

On dynamical zeta functions for 3-dimensional Riemannian foliated dynamical systems

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On dynamical zeta functions for 3-dimensional Riemannian foliated dynamical systems

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Abstract

In this thesis, we study 3-dimensional (Riemannian) foliated dynamical systems and their dynamical zeta functions. We firstly show a topological classification of 3-dimensional foliated dynamical systems, FDS^3 for short, and construct concrete examples of FDS^3 's for each type of our classification. Our main theorem is concerned with a cohomological description of the dynamical zeta function of a 3-dimensional Riemannian foliated dynamical system, called RFDS^3 for short. We prove a formula expressing the dynamical zeta function of an RFDS^3 as the regularized determinant of an infinitesimal operator acting on the leafwise cohomology associated to the RFDS^3 .

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Introduction

This thesis is motivated by the intention to deepen *arithmetic topology* in the analytic direction. Arithmetic topology is a new area in mathematics, which explores striking relationship between concepts and ideas in number theory and 3-dimensional topology (cf. [Mo12]). The basic idea is to regard a maximal ideal of a number ring as an arithmetic analog of a knot in a 3-manifold. Here are some analogies between basic concepts in both fields:

compactified spectrum of number ring $\overline{X}_k = \overline{\text{Spec}(\mathcal{O}_k)}$	oriented, connected, closed 3-manifold M
maximal ideal $\{\mathfrak{p}\} = \text{Spec}(\mathcal{O}_k/\mathfrak{p}) \hookrightarrow \overline{X}_k$	knot $\mathcal{K} : S^1 \hookrightarrow M$
finite set of maximal ideals $S = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$	link $\mathcal{L} = \mathcal{K}_1 \sqcup \dots \sqcup \mathcal{K}_r$
$\mathfrak{p}$ -adic integer ring $V_{\mathfrak{p}} = \text{Spec}(\mathcal{O}_{\mathfrak{p}})$ $\mathfrak{p}$ -adic field $\partial V_{\mathfrak{p}} = \text{Spec}(k_{\mathfrak{p}})$ local absolute Galois group $\text{Gal}(\overline{k}_{\mathfrak{p}}/k_{\mathfrak{p}})$	tubular n.b.d of a knot $V_{\mathcal{K}}$ boundary torus $\partial V_{\mathcal{K}}$ peripheral group $\pi_1(\partial V_{\mathcal{K}})$
union of $\mathfrak{p}_i$ -adic integer rings $V_S = \text{Spec}(\mathcal{O}_{\mathfrak{p}_1}) \sqcup \dots \sqcup \text{Spec}(\mathcal{O}_{\mathfrak{p}_r})$ union of $\mathfrak{p}_i$ -adic fields $\partial V_S = \text{Spec}(k_{\mathfrak{p}_1}) \sqcup \dots \sqcup \text{Spec}(k_{\mathfrak{p}_r})$	tubular n.b.d of a link $V_{\mathcal{L}} = V_{\mathcal{K}_1} \sqcup \dots \sqcup V_{\mathcal{K}_r}$ boundary tori $\partial V_{\mathcal{L}} = \partial V_{\mathcal{K}_1} \sqcup \dots \sqcup \partial V_{\mathcal{K}_r}$
complement of a finite set of primes $\overline{X}_S = \overline{X}_k \setminus S$ maximal Galois group with given ramification $\text{Gal}(k_S/k)$	link complement $X_{\mathcal{L}} = M \setminus \text{Int}(V_{\mathcal{L}})$ link group $\pi_1(X_{\mathcal{L}})$
ideal class group $Cl(k)$	1st homology group $H_1(\overline{M}; \mathbb{Z})$

Foundational analogies in arithmetic topology have been investigated by Mazur, Morishita, Kapranov and Reznikov etc systematically. So far, algebraic and topological aspects in arithmetic topology have been developed well. For example, as recent interesting works, we note the works by Niibo-Ueki [NU19] and Mihara

[Mi19] on idelic class field theory for 3-manifolds. Arithmetic topology is now a driving force to obtain analogous results in number theory and 3-dimensional topology.

In number theory, there is the deep analytic aspect such as the theory of zeta functions. So it would be desirable to develop a corresponding theory in 3-dimensional geometry. This would provide an analytic theory which refines, in dimension 3, the previous works by Selberg and Sunada etc ([Se56], [Su88a], [Su88b], [KS90]). For this problem, we may need to add geometric/analytic structures on a 3-manifold. As such approaches, we refer to [Ma12] and [Mc13] etc. In his monumental program motivated by the cohomological interpretation of arithmetic zeta functions ([D98], [D00], [D02a], [D02b], [D05], [D06], [D08]), Deninger considered more precise structures on a 3-manifold M , namely, a 1-codimensional Riemannian foliation $(\mathcal{F}, g_{\mathcal{F}})$ and a transverse dynamical system ϕ^t . It is called a Riemannian foliated dynamical system and we abbreviate it to an RFDS³. Here are some basic analogies in Deninger theory:

arithmetic curve $\overline{\text{Spec}(\mathcal{O}_k)}$	RFDS ³ $\mathfrak{S} = (M, \mathcal{F}, \phi^t, g_{\mathcal{F}})$
closed point $\text{Spec}(\mathcal{O}_k/\mathfrak{p})$	closed orbit γ
norm $N(\mathfrak{p}) := \#(\mathcal{O}_k/\mathfrak{p})$	exponential length $N(\gamma) := \exp(l(\gamma))$
Dedekind zeta function $\zeta_k(s) = \prod_{\mathfrak{p}} (1 - N(\mathfrak{p})^{-s})^{-1}$	dynamical zeta function $\zeta_{\mathfrak{S}}(s) = \prod_{\gamma} (1 - e^{-s \cdot l(\gamma)})^{-\epsilon_{\gamma}}$
hypothetical arithmetic cohomology $H_{\text{dyn}}^{\bullet}(\overline{\text{Spec}(\mathcal{O}_k)}, \mathbb{R})$	leafwise cohomology $\overline{H}_{\mathcal{F}}^{\bullet}(M)$

So the notion of an RFDS³ would be a suitable object for which precise analogies of analytic number theory could be developed. In fact, among other things, Deninger showed a geometric analog for an RFDS³ of the explicit formula in analytic number theory. Along this line of thoughts, in this thesis, we study dynamical zeta functions of 3-dimensional Riemannian foliated dynamical systems. Our main result is concerned with a cohomological description of the dynamical zeta function. We introduce an RFDS³-analog of the ξ -function and prove its analytic continuation. We then prove a formula expressing the dynamical zeta function in terms of the regularized determinant of a certain infinitesimal operator acting on the leafwise cohomology groups associated to the RFDS³.

The contents of this paper are organized with 2 chapters and each chapter is organized as follows: In Chapter 1, we introduce a foliated dynamical system of 3 dimension and a Riemannian foliated dynamical system of 3 dimension. In Section 1.1, we recall the notions of foliation and define a (Riemannian) foliated dynamical system of 3 dimension. In Section 1.2, we show a topological classification for FDS^3 's into 3 types. In Section 1.3, we give concrete examples for each type of the classification for FDS^3 's. Throughout Chapter 2, we show a cohomological expression of the dynamical zeta function of an RFDS^3 . In Section 2.1, we introduce basic notions for an RFDS^3 : leafwise cohomology and infinitesimal generator. In Section 2.2, we give a proof of the first main Theorem A for this chapter. In Section 2.3, we recall the zeta-regularized determinant and dynamical zeta function of an RFDS^3 . In Section 2.4, we show a leafwise cohomological expression of the dynamical zeta function of an RFDS^3 which is the second main Theorem B of this chapter.

Chapter 1 is based on [KMNT21] and Chapter 2 is based on [Kim17] and [Kim19].

Chapter 1. 3-dimensional foliated dynamical systems and classification

In this chapter, we introduce the notion of a 3-dimensional foliated dynamical system, called FDS³ for short, and show a topological classification of FDS³'s. We then give concrete examples of FDS³'s for each type of our classification. This chapter is based on [KMNT21].

1.1. 3-dimensional foliated dynamical systems

In this section, following Deninger and Kopei ([D98], [D00], [D02a], [D02b], [D05], [D06], [D08], [Ko06], [Ko11]), we introduce the notion of a 3-dimensional foliated dynamical system, called an FDS³ for short, and prepare some basic notions and properties. Although a foliated dynamical system can be introduced in any odd dimension, we consider only the 3-dimensional case in this paper, since we are concerned with analogies with arithmetic curves. For general materials in foliation theory, we refer to [CCF00] and [Tam92].

We begin to recall the notion of a complex foliation by Riemann surfaces (cf. [G99]).

Smooth foliation of 2-dimension. Let M be a smooth 3-manifold whose boundary ∂M may be non-empty. A *2-dimensional smooth foliation* $\mathcal{F}$ on M is a partition of sub-manifolds of 2-dimension. It is defined by a family of immersed 2-dimensional manifolds $\{L_a\}_{a \in A}$ satisfying the following conditions:

- (1) $M = \sqcup_{a \in A} L_a$ (disjoint union).
- (2) there is a system of foliated local coordinates $(U_i, \varphi_i)_{i \in I}$ where U_i is a foliated open subset of M and $\varphi_i : U_i \xrightarrow{\sim} \varphi_i(U_i) \subset \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$ is a diffeomorphism ($\mathbb{R}_{\geq 0} := \{t \in \mathbb{R} | t \geq 0\}$) such that if $U_i \cap L_a$ is non-empty, $\varphi(U_i \cap L_a) = \text{Int}(D) \times \{c_a\}$. Note that $\text{Int}(D)$ is the interior of a 2-disc $D \subset \mathbb{R}^2$ and $c_a \in \mathbb{R}_{\geq 0}$. Each L_a is called a *leaf* of the foliation $\mathcal{F}$.

Complex foliation. We call a foliation $\mathcal{F}$ on M a *complex foliation by Riemann surfaces* if we require further the following conditions:

- (3) each leaf L_a is a Riemann surface, namely, L_a has a complex structure.
- (4) the above condition with replacing $\mathbb{R}^2$ (leaf coordinate) by $\mathbb{C}$.
- (5) $(\varphi_j \circ \varphi_i^{-1})(z, t) = (f_{ij}(z, t), g_{ij}(t))$ for $(z, t) \in \varphi_i(U_i \cap U_j) \subset \mathbb{C} \times \mathbb{R}_{\geq 0}$, where f_{ij} is holomorphic in z and g_{ij} is independent of z .

Foliated 3-manifold. We call a pair of 3-manifold with a foliation $(M, \mathcal{F})$ simply a *foliated 3-manifold*.

Foliated 3-manifold by Riemann surfaces. We call a pair of 3-manifold with a complex foliation by Riemann surfaces $(M, \mathcal{F})$ a *foliated 3-manifold by Riemann surfaces*.

Remark 1.1.1. (1) By Frobenius' theorem, the following notions are equivalent if M is closed (cf. [CCF; §1.3], [Ta92; §28]):

- 2-dimensional smooth foliation on M ,
- 2-plane field distribution E (rank 2 subbundle of the tangent bundle TM) which is involutive,
- 2-plane field distribution E which is completely integrable.

(2) If the tangent bundle to a 2-dimensional smooth foliation $\mathcal{F}$ is orientable, $\mathcal{F}$ admits a structure of a complex foliation (cf. [HM16, Remark 1.2], [Hur06, Lemma A.3.1]). In fact, each orientable 2-dimensional leaf is equipped with a Riemannian metric and a smooth almost complex structure.

Morphism between foliated 3-manifolds. For foliated 3-manifolds $(M, \mathcal{F})$ and $(M', \mathcal{F}')$, a *morphism* $(M, \mathcal{F}) \rightarrow (M', \mathcal{F}')$ is defined to be a smooth map $f : M \rightarrow M'$ such that f maps any leaf L of $\mathcal{F}$ into a leaf L' of $\mathcal{F}'$. When $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ are foliated 3-manifolds by Riemann surfaces, we require further $f|_L : L \rightarrow L'$ to be holomorphic for any leaf L of $\mathcal{F}$. In terms of local coordinates, this is equivalent to saying that for any $p \in M$, there are foliated local coordinates (U, φ) and (U', φ') around p and $f(p)$, respectively, such that $(\varphi' \circ f|_{U \circ \varphi^{-1}})(z, t) = (f^t(z), t)$, where $f^t(z)$ is holomorphic for fixed t .

