Estimation of Occurrence of Jumps in Time Series Models Including Jump Diffusion Processes and Their Applications to Optimal Portfolio Formations

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Shozo Tokinaga
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1 Introduction

There exists various works related to modeling of time series generation in diverse fields, such as prices and demnads of production goods. Among them, works are shown for modeling where the time series includes jump diffusion processes, and these works have been successfully applied to the optimisation of stock exchanges[1]-[5].

Conventional works related to modeling of occurrence of jumps in time series have two serious problems; firstly jumps are defined as stochastic phenomena independent from the value of time series, and secondary occurrence of jumps are assumed to be well known and then used for the optimization.

For the first problem we use the model where the jumps are induced if the value of time series rise (fall) relatively larger then the probability of jumps become larger. The models were successfully applied to the optimal operations of power plants and the optimal valuation of option prices of production/sales of goods. We especially focus on the time lag when the value of time series returns to the previous level of time series after the occurrence of jumps so that we analyze more realistic cases[6]-[9].

For the second problem, we use the forecast of fractal time series which we earlier proposed so that we can extract jumps by comparing difference between time series and forecasts[10]-[12]. In the reduction of optimization of production and allocation of investment, we use models based on the stochastic dynamics programming.

The procedure to optimize the investment is summarized as follows. Based on the forecast of fractal time series, we find the occurrence of jumps by comparing the diagram of probability map. Namely, if we expect the future price exceeds a threshold, we decide to change the allocation of fund so that we can increase the profit (if price rises), or can decrease the loss (if price falls). We then get better profit compared to ordinary investment where no relocation of fund is exercised. As applications, we show the optimal allocation of investment for production/sales model of goods[6]-[9].

In Section 2 we treat modeling of time series generation including jumps. Section 3 shows the estimation of occurrence of jumps using the fractal forecast, In Section 4 we show the description of optimal investment.

Here we extend the simple time series model based on the Brownian motion to the cases in which time series includes jumps.

2 Modeling of time series generation including jumps

2.1 Time series model including jumps

Here we extend the simple time series model based on the Brownian motions to the cases time series include jumps[6]-[9].

We assume that the price process P(t) basically follow to the Brownian motion, but several fluctuations are imposed such as the sudden rise of value (called as upward jump) , or the sudden fall of value (called as downward jump). For explanation we used the words defined as the start of jumps as "go" process in upward of downward jumps, and also the start of the returning process to the previous level before jump as the "back" process.

In the time range dt, price process $P(t)$ follows as

$$\begin{aligned} dP &= (J_1 - P)\lambda_{go}dt + (J_2 - P)\lambda_{back}dt \\ &\quad +(K_1 - P)\gamma_{go}dt + (K_2 - P)\gamma_{back}dt + \sigma P\omega \end{aligned} \tag{1}$$

where ω is the derivative of the Brownian motion.

The first and second term in the right hand side of the equation means the occurrence of the upward jump and the returning process to the pervious value of P(t), in the same way, the third and fourth terms means the occurrence of the downward jump and the returning process to the previous value of P(t). Terms describing these processes are defined as follows. Firstly, $\lambda_{go}, \lambda_{back}$ are the probability of occurrence of "go" and "back" process in unit time span. Two values J_1, J_2 are the range of these two fluctuation assumed to be follow normal distributions $N(a, s)$ having the mean value a, and the standard deviation s. namely $J_1 \sim N(a_{11}, s_{11}), J_2 \sim N(a_{12}, s_{12})$.

In the downward jumps, the values $\gamma_{go}, \gamma_{back}$ are the probability of "go" and "back" process of jumps for unit of time, and the values K_1, K_2 correspond to the size of these fluctuations, Namely, $K_1 \sim N(a_{21}, s_{21}), K_2 \sim N(a_{22}, s_{22})$.

The probability of occurrence for beginning of jumps, and also the start of two processes "go, back" denoted as $\lambda_{go}(P), \lambda_{back}(P), \gamma_{go}(P), \gamma_{back}(P)$ are defined as simple piecewise linear function as shown in Fig.1. The reason to take simple forms of functions comes from the fact that we have no serious changes of the results even by using more complicated forms.

Fig.1 shows the probability of occurrence of jumps (upward and Downward) as the functions of P(T). As is seen, probabilities $\lambda_{go}(P), \lambda_{back}(P)$ are changed at the threshold values (such as DT_{11}), and the besides the transition areas these values are constant (such as θ_{11}).

