{aligned}$$

Since $\widehat{g\ddot{\Psi}} \in S(\mathbb{A}^2)$, the integral (4.69) converges if $\mathrm{Re}(z) > 1$ and $\mathrm{Re}(u) > \frac{1}{2} - \frac{1}{2}\mathrm{Re}(z)$ and is a product of an entire function and $L_F(u + \frac{z+1}{2}, \chi^2)\zeta_F(\frac{z+1}{2})$.

The second claim can be proved by a direct calculation. From (4.69), we have

$$\begin{aligned} Vf_\iota^\chi(w_0;u,z) &= C_1(\chi, \mathfrak{n}, u) \int_{\mathbb{A}^\times}\int_{\mathbb{A}^\times} \widehat{w_0\ddot{\Psi}_\iota}[(a, t)]\chi^2(a)|a|_{\mathbb{A}}^{u+\frac{z+1}{2}}|t|_{\mathbb{A}}^{\frac{z+1}{2}} d^\times t d^\times a \\ &= C_1(\chi, \mathfrak{n}, u) \int_{\mathbb{A}^\times}\int_{\mathbb{A}^\times} \widehat{\dot{\Psi}_\iota}[(t, -a)]\chi^2(a)|a|_{\mathbb{A}}^{u+\frac{z+1}{2}}|t|_{\mathbb{A}}^{\frac{z+1}{2}} d^\times t d^\times a \end{aligned}$$

because $\widehat{w_0\ddot{\Psi}}[(x,y)] = \widehat{\dot{\Psi}}[(y, -x)]$ ($\Psi \in \mathscr{S}(\mathbb{A}^2)$). We recall that an explicit form of $\widehat{\dot{\Psi}_\iota} = \prod_{v \in \Sigma_F} \widehat{\dot{\Psi}_{\iota,v}}$ is given in Lemma 3.3 as

$$\widehat{\dot{\Psi}_{\iota,v}}[(x_v, y_v)] = \begin{cases} q_v^{-\frac{\mathrm{d}_v}{2}} \mathbf{1}_{\mathfrak{p}_v^{-\mathrm{d}_v} \times \mathfrak{o}_v}[(x_v, y_v)] & (v \in \Sigma_{\mathrm{fin}} - S(\mathfrak{n})) \\ q_v^{-\frac{\mathrm{d}_v}{2}} (q_v+1)^{\frac{1}{2}} q_v^{-1} \mathbf{1}_{\mathfrak{p}_v^{-\mathrm{d}_v - 1} \times \mathfrak{o}_v^\times}[(x_v, y_v)] & (v \in S(\mathfrak{n}), \iota_v = 0) \\ q_v^{-\frac{\mathrm{d}_v}{2}} (1 + q_v^{-1})^{\frac{1}{2}} \left(\mathbf{1}_{\mathfrak{p}_v^{-\mathrm{d}_v} \times \mathfrak{o}_v} - q_v^{-1}\mathbf{1}_{\mathfrak{p}_v^{-\mathrm{d}_v - 1} \times \mathfrak{o}_v}\right)[(x_v, y_v)] & (v \in S(\mathfrak{n}), \iota_v = 1) \\ e^{-\pi(x_v^2 + y_v^2)} & (v \in \Sigma_\infty) \end{cases}.$$

Hence, $Vf_\iota^\chi(w_0;u,z)$ becomes the product of local integrals. We get (4.72) by a short calculation. □

Lemma 4.26. *For $g \in G_{\mathbb{A}}$, we have*

$$Vf_\iota^\chi(g;u,z) = \sum_{f \in B(\chi)} \left(\int_{\mathbf{K}} y(kg)^{\frac{u}{2}} f_\iota^\chi(kg)\overline{f(kw_0)} dk\right) Vf(w_0;u,z).$$

Proof. By noting (3.1), we have

$$\begin{aligned}
S_u f_\iota^\chi(g) &= S_u f_\iota^\chi(w_0 w_0^{-1} g)\\
&= (R(w_0^{-1}g) S_u f_\iota^\chi)(w_0)\\
&= S_u(\pi_u(w_0^{-1}g) f_\iota^\chi)(w_0)\\
&= \sum_{f \in B(\chi)} \langle \pi_u(w_0^{-1}g) f_\iota^\chi | f \rangle S_u f(w_0)
\end{aligned}$$

and

$$\langle \pi_u(w_0^{-1}g) f_\iota^\chi | f\rangle = \int_{\mathbf{K}} y(kw_0^{-1}g)^{\frac{u}{2}} y(k)^{-\frac{u}{2}} f_\iota^\chi(kw_0^{-1}g)\overline{f(k)} dk = \int_{\mathbf{K}} y(kg)^{\frac{u}{2}} f_\iota^\chi(kg)\overline{f(kw_0)} dk.$$

Hence the claim follows. □

Lemma 4.27. *It holds that*

$$J_\iota^\chi(u,z) = C_1(\chi^{-1}, \mathfrak{n}, -u) \sum_{\kappa \in A(\chi)_{\mathfrak{n}}} V f_\kappa^\chi(w_0; u, z) D_{\iota,\kappa}(u,z)$$

where

$$\begin{aligned}
D_{\iota,\kappa}(u,z) &= D_F^{-\frac{3}{2}} [\mathbf{K}_{\mathrm{fin}} : \mathbf{K}_0(\mathfrak{n})]^{-1} \zeta_F(\tfrac{z+1}{2}) L_F(-u + \tfrac{z+1}{2}, \chi^{-2})\\
&\times \prod_{\substack{v \in S(\mathfrak{n})\\ \iota_v = 0, \kappa_v = 0}} L_v(-u + \tfrac{z+1}{2}, \chi_v^{-2})^{-1} \prod_{\substack{v \in S(\mathfrak{n})\\ \iota_v = 0, \kappa_v = 1}} (1 - q_v^{-1}) q_v^{-\frac{z}{2}}\\
&\times \prod_{\substack{v \in S(\mathfrak{n})\\ \iota_v = 1, \kappa_v = 0}} q_v^{\frac{1}{2}} L_v(-u + \tfrac{z+1}{2}, \chi_v^{-2})^{-1} \prod_{\substack{v \in S(\mathfrak{n})\\ \iota_v = 1, \kappa_v = 1}} (q_v - 1).
\end{aligned}$$

Proof. Applying Lemma 4.26 to (4.67), we obtain

$$\begin{aligned}
J_\iota^\chi(u,z) &= C_1(\chi^{-1}, \mathfrak{n}, -u) \sum_{f \in B(\chi)} V f(w_0; u, z)\\
&\times \int_{G_{\mathbb{A}}} \Upsilon_\iota(g) \chi^{-1}(\det g) |\det g|_{\mathbb{A}}^{\frac{-u+z}{2}+1} \left(\int_{\mathbf{K}} y(kg)^{\frac{u}{2}} f_\iota^\chi(kg) \overline{f(kw_0)} dk \right) dg. \qquad (4.70)
\end{aligned}$$

By the change of variables $g \to k^{-1}g$ and $k \to kw_0^{-1}$, the integral (4.70) becomes

$$\int_{\mathbf{K}} \left(\int_{G_{\mathbb{A}}} \Upsilon_\iota(w_0 k^{-1} g) \chi^{-1}(\det g) |\det g|_{\mathbb{A}}^{\frac{-u+z}{2}+1} y(g)^{\frac{u}{2}} f_\iota^\chi(g) dg \right) \chi(\det k) \overline{f(k)} dk. \qquad (4.71)$$

It is easy to check that the integral inside the bracket on (4.71) belongs to the space $H(\chi)_{\mathfrak{n}}|_{\mathbf{K}}$ defined in the proof of Lemma 4.3 as the function of $k \in \mathbf{K}$. Therefore, the integral (4.71) is equal to the inner product of some function in $H(\chi)_{\mathfrak{n}}$ and f, hence is equal to 0 unless $f = f_\kappa^\chi \in B(\chi)_{\mathfrak{n}}$. To prove the claim, it only remains to verify that the integral (4.71) coincides with $D_{\iota,\kappa}(u,z)$. We denote the integral (4.71) by $D'_{\iota,\kappa}(u,z)$. By means of the Iwasawa decomposition of $G_{\mathbb{A}}$, we have

$$\begin{aligned}
D'_{\iota,\kappa}(u,z) = \int_{\mathbb{A}^\times} \int_{\mathbb{A}^\times} \int_{\mathbb{A}} \int_{\mathbf{K}} \int_{\mathbf{K}} & \Upsilon_\iota \left(w_0 k^{-1} \left(\begin{smallmatrix} a & n \\ 0 & b \end{smallmatrix} \right) k' \right) \chi(\det kk'^{-1})\\
& \times f_\iota^\chi(k') f_\kappa^\chi(k) \chi^{-2}(b) |a|_{\mathbb{A}}^{\frac{z+1}{2}} |b|^{-u+\frac{z+1}{2}} dk dk' dn d^\times a d^\times b.
\end{aligned}$$

If $\mathrm{Re}(z) > 1$ and $\mathrm{Re}(u) < -\frac{1}{2} + \frac{1}{2}\mathrm{Re}(z)$, this integral converges absolutely and decomposes to the product of local integrals

$$D'_{\iota,\kappa}(u,z) = \prod_{v \in \Sigma_F} D'_{\iota,\kappa}(u,z)_v$$

We calculate $D'_{\iota,\kappa}(u,z)_v$ by noting that the function $g \mapsto \Upsilon_\iota(w_0 g) = \Psi_0[(1,0)g]\Psi_\iota[(0,-1)g]$ is bi-$\mathbf{K}_0(\mathfrak{n}) \times O(2)^{\Sigma_\infty}$-invariant.

(i) When $v \in \Sigma_{\mathrm{fin}} - S(\mathfrak{n})$, we have

$$\begin{aligned}
D'_{\iota,\kappa}(u,z)_v &= \int_{F_v^\times}\int_{F_v^\times}\int_{F_v} \mathbf{1}_{\mathfrak{o}_v^2}[(a,n)]\mathbf{1}_{\mathfrak{o}_v^2}[(0,-b)]\chi_v^{-2}(b)|a|_v^{\frac{z+1}{2}}|b|_v^{-u+\frac{z+1}{2}}dnd^\times ad^\times b\\
&= q_v^{-\frac{\mathrm{d}_v}{2}}\int_{\mathfrak{o}_v-\{0\}}\int_{\mathfrak{o}_v-\{0\}}\chi_v^{-2}(b)|a|_v^{\frac{z+1}{2}}|b|_v^{-u+\frac{z+1}{2}}dnd^\times ad^\times b\\
&= q_v^{-\frac{3}{2}\mathrm{d}_v}\zeta_v(\tfrac{z+1}{2})L_v(-u+\tfrac{z+1}{2},\chi_v^{-2}).
\end{aligned}$$

(ii) When $v \in S(\mathfrak{n})$, we calculate in four cases by using the decomposition (4.16).

(ii-i) If $\iota_v = 0$, $\kappa_v = 0$,

$$\begin{aligned}
D'_{\iota,\kappa}(u,z)_v &= (q_v+1)^{-1}\int_{F_v^\times}\int_{F_v^\times}\int_{F_v}\mathbf{1}_{\mathfrak{o}_v^2}[(a,n)]\mathbf{1}_{\mathfrak{p}_v\times\mathfrak{o}_v^\times}[(0,-b)]\\
&\qquad\times\chi_v^{-2}(b)|a|_v^{\frac{z+1}{2}}|b|_v^{-u+\frac{z+1}{2}}dnd^\times ad^\times b\\
&= q_v^{-\frac{\mathrm{d}_v}{2}}(q_v+1)^{-1}\int_{\mathfrak{o}_v^\times}\int_{\mathfrak{o}_v-\{0\}}\chi_v^{-2}(b)|a|_v^{\frac{z+1}{2}}|b|_v^{-u+\frac{z+1}{2}}d^\times ad^\times b\\
&= q_v^{-\frac{3}{2}\mathrm{d}_v}(q_v+1)^{-1}\zeta_v(\tfrac{z+1}{2}).
\end{aligned}$$

(ii-ii) If $\iota_v = 0$, $\kappa_v = 1$, substituting the identity $\left(\begin{smallmatrix}\xi & -1\\ 1 & 0\end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix}a & n\\ 0 & b\end{smallmatrix}\right) = \left(\begin{smallmatrix}0 & b\\ -a & \xi b-n\end{smallmatrix}\right)$, we have

$$\begin{aligned}
D'_{\iota,\kappa}(u,z)_v &= q_v^{-\frac{1}{2}}(q_v+1)^{-1}\sum_{\xi\in\mathfrak{o}_v/\mathfrak{p}_v}\int_{F_v^\times}\int_{F_v^\times}\int_{F_v}\mathbf{1}_{\mathfrak{o}_v^2}[(0,b)]\mathbf{1}_{\mathfrak{p}_v\times\mathfrak{o}_v^\times}[(a,-\xi b+n)]\\
&\qquad\times\chi_v^{-2}(b)|a|_v^{\frac{z+1}{2}}|b|_v^{-u+\frac{z+1}{2}}dnd^\times ad^\times b\\
&= q_v^{-\frac{\mathrm{d}_v}{2}}q_v^{\frac{1}{2}}(q_v+1)^{-1}(1-q_v^{-1})\\
&\qquad\times\int_{\mathfrak{o}_v-\{0\}}\int_{\mathfrak{p}_v-\{0\}}\chi_v^{-2}(b)|a|_v^{\frac{z+1}{2}}|b|_v^{-u+\frac{z+1}{2}}d^\times ad^\times b\\
&= q_v^{-\frac{3}{2}\mathrm{d}_v}(1-q_v^{-1})(q_v+1)^{-1}q_v^{-\frac{z}{2}}\zeta_v(\tfrac{z+1}{2})L_v(-u+\tfrac{z+1}{2},\chi_v^{-2}).
\end{aligned}$$

(ii-iii) If $\iota_v = 1$, $\kappa_v = 0$, substituting the identity $\left(\begin{smallmatrix}a & n\\ 0 & b\end{smallmatrix}\right)\left(\begin{smallmatrix}\xi & -1\\ 1 & 0\end{smallmatrix}\right) = \left(\begin{smallmatrix}\xi a+n & -a\\ b & 0\end{smallmatrix}\right)$, we have

$$\begin{aligned}
D'_{\iota,\kappa}(u,z)_v &= q_v^{-\frac{1}{2}}(q_v+1)^{-1}\sum_{\xi\in\mathfrak{o}_v/\mathfrak{p}_v}\int_{F_v^\times}\int_{F_v^\times}\int_{F_v}\mathbf{1}_{\mathfrak{o}_v^2}[(\xi a+n,-a)]\mathbf{1}_{\mathfrak{o}_v^\times\times\mathfrak{o}_v}[(-b,0)]\\
&\qquad\times\chi_v^{-2}(b)|a|_v^{\frac{z+1}{2}}|b|_v^{-u+\frac{z+1}{2}}dnd^\times ad^\times b\\
&= q_v^{-\frac{\mathrm{d}_v}{2}}q_v^{\frac{1}{2}}(q_v+1)^{-1}\int_{\mathfrak{o}_v^\times}\int_{\mathfrak{o}_v-\{0\}}\chi_v^{-2}(b)|a|_v^{\frac{z+1}{2}}|b|_v^{-u+\frac{z+1}{2}}d^\times ad^\times b\\
&= q_v^{-\frac{3}{2}\mathrm{d}_v}(q_v+1)^{-1}q_v^{\frac{1}{2}}\zeta_v(\tfrac{z+1}{2}).
\end{aligned}$$

(ii-iv) If $\iota_v = 1$, $\kappa_v = 1$, substituting the identity $\left(\begin{smallmatrix}\xi & -1\\ 1 & 0\end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix}a & n\\ 0 & b\end{smallmatrix}\right)\left(\begin{smallmatrix}\xi' & -1\\ 1 & 0\end{smallmatrix}\right) = \left(\begin{smallmatrix}b & 0\\ -\xi' a+\xi b-n & a\end{smallmatrix}\right)$, we have

$$\begin{aligned}
D'_{\iota,\kappa}(u,z)_v &= q_v^{-1}(q_v+1)^{-1}\sum_{\xi,\xi'\in\mathfrak{o}_v/\mathfrak{p}_v}\int_{F_v^\times}\int_{F_v^\times}\int_{F_v}\mathbf{1}_{\mathfrak{o}_v^2}[(b,0)]\mathbf{1}_{\mathfrak{o}_v^\times\times\mathfrak{o}_v}[(\xi' a-\xi b+n,-a)]\\
&\qquad\times\chi_v^{-2}(b)|a|_v^{\frac{z+1}{2}}|b|_v^{-u+\frac{z+1}{2}}dnd^\times ad^\times b\\
&= q_v^{-\frac{\mathrm{d}_v}{2}}(1-q_v^{-1})q_v(q_v+1)^{-1}\int_{\mathfrak{o}_v-\{0\}}\int_{\mathfrak{o}_v-\{0\}}\chi_v^{-2}(b)|a|_v^{\frac{z+1}{2}}|b|_v^{-u+\frac{z+1}{2}}d^\times ad^\times b\\
&= q_v^{-\frac{3}{2}\mathrm{d}_v}(q_v-1)(q_v+1)^{-1}\zeta_v(\tfrac{z+1}{2})L_v(-u+\tfrac{z+1}{2},\chi_v^{-2}).
\end{aligned}$$

(iii) When $v\in\Sigma_\infty$, since the integrand is independent of $k,k'\in O(2)$, we have

$$\begin{aligned}
D'_{\iota,\kappa}(u,z)_v &= \int_{\mathbb{R}^\times}\int_{\mathbb{R}^\times}\int_{\mathbb{R}} e^{-\pi(a^2+b^2+n^2)}|a|^{\frac{z+1}{2}}|b|_v^{-u+\frac{z+1}{2}+\sqrt{-1}a(\chi_v^{-2})}dnd^\times ad^\times b\\
&= \int_{\mathbb{R}^\times}\int_{\mathbb{R}^\times} e^{-\pi(a^2+b^2)}|a|^{\frac{z+1}{2}}|b|_v^{-u+\frac{z+1}{2}+\sqrt{-1}a(\chi_v^{-2})}d^\times ad^\times b\\
&= \Gamma_{\mathbb{R}}(\tfrac{z+1}{2})\Gamma_{\mathbb{R}}(-u+\tfrac{z+1}{2}+\sqrt{-1}a(\chi_v^{-2})).
\end{aligned}$$

This completes the proof. $\square$

Now we prove Proposition 4.24.

Proof of Proposition 4.24.
By Lemmas 4.25 and 4.27,

$$\begin{aligned}
J_\iota^\chi(u,z) =& D_F^{\frac{z-5}{2}}[\mathbf{K}_{\mathrm{fin}}:\mathbf{K}_0(\mathfrak{n})]^{-\frac{1}{2}}C_1(\chi,\mathfrak{n},u)C_1(\chi^{-1},\mathfrak{n},-u)\\
&\times\zeta_F(\tfrac{z+1}{2})^2L_F(u+\tfrac{z+1}{2},\chi^2)L_F(-u+\tfrac{z+1}{2},\chi^{-2})\\
&\times\sum_{\kappa\in A(\chi)_\mathfrak{n}}\Bigg\{\prod_{\substack{v\in S(\mathfrak{n})\\ \iota_v=0,\kappa_v=0}} q_v^{\frac{z-1}{2}}L_v(u+\tfrac{z+1}{2},\chi_v^2)^{-1}L_v(-u+\tfrac{z+1}{2},\chi_v^{-2})^{-1}\\
&\qquad\times\prod_{\substack{v\in S(\mathfrak{n})\\ \iota_v=0,\kappa_v=1}}(1-q_v^{\frac{z-1}{2}})(1-q_v^{-1})q_v^{-\frac{z+1}{2}}\\
&\qquad\times\prod_{\substack{v\in S(\mathfrak{n})\\ \iota_v=1,\kappa_v=0}} q_v^{\frac{z}{2}}L_v(u+\tfrac{z+1}{2},\chi_v^2)^{-1}L_v(-u+\tfrac{z+1}{2},\chi_v^{-2})^{-1}\\
&\qquad\times\prod_{\substack{v\in S(\mathfrak{n})\\ \iota_v=1,\kappa_v=1}} q_v^{-\frac{1}{2}}(1-q_v^{\frac{z-1}{2}})(q_v-1)\Bigg\}. \qquad (4.72)
\end{aligned}$$

Since the index $\kappa=(\kappa_v)_{v\in S(\mathfrak{n})}$ runs over all family $\{0,1\}^{S(\mathfrak{n})}$, the sum in (4.72) equals

$$\begin{aligned}&\prod_{\substack{v\in S(\mathfrak{n})\\ \iota_v=0}}\left\{q_v^{\frac{z-1}{2}}L_v(u+\tfrac{z+1}{2},\chi_v^2)^{-1}L_v(-u+\tfrac{z+1}{2},\chi_v^{-2})^{-1}+(1-q_v^{\frac{z-1}{2}})(1-q_v^{-1})q_v^{-\frac{z+1}{2}}\right\}\\
&\quad\times\prod_{\substack{v\in S(\mathfrak{n})\\ \iota_v=1}}\left\{q_v^{\frac{z}{2}}L_v(u+\tfrac{z+1}{2},\chi_v^2)^{-1}L_v(-u+\tfrac{z+1}{2},\chi_v^{-2})^{-1}+q_v^{-\frac{1}{2}}(1-q_v^{\frac{z-1}{2}})(q_v-1)\right\}\\
=&\prod_{\substack{v\in S(\mathfrak{n})\\ \iota_v=0}}\left\{-q_v^{-1}+q_v^{-2}+q_v^{\frac{z-1}{2}}+q_v^{-\frac{z+1}{2}}-q_v^{-u-1}\chi_v(\varpi_v)^2-q_v^{u-1}\chi_v(\varpi_v)^{-2}\right\}\\
&\quad\times\prod_{\substack{v\in S(\mathfrak{n})\\ \iota_v=1}}\left\{q_v^{\frac{1}{2}}-q_v^{-\frac{1}{2}}+q_v^{\frac{z}{2}-1}+q_v^{-\frac{z}{2}-1}-q_v^{-u-\frac{1}{2}}\chi_v(\varpi_v)^2-q_v^{u-\frac{1}{2}}\chi_v(\varpi_v)^{-2}\right\}\end{aligned}$$

by a simple calculation. Hence the claim follows. □

Finally, we get an explicit formula of $\mathbb{I}_2(\beta)$. For $u\in\sqrt{-1}\mathbb{R}$ and $z\in\mathbb{C}$ with $\mathrm{Re}(z)>1$, it can be easily checked that the dominating function of the integrand of (4.66) can be obtained by replacing z by $\mathrm{Re}(z)$. Therefore $J_\iota^\chi(u,z)$ is bounded on the vertical line $\mathrm{Re}(z)=\sigma$ with $\sigma>1$. Due to $\int_{L_\sigma}|\beta(z)D_F^{\frac{z+1}{2}}||dz|<+\infty$, we can exchange the order of integral in (4.65) as

$$\mathbb{J}_\iota^\chi(\beta;u)=\int_{L_\sigma}\beta(z)D_F^{\frac{z+1}{2}}J_\iota^\chi(u,z)dz\quad(\sigma>1).$$

Due to the formula of $J_\iota^\chi(u,z)$ in Proposition 4.24, a short calculation shows that

$$D_F^{\frac{z+1}{2}}D'_\chi(\mathfrak{n}|\mathbf{s}_S,\mathfrak{s}_\infty,u)\sum_{\iota\in A(\chi)_\mathfrak{n}}J_\iota^\chi(u,z)=\zeta_F(\tfrac{z+1}{2})I^0_{\mathrm{Eis},\chi}(u,z)\tag{4.73}$$

where $I^0_{\mathrm{Eis},\chi}(u,z)$ is defined as in (1.7).