Isomorphism between foliated 3-manifolds. An *isomorphism* $(M, \mathcal{F}) \xrightarrow{\sim} (M', \mathcal{F}')$ of foliated 3-manifolds (resp. foliated 3-manifolds by Riemann surfaces) is a diffeomorphism $f : M \rightarrow M'$ such that $f|_L : L \rightarrow L'$ is diffeomorphic (resp. biholomorphic) for any leaf L of $\mathcal{F}$. We identify $(M, \mathcal{F})$ with $(M', \mathcal{F}')$ if there is an isomorphism between them.

Transverse flow. A *smooth dynamical system* (or *smooth flow*) on M is a smooth $\mathbb{R}$ -action

$$\phi : \mathbb{R} \times M \rightarrow M$$

such that $\phi^t := \phi(t, \cdot)$ is a diffeomorphism of M for each $t \in \mathbb{R}$ which satisfies $\phi^0 = \text{id}_M$, $\phi^{t+t'} = \phi^t \circ \phi^{t'}$ for any $t, t' \in \mathbb{R}$. A *transverse flow* ϕ is a smooth dynamical system on M which maps leaves of a foliation to leaves.

Now we introduce the main object in this paper (cf. [Ko11]).

Definition 1.1.2 (Foliated dynamical system). We define a 3-dimensional *foliated dynamical system* by a triple $\mathfrak{S} = (M, \mathcal{F}, \phi)$, where

- (1) M is a connected, closed smooth 3-manifold,
 - (2) $\mathcal{F}$ is a complex foliation by Riemann surfaces on M ,
 - (3) ϕ is a flow on M ,
- and these data must satisfy the following conditions:
- (i) there are finite number of compact leaves $L_1^\infty, \dots, L_r^\infty$, which may be empty, such that for any i and t we have $\phi^t(L_i^\infty) = L_i^\infty$ and that any orbit of the flow ϕ is transverse to leaves in $M \setminus \cup_{i=1}^r L_i^\infty$.
 - (ii) for each $t \in \mathbb{R}$ the diffeomorphism ϕ^t of M maps leaves to leaves, i.e. ϕ^t is a transverse flow.

In this paper, a 3-dimensional foliated dynamical system is called an FDS³ for short.

Throughout this paper, we shall use the following notations. For an FDS³ $\mathfrak{S} = (M, \mathcal{F}, \phi)$, we set

$$\begin{aligned}
\mathcal{F}^c &:= \{L \in \mathcal{F} \mid L : \text{compact}\}. \\
\mathcal{P}_\mathfrak{S}^\infty &:= \{L \in \mathcal{F}^c \mid L : \text{non-transverse}\} \\
&= \{L_1^\infty, \dots, L_r^\infty\} \text{ (not necessarily non-empty).} \\
L^\infty &:= \cup_{i=1}^r L_i^\infty. \\
M_0 &:= M \setminus L^\infty. \\
\mathcal{P}_\mathfrak{S} &:= \text{the set of closed orbits of the flow } \phi \text{ which are transverse to} \\
&\text{leaves in } M_0. \\
\overline{\mathcal{P}_\mathfrak{S}} &:= \mathcal{P}_\mathfrak{S} \sqcup \mathcal{P}_\mathfrak{S}^\infty.
\end{aligned}$$

Remark 1.1.3. As Deninger and Kopei suggested (cf. [D98] $\sim$ [D08], [Ko06], [Ko11]), an FDS³ $\mathfrak{S} = (M, \mathcal{F}, \phi)$ may be regarded as a geometric analogue of a compact arithmetic curve, namely, a smooth proper algebraic curve over a finite field of $\overline{\text{Spec}(\mathcal{O}_k)} = \text{Spec}(\mathcal{O}_k) \cup \mathcal{P}_k^\infty$ for the ring $\mathcal{O}_k$ of integers of a number field k , where $\mathcal{P}_k^\infty$ is the set of infinite primes of k . The set $\mathcal{P}_\mathfrak{S}$ corresponds to the set of closed points (finite primes) of C or $\text{Spec}(\mathcal{O}_k)$. When $\mathcal{P}_\mathfrak{S}^\infty$ is non-empty, it corresponds to $\mathcal{P}_k^\infty$. So the analogy is closer if $\mathcal{P}_\mathfrak{S}$ is countably infinite set. We give such examples in the section 2.

FDS³-morphism. Let $\mathfrak{S} = (M, \mathcal{F}, \phi)$ and $\mathfrak{S}' = (M', \mathcal{F}', \phi')$ be an FDS³'s. An *FDS³-morphism* $\mathfrak{S} \rightarrow \mathfrak{S}'$ is defined by a morphism of foliated 3-manifolds by Riemann surfaces such that $f : M \rightarrow M'$ commutes with the flows, namely, $f \circ \phi^t = \phi'^t \circ f$ for any $t \in \mathbb{R}$.

FDS³-isomorphism. An *FDS³-isomorphism* $\mathfrak{S} \xrightarrow{\sim} \mathfrak{S}'$ is an FDS³-morphism which is an isomorphism of foliated 3-manifolds by Riemann surfaces. We identify $\mathfrak{S}$ with $\mathfrak{S}'$ if there is an FDS³-isomorphism between them.

Leafwise tangent bundle. For an FDS³ $\mathfrak{S} = (M, \mathcal{F}, \phi)$, let $T\mathcal{F}$ denote the subbundle of the tangent bundle TM whose total space is the union of the tangent spaces of leaves. We call the subbundle $T\mathcal{F}$ a *Leafwise tangent bundle*. Let $\dot{\phi}^t = \frac{d}{dt}\phi^t$ be the vector field on M which generates the flow ϕ .

Lemma 1.1.4. *For an FDS³ $\mathfrak{S} = (M, \mathcal{F}, \phi)$, there is the unique closed smooth 1-form ω on M_0 satisfying*

$$\begin{aligned}\omega|_{T\mathcal{F}} &= 0 \\ \omega(\dot{\phi}^t) &= 1.\end{aligned}$$

More precisely, let $(M, \mathcal{F}, \phi)$ be a triple satisfying (1), (2), (3), and (i) in Definition 1.1.2. Then there is the unique smooth 1-form ω on M_0 satisfying the above conditions, and the condition (ii) is equivalent to that ω is closed.

PROOF. Let $(M, \mathcal{F}, \phi)$ be a triple satisfying (1), (2), (3), and (i) in Definition 1.1.2. Then we can take a smooth 1-form ω on M_0 such that for each foliated coordinate $(z, t) \in U$, $\omega|_U = h(z, t)dt$. Writing $\dot{\phi}^t = a(z, t)\partial_z + b(z, t)\partial_{\bar{z}} + c(z, t)\partial_t$, we have $\omega(\dot{\phi}^t) = c(z, t)h(z, t)$ and so $h(z, t)$ is uniquely determined by $\omega(\dot{\phi}^t) = 1$. This yields the first assertion.

Now, assume the condition (ii). For each point $p \in M_0$, there is a foliated local coordinates $(U_i; \varphi_i)_{i \in I}$ of M_0 such that if $U_i \cap U_j$ is not empty, $(\varphi_j \circ \varphi_i^{-1})(z_i, t_i) = (f_{ij}(z, t), t + c_{ij})$ for some constant $c_{ij} \in \mathbb{R}$. Hence $\omega|_{U_i} = dt_i$ and so ω is closed.

Conversely, assume that $d\omega = 0$. We may take each U_i is simply connected so that $\omega|_{U_i}$ by Poincaré's Lemma. Then we can follow the above argument in the reverse way to obtain the condition (ii). $\square$

Definition 1.1.5 (Canonical 1-form). Let $\mathfrak{S} = (M, \mathcal{F}, \phi)$ be an FDS³. We call the smooth closed 1-form in Lemma 1.1.4 the *canonical 1-form* of $\mathfrak{S}$ and denote it by $\omega_{\mathfrak{S}}$. The de Rham cohomology class of $\omega_{\mathfrak{S}}$ defines the *period homomorphism*

$$[\omega_{\mathfrak{S}}] : H_1(M_0; \mathbb{Z}) \rightarrow \mathbb{R}; [l] \mapsto \int_l \omega_{\mathfrak{S}},$$

and the *period group* of $\mathfrak{S}$ is defined by the image of $[\omega_{\mathfrak{S}}]$, which we denote by $\Lambda_{\mathfrak{S}}$

Lemma 1.1.6. *Let $\mathfrak{S}_1 = (M, \mathcal{F}, \phi_1)$ and $\mathfrak{S}_2 = (M, \mathcal{F}, \phi_2)$ be FDS³'s. Then we have $\omega_{\mathfrak{S}_1} = \omega_{\mathfrak{S}_2}$ if and only if $(\dot{\phi}_1^t - \dot{\phi}_2^t)_p \in T_p\mathcal{F}$ for any $p \in M_0$.*

PROOF. Since $\omega_{\mathfrak{S}_i}|_{T\mathcal{F}} = 0$, $\omega_{\mathfrak{S}_i}(\dot{\phi}_i^t) = 1$ for $i = 1, 2$, we have

$$\begin{aligned} (\dot{\phi}_1^t - \dot{\phi}_2^t)_p \in T_p\mathcal{F} \text{ for any } p \in M_0 &\Leftrightarrow \omega_{\mathfrak{S}_1}(\dot{\phi}_1^t - \dot{\phi}_2^t) = 0 \\ &\Leftrightarrow \omega_{\mathfrak{S}_1}(\dot{\phi}_1^t) = \omega_{\mathfrak{S}_1}(\dot{\phi}_2^t) = 1 \\ &\Leftrightarrow \omega_{\mathfrak{S}_1}|_{T\mathcal{F}} = 0, \omega_{\mathfrak{S}_1}(\dot{\phi}_2^t) = 1 \\ &\Leftrightarrow \omega_{\mathfrak{S}_1} = \omega_{\mathfrak{S}_2}. \end{aligned}$$

□

1.2. Topological classification of FDS³'s

In this section, we show a decomposition theorem for an FDS³, which yields a topological classification of FDS³'s. We start with some preparation about the holonomy of a leaf and Tischler's theorem.

Foliations without holonomy and Tischler's theorem. First, we recall the notion of the holonomy group of a leaf (cf. [CCF; §2.2, §2.3], [Ta92; §22]). Let $(M, \mathcal{F})$ be a foliated 3-manifold and let L be a leaf of $\mathcal{F}$. Choose $p \in L$ and let $[c] \in \pi_1(L, p)$ be represented by a loop $c : [0, 1] \rightarrow L$ with $c(0) = c(1) = p$. We may choose a subdivision $0 = t_0 < t_1 < \cdots < t_{m+1} = 1$ and a chain of foliated charts $(U_0; \varphi_0), \dots, (U_m; \varphi_m)$ such that $c(t_i, t_{i+1}) \subset U_i$ for $0 \leq i \leq m$. Then we have

$$(\varphi_{i+1} \circ \varphi_i^{-1})(z, t) = (f_{i,i+1}(z, t), g_{i,i+1}(t))$$

for some smooth function $f_{i,i+1}, g_{i,i+1}$. We set

$$h_c := g_{m-1,m} \circ \cdots \circ g_{0,1},$$

which is a local homeomorphism of $\mathbb{R}$ fixing 0. Let G be the group of germs of homeomorphisms of $\mathbb{R}$ fixing 0 and let $\hat{h}_c \in G$ be the germ of h_c . Then the corresponding $[c] \mapsto \hat{h}_c$ gives a well-defined homomorphism $\Psi : \pi_1(L, p) \rightarrow G$. The holonomy group $\mathcal{H}(L)$ of L is defined by the image of Ψ , which is independent, up to conjugation, of all choices. A foliated 3-manifold $(M, \mathcal{F})$ is said to be *without holonomy* if the holonomy group $\mathcal{H}(L)$ is trivial for any leaf L of $\mathcal{F}$.

The next theorem, due to Candel and Conlon, is an important ingredient for our decomposition theorem given in the following subsection. For the proof, we refer to [CCF; §9.1]. For a foliated 3-manifold $(M, \mathcal{F})$, a subset X of M is said to be *$\mathcal{F}$ -saturated* if X is a union of leaves of $\mathcal{F}$.

