PARAMETRIC ESTIMATION FOR SPDES DRIVEN BY AN INFINITE DIMENSIONAL MIXED FRACTIONAL BROWNIAN MOTION

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PARAMETRIC ESTIMATION FOR SPDES DRIVEN BY AN INFINITE DIMENSIONAL MIXED FRACTIONAL BROWNIAN MOTION

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Abstract

Parametric and nonparametric inference for stochastic processes driven by a fractional Brownian motion were investigated by Mishura and Prakasa Rao among others. Similar problems for processes driven by an infinite-dimensional fractional Brownian motion were studied by Huebner, Rozovskii, Prakasa Rao, Cialenco and others. Parametric estimation for stochastic partial differential equations driven by driven by an infinite-dimensional mixed fractional Brownian motion is discussed in this article.

Key Words and Phrases: Stochastic differential equation, Parametric estimation, Infinite-dimensional mixed fractional Brownian motion.

1. Introduction

Statistical inference for diffusion type processes satisfying stochastic differential equations driven by Wiener processes has been studied earlier and a comprehensive survey of various methods is given in Prakasa Rao (1999a). There has been a recent interest to study similar problems for stochastic processes driven by a fractional Brownian motion to model processes having long range dependence. Le Breton (1998) studied parameter estimation and filtering in a simple linear model driven by a fractional Brownian motion. Kleptsyna and Le Breton (2002) studied parameter estimation problems for fractional Ornstein-Uhlenbeck type process driven by a fractional Brownian motion. This is a fractional analogue of the Ornstein-Uhlenbeck process driven by a standard Wiener process. It is a continuous time first order auto-regressive process $X = \{X_t, t \geq 0\}$ which is the solution of a one-dimensional homogeneous linear stochastic differential equation driven by a fractional Brownian motion (fBm) $W^H = \{W_t^H, t \geq 0\}$ with Hurst parameter $H \in [1/2, 1)$. Such a process is the unique Gaussian process satisfying the linear integral equation

$$X_t = \theta \int_0^t X_s ds + \sigma W_t^H, t \geq 0. \tag{1}$$

They investigated the problem of estimation of the parameters θ and σ^2 based on the observation $\{X_s, 0 \leq s \leq T\}$ and proved that the maximum likelihood estimator $\hat{\theta}_T$ is strongly consistent as $T \to \infty$. More general classes of stochastic processes satisfying linear stochastic differential equations driven by a fractional Brownian motion were studied

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and the asymptotic properties of the maximum likelihood and the Bayes estimators for parameters involved in such processes is investigated in Prakasa Rao (2010). Prakasa Rao (2010) gives a comprehensive discussion on problems of estimation for processes driven by a fractional Brownian motion.

Geometric Brownian motion driven by a standard Brownian motion has been widely used for modeling fluctuations of share prices in a stock market using Black-Scholes model. However efforts to model fluctuations in financial markets with long range dependence through processes driven by a fractional Brownian motion were not successful as it was noted that such a modeling creates arbitrage opportunities contrary to the fundamental assumption of no arbitrage opportunity for modeling rational market behaviour. Cheridito (2001) proposed modeling through processes driven by a mixed fractional Brownian motion. It was shown by Cheridito (2001) that a mixed fractional Brownian motion is a semimartingale if and only if the Hurst index H is either equal to $\frac{1}{2}$ reducing the process to a Wiener process or $H \in (3/4, 1)$. Furthermore the probability measure generated by such a process is absolutely continuous with respect to the probability measure generated by a Wiener process if $H = 1/2$ or $H \in (3/4, 1)$. This in turn will lead to no arbitrage opportunities for modeling financial market behaviour through processes driven by a mixed fractional Brownian motion. This discussion is to motivate the study of processes driven by a mixed fractional Brownian motion.

The problem of estimation of parameters for processes driven by processes which are mixtures of independent Brownian and fractional Brownian motions started from the works of Cheridito (2001), Rudomino-Dusyatska (2003) and more recently in Prakasa Rao (2015a,b;2017a,b;2018a,b;2019;2021a,b,c) among others. Mixed fractional Brownian models were studied in Mishura (2008) and Prakasa Rao (2010). Cai et al. (2016) present a new approach via filtering for analysis of mixed processes of type $\{X_t = B_t + G_t, 0 \leq t \leq T\}$ where $\{B_t, 0 \leq t \leq T\}$ is a Brownian motion and $\{G_t, 0 \leq t \leq T\}$ is an independent Gaussian process. Statistical analysis of mixed fractional Ornstein-Uhlenbeck process was investigated in Chigansky and Kleptsyna (2019). Fractional Ornstein-Uhlenbeck type process driven a mixed fractional Brownian motion has also been termed as "mixed fractional Ornstein-Uhlenbeck process" in Marushkevych (2016). Large deviations for drift parameter estimator of a mixed fractional Ornstein-Uhlenbeck process were studied by Marushkevych (2016).