Let $\mathcal{V}$ be a subset of $\mathbb{C}$ of the form $\{z\in\mathbb{C}\mid\sigma_1\le\mathrm{Re}(z)\le\sigma_2\text{ or }\mathrm{Re}(z)\ge\sigma_3\}$ for some positive real numbers σ_1, σ_2, and σ_3 such that $0<\sigma_1<\sigma_2<1<\sigma_3$. Then, by (1.1), (1.2) and Stirling's formula, we have

$$\begin{aligned}&\frac{L_F(\frac{z+1}{2}+\sqrt{-1}u,\chi^2)L_F(\frac{z+1}{2}-\sqrt{-1}u,\chi^{-2})}{L_F(1+\sqrt{-1}u,\chi^2)L_F(1-\sqrt{-1}u,\chi^{-2})}\\
\ll_{\mathcal{V}}&\prod_{v\in\Sigma_\infty}\left(1+\left|\tfrac{u+\mathrm{Im}(z)}{2}+a(\chi_v)\right|\right)^{2N_0+\varepsilon}\\
\ll_{\mathcal{V}}&\prod_{v\in\Sigma_\infty}(1+|\mathrm{Im}(z)|)^{2N_0+\varepsilon}\prod_{v\in\Sigma_\infty}\left(1+\left|\tfrac{u}{2}+a(\chi_v)\right|\right)^{2N_0+\varepsilon},\quad u\in\mathbb{R},z\in\mathcal{V}\end{aligned}$$

for some $\varepsilon>0$, and hence we obtain

$$\begin{aligned}&\left|D_F^{\frac{z+1}{2}}D'_\chi(\mathfrak{n}|\mathbf{s}_S,\mathfrak{s}_\infty,\sqrt{-1}u)\right|\sum_{\iota\in A(\chi)_\mathfrak{n}}|J_\iota^\chi(u,z)|\\
\ll_{\mathcal{V}}&(1+|\mathrm{Im}(z)|)^{2N_0+\varepsilon}\prod_{v\in\Sigma_\infty}(1+|\tfrac{u}{2}+a(\chi_v)|)^{-2N+2N_0+\varepsilon},\quad u\in\mathbb{R},z\in\mathcal{V}\end{aligned}$$

from (4.27). We remark that the implied constants of the above estimates are independent of $\chi \in \Xi(\mathfrak{o})$. Therefore, by the same way as in the proof of Lemma 4.9 and the assumption of $\beta \in \mathcal{B}_1$, we can check that the order of all the summations and the integrals in (4.62) can be exchanged freely whenever $\sigma > 1$ and the function

$$\hat{I}^0_{\mathrm{Eis}}(\mathbf{s}_S, \mathfrak{s}_\infty, z) = \zeta_F(\tfrac{z+1}{2})\frac{\mathrm{vol}(\mathbb{A}^1/F^\times)}{8\pi\sqrt{-1}} \sum_{\chi\in\Xi(\mathfrak{o})} \int_{\sqrt{-1}\mathbb{R}} I^0_{\mathrm{Eis},\chi}(u,z)du$$

defines a holomorphic function on the region $\{z \in \mathbb{C} \mid -1 < \mathrm{Re}(z) < 1 \text{ or } \mathrm{Re}(z) > 1\}$. By changing the order of the summations and the integrals in (4.62), we have

$$\mathbb{I}_2(\beta) = \int_{L_\sigma} \beta(z)\hat{I}^0_{\mathrm{Eis}}(\mathbf{s}_S, \mathfrak{s}_\infty, z)dz, \quad (\sigma > 1). \tag{4.74}$$

Our aim is to shift the contour L_σ with $\sigma > 1$ to L_σ with $-1 < \sigma < 1$. By considering the same modification of contour and calculations studied in ([10, pp. 32–35]), we can show that there exists a holomorphic function $\mathscr{I}$ on the region $\mathrm{Re}(z) > -1$ such that

(a) $\mathscr{I}(z) = \hat{I}^0_{\mathrm{Eis}}(\mathbf{s}_S, \mathfrak{s}_\infty, z)$ if $\mathrm{Re}(z) > 1$.
(b) $\mathscr{I}(z) = \hat{I}^0_{\mathrm{Eis}}(\mathbf{s}_S, \mathfrak{s}_\infty, z) + \frac{1}{2}\Lambda_F(z)\sum_{\substack{\chi\in\Xi(\mathfrak{o})\\ \chi^2=1}} \hat{I}_{\mathrm{res}}(-z)$ if $-1 < \mathrm{Re}(z) < 1$.
(c) $\mathscr{I}(z)$ is vertically of moderate growth.

where we recall the function $\hat{I}_{\mathrm{res}}(z)$ defined in (4.57). By using $\mathscr{I}$, we can shift the contour in (4.74) as

$$\mathbb{I}_2(\beta) = \int_{L_\sigma} \beta(z)\left(\hat{I}^0_{\mathrm{Eis}}(\mathbf{s}_S, \mathfrak{s}_\infty, z) + \frac{1}{2}\Lambda_F(-z+1)\sum_{\substack{\chi\in\Xi(\mathfrak{o})\\ \chi^2=1}} \hat{I}_{\mathrm{res},\chi}(-z)\right)dz, \quad (-1 < \sigma < 1). \tag{4.75}$$

Remark 4.28. We note that the complex variable s in ([10]) coincides with $\frac{z+1}{2}$ in our notation and the Haar measure on $\mathbb{A}^\times$ in ([10]) is normalized so that the volume of $\mathbb{A}^1/F^\times$ equals 1.

Set

$$\hat{I}^0_{\mathrm{res}}(\mathbf{s}_S, \mathfrak{s}_\infty, z) = \frac{1}{2}\sum_{\substack{\chi\in\Xi(\mathfrak{o})\\ \chi^2=1}} \Big(\Lambda_F(z+1)\hat{I}^0_{\mathrm{res},\chi}(z) + \Lambda_F(-z+1)\hat{I}^0_{\mathrm{res},\chi}(-z)\Big)$$

Then $\hat{I}^0_{\mathrm{res}}(\mathbf{s}_S, \mathfrak{s}_\infty, z)$ and $\hat{I}^0_{\mathrm{Eis}}(\mathbf{s}_S, \mathfrak{s}_\infty, z)$ are holomorphic on the region $-1 < \mathrm{Re}(z) < 1$ and vertically of moderate growth satisfying

$$\mathbb{I}_{\mathrm{Eis}}(\mathfrak{n}|\mathbf{s}_S, \mathfrak{s}_\infty; \beta) = \int_{L_\sigma} \beta(z)\left(\hat{I}^0_{\mathrm{Eis}}(\mathbf{s}_S, \mathfrak{s}_\infty, z) + \hat{I}^0_{\mathrm{res}}(\mathbf{s}_S, \mathfrak{s}_\infty, z)\right)dz$$

by (4.58) and (4.75).

5 The geometric side

We recall the notations used in the definition of the test function. Let $\mathfrak{n} \subset \mathfrak{o}$ be a non-zero ideal and S be a finite subset of Σ_{fin} such that $S \cap S(\mathfrak{n}) = \emptyset$ and $|2|_v = 1$ for any $v \in S \cup S(\mathfrak{n})$. For a large enough positive integer N, the parameters $\mathbf{s}_S \in \mathfrak{X}_S$ and $\mathfrak{s}_\infty = (\mathfrak{s}_v = (s_{i,v})_{1\le i\le N})_{v\in\Sigma_\infty} \in (\mathbb{C}^N)^{\Sigma_\infty}$

such that $\mathrm{Re}(s_v) > 0$ for $v \in S$ and $\mathrm{Re}(s_{i,v}) > 1$ and $\mathrm{Re}(s_{i,v}) \neq \mathrm{Re}(s_{j,v})$ if $i \neq j$ for $v \in \Sigma_\infty$ are fixed. The functions $\Phi(g) = \Phi(\mathfrak{n}|\mathbf{s}_S, \mathfrak{s}_\infty; g)$ and $\boldsymbol{\Phi}(g,h) = \boldsymbol{\Phi}(\mathfrak{n}|\mathbf{s}_S, \mathfrak{s}_\infty; g, h)$ are defined as in (2.15) and (2.16). The notations and calculations in the following sections, for the most part, are based on ([16]).

From the classification of the conjugacy classes of $Z_F\backslash G_F$, we have

$$\boldsymbol{\Phi}(g,g) = J_{\mathrm{id}}(g) + J_{\mathrm{uni}}(g) + J_{\mathrm{hyp}}(g) + J_{\mathrm{ell}}(g)$$

where the four terms on the right-hand side is defined as follows (see [7], [10]).

$$\begin{aligned}
J_{\mathrm{id}}(g) &= \Phi(1), \\
J_{\mathrm{uni}}(g) &= \sum_{\xi \in Z_F N_F \backslash G_F} \Phi\left(g^{-1}\xi^{-1}\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\xi g\right), \\
J_{\mathrm{hyp}}(g) &= \frac{1}{2}\sum_{\xi \in H_F\backslash G_F}\sum_{a \in F^\times - \{1\}} \Phi\left(g^{-1}\xi^{-1}\left(\begin{smallmatrix}a&0\\0&1\end{smallmatrix}\right)\xi g\right), \\
J_{\mathrm{ell}}(g) &= \frac{1}{2}\sum_{E \in \mathfrak{E}}\sum_{\xi \in T_E\backslash G_F}\sum_{\gamma \in Z_F\backslash(T_E - Z_F)} \Phi\left(g^{-1}\xi^{-1}\gamma\xi g\right),
\end{aligned}$$

where $\mathfrak{E}$ is the set of all quadratic division algebra $E \subset M_2(F)$ and $T_E = E^\times$ viewed as an F-elliptic torus of GL(2). For each conjugacy class $\natural \in \{\mathrm{id}, \mathrm{uni}, \mathrm{hyp}, \mathrm{ell}\}$ we calculate the integral

$$\mathbb{J}_\natural(\beta) = \int_{Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}}} \mathcal{E}^*_\beta(g) J_\natural(g)\,dg$$

explicitly in the succeeding three sections. We shall show the following theorem.

Theorem 5.1. *For each $\natural \in \{\mathrm{id}, \mathrm{uni}, \mathrm{hyp}, \mathrm{ell}\}$, there exists a holomorphic function $\hat{J}^0_\natural(\mathbf{s}_S, \mathfrak{s}_\infty, z)$ on the region $-1 < \mathrm{Re}(z) < 1$ which is vertically of moderate growth and satisfies*

$$\mathbb{J}_\natural(\beta) = \int_{L_\sigma} \beta(z)\hat{J}^0_\natural(\mathbf{s}_S, \mathfrak{s}_\infty, z)\,dz$$

for $-1 < \sigma < 1$.

5.1 The identity term

It is quite easy to show that $\mathbb{J}_{\mathrm{id}}(\beta) = 0$ from Lemma 4.12. If we put $\hat{J}^0_{\mathrm{id}}(\mathbf{s}_S, \mathfrak{s}_\infty, z) = 0$, Theorem 5.1 is valid for the identity term.

5.2 The unipotent term

By noting $\mathrm{vol}(N_F\backslash N_{\mathbb{A}}) = 1$, we have

$$\begin{aligned}
\mathbb{J}_{\mathrm{uni}}(\beta) &= \int_{Z_{\mathbb{A}}N_F\backslash G_{\mathbb{A}}} \Phi(g^{-1}\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)g)\mathcal{E}^*_\beta(g)\,dg \\
&= \int_{Z_{\mathbb{A}}N_{\mathbb{A}}\backslash G_{\mathbb{A}}} \Phi(g^{-1}\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)g)\mathcal{E}^*_{\beta,\circ}(g)\,dg \qquad (5.1)
\end{aligned}$$

where $\mathcal{E}^*_{\beta.\circ}(g) = \int_{N_F\backslash N_{\mathbb{A}}} \mathcal{E}^*_\beta(ng)dn$ is the constant term which is given as

$$\mathcal{E}^*_{\beta.\circ}(g) = \int_{L_\sigma} \beta(z)\left(\Lambda_F(-z)y(g)^{\frac{z+1}{2}} + \Lambda_F(z)y(g)^{\frac{-z+1}{2}}\right)dz.$$

To compute $\mathbb{J}_{\mathrm{uni}}(\beta)$, we consider the integral

$$U(w) = \int_{Z_{\mathbb{A}}N_{\mathbb{A}}\backslash G_{\mathbb{A}}} \Phi(g^{-1}\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)g)y(g)^w dg, \quad (w\in\mathbb{C}) \tag{5.2}$$

and the local integrals

$$U_v(w) = \int_{Z_vN_v\backslash G_v} \Phi_v(g^{-1}\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)g)y(g)_v^w dg = \int_{F_v^\times}\int_{\mathbf{K}_v} \Phi_v(k^{-1}\left(\begin{smallmatrix}1&a\\0&1\end{smallmatrix}\right)k)|a|_v^{1-w}d^\times a dk$$

for any $v\in\Sigma_F$.

Lemma 5.2.
(i) For any $v\in S$ and $w\in\mathbb{C}$ with $-\mathrm{Re}(s_v)<\mathrm{Re}(w)<1$, we have

$$U_v(w) = q_v^{-\frac{\mathrm{d}_v}{2}} \times (-q_v^{-\frac{s_v+1}{2}})(1-q_v^{w-1})^{-1}(1-q_v^{-s_v-w})^{-1}.$$

(ii) For any $v\in S(\mathfrak{n})$ and $w\in\mathbb{C}$ with $\mathrm{Re}(w)<1$, we have

$$U_v(w) = q_v^{-\frac{\mathrm{d}_v}{2}}(1+q_v^w)(1+q_v)^{-1}(1-q_v^{w-1})^{-1}.$$

(iii) For any $v\in\Sigma_{\mathrm{fin}}-(S\cup S(\mathfrak{n}))$ and $w\in\mathbb{C}$ with $\mathrm{Re}(w)<1$, we have

$$U_v(w) = q_v^{-\frac{\mathrm{d}_v}{2}}(1-q_v^{w-1})^{-1}.$$

(iv) For any $v\in\Sigma_\infty$ and $w\in\mathbb{C}$ with $-\min\{\mathrm{Re}(s_{i,v})\mid 1\le i\le N\}<\mathrm{Re}(w)<1$, we have

$$U_v(w) = 2^{-w-1}\pi^{-\frac{w+1}{2}}\Gamma_{\mathbb{R}}(1-w)\Gamma(\tfrac{1-w}{2})\left\{\sum_{i=1}^N \mu_v^{(i)}(\mathfrak{s}_\infty)\frac{\Gamma(\frac{s_{i,v}+w}{2})}{\Gamma(\frac{s_{i,v}-w}{2}+1)}\right\}.$$

Proof. The formulas (i), (ii), and (iii) are proved in ([16, Lemma 5.1]). Thus, it only remains to verify (iv). For $v\in\Sigma_\infty$, note that $\Phi_v(k^{-1}gk)=\Phi_{\mathfrak{s}_v}(k^{-1}gk)=\Phi_{\mathfrak{s}_v}(g)$ for any $g\in G_v$ and $k\in\mathbf{K}_v=O(2)$ by the formula (2.10). Hence we have

$$\begin{aligned}
\Phi_v(k^{-1}\left(\begin{smallmatrix}1&a\\0&1\end{smallmatrix}\right)k) &= \Phi_v(\left(\begin{smallmatrix}1&a\\0&1\end{smallmatrix}\right))\\
&= \frac{1}{4\pi}\sum_{i=1}^N \mu_v^{(i)}(\mathfrak{s}_\infty)\frac{\Gamma(\frac{s_{i,v}+1}{2})^2}{\Gamma(s_{i,v}+1)}\left\{\tfrac{4}{a^2+4}\right\}^{\frac{s_{i,v}+1}{2}} {}_2F_1\left(\tfrac{s_{i,v}+1}{2},\tfrac{s_{i,v}+1}{2};s_{i,v}+1;\tfrac{4}{a^2+4}\right)\\
&= \frac{1}{4\pi}\sum_{i=1}^N \mu_v^{(i)}(\mathfrak{s}_\infty)\frac{\Gamma(\frac{s_{i,v}+1}{2})^2}{\Gamma(s_{i,v}+1)}\times 4^{\frac{s_{i,v}+1}{2}}|a|^{-s_{i,v}-1}{}_2F_1\left(\tfrac{s_{i,v}+1}{2},\tfrac{s_{i,v}+1}{2};s_{i,v}+1;-4a^{-2}\right)
\end{aligned}$$

Here, we use the second equation of ([12, p. 47, (2.4.1)]) in the last line. Therefore we need to compute the following integral

$$\begin{aligned}
\mathcal{I}(s,w) &= \int_{\mathbb{R}^\times} {}_2F_1(\tfrac{s+1}{2},\tfrac{s+1}{2};s+1,-4a^{-2})|a|^{-w-s}d^\times a\\
&= 2\int_0^\infty {}_2F_1(\tfrac{s+1}{2},\tfrac{s+1}{2};s+1,-4a^{-2})a^{-w-s}d^\times a\\
&= 2^{-w-s}\int_0^\infty {}_2F_1(\tfrac{s+1}{2},\tfrac{s+1}{2};s+1,-a)a^{\frac{s+w}{2}}d^\times a \quad (\text{substituting } a\to 2a^{-\frac{1}{2}})
\end{aligned}$$

with $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > 1$.

We can show that this integral converges absolutely if $-\mathrm{Re}(s) < \mathrm{Re}(w) < 1$ by means of the well-known integral formula:

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt, \tag{5.3} $$
$$ (\mathrm{Re}(c) > \mathrm{Re}(b) > 0, z \in \mathbb{C} - [1,+\infty)). $$

Moreover, from ([3, p. 336 (3)]), we obtain

$$ \begin{aligned} \mathcal{I}(s,w) &= 2^{-w-s} \frac{\Gamma(s+1)}{\Gamma(\frac{s+1}{2})^2} \frac{\Gamma(\frac{1-w}{2})^2 \Gamma(\frac{s+w}{2})}{\Gamma(\frac{s-w}{2}+1)} \\ &= 2^{-w-s} \pi^{\frac{-w+1}{2}} \frac{\Gamma(s+1)}{\Gamma(\frac{s+1}{2})^2} \times \Gamma_{\mathbb{R}}(\tfrac{1-w}{2}) \Gamma(\tfrac{1-w}{2}) \frac{\Gamma(\frac{s+w}{2})}{\Gamma(\frac{s-w}{2}+1)} \end{aligned} $$

By applying this formula to the obvious identity

$$ U_v(w) = \frac{1}{4\pi} \sum_{i=1}^N \mu_v^{(i)}(\mathfrak{s}_\infty) \frac{\Gamma(\frac{s_{i,v}+1}{2})^2}{\Gamma(s_{i,v}+1)} \times 4^{\frac{s_{i,v}+1}{2}} \mathcal{I}(s_{i,v}, w), $$

we are done. □

Suppose that $-1 < \mathrm{Re}(w) < 0$, from Lemma 5.2, the integral (5.2) converges absolutely and equals

$$ \begin{aligned} U(w) = D_F^{-\frac{1}{2}} \zeta_F(1-w) &\prod_{v \in S} \frac{-q_v^{-\frac{s_v+1}{2}}}{1 - q_v^{-s_v-w}} \prod_{v \in S(\mathfrak{n})} \frac{1+q_v^w}{1+q_v} \\ &\times \prod_{v \in \Sigma_\infty} 2^{-w-1} \pi^{-\frac{w+1}{2}} \Gamma\left(\tfrac{-w+1}{2}\right) \left\{ \sum_{i=1}^N \mu_v^{(i)}(\mathfrak{s}_\infty) \frac{\Gamma(\frac{s_{i,v}+w}{2})}{\Gamma(\frac{s_{i,v}-w}{2}+1)} \right\}. \end{aligned} $$

Then $U(w)$ has a meromorphic continuation to $\mathbb{C}$ and is holomorphic on $0 < \mathrm{Re}(w) < 1$. By changing the order of the integral (5.1), we obtain

$$ \begin{aligned} \mathbb{J}_{\mathrm{uni}}(\beta) &= \int_{L_\sigma} \beta(z) \left(\Lambda_F(-z) U(\tfrac{z+1}{2}) + \Lambda_F(z) U(\tfrac{-z+1}{2}) \right) dz \\ &= \int_{L_\sigma} \beta(z) \hat{J}^0_{\mathrm{uni}}(\mathbf{s}_S, \mathfrak{s}_\infty, z) dz \end{aligned} $$

where we set

$$ \hat{J}^0_{\mathrm{uni}}(\mathbf{s}_S, \mathfrak{s}_\infty, z) = D_F^{\frac{z}{4}} \zeta_F(\tfrac{z+1}{2}) \left\{ \hat{J}^1_{\mathrm{uni}}(\mathbf{s}_S, \mathfrak{s}_\infty, z) + \hat{J}^1_{\mathrm{uni}}(\mathbf{s}_S, \mathfrak{s}_\infty, -z) \right\} \tag{5.4} $$

with

$$ \begin{aligned} \hat{J}^1_{\mathrm{uni}}(\mathbf{s}_S, \mathfrak{s}_\infty, z) = D_F^{\frac{z-2}{4}} \Lambda_F(-z) &\prod_{v \in S} \frac{-q_v^{-\frac{s_v+1}{2}}}{1 - q_v^{-s_v - \frac{z+1}{2}}} \prod_{v \in S(\mathfrak{n})} \frac{1+q_v^{\frac{z+1}{2}}}{1+q_v} \\ &\times \prod_{v \in \Sigma_\infty} 2^{-\frac{z+3}{2}} \pi^{-\frac{z+3}{4}} \Gamma\left(\tfrac{-z+1}{4}\right) \left\{ \sum_{i=1}^N \mu_v^{(i)}(\mathfrak{s}_\infty) \frac{\Gamma(\frac{s_{i,v}}{2} + \frac{z+1}{4})}{\Gamma(\frac{s_{i,v}}{2} - \frac{z-3}{4})} \right\}. \end{aligned} $$

Then $\hat{J}^0_{\mathrm{uni}}(\mathbf{s}_S, \mathfrak{s}_\infty, z)$ is the desired function in Theorem 5.1 for the unipotent term.

5.3 The hypergeometric series in two variables

In this subsection, we state a brief explanation of a certain hypergeometric series in two variables to describe the hyperbolic term and the elliptic term. In ([14], see also [11]), the Kampé de Fériet function is defined as follows:

$$F^{\lambda:r;s}_{\nu:u;v}\left({\alpha_1,\dots,\alpha_\lambda:a_1,\dots,a_r;c_1,\dots,c_s \atop \beta_1,\dots,\beta_\mu:b_1,\dots,b_u;d_1,\dots,d_v};x,y\right)=\sum_{m,n=0}^{\infty}\frac{(\alpha_1,\dots,\alpha_\lambda)_{m+n}(a_1,\dots,a_r)_m(c_1,\dots,c_s)_n}{m!n!(\beta_1,\dots,\beta_\mu)_{m+n}(b_1,\dots,b_u)_m(c_1,\dots,c_v)_n}x^m y^n$$

where $(a_1,\dots,a_r)_n=(a_1)_n\dots(a_r)_n$ and $(a)_n$ is the Pochhammer symbol given by $(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}$. It is known that the convergence condition of this series depends on the number of parameters.