The effects of changes of parameters are explained as follows. The mean value a_{11} of normal distribution determine the size of jumps, and the probabilities of occurrence of jumps $\theta_{11}(\theta_{21})$ define the number of jumps, If these values are large, jumps are generated more often. Similarly,

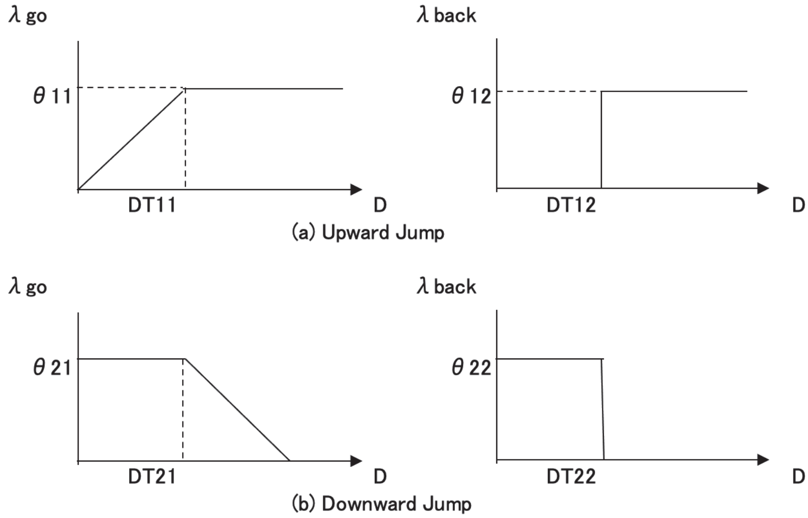

Figure 1: Probabilities of upward and downward jumps $\lambda_{go}, \lambda_{back}, \gamma_{go}, \gamma_{back}$ (Upper:upward, Lower:downward)

the probabilities $\theta_{12}(\theta_{22})$ determine the time necessary to return the time series back to the previous level. If theses values are small, the time series takes longer time to return previous levels.

Fig.2 shows examples of occurrence of upward and downward jumps.

Now we must note that if the probability θ_{12} and θ_{22} are low, the time become larger necessary for the time series to return to previous level before the occurrence of jumps, namely the 'back' process happens slowly and need more time to return to previous level. In these cases, the time series maintains the same higher level in the upward jumps (lower level in the downward jumps) for longer time. The duration time affects the optimization of investment. Fig.3 shows examples of slow 'back' process returning to previous level.

2.2 Two way application of time series models

We use the underlying time series model to two cases of optimal problems, namely, the optimal production and the optimal portfolio (two way applications). In the first optimality problem, we observe a relevant length of behavior of price changes of products, and decide the allocation of resources (such as capital). Therefore the integration of profit (value) is utilized for the decision. On the other hand, in the second optimality problem like securities investment, we observe only small range of time (such as from time t to $t + dt$), and then we change the allocation among them (securities). In this case we do not need the integration of price process, and use the price change itself described in the equation. Hereafter, we describe the optimality problem, and move to the second problem.

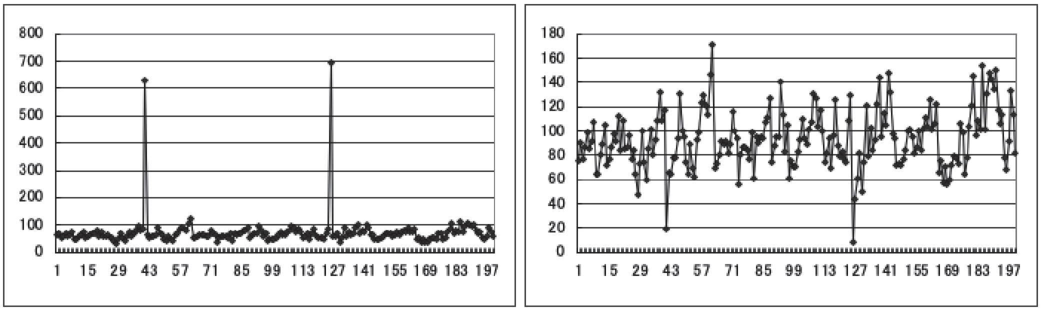

Figure 2: Example of upward and downward jumps (Upper:upward jumps, Lower:downward jumps))