Stochastic partial differential equations can be used for modeling the evolution of dynamical systems in the presence of spatial-temporal noises with applications in fluid mechanics, oceanography, study of temperature changes, finance, economics, biological, ecological systems and many other disciplines (cf. Cialenco et al. (2019)). Huebner et al. (1993) initiated the study of parametric estimation for a class of stochastic partial differential equations (SPDEs) in the presence of white noise or the driving force is an infinite-dimensional Wiener process. These results were extended to parabolic stochastic partial differential equations in Huebner and Rozovskii (1995). Prakasa Rao (2000) studied Bayes estimation for stochastic partial differential equations in the white noise case. For other results on parametric inference for SPDEs, see Prakasa Rao (2000;2001;2002a,b;2004;2013). A comprehensive survey of results is given in Prakasa Rao (2001;2002a). Parameter estimation for a two-dimensional stochastic Navier-Stokes equation, driven by infinite-dimensional fractional Brownian motion, was studied in Prakasa Rao (2013). Lototsky and Rozovsky (2017) give an extensive survey of theory of SPDE and a discussion on parametric inference for such processes. Cialenco and his coauthors obtained several results dealing with parametric inference for SPDE based on continuous observation or discrete sampling of the processes. Cialenco (2018) gives a comprehensive survey of their results.

Our aim in this paper is to initiate the study of parametric inference for processes driven by an infinite-dimensional mixed fractional Brownian motion. As far as we are aware, this problem has not been investigated earlier. As was mentioned above, modeling by processes driven by a mixed fractional Brownian motion plays a major role in finance. The infinite-dimensional space-time version of this process will be helpful in modeling changes in finance structure locally on a global scale just as study of changes in the sea surface temperature is the motivation for the study of SPDEs driven by infinite-dimensional Brownian motion. Results obtained in this paper and proofs are analogous to those obtained earlier for SPDEs driven by an infinite-dimensional fractional Brownian motion but they are not consequences of those results. Following the results obtained by Chigansky and Kleptsyna (2019) and Cai et al. (2016) for processes driven by a mixed fractional Brownian motion, we are able to generalize the results for SPDEs driven by an infinite-dimensional mixed fractional Brownian motion.

Our study of the asymptotic properties of the estimators are based on the availability of continuous path data on an interval $[0.T]$. We study the asymptotic properties of the maximum likelihood estimators (MLE) as $T \to \infty$. The MLEs are based on the coordinate processes $\{u_{k\epsilon}(t), t \in [0, T], k = 1, \ldots, N\}$ to be defined later. However these coordinate processes are not observable in practice. It is interesting to investigate the asymptotic properties of the MLE based on discrete sampling, by assuming that the first N coordinate processes are observed at M time grid points uniformly spaced over the time interval $[0, T]$ as $N \to \infty$ and or as $T, M \to \infty$ and investigate sufficient conditions on the growth rate of N, M and T which ensure consistency and asymptotic normality of the estimator. We will come back to this problem in future.

2. Properties of processes driven by a mfBm

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a stochastic basis satisfying the usual conditions. The natural filtration of a stochastic process is understood as the P-completion of the filtration generated by this process. Let $\{W_t, t \geq 0\}$ be a standard Wiener process and $W^H = \{W_t^H, t \geq 0\}$ be an independent normalized fractional Brownian motion with Hurst parameter $H \in (0, 1)$, that is, a Gaussian process with continuous sample paths such that $W_0^H = 0, E(W_t^H) = 0$ and

$$E(W_s^H W_t^H) = \frac{1}{2}[s^{2H} + t^{2H} - |s - t|^{2H}], t \geq 0, s \geq 0. \tag{2}$$

Let

$$\tilde{W}_t^H = W_t + W_t^H, t \geq 0.$$

The process $\{\tilde{W}_t^H, t \geq 0\}$ is called the *mixed fractional Brownian motion* with Hurst index H. We assume hereafter that Hurst index H is *known* and that $H \in (\frac{3}{4}, 1)$.