The Kampé de Fériet function can be regarded as a generalization of the Appell series ([1]). For example,

$$\begin{aligned} F^{0:2;2}_{1:0;0}\left({-:\alpha,\alpha';\beta,\beta' \atop \gamma:-;-};x,y\right) &= \sum_{m,n=0}^{\infty}\frac{(\alpha)_m(\alpha')_m(\beta)_n(\beta')_n}{m!n!(\gamma)_{m+n}}x^m y^n \\ &= F_3(\alpha,\alpha',\beta,\beta',\gamma,x,y) \end{aligned}$$

for $|x|,|y|<1$ where F_3 is the Appell series of the third kind. Let p and q be non-negative integrers. To simplify notation, we set

$$F_3^{(p,q)}\left({a_1,\dots,a_{p+2};b_1,\dots,b_{q+2} \atop a'_1,\dots,a'_p;b'_1,\dots,b'_q;c};x,y\right)=F^{0:p+2;q+2}_{1:p;q}\left({-:a_1,\dots,a_{p+2};b_1,\dots,b_{q+2} \atop c:a'_1,\dots,a'_p;b_1,\dots,b'_q};x,y\right) \tag{5.5}$$

which is a generalization of F_3. We remark that $F_3^{(0,0)}$ coincides with F_3. Let us see some properties of $F_3^{(p,q)}$ we need in this article.

First, we see several convergence conditions of $F_3^{(p,q)}$.

Proposition 5.3. *Let*

$$\sigma=\mathrm{Re}(c+a'_1+\dots+a'_p-a_1-\dots-a_{p+2})$$

and

$$\tau=\mathrm{Re}(c+b'_1+\dots+b'_q-b_1-\dots-b_{q+2}).$$

(i) The series $F_3^{(p,q)}$ always converges absolutely for $|x|<1$, $|y|<1$.
(ii) The series $F_3^{(p,q)}$ converges absolutely for

$$\begin{aligned} &|x|\le 1, |y|\le 1 \text{ if } \sigma>0,\tau>0,\\ &|x|\le 1, |y|< 1 \text{ if } \sigma>0,\tau\le 0,\\ &|x|< 1, |y|\le 1 \text{ if } \sigma\le 0,\tau>0. \end{aligned}$$

Proof. Stirling's formula tells us that

$$\frac{\Gamma(\alpha+m)}{\Gamma(\beta+m)}\sim(m+1)^{\alpha-\beta}\quad(m\to+\infty)$$

For all $\alpha, \beta \in \mathbb{C}$. Hence we have

$$\begin{aligned}
&\frac{(a_1)_m \cdots (a_{p+2})_m (b_1)_n \dots (b_{q+2})_n}{m!n!(a'_1)_m \dots (a'_p)_m (b'_1)_n \dots (b'_q)_n (c)_{m+n}} \\
&\ll \left| \tfrac{\Gamma(a_1+m)\dots\Gamma(a_{p+2}+m)}{\Gamma(1+m)\Gamma(a'_1+m)\dots\Gamma(a'_p+m)\Gamma(c+m)} \right| \left| \tfrac{\Gamma(b_1+n)\dots\Gamma(b_{q+2}+n)}{\Gamma(1+n)\Gamma(b'_1+n)\dots\Gamma(b'_q+n)\Gamma(c+n)} \right| \left| \tfrac{\Gamma(c+m)\Gamma(c+n)}{\Gamma(c+m+n)} \right| \\
&\ll (m+1)^{-\sigma-1}(n+1)^{-\tau-1}. \qquad (5.6)
\end{aligned}$$

This shows the claim. □

Second, we prove the following integral formula of $F_3^{(p,q)}$ which gives us an analytic continuation of $F_3^{(p,q)}$ to the region $x, y \in \mathbb{C} - [1, +\infty)$.

Proposition 5.4. *Suppose $x, y \in \mathbb{C}$ satisfy the convergence condition in Proposition 5.3. Then, if $\mathrm{Re}(c) > \mathrm{Re}(a_{p+2}) > 0$, we have*

$$\begin{aligned}
&F_3^{(p,q)} \left({a_1,\dots,a_{p+2};b_1,\dots,b_{q+2} \atop a'_1,\dots,a'_p;b'_1,\dots,b'_q;c} ; x, y \right) \\
=&\frac{\Gamma(c)}{\Gamma(a_{p+2})\Gamma(c-a_{p+2})} \\
&\times \int_0^1 t^{a_{p+2}-1}(1-t)^{c-a_{p+2}-1} {}_{p+1}F_p \left({a_1,\dots,a_{p+1} \atop a'_1,\dots,a'_p} ; xt \right) {}_{q+2}F_{q+1} \left({b_1,\dots,b_{q+2} \atop b'_1,\dots,b'_q,c-a_{p+2}} ; y(1-t) \right) dt.
\end{aligned}$$

where ${}_rF_s$ is the generalized hypergeometric function.

Proof. By means of the well-known integral formula for ${}_{p+2}F_{p+1}$ given as

$$\begin{aligned}
{}_{p+2}F_{p+1} \left({\alpha_1,\dots,\alpha_{p+2} \atop \alpha'_1,\dots,\alpha'_{p+1}} ; z \right) =& \frac{\Gamma(\alpha'_{p+1})}{\Gamma(\alpha_{p+2})\Gamma(\alpha'_{p+1}-\alpha_{p+2})} \\
&\times \int_0^1 t^{\alpha_{p+2}-1}(1-t)^{\alpha'_{p+1}-\alpha_{p+2}-1} {}_{p+1}F_p \left({\alpha_1,\dots,\alpha_{p+1} \atop \alpha'_1,\dots,\alpha'_p} ; zt \right) dt \\
&(\mathrm{Re}(\alpha'_{p+1}) > \mathrm{Re}(\alpha_{p+2}) > 0),
\end{aligned}$$

we obtain

$$
\begin{aligned}
&F_3^{(p,q)}\left({a_1,\dots,a_{p+2};b_1,\dots,b_{q+2} \atop a'_1,\dots,a'_p;b'_1,\dots,b'_q;c};x,y\right)\\
=&\sum_{m,n\geq 0}\frac{(a_1)_m\cdots(a_{p+2})_m(b_1)_n\dots(b_{q+2})_n}{m!n!(a'_1)_m\dots(a'_p)_m(b'_1)_n\dots(b'_q)_n(c)_{m+n}}x^m y^n\\
=&\sum_{m,n\geq 0}\frac{(a_1)_m\cdots(a_{p+2})_m(b_1)_n\dots(b_{q+2})_n}{m!n!(a'_1)_m\dots(a'_p)_m(b'_1)_n\dots(b'_q)_n(c+n)_m(c)_n}x^m y^n\\
=&\sum_{n\geq 0}\frac{(b_1)_n\dots(b_{q+2})_n}{(b'_1)_n\dots(b'_q)_n(c)_n n!}y^n\cdot {}_{p+2}F_{p+1}\left({a_1,\dots,a_{p+2} \atop a'_1,\dots,a'_p,c+n};x\right)\\
=&\sum_{n\geq 0}\frac{(b_1)_n\dots(b_{q+2})_n}{(b'_1)_n\dots(b'_q)_n(c)_n n!}y^n\cdot\frac{\Gamma(c+n)}{\Gamma(a_{p+2})\Gamma(c-a_{p+2}+n)}\\
&\qquad\times\int_0^1 t^{a_{p+2}-1}(1-t)^{c-a_{p+2}+n-1}{}_{p+1}F_p\left({a_1,\dots,a_{p+1} \atop a'_1,\dots,a'_p};xt\right)dt\\
=&\frac{\Gamma(c)}{\Gamma(a_{p+2})\Gamma(c-a_{p+2})}\sum_{n\geq 0}\frac{(b_1)_n\dots(b_{q+2})_n}{(b'_1)_n\dots(b'_q)_n(c-a_{p+2})_n n!}y^n\\
&\qquad\times\int_0^1 t^{a_{p+2}-1}(1-t)^{c-a_{p+2}+n-1}{}_{p+1}F_p\left({a_1,\dots,a_{p+1} \atop a'_1,\dots,a'_p};xt\right)dt\\
=&\frac{\Gamma(c)}{\Gamma(a_{p+2})\Gamma(c-a_{p+2})}\\
&\times\int_0^1 t^{a_{p+2}-1}(1-t)^{c-a_{p+2}-1}{}_{p+1}F_p\left({a_1,\dots,a_{p+1} \atop a'_1,\dots,a'_p};xt\right){}_{q+2}F_{q+1}\left({b_1,\dots,b_{q+2} \atop b'_1,\dots,b'_q,c-a_{p+2}};y(1-t)\right)dt.
\end{aligned}
$$

□

Finally, we show the following lemma for later use.

Lemma 5.5. *Suppose* $y\in\mathbb{C}-[1,+\infty)$ *and* $\mathrm{Re}(b_1)>0$ *or* $\mathrm{Re}(b_2)>0$.
(i) For $|x|<1$, *We have*

$$\sum_{m=0}^{\infty}\frac{(a_1)_m(a_2)_m}{m!(a'_1)_m}x^m{}_2F_1(b_1,b_2;c+m;y)=F_3^{(1,0)}\left({a_1,a_2,c;b_1;b_2 \atop a'_1;c};x,y\right). \tag{5.7}$$

(ii) If $\mathrm{Re}(c+a'_1-a_1-a_2-a_3)>0$, *we have*

$$\sum_{m=0}^{\infty}\frac{(a_1)_m(a_2)_m(a_3)_m}{m!(a'_1)_m(c)_m}{}_2F_1(b_1,b_2;c+m;y)=F_3^{(1,0)}\left({a_1,a_2,a_3;b_1;b_2 \atop a'_1;c};1,y\right). \tag{5.8}$$

Proof. The identities (5.7) and (5.8) are obvious if $|y|<1$ by Proposition 5.3. We assume $\mathrm{Re}(b_2)>0$. Let $\mathcal{U}\subset\mathbb{C}-[1,+\infty)$ be a compact subset and $y\in\mathcal{U}$. By the integral formula (5.3) and Stirling's formula, we have the estimate

$$\begin{aligned}
|{}_2F_1(b_1,b_2;c+m;y)| &\leq \left|\frac{\Gamma(c+m)}{\Gamma(b_2)\Gamma(c-b_2+m)}\right| \left|\int_0^1 t^{\mathrm{Re}(b_2)-1}(1-t)^{\mathrm{Re}(c-b_2)+m-1}|1-yt|^{-\mathrm{Re}(b_1)}dt\right.\\
&\ll_{\mathcal{U}} \left|\frac{\Gamma(c+m)}{\Gamma(b_2)\Gamma(c-b_2+m)}\right| \left|\int_0^1 t^{\mathrm{Re}(b_2)-1}(1-t)^{\mathrm{Re}(c-b_2)+m-1}dt\right.\\
&= \left|\frac{\Gamma(c+m)}{\Gamma(b_2)\Gamma(c-b_2+m)}\right| \frac{\Gamma(\mathrm{Re}(c-b_2)+m)\Gamma(\mathrm{Re}(b_2))}{\Gamma(\mathrm{Re}(c)+m)}\\
&\ll_{\mathcal{U}} \left|\frac{\Gamma(c+m)}{\Gamma(\mathrm{Re}(c)+m)}\right| \left|\frac{\Gamma(\mathrm{Re}(c-b_2)+m)}{\Gamma(c-b_2+m)}\right|\\
&\ll_{\mathcal{U}} 1
\end{aligned}$$

for large $m \in \mathbb{Z}_{\geq 0}$. Since $|x| < 1$ and $\mathrm{Re}(c + a_1' - a_1 - a_2 - a_3) > 0$, we have the dominating series

$$\sum_{m=0}^{\infty} \left|\frac{(a_1)_m(a_2)_m}{m!(a_1')_m} x^m {}_2F_1(b_1,b_2;c+m;y)\right| \ll_{\mathcal{U}} \sum_{m=0}^{\infty} \left|\frac{(a_1)_m(a_2)_m}{m!(a_1')_m} x^m\right| < +\infty$$

and

$$\sum_{m=0}^{\infty} \left|\frac{(a_1)_m(a_2)_m(a_3)_m}{m!(a_1')_m(c)_m} {}_2F_1(b_1,b_2;c+m;y)\right| \ll_{\mathcal{U}} \sum_{m=0}^{\infty} \left|\frac{(a_1)_m(a_2)_m(a_3)_m}{m!(a_1')_m(c)_m}\right| < +\infty.$$

Therefore, the series (5.7) and (5.8) define holomorphic functions with respect to $y \in \mathbb{C} - [1, +\infty)$. Hence, (i) and (ii) follow from analytic continuation. □

6 The F-hyperbolic term

In this subsection, we compute the integral $\mathbb{J}_{\mathrm{hyp}}(\beta)$ and estimate its absolute convergence.

6.1 An overview of the F-hyperbolic term

From Lemmas 2.9 and 3.8, we get the following estimation:

$$\begin{aligned}
&\int_{Z_{\mathbb{A}}G_F\backslash G_F} |\mathcal{E}_\beta^*(g)| \sum_{\xi \in H_F\backslash G_F} \sum_{a \in F^\times - \{1\}} \left|\Phi(g^{-1}\xi^{-1} \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \xi g)\right| dg\\
\leq &\int_{Z_{\mathbb{A}}G_F\backslash G_F} |\mathcal{E}_\beta^*(g)| \sum_{\gamma \in Z_F\backslash G_F} \left|\Phi(g^{-1}\gamma g)\right| dg < +\infty.
\end{aligned}$$

Hence by changing the order of the summations and the integral, we have

$$\begin{aligned}
\mathbb{J}_{\mathrm{hyp}}(\beta) &= \frac{1}{2} \sum_{a \in F^\times - \{1\}} \int_{Z_{\mathbb{A}}H_F\backslash G_{\mathbb{A}}} \mathcal{E}_\beta^*(g)\Phi(g^{-1} \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) g)dg\\
&= \frac{1}{2} \sum_{a \in F^\times - \{1\}} \int_{\mathbf{K}} \int_{\mathbb{A}} \int_{\mathbb{A}^\times/F^\times} \mathcal{E}_\beta^*(\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)) \Phi(k^{-1} \left(\begin{smallmatrix} 1 & -x \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) k)d^\times t dx dk\\
&= \frac{1}{2} \sum_{a \in F^\times - \{-1\}} \int_{\mathbf{K}} \int_{\mathbb{A}} \Phi(k^{-1} \left(\begin{smallmatrix} a & (a-1)x \\ 0 & 1 \end{smallmatrix}\right) k) P_\beta(0; \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)) dx dk \qquad (6.1)
\end{aligned}$$

where we set
$$P_\beta(w;g)=\int_{\mathbb{A}^\times/F^\times}\mathcal{E}^*_\beta((\begin{smallmatrix}t&0\\0&1\end{smallmatrix})g)|t|_{\mathbb{A}}^w d^\times t$$
for $w\in\mathbb{C}$ and $g\in G_{\mathbb{A}}$. Note that Proposition 3.8 ensures the absolute convergence of this integral. From the Fourier expansion of $E^*(z;g)$ in (3.18), the value $\mathcal{E}^*_\beta((\begin{smallmatrix}t&0\\0&1\end{smallmatrix})(\begin{smallmatrix}1&x\\0&1\end{smallmatrix}))$ can be written as
$$\mathcal{E}^*_\beta((\begin{smallmatrix}t&0\\0&1\end{smallmatrix})(\begin{smallmatrix}1&x\\0&1\end{smallmatrix}))=C^+_\beta(t)+C^-_\beta(t)+\mathcal{W}_\beta(t,x)$$
where
$$C^\pm_\beta(t)=\int_{L_\sigma}\beta(z)\Lambda_F(\mp z)|t|_{\mathbb{A}}^{\frac{\pm z+1}{2}}dz,$$
$$\mathcal{W}_\beta(t,x)=\int_{L_\sigma}\beta(z)\Lambda_F(-z)\sum_{a\in F^\times}W_\psi\left(z;(\begin{smallmatrix}at&0\\0&1\end{smallmatrix})(\begin{smallmatrix}1&x\\0&1\end{smallmatrix})\right)dz$$
with $\sigma>1$. Then we have
$$P_\beta(w;(\begin{smallmatrix}1&x\\0&1\end{smallmatrix}))=\mathfrak{C}^+_\beta(w)+\mathfrak{C}^-_\beta(w)+\mathfrak{W}_\beta(w;x)$$
where
$$\mathfrak{C}^\pm_\beta(w)=\int_{\mathbb{A}^\times/F^\times}C^\pm_\beta(t)|t|_{\mathbb{A}}^w d^\times t=\mathrm{vol}(\mathbb{A}^1/F^\times)\int_{\mathbb{R}_+}C^\pm_\beta(t)|t|^w d^\times t,\tag{6.2}$$
$$\begin{aligned}\mathfrak{W}_\beta(w;x)&=\int_{\mathbb{A}^\times/F^\times}\mathcal{W}_\beta(t,x)|t|_{\mathbb{A}}^w d^\times t\\&=\int_{L_\sigma}\beta(z)\Lambda_F(-z)\left(\int_{\mathbb{A}^\times}W_\psi\left(z;(\begin{smallmatrix}t&0\\0&1\end{smallmatrix})(\begin{smallmatrix}1&x\\0&1\end{smallmatrix})\right)|t|_{\mathbb{A}}^w d^\times t\right)dz.\end{aligned}\tag{6.3}$$
By Lemma 4.14 and the estimate
$$\int_{\mathbb{A}^\times}\left|W_\psi\left(z;(\begin{smallmatrix}t&0\\0&1\end{smallmatrix})(\begin{smallmatrix}1&x\\0&1\end{smallmatrix})\right)\right||t|_{\mathbb{A}}^{\mathrm{Re}(w)}d^\times t\ll 1,\quad \mathrm{Re}(w)>\tfrac{1}{2}|\mathrm{Re}(z)|+\tfrac{1}{2}$$
which follows from (3.19), the integrals (6.2) and (6.3) converge absolutely for all $w\in\mathbb{C}$ and for $\mathrm{Re}(w)>\frac{\sigma+1}{2}$ respectively.

For $\sigma,\sigma_1\in\mathbb{R}$ such that $\sigma_1<-2\mathrm{Re}(w)-1<\sigma$, by the same way as the proof of Lemma 4.14 and the residue theorem, we have
$$\begin{aligned}\int_{\mathbb{R}_+}C^+_\beta(t)|t|_{\mathbb{A}}^w d^\times t&=\int_0^\infty\int_{L_\sigma}\beta(z)\Lambda_F(-z)t^{w+\frac{z+1}{2}}dz\,d^\times t\\&=\int_{L_\sigma}\left(\int_0^1\beta(z)\Lambda_F(-z)t^{w+\frac{z+1}{2}}d^\times t\right)dz\\&\qquad+\int_{L_{\sigma_1}}\left(\int_1^\infty\beta(z)\Lambda_F(-z)t^{w+\frac{z+1}{2}}d^\times t\right)dz\\&=\int_{L_\sigma}\beta(z)\Lambda_F(-z)\frac{1}{w+\frac{z+1}{2}}dz-\int_{L_{\sigma_1}}\beta(z)\Lambda_F(-z)\frac{1}{w+\frac{z+1}{2}}dz\\&=2\pi\sqrt{-1}\operatorname*{Res}_{z=-2w-1}\left(\beta(z)\Lambda_F(-z)\frac{1}{w+\frac{z+1}{2}}\right)\\&=4\pi\sqrt{-1}\beta(-2w-1)\Lambda_F(2w+1).\end{aligned}$$

Hence we get

$$\mathfrak{C}_\beta^+(w) = 4\pi\sqrt{-1}\mathrm{vol}(\mathbb{A}^1/F^\times)\beta(-2w-1)\Lambda_F(2w+1).$$

In a similar manner, we can also calculate the integral $\mathfrak{C}_\beta^-(w)$ as

$$\mathfrak{C}_\beta^-(w) = -4\pi\sqrt{-1}\mathrm{vol}(\mathbb{A}^1/F^\times)\beta(2w+1)\Lambda_F(2w+1).$$

Because $\beta(\pm1)=\beta'(\pm1)=0$, the terms $\mathfrak{C}_\beta^+(0)$ and $\mathfrak{C}_\beta^-(0)$ do not contribute to $P_\beta(0;(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}))$.

Next, we observe the integral (6.3). Referring to ([16, Lemma 6.1]) and ([16, p. 3005]), we have the following Lemma.

Lemma 6.1. *For any $g=(g_v)_{v\in\Sigma_F}$ and $z,w\in\mathbb{C}$ with $\mathrm{Re}(w)>\frac{1}{2}\mathrm{Re}(z)+\frac{1}{2}$, we have*

$$\int_{\mathbb{A}^\times} W_\psi\left(z;(\begin{smallmatrix}t & 0\\ 0 & 1\end{smallmatrix})g\right)|t|_{\mathbb{A}}^w d^\times t = D_F^{w-\frac{z+1}{2}}\frac{\zeta_F(w+\frac{z+1}{2})\zeta_F(w+\frac{-z+1}{2})}{\zeta_F(z+1)}\prod_{v\in\Sigma_F}\varphi_v^{(w,z)}(g_v) \tag{6.4}$$

where $\varphi_v^{(w,z)}$ is the $\mathbf{K}_v$-spherical function on G_v such that

$$\varphi_v^{(w,z)}\left((\begin{smallmatrix}t_1 & 0\\ 0 & t_2\end{smallmatrix})(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix})k\right) = \left|\frac{t_1}{t_2}\right|_v^{-w}\left\{A_v(w,z)h_v^{(w,z)}(x)+A_v(w,-z)h_v^{(w,-z)}(x)\right\} \tag{6.5}$$

with

$$h_v^{(w,z)}(x)=\begin{cases}\max\{1,|x|_v\}^{-\frac{z+2w+1}{2}} & (v\in\Sigma_{\mathrm{fin}})\\ (1+x^2)^{-\frac{z+2w+1}{4}}{}_2F_1\left(\frac{z+2w+1}{4},\frac{z-2w+1}{4};\frac{z+2}{2};\frac{1}{1+x^2}\right) & (v\in\Sigma_\infty),\end{cases}$$
$$A_v(w,z)=\zeta_v(1)\zeta_v(-z)\zeta_v(\tfrac{-z+2w+1}{2})^{-1}\zeta_v(\tfrac{-z-2w+1}{2})^{-1}.$$

We denote the right-hand side of (6.4) by $F(w,z)$. Suppose $\sigma>1$ and $\mathrm{Re}(w)>\frac{\sigma+1}{2}$. We fix $\sigma'>\sigma$ such that $\frac{\sigma'-1}{2}<\mathrm{Re}(w)<\frac{\sigma'+1}{2}$. Then, the possible pole occurs only at $z=2w-1$ on the region $\sigma<\mathrm{Re}(z)<\sigma'$. Therefore, by Lemma 6.1 and shifting the contour L_σ to $L_{\sigma'}$ we have

$$\begin{aligned}\mathfrak{W}_\beta(w;x) &= \int_{L_\sigma}\beta(z)\Lambda_F(-z)F(w,z)dz\\ &= \int_{L_{\sigma'}}\beta(z)\Lambda_F(-z)F(w,z)dz\\ &\qquad\qquad -2\pi\sqrt{-1}\operatorname*{Res}_{z=2w-1}\beta(z)\Lambda_F(-z)F(w,z)\\ &= \int_{L_{\sigma'}}\beta(z)\Lambda_F(-z)F(w,z)dz\\ &\qquad\qquad +4\pi\sqrt{-1}\left\{\operatorname*{Res}_{s=1}\zeta_F(s)\right\}\beta(2w-1)\Lambda_F(1-2w).\end{aligned}$$

The first term is a holomorphic function on $\frac{1-\sigma'}{2}<\mathrm{Re}(w)<\frac{\sigma'+1}{2}$, and the second term is entire due to $\beta(0)=\beta(\pm1)=\beta'(\pm1)=0$. Hence by substituting $w=0$ and shifting the contour $L_{\sigma'}$ to L_σ for $-1<\sigma<1$, we have

$$P_\beta(0;(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}))=\mathfrak{W}_\beta(0;x)=\int_{L_\sigma}\beta(z)\zeta_F(\tfrac{z+1}{2})\zeta_F(\tfrac{-z+1}{2})\prod_{v\in\Sigma_F}\varphi_v^{(0,z)}\left((\begin{smallmatrix}1 & x_v\\ 0 & 1\end{smallmatrix})\right)dz. \tag{6.6}$$

6.2 The orbital integral

By the expressions (6.1), (6.6), and formally changing the order of integrals, the integral

$$\mathfrak{F}^{(z)}(a) = \int_{\mathbf{K}} \int_{\mathbb{A}} \Phi\left(k^{-1} \left(\begin{smallmatrix} a & (a-1)x \\ 0 & 1 \end{smallmatrix}\right) k\right) \prod_{v \in \Sigma_F} \varphi_v^{(0,z)}\left(\left(\begin{smallmatrix} 1 & x_v \\ 0 & 1 \end{smallmatrix}\right)\right) dx dk, \quad a \in F^\times - \{1\} \tag{6.7}$$

arises naturally. We set its local integrals as

$$\mathfrak{F}_v^{(z)}(a) = \int_{\mathbf{K}_v} \int_{F_v} \Phi_v\left(k^{-1} \left(\begin{smallmatrix} a & (a-1)x \\ 0 & 1 \end{smallmatrix}\right) k\right) \varphi_v^{(0,z)}\left(\left(\begin{smallmatrix} 1 & x_v \\ 0 & 1 \end{smallmatrix}\right)\right) dx dk, \quad a \in F_v^\times - \{1\} \tag{6.8}$$

for $v \in \Sigma_F$. The following proposition is sufficient to guarantee the formal computation explained above.