Theorem 1.2.1 ([CCF; Theorem 9.1.4]). *Let $(M, \mathcal{F})$ be a foliated 3-manifold with M being closed. Let X be a connected $\mathcal{F}$ -saturated open subset of M . Assume that the foliated 3-manifold $(X, \mathcal{F}|_X)$ is without holonomy. Then one of the following holds:*

- (1) X is a surface bundle over S^1 or an open interval, and $\mathcal{F}|_X$ is the bundle foliation.
- (2) Each leaf of $\mathcal{F}|_X$ is dense in X .

Next, we recall Tischler's theorem ([Tis70]) and its generalization ([CC81]).

Theorem 1.2.2 ([Ti70; Theorem 1], [CC81; Theorem 2.1]). *Let $(M, \mathcal{F})$ be a foliated 3-manifold with M being closed. Let X be an $\mathcal{F}$ -saturated open set of M . Assume that there is a non-vanishing, closed 1-form ω on X such that $\text{Ker}(\omega)$ defines $\mathcal{F}|_X$. Then X is a surface bundle over S^1 .*

Remark 1.2.3. A non-vanishing, closed 1-form assumed in Theorem 1.2.2 is approximated (in the C^∞ -topology) by the 1-form $\varpi^*(d\theta)$, where $\varpi : X \rightarrow S^1$ is the fibration and $d\theta$ is an angular 1-form on S^1 (cf. [ibid]).

A decomposition theorem. Let $\mathfrak{S} = (M, \mathcal{F}, \phi)$ be an FDS³. Let $X_1, \dots, X_d$ be connected components of $M_0 := M \setminus L^\infty$. Each $(X_a, \mathcal{F}|_{X_a})$ is a foliated 3-manifold for $1 \leq a \leq d$.

Lemma 1.2.4. *The foliated 3-manifold $(X_a, \mathcal{F}|_{X_a})$ is without holonomy.*

PROOF. Let L be any leaf of $\mathcal{F}|_{X_a}$. Since ϕ^t maps any leaf to a leaf, there is a system foliated charts $\{(U_i; \varphi_i)\}$ which covers L and satisfies

$$\varphi_i(\phi^t(p)) = (z_i(p), t)$$

for any $p \in L$. Therefore, if $U_i \cap U_j$ is non-empty, we have

$$(\varphi_i \circ \varphi_j^{-1})(z_j, t) = (z_i(\varphi_j^{-1}(z_j, t)), t)$$

and so the holonomy group $\mathcal{H}(L)$ is trivial. Hence $(X_a, \mathcal{F}|_{X_a})$ is without holonomy. $\square$

Theorem 1.2.5. (A decomposition theorem). *Let $\mathfrak{S} = (M, \mathcal{F}, \phi)$ be an FDS³ and let X_a be a connected components of $M_0 := M \setminus L^\infty$. The foliated 3-manifold $(X_a, \mathcal{F}|_{X_a})$ is one of the followings:*

- (1) X_a is a surface bundle over S^1 or an open interval, and $\mathcal{F}|_{X_a}$ is the bundle foliation.
- (2) X_a is a surface bundle over S^1 and any leaf in $\mathcal{F}|_{X_a}$ is dense in X_a .

PROOF. By Definition 1.1.2 of an FDS³, X_a is a connected $\mathcal{F}$ -saturated open subset of M . By Theorem 1.2.1, (1) X_a is a surface bundle over S^1 or an open interval, and $\mathcal{F}|_{X_a}$ is the bundle foliation, or (2) each leaf of $\mathcal{F}|_{X_a}$ is dense in X_a . For the case (2), X_a is a surface bundle over S^1 by Lemma 1.1.4 and Theorem 1.2.2. $\square$

Remark 1.2.6. For an FDS³ $\mathfrak{S} = (M, \mathcal{F}, \phi)$, if M is cut along L^∞ , then M_0 is decomposed into connected components $M_0 = X_1 \sqcup \cdots \sqcup X_d$, where each foliated 3-manifold $(X_a, \mathcal{F}|_{X_a})$ has the structure described in Theorem 1.2.5 and the flow $\phi|_{X_a}$ is transverse to leaves of $\mathcal{F}|_{X_a}$. This is the reason that we call Theorem 1.2.5 a *decomposition theorem*. It may remind us of the JSJ decomposition of a 3-manifold ([JS79], [Jo79]).

Theorem 1.2.5 can be restated as the following classification of FDS³'s. We use the same notations as in Section 1.1.

Corollary 1.2.7 (A classification). An FDS³ $\mathfrak{S} = (M, \mathcal{F}, \phi)$ is classed as one of the following types:

- I. $\mathcal{F} = \mathcal{F}^c$ and $\mathcal{P}_\mathfrak{S}^\infty$ is empty.
Then M is a fiber bundle over S^1 and $\mathcal{F}$ is the bundle foliation.
- II. $\mathcal{F}^c$ is empty.
Then M is a surface bundle over S^1 and any leaf of $\mathcal{F}$ is dense in M .
- III. $\mathcal{P}_\mathfrak{S}^\infty$ is a non-empty (finite) set.
Let X_a be a connected component of M_0 . Then the foliated 3-manifold $(X_a, \mathcal{F}|_{X_a})$ is one of the followings.
 - (a) X_a is a surface bundle over S^1 and $\mathcal{F}|_{X_a}$ is the bundle foliation.
 - (b) X_a is a surface bundle over an open interval and $\mathcal{F}|_{X_a}$ is the bundle foliation.
 - (c) X_a is a surface bundle over S^1 and any leaf in $\mathcal{F}|_{X_a}$ is dense in X_a .

We note that the class of FDS³'s of bundle foliation over S^1 is characterized by the period group. Although this may be known (cf. [CCF00; §9.3], [F04; §2.1]), we give a proof in the following, for the sake of readers.

Proposition 1.2.8. *Let $\mathfrak{S}$ be an FDS³. If $\mathfrak{S}$ is of type I or of type III-(a), then the period group $\Lambda_\mathfrak{S} = \mathbb{Z}$. Conversely, if the $\Lambda_\mathfrak{S}$ has rank one (namely, $\Lambda_\mathfrak{S} \simeq \mathbb{Z}$) and M_0 is connected, then $\mathfrak{S}$ is of type I or of type III-(a).*

PROOF. Suppose that $\mathfrak{S} = (M, \mathcal{F}, \phi)$ is an FDS³ of type I or of type III-(a). Then, for any connected component X_a of M_0 , there is a fibration $\varpi_a : X_a \rightarrow S^1$. Let $d\theta$ be the angular form 1-form on S^1 such that $\int_{S^1} d\theta = 1$. Then the canonical 1-form $\omega_\mathfrak{S}$ is given by $\omega_\mathfrak{S}|_{X_a} = \varpi_a^*(d\theta)$ for any a . For $[l] \in H_1(X_a; \mathbb{Z})$, we have the degree of $\varpi_*(l)$ on S^1

$$[\omega_\mathfrak{S}]([l]) = \int_{(\varpi_a)_*(l)} d\theta$$

for any a and hence $\Lambda_{\mathfrak{S}} = \mathbb{Z}$. Conversely, suppose that $\mathfrak{S} = (M, \mathcal{F}, \phi)$ is an FDS³ with M_0 begin connected and $\Lambda_{\mathfrak{S}} = \lambda\mathbb{Z}$ for some $\lambda \in \mathbb{R}^\times$ so that $[\omega_{\mathfrak{S}}](H_1(M_0; \mathbb{Z})) = \lambda\mathbb{Z}$. Fix a base point $p_0 \in M_0$ and define the map $\varpi : M_0 \rightarrow S^1$ by

$$\varpi(p) := \exp\left(\frac{2\pi i}{\lambda} \int_{\gamma} \omega_{\mathfrak{S}}\right),$$

where γ is a path from p_0 to p . Then we see easily that $\lambda\varpi^*(d\theta) = \omega_{\mathfrak{S}}$. By Definition 1.1.5 of $\omega_{\mathfrak{S}}$, ϖ is a fibration and $\mathcal{F}$ consists of fibers of ϖ . Hence $\mathfrak{S}$ is of type I or of type III-(a). $\square$

Finally, we note that an FDS³ of type III-(b) has no transverse closed orbits.

Proposition 1.2.9. *Let $\mathfrak{S} = (M, \mathcal{F}, \phi)$ be an FDS³ of type III-(b). Then $\mathcal{P}_{\mathfrak{S}}$ is empty and $\Lambda_{\mathfrak{S}} = \{0\}$.*

PROOF. Let X_a be a connected component of M_0 . We may identify $(X_a, \mathcal{F}|_{X_a})$ with $(L \times (0, 1), \{L \times \{t\}\}_{t \in (0, 1)})$, where $L \in \mathcal{F}|_{X_a}$. Let $\varpi : X_a \rightarrow (0, 1)$ be the projection. For any closed curve $l : S^1 \rightarrow X_a$, $\varpi \circ l : S^1 \rightarrow (0, 1)$ has the maximum μ and so l is not transverse to the leaf $L \times \{\mu\}$. Hence $\mathcal{P}_{\mathfrak{S}}$ is empty. Next, since X_a is homotopy equivalent to the leaf $L \times \{\frac{1}{2}\}$, l is homotopic to a closed curve l' in $L \times \{\frac{1}{2}\}$. Since $\omega_{\mathfrak{S}}|_{T\mathcal{F}} = 0$, we have $[\omega_{\mathfrak{S}}](l) = [\omega_{\mathfrak{S}}](l') = 0$. Hence $\Lambda_{\mathfrak{S}} = \{0\}$. $\square$

1.3. Examples of FDS³'s

In this section, we construct concrete examples of FDS³'s for each type in Corollary 1.2.7.

Type I. We give an example of an FDS³ of type I. Note that any smooth surface bundle over S^1 is obtained by the mapping torus of a surface diffeomorphism.

Example 1.3.I (Mapping torus and pseudo-Anosov flow). Let R_g be a connected, closed smooth surface of genus $g \geq 1$ and let φ be a diffeomorphism of R_g . Let M be the *mapping torus* $M(R_g, \varphi)$ defined by

$$M(R_g, \varphi) := (R_g \times [0, 1]) / (z, 1) \sim (\varphi(z), 0).$$

Then the projection

$$\varpi : M \rightarrow S^1 = \mathbb{R}/\mathbb{Z}; \varpi([z, s]) = s(\text{mod } \mathbb{Z})$$

is a fibration, where each fiber $\varpi^{-1}(\theta)$ over $\theta \in S^1$ is diffeomorphic to R_g . The set of fibers $\mathcal{F} := \{\varpi^{-1}(\theta)\}_{\theta \in S^1}$ defines a 2-dimensional foliation, the bundle

foliation. Since $T\mathcal{F}$ is orientable, $\mathcal{F}$ admits a complex foliation structure (cf. Remark 1.1.1 (2)). Let ϕ be the suspension flow defined by

$$\phi^t([z, s]) := [z, s + t].$$

Then $\mathfrak{S} := (M, \mathcal{F}, \phi)$ forms an FDS³. Suppose further that the diffeomorphism φ is of pseudo-Anosov type (cf. [Ca07; §1.11]). Note that φ is an Anosov diffeomorphism when R_g is a torus. Then, since φ has countably infinite periodic points, $\mathcal{P}_{\mathfrak{S}}$ is a countably infinite set. By Proposition 1.2.8, the canonical 1-form $\omega_{\mathfrak{S}}$ is $\varpi^*(d\theta)$ and the period group $\Lambda_{\mathfrak{S}} = \mathbb{Z}$.

Remark 1.3.1. An FDS³ of type I may be regarded as an analogue of a smooth proper algebraic curve C over a finite field $\mathbb{F}_q$, where the 2-dimensional foliation corresponds to the geometric fiber $C \otimes \overline{\mathbb{F}_q}$ and the monodromy φ corresponds to the Frobenius automorphism in $\text{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$.