Let us consider a stochastic process $Y = \{Y_t, t \geq 0\}$ defined by the stochastic integral equation

$$Y_t = \int_0^t C(s)ds + \tilde{W}_t^H, t \geq 0 \tag{3}$$

where the process $C = \{C(t), t \geq 0\}$ is an $(\mathcal{F}_t)$-adapted process. For convenience, we write the above integral equation in the form of a stochastic differential equation

$$dY_t = C(t)dt + d\tilde{W}_t^H, t \geq 0 \tag{4}$$

driven by the mixed fractional Brownian motion $\tilde{W}^H$. Following the recent works by Cai et al. (2016) and Chigansky and Kleptsyna (2019), one can construct an integral transformation that transforms the mixed fractional Brownian motion $\tilde{W}^H$ into a martingale M^H. Let $g_H(s,t)$ be the solution of the integro-differential equation

$$g_H(s,t) + H\frac{d}{ds}\int_0^t g_H(r,t)|s-r|^{2H-1} sign(s-r)dr = 1, 0 < s < t. \tag{5}$$

Cai et al. (2016) proved that the process

$$M_t^H = \int_0^t g_H(s,t)d\tilde{W}_s^H, t \geq 0 \tag{6}$$

is a Gaussian martingale with quadratic variation

$$< M^H >_t = \int_0^t g_H(s,t)ds, t \geq 0 \tag{7}$$

Let w_t^H denote the quadratic variation $< M^H >_t$ over the interval $[0,t]$. It is known that the natural filtration of the martingale M^H coincides with that of the mixed fractional Brownian motion $\tilde{W}^H$. Suppose that, for the martingale M^H defined by the equation (6), the sample paths of the process $\{C(t), t \geq 0\}$ are smooth enough in the sense that the process

$$Q_H(t) = \frac{d}{d < M^H >_t}\int_0^t g_H(s,t)C(s)ds, t \geq 0 \tag{8}$$

is well defined. Define the process

$$Z_t = \int_0^t g_H(s,t)dY_s, t \geq 0. \tag{9}$$

As a consequence of the results in Cai et al. (2016), it follows that the process Z is a fundamental semimartingale associated with the process Y in the following sense.

Theorem 2.1: *Let $g_H(s,t)$ be the solution of the equation (5). Define the process Z as given in the equation (9). Then the following relations hold.*
(i) The process Z is a semimartingale with the decomposition

$$Z_t = \int_0^t Q_H(t)d < M^H >_s + M_t^H, t \geq 0 \tag{10}$$

where M^H is the martingale defined by the equation (6).
(ii) The process Y admits the representation

$$Y_t = \int_0^t \hat{g}_H(s,t)dZ_s, t \geq 0 \tag{11}$$

where

$$\hat{g}_H(s,t) = 1 - \frac{d}{d < M^H >_s} \int_0^t g_H(r,s)dr. \tag{12}$$

(iii) The natural filtrations $(\mathcal{Y}_t)$ *and* $(\mathcal{Z}_t)$ *of the processes* Y *and* Z *respectively coincide.*

Applying Corollary 2.9 in Cai et al. (2016), it follows that the probability measures μ_Y and $\mu_{\tilde{W}^H}$ generated by the processes Y and $\tilde{W}^H$ on an interval $[0,T]$ are absolutely continuous with respect to each other and the Radon-Nikodym derivative is given by

$$\frac{d\mu_Y}{d\mu_{\tilde{W}^H}}(Y) = \exp[\int_0^T Q_H(s)dZ_s - \frac{1}{2}\int_0^T [Q_H(s)]^2 d < M^H >_s] \tag{13}$$

which is also the likelihood function based on the observation $\{Y_s, 0 \le s \le T.\}$ Since the filtrations generated by the processes Y and Z are the same, the information contained in the families of σ-algebras $(\mathcal{Y}_t)$ and $(\mathcal{Z}_t)$ is the same and hence the problem of the estimation of the parameters involved based on the observation $\{Y_s, 0 \le s \le T\}$ and $\{Z_s, 0 \le s \le T\}$ are equivalent. Since the process $\{Z_s, 0 \le s \le T\}$ is driven by a martingale, it is convenient to discuss asymptotic behaviour of the estimators through limit theorems available for martingales. This explanation motivates the study of problem of estimation through the process Z instead of the original process Y.