Proposition 6.2. *Let $a \in F^\times - \{1\}$, then the integral (6.7) converges absolutely in the region*

$$|\mathrm{Re}(z)| < \min\{2\mathrm{Re}(s_v) - 1 | v \in S\} \cup \{2\mathrm{Re}(s_{i,v}) + 1 | v \in \Sigma_\infty, 1 \le v \le N\}.$$

On this region, we have the product formula $\mathfrak{F}^{(z)}(a) = \prod_{v \in \Sigma_F} \mathfrak{F}_v^{(z)}(a)$. Moreover, it holds that $\mathfrak{F}^{(z)}(a) = 0$ unless $a \in \mathfrak{o}(S)^+$.

We shall compute explicit formulas of local integrals (6.8). The proof of Proposition 6.2 is given in the proof of the following Proposition for $v \in \Sigma_\infty$ and the calculations in ([16, §9.1]) for $v \in \Sigma_{\mathrm{fin}}$.

Proposition 6.3. *Let $v \in \Sigma_F$ and $a \in F_v^\times - \{1\}$.*

(i) For $v \in \Sigma_{\mathrm{fin}} - (S \cup S(\mathfrak{n}))$, we have

$$\mathfrak{F}_v^{(z)}(a) = \delta(a \in \mathfrak{o}_v^\times) q_v^{-\frac{d_v}{2}} \mathcal{O}_{v,0}^{1,(z)}((a-1)^{-2}a).$$

In particular, $\mathfrak{F}_v^{(z)}(a) = q_v^{-\frac{d_v}{2}}$ if $|a|_v = |a-1|_v = 1$.

(ii) For $v \in S(\mathfrak{n})$, we have

$$\mathfrak{F}_v^{(z)}(a) = \delta(a \in \mathfrak{o}_v^\times) q_v^{-\frac{d_v}{2}} \mathcal{O}_{v,1}^{1,(z)}((a-1)^{-2}a).$$

(iii) Suppose $v \in S$. For $z \in \mathbb{C}$ with $|\mathrm{Re}(z)| < 2\mathrm{Re}(s_v) + 1$, we have

$$\mathfrak{F}_v^{(z)}(a) = q_v^{-\frac{d_v}{2}} \mathcal{S}_v^{1,(z)}(s_v; (a-1)^{-2}a).$$

(iv) Suppose $v \in \Sigma_\infty$. For $z \in \mathbb{C}$ with $|\mathrm{Re}(z)| < 2\min\{\mathrm{Re}(s_{i,v}) | 1 \le i \le N\} + 1$, we have

$$\begin{aligned}
\mathfrak{F}_v^{(z)}(a) &= \delta(a > 0) \frac{\pi^{-\frac{1}{2}}}{4} \sum_{i=1}^N \mu_v^{(i)}(\mathfrak{s}_\infty) \left|\tfrac{a+1}{a-1}\right| \left\{\tfrac{4a}{(a+1)^2}\right\}^{\frac{s_{i,v}+1}{2}} \frac{\Gamma(\frac{s_{i,v}}{2} + \frac{z+1}{4})\Gamma(\frac{s_{i,v}}{2} - \frac{z-1}{4})}{\Gamma(s_{i,v}+1)} \\
&\quad \times F_3^{(1,0)}\left(\begin{matrix} \frac{s_{i,v}}{2} + \frac{z+1}{4}, \frac{s_{i,v}}{2} - \frac{z-1}{4}, \frac{s_{i,v}+1}{2}; \frac{z+1}{4}, \frac{-z+1}{4} \\ s_{i,v}+1; \frac{s_{i,v}+1}{2} \end{matrix}; \frac{4a}{(a+1)^2}, -\frac{4a}{(a-1)^2}\right) \\
&= \sum_{i=1}^N \mu_v^{(i)}(\mathfrak{s}_\infty) \mathcal{O}_v^{+,(z)}\left(s_{i,v}; \frac{a+1}{a-1}\right)
\end{aligned}$$

where $F_3^{(1,0)}$ is the hypergeometric series in two variables defined in (5.5)

Proof. The identities (i), (ii), and (iii) have been already proved in ([16, Theorem 6.4 and §9]). Here, it suffices to verify the formula (iv). Suppose $v \in \Sigma_\infty$. We recall that $\Phi_{\mathfrak{s}_v}(k^{-1}gk) = \Phi_{\mathfrak{s}_v}(g)$ holds for any $g \in G_v = G_{\mathbb{R}}$ and $k \in \mathbf{K}_v = O(2)$ and $\Phi_{\mathfrak{s}_v}(g) = 0$ if $\det g < 0$. Hence, we have $\mathfrak{F}_v^{(z)}(a) = 0$ for $a < 0$.

Suppose $a > 0$. Then, by (2.10) and (6.5), the local integral $\mathfrak{F}_v^{(z)}(a)$ becomes

$$\begin{aligned}\mathfrak{F}_v^{(z)}(a) &= \int_{F_v} \Phi_v\left(\left(\begin{smallmatrix} a & (a-1)x \\ 0 & 1\end{smallmatrix}\right)\right)\varphi_v^{(0,z)}\left(\left(\begin{smallmatrix} 1 & x_v \\ 0 & 1\end{smallmatrix}\right)\right)dx \\ &= \frac{\pi^{-\frac{1}{2}}}{4}\sum_{i=1}^{N}\mu_v^{(i)}(\mathfrak{s}_\infty)\left\{\mathcal{I}(s_{i,v}, z; a) + \mathcal{I}(s_{i,v}, -z; a)\right\} \qquad (6.9)\end{aligned}$$

where we set

$$\mathcal{I}(s, z; a) = \frac{\Gamma(\frac{s+1}{2})^2}{\Gamma(s+1)}\frac{\Gamma(-\frac{z}{2})}{\Gamma(\frac{-z+1}{4})^2}\int_{\mathbb{R}}\mathcal{F}(s, z, x; a)dx \qquad (6.10)$$

with

$$\begin{aligned}\mathcal{F}(s, z, x; a) = &\left\{\tfrac{4a}{(a+1)^2+(a-1)^2x^2}\right\}^{\frac{s+1}{2}}\left(\tfrac{1}{1+x^2}\right)^{\frac{z+1}{4}} \\ &\times {}_2F_1\left(\tfrac{s+1}{2}, \tfrac{s+1}{2}; s+1; \tfrac{4a}{(a+1)^2+(a-1)^2x^2}\right){}_2F_1\left(\tfrac{z+1}{4}, \tfrac{z+1}{4}; \tfrac{z}{2}+1; \tfrac{1}{1+x^2}\right)\end{aligned}$$

for $x \in \mathbb{R}$ and $s, z \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. Because $\mathrm{Re}(\frac{z}{2}+1) - \mathrm{Re}(\frac{z+1}{4}) - \mathrm{Re}(\frac{z+1}{4}) = \frac{1}{2} > 0$ and $a \neq 1$, We have the estimate

$$|\mathcal{F}(s, z, x; a)| \ll_{s,z,a} (1+|x|)^{-\mathrm{Re}(s) - \frac{\mathrm{Re}(z)+3}{2}}, \quad x \in \mathbb{R},$$

which shows the absolute convergence of the integral $\mathcal{I}(s, z; a)$ when $\mathrm{Re}(s) + \frac{\mathrm{Re}(z)+3}{2} > 1$. By the condition $\mathrm{Re}(s) > 1$, we may assume that $0 < \mathrm{Re}(z) < 1$ from now on. By substituting the uniform absolute convergence sum

$$\begin{aligned}\mathcal{F}(s, z, x; a) = &\frac{\Gamma(s+1)}{\Gamma(\frac{s+1}{2})^2}\frac{\Gamma(\frac{z}{2}+1)}{\Gamma(\frac{z+1}{4})^2} \\ &\times \sum_{m,n=0}^{\infty}\frac{\Gamma(\frac{s+1}{2}+m)^2}{m!\Gamma(s+1+m)}\frac{\Gamma(\frac{z+1}{4}+n)^2}{n!\Gamma(\frac{z}{2}+1+n)}\left\{\tfrac{4a}{(a+1)^2+(a-1)^2x^2}\right\}^{\frac{s+1}{2}+m}\left(\tfrac{1}{1+x^2}\right)^{\frac{z+1}{4}+n}\end{aligned}$$

into (6.10), we get

$$\begin{aligned}\mathcal{I}(s, z; a) = &\frac{\Gamma(\frac{z}{2}+1)\Gamma(-\frac{z}{2})}{\Gamma(\frac{z+1}{4})^2\Gamma(\frac{-z+1}{4})^2}\sum_{m,n=0}^{\infty}\frac{\Gamma(\frac{s+1}{2}+m)^2}{m!\Gamma(s+1+m)}\frac{\Gamma(\frac{z+1}{4}+n)^2}{n!\Gamma(\frac{z}{2}+1+n)}\left\{\tfrac{4a}{(a-1)^2}\right\}^{\frac{s+1}{2}+m} \\ &\times \int_{\mathbb{R}}\left\{x^2 + \left(\tfrac{a+1}{a-1}\right)^2\right\}^{-\frac{s+1}{2}-m}(x^2+1)^{-\frac{z+1}{4}-n}dx. \qquad (6.11)\end{aligned}$$

For $c, \alpha \in \mathbb{C}$ with $c \notin \sqrt{-1}\mathbb{R}$ and $\mathrm{Re}(\alpha) > \frac{1}{4}$, the function on $\mathbb{R}$ defined by $x \to (x^2+c^2)^{-\alpha}$ belongs to $\mathcal{L}^2(\mathbb{R})$. From ([12, p. 401]), its Fourier transform is given as follows:

$$\int_{\mathbb{R}}(x^2+c^2)^{-\alpha}e^{-2\pi\sqrt{-1}xy}dx = 2\left(\tfrac{\pi|y|}{c}\right)^{\alpha-\frac{1}{2}}\pi^{\frac{1}{2}}\Gamma(\alpha)^{-1}K_{\alpha-\frac{1}{2}}(2\pi c|y|). \qquad (6.12)$$

By means of the Plancherel theorem for $\mathcal{L}^2(\mathbb{R})$, the integral in (6.11) equals

$$\begin{aligned}
&\int_{\mathbb{R}} \left\{x^2+(\tfrac{a+1}{a-1})^2\right\}^{-\frac{s+1}{2}-m}\overline{(x^2+1)^{-\frac{\bar{z}+1}{4}-n}}dx\\
=&4\pi^{\frac{s+1}{2}+m}\overline{\pi^{\frac{\bar{z}+1}{4}}}\left|\tfrac{a-1}{a+1}\right|^{\frac{s}{2}+m}\Gamma(\tfrac{s+1}{2}+m)^{-1}\overline{\Gamma(\tfrac{\bar{z}+1}{4}+n)}^{-1}\\
&\qquad\times\int_{\mathbb{R}} K_{\frac{s}{2}+m}\left(2\pi\left|\tfrac{a+1}{a-1}\right||y|\right)\overline{K_{\frac{\bar{z}-1}{4}+n}(2\pi|y|)}|y|^{\frac{s}{2}+m}\overline{|y|^{\frac{\bar{z}+1}{4}+n-\frac{1}{2}}}dy\\
=&8\pi^{\frac{s}{2}+\frac{z+3}{4}+m+n}\left|\tfrac{a-1}{a+1}\right|^{\frac{s}{2}+m}\Gamma(\tfrac{s+1}{2}+m)^{-1}\Gamma(\tfrac{z+1}{4}+n)^{-1}\\
&\qquad\times\int_0^\infty K_{\frac{s}{2}+m}\left(2\pi\left|\tfrac{a+1}{a-1}\right|y\right)K_{\frac{z-1}{4}+n}(2\pi y)y^{\frac{s}{2}+\frac{z+3}{4}+m+n-1}dy. \qquad (6.13)
\end{aligned}$$

Substituting the above integral into (6.11) and changing variable $y \to \pi^{-1}y$, we have

$$\begin{aligned}
\mathcal{I}(s,z;a)=&\frac{8\Gamma(\frac{z}{2}+1)\Gamma(-\frac{z}{2})}{\Gamma(\frac{z+1}{4})^2\Gamma(\frac{-z+1}{4})^2}\left|\tfrac{a+1}{a-1}\right|^{\frac{1}{2}}\sum_{m,n=0}^{\infty}\frac{\Gamma(\frac{s+1}{2}+m)}{m!\Gamma(s+1+m)}\frac{\Gamma(\frac{z+1}{4}+n)}{n!\Gamma(\frac{z}{2}+1+n)}\left\{\tfrac{4a}{|a^2-1|}\right\}^{\frac{s+1}{2}+m}\\
&\qquad\times\int_0^\infty K_{\frac{s}{2}+m}\left(2\left|\tfrac{a+1}{a-1}\right|y\right)K_{\frac{z-1}{4}+n}(2y)y^{\frac{s}{2}+\frac{z+3}{4}+m+n-1}dy. \qquad (6.14)
\end{aligned}$$

By $K_\alpha(z)=\frac{1}{2}\int_0^\infty \exp(-\frac{z}{2}(u+u^{-1}))y^{-\alpha-1}du$, the integral in (6.14) equals

$$\begin{aligned}
&\int_0^\infty K_{\frac{s}{2}+m}\left(2\left|\tfrac{a+1}{a-1}\right|y\right)K_{\frac{z-1}{4}+n}(2y)y^{\frac{s}{2}+\frac{z+3}{4}+m+n-1}dy\\
=&\frac{1}{4}\int_0^\infty\int_0^\infty \exp(-\left|\tfrac{a+1}{a-1}\right|y(u+u^{-1}))u^{-\frac{s}{2}-m-1}du\\
&\qquad\times\int_0^\infty \exp(-y(v+v^{-1}))v^{-\frac{z+3}{4}-n}dv y^{\frac{s}{2}+\frac{z+3}{4}+m+n-1}dy\\
=&\frac{1}{4}\int_0^\infty\int_0^\infty\int_0^\infty \exp\left(-\left(\left|\tfrac{a+1}{a-1}\right|u^{-1}+v^{-1}\right)y^2\right)dy\,\exp(-\left|\tfrac{a+1}{a-1}\right|u)u^{\frac{s}{2}+m-1}du\\
&\qquad\times\exp(-v)v^{\frac{z-5}{4}+n}dv \qquad (\text{substituting } u\to yu^{-1}, v\to yv^{-1}).
\end{aligned}$$

Since $\int_0^\infty e^{-tx^2}dx=\frac{1}{2}\pi^{\frac{1}{2}}t^{-\frac{1}{2}}$ for $t>0$, the above integral becomes

$$\begin{aligned}
&\frac{\pi^{\frac{1}{2}}}{8}\int_0^\infty\int_0^\infty\left(\left|\tfrac{a+1}{a-1}\right|u^{-1}+v^{-1}\right)^{-\frac{1}{2}}\exp(-\left|\tfrac{a+1}{a-1}\right|u)u^{\frac{s}{2}+m-1}du\\
&\qquad\times\exp(-v)v^{\frac{z-5}{4}+n}dv\\
=&\frac{\pi^{\frac{1}{2}}}{8}\int_0^\infty\int_0^\infty\left(u+\left|\tfrac{a+1}{a-1}\right|v\right)^{-\frac{1}{2}}\exp(-\left|\tfrac{a+1}{a-1}\right|u)u^{\frac{s}{2}+m-\frac{1}{2}}du\\
&\qquad\times\exp(-v)v^{\frac{z-3}{4}+n}dv\\
=&\frac{\pi^{\frac{1}{2}}}{8}\left|\tfrac{a-1}{a+1}\right|^{\frac{z+1}{4}+n}\int_0^\infty\int_0^\infty(u+v)^{-\frac{1}{2}}\exp(-\left|\tfrac{a+1}{a-1}\right|u)u^{\frac{s}{2}+m-\frac{1}{2}}du\\
&\qquad\times\exp\left(-\left|\tfrac{a-1}{a+1}\right|v\right)v^{\frac{z-3}{4}+n}dv \qquad (\text{substituting } v\to\left|\tfrac{a-1}{a+1}\right|v).
\end{aligned}$$

Substituting this into (6.14) and changing the order of the integrals and the summation, we get

$$\begin{aligned}\mathcal{I}(s,z;a) = &\frac{\pi^{\frac{1}{2}}\Gamma(\frac{s+1}{2})\Gamma(-\frac{z}{2})}{\Gamma(s+1)\Gamma(\frac{z+1}{4})\Gamma(\frac{-z+1}{4})^2}\left(\tfrac{4a}{|a^2-1|}\right)^{\frac{s+1}{2}}\left|\tfrac{a-1}{a+1}\right|^{\frac{z-1}{4}} \\ &\times \int_0^\infty\int_0^\infty (u+v)^{-\frac{1}{2}}u^{\frac{s-1}{2}}\exp\left(-\left|\tfrac{a+1}{a-1}\right|u\right){}_1F_1\left(\tfrac{s+1}{2};s+1;\tfrac{4a}{|a^2-1|}u\right)du \\ &\qquad \times v^{\frac{z-3}{4}}\exp\left(-\left|\tfrac{a-1}{a+1}\right|v\right){}_1F_1\left(\tfrac{z+1}{4};\tfrac{z}{2}+1;\left|\tfrac{a-1}{a+1}\right|v\right)dv. \end{aligned} \tag{6.15}$$

This calculation is guaranteed as follows. The dominating function of the integrand of (6.15) can be obtained by replacing s and z by $\mathrm{Re}(s)$ and $\mathrm{Re}(z)$ respectively from Stirling's formula. Moreover, it holds that

$$\begin{aligned}&(u+v)^{-\frac{1}{2}}u^{\frac{\mathrm{Re}(s)-1}{2}}\exp\left(-\left|\tfrac{a+1}{a-1}\right|u\right){}_1F_1\left(\tfrac{\mathrm{Re}(s)+1}{2};\mathrm{Re}(s)+1;\tfrac{4a}{|a^2-1|}u\right) \\ &\qquad\times v^{\frac{\mathrm{Re}(z)-3}{4}}\exp\left(-\left|\tfrac{a-1}{a+1}\right|v\right){}_1F_1\left(\tfrac{\mathrm{Re}(z)+1}{4};\tfrac{\mathrm{Re}(z)}{2}+1;\left|\tfrac{a-1}{a+1}\right|v\right) \\ \le&2^{-\frac{1}{2}}u^{\frac{\mathrm{Re}(s)}{2}-\frac{3}{4}}\exp\left(-\left|\tfrac{a+1}{a-1}\right|u\right){}_1F_1\left(\tfrac{\mathrm{Re}(s)+1}{2};\mathrm{Re}(s)+1;\tfrac{4a}{|a^2-1|}u\right) \\ &\qquad\times v^{\frac{\mathrm{Re}(z)}{4}-1}\exp\left(-\left|\tfrac{a-1}{a+1}\right|v\right){}_1F_1\left(\tfrac{\mathrm{Re}(z)+1}{4};\tfrac{\mathrm{Re}(z)}{2}+1;\left|\tfrac{a-1}{a+1}\right|v\right),\end{aligned}$$

which is integrable on $\mathbb{R}_+^2$ from the assumptions $\mathrm{Re}(s)>1$ and $0<\mathrm{Re}(z)<1$ and the asymptotic formula

$$ {}_1F_1(a;c;z) = \frac{\Gamma(c)}{\Gamma(a)}e^z z^{a-c}\left\{1+O(z^{-1})\right\}, \qquad (\mathrm{Re}(z)\to+\infty) \tag{6.16}$$

stated in ([12, p. 289]). Hence the above formal computation is justified.