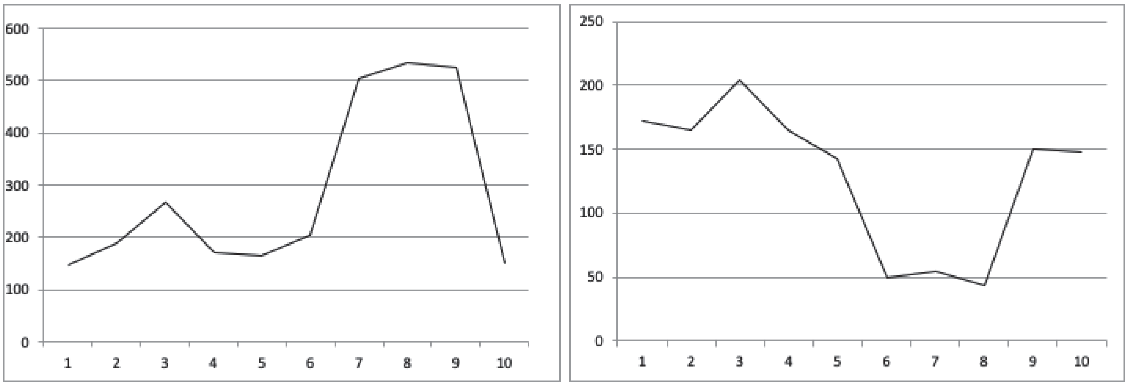

Figure 3: Examples of slow ’back’ process returning to previous level (Left:upward jumps, Right:downward jumps)

3 Estimation of occurrence of jumps using the fractal forecast

3.1 Forecast of fractal time series

As described above, we introduce the time series model where the model of jump occurrence. Namely, if the time series rise (fall) over the ordinary levels, then we notify that the beginning of jumps are found. However, there exist two problems, usage of detection of jumps and the forecast (detection) of jumps. In works describing the time series model including jump processes, we found that the occurrence of jumps are known a priori. The value functions are evaluated under assumption that time series have several jumps at known location. The assumption postulate that the jumps last a time span and the policy postulate known jumps are effective. However, in real world, we usually cannot know the appearance of jump, and don't have confidence in that these are jumps.

Then, we use our proposed method to detect the jumps in the time series. We summarize the method of forecast of fractal time series(Brownian motion) as follows.

In conventional works, we demonstrated extractions of jump diffusion processes by using the multi state fuzzy inference systems. However, we need many computation time to identify the fuzzy system and we restrict to fractal time series, then we use the simple method to detect the jumps in fractal time series. we show only outline of the method.

Firstly we estimate the impulse response function h_{ij} from the original fractional time series. Here, we assume that $T_s < t < T_e$, and define $T_1 = T_e - T_s$. And we also define $b = a^D, a = T_2/T_1$ where D is the fractal dimension. Then we have following successive approximation for the range $0 < t < T_2$.

$$y(t) = b^{-1} \int_0^{b_{t0}} h(\frac{t}{b}, \frac{t-\tau}{b}) x(\tau) d\tau \tag{2}$$

Namely, in the range $T_s < t < T_e$ where the range is extended a times, we find b pieces of fractal figures, and then it is possible to get the above expressions. Namely, we observe b pieces of fractal figures in $T_s < t < T_e$ in the expanded time range by a times in this range. We also have next expression.

Now, we discuss the usage of forecast to detect jumps in investment.

(Step 1) Time series forecast

Based on the forecast of fractal time series, we have the estimation of future value of $P + dP$ (denoted as $\hat{P}(t + dt)$).

(Step 2) Estimation of occurrence of jumps

We compare the estimation $\hat{P}(t + dt)$ with the diagram of probability map shown in Fig.1. We decide the jump will be induced if $\hat{P}(t + dt) > P_r$ where P_r is a value of P in the diagram corresponds to the probability $\theta_{11}/3$ (upward jump). Similarly, in case of downward jump, we check whether $\hat{P}(t + dt) < P_r$ where P_r is a value of P in the diagram corresponds to the probability $\theta_{21}/3$.

(Step 3) Change of allocation)

If the jump is expected, then we decide to change the allocation of fund $(x_1, x_2, ..., x_N)$, so that we can increase the profit (if price rises), or can decrease the loss (if price falls). Then, we get better profit compared to ordinary investment where no reallocation of fund is exercised.