3. Parametric estimation for SPDE driven by infinite-dimensional mfBm

Kallianpur and Xiong (1995) discussed the properties of solutions of stochastic partial differential equations (SPDE) driven by infinite-dimensional Brownian motion. They indicate that SPDE's are being used for stochastic modeling, for instance, for the study of neuronal behaviour in neurophysiology and in building stochastic models of turbulence. The theory of SPDE's is investigated in Ito (1984), Rozovskii (1990) and Da Prato and Zabczyk (1992). Huebner et al. (1993) started the investigation of maximum likelihood estimation of parameters of two types of SPDE's and extended their results for a class of parabolic SPDE's in Huebner and Rozovskii (1995). Asymptotic properties of Bayes estimators for such problems were discussed in Prakasa Rao (2000). A short review and a comprehensive survey of these results are given in Prakasa Rao (2001,2002a). Our aim in this section is to study the problems of parameter estimation for some SPDE driven by an infinite-dimensional mixed fractional Brownian motion.

Stochastic PDE with linear drift (absolutely continuous case)

Let U be a real separable Hilbert space and Q be a self-adjoint positive operator. Further suppose that the operator Q is nuclear. Then Q admits a sequence of eigenvalues $\{q_n, n \ge 1\}$ with $0 < q_n$ decreasing to zero as $n \to \infty$ and $\sum_{n=0}^{\infty} q_n < \infty$. In addition the corresponding eigen vectors $\{e_n, n \ge 1\}$ form an orthonormal basis in U. We define the *infinite-dimensional mixed fractional Brownian motion* on U with covariance Q as

$$\tilde{\mathcal{W}}_Q^H(t) = \sum_{n=0}^{\infty} \sqrt{q_n} e_n \tilde{W}_n^H(t) \tag{14}$$

where $\tilde{W}_n^H, n \ge 1$ are real independent mfBm's with Hurst index H . Formal definition is given in Section 2.

Let $U = L_2[0,1]$ and $\mathcal{W}_Q^H$ be the infinite dimensional mfBm on U with the Hurst index H and with the nuclear covariance operator Q.

Consider the process $u_\varepsilon(t,x), 0 \le x \le 1, 0 \le t \le T$ governed by the stochastic partial differential equation

$$du_\varepsilon(t,x) = (\triangle u_\varepsilon(t,x) + \theta u_\varepsilon(t,x))dt + \varepsilon \; d\tilde{\mathcal{W}}_Q^H(t,x) \tag{15}$$

where $\triangle = \frac{\partial^2}{\partial x^2}$. Suppose that $\varepsilon \to 0$ and $\theta \in \Theta \subset R$. Suppose the initial and the boundary conditions are given by

$$u_\varepsilon(0,x) = f(x), f \in L_2[0,1] \tag{16}$$

$$u_\varepsilon(t,0) = u_\varepsilon(t,1) = 0, 0 \le t \le T. \tag{17}$$

Let us consider a special covariance operator Q with $e_k = \sin k\pi x, k \ge 1$ and $\lambda_k = (\pi k)^2, k \ge 1$. Then $\{e_k\}$ is a complete orthonormal system with the eigenvalues $q_i = (1+\lambda_i)^{-1}, i \ge 1$ for the operator Q and $Q = (I - \triangle)^{-1}$.

Guerra and Nualart (2008) proved an existence and uniqueness theorem for solutions of multidimensional time dependent stochastic differential equations driven by a multidimensional fractional Brownian motion with Hurst index $H > \frac{1}{2}$ and a multidimensional standard Brownian motion. Similar results were obtained by Mishura and Shevchenko (2011) and da Silva et al. (2018) under weaker conditions. Mishura et al. (2019) has given sufficient conditions for the existence and uniqueness of a mild solution $u_\varepsilon(t,x)$ for stochastic differential equation driven by an infinite-dimensional mfBm.