By the change of variable $v\to uv$, the integral in (6.15) equals

$$\begin{aligned}&\int_0^\infty\int_0^\infty (u+v)^{-\frac{1}{2}}u^{\frac{s-1}{2}}\exp\left(-\left|\tfrac{a+1}{a-1}\right|u\right){}_1F_1\left(\tfrac{s+1}{2};s+1;\tfrac{4a}{|a^2-1|}u\right)du \\ &\qquad\times v^{\frac{z-3}{4}}\exp\left(-\left|\tfrac{a-1}{a+1}\right|v\right){}_1F_1\left(\tfrac{z+1}{4};\tfrac{z}{2}+1;\left|\tfrac{a-1}{a+1}\right|v\right)dv \\ =&\int_0^\infty u^{\frac{s}{2}+\frac{z-3}{4}}\exp\left(-\left|\tfrac{a+1}{a-1}\right|u\right){}_1F_1\left(\tfrac{s+1}{2};s+1;\tfrac{4a}{|a^2-1|}u\right) \\ &\qquad\times\int_0^\infty (1+v)^{-\frac{1}{2}}v^{\frac{z-3}{4}}\exp\left(-\left|\tfrac{a-1}{a+1}\right|uv\right){}_1F_1\left(\tfrac{z+1}{4};\tfrac{z}{2}+1;\left|\tfrac{a-1}{a+1}\right|uv\right)dvdu.\end{aligned} \tag{6.17}$$

By using the integral formulas

$$\begin{aligned}{}_1F_1(a;c;z) &= \frac{\Gamma(c)e^z}{\Gamma(a)\Gamma(c-a)}\int_0^1 e^{-zt}t^{c-a-1}(1-t)^{a-1}dt, \quad (\mathrm{Re}(c)>\mathrm{Re}(a)>0), \\ U(a,c,z) &= \frac{1}{\Gamma(a)}\int_0^\infty e^{-zt}t^{a-1}(1+t)^{c-a-1}dt, \quad (\mathrm{Re}(a)>0, \mathrm{Re}(z)>0)\end{aligned}$$

from ([12, p. 275, 277]), we have the following computation:

$$\begin{aligned}
&\int_0^\infty (1+v)^{-\frac{1}{2}} v^{\frac{z-3}{4}} \exp\left(-\left|\tfrac{a-1}{a+1}\right| uv\right) {}_1F_1\left(\tfrac{z+1}{4}; \tfrac{z}{2}+1; \left|\tfrac{a-1}{a+1}\right| uv\right) dv \\
=&\frac{\Gamma(\frac{z}{2}+1)}{\Gamma(\frac{z+1}{4})\Gamma(\frac{z+3}{4})} \int_0^\infty (1+v)^{-\frac{1}{2}} v^{\frac{z-3}{4}} \int_0^1 \exp\left(-\left|\tfrac{a-1}{a+1}\right| uvt\right) t^{\frac{z-1}{4}} (1-t)^{\frac{z-3}{4}} dtdv \\
=&\frac{\Gamma(\frac{z}{2}+1)}{\Gamma(\frac{z+1}{4})\Gamma(\frac{z+3}{4})} \int_0^1 t^{\frac{z-1}{4}} (1-t)^{\frac{z-3}{4}} \int_0^\infty \exp\left(-\left|\tfrac{a-1}{a+1}\right| uvt\right) v^{\frac{z-3}{4}} (1+v)^{-\frac{1}{2}} dvdt \\
=&\frac{\Gamma(\frac{z}{2}+1)}{\Gamma(\frac{z+3}{4})} \int_0^1 t^{\frac{z-1}{4}} (1-t)^{\frac{z-3}{4}} U\left(\tfrac{z+1}{4}, \tfrac{z+3}{4}, \left|\tfrac{a-1}{a+1}\right| ut\right) dt. \qquad (6.18)
\end{aligned}$$

Here, we can easily check the absolute convergence of the integral in the second line which ensures the change of the order of the integrals. By (6.15), (6.17), and (6.18), the integral $\mathcal{I}(s,z;a)$ is equal to

$$\begin{aligned}
\mathcal{I}(s,z;a) = &\frac{\pi^{\frac{1}{2}}\Gamma(\frac{s+1}{2})\Gamma(\frac{z}{2}+1)\Gamma(-\frac{z}{2})}{\Gamma(s+1)\Gamma(\frac{z+1}{4})\Gamma(\frac{z+3}{4})\Gamma(\frac{-z+1}{4})^2} \left(\tfrac{4a}{|a^2-1|}\right)^{\frac{s+1}{2}} \left|\tfrac{a-1}{a+1}\right|^{\frac{z-1}{4}} \\
&\times \int_0^\infty u^{\frac{s}{2}+\frac{z-3}{4}} \exp\left(-\left|\tfrac{a+1}{a-1}\right| u\right) {}_1F_1\left(\tfrac{s+1}{2}; s+1; \tfrac{4a}{|a^2-1|}u\right) \\
&\times \int_0^1 t^{\frac{z-1}{4}} (1-t)^{\frac{z-3}{4}} U\left(\tfrac{z+1}{4}, \tfrac{z+3}{4}, \left|\tfrac{a-1}{a+1}\right| ut\right) dtdu. \qquad (6.19)
\end{aligned}$$

The identity (6.19) holds for $z \in \mathbb{C}$ such that $-1 < \mathrm{Re}(z) < 1$ by analytic continuation. By applying the equation

$$U(a,c,z) = \frac{\Gamma(1-c)}{\Gamma(1+a-c)} {}_1F_1(a;c;z) + \frac{\Gamma(c-1)}{\Gamma(a)} z^{1-c} {}_1F_1(1+a-c; 2-c; z) \qquad (6.20)$$

from ([12, p. 264]), (6.19) can be rewritten as

$$\begin{aligned}
\mathcal{I}(s,z;a) =&\frac{\Gamma(\frac{s+1}{2})\Gamma(\frac{z}{2}+1)\Gamma(-\frac{z}{2})}{\Gamma(s+1)\Gamma(\frac{z+1}{4})\Gamma(\frac{z+3}{4})\Gamma(\frac{-z+1}{4})} \left(\tfrac{4a}{|a^2-1|}\right)^{\frac{s+1}{2}} \left|\tfrac{a-1}{a+1}\right|^{\frac{z-1}{4}} \\
&\times \int_0^\infty u^{\frac{s}{2}+\frac{z-3}{4}} \exp\left(-\left|\tfrac{a+1}{a-1}\right| u\right) {}_1F_1\left(\tfrac{s+1}{2}; s+1; \tfrac{4a}{|a^2-1|}u\right) \\
&\times \int_0^1 t^{\frac{z-1}{4}} (1-t)^{\frac{z-3}{4}} {}_1F_1\left(\tfrac{z+1}{4}; \tfrac{z+3}{4}; \left|\tfrac{a-1}{a+1}\right| ut\right) dtdu \\
&+\frac{\pi^{\frac{1}{2}}\Gamma(\frac{s+1}{2})\Gamma(\frac{z}{2}+1)\Gamma(\frac{z-1}{4})\Gamma(-\frac{z}{2})}{\Gamma(s+1)\Gamma(\frac{z+1}{4})^2\Gamma(\frac{z+3}{4})\Gamma(\frac{-z+1}{4})^2} \left(\tfrac{4a}{|a^2-1|}\right)^{\frac{s+1}{2}} \\
&\times \int_0^\infty u^{\frac{s-1}{2}} \exp\left(-\left|\tfrac{a+1}{a-1}\right| u\right) {}_1F_1\left(\tfrac{s+1}{2}; s+1; \tfrac{4a}{|a^2-1|}u\right) \\
&\times \int_0^1 (1-t)^{\frac{z-3}{4}} {}_1F_1\left(\tfrac{1}{2}; \tfrac{-z+5}{4}; \left|\tfrac{a-1}{a+1}\right| ut\right) dtdu.
\end{aligned}$$

We denote the first term by $\mathcal{I}_1(s,z;a)$ and the second term by $\mathcal{I}_2(s,z;a)$. Then the absolute convergence of the integrals in $\mathcal{I}_1(s,z;a)$ and $\mathcal{I}_2(s,z;a)$ for $\mathrm{Re}(s) > 1$ and $-1 < \mathrm{Re}(z) < 1$ can by checked by using the asymptotic formula (6.16).

First, we compute the value of $\mathcal{I}_2(s,z;a)$. By ([4, p. 200, (95)]), the integral in $\mathcal{I}_2(s,z;a)$ with respect to t becomes

$$\int_0^1 (1-t)^{\frac{z-3}{4}} {}_1F_1\left(\tfrac{1}{2}; \tfrac{-z+5}{4}; \left|\tfrac{a-1}{a+1}\right| ut\right) dt = \frac{\Gamma(\frac{z+1}{4})}{\Gamma(\frac{z+5}{4})} {}_2F_2\left(1, \tfrac{1}{2}; \tfrac{z+5}{4}, \tfrac{-z+5}{4}; \left|\tfrac{a-1}{a+1}\right| u\right).$$

Hence we have

$$\begin{aligned}
\mathcal{I}_2(s,z;a) &= \frac{\pi^{\frac{1}{2}}\Gamma(\frac{s+1}{2})\Gamma(\frac{z}{2}+1)\Gamma(\frac{z-1}{4})\Gamma(-\frac{z}{2})}{\Gamma(s+1)\Gamma(\frac{z+1}{4})\Gamma(\frac{z+3}{4})\Gamma(\frac{z+5}{4})\Gamma(\frac{-z+1}{4})^2} \left(\tfrac{4a}{|a^2-1|}\right)^{\frac{s+1}{2}} \\
&\quad \times \int_0^\infty u^{\frac{s-1}{2}} \exp\left(-\left|\tfrac{a+1}{a-1}\right| u\right) {}_1F_1\left(\tfrac{s+1}{2}; s+1; \tfrac{4a}{|a^2-1|}u\right) {}_2F_2\left(1, \tfrac{1}{2}; \tfrac{z+5}{4}, \tfrac{-z+5}{4}; \left|\tfrac{a-1}{a+1}\right| u\right) du \\
&= \frac{8\pi^{\frac{1}{2}}\Gamma(\frac{s+1}{2})}{\Gamma(s+1)} \frac{z\Gamma(\frac{z}{2})\Gamma(-\frac{z}{2})}{(z^2-1)\Gamma(\frac{z+1}{4})^2\Gamma(\frac{-z+1}{4})^2} \left(\tfrac{4a}{|a^2-1|}\right)^{\frac{s+1}{2}} \\
&\quad \times \int_0^\infty u^{\frac{s-1}{2}} \exp\left(-\left|\tfrac{a+1}{a-1}\right| u\right) {}_1F_1\left(\tfrac{s+1}{2}; s+1; \tfrac{4a}{|a^2-1|}u\right) {}_2F_2\left(1, \tfrac{1}{2}; \tfrac{z+5}{4}, \tfrac{-z+5}{4}; \left|\tfrac{a-1}{a+1}\right| u\right) du
\end{aligned}$$

because $\Gamma(x+1) = x\Gamma(x)$. The above formula shows that

$$\mathcal{I}_2(s,z;a) + \mathcal{I}_2(s,-z;a) = 0 \tag{6.21}$$

on $-1 < \mathrm{Re}(z) < 1$ (the singularity at $z=0$ is removable).

Second, we calculate the value $\mathcal{I}_1(s,z;a)$. By ([4, p. 200, (94)]), the integral in $\mathcal{I}_1(s,z;a)$ with respect to t equals

$$\int_0^1 t^{\frac{z-1}{4}} (1-t)^{\frac{z-3}{4}} {}_1F_1\left(\tfrac{z+1}{4}; \tfrac{z+3}{4}, \left|\tfrac{a-1}{a+1}\right| ut\right) dt = \frac{\Gamma(\frac{z+1}{4})\Gamma(\frac{z+3}{4})}{\Gamma(\frac{z}{2}+1)} {}_1F_1\left(\tfrac{z+1}{4}; \tfrac{z}{2}+1; \left|\tfrac{a-1}{a+1}\right| u\right).$$

Hence we have

$$\begin{aligned}
\mathcal{I}_1(s,z;a) =& \frac{\Gamma(\frac{s+1}{2})\Gamma(-\frac{z}{2})}{\Gamma(s+1)\Gamma(\frac{-z+1}{4})} \left(\tfrac{4a}{|a^2-1|}\right)^{\frac{s+1}{2}} \left|\tfrac{a-1}{a+1}\right|^{\frac{z-1}{4}} \\
&\times \int_0^\infty u^{\frac{s}{2}+\frac{z-3}{4}} \exp\left(-\left|\tfrac{a+1}{a-1}\right| u\right) {}_1F_1\left(\tfrac{s+1}{2}; s+1; \tfrac{4a}{|a^2-1|}u\right) {}_1F_1\left(\tfrac{z+1}{4}; \tfrac{z}{2}+1; \left|\tfrac{a-1}{a+1}\right| u\right) du \\
=& \frac{\Gamma(\frac{s+1}{2})\Gamma(-\frac{z}{2})}{\Gamma(s+1)\Gamma(\frac{-z+1}{4})} \left\{\tfrac{4a}{(a+1)^2}\right\}^{\frac{s+1}{2}} \left|\tfrac{a-1}{a+1}\right|^{\frac{z-1}{2}} \\
&\times \int_0^\infty u^{\frac{s}{2}+\frac{z-3}{4}} \exp(-u) {}_1F_1\left(\tfrac{s+1}{2}; s+1; \tfrac{4a}{(a+1)^2}u\right) {}_1F_1\left(\tfrac{z+1}{4}; \tfrac{z}{2}+1; (\tfrac{a-1}{a+1})^2 u\right) du
\end{aligned} \tag{6.22}$$

by changing the variable $u \to \left|\frac{a-1}{a+1}\right| u$. Let $z \in \mathbb{C}$ such that $z \neq 0$ and $-1 < \mathrm{Re}(z) < 1$. Then, from (6.20), (6.21), and (6.22), we get

$$\begin{aligned}
&\mathcal{I}(s,z;a)+\mathcal{I}(s,-z;a)\\
=&\mathcal{I}_1(s,z;a)+\mathcal{I}_1(s,-z;a)\\
=&\frac{\Gamma(\frac{s+1}{2})}{\Gamma(s+1)}\left\{\tfrac{4a}{(a+1)^2}\right\}^{\frac{s+1}{2}}\left|\tfrac{a-1}{a+1}\right|^{\frac{z-1}{2}}\int_0^\infty u^{s+\frac{z-5}{4}}\exp\left(-u\right){}_1F_1\left(\tfrac{s+1}{2};s+1;\tfrac{4a}{(a+1)^2}u\right)\\
&\times\left[\frac{\Gamma(-\frac{z}{2})}{\Gamma(\frac{-z+1}{4})}{}_1F_1(\tfrac{z+1}{4};\tfrac{z}{2}+1;(\tfrac{a-1}{a+1})^2u)+\left\{(\tfrac{a-1}{a+1})^2u\right\}^{-\frac{z}{2}}\frac{\Gamma(\frac{z}{2})}{\Gamma(\frac{z+1}{4})}{}_1F_1(\tfrac{-z+1}{4};-\tfrac{z}{2}+1;(\tfrac{a-1}{a+1})^2u)\right]du\\
=&\frac{\Gamma(\frac{s+1}{2})}{\Gamma(s+1)}\left\{\tfrac{4a}{(a+1)^2}\right\}^{\frac{s+1}{2}}\left|\tfrac{a-1}{a+1}\right|^{\frac{z-1}{2}}\\
&\times\int_0^\infty u^{\frac{s}{2}+\frac{z-3}{4}}\exp\left(-u\right){}_1F_1\left(\tfrac{s+1}{2};s+1;\tfrac{4a}{(a+1)^2}u\right)U(\tfrac{z+1}{4},\tfrac{z}{2}+1,(\tfrac{a-1}{a+1})^2u)du. \qquad (6.23)
\end{aligned}$$

From this equation, the singularity at $z=0$ is removable. By applying the integral formula

$$U(a,c,z)=\frac{e^z}{\Gamma(a)}\int_1^\infty e^{-zt}(t-1)^{a-1}t^{c-a-1}dt$$

from ([12, p. 277]) to the integral in (6.23), we obtain

$$\begin{aligned}
&\int_0^\infty u^{\frac{s}{2}+\frac{z-3}{4}}\exp\left(-u\right){}_1F_1\left(\tfrac{s+1}{2};s+1;\tfrac{4a}{(a+1)^2}u\right)U(\tfrac{z+1}{4},\tfrac{z}{2}+1,(\tfrac{a-1}{a+1})^2u)du\\
=&\Gamma(\tfrac{z+1}{4})^{-1}\int_0^\infty u^{\frac{s}{2}+\frac{z-3}{4}}\exp\left(-\tfrac{4a}{(a+1)^2}u\right){}_1F_1\left(\tfrac{s+1}{2};s+1;\tfrac{4a}{(a+1)^2}u\right)\\
&\qquad\times\int_1^\infty\exp(-(\tfrac{a-1}{a+1})^2ut)(t-1)^{\frac{z-3}{4}}t^{\frac{z-1}{4}}dtdu\\
=&\Gamma(\tfrac{z+1}{4})^{-1}\int_1^\infty(t-1)^{\frac{z-3}{4}}t^{\frac{z-1}{4}}\int_0^\infty u^{\frac{s}{2}+\frac{z-3}{4}}{}_1F_1\left(\tfrac{s+1}{2};s+1;\tfrac{4a}{(a+1)^2}u\right)\\
&\qquad\times\exp\left(-\left\{\tfrac{4a}{(a+1)^2}+(\tfrac{a-1}{a+1})^2t\right\}u\right)dudt \qquad (\because (6.16))\\
=&\frac{\Gamma(\frac{s}{2}+\frac{z+1}{4})}{\Gamma(\frac{z+1}{4})}\int_1^\infty(t-1)^{\frac{z-3}{4}}t^{\frac{z-1}{4}}\left\{\tfrac{4a}{(a+1)^2}+(\tfrac{a-1}{a+1})^2t\right\}^{-\frac{s}{2}-\frac{z+1}{4}}\\
&\qquad\times{}_2F_1\left(\tfrac{s+1}{2},\tfrac{s}{2}+\tfrac{z+1}{4};s+1;\frac{\frac{4a}{(a+1)^2}}{\frac{4a}{(a+1)^2}+(\frac{a-1}{a+1})^2t}\right)dudt. \qquad (6.24)
\end{aligned}$$

In the above computation, we use the integral formula

$$\int_0^\infty t^{\lambda-1}e^{-zt}{}_1F_1(a;c;kt)dt=\Gamma(\lambda)z^{-\lambda}{}_2F_1(a,\lambda;c;\tfrac{k}{z}),$$
$$(\mathrm{Re}(z)>\mathrm{Re}(k),|z|>|k|,\mathrm{Re}(\lambda)>0)$$

from ([12, p. 278]). Since $0<\frac{\frac{4a}{(a+1)^2}}{\frac{4a}{(a+1)^2}+(\frac{a-1}{a+1})^2t}\leq\frac{4a}{(a+1)^2}<1$ for $t\geq 1$, (6.24) equals the following termwise integral:

$$\frac{\Gamma(s+1)}{\Gamma(\frac{s+1}{2})\Gamma(\frac{z+1}{4})}\sum_{m=0}^{\infty}\frac{\Gamma(\frac{s+1}{2}+m)\Gamma(\frac{s}{2}+\frac{z+1}{4}+m)}{m!\Gamma(s+1+m)}$$
$$\times\int_1^\infty t^{\frac{z-1}{4}}(t-1)^{\frac{z-3}{4}}\left\{\tfrac{4a}{(a+1)^2}+(\tfrac{a-1}{a+1})^2t\right\}^{-\frac{s}{2}-\frac{z+1}{4}}\left\{\frac{\frac{4a}{(a+1)^2}}{\frac{4a}{(a+1)^2}+(\frac{a-1}{a+1})^2t}\right\}^m dt$$
$$=\frac{\Gamma(s+1)}{\Gamma(\frac{s+1}{2})\Gamma(\frac{z+1}{4})}\sum_{m=0}^{\infty}\frac{\Gamma(\frac{s+1}{2}+m)\Gamma(\frac{s}{2}+\frac{z+1}{4}+m)}{m!\Gamma(s+1+m)}\left\{\tfrac{4a}{(a+1)^2}\right\}^m\left|\tfrac{a+1}{a-1}\right|^{s+\frac{z+1}{2}+2m}$$
$$\times\int_1^\infty t^{\frac{z-1}{4}}(t-1)^{\frac{z-3}{4}}\left\{\tfrac{4a}{(a-1)^2}+t\right\}^{-\frac{s}{2}-\frac{z+1}{4}-m}dt$$
$$=\frac{\Gamma(s+1)}{\Gamma(\frac{s+1}{2})\Gamma(\frac{z+1}{4})}\sum_{m=0}^{\infty}\frac{\Gamma(\frac{s+1}{2}+m)\Gamma(\frac{s}{2}+\frac{z+1}{4}+m)}{m!\Gamma(s+1+m)}\left\{\tfrac{4a}{(a+1)^2}\right\}^m\left|\tfrac{a+1}{a-1}\right|^{s+\frac{z+1}{2}+2m}$$
$$\times\int_0^1 t^{\frac{s}{2}-\frac{z-1}{4}+m-1}(1-t)^{\frac{z-3}{4}}\left\{1+\tfrac{4a}{(a-1)^2}t\right\}^{-\frac{s}{2}-\frac{z+1}{4}-m}dt\qquad(\text{substituting } t\to t^{-1})$$
$$=\frac{\Gamma(s+1)}{\Gamma(\frac{s+1}{2})}\sum_{m=0}^{\infty}\frac{\Gamma(\frac{s}{2}-\frac{z-1}{4}+m)\Gamma(\frac{s}{2}+\frac{z+1}{4}+m)}{m!\Gamma(s+1+m)}\left\{\tfrac{4a}{(a+1)^2}\right\}^m\left|\tfrac{a+1}{a-1}\right|^{s+\frac{z+1}{2}+2m}$$
$$\times{}_2F_1\left(\tfrac{s}{2}+\tfrac{z+1}{4}+m,\tfrac{s}{2}-\tfrac{z-1}{4}+m;\tfrac{s+1}{2}+m;-\tfrac{4a}{(a-1)^2}\right)$$
$$=\frac{\Gamma(s+1)}{\Gamma(\frac{s+1}{2})}\sum_{m=0}^{\infty}\frac{\Gamma(\frac{s}{2}-\frac{z-1}{4}+m)\Gamma(\frac{s}{2}+\frac{z+1}{4}+m)}{m!\Gamma(s+1+m)}\left\{\tfrac{4a}{(a+1)^2}\right\}^m\left|\tfrac{a+1}{a-1}\right|^{\frac{z+1}{2}}$$
$$\times{}_2F_1\left(\tfrac{z+1}{4},\tfrac{-z+1}{4};\tfrac{s+1}{2}+m;-\tfrac{4a}{(a-1)^2}\right)\tag{6.25}$$

by means of the integral formula (5.3) and the transformation formula ([12, (2.4.1), 1st line]). Therefore, due to (6.23), (6.24), and (6.25), we have

$$\mathcal{I}(s,z;a)+\mathcal{I}(s,-z;a)$$
$$=\left|\tfrac{a+1}{a-1}\right|\left\{\tfrac{4a}{(a+1)^2}\right\}^{\frac{s+1}{2}}\sum_{m=0}^{\infty}\frac{\Gamma(\frac{s}{2}-\frac{z-1}{4}+m)\Gamma(\frac{s}{2}+\frac{z+1}{4}+m)}{m!\Gamma(s+1+m)}\left\{\tfrac{4a}{(a+1)^2}\right\}^m$$
$$\times{}_2F_1\left(\tfrac{z+1}{4},\tfrac{-z+1}{4};\tfrac{s+1}{2}+m;-\tfrac{4a}{(a-1)^2}\right)$$
$$=\left|\tfrac{a+1}{a-1}\right|\left\{\tfrac{4a}{(a+1)^2}\right\}^{\frac{s+1}{2}}\frac{\Gamma(\frac{s}{2}+\frac{z+1}{4})\Gamma(\frac{s}{2}-\frac{z-1}{4})}{\Gamma(s+1)}$$
$$\times F_3^{(1,0)}\left(\substack{\frac{s+1}{2},\frac{s}{2}+\frac{z+1}{4},\frac{s}{2}-\frac{z-1}{4};\frac{z+1}{4},\frac{-z+1}{4}\\ s+1;\frac{s+1}{2}};\tfrac{4a}{(a+1)^2},-\tfrac{4a}{(a-1)^2}\right)$$
$$=4\pi^{\frac{1}{2}}\mathcal{O}_v^{+,(z)}\left(s;\frac{a+1}{a-1}\right)\tag{6.26}$$

from Lemma 5.5. This equation holds for all $z\in\mathbb{C}$ such that $|\mathrm{Re}(z)|<2\mathrm{Re}(s)+1$ by analytic continuation. Hence, the claim follows by (6.9) and (6.26). □

Next we discuss the convergence property of the sum $\sum_{a\in F^\times-\{1\}}\mathfrak{F}^{(z)}(a)$.

Proposition 6.4. *Suppose* $\mathrm{Re}(s_v)>2$ *for all* $v\in S$. *Let* $0<\sigma<1$. *The series*

$$\sum_{a\in F^\times-\{1\}}\mathfrak{F}^{(z)}(a)\tag{6.27}$$

converges absolutely and uniformly for $|\mathrm{Re}(z)| \le \sigma$*, locally uniformly in* s_v $(v \in S)$*, and uniformly with restect to the imaginary parts of* $s_{i,v}$ $(1 \le i \le N, v \in \Sigma_\infty)$*. Moreover, the series (6.27) defines a holomorphic function on the region*

$$|\mathrm{Re}(z)| < \min\{1\} \cup \{\mathrm{Re}(s_v) - 2 | v \in S\}.$$

The estimation of the upper bounds of the local integrals $\mathfrak{F}_v^{(z)}(a)$ for $v \in \Sigma_{\mathrm{fin}}$ have been completed in ([16, §6]). Proposition 6.4 can be proved by simulating their method if we get a "good" upper bound of $\mathfrak{F}_v^{(z)}(a)$ for $v \in \Sigma_\infty$. The following Lemmas enables us to use the same way as the proof of ([16, Proposition 6.12]).