Type II. We construct two examples of FDS³'s of type II. First, we give a method to construct countably infinitely many closed orbits using the horseshoe map. Let $\mathfrak{S} = (M, \mathcal{F}, \phi)$ be an FDS³ and $\gamma \in \mathcal{P}_{\mathfrak{S}}$. We say that the flow ϕ is of *contraction type* around γ if there is a tubular neighborhood $V = D \times S^1$ of γ , where $D(\subset C)$ is a 2-disc centered at 0 and $\gamma = \{0\} \times S^1$, such that for any $t \in \mathbb{R}$ and $p \in \gamma$, $\phi^t \subset V$ and

$$\phi_{D,p}^t := \phi^t|_{D \times \{p\}} : D = D \times \{p\} \xrightarrow{\phi^t} \phi^t(D) \times \{\phi^t(p)\} \subset D$$

is a contraction map, namely, $T_z(\phi_{D,p}^t)$ has the eigenvalues λ_1, λ_2 satisfying $0 < |\lambda_1|, |\lambda_2| < 1$ for $z \in \text{Int}(D) \setminus \{0\}$. Let $h : \text{Int}(D) \rightarrow \text{Int}(D)$ be the *horseshoe diffeomorphism* ([Sm00]) and let $U = M(\text{Int}, h)$ be the mapping torus of h equipped with the suspension flow ϕ_h . Note that there is a diffeomorphism $\psi : U \xrightarrow{\sim} \text{Int}(V)$.

Lemma 1.3.2. *Notations being as above, let us replace $(\text{Int}(V), \phi|_{\text{Int}(V)})$ by (U, ϕ_h) via ψ so that the resulting 3-manifold $M_{\gamma,h}$ is equipped with a new smooth flow $\phi_{\gamma,h}$ satisfying $\phi_{\gamma,h} = \phi_h$ in U and $\phi_{\gamma,h} = \phi$ in $M \setminus \text{Int}(V)$. We define the foliation $\mathcal{F}_{\gamma,h}$ on $M_{\gamma,h}$ by the foliation $\mathcal{F}$ on M via ψ . Then the triple $\mathfrak{S}_{\gamma,h} = (M_{\gamma,h}, \mathcal{F}, \phi_{\gamma,h})$ forms an FDS³ and the flow $\phi_{\gamma,h}$ has countably infinitely many closed orbits around γ . Moreover, the canonical 1-forms of $\mathfrak{S}$ and $\mathfrak{S}_{\gamma,h}$ are same, $\omega_{\mathfrak{S}} = \omega_{\mathfrak{S}_{\gamma,h}}$.*

PROOF. The triple $\mathfrak{S}_{\gamma,h} = (M_{\gamma,h}, \mathcal{F}, \phi_{\gamma,h})$ forms an FDS³ is easily seen by the construction. The latter property follows from that the horseshoe map has countably infinitely many periodic points ([KH97; Corollary 2.5.1]). For any $p \in \gamma$, we may assume, by the change of parameters, that there is $t_0 > 0$ such that $\phi^{t_0}|_{\text{Int}(D) \times \{p\}} : \text{Int}(D) \times \{p\} \rightarrow \phi^{t_0}(\text{Int}(D)) \times \{p\}$ and $\phi_h^{t_0}|_{\text{Int}(D) \times \{p\}} : \text{Int}(D) \times \{p\} \rightarrow \phi_h^{t_0}(\text{Int}(D)) \times \{p\}$ are the first return diffeomorphisms. Since these are isotopic

and an isotopy is realized as a flow in $\text{Int}(V)$, we see that the canonical 1-forms are unchanged. $\square$

Example 1.3.II.1. The following example of a foliated 3-manifold was considered in [Ca07; S4.2] and [Hu06; § A.5]. (See also Álvarez-López's example in [D06; page 10].) Let T^2 be the 2-dimensional torus $\mathbb{C}/\mathbb{Z}^2$ so that the fundamental group $\pi_1(T^2)$ is generated by the homotopy classes of a meridian and a longitude, say $\mathfrak{m}$ and $\mathfrak{l}$, respectively. Let $\rho : \pi_1(T^2) \rightarrow \mathbb{R}$ be a given homomorphism such that $\rho(\mathfrak{m}) \notin \mathbb{Q}$ or $\rho(\mathfrak{l}) \notin \mathbb{Q}$, and we set $\bar{\rho}(g) := \rho(g) \bmod \mathbb{Z} \in S^1$ for $g \in \pi_1(T^2)$. Let M be the quotient 3-manifold of $\mathbb{C} \times S^1$ by the action of $\pi_1(T^2)$

$$M := (\mathbb{C} \times S^1)/\pi_1(T^2),$$

where $\pi_1(T^2)$ acts on the universal cover $\mathbb{C}$ of T^2 as the monodromy and on S^1 by $\theta \mapsto \theta + \bar{\rho}(g)$ for $\theta \in S^1$ and $g \in \pi_1(T^2)$. Let L_θ denote the image in M of $\mathbb{C} \times \{\theta\}$ in M . Then $\mathcal{F} := \{L_\theta\}_{\theta \in S^1}$ forms a 2-dimensional foliation on M . Here we see that

$$L_\theta = \begin{cases} S^1 \times \mathbb{R} & \text{if } \rho(\mathfrak{m}) \notin \mathbb{Q}, \rho(\mathfrak{l}) \in \mathbb{Q} \text{ or } \rho(\mathfrak{m}) \in \mathbb{Q}, \rho(\mathfrak{l}) \notin \mathbb{Q}, \\ \mathbb{R}^2 & \text{if } \rho(\mathfrak{m}), \rho(\mathfrak{l}) \notin \mathbb{Q} \text{ and } \rho(\mathfrak{m})/\rho(\mathfrak{l}) \notin \mathbb{Q}, \end{cases}$$

and that any leaf L_θ is dense in M . Since $T\mathcal{F}$ is orientable, $\mathcal{F}$ admits a complex foliation structure (Remark 1.1.1 (2)). Let ϕ_1 be the flow defined by

$$\phi_1^t([z, \theta]) := [z, \theta + \bar{t}]$$

where $\bar{t} := t \bmod \mathbb{Z}$. Then we obtain an FDS³ $\mathfrak{S}_1 := (M, \mathcal{F}, \phi_1)$ of type II. The canonical 1-form $\omega_{\mathfrak{S}_1}$ is given by $\omega_{\mathfrak{S}_1}|_{T\mathcal{F}} = 0$ and $\omega_{\mathfrak{S}_1}|_{TS^1} = d\theta$, and the period group $\Lambda_{\mathfrak{S}_1} = \mathbb{Z} + \rho(\mathfrak{m})\mathbb{Z} + \rho(\mathfrak{l})\mathbb{Z}$. For this example, however, any orbit of the flow ϕ_1 is closed with period 1 and hence $\mathcal{P}_{\mathfrak{S}}$ is uncountable.

In order to obtain an FDS³ $\mathfrak{S}$ with countably infinite $\mathcal{P}_{\mathfrak{S}}$, we interpret the above $(M, \mathcal{F})$ from a different view and define a new dynamical system. In fact, the natural map $\mathbb{C} \times S^1 \rightarrow T^2 \times S^1$ induces the diffeomorphism

$$M \xrightarrow{\sim} T^3 = (\mathbb{R}/\mathbb{Z})^3.$$

Then the leaf L_θ is given by

$$L_\theta := \{(\theta_1, \theta_2, \theta - \rho(\mathfrak{m})\theta_1 - \rho(\mathfrak{l})\theta_2) | \theta_1, \theta_2 \in S^1\}$$

for $p = (\theta_1, \theta_2, \theta_3) \in T^3$. Let $V_{\mathfrak{m}} := (1, 0, -\rho(\mathfrak{m}))$, $V_{\mathfrak{l}} := (0, 1, -\rho(\mathfrak{l}))$ be vector fields on T^3 , and we define the smooth dynamical system ϕ_2 on T^3 by the equation

$$\begin{aligned} \frac{d}{dt}\phi_2^t(p) &:= (0, 0, 1) + \sin(2\pi\theta_1)V_{\mathfrak{m}} + \sin(2\pi\theta_2)V_{\mathfrak{l}} \\ &= (\sin(2\pi\theta_1), \sin(2\pi\theta_2), 1 - \rho(\mathfrak{m})\sin(2\pi\theta_1) - \rho(\mathfrak{l})\sin(2\pi\theta_2)). \end{aligned}$$

Then we obtain an FDS³ $\mathfrak{S}_2 := (T^3, \mathcal{F}, \phi_2)$ of type II. Since V_m and V_l are tangent to $\mathcal{F}$ and so $(\dot{\phi}_1^t - \dot{\phi}_2^t)_p \in T_p \mathcal{F}$ for $p \in M = T^3$, we have $\omega_{\mathfrak{S}_2} = \omega_{\mathfrak{S}_1}$, $\Lambda_{\mathfrak{S}_2} = \Lambda_{\mathfrak{S}_1}$ by Lemma 1.1.6. We see that $\mathcal{P}_{\mathfrak{S}_2}$ consists of the following four closed orbits

$$\begin{cases} \gamma_1 = \{(0, 0, \theta) | \theta \in S^1\}, \\ \gamma_2 = \{(\frac{1}{2}, 0, \theta) | \theta \in S^1\}, \gamma'_2 = \{(0, \frac{1}{2}, \theta) | \theta \in S^1\}, \\ \gamma_3 = \{(\frac{1}{2}, \frac{1}{2}, \theta) | \theta \in S^1\}, \end{cases}$$

and that ϕ_2 is of contracting type around γ_3 . We replace a tubular neighborhood of γ_3 by the suspension of the horseshoe map. By Lemma 1.3.2, we obtain an FDS³ $\mathfrak{S} = (M, \mathcal{F}, \phi)$ of type II such that $\mathcal{P}_{\mathfrak{S}}$ is countably infinite, and the canonical 1-form $\omega_{\mathfrak{S}}$ and the period group $\Lambda_{\mathfrak{S}}$ are same as $\omega_{\mathfrak{S}_i}$ and $\Lambda_{\mathfrak{S}_i}$ ($i = 1, 2$), respectively.

Example 1.3.II.2. Let $T^2 := (\mathbb{R}/\mathbb{Z})^2$ be the 2-dimensional torus and let $\varphi_A : T^2 \rightarrow T^2$ be the linear Anosov diffeomorphism of T^2 defined by the matrix

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}.$$

Then there are two fixed points $p_1 = (0, 0)$ and $p_2 = (\frac{1}{2}, 0)$ of φ_A . For each p_i from T^2 and glue $(T_{p_i}(T^2) \setminus \{0\})/\mathbb{R}_+ = S^1$ on it. Then we obtain a twice-punctured torus T_*^2 with $\partial T_*^2 = S_1^1 \sqcup S_2^1$, $S_i^1 = S^1$, and the diffeomorphism $\varphi_A^* : T_*^2 \rightarrow T_*^2$.

Let $M^* = M(T_*^2, \varphi_A^*)$ be the mapping torus of φ_A^* , which has two boundary components, say T_1 , and T_2 , which correspond to p_1 and p_2 , respectively. Let $\mathcal{F}^* = \{L_\theta\}_{\theta \in S^1}$ be the bundle foliation of the fibration $\varpi_* : M^* \rightarrow S^1$ and let ϕ_* be the suspension flow of φ_A^* . For each i , $\mathcal{F}^*|_{T_i}$ defines a foliation on T_i whose leaves are S^1 . The 1-form $\omega_* := \varpi_*^{-1}(d\theta)$ satisfies $\omega_*|_{T\mathcal{F}^*} = 0$, $\omega_*(\dot{\phi}_*^t) = 1$.

Finally, let $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ and consider the diffeomorphism $\psi_\lambda : T_1 \rightarrow T_2$ defined by sending L_θ to $L_{\theta+\lambda}$. We define M to be the 3-manifold obtained from M^* by gluing T_1 and T_2 via ψ_λ . In fact, M is a Σ_2 -bundle over S^1 . The foliation $\mathcal{F}$ and the flow ϕ on M are induced by $\mathcal{F}^*$ and ϕ_* , and $\mathcal{F}$ has a complex structure. Then any leaf in $\mathcal{F}$ is dense in M by the way of the gluing. Thus we obtain the FDS³ $\mathfrak{S} = (M, \mathcal{F}, \phi)$ of type II. Further, since ϕ comes from the suspension of the Anosov diffeomorphism, $\mathcal{P}_{\mathfrak{S}}$ is a countably infinite set. The canonical 1-form $\omega_{\mathfrak{S}}$ is induced by ω_* and the period group $\Lambda_{\mathfrak{S}} = \mathbb{Z} + \lambda\mathbb{Z}$.