We assume that sufficient conditions hold so that there exists a unique square integrable solution $u_\varepsilon(t,x)$ of (15) under the conditions (16)-(17) and consider it as a formal sum

$$u_\varepsilon(t,x) = \sum_{i=1}^{\infty} u_{i\varepsilon}(t) e_i(x). \tag{18}$$

It can be checked that the Fourier coefficient $u_{i\varepsilon}(t)$ satisfies the stochastic differential equation

$$du_{i\varepsilon}(t) = (\theta - \lambda_i) u_{i\varepsilon}(t)dt + \frac{\varepsilon}{\sqrt{\lambda_i + 1}} d\tilde{W}_i^H(t), \; 0 \le t \le T \tag{19}$$

with the initial condition

$$u_{i\varepsilon}(0) = v_i, \; v_i = \int_0^1 f(x) e_i(x) dx. \tag{20}$$

Let $P_\theta^{(\varepsilon)}$ be the probability measure generated by u_ε when θ is the true parameter. Suppose θ_0 is the true parameter. Observe that the process $\{u_{i\varepsilon}(t), 0 \le t \le T\}$ is a mixed fractional Ornstein-Uhlenbeck type process (cf. Marushkevych (2016), Chigansky and Kleptsyna (2019), Cai et al. (2016)).

Following the notation given in the previous section, define

$$M_i^H(t) = \int_0^t g_H(s,t) d\tilde{W}_i^H(s), 0 \le t \le T, \tag{21}$$

$$Q_{i\varepsilon}(t) = \frac{\sqrt{\lambda_i + 1}}{\varepsilon} \frac{d}{dw_t^H} \int_0^t g_H(s,t) u_{i\varepsilon}(s) ds, t \in [0,T], \tag{22}$$

$$Z_{i\varepsilon}(t) = (\theta - \lambda_i) \int_0^t Q_{i\varepsilon}(s) dw_s^H + M_i^H(t), 0 \le t \le T. \tag{23}$$

These processes are constructed using the equations (3), (8) and (10) with $C(s) = \frac{\sqrt{\lambda_i+1}}{\epsilon} u_{i\epsilon}(s), 0 \le s \le T$.

Observe that M_i^H is a zero mean Gaussian martingale. Furthermore, it follow that the process $\{Z_{i\varepsilon}(t)\}$ is a semimartingale and the natural filtrations $(\mathcal{Z}_{i\varepsilon_t})$ and $(\mathcal{U}_{i\varepsilon_t})$ of the processes $Z_{i\varepsilon}$ and $u_{i\varepsilon}$ respectively coincide. Let $P_{i\theta}^{T,\varepsilon}$ be the probability measure generated by the process $\{u_{i\varepsilon}(t), 0 \le t \le T\}$ when θ is the true parameter. Let θ_0 be the true parameter. It follows, by the Girsanov type theorem (see equation (13)), it follows that the log-likelihood ratio process is given by

$$\begin{aligned} \log \frac{dP_{i\theta}^{T,\varepsilon}}{dP_{i\theta_0}^{T,\varepsilon}} &= \frac{\lambda_i + 1}{\varepsilon^2} [(\theta - \theta_0) \int_0^T Q_{i\varepsilon}(t) dZ_{i\varepsilon}(t) \\ &\quad - \frac{1}{2} \{(\theta - \lambda_i)^2 - (\theta_0 - \lambda_i)^2\} \int_0^T Q_{i\varepsilon}^2(t) dw_t^H]. \end{aligned} \tag{24}$$

Let $u_\varepsilon^N(t,x)$ be the projection of the solution $u_\varepsilon(t,x)$ onto the subspace spanned by the eigen vectors $\{e_i, 1 \le i \le N\}$. Then

$$u_\varepsilon^N(t,x) = \sum_{i=1}^N u_{i\varepsilon}(t) e_i(x) \tag{25}$$

From the independence of the processes $\tilde{W}_i^H, 1 \le i \le N$ and hence of the processes $u_{i\varepsilon}, 1 \le i \le N$, it follows that the Radon-Nikodym derivative, of the probability measure $P_\theta^{N,T,\varepsilon}$ generated by the process $u_\varepsilon^N, 0 \le t \le T$ when θ is the true parameter with respect to the probability measure $P_{\theta_0}^{N,T,\varepsilon}$ generated by the process $u_\varepsilon^n, 0 \le t \le T$ when θ_0 is the true parameter, is given by

$$\begin{aligned} \log \frac{dP_\theta^{N,T,\varepsilon}}{dP_{\theta_0}^{N,T,\varepsilon}}(u_\varepsilon^N) &= \sum_{i=1}^N \frac{\lambda_i + 1}{\varepsilon^2} [(\theta - \theta_0) \int_0^T Q_{i\varepsilon}(t) dZ_{i\varepsilon}(t) \\ &\quad - \frac{1}{2} \{(\theta - \lambda_i)^2 - (\theta_0 - \lambda_i)^2\} \int_0^T Q_{i\varepsilon}^2(t) dw_t^H]. \end{aligned} \tag{26}$$