4 Description of optimal investments

4.1 Partial differential equations describing optimal production

Under the assumption of price change stated above, we derive a set of partial differential equations (PDF) giving optima investment. We assume that we invest total fund F proportional to the profit V_i obtained from ith product $(i = 1, 2, ..., N)$. Originally, we must formulize total optimization including N products, but by iterative approximations, we reduce the problem into an optimization including only two products. Because, in the total optimization, we at first select two products and the allocation to other products are fixed, then to optimize the profit for two products. Then, in the next step, we select another two products, and trace the same procedure to these new products. After sufficient time of iterations, we can get almost optimal allocation of fund to all N products.

We assume x_1, x_2 are the allocation of F to two products, and impose restriction $x_L < x_i < x_H$. The demands for two products are denoted as D_1, D_2, and they are sold in the amount $x_1 D_1, x_2 D_2$. The allocations x_i are given by $x_i = V_i/(V_1 + V_2)$. where V_i is the expected profits obtained from ith product.

The expected profits from products are given as follows(see Appendix A). However, for simplicity, we omit the subject i and $D_i = 1$ in the reduction.

$$V = \int_t^T \exp\{-\rho(t-\tau)\} P d\tau \tag{3}$$

By dividing the time span into two parts and by applying the Ito Lemma, then we obtain following partial differential equation. The solution for the equation give the profit V.

$$0 = L(V) + P + \sum_{k=1}^{2} E[V_k^{(+,P)} - V]\lambda_k(P) \tag{4}$$

$$+ \sum_{k=1}^{2} E[V_k^{(-,P)} - V]\gamma_k(P) \tag{5}$$

$$L(V) = V_t + 0.5\sigma^2 V_{PP} - \rho V \tag{6}$$

where subjects $k = 1, 2$ correspond to "go, back" processes and $V_k^{(+,P)} V_k^{(-,P)}$ are the value of V when upward or downward jumps occur in P. We denote

$$V_P = \partial V/\partial P, V_{PP} = \partial V^2/\partial P^2. \tag{7}$$

In general, the form of the equation is complicated and is hard to get analytical solution, then we apply successive numerical solutions based on finite difference equations. Important thing is the contributions (deviations) terms in equations given by jumps are only estimation based on the forecast using fractal. If we find by the forecast a kind of rise or fall of price, then we include corresponding them in PDFs.

4.2 Optimal portfolio of several securities

In case of portfolio selection of securities we simplify the scheme while we only need to see the prices in the next time point $t + dt$. If the jump is expected by using the method described above, then we estimate the price in the next time point as

$$\begin{aligned} P + dP &= P + E[(J_1 - P)\lambda_{go}dt + (J_2 - P)\lambda_{back}dt \\ &\quad +(K_1 - P)\gamma_{go}dt + (K_2 - P)\gamma_{back}dt + \sigma P\omega] \end{aligned} \tag{8}$$

We have

$$P + dP = [J_1\lambda_{go} + J_2\lambda_{back} + K_1\gamma_{go} + K_2\gamma_{back}]dt \tag{9}$$

These terms are easily evaluated by using probability distributions (we use only one term depending on the upward/downward jump and 'go'/'back' process).

5 Applications

5.1 Applications to optimal production

We show the comparison of rewards between two cases in the optimization of investment depending on the usage of forecast of occurrence of jumps. Namely, one of them is defined as the investment without consideration about jumps (called as Case N) delivering reward R_N, and another case includes forecast and the usage of effects of jumps (Case Y) and its reward (R_Y). In the simulation studies we define the ration r as follows

$$r = R_Y/R_N \tag{10}$$

Parameters used in the model are changed in the following rages.
$a_{11} = 200 \sim 1500, s_{11} = 50 \sim 200, a_{12} = 150 \sim 250, s_{12} = 2 \sim 20.$
$\theta_{11} = 0.01 \sim 0.05, \theta_{12} = 0.30 \sim 0.85.$
$a_{21} = 30 \sim 100, s_{21} = 3 \sim 15, a_{22} = 70 \sim 120, s_{22} = 3 \sim 12.$
$\theta_{21} = 0.01 \sim 0.05, \theta_{22} = 0.3 \sim 0.85.$

On the other hand, for simplicity, we carry out our simulation studies when the following parameters are fixed to constant values.

$\sigma = 0.2, PT_{11} = 450, PT_{12} = 300, PT_{21} = 100, PT_{22} = 50$. And the initial value of $P(t)$ is $150 (P(0) = 0)$.