Lemma 6.5. *Let* $\alpha, \beta, \gamma \in \mathbb{C}$ *with* $\mathrm{Re}(\gamma) > \mathrm{Re}(\beta) > 0$*. Then we have*

$$|{}_2F_1(\alpha, \beta; \gamma; u)| \ll \left|\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\right| \frac{\Gamma(\mathrm{Re}(\gamma-\beta))}{\Gamma(\mathrm{Re}(\alpha))} \sum_{n=0}^{\infty} (n+1)^{\mathrm{Re}(\alpha+\beta-\gamma)-1} u^n \qquad (0 \le u < 1).$$

with the implied constant independent of the imaginary parts of α*,* β*, and* γ*.*

Proof. From the integral formula (5.3), we have

$$\begin{aligned}
&|{}_2F_1(\alpha, \beta; \gamma; u)| \\
\le& \left|\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\right| \int_0^1 t^{\mathrm{Re}(\beta)-1}(1-t)^{\mathrm{Re}(\gamma-\beta)-1}(1-ut)^{-\mathrm{Re}(\alpha)} dt \\
=& \left|\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\right| \frac{\Gamma(\mathrm{Re}(\beta))\Gamma(\mathrm{Re}(\gamma-\beta))}{\Gamma(\mathrm{Re}(\gamma))} {}_2F_1(\mathrm{Re}(\alpha), \mathrm{Re}(\beta); \mathrm{Re}(\gamma); u) \\
=& \left|\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\right| \frac{\Gamma(\mathrm{Re}(\gamma-\beta))}{\Gamma(\mathrm{Re}(\alpha))} \sum_{n=0}^{\infty} \frac{\Gamma(\mathrm{Re}(\alpha)+n)\Gamma(\mathrm{Re}(\beta)+n)}{n!\Gamma(\mathrm{Re}(\gamma)+n)} u^n \qquad (0 \le u < 1).
\end{aligned}$$

By means of Stirling's formula, we have

$$\frac{\Gamma(\mathrm{Re}(\alpha)+n)\Gamma(\mathrm{Re}(\beta)+n)}{n!\Gamma(\mathrm{Re}(\gamma)+n)} \ll (n+1)^{\mathrm{Re}(\alpha+\beta-\gamma)-1} \qquad (n \in \mathbb{Z}_{\ge 0})$$

with the implied constant independent of the imaginary parts of α, β, and γ. We complete the proof. □

Lemma 6.6. *Let* $v \in \Sigma_\infty$*,* $s \in \mathbb{C}$ *with* $\mathrm{Re}(s) > 1$*, and* $0 < \sigma < 1$*. Then we have*

$$\begin{aligned}
\left|\mathcal{O}_v^{+,(z)}\left(s; \frac{a+1}{a-1}\right)\right| \ll_\sigma& \delta(a>0) \left|\Gamma(\tfrac{z+1}{4})\Gamma(\tfrac{-z+1}{4})\right|^{-1} \\
&\times (1+a)^{\frac{-\mathrm{Re}(s)+|\mathrm{Re}(z)|}{2}} |a-1|_v^{-\frac{|\mathrm{Re}(z)|+1}{2}} \left\{\log\left(3 + \frac{1}{|a-1|_v}\right)\right\}^2 \qquad (6.28)
\end{aligned}$$

uniformly for $a \in F_v^\times - \{1\}$ *and* $|\mathrm{Re}(z)| \le \sigma$*. Moreover, the implied constant is independent of the imaginary part of* s*.*

Proof. Let $a > 0$ and $z \in \mathbb{C}$ with $a \ne 1$ and $|\mathrm{Re}(z)| \le \sigma$. By the transformation formula ([12, (2.4.1), 4th line]), we have

$$\begin{aligned}
(1+x^2)^{-\frac{z+1}{4}} {}_2F_1(\tfrac{z+1}{4}, \tfrac{z+1}{4}; \tfrac{1}{2}; \tfrac{x^2}{1+x^2}) =& (1+x^2)^{-\frac{z+1}{4}} \frac{\Gamma(\frac{1}{2})\Gamma(-\frac{z}{2})}{\Gamma(\frac{-z+1}{4})^2} {}_2F_1(\tfrac{z+1}{4}, \tfrac{z+1}{4}; \tfrac{z}{2}+1; \tfrac{1}{1+x^2}) \\
&+ (1+x^2)^{\frac{z-1}{4}} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{z}{2})}{\Gamma(\frac{z+1}{4})^2} {}_2F_1(\tfrac{-z+1}{4}, \tfrac{-z+1}{4}; -\tfrac{z}{2}+1; \tfrac{1}{1+x^2}) \\
=& \varphi_v^{(0,z)}\left(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\right).
\end{aligned}$$

Hence, by (6.9) and (6.26), we get

$$\mathcal{O}_v^{+,(z)}\left(s;\frac{a+1}{a-1}\right)=\frac{1}{4\pi}\frac{\Gamma(\frac{s+1}{2})^2}{\Gamma(s+1)}\int_{\mathbb{R}}\left\{\tfrac{4a}{(a+1)^2+(a-1)^2x^2}\right\}^{\frac{s+1}{2}}(\tfrac{1}{1+x^2})^{\frac{z+1}{4}} \times {}_2F_1\left(\tfrac{s+1}{2},\tfrac{s+1}{2};s+1;\tfrac{4a}{(a+1)^2+(a-1)^2x^2}\right){}_2F_1\left(\tfrac{z+1}{4},\tfrac{z+1}{4};\tfrac{1}{2};\tfrac{x^2}{1+x^2}\right)dx. \tag{6.29}$$

By Lemma 6.5 and noting $u^{-1}\log\frac{1}{1-u}\ll\log(1+\frac{1}{1-u})$ $(0\le u<1)$, we have the estimate

$$\begin{aligned}
&\left|{}_2F_1\left(\tfrac{s+1}{2},\tfrac{s+1}{2};s+1;\tfrac{4a}{(a+1)^2+(a-1)^2x^2}\right)\right|\\
\ll&\left|\frac{\Gamma(s+1)}{\Gamma(\frac{s+1}{2})^2}\right|\sum_{n=0}^{\infty}(n+1)^{-1}\left\{\tfrac{4a}{(a+1)^2+(a-1)^2x^2}\right\}^n\\
=&\left|\frac{\Gamma(s+1)}{\Gamma(\frac{s+1}{2})^2}\right|\left\{\tfrac{4a}{(a+1)^2+(a-1)^2x^2}\right\}^{-1}\log\left(\frac{1}{1-\frac{4a}{(a+1)^2+(a-1)^2x^2}}\right)\\
\ll&\left|\frac{\Gamma(s+1)}{\Gamma(\frac{s+1}{2})^2}\right|\log\left(1+\frac{1}{1-\frac{4a}{(a+1)^2+(a-1)^2x^2}}\right)\\
\le&\left|\frac{\Gamma(s+1)}{\Gamma(\frac{s+1}{2})^2}\right|\log\left(1+\frac{1}{1-\frac{4a}{(a+1)^2}}\right)\\
\ll&\left|\frac{\Gamma(s+1)}{\Gamma(\frac{s+1}{2})^2}\right|\log\left(3+\frac{1}{|a-1|}\right).
\end{aligned}$$

Moreover, if $-\sigma\le\mathrm{Re}(z)\le 0$, we have

$$\begin{aligned}
&\left|{}_2F_1\left(\tfrac{z+1}{4},\tfrac{z+1}{4};\tfrac{1}{2};\tfrac{x^2}{1+x^2}\right)\right|\\
\ll&\left|\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{z+1}{4})\Gamma(\frac{-z+1}{4})}\right|\frac{\Gamma(\frac{-\mathrm{Re}(z)+1}{4})}{\Gamma(\frac{\mathrm{Re}(z)+1}{4})}\sum_{n=0}^{\infty}(n+1)^{\frac{\mathrm{Re}(z)}{2}-1}(\tfrac{x^2}{1+x^2})^2\\
\le&\left|\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{z+1}{4})\Gamma(\frac{-z+1}{4})}\right|\frac{\Gamma(\frac{1+\sigma}{4})}{\Gamma(\frac{1-\sigma}{4})}\sum_{n=0}^{\infty}(n+1)^{-1}(\tfrac{x^2}{1+x^2})^2\\
\ll_\sigma&\left|\Gamma(\tfrac{z+1}{4})\Gamma(\tfrac{-z+1}{4})\right|^{-1}(\tfrac{x^2}{1+x^2})^{-1}\log\left(\frac{1}{1-\frac{x^2}{1+x^2}}\right)\\
\ll_\sigma&\left|\Gamma(\tfrac{z+1}{4})\Gamma(\tfrac{-z+1}{4})\right|^{-1}\log\left(1+\frac{1}{1-\frac{x^2}{1+x^2}}\right)\\
\ll_\sigma&\left|\Gamma(\tfrac{z+1}{4})\Gamma(\tfrac{-z+1}{4})\right|^{-1}\log(3+|x|)
\end{aligned}$$

for $a>0$ and $x\in\mathbb{R}$. From these estimates and (6.29), we get

$$\left|\mathcal{O}_v^{+,(z)}\left(s;\frac{a+1}{a-1}\right)\right|\ll_\sigma\left|\Gamma(\tfrac{z+1}{4})\Gamma(\tfrac{-z+1}{4})\right|^{-1}\log\left(3+\frac{1}{|a-1|}\right)\times\int_{\mathbb{R}}\left\{\tfrac{4a}{(a+1)^2+(a-1)^2x^2}\right\}^{\frac{\mathrm{Re}(s)+1}{2}}(\tfrac{1}{1+x^2})^{\frac{\mathrm{Re}(z)+1}{4}}\log(3+|x|)\,dx.$$

By the change of variable $x \to |\frac{a+1}{a-1}|x$, the following estimate holds.

$$\begin{aligned}
&\left|\mathcal{O}_v^{+,(z)}\left(s; \frac{a+1}{a-1}\right)\right| \\
&\ll_\sigma \left|\Gamma(\tfrac{z+1}{4})\Gamma(\tfrac{-z+1}{4})\right|^{-1} \left|\frac{a-1}{a+1}\right|^{\frac{-1+\mathrm{Re}(z)}{2}} \left\{\frac{4a}{(a+1)^2}\right\}^{\frac{\mathrm{Re}(s)+1}{2}} \log\left(3+\frac{1}{|a-1|}\right) \\
&\quad \times \int_{\mathbb{R}} \left(\tfrac{1}{1+x^2}\right)^{\frac{\mathrm{Re}(s)+1}{2}} \left(\frac{1}{\frac{(a-1)^2}{(a+1)^2}+x^2}\right)^{\frac{\mathrm{Re}(z)+1}{4}} \log\left(3+|\tfrac{a+1}{a-1}||x|\right) dx \\
&\ll_\sigma \left|\Gamma(\tfrac{z+1}{4})\Gamma(\tfrac{-z+1}{4})\right|^{-1} \left|\frac{a-1}{a+1}\right|^{\frac{-1+\mathrm{Re}(z)}{2}} \left\{\frac{4a}{(a+1)^2}\right\}^{\frac{\mathrm{Re}(s)+1}{2}} \left\{\log\left(3+\frac{1}{|a-1|}\right)\right\}^2 \\
&\quad \times \int_{\mathbb{R}} \left(\tfrac{1}{1+x^2}\right)^{\frac{\mathrm{Re}(s)+1}{2}} x^{-\frac{\mathrm{Re}(z)+1}{2}} \log\left(3+|x|\right) dx \\
&\ll_\sigma \left|\Gamma(\tfrac{z+1}{4})\Gamma(\tfrac{-z+1}{4})\right|^{-1} (1+|a|)^{\frac{-\mathrm{Re}(s)-\mathrm{Re}(z)}{2}} |a-1|^{\frac{\mathrm{Re}(z)-1}{2}} \left\{\log\left(3+\frac{1}{|a-1|}\right)\right\}^2.
\end{aligned}$$

By the symmetry of $\mathcal{O}_v^{+,(z)}\left(s; \frac{a+1}{a-1}\right)$ between z and $-z$, the statement holds. □

By Proposition 6.4, we can change the order of integrals in (6.1) as

$$\mathbb{J}_{\mathrm{hyp}}(\beta) = \frac{1}{2}\int_{L_\sigma} \beta(z)\zeta_F(\tfrac{z+1}{2})\zeta_F(\tfrac{-z+1}{2}) \left\{\sum_{a\in F^\times-\{1\}} \mathfrak{F}^{(z)}(a)\right\} dz.$$

If we put

$$\hat{J}^0_{\mathrm{hyp}}(\mathbf{s}_S, \mathfrak{s}_\infty, z) = \frac{1}{2}\zeta_F(\tfrac{z+1}{2})\zeta_F(\tfrac{-z+1}{2}) \left\{\sum_{a\in F^\times-\{1\}} \mathfrak{F}^{(z)}(a)\right\}, \tag{6.30}$$

then this is the desired function in Theorem 5.1 for the hyperbolic term. In fact, we can check that the upper bounds for the local integrals investigated in ([16, §6]) for $v \in \Sigma_{\mathrm{fin}}$ are independent of the imaginary part of $z \in \mathbb{C}$. Moreover, the upper bound of (6.28) depends only on the real part of $z \in \mathbb{C}$ and the gamma factors. Hence $\hat{J}^0_{\mathrm{hyp}}(\mathbf{s}_S, \mathfrak{s}_\infty, z)$ is vertically of moderate growth from (1.2) and Stirling's formula.

7 The F-elliptic term

In this section, we calculate the integral $\mathbb{J}_{\mathrm{ell}}(\beta)$. We recall the assumption that the the prime 2 splits completely in $F/\mathbb{Q}$, i.e. $F_v \cong \mathbb{Q}_2$ for $v \in \Sigma_{\mathrm{dyadic}}$ where Σ_{dyadic} denotes the set of all $v \in \Sigma_{\mathrm{fin}}$ such that $|2|_v < 1$.

7.1 Preliminaries for computation

We recall the notations of the parametrization of elliptic elements defined in ([16, §7.1]). Set $Q_F = \{(t,n) \in F^2 | t^2 - 4n \neq 0\}$. Two elements $(t,n), (t',n') \in Q_F$ are said to be F-equivalent if there exists $c \in F^\times$ such that $(t', n') = (ct, c^2 n)$. The F-equivalence class of an element $(t, n) \in Q_F$ is denoted by $(t : n)_F$. The quotient set of Q_F by the F-equivalence relation is denoted by $\mathcal{Q}_F$. Set

$$\mathcal{Q}_F^{\mathrm{Irr}} = \{(t : n)_F \in \mathcal{Q}_F | t^2 - 4n \notin (F^\times)^2\}.$$

For $\widetilde{\gamma} \in \mathcal{Q}_F^{\mathrm{Irr}}$, we fix its representative $(t, n) \in Q_F$ once and for all and set $\gamma = \begin{pmatrix} \frac{t}{2} & 1 \\ \frac{\Delta}{4} & \frac{t}{2} \end{pmatrix} \in G_F$ and $\Delta = t^2 - 4n$. Then, the G_F-conjugacy class with the characteristic polynomial $X^2 - tX + n$ is represented by the element γ. Let E be a quadratic extension of F such that $E \cong F[x]/(x^2 - \frac{\Delta}{4})$ and $\iota_\Delta : E \hookrightarrow \mathrm{M}_2(F)$ be the F-algebra embedding given by

$$\iota_\Delta(a + \tfrac{b}{2}\sqrt{\Delta}) = \begin{pmatrix} a & b \\ \frac{b}{4}\Delta & a \end{pmatrix}, \quad a, b \in F.$$

Then the centralizer of γ in G_F is $G_{\gamma,F} = \iota_\Delta(E^\times)$.

For $v \in \Sigma_F$, the element $4^{-1}\Delta$ can be written in F_v as $4^{-1}\Delta = \Delta_v^0 m_v^2$ where $m_v \in F_v^\times$ and $\Delta_v^0 \in F^\times - (F^\times)^2$. We may assume that $\Delta_v^0 \in (\mathfrak{p}_v - \mathfrak{p}_v^2) \cup \{1\} \cup (\mathfrak{o}_v^\times - (\mathfrak{o}_v^\times)^2)$ for $v \in \Sigma_{\mathrm{fin}}$ and $\Delta_v^0 \in \{1, -1\}$ for $v \in \Sigma_\infty$. We fix such a factorization of $4^{-1}\Delta$ in $F_v^\times$. Since the prime 2 splits completely in F, we have $\mathfrak{o}_v^\times/(\mathfrak{o}_v^\times)^2 \cong \mathbb{Z}_2^\times/(1 + 8\mathbb{Z}_2) = \{\pm 5, \pm 1\}$, thus we may suppose $\Delta_v^0 \in \{\pm 5, \pm 1, \pm 10, \pm 2\}$ for all $v \in \Sigma_{\mathrm{dyadic}}$.

We have the direct sum decomposition $E_v = F_v + \sqrt{\Delta_v^0} F_v$ as vector spaces, which determine an F_v-algebra embedding $\iota_{\Delta_v^0} : E_v \hookrightarrow \mathrm{M}_2(F_v)$ as

$$\iota_{\Delta_v^0}(a + b\sqrt{\Delta_v^0}) = \begin{pmatrix} a & b \\ b\Delta_v^0 & a \end{pmatrix}, \quad a, b \in F_v, \text{ if } \Delta_v^0 \neq 1,$$

$$\iota_{\Delta_v^0}(a + b\sqrt{\Delta_v^0}) = \begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix}, \quad a, b \in F_v, \text{ if } \Delta_v^0 = 1.$$

We set $\mathfrak{T}_{\Delta,v} = \iota_{\Delta_v^0}(E_v^\times)$ and $R_{\Delta,v} \in G_v$ as

$$R_{\Delta,v} = \begin{cases} \begin{pmatrix} m_v & 0 \\ 0 & 1 \end{pmatrix} & \text{if } \Delta_v^0 \neq 1 \\ \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} m_v & 0 \\ 0 & 1 \end{pmatrix} & \text{if } \Delta_v^0 = 1. \end{cases}$$

Then we have the relation

$$R_{\Delta,v}^{-1} \mathfrak{T}_{\Delta,v} R_{\Delta,v} = G_{\gamma,v}.$$

Since $\Delta \in \mathfrak{o}_v^\times$ for almost all $v \in \Sigma_{\mathrm{fin}}$, the system $R_\Delta = (R_{\Delta,v})_{v\in\Sigma_F}$ belongs to $G_\mathbb{A}$. We define a closed subgroup $\mathfrak{T}_\Delta$ of $G_\mathbb{A}$ by

$$\mathfrak{T}_\Delta = \{h = (h_v)_{v\in\Sigma_F} \in G_\mathbb{A} | h_v \in \mathfrak{T}_\Delta \cap \mathbf{K}_v \text{ for almost all } v \in \Sigma_{\mathrm{fin}}\}.$$

Then, we have $R_\Delta^{-1}\mathfrak{T}_\Delta R_\Delta = G_{\gamma,\mathbb{A}}$. By a similar calculation in ([16, p. 3005]), we obtain

$$\begin{aligned}
\mathbb{J}_{\mathrm{ell}}(\beta) &= \frac{1}{2}\sum_{\widetilde{\gamma}\in\mathcal{Q}_F^{\mathrm{Irr}}} \int_{Z_\mathbb{A} G_F\backslash G_\mathbb{A}} \sum_{\xi\in G_{\gamma,F}\backslash G_F} \Phi(g^{-1}\xi^{-1}\gamma\xi g)\mathcal{E}_\beta^*(g)dg \\
&= \frac{1}{2}\sum_{\widetilde{\gamma}\in\mathcal{Q}_F^{\mathrm{Irr}}} \int_{G_{\gamma,\mathbb{A}}\backslash G_\mathbb{A}} \Phi(g^{-1}\gamma g)\left\{\int_{Z_\mathbb{A} G_{\gamma,F}\backslash G_{\gamma,\mathbb{A}}} \mathcal{E}_\beta^*(hg)dh\right\}dg \\
&= \frac{1}{2}\sum_{\widetilde{\gamma}\in\mathcal{Q}_F^{\mathrm{Irr}}} \int_{\mathfrak{T}_\Delta\backslash G_\mathbb{A}} \Phi(g^{-1}R_\Delta\gamma R_\Delta^{-1}g)(\mathcal{E}_\beta^*)^\Delta(g)dg \\
&= \frac{1}{2}\sum_{\widetilde{\gamma}\in\mathcal{Q}_F^{\mathrm{Irr}}} \int_{\mathfrak{T}_\Delta\backslash G_\mathbb{A}} \Phi(g^{-1}\hat{\gamma}g)(\mathcal{E}_\beta^*)^\Delta(g)dg \qquad (7.1)
\end{aligned}$$

with a certain Haar measure on $G_{\gamma,\mathbb{A}}$ where $\hat{\gamma} = (\hat{\gamma}_v)_{v\in\Sigma_F} \in G_{\mathbb{A}}$ such that

$$\hat{\gamma}_v = \begin{pmatrix} \frac{t}{2m_v} & 1 \\ \Delta_v^0 & \frac{t}{2m_v} \end{pmatrix} \quad \text{if} \quad \Delta_v^0 \neq 1 \tag{7.2}$$

$$\hat{\gamma}_v = \begin{pmatrix} \frac{t}{2m_v}+1 & 0 \\ 0 & \frac{t}{2m_v}-1 \end{pmatrix} \quad \text{if} \quad \Delta_v^0 = 1 \tag{7.3}$$

and

$$(\mathcal{E}_\beta^*)^\Delta(g) = \int_{Z_{\mathbb{A}}G_{\gamma,F}\backslash G_{\gamma,\mathbb{A}}} \mathcal{E}_\beta^*(hR_\Delta^{-1}g)dh.$$

This integral is the periods of Eisenstein series along with elliptic tori. Let $\mathfrak{d}_{E/F}$ be the relative discriminant of E/F and ε_Δ be the idele class character corresponding to the quadratic extension E/F by class field theory.

By ([16, (7.5)]) and ([16, Proposition 7.7]), we have

$$(\mathcal{E}_\beta^*)^\Delta(g) = \int_{L_\sigma} \beta(z)\Lambda_F(z+1)E^\Delta(z;g) \tag{7.4}$$

where

$$E^\Delta(z;g) = E^\Delta(z;1)\prod_{v\in\Sigma_F}\varphi_v^{\Delta,(z)}(g_v), \quad g \in G_{\mathbb{A}} \tag{7.5}$$

with

$$\begin{aligned} E^\Delta(z;1) =& D_F^{-\frac{1}{2}}\mathrm{N}(\mathfrak{d}_{E/F})^{\frac{z+1}{4}}\prod_{v\in\Sigma_F}|m_v|^{-1}\prod_{\substack{v\in\Sigma_F\\ \Delta_v^0=1}}|2|_v^{-1} \\ &\times \frac{\zeta(\frac{z+1}{2})L_F(\frac{z+1}{2},\varepsilon_\Delta)}{\zeta_F(z+1)}\prod_{\substack{v\in\Sigma_\infty\\ \Delta_v^0=-1}}2^{-1}\prod_{\substack{v\in\Sigma_{\mathrm{dyadic}}\\ \Delta_v^0=5}}3^{-1}2^{\frac{z+1}{2}+1}(1+2^{-z}) \end{aligned} \tag{7.6}$$

and $\varphi_v^{\Delta,(z)}$ ($v \in \Sigma_F$) is a unique smooth function on G_v satisfying the properties in ([16, Lemma 7.5]).