Furthermore the Fisher information is given by

$$\begin{aligned} I_{N\varepsilon}(\theta) &= E_\theta[\frac{\partial \log \frac{dP_\theta^{N,T\varepsilon}}{dP_{\theta_0}^{N,T,\varepsilon}}}{\partial \theta}]^2 \\ &= \sum_{i=1}^N \frac{\lambda_i + 1}{\varepsilon^2} E_\theta\{\int_0^T Q_{i\varepsilon}^2(t) dw_t^H\}. \end{aligned} \tag{27}$$

It is easy to check that the maximum likelihood estimator $\hat{\theta}_{N,\varepsilon}$ of the parameter θ based on the projection u_ε^N of u_ε is given by

$$\hat{\theta}_{N,\varepsilon} = \frac{\sum_{i=1}^N (\lambda_i + 1) \int_0^T Q_{i\varepsilon}(t) dZ_{i\varepsilon}(t)}{\sum_{i=1}^N (\lambda_i + 1) \int_0^T Q_{i\varepsilon}^2(t) dw_t^H}. \tag{28}$$

Suppose θ_0 is the true parameter. It is easy to see that

$$\varepsilon^{-1}(\hat{\theta}_{N,\varepsilon} - \theta_0) = \frac{\sum_{i=1}^N \sqrt{\lambda_i + 1} \int_0^T Q_{i\varepsilon}(t) dM_i^H(t)}{\sum_{i=1}^N (\lambda_i + 1) \int_0^T Q_{i\varepsilon}^2(t) dw_t^H}. \tag{29}$$

Observe that $M_i, 1 \leq i \leq N$ are independent zero mean Gaussian martingales with $< M_i >= w^H, 1 \leq i \leq N$.

Theorem 3.1 : *The maximum likelihood estimator $\hat{\theta}_{N,\varepsilon}$ is strongly consistent, that is,*

$$\hat{\theta}_{N,\varepsilon} \to \theta_0 \;\; a.s \;\; [P_{\theta_0}] \; as \;\; \varepsilon \to 0 \tag{30}$$

provided

$$\sum_{i=1}^N \int_0^T (\lambda_i + 1) Q_{i\varepsilon}^2(t) dw_t^H \to \infty \;\; \text{a.s} \;\; [P_{\theta_0}] \;\; \text{as} \;\; \varepsilon \to 0. \tag{31}$$

Proof : This theorem follows by observing that the process

$$R_\varepsilon^N \equiv \sum_{i=1}^N \varepsilon \int_0^T \sqrt{\lambda_i + 1} Q_{i\varepsilon}(t) dM_t^H, T \geq 0 \tag{32}$$

is a local martingale with the quadratic variation process

$$< R_\varepsilon^N >_T = \sum_{i=1}^N \int_0^T \varepsilon^2 (\lambda_i + 1) Q_{i\varepsilon}^2(t) dw_t^H \tag{33}$$

and applying the Strong law of large numbers (cf. Liptser (1980); Prakasa Rao (1999b), p. 61) under the condition (31) stated above.

Limiting distribution :

We now discuss the limiting distribution of the MLE $\hat{\theta}_{N\varepsilon}$ as $\varepsilon \to 0$.

Theorem 3.2 : *Assume that the process $\{R_\varepsilon^N, \varepsilon \geq 0\}$ is a local continuous martingale and that there exists a norming function $I_\varepsilon^N, \varepsilon \geq 0$ such that*

$$(I_\varepsilon^N)^2 < R_\varepsilon^N >_T = (I_\varepsilon^N)^2 \sum_{i=1}^N \int_0^T \varepsilon^2 (\lambda_i + 1) Q_{i\varepsilon}^2(t) dw_t^H \to \eta^2 \;\; in\ probability \;\; as \;\; \varepsilon \to 0 \tag{34}$$

where η is a random variable such that $P(\eta > 0) = 1$. Then

$$(I_\varepsilon^N R_\varepsilon^N, (I_\varepsilon^N)^2 < R_\varepsilon^N >_T) \to (\eta Z, \eta^2) \text{ in law as } \epsilon \to 0 \tag{35}$$

where the random variable Z has the standard Gaussian distribution and the random variables Z and η are independent.

Proof : This theorem follows as a consequence of the central limit theorem for local martingales (cf. Theorem 1.49 ; Remark 1.47 , Prakasa Rao (1999b), p. 65).