We change only one parameter in a range to study the effects to optimal investment. Then we give the ordinary (standard) value of parameters as follows. We examine the effects of one parameter on the optimal investment by changing the focused parameter in a range while keeping other parameters to theses standard values.

$a_{11} = 700, s_{11} = 100, a_{12} = 200, s_{12} = 10,\ \theta_{11} = 0.01, \theta_{12} = 0.85,\ a_{21} = 30, s_{21} = 10, a_{22} = 100, s_{22} = 10,\ \theta_{21} = 0.04, \theta_{22} = 0.85.$

We also assume that we are facing to the optimal investment (allocation of production) among ten products. Each time series of prices of these products include similar jump processes superposed on fractal time series. Then, we evaluate the effects of forecast of jumps and calculate the profit including terms contributed from jumps by comparing the cases without forecast. Then, we have the ratio $r = R_Y/R_N$ as mentioned above.

Even though there are various aspects of simulation results, however followings are easily expected from the first consideration.

(Case 1) Jump size

If the size of jumps is relatively large, the effects of forecast of jumps increase the effects of optimal allocation of investment.

(Case 2) Probability of occurrence of jumps

In the same way, if the probability of jump occurrence is large the effects of forecast of jumps increase the result of optimization.

(Case 3) Time to return to previous levels

On the other hand, the effects of the parameter θ_{12}, θ_{22} defining the time to return to previous levels are not self-evident. Namely, if the time needed to return the time series to the previous level at the end of jumps is long, the changed allocation of investment is kept longer, and the effects last longer.

At first, we quickly summarize the simulation results for Case 1, Fig.4 shows the change of the ratio r depending on the parameter a_{11} (used for upward jump, left figure) and a_{21} (used for downward jump , right figure), respectively. In the same way, for Case 2, in Fig.5 we show the change of the ratio r depending on the parameter θ_{11} (used for upward jump, left figure) and θ_{21} (used for downward jump right figure), respectively.

As is seen from the results, in Case 1 the ration r monotonically increases (decreases) for upward jumps (for downward jumps) along the change of parameter. We notify that in cases of occurrence of downward jumps, if a_{11} is smaller then the gap between current value and the next value of P becomes larger.

In the same way, the ratio r increases for upward and downward jumps along the change of parameter. Then it is clear that the size of jump and the occurrence probability of jumps affect direct the ratio r, and we find the advantage of time series forecast on optimal production.

Then, we show the simulation study for Case 3. In Fig.6, we summarize the results of effects of parameters θ_{12} (for upward jump), and θ_{22} for downward jump) on the ratio r.

As shown in the figures, the probability of occurrence of 'back' jumps becomes smaller, then the ratio r becomes larger, since the time series remains around the same level, it then emphasizes the discrepancy between ordinary allocation considering no jumps and our adaptive allocation using forecast. The time to be needed for returning previous levels of the time series becomes longer, the optimal investment dominates on ordinary allocation.

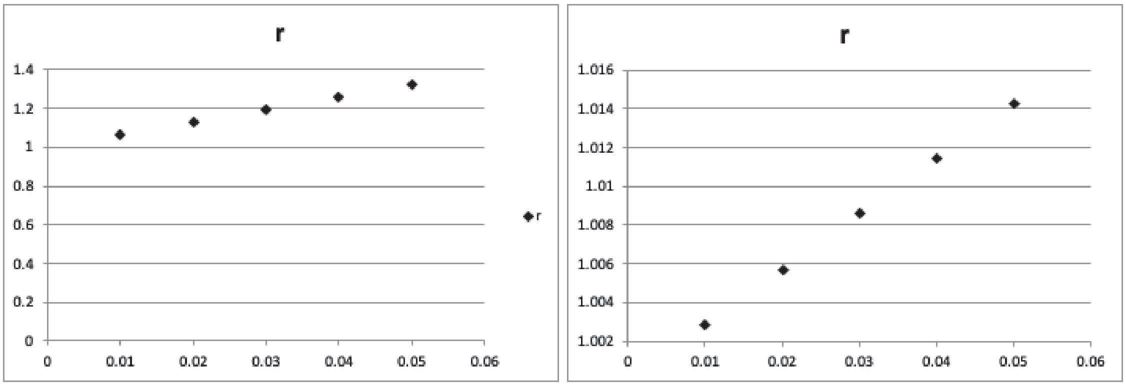

Figure 4: Change of ratio r depending on a_{11}(Left:upward jump) and a_{21}(Right:downward jump)