7.2 The local orbital integral

For $v \in \Sigma_F$ and $z \in \mathbb{C}$, we set

$$\mathfrak{E}_v^{(z)}(\hat{\gamma}_v) = \int_{\mathfrak{T}_{\Delta,v}\backslash G_v} \Phi_v(g^{-1}\hat{\gamma}_v g)\varphi_v^{\Delta,(z)}(g)dg$$

where $\hat{\gamma}_v$ is given as in (7.2) and (7.3). Explicit formulas of this integral for $v \in \Sigma_{\mathrm{fin}}$ are calculated in ([16, Theorem 7.9]). It remains to compute the integral for $v \in \Sigma_\infty$.

Proposition 7.1. *For $v \in \Sigma_F$, we have the following equations.*
(i) When $v \in \Sigma_{\mathrm{fin}} - (S \cup S(\mathfrak{n}) \cup \Sigma_{\mathrm{dyadic}})$ or $v \in \Sigma_{\mathrm{dyadic}}$ with $\Delta_v^0 \neq 5$, we have

$$\mathfrak{E}_v^{(z)}(\hat{\gamma}_v) = q_v^{-\frac{\mathrm{d}_v}{2}}|2m_v|_v\mathcal{O}_{v,0}^{\Delta_v^0,(z)}\left(\tfrac{n}{4m_v^2}\right)\delta\left(\tfrac{t}{2m_v} \notin \mathfrak{o}_v^\times \text{ or } \tfrac{n}{4m_v^2} \notin \mathfrak{p}_v\right)$$

if $\Delta_v^0 = 1$, *and otherwise*

$$\begin{aligned}\mathfrak{E}_v^{(z)}(\hat{\gamma}_v) =& q_v^{-\frac{\mathrm{d}_v}{2}} |m_v|_v \mathcal{O}_{v,0}^{\Delta_v^0,(z)}\left(\tfrac{n}{m_v^2}\right) \\ &\times \begin{cases} 1 & (v \notin \Sigma_{\mathrm{dyadic}},\ \Delta_v^0 \in \mathfrak{o}_v^\times - (\mathfrak{o}_v^\times)^2) \\ \delta\left(\tfrac{t}{2m_v} \notin \mathfrak{o}_v^\times\right) & (v \in \Sigma_{\mathrm{dyadic}},\ \Delta_v^0 \in \{-1,-5\}) \\ \delta\left(\tfrac{t}{2m_v} \notin \mathfrak{p}_v\right) & (\Delta_v^0 \in \mathfrak{p}_v - \mathfrak{p}_v^2). \end{cases}\end{aligned}$$

(ii) When $v \in \Sigma_{\mathrm{dyadic}}$ *with* $\Delta_v^0 = 5$, *we have*

$$\mathfrak{E}_v^{(z)}(\hat{\gamma}_v) = q_v^{-\frac{\mathrm{d}_v}{2}} |2m_v|_v 2^{\frac{-z-1}{2}} 3(1+2^{-z})^{-1} \mathcal{O}_{v,0}^{\Delta_v^0,(z)}\left(\tfrac{n}{4m_v^2}\right).$$

(iii) When $v \in S(\mathfrak{n})$, *we have*

$$\mathfrak{E}_v^{(z)}(\hat{\gamma}_v) = q_v^{-\frac{\mathrm{d}_v}{2}} |m_v|_v \mathcal{O}_{v,1}^{\Delta_v^0,(z)}\left(\tfrac{n}{m_v^2}\right) \times \begin{cases} \delta\left(\tfrac{t}{2m_v} \notin \mathfrak{o}_v^\times \text{ or } \tfrac{n}{m_v^2} \notin \mathfrak{p}_v\right) & (\Delta_v^0 = 1) \\ \delta\left(\tfrac{n}{m_v^2} \notin \mathfrak{o}_v\right) & (\Delta_v^0 \in \mathfrak{o}_v^\times - (\mathfrak{o}_v^\times)^2) \\ \delta\left(\tfrac{n}{m_v^2} \notin \mathfrak{p}_v\right) & (\Delta_v^0 \in \mathfrak{p}_v - \mathfrak{p}_v^2). \end{cases}$$

(iv) When $v \in S$, *for* $|\mathrm{Re}(z)| < 2\mathrm{Re}(s_v) + 1$, *we have*

$$\mathfrak{E}_v^{(z)}(\hat{\gamma}_v) = q_v^{-\frac{\mathrm{d}_v}{2}} |m_v|_v \mathcal{S}_v^{\Delta_v^0,(z)}\left(s_v; \tfrac{n}{m_v^2}\right).$$

(v) When $v \in \Sigma_\infty$, *for* $|\mathrm{Re}(z)| < 1$, *we have*

$$\mathfrak{E}_v^{(z)}(\hat{\gamma}_v) = 2|m_v|_v \sum_{i=1}^{N} \mu_v^{(i)}(\mathfrak{s}_\infty) \mathcal{O}_v^{\mathrm{sgn}(\Delta_v^0),(z)}\left(s_{i,v}; \tfrac{t}{2m_v}\right).$$

In preparation for proving the above Proposition, we need the following lemma.

Lemma 7.2. *Let* $s \in \mathbb{C}$ *such that* $\mathrm{Re}(s) > 1$.
(i) For a non-negative integer m, *we have*

$$\int_0^\infty x^{s-1}(1+x)^{-2s-2m}(1-x)^{2m} dx = \frac{(2m)!}{m!} \frac{\Gamma(s)\Gamma(s+m)}{\Gamma(2s+2m)}. \tag{7.7}$$

(ii) For any $z \in \mathbb{C} - [1, +\infty)$, *we have*

$$\int_0^\infty x^{s-1}(1+x)^{-2s} {}_2F_1\left(a, b; \tfrac{1}{2}; z(\tfrac{1-x}{1+x})^2\right) dx = \pi^{\frac{1}{2}} 2^{-2s+1} \frac{\Gamma(s)}{\Gamma(s+\frac{1}{2})} {}_2F_1(a, b; s+\tfrac{1}{2}; z). \tag{7.8}$$

Proof. We prove (i) by induction on m. The case of $m = 0$ is immediate from ([3, p. 310, (19)]).

Suppose (7.7) holds for m and any s with $\mathrm{Re}(s) > 1$. Then we have

$$\begin{aligned}
&\int_0^\infty x^{s-1}(1+x)^{-2s-2(m+1)}(1-x)^{2(m+1)}dx \\
=&\int_0^\infty x^{s-1}(1+x)^{-2s-2m-2}(1-x)^{2m}\left\{(1+x)^2-4x\right\}dx \\
=&\int_0^\infty x^{s-1}(1+x)^{-2s-2m}(1-x)^{2m}dx \\
&\qquad\qquad -4\int_0^\infty x^{(s+1)-1}(1+x)^{-2(s+1)-2m}(1-x)^{2m}dx \\
=&\frac{(2m)!}{m!}\left\{\frac{\Gamma(s)\Gamma(s+m)}{\Gamma(2s+2m)}-4\frac{\Gamma(s+1)\Gamma(s+m+1)}{\Gamma(2s+2m+2)}\right\}
\end{aligned}$$

by the induction hypothesis. By an easy calculation, this equals $\frac{(2m+2)!}{(m+1)!}\frac{\Gamma(s)\Gamma(s+m+1)}{\Gamma(2s+2m+2)}$. Hence (i) follows.

Next, we show (ii). Since the integral (7.8) converges locally uniformly on $z \in \mathbb{C} - [1, +\infty)$, it suffice to check that (7.8) holds for $|z| < 1$ by analytic continuation. When $|z| < 1$, we get the following termwise integral:

$$\begin{aligned}
&\int_0^\infty x^{s-1}(1+x)^{-2s}{}_2F_1\left(a,b;\tfrac{1}{2};z(\tfrac{1-x}{1+x})^2\right)dx \\
=&\frac{\Gamma(\frac{1}{2})}{\Gamma(a)\Gamma(b)}\sum_{m=0}^\infty\frac{\Gamma(a+m)\Gamma(b+m)}{m!\Gamma(\frac{1}{2}+m)}z^m\int_0^\infty x^{s-1}(1+x)^{-2s-2m}(1-x)^{2m}dx \\
=&\frac{\pi^{\frac{1}{2}}}{\Gamma(a)\Gamma(b)}\sum_{m=0}^\infty\frac{\Gamma(a+m)\Gamma(b+m)}{m!\Gamma(\frac{1}{2}+m)}z^m\frac{(2m)!}{m!}\frac{\Gamma(s)\Gamma(s+m)}{\Gamma(2s+2m)}\qquad(\because \text{(i)}). \qquad (7.9)
\end{aligned}$$

Since $m!\Gamma(\frac{1}{2}+m) = 2^{-2m}\pi^{\frac{1}{2}}(2m)!$ and $\Gamma(2s+2m) = 2^{2s+2m-1}\pi^{-\frac{1}{2}}\Gamma(s+m)\Gamma(s+\frac{1}{2}+m)$, (7.9) becomes

$$\begin{aligned}
&\pi^{\frac{1}{2}}2^{-2s+1}\frac{\Gamma(s)}{\Gamma(a)\Gamma(b)}\sum_{m=0}^\infty\frac{\Gamma(a+m)\Gamma(b+m)}{m!\Gamma(s+\frac{1}{2}+m)}z^m \\
=&\pi^{\frac{1}{2}}2^{-2s+1}\frac{\Gamma(s)}{\Gamma(s+\frac{1}{2})}{}_2F_1(a,b;s+\tfrac{1}{2};z).
\end{aligned}$$

We complete the proof. □

Proof of Proposition 7.1. We repeat that the estimates of formulas (i), (ii), (iii), and (iv) have been done in ([16, Theorem 7.9]), we only prove (v).

Suppose $\Delta_v^0 = 1$, then $\mathfrak{T}_{\Delta,v} = H_v$. We note that $n \neq 0$ and $t \neq 2m_v$ because $t^2 - 4n \in F^\times - (F^\times)^2$. The proofs of ([16, Lemma 7.5]) and ([16, Lemma 6.1]) say that the function $\varphi_v^{\Delta,(z)}$ coincides with $\varphi_v^{(0,z)}$ defined in (6.5). To simplify notations, set $a = \frac{t}{2m_v} \in F_v - \{1\}$ and $b = \frac{a+1}{a-1}$. Then we have

$$\mathfrak{E}_v^{(z)}(\hat{\gamma}_v) = |2m_v|\mathfrak{F}_v^{(z)}(b)$$

by the same reason as in ([16, p. 3021]) where $\mathfrak{F}_v^{(z)}$ is given by (6.8). Hence the claim follows from Proposition 6.3.

Next, suppose $\Delta_v^0 = -1$. In a similar manner as ([16, (9.11)]), we get

$$\mathfrak{E}_v^{(z)}(\hat{\gamma}_v) = |m_v|^{-1} \int_{\mathbb{R}^\times} \int_{\mathbb{R}} \int_{O(2)} \Phi_{\mathfrak{s}_v} \left(k^{-1} \left(\begin{smallmatrix} a+x & t^{-1}(1+x^2) \\ -t & a-x \end{smallmatrix} \right) k \right) |t|^{\frac{z-1}{2}} dk dx d^\times t$$
$$= |m_v|^{-1} \int_{\mathbb{R}^\times} \int_{\mathbb{R}} \Phi_{\mathfrak{s}_v} \left(\left(\begin{smallmatrix} a+x & t^{-1}(1+x^2) \\ -t & a-x \end{smallmatrix} \right) \right) |t|^{\frac{z-1}{2}} dx d^\times t$$

because $\Phi_{\mathfrak{s}_v}(k^{-1}gk) = \Phi_{\mathfrak{s}_v}(g)$ for $g \in G_{\mathbb{R}}$ and $k \in O(2)$. Then, from (2.10), we have

$$\mathfrak{E}_v^{(z)}(\hat{\gamma}_v) = |m_v|^{-1} \sum_{i=1}^{N} \mu_v^{(i)}(\mathfrak{s}_\infty) \mathfrak{I}(s_{i,v}, z) \tag{7.10}$$

where

$$\mathfrak{I}(s,z) = \frac{1}{4\pi} \frac{\Gamma(\frac{s+1}{2})^2}{\Gamma(s+1)} \int_{\mathbb{R}^\times} \int_{\mathbb{R}} \left\{ \tfrac{4(a^2+1)t^2}{(x^2+c^2)(x^2+\bar{c}^2)} \right\}^{\frac{s+1}{2}}$$
$$\times {}_2F_1\left(\tfrac{s+1}{2}, \tfrac{s+1}{2}; s+1; \tfrac{4(a^2+1)t^2}{(x^2+c^2)(x^2+\bar{c}^2)} \right) |t|^{\frac{z-1}{2}} dx d^\times t \tag{7.11}$$

for $\mathrm{Re}(s) > 1$ with $c = (t^2+1+2\sqrt{-1}at)^{\frac{1}{2}}$. Suppose that $|\mathrm{Re}(z)| < 2\mathrm{Re}(s)+1$. Then by exchanging the order of the integrals and the summation formally, we have

$$\mathfrak{I}(s,z) = \frac{1}{4\pi} \sum_{m=0}^{\infty} \frac{\Gamma(\frac{s+1}{2}+m)^2}{m!\Gamma(s+1+m)} \int_{\mathbb{R}^\times} \int_{\mathbb{R}} \left\{ \tfrac{4(a^2+1)t^2}{(x^2+c^2)(x^2+\bar{c}^2)} \right\}^{\frac{s+1}{2}+m} |t|^{\frac{z-1}{2}} dx d^\times t$$
$$= \frac{1}{4\pi} \sum_{m=0}^{\infty} \frac{\Gamma(\frac{s+1}{2}+m)^2}{m!\Gamma(s+1+m)} \left\{ 4(a^2+1) \right\}^{\frac{s+1}{2}+m}$$
$$\times \int_{\mathbb{R}^\times} |t|^{s+\frac{z+1}{2}+2m} \int_{\mathbb{R}} (x^2+c^2)^{-\frac{s+1}{2}-m} (x^2+\bar{c}^2)^{-\frac{s+1}{2}-m} dx d^\times t. \tag{7.12}$$

We can obtain a dominating function by replacing s and z by $\mathrm{Re}(s)$ and $\mathrm{Re}(z)$ respectively. Thus, the justification of the above computation is obtained by computing an explicit formula of (7.12). By the Plancherel theorem for $\mathcal{L}^2(\mathbb{R})$, (6.12), and a similar way as the calculation in (6.13), we have

$$\int_{\mathbb{R}} (x^2+c^2)^{-\frac{s+1}{2}-m} (x^2+\bar{c}^2)^{-\frac{s+1}{2}-m} dx$$
$$= 8\pi^{s+2m+1} \Gamma(\tfrac{s+1}{2}+m)^{-2} |c|^{-s-2m} \int_0^\infty K_{\frac{s}{2}+m}(2\pi c y) K_{\frac{s}{2}+m}(2\pi \bar{c} y) y^{s+2m} dy. \tag{7.13}$$

By substituting (7.13) into (7.12), we get

$$\mathfrak{I}(s,z) = 2 \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(s+1+m)} \pi^{s+2m} \left\{ 4(a^2+1) \right\}^{\frac{s+1}{2}+m}$$
$$\times \int_{\mathbb{R}^\times} |t|^{s+\frac{z+1}{2}+2m} |c|^{-s-2m} \int_0^\infty K_{\frac{s}{2}+m}(2\pi c y) K_{\frac{s}{2}+m}(2\pi \bar{c} y) y^{s+2m} dy d^\times t. \tag{7.14}$$

By $K_\alpha(z) = \frac{1}{2}\int_0^\infty \exp(-\frac{z}{2}(u+u^{-1}))u^{-\alpha-1}du$, the integral with respect to y becomes

$$\begin{aligned}
&\int_0^\infty K_{\frac{s}{2}+m}(2\pi cy)K_{\frac{s}{2}+m}(2\pi\overline{c}y)y^{s+2m}dy \\
=&\frac{1}{4}\int_0^\infty y^{s+2m}\int_0^\infty \exp(-\pi cy(u+u^{-1}))u^{-\frac{s}{2}-m-1}du \\
&\qquad\qquad \times\int_0^\infty \exp(-\pi\overline{c}y(v+v^{-1}))v^{-\frac{s}{2}-m-1}dvdy \\
=&\frac{1}{4}\int_0^\infty\int_0^\infty\int_0^\infty \exp(-\pi(cu+\overline{c}v)y^2)dy\ \exp(-\pi cu^{-1})u^{-\frac{s}{2}-m-1}du \\
&\qquad\qquad \times\exp(-\pi\overline{c}v^{-1})v^{-\frac{s}{2}-m-1}dv \qquad (u\to yu, v\to yv) \\
=&\frac{1}{4}\int_0^\infty\int_0^\infty (cu+\overline{c}v)^{-\frac{1}{2}}\ \exp(-\pi cu^{-1})u^{-\frac{s}{2}-m-1}du\ \exp(-\pi\overline{c}v^{-1})v^{-\frac{s}{2}-m-1}dv \\
=&\frac{1}{4}\int_0^\infty\int_0^\infty (\overline{c}u+cv)^{-\frac{1}{2}}\ \exp(-\pi cu)u^{\frac{s}{2}+m-\frac{1}{2}}\ \exp(-\pi\overline{c}v)v^{\frac{s}{2}+m-\frac{1}{2}}dudv \qquad (7.15) \\
&\qquad\qquad (u\to u^{-1}, v\to v^{-1}).
\end{aligned}$$

As a function of complex variables u, v, the integrand of (7.15) is holomorphic on the region

$$\{(u,v)\in\mathbb{C}^2 \mid u,v\neq 0,\ |\mathrm{Arg}(u)|, |\mathrm{Arg}(v)| < \tfrac{\pi}{4}\}.$$

As long as $|\mathrm{Arg}(u)| < \frac{\pi}{4}$ and $|\mathrm{Arg}(v)| < \frac{\pi}{4}$, the integrand decays exponentially as $|u|, |v| \to +\infty$ and that is bounded above by $O(|u|^{\frac{\mathrm{Re}(s)}{2}-1})$ as $|u|\to 0$ and $O(|v|^{\frac{\mathrm{Re}(s)}{2}-1})$ as $|v|\to 0$ because $|\mathrm{Arg}(c)| < \frac{\pi}{4}$. Hence we can shift the contours $\{u|u>0\}$ to $\{cu|u>0\}$ and $\{v|v>0\}$ to $\{\overline{c}v|v>0\}$ by Cauchy's integral theorem. Then, (7.15) becomes

$$\begin{aligned}
=&\frac{1}{4}|c|^{s+2n}\int_0^\infty\int_0^\infty (u+v)^{-\frac{1}{2}}\ \exp(-\pi c^2u)u^{\frac{s}{2}+m-\frac{1}{2}}\ \exp(-\pi\overline{c}^2v)v^{\frac{s}{2}+m-\frac{1}{2}}dudv \\
=&\frac{1}{4}\pi^{-s-2n-\frac{1}{2}}|c|^{s+2n}\int_0^\infty\int_0^\infty (u+v)^{-\frac{1}{2}}\ \exp(-c^2u)u^{\frac{s}{2}+m-\frac{1}{2}}\ \exp(-\overline{c}^2v)v^{\frac{s}{2}+m-\frac{1}{2}}dudv \\
&\qquad\qquad (u\to\pi^{-1}u, v\to\pi^{-1}v).
\end{aligned}$$

Recall that $c = (t^2+1+2\sqrt{-1}at)^{\frac{1}{2}}$. Substituting the above equation into (7.14), we have

$$\begin{aligned}
\mathfrak{I}(s,z) =&\pi^{-\frac{1}{2}}\sum_{m=0}^\infty \frac{\{4(a^2+1)\}^{\frac{s+1}{2}+m}}{m!\Gamma(s+1+m)}\int_0^\infty e^{-(u+v)t^2}\cos(2a(u-v)t)t^{s+\frac{z+1}{2}+2m-1}dt \\
&\times\int_0^\infty\int_0^\infty (u+v)^{-\frac{1}{2}}e^{-u-v}u^{\frac{s}{2}+m-\frac{1}{2}}v^{\frac{s}{2}+m-\frac{1}{2}}dudv. \qquad (7.16)
\end{aligned}$$

By the integral formula

$$\begin{aligned}
\int_0^\infty e^{-\alpha x^2}\cos(\beta x)x^{s-1}dx &= \tfrac{1}{2}\alpha^{-\frac{s}{2}}\Gamma(\tfrac{s}{2})\exp(-\tfrac{\beta^2}{4\alpha}){}_1F_1(-\tfrac{1}{2}s+\tfrac{1}{2};\tfrac{1}{2};\tfrac{\beta^2}{4\alpha}) \\
&= \tfrac{1}{2}\alpha^{-\frac{s}{2}}\Gamma(\tfrac{s}{2}){}_1F_1(-\tfrac{s}{2};\tfrac{1}{2};-\tfrac{\beta^2}{4\alpha}) \qquad (\mathrm{Re}(s)>0, \mathrm{Re}(\alpha)>0)
\end{aligned}$$

from ([3, p. 320, (30)]), we can rewrite (7.16) as

$$\begin{aligned}\mathfrak{I}(s,z) =&\pi^{-\frac{1}{2}}\sum_{m=0}^{\infty}\frac{\Gamma(\frac{s}{2}+\frac{z+1}{4}+m)}{m!\Gamma(s+1+m)}\left\{4(a^2+1)\right\}^{\frac{s+1}{2}+m}\\ &\times\int_0^\infty\int_0^\infty(u+v)^{-\frac{s}{2}-\frac{z+3}{4}-m}e^{-u-v}u^{\frac{s}{2}+m-\frac{1}{2}}v^{\frac{s}{2}+m-\frac{1}{2}}{}_1F_1\left(\tfrac{s}{2}+\tfrac{z+1}{4}+m;\tfrac{1}{2};-\tfrac{a^2(u-v)^2}{u+v}\right)dudv\\ =&\pi^{-\frac{1}{2}}\sum_{m=0}^{\infty}\frac{\Gamma(\frac{s}{2}+\frac{z+1}{4}+m)}{m!\Gamma(s+1+m)}\left\{4(a^2+1)\right\}^{\frac{s+1}{2}+m}\\ &\times\int_0^\infty\int_0^\infty u^{\frac{s}{2}-\frac{z-1}{4}+m-1}e^{-(1+v)u}{}_1F_1\left(\tfrac{s}{2}+\tfrac{z+1}{4}+m;\tfrac{1}{2};-\tfrac{a^2(1-v)^2}{1+v}u\right)du\\ &\qquad\times(1+v)^{-\frac{s}{2}-\frac{z+3}{4}-m}v^{\frac{s}{2}+m-\frac{1}{2}}dv\qquad(v\to uv).\end{aligned}\tag{7.17}$$

By applying the equation

$$\int_0^\infty x^{s-1}e^{-\alpha x}{}_1F_1(\beta;\rho;\lambda x)dx=\alpha^{-s}\Gamma(s){}_2F_1(\beta,s;\rho;\tfrac{\lambda}{\alpha})$$
$$(\mathrm{Re}(s)>0,\mathrm{Re}(\alpha)>\max\{0,\mathrm{Re}(\lambda)\})$$

from ([3, p. 337, (9)]), the value (7.17) becomes

$$\begin{aligned}\mathfrak{I}(s,z) =&\pi^{-\frac{1}{2}}\sum_{m=0}^{\infty}\frac{\Gamma(\frac{s}{2}+\frac{z+1}{4}+m)\Gamma(\frac{s}{2}-\frac{z-1}{4}+m)}{m!\Gamma(s+1+m)}\left\{4(a^2+1)\right\}^{\frac{s+1}{2}+m}\\ &\times\int_0^\infty(1+v)^{-s-2m-1}v^{\frac{s}{2}+m-\frac{1}{2}}{}_2F_1\left(\tfrac{s}{2}+\tfrac{z+1}{4}+m,\tfrac{s}{2}-\tfrac{z-1}{4}+m;\tfrac{1}{2};-a^2(\tfrac{1-v}{1+v})^2\right)dv\\ =&2\sum_{m=0}^{\infty}\frac{\Gamma(\frac{s}{2}+\frac{z+1}{4}+m)\Gamma(\frac{s}{2}-\frac{z-1}{4}+m)\Gamma(\frac{s+1}{2}+m)}{m!\Gamma(\frac{s}{2}+1+m)\Gamma(s+1+m)}(a^2+1)^{\frac{s+1}{2}+m}\\ &\qquad\times{}_2F_1\left(\tfrac{s}{2}+\tfrac{z+1}{4}+m,\tfrac{s}{2}-\tfrac{z-1}{4}+m;\tfrac{s}{2}+1+m;-a^2\right)\\ =&2(a^2+1)\sum_{m=0}^{\infty}\frac{\Gamma(\frac{s}{2}+\frac{z+1}{4}+m)\Gamma(\frac{s}{2}-\frac{z-1}{4}+m)\Gamma(\frac{s+1}{2}+m)}{m!\Gamma(\frac{s}{2}+1+m)\Gamma(s+1+m)}\\ &\qquad\times{}_2F_1\left(\tfrac{z+3}{4},\tfrac{-z+3}{4};\tfrac{s}{2}+1+m;-a^2\right)\end{aligned}\tag{7.18}$$

by Lemma 7.2 (ii) and ([12, (2.4.1), 1st line]). Then, from Lemma 5.5, the sum (7.18) equals

$$\begin{aligned}\mathfrak{I}(s,z) =&2(a^2+1)\frac{\Gamma(\frac{s}{2}+\frac{z+1}{4})\Gamma(\frac{s}{2}-\frac{z-1}{4})\Gamma(\frac{s+1}{2})}{\Gamma(s+1)\Gamma(\frac{s}{2}+1)}\\ &\qquad\times F_3^{(1,0)}\left({\tfrac{s}{2}+\tfrac{z+1}{4},\tfrac{s}{2}-\tfrac{z-1}{4},\tfrac{s+1}{2};\tfrac{z+3}{4},\tfrac{-z+3}{4}\atop s+1;\tfrac{s}{2}+1};1,-a^2\right)\\ =&2\times\mathcal{O}_v^{-,(z)}(s;a).\end{aligned}\tag{7.19}$$

The claim follows from (7.10) and (7.19). □

It remains to discuss the absolute convergence of the series $\sum_{\gamma\in\mathcal{Q}_F^{\mathrm{Irr}}}\left(\prod_{v\in\Sigma_F}\mathfrak{E}_v^{(z)}(\hat{\gamma}_v)\right)$.