Observe that

$$(I_\varepsilon^N)^{-1}(\hat{\theta}_{N\varepsilon} - \theta_0) = \frac{I_\varepsilon^N R_\varepsilon^N}{(I_\varepsilon^N)^2 < R_\varepsilon^N >}. \tag{36}$$

Applying the above theorem, we obtain the following result.

Theorem 3.3 : *Suppose the conditions stated in Theorem 3.2 hold. Then*

$$(I_\varepsilon^N)^{-1}(\hat{\theta}_{N\varepsilon} - \theta_0) \to \frac{Z}{\eta} \text{ in law as } \varepsilon \to 0 \tag{37}$$

where the random variable Z has the standard Gaussian distribution and the random variables Z and η are independent.

Remarks : (i) If the random variable η is a constant with probability one, then the limiting distribution of the maximum likelihood estimator is Gaussian with mean 0 and variance η^{-2}. Otherwise it is a mixture of the Gaussian distributions with mean zero and variance η^{-2} with the mixing distribution as that of η.

(ii) Suppose that

$$\lim_{N\to\infty} \lim_{\varepsilon\to 0} \varepsilon^2 I_\epsilon^N = I(\theta) \tag{38}$$

exists and is positive. Since the sequence of Radon-Nikodym derivatives

$$\{\frac{dP_\theta^{N,T,\varepsilon}}{dP_{\theta_0}^{N,T,\varepsilon}}, n \geq 1\}$$

form a non-negative martingale with respect to the filtration generated by the sequence of random variables $\{u_\varepsilon^N, N \geq 1\}$, it converges almost surely to a random variable $\nu_{\varepsilon,\theta,\theta_0}$ as $N \to \infty$ for every $\varepsilon > 0$. It is easy to see that the limiting random variable is given by

$$\begin{aligned} &\nu_{\varepsilon,\theta,\theta_0}(u_\varepsilon) \\ &= \exp\{\sum_{i=1}^{\infty} \frac{\lambda_i + 1}{\varepsilon^2}[(\theta - \theta_0) \int_0^T Q_{i\varepsilon}(t) dZ_{i\varepsilon}(t) \\ &\quad -\frac{1}{2}\{(\theta - \lambda_i)^2 - (\theta_0 - \lambda_i)^2\} \int_0^T Q_{i\varepsilon}^2(t) dw_t^H]\}. \end{aligned} \tag{39}$$

Furthermore the sequence of random variables $u_\varepsilon^N(t)$ converge in probability to the random variable $u_\varepsilon(t)$ as $N \to \infty$ for every $\varepsilon > 0$. Hence, by Lemma 4 in Skorokhod (1965, p. 100), it follows that the measures P_θ^ε generated by the processes $u_\varepsilon$for different values of θ, are absolutely continuous with respect to each other and the Radon-Nikodym derivative of the probability measure P_θ^ε with respect to the probability measure $P_{\theta_0}^\varepsilon$ is given by

$$\begin{aligned} \frac{dP_\theta^\varepsilon}{dP_{\theta_0}^\varepsilon}(u_\varepsilon) &= \nu_{\varepsilon,\theta,\theta_0}(u_\varepsilon) \\ &= \exp\{\sum_{i=1}^{\infty} \frac{\lambda_i+1}{\varepsilon^2}[(\theta-\theta_0)\int_0^T Q_{i\varepsilon}(t)dZ_{i\varepsilon}(t) \\ &\quad -\frac{1}{2}\{(\theta-\lambda_i)^2-(\theta_0-\lambda_i)^2\}\int_0^T Q_{i\varepsilon}^2(t)dw_t^H]\}. \end{aligned} \tag{40}$$

It can be checked that the MLE $\hat{\theta}_\varepsilon$ of θ based on u_ε satisfies the likelihood equation

$$\alpha_\varepsilon = (\hat{\theta}_\varepsilon - \theta_0)\beta_\varepsilon \tag{41}$$

when θ_0 is the true parameter where

$$\alpha_\varepsilon = \sum_{i=1}^{\infty} \sqrt{\lambda_i+1} \int_0^T Q_{i\varepsilon}(t)dM_i^H(t) \tag{42}$$

and

$$\beta_\varepsilon = \sum_{i=1}^{\infty}(\lambda_i+1)\int_0^T Q_{i\varepsilon}^2(t)dw_t^H. \tag{43}$$

One can obtain sufficient conditions for studying the asymptotic behaviour of the estimator $\hat{\theta}_\varepsilon$ as in the finite projection case discussed above. We omit the details.