Type III. We construct examples of FDS³'s of type III. This type of FDS³ may be regarded as an analogue of a number ring $\text{Spec}(\overline{\mathcal{O}_k}) = \text{Spec}(\mathcal{O}_k) \cup \mathcal{P}_k^\infty$ in the respect that $\mathcal{P}_{\mathfrak{S}}^\infty$ corresponds to the set $\mathcal{P}_k^\infty$ of infinite primes of a number field k (cf. Remark 1.1.3).

Type III-(a). We give two examples of FDS³'s of type III-(a).

Example 1.3.III-(a).1 (Reeb foliation on S^3 and the horseshoe flow). Let M be the 3-sphere S^3 . Let $M := S^3 = V_1 \cup V_2$ be the Heegaard splitting of genus one, where V_i is a solid torus $D^2 \times S^1$ ([He04; Chapter 2]). Consider the *Reeb foliation* $\mathcal{F}_i$ on each V_i and so the 2-dimensional foliation $\mathcal{F}$ on S^3 by getting $\mathcal{F}_1$ and $\mathcal{F}_2$ together, where any leaf is diffeomorphic to $\mathbb{R}^2$ besides the only one compact leaf $L^\infty = \partial V_1 = \partial V_2$ ([CCF00; §1.1], [Ta92; §1]). We define the dynamical system ϕ as follows.

First, we consider a flow ϕ_1 on M such that any orbit of ϕ_1 is transverse to leaves in $\mathcal{F} \setminus L^\infty$ and $\phi_1^t(L^\infty) = L^\infty$ for $t \in \mathbb{R}$. Then there is the only one closed orbit $\gamma_i = \{0\} \times S^1$ in each V_i (0 being the center of D^2). In fact, the flow ϕ_1 on $\text{Int}(V_i)$ is the suspension flow of a contraction map $\varphi_i : \text{Int}(D^2) \ni z \mapsto az \in \text{Int}(D^2)$ ($0 < a < 1$) and so ϕ_1 is of contraction type around γ_i .

Next, we replace the contraction map φ_i by the horseshoe map h and the flow ϕ_1 around γ_i by the suspension of h . Let ϕ be the resulting flow on all of M . By Lemma 1.3.2, we have an FDS³ $\mathfrak{S} = (M, \mathcal{F}, \phi)$ of type III-(a) such that $\mathcal{P}_{\mathfrak{S}}$ is a countably infinite set and $\mathcal{P}_{\mathfrak{S}}^\infty$ consists of the only one non-transverse compact leaf L^∞ . We have the decomposition of $M_0 = S^3 \setminus L^\infty$ into connected components

$$M_0 = \text{Int}(V_1) \sqcup \text{Int}(V_2),$$

and each $(\text{Int}(V_i), \mathcal{F}|_{\text{Int}(V_i)})$ ($i = 1, 2$) is the bundle foliation of the $\text{Int}(D^2)$ -bundle over S^1 . By Proposition 1.2.8, the canonical 1-form $\omega_{\mathfrak{S}}$ restricted on each $\text{Int}(V_i)$ is the pull-back of the angular 1-form of S^1 under the fibration, and so the periodic group $\Lambda_{\mathfrak{S}} = \mathbb{Z}$.

In view of the analogy in arithmetic topology ([Mo12; Chapter 3]), the 3-sphere S^3 may be regarded as an analogue of $\overline{\text{Spec}(\mathbb{Z})} = \text{Spec}(\mathbb{Z}) \cup \mathcal{P}_{\mathbb{Q}}^\infty$, where $\mathcal{P}_{\mathbb{Q}}^\infty$ consists of the only one infinite prime of $\mathbb{Q}$.

Example 3.III-(a).2 (Open book decomposition). Let M be a closed 3-manifold. It is known that M contains a fibered link $L = K_1 \cup \cdots \cup K_r$, namely, there is a fibration $\varpi : M \setminus \text{Int}(V(L)) \rightarrow S^1$, where $\text{Int}(V(L)) = \sqcup_{i=1}^r \text{Int}(V(K_i))$ is the interior of a tubular neighborhood of L and any fiber of ϖ is a surface with r boundary components. We have the foliation on $M \setminus \text{Int}(V(L))$ by turbularizing the fibers of ϖ around $\partial V(L) = \sqcup_{i=1}^r \partial \text{Int}(V(K_i))$ ([CCF00; Example 3.3.11]). The structure this induces on M is called an *open book decomposition* ([Ca07; Example 4.11]). We fill in $V(L)$ with the Reeb component to obtain the foliation $\mathcal{F}$ on all of M . We define the flow on $M \setminus \text{Int}(V(L))$ by the suspension of the monodromy φ of the fibration ϖ . We suppose that φ is of pseudo-Anosov type (for example, this is the case if L is a hyperbolic link). The flow on $\text{Int}(V(L))$ is defined to be the one transverse to any leaf of the Reeb foliation. Thus we have an FDS³ $\mathfrak{S} = (M, \mathcal{F}, \phi)$ of type III-(a) such that $\mathcal{P}_{\mathfrak{S}}^\infty = \{\partial V(K_1), \dots, \partial V(K_r)\}$

and $\mathcal{P}_\mathfrak{E}$ is a countably infinite set. We have the decomposition of $M_0 = M \setminus \partial V(L)$ into connected components

$$M_0 = \sqcup_{i=1}^r \text{Int}(V(K_i)) \sqcup (M \setminus V(L)).$$

Here, $\text{Int}(V(K_i)) = \mathbb{R}^2 \times S^1$ and $\mathcal{F}|_{\text{Int}(V(K_i))}$ is the bundle foliation over S^1 , and $M \setminus V(L)$ is a surface bundle over S^1 and $\mathcal{F}|_{M \setminus V(L)}$ is also the bundle foliation over S^1 . By Proposition 1.2.8, the canonical 1-form $\omega_\mathfrak{E}$ restricted on $\text{Int}(V(K_i))$ or $M \setminus V(L)$ is the pull-back of the angular 1-form of S^1 under the fibration, and so the period group $\Lambda_\mathfrak{E} = \mathbb{Z}$.

Remark 1.3.3. Example 3.III-(a).2 shows that any closed smooth 3-manifold M admit a structure of an FDS³ with non-empty $\mathcal{P}_\mathfrak{E}^\infty$. So the notion of an FDS³ is generic in this sense. Moreover, the fibration $\varpi : M \setminus \text{Int}(V(L)) \rightarrow S^1$ above induces the surjective homomorphism $\pi_1(M \setminus L) \rightarrow \mathbb{Z}$ and hence M has a $\mathbb{Z}$ -covering ramified over L . This may be analogous to that any number field k has a $\mathbb{Z}_l$ -extension ramified over primes $\mathfrak{l}_1, \dots, \mathfrak{l}_n$ lying above (l) , where l is a prime number.

Type III-(b). We give an example of FDS³'s of type III-(b).

Example 1.3.III-(b). Let $T^3 := (\mathbb{R}/\mathbb{Z})^3 = \{(\theta_1, \theta_2, \theta_3) | \theta_i \in \mathbb{R}/\mathbb{Z}\}$ be the 3-dimensional torus and let $\mathcal{F} = \{T^2 \times \{\theta_3\}\}_{\theta_3 \in \mathbb{R}/\mathbb{Z}}$ be the linear foliation which admits a complex structure. We define the smooth dynamical system ϕ on T^3 by

$$\frac{d}{dt}\phi^t(p) := (\cos(2\pi\theta_3), 0, \sin(2\pi\theta_3))$$

for $p = (\theta_1, \theta_2, \theta_3)$. Then we obtain an FDS³ $(T^3, \mathcal{F}, \phi)$ which is equipped with $\mathcal{P}_\mathfrak{E}^\infty = \{L_1^\infty, L_2^\infty\}$, where

$$L_1^\infty := \{(\theta_1, \theta_2, 0) | \theta_i \in \mathbb{R}/\mathbb{Z}\},$$

$$L_2^\infty := \{(\theta_1, \theta_2, \frac{1}{2}) | \theta_i \in \mathbb{R}/\mathbb{Z}\}.$$

We have the decomposition of $M_0 := T^3 \setminus (L_1^\infty \cup L_2^\infty)$ into connected components

$$M_0 = X_1 \sqcup X_2,$$

$$X_1 := \{(\theta_1, \theta_2, \theta_3) | \theta_1, \theta_2 \in \mathbb{R}/\mathbb{Z}, 0 < \theta_3 < \frac{1}{2}\},$$

$$X_2 := \{(\theta_1, \theta_2, \theta_3) | \theta_1, \theta_2 \in \mathbb{R}/\mathbb{Z}, \frac{1}{2} < \theta_3 < 1\}.$$

Here each X_a ($a = 1, 2$) is a T^2 -bundle over an open interval and $\mathcal{F}|_{X_a}$ is the bundle foliation. The canonical 1-form $\omega_\mathfrak{E}$ is given by $\omega_\mathfrak{E}|_{X_a} = \frac{d\theta_3}{\sin(2\pi\theta_3)}$ and the period group $\Lambda_\mathfrak{E} = \{0\}$ by Proposition 1.2.9.

Type III-(c). Finally, We give an example of FDS³'s of type III-(c).

Example 1.3.III-(c). Let $T^3 := (\mathbb{R}/\mathbb{Z})^3 = \{(\theta_1, \theta_2, \theta_3) | \theta_i \in \mathbb{R}/\mathbb{Z}\}$. Fix an irrational number $\rho \in \mathbb{R} \setminus \mathbb{Q}$. Let ω_0 be the smooth 1-form on T^3 defined by

$$\omega_0 := \sin(2\pi\theta_3)(d\theta_1 + \rho d\theta_2) + d\theta_3.$$

Since we see $\omega_0 \wedge d\omega_0 = 0$, $\text{Ker}(\omega_0)$ defines the foliation $\mathcal{F}$ on T^3 by Frobenius' theorem (cf. Remark 1.1.1 (1)). Let ϕ be the smooth dynamical system defined by

$$\frac{d}{dt}\phi_1^t(p) := (1, 0, 0)$$

for $p = (\theta_1, \theta_2, \theta_3)$. Then we obtain an FDS³ $\mathfrak{S}_1 := (T^3, \mathcal{F}, \phi_1)$ which is equipped with $\mathcal{P}_{\mathfrak{S}_1}^\infty = \{L_1^\infty, L_2^\infty\}$, where

$$L_1^\infty := \{(\theta_1, \theta_2, 0) | \theta_i \in \mathbb{R}/\mathbb{Z}\},$$

$$L_2^\infty := \{(\theta_1, \theta_2, \frac{1}{2}) | \theta_i \in \mathbb{R}/\mathbb{Z}\}.$$

We have the decomposition of $M_0 := T^3 \setminus (L_1^\infty \cup L_2^\infty)$ into connected components

$$M_0 = X_1 \sqcup X_2,$$

$$X_1 := \{(\theta_1, \theta_2, \theta_3) | \theta_1, \theta_2 \in \mathbb{R}/\mathbb{Z}, 0 < \theta_3 < \frac{1}{2}\},$$

$$X_2 := \{(\theta_1, \theta_2, \theta_3) | \theta_1, \theta_2 \in \mathbb{R}/\mathbb{Z}, \frac{1}{2} < \theta_3 < 1\}.$$

Here, any leaf of $\mathcal{F}|_{X_a}$ is dense in X_a for each $a = 1, 2$. So $\mathfrak{S}_1$ is of type III-(c). Let $\omega_{\mathfrak{S}_1}$ be the smooth 1-form defined by

$$\omega_{\mathfrak{S}_1} := \frac{\omega_0}{\sin(2\pi\theta_3)} = d\theta_1 + \rho d\theta_2 + \frac{d\theta_3}{\sin(2\pi\theta_3)}.$$

Then we see

$$\text{Ker}(\omega_{\mathfrak{S}_1}) = \text{Ker}(\omega_0)$$

$$\omega_{\mathfrak{S}_1}(\dot{\phi}^t) = 1$$

and so $\omega_{\mathfrak{S}_1}$ is indeed the canonical 1-form of $\mathfrak{S}_1$. Let γ_a^1, γ_a^2 be closed curves in X_a defined by

$$\gamma_a^1 := \left\{ \left(t, 0, \frac{2a-1}{4} \right) | t \in \mathbb{R}/\mathbb{Z} \right\}$$

$$\gamma_a^2 := \left\{ \left(0, t, \frac{2a-1}{4} \right) | t \in \mathbb{R}/\mathbb{Z} \right\}.$$