Proposition 7.3. *For any $\sigma\in\mathbb{R}$ such that $0<\sigma<1$, we have*

$$\sum_{\gamma=(t:n)_F\in\mathcal{Q}_F^{\mathrm{Irr}}}|\Lambda_F(z+1)||E^{\Delta}(z;1)|\prod_{v\in\Sigma_F}|\mathfrak{E}_v^{(z)}(\hat{\gamma}_v)|<+\infty\tag{7.20}$$

uniformly for z, s_v $(v \in S)$, *and the imaginary parts of* $s_{i,v}$ $(1 \le i \le N, v \in \Sigma_\infty)$ *such that* $\min\{\mathrm{Re}(s_v) | v \in S\} > 2\max\left\{\mathrm{Re}(s_{i,v})|_{1\le i\le N}^{v\in\Sigma_\infty;}\right\} + 1$ *and* $|\mathrm{Re}(z)| < \delta$.

It suffices to obtain the upper bound of $\mathfrak{E}_v^{(z)}(\hat{\gamma}_v)$ for $v \in \Sigma_\infty$ since the estimations of upper bounds of $\mathfrak{E}_v^{(z)}(\hat{\gamma}_v)$ for $v \in \Sigma_{\mathrm{fin}}$ has been already done in ([16, Lemma 7.14]). The following lemma is a substitution of ([16, Lemma 7.14, (4) and (5)]) which allows us to apply the same way as the proof of the absolute convergence of (7.20) investigated in ([16, §7]).

Lemma 7.4. *Let* $v \in \Sigma_\infty$, $s \in \mathbb{C}$ *with* $\mathrm{Re}(s) > 1$, *and* $0 < \sigma < 1$. *Then we have the following estimates.*
(i)

$$|\mathcal{O}_v^{+,(z)}(s;a)| \ll_\sigma \delta(a>1) \left|\Gamma(\tfrac{z+1}{4})\Gamma(\tfrac{-z+1}{4})\right|^{-1} \times \min\{1, |a-1|_v\}^{-\frac{\mathrm{Re}(s)+1}{2}} |a|_v^{\frac{|\mathrm{Re}(z)|+1}{2}} \left\{\log(3+|a|_v)\right\}^2,$$

uniformly on $a \in F_v$ *and* $|\mathrm{Re}(z)| \le \sigma$. *Moreover, the implied constant is independent of the imaginary part of* s.

(ii)

$$|\mathcal{O}_v^{-,(z)}(s;a)| \ll_\sigma (1+|a|_v)^{\frac{|\mathrm{Re}(z)|+1}{2}} \log(3+|a|_v)$$

uniformly on $a \in F_v$ *and* $|\mathrm{Re}(z)| \le \sigma$. *Moreover, the implied constant is independent of the imaginary part of* s.

Proof. The estimate (i) is a consequence of Lemma 6.6. It suffices to check (ii). From the proof of Lemma 6.5, we have the inequality

$$\left|\frac{\Gamma(\frac{s+1}{2})^2}{\Gamma(s+1)}\right| \left|{}_2F_1(\tfrac{s+1}{2}, \tfrac{s+1}{2}; s+1; u)\right| \le \frac{\Gamma(\frac{\mathrm{Re}(s)+1}{2})^2}{\Gamma(\mathrm{Re}(s)+1)} {}_2F_1(\tfrac{\mathrm{Re}(s)+1}{2}, \tfrac{\mathrm{Re}(s)+1}{2}; \mathrm{Re}(s)+1; u)$$

for $0 \le u < 1$. Hence, by (7.11) and (7.19), we have

$$\left|\mathcal{O}_v^{-,(z)}(s;a)\right| \le \mathcal{O}_v^{-,(\mathrm{Re}(z))}(\mathrm{Re}(s);a).$$

Therefore, we may assume that $s, z \in \mathbb{R}$. By (7.18) and (7.19),

$$\mathcal{O}_v^{-,(z)}(s;a) = (a^2+1) \sum_{m=0}^{\infty} \frac{\Gamma(\frac{s}{2}+\frac{z+1}{4}+m)\Gamma(\frac{s}{2}-\frac{z-1}{4}+m)\Gamma(\frac{s+1}{2}+m)}{m!\Gamma(\frac{s}{2}+1+m)\Gamma(s+1+m)} \times {}_2F_1\left(\tfrac{z+3}{4}, \tfrac{-z+3}{4}; \tfrac{s}{2}+1+m; -a^2\right). \tag{7.21}$$

Suppose $0 \le z \le \delta$. From the third line of ([12, (2.4.1)]), the series (7.21) equals

$$\begin{aligned}\mathcal{O}_v^{-,(z)}(s;a) =& (a^2+1)^{\frac{z+1}{4}} \sum_{m=0}^{\infty} \frac{\Gamma(\frac{s}{2}+\frac{z+1}{4}+m)\Gamma(\frac{s}{2}-\frac{z-1}{4}+m)\Gamma(\frac{s+1}{2}+m)}{m!\Gamma(\frac{s}{2}+1+m)\Gamma(s+1+m)} \\ & \times {}_2F_1\left(\tfrac{-z+3}{4}, \tfrac{s}{2}-\tfrac{z-1}{4}+m; \tfrac{s}{2}+1+m; \tfrac{a^2}{a^2+1}\right) \\ =& \Gamma(\tfrac{-z+3}{4})^{-1}(a^2+1)^{\frac{z+1}{4}} \\ & \times \sum_{m,n=0}^{\infty} \frac{\Gamma(\frac{s}{2}+\frac{z+1}{4}+m)\Gamma(\frac{s+1}{2}+m)\Gamma(\frac{-z+3}{4}+n)\Gamma(\frac{s}{2}-\frac{z-1}{4}+m+n)}{m!n!\Gamma(s+1+m)\Gamma(\frac{s}{2}+1+m+n)} (\tfrac{a^2}{a^2+1})^n.\end{aligned}$$

By Stirling's formula, we have

$$\begin{aligned}&\left|\frac{\Gamma(\frac{s}{2}+\frac{z+1}{4}+m)\Gamma(\frac{s+1}{2}+m)\Gamma(\frac{-z+3}{4}+n)\Gamma(\frac{s}{2}-\frac{z-1}{4}+m+n)}{m!n!\Gamma(s+1+m)\Gamma(\frac{s}{2}+1+m+n)}\right|\\ \ll&(m+1)^{\frac{z-5}{4}}(n+1)^{\frac{-z-1}{4}}(m+n+1)^{\frac{-z-3}{4}}\\ \leq&(m+1)^{\frac{\sigma-5}{4}}(n+1)^{-1}\qquad(m,n\geq 0)\end{aligned}\tag{7.22}$$

uniformly on $0\leq z\leq\sigma$. Hence we obtain

$$\begin{aligned}|\mathcal{O}_v^{-,(z)}(s;a)|\ll&|\Gamma(\tfrac{-\sigma+3}{4})|^{-1}(a^2+1)^{\frac{z+1}{4}}\sum_{m,n=0}^{\infty}(m+1)^{\frac{\sigma-5}{4}}(n+1)^{-1}(\tfrac{a^2}{a^2+1})^n\\ \ll_\sigma&(a^2+1)^{\frac{z+1}{4}}\left\{-(\tfrac{a^2}{a^2+1})^{-1}\log\left(1-\tfrac{a^2}{a^2+1}\right)\right\}\\ \ll_\sigma&(1+|a|)^{\frac{z+1}{2}}\log(3+|a|),\qquad a\in\mathbb{R}.\end{aligned}$$

By the symmetry of $\mathcal{O}_v^{-,(z)}(s;a)$ between z and $-z$, we are done. □

Recall that

$$(\mathcal{E}_\beta^*)^\Delta(g)=\int_{L_\sigma}\beta(z)\Lambda_F(z+1)E^\Delta(z;1)\prod_{v\in\Sigma_F}\varphi_v^{\Delta,(z)}(g_v)dz,\quad g=(g_v)_{v\in\Sigma_F}$$

from (7.4) and (7.5). By substituting this into (7.1) and changing the order of integrals, we obtain

$$\mathbb{J}_{\rm ell}(\beta)=\frac{1}{2}\sum_{\tilde\gamma\in\mathcal{Q}_F^{\rm Irr}}\int_{L_\sigma}\beta(z)\Lambda_F(z+1)E^\Delta(z;1)\{\prod_{v\in\Sigma_F}\mathfrak{E}_v^{(z)}(\hat\gamma_v)\}dz.$$

Here, the change of the order of the integrals is guaranteed by (1.2), Stirling's formula and a similar manner as in ([16, Lemma 7.21]). We set the function

$$\hat J^0_{\rm ell}(\mathbf{s}_S,\mathfrak{s}_\infty,z)=\frac{1}{2}\sum_{\tilde\gamma\in\mathcal{Q}_F^{\rm Irr}}\Lambda_F(z+1)E^\Delta(z;1)\{\prod_{v\in\Sigma_F}\mathfrak{E}_v^{(z)}(\hat\gamma_v)\}$$

which is vertically of moderate growth on $-1<\mathrm{Re}(z)<1$ from Proposition 7.3. Then, this is the desired function in Theorem 5.1 for the elliptic term.

8 Proofs of the main results

Proof of Theorem 1.1. For $\flat\in\{\mathrm{cus},\mathrm{Eis},\mathrm{res}\}$ and $\natural\in\{\mathrm{uni},\mathrm{hyp},\mathrm{ell}\}$, let $\hat I^0_\flat(\mathbf{s}_S,\mathfrak{s}_\infty,z)$ and $\hat J^0_\natural(\mathbf{s}_S,\mathfrak{s}_\infty,z)$ be the holomorphic functions defined from §4 to §7. Then, by Proposition 3.9, we have the identity

$$\begin{aligned}&\hat I^0_{\rm cus}(\mathbf{s}_S,\mathfrak{s}_\infty,z)+\hat I^0_{\rm Eis}(\mathbf{s}_S,\mathfrak{s}_\infty,z)+\hat I^0_{\rm res}(\mathbf{s}_S,\mathfrak{s}_\infty,z)\\ &\qquad=\hat J^0_{\rm uni}(\mathbf{s}_S,\mathfrak{s}_\infty,z)+\hat J^0_{\rm hyp}(\mathbf{s}_S,\mathfrak{s}_\infty,z)+\hat J^0_{\rm ell}(\mathbf{s}_S,\mathfrak{s}_\infty,z)\end{aligned}\tag{8.1}$$

for $-1<\mathrm{Re}(z)<1$ and $\min\{\mathrm{Re}(s_v)|v\in S\}>2\max\left\{\mathrm{Re}(s_{i,v})\Big|{}^{v\in\Sigma_\infty,}_{1\leq i\leq N}\right\}+1$ (cf. Proposition 7.3). By the same discussion for the elliptic term done in ([16, §8.1]), we can check that

$$\hat I^0_\flat(\mathbf{s}_S,\mathfrak{s}_\infty,z)=\zeta_F(\tfrac{z+1}{2})I^0_\flat(\mathbf{s}_S,\mathfrak{s}_\infty,z),\quad(\flat\in\{\mathrm{cus},\mathrm{Eis},\mathrm{res}\})$$

$$\hat{J}^0_\natural(\mathbf{s}_S, \mathfrak{s}_\infty, z) = \zeta_F(\tfrac{z+1}{2}) J^0_\natural(\mathbf{s}_S, \mathfrak{s}_\infty, z), \quad (\natural \in \{\mathrm{uni}, \mathrm{hyp}, \mathrm{ell}\})$$

where $I^0_\flat(\mathbf{s}_S, \mathfrak{s}_\infty, z)$ and $J^0_\natural(\mathbf{s}_S, \mathfrak{s}_\infty, z)$ are the function defined in (1.5), (1.6), (1.8), and Theorem 1.1. Dividing (8.1) by $\zeta_F(\frac{z+1}{2})$, we complete the proof. □

Proof of Corollary 1.2. Recall the notations $\mathbf{s}_S = (s_v)_{v\in S}$ and $\mathfrak{s}_\infty = ((s_{i,v})_{1\le i\le N})_{v\in\Sigma_\infty}$. From now on, we fix the parameters $s_{i,v}$ for $v \in \Sigma_\infty$, $i > 1$ and regard $\mathfrak{s}_{1,\infty} := (s_{1,v})_{v\in\Sigma_\infty} \in \mathbb{C}^{\Sigma_\infty}$ as complex variables. For $v \in \Sigma_\infty$ and $\alpha_v \in \mathcal{A}_v$, we define $\alpha_v^{(1)} \in \mathcal{A}_v$ by

$$\alpha_v^{(1)}(s_{1,v}) = \left\{ \prod_{1<j\le N} \left(\frac{s_{j,v}^2 - s_{1,v}^2}{4} \right) \right\} \alpha_v(s_{1,v}) = \mu_v^{(1)}(\mathfrak{s}_\infty)^{-1} \alpha_v(s_{1,v}).$$

For $\alpha = \otimes_{v\in S\cup\Sigma_\infty} \alpha_v \in \mathcal{A}_{S\cup\Sigma_\infty}$, $\flat \in \{\mathrm{cus}, \mathrm{Eis}, \mathrm{res}\}$, and $\natural \in \{\mathrm{uni}, \mathrm{hyp}, \mathrm{ell}\}$, we put

$$\alpha^{(1)} = (\otimes_{v\in S}\alpha_v) \otimes (\otimes_{v\in\Sigma_\infty} \alpha_v^{(1)}),$$

$$\tilde{\mathbb{I}}^0_\flat(\alpha|\mathfrak{n}; z) = \left(\tfrac{1}{2\pi\sqrt{-1}}\right)^{\#S\cup\Sigma_\infty} \int_{L(\mathbf{c})} I^0_\flat(\mathbf{s}_S, \mathfrak{s}_\infty, z) \alpha^{(1)}(\mathbf{s}_S, \mathfrak{s}_{1,\infty}) d\mu(\mathbf{s}_S, \mathfrak{s}_{1,\infty}),$$

and

$$\tilde{\mathbb{J}}^0_\natural(\alpha|\mathfrak{n}; z) = \left(\tfrac{1}{2\pi\sqrt{-1}}\right)^{\#S\cup\Sigma_\infty} \int_{L(\mathbf{c})} J^0_\natural(\mathbf{s}_S, \mathfrak{s}_\infty, z) \alpha^{(1)}(\mathbf{s}_S, \mathfrak{s}_{1,\infty}) d\mu(\mathbf{s}_S, \mathfrak{s}_{1,\infty})$$

where $\mathbf{c} = (c_v)_{v\in S} \in \mathbb{R}^{S\cup\Sigma_\infty}$ such that $c_v > 1$ is sufficiently large for each $v \in S \cup \Sigma_\infty$, $L(\mathbf{c}) = \prod_{v\in S\cup\Sigma_\infty} L_v(c_v)$, and $d\mu(\mathbf{s}_S, \mathfrak{s}_{1,\infty}) = \prod_{v\in S} d\mu_v(s_v) \prod_{v\in\Sigma_\infty} d\mu_v(s_{1,v})$. Then, it can be checked that

$$\tilde{\mathbb{I}}^0_\flat(\alpha|\mathfrak{n}; z) = \mathbb{I}^0_\flat(\alpha|\mathfrak{n}; z)$$

and

$$\tilde{\mathbb{J}}^0_\natural(\alpha|\mathfrak{n}; z) = \mathbb{J}^0_\natural(\alpha|\mathfrak{n}; z)$$

by the following formulas:

$$\frac{1}{2\pi\sqrt{-1}} \int_{L_v(c_v)} \{q_v^{\frac{\nu+1}{2}} + q_v^{\frac{-\nu+1}{2}} - q_v^{\frac{s_v+1}{2}} - q_v^{\frac{s_v+1}{2}}\}^{-1} \alpha_v(s_v) d\mu(s_v) = -\alpha_v(\nu), \quad (v \in S,\ |\mathrm{Re}(\nu)| < c_v),$$

$$\frac{1}{2\pi\sqrt{-1}} \int_{L_v(c_v)} \{s_{1,v}^2 - \lambda^2\}^{-1} \alpha_v^{(1)}(s_{1,v}) d\mu_v(s_{1,v}) = \frac{1}{2} \alpha_v^{(1)}(\nu), \quad (v \in \Sigma_\infty,\ |\mathrm{Re}(\nu)| < c_v),$$

$$\frac{1}{2\pi\sqrt{-1}} \int_{L_v(c_v)} \mu_v^{(i)}(\mathfrak{s}_\infty) \alpha_v^{(1)}(s_{1,v}) d\mu_v(s_{1,v}) = 0, \quad (v \in \Sigma_\infty,\ i > 1).$$

Thus, Corollary 1.2 follows from Theorem 1.1. □

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References

[1] P. Appell, *Sur les séries hypergéométriques de deux variables et sur des équations différentielles linéaires aux dérivées partielles.* C. R. Acad. Sci. **90** (1880) 296–298, 731–735.

[2] D. Bump, *Automorphic forms and representations.* Cambridge Studies in Advanced Mathematics, vol. 55, Cambridge University Press, Cambridge, 1997.

[3] A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, *Tables of Integral Transforms.* Vol. I. Based, in part, on notes left by Harry Bateman, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954.

[4] A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, *Tables of Integral Transforms.* Vol. II. Based, in part, on notes left by Harry Bateman, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954.

[5] S. Gelbert, *Automorphic forms on adele groups.* Ann. of Math. Studies, No. 83, Princeton Univ. Press, Princeton, N. J., 1975.

[6] S. Gelbart, H. Jacquet, *A relation between automorphic representations of $GL(2)$ and $GL(3)$.* Ann. Sci. École Norm. Sup. (4) **11** (1978) no. 4, 471–542.

[7] S. Gelbart, H. Jacquet, *Forms of $GL(2)$ from the analytic point of view.* In Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, RI, (1979) 213–251.

[8] H. Iwaniec, *Spectral Methods of Automorphic Forms.* Graduate Studies in Mathematics, 53, 2nd ed. Amer. Math. Soc., Providence (2002)

[9] H. Iwaniec, E. Kowalski, *Analytic number theory.* American Mathematical Society Colloquium Publications, vol. 53, American Mathematical Society, Providence, RI, 2004.

[10] H. Jacquet, D. Zagier, *Eisenstein series and the Selberg trace formula II.* Trans. Amer. Math. Soc. **300** (1) (1987) 1–48.

[11] H. Liu, W. Wang, *Transformation and summation formulae for Kampé de Fériet series.* (English summary) J. Math. Anal. Appl. **409** (2014), no. 1, 100–110.

[12] W. Magnus, F. Oberhettinger, R. P. Soni, *Formulas and theorems for the special functions of mathematical physics.* (3rd ed. of the English translation, Springer, New York, 1966) Math. Comp. **23** (1969), no. 106, 471.

[13] S. Mizumoto, *On the second L-functions attached to Hilbert modular forms.* Math. Ann. **269** (1984) 191–216.

[14] H. M. Srivastava, P. W. Karlsson, *Multiple Gaussian Hypergeometric Series.* Halsted Press, Ellis Horwood, John Wiley and Sons, Chichester, New York, Chichester, Brisbane, Toronto, 1985.

[15] S. Sugiyama, *Regularized periods of automorphic forms on $GL(2)$.* Tohoku Math. J. **65** (2013), no. 3, 373–409.

[16] S. Sugiyama, M. Tsuzuki, *An explicit trace formula of Jacquet-Zagier type for Hilbert modular forms.* J. Funct. Anal. **275** (2018), no. 11, 2978–3064.

[17] K. Takase, *On the trace formula of the Hecke operators and the special values of the second L-functions attached to the Hilbert modular forms.* Manuscripta Math. **55** (1986) 137–170.

[18] E. C. Titchmarsh, *The theory of the Riemann zeta-function.* 2nd. ed., The Clarendon Press Oxford University Press, New York, 1986. Edited and with a preface by D. R. Heath-Brown.

[19] M. Tsuzuki, *Spectral means of central values of automorphic L-functions for $GL(2)$*. Mem. Amer. Math. Soc. **235** (2015), no. 1110.

[20] H. Wu, *Deducing Selberg trace formula via Rankin-Selberg method for GL_2*. (English summary) Trans. Amer. Math. Soc. **372** (2019), no. 12, 8507–8551.

[21] D. Zagier, *Modular Forms Whose Fourier Coefficients Involve Zeta-Functions of Quadratic Fields*, Lecture Notes in Math., vol. 627, Springer, 1977, 105–169.

[22] D. Zagier, *Eisenstein series and the Selberg trace formula I*, in: Automorphic Forms, Representation Theory and Arithmetic, Tata Institute, Bombay, 1979, 303–355.