Stochastic PDE with linear drift (singular case) :

Let $(\Omega, \mathcal{F}, P)$ be a probability space and consider the process $u_\varepsilon(t,x), 0 \le x \le 1, 0 \le t \le T$ governed by the stochastic partial differential equation

$$du_\varepsilon(t,x) = \theta \triangle u_\varepsilon(t,x)dt + \varepsilon(I-\triangle)^{-1/2}d\tilde{W}(t,x) \tag{44}$$

where $\theta > 0$ satisfying the initial and the boundary conditions

$$\begin{aligned} u_\varepsilon(0,x) &= f(x),\ 0<x<1,\ f \in L_2[0,1], \\ u_\varepsilon(t,0) &= u_\varepsilon(t,1) = 0,\ 0 \le t \le T. \end{aligned} \tag{45}$$

Here I is the identity operator, $\triangle = \frac{\partial^2}{\partial x^2}$ as defined above and the process $\tilde{W}(t,x)$ is the cylindrical infinite dimensional mfBm with $H \in [\frac{1}{2}, 1)$. Following the discussion in the previous section, we assume the existence of a square integrable solution $u_\varepsilon(t,x)$ for the equation (44) subject to the boundary conditions (45). Then the Fourier coefficients $u_{i\varepsilon}(t)$ of satisfy the stochastic differential equations

$$du_{i\varepsilon}(t) = -\theta\lambda_i u_{i\varepsilon}(t)dt + \frac{\varepsilon}{\sqrt{\lambda_i+1}}d\tilde{W}_i^H(t),\ 0 \le t \le T, \tag{46}$$

with

$$u_{i\varepsilon}(0) = v_i, v_i = \int_0^1 f(x)e_i(x)dx. \tag{47}$$

Let $u_\varepsilon^{(N)}(t,x)$ be the projection of $u_\varepsilon(t,x)$ onto the subspace spanned by $\{e_1, \cdots, e_N\}$ in $L_2[0,1]$. In other words

$$u_\varepsilon^{(N)}(t,x) = \sum_{i=1}^N u_{i\varepsilon}(t)e_i(x). \tag{48}$$

Let $P_\theta^{(\varepsilon,N)}$ be the probability measure generated by $u_\varepsilon^{(N)}$ on the subspace spanned by $\{e_1, \cdots, e_N\}$ in $L_2[0,1]$. It can be shown that the measures $\{P_\theta^{(\varepsilon,N)}, \theta \in \Theta\}$ form an equivalent family and

$$\begin{aligned}\log \frac{dP_\theta^{(\varepsilon,N)}}{dP_{\theta_0}^{(\varepsilon,N)}}(u_\varepsilon^{(N)}) \\ = -\frac{1}{\varepsilon^2}\sum_{i=1}^N \lambda_i(\lambda_i+1)[(\theta-\theta_0)\int_0^T Q_{i\varepsilon}(t)dZ_{i\varepsilon}(t) - \frac{1}{2}(\theta-\theta_0)^2\lambda_i\int_0^T Q_{i\varepsilon}^2(t)dw_t^H].\end{aligned} \tag{49}$$

It can be checked that the MLE $\hat{\theta}_{\varepsilon,N}$ of θ based on $u_\varepsilon^{(N)}$ satisfies the likelihood equation

$$\alpha_{\varepsilon,N} = -\varepsilon^{-1}(\hat{\theta}_{\varepsilon,N} - \theta_0)\beta_{\varepsilon,N} \tag{50}$$

when θ_0 is the true parameter where

$$\alpha_{\varepsilon,N} = \sum_{i=1}^N \lambda_i\sqrt{\lambda_i+1}\int_0^T Q_{i\varepsilon}(t)dM_i^H(t) \tag{51}$$

and

$$\beta_{\varepsilon,N} = \sum_{i=1}^N (\lambda_i+1)\lambda_i^2\int_0^T Q_{i,\varepsilon}^2(t)dw_t^H. \tag{52}$$

Asymptotic properties of these estimators can be investigated as in the previous example. We do not go into the details as the arguments are similar.

Remarks : One can study the local asymptotic mixed normality (LAMN) of the family of probability measures generated by the log-likelihood ratio processes by the standard arguments as in Prakasa Rao (1999a,b) and hence investigate the asymptotic efficiency of the MLE using Hájek-Le Cam type bounds.

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