Then we see that $H_1(X_a, \mathbb{Z})$ is generated by the homology classes of γ_a^1, γ_a^2 and that $[\omega_{\mathfrak{S}_1}](\gamma_a^1) = 1, [\omega_{\mathfrak{S}_1}](\gamma_a^2) = \rho$. Hence the period group $\lambda_{\mathfrak{S}_1}$ is $\mathbb{Z} + \rho\mathbb{Z}$. For this example $\mathfrak{S}_1$, any orbits in M_0 is closed and so $\mathcal{P}_{\mathfrak{S}_1}$ is uncountable. However, as in Example 3.II.1, we can change ϕ_1 to obtain an FDS³ having countably infinitely many closed orbits in M_0 as follows. Let $V_2 := (-\rho, 1, 0)$, $V_3 := (-1, 0, \sin(2\pi\theta_3))$

be vector fields on T^3 and we consider the smooth dynamical system ϕ_2 defined by

$$\begin{aligned}\frac{d}{dt}\phi_2^t(p) &:= (1, 0, 0) + \sin(2\pi\theta_2)V_2 + \cos(2\pi\theta_3)V_3 \\ &= (1 - \rho \sin(2\pi\theta_2) - \cos(2\pi\theta_3), \sin(2\pi\theta_2), \frac{1}{2} \sin(4\pi\theta_3)).\end{aligned}$$

Then we obtain an FDS³ $\mathfrak{S}_2 := (T^3, \mathcal{F}, \phi_2)$ of type III-(c). Since V_2 and V_3 are tangent to $\mathcal{F}$ and so $(\dot{\phi}_1^t - \dot{\phi}_2^t)_p \in T_p\mathcal{F}$ for $p \in T^3$, we have $\omega_{\mathfrak{S}_2} = \omega_{\mathfrak{S}_1}$, $\Lambda_{\mathfrak{S}_2} = \Lambda_{\mathfrak{S}_1}$ by Lemma 1.1.6. We see that $\mathcal{P}_{\mathfrak{S}_2}$ consists of the following four closed orbits in M_0

$$\begin{cases} \gamma_1 = \{(\theta_1, 0, \frac{1}{4}) | \theta_1 \in S^1\}, \\ \gamma_2 = \{(\theta_1, 0, \frac{3}{4}) | \theta_1 \in S^1\}, \\ \gamma_3 = \{(\theta_1, \frac{1}{2}, \frac{1}{4}) | \theta_1 \in S^1\}, \\ \gamma_4 = \{(\theta_1, \frac{1}{4}, \frac{3}{4}) | \theta_1 \in S^1\}, \end{cases}$$

and that ϕ_2 is of contracting type around γ_3 and γ_4 . We replace a tubular neighborhood of γ_3 or γ_4 by the suspension of the horseshoe map. By Lemma 1.3.2, we obtain an FDS³ $\mathfrak{S} = (M, \mathcal{F}, \phi)$ of type III-(c) such that $\mathcal{P}_{\mathfrak{S}}$ is countably infinite, and the canonical 1-form $\omega_{\mathfrak{S}}$ and the period group $\Lambda_{\mathfrak{S}}$ are same as $\omega_{\mathfrak{S}_i}$ and $\Lambda_{\mathfrak{S}_i}$ ($i = 1, 2$), respectively.

Chapter 2. Leafwise cohomological expression of dynamical zeta functions

In a series of papers (c.f. [D98], [D02b], [D05], [D06], [D08]), Deninger considered a conjectural dynamical system associated to a complete arithmetic curves $\overline{\text{Spec}(\mathcal{O}_k)}$, where $\mathcal{O}_k$ is the ring of integers of a number field k , and he proposed a hypothetical formula expressing the completed Dedekind zeta function $\hat{\zeta}_K(s)$ of K in terms of the regularized determinant of an infinitesimal generator of the flow acting on infinite dimensional cohomology groups $H_{\text{dyn}}^\bullet(\overline{\text{Spec}(\mathcal{O}_K)}, \mathcal{R})$:

$$(2.0.1) \quad \hat{\zeta}_K(s) = \prod_{i=0}^2 \det_\infty \left(\frac{1}{2\pi} (s - \Theta) | H_{\text{dyn}}^i(\overline{\text{Spec}(\mathcal{O}_K)}, \mathcal{R}) \right)^{(-1)^{i+1}}.$$

The main object of this chapter is to prove a formula analogous to the formula (2.0.1) for the dynamical zeta function of a 3-dimensional Riemannian foliated dynamical system (cf. Theorem B below). Here a Riemannian foliated dynamical system is an FDS³ equipped with bundle-like Riemannian metric $g_{\mathcal{F}}$ (cf. Section 2.1) and we abbreviate it to RFDS³.

Let $(M, \mathcal{F}, \phi, g_{\mathcal{F}})$ be a RFDS³. Then we can define the *reduced leafwise cohomology* $\bar{H}_{\mathcal{F}}^\bullet(M)$ on which an *infinitesimal generator* Θ acts. Here, we assume that the operator ϕ^{t*} which is the pullback of the flow ϕ on the leafwise forms is unitary. We then consider the infinite series

$$\xi_p(s, z) := \sum_{\rho \in \text{Sp}(\Theta_p)} (s - \rho)^{-z},$$

where Θ_p denotes the operator Θ acting on $\bar{H}_{\mathcal{F}}^p(M)$. We note that $\xi_p(s, z)$ is an analog of the ξ -function in analytic number theory.

Here is our first main theorem.

Theorem A. *The following assertions hold:*

- (1) *The series $\xi_p(s, z)$ is absolutely convergent on $\text{Re}(z) \gg 0$ for any $s \in \mathbb{C}$.*
- (2) *It extends to a meromorphic function of $z \in \mathbb{C}$ and $s \in \mathbb{C}$ which is holomorphic at $z = 0$.*

As a consequence of the theorem, the series $\xi_p(s, z)$ defines a Hurwitz-type spectral zeta function associated with the infinitesimal generator. It follows that a zeta-regularized determinant can be defined since the spectral zeta function is

regular at $z = 0$.

For the second result, we define the dynamical zeta function for RFDS³.

$$\zeta_{\mathcal{F}}(s) = \prod_{\gamma} (1 - e^{-s \cdot l(\gamma)})^{-\epsilon_{\gamma}},$$

where γ runs over closed orbits of ϕ and $l(\gamma)$ is the length of γ . Here, ϵ_{γ} denotes the index of a closed orbit, and we assume that the set of closed orbits is a countable set and that all closed orbits are non-degenerate, namely, ϵ_{γ} is non-zero.

We give our second main theorem as follows:

Theorem B. *The dynamical zeta function on a Riemannian foliated dynamical system of 3 dimension has the following leafwise cohomological expression*

$$\zeta_{\mathcal{F}}(s) = \prod_{i=0}^2 \det_{\infty}(s - \Theta|\bar{H}_{\mathcal{F}}^i(M))^{(-1)^{i+1}}.$$

This chapter is based on [Kim17] and [Kim19].

2.1. Bundle-like metric, leafwise cohomology and infinitesimal generator

In this section, we recall basic concepts and materials concerning 3-dimensional Riemannian foliated dynamical systems, which will be used in the subsequent sections.

Bundle-like metric. A Riemannian metric $g_{\mathcal{F}}$ on an FDS³ $\mathfrak{S} = (M, \mathcal{F}, \phi)$ is called a *bundle-like metric* whose geodesics are perpendicular to all leaves whenever they are perpendicular to one leaf. Note that any 1-codimensional foliation without singularities is Riemannian.

Definition 2.1.1. We define a 3-dimensional *Riemannian foliated dynamical system* by an FDS³ $\mathfrak{S} = (M, \mathcal{F}, \phi)$ with a bundle-like metric $g_{\mathcal{F}}$. We denote it by $\mathfrak{S} = (M, \mathcal{F}, \phi, g_{\mathcal{F}})$ and call it an RFDS³ for short.

Leafwise de Rham complex. For the triple $(M, \mathcal{F}, \phi)$, let $T\mathcal{F}$ be a sub-bundle of the tangent bundle TM which is tangent to the leaves of the foliation. The restriction of $T\mathcal{F}$ on a leaf $\mathcal{L}$ is identified with the tangent bundle $T\mathcal{L}$ of the leaf. We call $T\mathcal{F}$ the *leafwise tangent bundle*.

We define the space of leafwise i -forms by

$$\Omega_{\mathcal{F}}^i(M) := \Gamma(M, \wedge^i T^* \mathcal{F}) \subset \Omega^i(M).$$

Let $d_{\mathcal{F}}$ (resp. d_0) be the exterior derivative acting only along leaves (resp. the flow). Then the de Rham complex $(\Omega^i(M), d^i)$ has a decomposition:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \Omega_{\mathcal{F}}^i(M) & \xrightarrow{d_{\mathcal{F}}} & \Omega_{\mathcal{F}}^{i+1}(M) & \longrightarrow & \cdots \\ & & \searrow & \downarrow & \searrow & \downarrow & \searrow \\ \cdots & \longrightarrow & \Omega_0^i(M) & \xrightarrow{d_{\mathcal{F}}} & \Omega_0^{i+1}(M) & \longrightarrow & \cdots \end{array}$$

where $\Omega_0^i(M)$ is the complement of $\Omega_{\mathcal{F}}^i(M)$.

We simply denote the restriction $d^i|_{\Omega_{\mathcal{F}}^i(M)}$ by $d_{\mathcal{F}}^i$. Since we have $d_{\mathcal{F}}^{i+1} \circ d_{\mathcal{F}}^i = 0$ on $\Omega_{\mathcal{F}}^i(M)$, the pairs $\{(\Omega_{\mathcal{F}}^i(M), d_{\mathcal{F}}^i)\}_i$ form a cochain complex:

$$0 \rightarrow \Omega_{\mathcal{F}}^0(M) \xrightarrow{d_{\mathcal{F}}^0} \Omega_{\mathcal{F}}^1(M) \xrightarrow{d_{\mathcal{F}}^1} \Omega_{\mathcal{F}}^2(M) \xrightarrow{d_{\mathcal{F}}^2} 0.$$

We call this complex the *leafwise de Rham complex*.

Inner product. We define an inner product on $\Omega_{\mathcal{F}}^i(M)$ by

$$(\alpha, \beta) := \int_M \langle \alpha, \beta \rangle_{\mathcal{F}} \text{vol}_M,$$

where $\langle \cdot, \cdot \rangle_{\mathcal{F}}$ is a metric $g_{\mathcal{F}}$ on $\wedge^i T^* \mathcal{F}$ induced by the metric g and vol_X is the volume form on X defined by the metric g .

Leafwise cohomology. We denote the kernel of $d_{\mathcal{F}}^i$ by $Z_{\mathcal{F}}^i(M)$ and the image of $d_{\mathcal{F}}^i$ by $B_{\mathcal{F}}^{i+1}(M)$. Note that a leafwise i -th form in $Z_{\mathcal{F}}^i(M)$ (resp. $B_{\mathcal{F}}^{i+1}(M)$) is called leafwise closed i -th form (resp. leafwise exact i -th form).

Definition 2.1.2. We define the i -th *leafwise cohomology group* (or *foliation cohomology group*) by

$$H_{\mathcal{F}}^i(M) := Z_{\mathcal{F}}^i(M) / B_{\mathcal{F}}^i(M).$$

The leafwise cohomology group is trivial for $i > 2$.

Unfortunately, the leafwise cohomology group is of infinite dimension in general and not a Hausdorff space. We modify it by taking a quotient with respect to the closure in the smooth topology.

$$\bar{H}_{\mathcal{F}}^i(M) := Z_{\mathcal{F}}^i(M) / \overline{B_{\mathcal{F}}^i(M)}.$$

We call it the reduced leafwise cohomology group.

Leafwise Hodge theorem. For a bundle-like metric $g_{\mathcal{F}}$, we have the Hodge

*-operator. If we denote by δ the adjoint operator of the exterior derivative d , it has a decomposition into $\delta_{\mathcal{F}}$ and δ_0

$$\begin{array}{ccccccc} \cdots & \longleftarrow & \Omega_{\mathcal{F}}^i(M) & \xleftarrow{\delta_{\mathcal{F}}^i} & \Omega_{\mathcal{F}}^{i+1}(M) & \longleftarrow & \cdots \\ & \swarrow & \downarrow & \searrow & \downarrow & \swarrow & \\ \cdots & \longleftarrow & \Omega_0^i(M) & \xleftarrow{\delta_{\mathcal{F}}^i} & \Omega_0^{i+1}(M) & \longleftarrow & \cdots \end{array}$$

Definition 2.1.3. An operator defined by

$$\Delta_{\mathcal{F}} := d_{\mathcal{F}}\delta_{\mathcal{F}} + \delta_{\mathcal{F}}d_{\mathcal{F}} \text{ on } \Omega_{\mathcal{F}}^i(M)$$

is called the *leafwise Laplacian*. A leafwise form $\omega \in \ker \Delta_{\mathcal{F}}$ is called a leafwise harmonic form.

If we define Δ_0 on $\Omega_{\mathcal{F}}^i(M)$ by $\delta_0 d_0$, the restriction of the Laplacian Δ on $\Omega_{\mathcal{F}}^i(M)$ can be represented by

$$\Delta|_{\Omega_{\mathcal{F}}^i(M)} = \Delta_{\mathcal{F}} + \Delta_0.$$

We have the following significant proposition by Álvarez López and Kordyukov ([LK01]).

Proposition 2.1.4 (*Leafwise Hodge theorem*). *Given a bundle-like metric, An leafwise cohomology class can be uniquely represented by a leafwise harmonic form. We have an isomorphism*

$$\bar{H}_{\mathcal{F}}^i(M) \cong \ker(\Delta_{\mathcal{F}}^i).$$

Infinitesimal generator. Assume that the $\mathbb{R}$ -action ϕ is conformal on $\Omega_{\mathcal{F}}^i(M)$ with respect to the bundle-like metric $g_{\mathcal{F}}$, i.e. the pullback ϕ^{t*} is a unitary operator acting on $\Omega_{\mathcal{F}}^i(M)$:

$$(\phi^{t*}\omega, \phi^{t*}\eta) = (\omega, \eta) \text{ for } \forall t \in \mathbb{R}.$$

It is easy to check that ϕ^{t*} on $\bar{H}_{\mathcal{F}}^i(M)$ is surjective and strongly continuous, i.e.

$$\lim_{t \rightarrow t_0} \phi^{t*}[h] = \phi^{t_0*}[h] \text{ for } \forall t_0 \in \mathbb{R}, [h] \in \bar{H}_{\mathcal{F}}^i(M).$$

We have that $(\phi^{t*})_{t \in \mathbb{R}}$ is the strongly continuous one-parameter unitary group on the Hilbert space $\bar{H}_{\mathcal{F}}^i(M)$.

Then the following lemma follows from the Stone's theorem.

Lemma 2.1.5 (Stone's theorem). *We define the infinitesimal generator of $(\phi^{t*})_{t \in \mathbb{R}}$ by*

$$\Theta := \lim_{t \rightarrow 0} \frac{\phi^{t*} - \text{id}}{t}.$$

By our assumption, $(\phi^{t*})_{t \in \mathbb{R}}$ is the strongly continuous one-parameter unitary group on the Hilbert space $\bar{H}_{\mathcal{F}}^i(M)$, then $A := -i\Theta$ is self-adjoint on $\bar{H}_{\mathcal{F}}^i(M)$ and we have

$$\phi^{t*} = e^{itA} = e^{t\Theta} \text{ for } \forall t \in \mathbb{R}.$$

The infinitesimal generator is a first-order differential operator along the transversal flow. Then the exterior derivative d_0 along the flow (resp. adjoint operator δ_0) can be represented by

$$\begin{aligned} d_0\omega &= \Theta\omega \wedge \omega_\phi, \\ \delta_0(\omega \wedge \omega_\phi) &= -\Theta\omega. \end{aligned}$$

It leads the following lemma.

Lemma 2.1.6. *The negative square of the infinitesimal generator on a leafwise cohomology group coincide with the Laplacian on a space of leafwise harmonic forms*

$$\begin{aligned} -\Theta^2|_{\bar{H}_{\mathcal{F}}^i(M)} &= \Delta_0|_{\ker(\Delta_{\mathcal{F}})} \\ &= \Delta|_{\ker(\Delta_{\mathcal{F}})}. \end{aligned}$$

Since M is compact and closed, the Laplacian has pure point spectrum which consists of non-negative eigenvalues with finite multiplicity. Hence the infinitesimal generator has pure imaginary eigenvalues with finite multiplicity.

For a leafwise harmonic form $\omega_{\mathcal{F}} \in \ker \Delta_{\mathcal{F}}$, a fundamental solution of the heat equation whose initial value is $\omega_{\mathcal{F}}$, i.e.

$$\begin{cases} \left(\frac{\partial}{\partial t} + \Delta_0\right) \omega_{\mathcal{F}}^t = 0, \\ \omega_{\mathcal{F}}^0 = \omega_{\mathcal{F}}, \end{cases}$$

is given by

$$\omega_{\mathcal{F}}^t(x, y, s) = \begin{cases} \int_{S^1} K(t, s, s') \omega_{\mathcal{F}}(x, y, s') ds' & \text{if } (x, y, s) \text{ is periodic,} \\ \int_{\mathbb{R}} K(t, s, s') \omega_{\mathcal{F}}(x, y, s') ds' & \text{otherwise} \end{cases}$$

where $K(t, s, s')$ is a factor of the heat kernel of the Laplacian Δ . Hence we have an asymptotic expansion for the heat kernel around $t = 0$

$$\text{tr}(e^{-\Delta_0 t}|_{\ker \Delta_{\mathcal{F}}}) \stackrel{t \downarrow 0}{\sim} t^{-\frac{1}{2}}(a_0 + a_1 t + a_2 t^2 + \cdots).$$

Then the spectral zeta function $\zeta_{\Delta_0}(s)$ associated with Δ_0 has only simple poles at $s = \frac{1}{2} - n$ ($n = 0, 1, 2, \dots$).

Fixing a positive number $T > 0$, we consider 2 series

$$\begin{aligned} V_p^+(t) &= \sum_{\text{Im}(\rho) > T} e^{\rho it} \\ V_p^-(t) &= \sum_{\text{Im}(\rho) < -T} e^{-\rho it} \end{aligned}$$

where ρ runs over the spectrum of Θ on $\bar{H}_{\mathcal{F}}^p(M)$. It follows from Lemma 2.1.6 that it is a partial sum of $\text{tr}(e^{-\sqrt{\Delta_0}t} | \ker \Delta_{\mathcal{F}})$. Since $\text{tr}(e^{-\sqrt{\Delta_0}t} | \ker \Delta_{\mathcal{F}})$ is the inverse Mellin transform of $\Gamma(s)\zeta_{\Delta_0}(\frac{s}{2})$ with simple poles at $s = 1, -2n$ and double poles at $s = -2n - 1$ ($n = 0, 1, 2, \dots$), we deduce that the series $V_p^+(t)$ converges absolutely and has an asymptotic expansion around $t = 0$ as follows:

$$V_p^+(t) \stackrel{t \downarrow 0}{\sim} a_p^+ t^{-1} + \sum_{k=0}^N (b_{p,k}^+ + c_{p,k}^+ t \log t) t^{2k} + \mathcal{O}_1(t^{N-1}) + \mathcal{O}_2(t^{N-1}) t \log t.$$

The same result holds for $V_p^-(t)$ by the same argument.

2.2. Proof of Theorem A

In this section, we give a proof of Theorem A.

Proof of Theorem A. We fix a positive number $T > 0$. We consider the 2 series for $s \in \mathbb{C}$ such that $|\text{Im}(s)| < T$

$$\begin{aligned} \theta_p^+(t) &= V_p^+(t) e^{-sit}, \\ \theta_p^-(t) &= V_p^-(t) e^{sit}. \end{aligned}$$

They play a role like a partition function.

We take the Mellin transform for the series and define the following functions

$$\begin{aligned} \xi_p^+(s, z) &= \frac{e^{\frac{\pi}{2}iz}}{\Gamma(z)} \int_0^\infty \theta_p^+(t) t^{z-1} dt, \\ \xi_p^-(s, z) &= \frac{e^{-\frac{\pi}{2}iz}}{\Gamma(z)} \int_0^\infty \theta_p^-(t) t^{z-1} dt. \end{aligned}$$

Since $V_p^\pm(t)$ are convergent, we have for $\text{Re}(z) > 1$

$$\begin{aligned} \xi_p^+(s, z) &= \sum_{\text{Im}(\rho) > T} (s - \rho)^{-z}, \\ \xi_p^-(s, z) &= \sum_{\text{Im}(\rho) < -T} (s - \rho)^{-z}. \end{aligned}$$

Next, we consider

$$\begin{aligned}
\xi_p^+(s, z) &= \frac{e^{\frac{\pi}{2}iz}}{\Gamma(z)} \int_0^\infty \theta_p^+(t) t^{z-1} dt \\
&= \frac{e^{\frac{\pi}{2}iz}}{\Gamma(z)} \left(\int_0^1 \theta_p^+(t) t^{z-1} dt + \int_1^\infty \theta_p^+(t) t^{z-1} dt \right), \\
\xi_p^-(s, z) &= \frac{e^{-\frac{\pi}{2}iz}}{\Gamma(z)} \int_0^\infty \theta_p^-(t) t^{z-1} dt \\
&= \frac{e^{-\frac{\pi}{2}iz}}{\Gamma(z)} \left(\int_0^1 \theta_p^-(t) t^{z-1} dt + \int_1^\infty \theta_p^-(t) t^{z-1} dt \right).
\end{aligned}$$

Since $V_p^\pm(t)$ are of rapid decay at infinity, the second terms are convergent for any $z \in \mathbb{C}$. Hence we have

$$\begin{aligned}
\xi_p^+(s, z) &= \frac{e^{\frac{\pi}{2}iz}}{\Gamma(z)} \left(\int_0^1 \theta_p^+(t) t^{z-1} dt + \int_1^\infty \theta_p^+(t) t^{z-1} dt \right) \\
&= \frac{e^{\frac{\pi}{2}iz}}{\Gamma(z)} \left(\frac{a_p^+}{z-1} - \frac{a_p^+ si + b_{p,0}^+}{z} - \frac{b_{p,0}^+ si}{z+1} - \frac{c_{p,0}^+}{(z+1)^2} + \dots \right) \\
&= \frac{e^{\frac{\pi}{2}iz}}{\Gamma(z)} \left(\frac{a_p^+}{z-1} + \eta_p^+(s, z) \right),
\end{aligned}$$

where $\eta_p^+(s, z)$ is a meromorphic function of (s, z) for $|\text{Im}(s)| < T$ and $z \in \mathbb{C}$ and is regular at $z = 0$. The meromorphic function $\eta_p^+(s, z)$ has only simple poles at $z = -2n - 1$ ($n = 0, 1, 2, \dots$). The same result holds for $\xi_p^-(s, z)$ by the same argument:

$$\xi_p^-(s, z) = \frac{e^{-\frac{\pi}{2}iz}}{\Gamma(z)} \left(\frac{-a_p^-}{z-1} + \eta_p^-(s, z) \right).$$

Note that $\eta_p^\pm(s, z)$ is meromorphic in $|\text{Im}(s)| < T$ and $z \in \mathbb{C}$ and is regular at $z = 0$. Since $\Gamma(z)^{-1}$ has the analytic continuation to the entire plane, $\xi_i^\pm(s, z)$ extend to a meromorphic function of $z \in \mathbb{C}$ and $s \in \mathbb{C}$. Since $\xi_p(s, z)$ differs from $\xi_p^+(s, z) + \xi_p^-(s, z)$ by the sum of the finite terms, we have that $\xi_p(s, z)$ is a meromorphic function for all $|\text{Im}(s)| < T$ and all $z \in \mathbb{C}$ and is regular at $z = 0$. $\square$

2.3. Dynamical zeta functions of RFDS³'s.

In this section, we recall the notion of the zeta regularized determinant and introduce the dynamical zeta function of an RFDS³.

Zeta-regularized determinant. We recall the notion of the zeta regularized determinant. Let $\Theta : V \rightarrow V$ be a linear operator acting on a complex vector space V of countable dimension. We assume that V is the direct sum of

finite-dimensional Θ -invariant sub-spaces. Let $\text{Sp}(\Theta)$ be the set of eigenvalues of Θ . The *spectral zeta function* associated with the operator Θ is defined by the analytic continuation of Dirichlet series

$$\zeta_{\Theta}(s) = \sum_{\lambda \neq 0 \in \text{Sp}(\Theta)} \lambda^{-s} \text{ with } \lambda^{-s} = |\lambda|^{-s} e^{-is(\text{Arg}\lambda)}, -\pi < \text{Arg}\lambda \leq \pi.$$

We assume that the Dirichlet series converges absolutely on some right-half plane and has an analytic continuation to the half plane $\text{Re}(s) > -\epsilon$ for some $\epsilon > 0$ which is holomorphic at $s = 0$. Under these conditions, we define the *zeta-regularized determinant* by

$$\det_{\infty}(\Theta|V) := \exp(-\partial_s \zeta_{\Theta}(0)).$$

The dynamical zeta function of an RFDS³. Let $(M, \mathcal{F}, \phi, g_{\mathcal{F}})$ be the foliated dynamical system with a bundle-like metric which we discussed above. We define the *dynamical zeta function* for a RFDS³ by the analytic continuation of the infinite product

$$\zeta_{\mathcal{F}}(s) = \prod_{\gamma} (1 - e^{-s \cdot l(\gamma)})^{-\epsilon_{\gamma}},$$

where γ runs over periodic orbits of ϕ and $l(\gamma)$ is the length of γ . Here, ϵ_{γ} is the index of a closed orbit.

Index of a closed orbit. For a closed orbit γ of ϕ , we set an index

$$\epsilon_{\gamma} := \text{sgn } \det(1 - T_x \phi^{l(\gamma)}|_{T_x \mathcal{F}}) \quad \text{for } x \in \gamma,$$

where $T_x \phi^t : T_x \mathcal{F} \rightarrow T_{\phi^t(x)} \mathcal{F}$ is the differential of ϕ^t . It does not depend on the choice of the point $x \in \gamma$. We call a closed orbit γ *non-degenerate* in a sense that ϵ_{γ} is non-zero.

Absolute convergent condition. It is known that the infinite product converges absolutely on $\text{Re}(s) > h(\phi)$ where $h(\phi)$ is *topological entropy* and only if it is finite. Note that the topological entropy $h(\phi)$ is defined by

$$h(\phi) := \lim_{T \rightarrow +\infty} \frac{1}{T} \log N(T) \geq 0,$$

where $N(T)$ denotes the cardinality of orbits whose length is less than or equal to T , i.e. $N(T) = \text{Card}\{\gamma | l(\gamma) \leq T\}$. We assume that the topological entropy $h(\phi)$ of a foliated dynamical system $(M, \mathcal{F}, \phi)$ is finite so that $\zeta(s)$ converges absolutely on the right-half plane.

2.4. Proof of Theorem B

In this section, we prove Theorem B by employing the dynamical Lefschetz trace formula.

Dynamical Lefschetz trace formula. Let $(M, \mathcal{F}, \phi, g_{\mathcal{F}})$ be a RFDS³. In the following, we assume that the set of closed orbits is a countable set and all closed orbits are non-degenerate. Alvarez-Lopez and Kordyukov developed the *dynamical Lefschetz trace formula*:

Proposition 2.4.1 ([LK02]). *For every test function $\varphi \in \mathcal{D}(\mathbb{R}) = C_0^\infty(\mathbb{R})$, the operator*

$$A_\varphi = \int_{\mathbb{R}} \varphi(t) \phi^{t*} dt$$

on $\bar{H}_{\mathcal{F}}^i(M)$ is of trace class. Setting:

$$\text{Tr}(\phi^{t*} | \bar{H}_{\mathcal{F}}^i(M))(\varphi) = \text{tr} A_\varphi$$

defines a distribution on $\mathbb{R}$. The following formula holds in $\mathcal{D}'(\mathbb{R})$:

$$\sum_{i=0}^{\dim \mathcal{F}} (-1)^i \text{Tr}(\phi^{t*} | \bar{H}_{\mathcal{F}}^i(M)) = \chi_{\text{Co}}(\mathcal{F}, g_{\mathcal{F}}) \delta_0 + \sum_{\gamma} l(\gamma) \sum_{k \in \mathbb{Z} \setminus 0} \epsilon_{\gamma} \delta_{kl(\gamma)}.$$

Here $\chi_{\text{Co}}(\mathcal{F}, \mu)$ denotes Connes' Euler characteristic of the foliation with respect to the bundle-like metric (cf. [Co00]) and δ_{τ} is the Dirac delta function in $\mathcal{D}'(\mathbb{R})$ which is non-zero at τ .

The following formula holds in $\mathcal{D}'(\mathbb{R}_{>0})$:

$$\sum_{i=0}^{\dim \mathcal{F}} (-1)^i \text{Tr}(\phi^{t*} | \bar{H}_{\mathcal{F}}^i(M)) = \sum_{\gamma} \sum_{n \in \mathbb{N}} l(\gamma) \epsilon_{\gamma} \delta_{nl(\gamma)}.$$

Lemma 2.1.5 (Stone's theorem) leads to the following corollary:

Corollary 2.4.2. *The following equality holds in $\mathcal{D}'(\mathbb{R}_{>0})$:*

$$\sum_{i=0}^2 (-1)^i \text{Tr}(\phi^{t*} | \bar{H}_{\mathcal{F}}^i(M)) = \sum_{i=0}^2 (-1)^i \sum_{\rho \in \text{Sp}(\Theta_i)} e^{\rho t}.$$

where Θ_i denotes the operator Θ acting on $\bar{H}_{\mathcal{F}}^i(M)$.

It is enough to show that the zeta-regularized determinant coincides with the infinite product of periodic orbits on some right-half plane of the topological

entropy, *i.e.*,

$$\begin{aligned}\zeta_{\mathcal{F}}(s) &= \prod_{\gamma} (1 - e^{-sl(\gamma)})^{-\epsilon_{\gamma}} \\ &= \prod_{i=0}^2 \det_{\infty}(s - \Theta|\bar{H}_{\mathcal{F}}^i(M))^{(-1)^{i+1}} \quad \text{for } \operatorname{Re}(s) > P,\end{aligned}$$

where P is a sufficiently large number $P > h(\phi)$. Then the assertion follows from the uniqueness of analytic continuation.

We apply the Laplace transform for the dynamical Lefschetz trace formula. We have

$$\mathcal{L} \left[\sum_{i=0}^2 (-1)^i \sum_{\rho \in \operatorname{Sp}(\Theta_i)} t^{z-1} e^{\rho t} \right] (s) = \Gamma(z) \sum_{i=0}^2 (-1)^i \sum_{\rho \in \operatorname{Sp}(\Theta_i)} (s - \rho)^{-z} \quad (1)$$

for the left hand side, and

$$\mathcal{L} \left[\sum_{\gamma} \sum_{n \in \mathbb{N}} l(\gamma) \epsilon_{\gamma} \delta_{nl(\gamma)} t^{z-1} \right] (s) = \sum_{\gamma} \sum_{n \in \mathbb{N}} \frac{l(\gamma) \epsilon_{\gamma}}{e^{nl(\gamma)s}} (nl(\gamma))^{z-1} \quad (2)$$

for the right hand side. Both sides are defined for $\operatorname{Re}(z) > 1$, where the former infinite series (1) is defined from the proof of Theorem A, and $\operatorname{Re}(s) > h(\phi)$ over where the latter infinite series (2) is defined. We denote by P a sufficiently large number bigger than $h(\phi)$.

Let $L_{\delta-}$ be a contour consisting of the lower edge of the cut from $-\infty$ to $-\delta$, the circle $t = \delta e^{i\phi}$ for $-\pi \leq \phi \leq \pi$ and the upper edge of the cut from $-\delta$ to $-\infty$.

$$\int_{L_{\delta-}} e^{\lambda t} t^{-z} dt = 2i \sin(z\pi) \int_{\delta}^{\infty} e^{-v} v^{-z} dv + I$$

where I denotes the integral along the circle $t = |\delta|$. Since I tends to zero as $\delta \rightarrow 0$, we have

$$\begin{aligned}\lim_{\delta \rightarrow 0} \int_{L_{\delta-}} e^{\lambda t} t^{-z} dt &= 2i \sin(z\pi) \Gamma(1-z) \\ &= \frac{2\pi i}{\Gamma(z)}\end{aligned}$$

Hence we have the formula for $\lambda > 0$

$$\frac{\lambda^{z-1}}{\Gamma(z)} = \frac{1}{2\pi i} \lim_{\delta \rightarrow 0} \int_{L_{\delta-}} e^{\lambda t} t^{-z} dt.$$

By applying the formula for the series (2), we get

$$\begin{aligned} \frac{1}{\Gamma(z)} \sum_{\gamma} \sum_{n \in \mathbb{N}} \frac{l(\gamma) \epsilon_{\gamma}}{e^{nl(\gamma)s}} (nl(\gamma))^{z-1} &= \frac{1}{2\pi i} \lim_{\delta \rightarrow 0} \int_{L_{\delta-}} \left(\sum_{\gamma} \sum_{n \in \mathbb{N}} l(\gamma) \epsilon_{\gamma} e^{-nl(\gamma)(s-t)} \right) t^{-z} dt \\ &= \frac{-1}{2\pi i} \lim_{\delta \rightarrow 0} \int_{L_{\delta-}} \frac{\zeta'_{\mathcal{F}}}{\zeta_{\mathcal{F}}}(s-t) t^{-z} dt. \end{aligned}$$

Since the series (1) has a meromorphic extension and is holomorphic at $z = 0$ from Theorem A, we obtain the two equalities for $-\pi \leq \arg(t) \leq \pi$

$$\begin{aligned} \sum_{i=0}^2 (-1)^i \xi_i(s, z) &= \frac{-1}{2\pi i} \lim_{\delta \rightarrow 0} \int_{L_{\delta-}} \frac{\zeta'_{\mathcal{F}}}{\zeta_{\mathcal{F}}}(s-t) t^{-z} dt, \\ \sum_{i=0}^2 (-1)^i \partial_z \xi_i(s, 0) &= \frac{1}{2\pi i} \lim_{\delta \rightarrow 0} \int_{L_{\delta-}} \frac{\zeta'_{\mathcal{F}}}{\zeta_{\mathcal{F}}}(s-t) \log(|t| e^{\arg(t)i}) dt \end{aligned}$$

It remains to see the following:

$$\begin{aligned} &\frac{1}{2\pi i} \lim_{\delta \rightarrow 0} \int_{L_{\delta-}} \frac{\zeta'_{\mathcal{F}}}{\zeta_{\mathcal{F}}}(s-t) \log(|t| e^{\arg(t)i}) dt \\ &= \frac{1}{2\pi i} \int_{-\infty}^0 \frac{\zeta'_{\mathcal{F}}}{\zeta_{\mathcal{F}}}(s-t) (\log(|t|) - \pi i) dt + \frac{1}{2\pi i} \int_0^{-\infty} \frac{\zeta'_{\mathcal{F}}}{\zeta_{\mathcal{F}}}(s-t) (\log(|t|) + \pi i) dt \\ &= \int_0^{-\infty} \frac{\zeta'_{\mathcal{F}}}{\zeta_{\mathcal{F}}}(s-t) dt = - \int_0^{\infty} \frac{\zeta'_{\mathcal{F}}}{\zeta_{\mathcal{F}}}(s+t) dt \\ &= \log(\zeta_{\mathcal{F}}(s)) \end{aligned}$$

Therefore we have

$$\begin{aligned} \zeta_{\mathcal{F}}(s) &= \prod_{i=0}^2 \exp(-\partial_z \xi(s, 0))^{(-1)^{i+1}} \\ &= \prod_{i=0}^2 \det_{\infty}(s - \Theta|\bar{H}_{\mathcal{F}}^i(M))^{(-1)^{i+1}} \end{aligned}$$

for $\operatorname{Re}(s) > P \geq h(\phi)$. Hence Theorem B follows from the uniqueness of the analytic continuation.

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