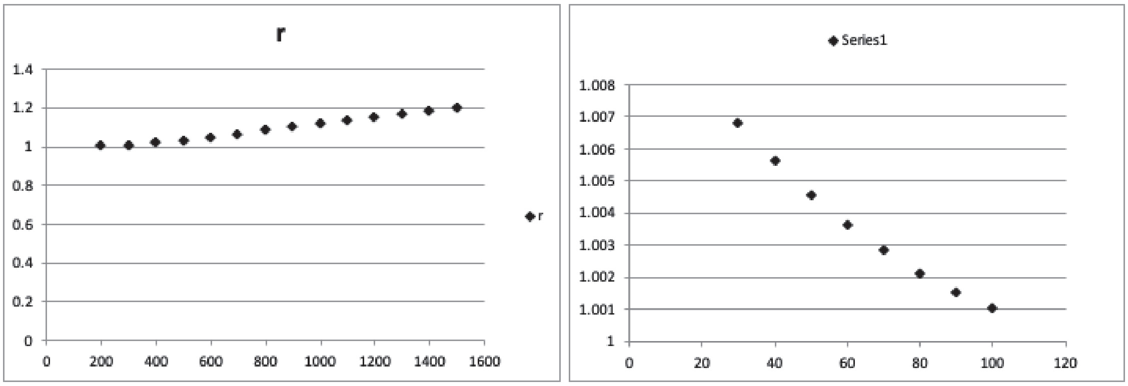

Figure 5: Change of ratio r depending on θ_{11}(Left:upward jump) and θ_{21}(Right:downward jump)

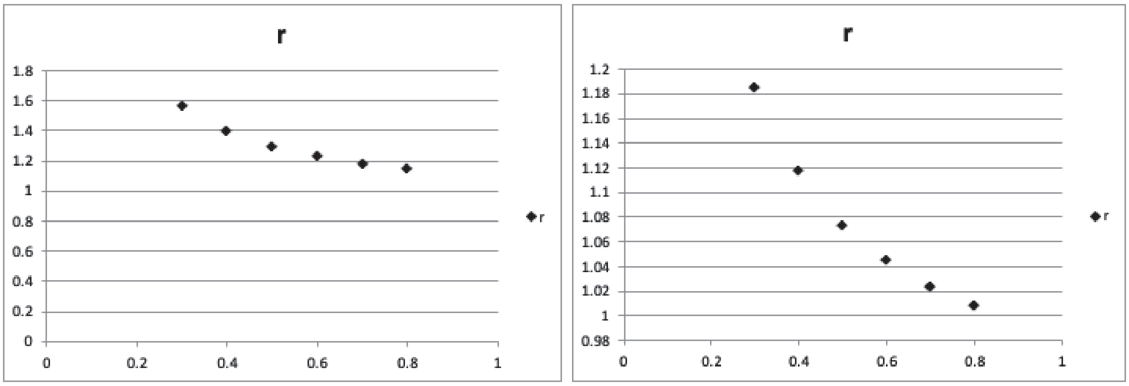

Figure 6: Change of ratio r depending on θ_{12}(Left:upward jump) and θ_{22}(Right:downward jump)

5.2 Applications to optimal portfolio

Then we apply the same forecast method of time series to optimize the portfolio of securities. Simply say, we change the weight of investment to securities depending on the future price, and especially we increase the weight of security expected to have higher price. However, the framework of the allocation of investment to each security is the same as treated in previous section for the production of goods. We also see many resemble characteristics in the treatment of optimal portfolio to the optimal production. Then, we simply summarize the simulation results in the following. To emphasize the effects to forecast the occurrence of jumps, we compare the value of investment only at the time point where jumps occur, and ignore the time points where a time series has no jumps and the same result is obtained between investment with and without forecast. Table 1 shows the change of ratio r related to the size of jumps depending on a_{11} (upward jump) and a_{21}(downward jump). Table 2 is the range of ratio r related to the probability of occurrence of jumps depending on θ_{11}(upward jump) and θ_{21}(downward jump). Table 3 denotes the Change of ratio r related to the probability of occurrence of 'back' process in jumps depending on θ_{12}(upward jump) and θ_{22}(downward jump).

Table 1: Change of ratio r depending on a_{11} (Upper:upward jump) and a_{21}(Lower:downward jump)

a_{11}	200	500	700	1000	1300
r	1.009	1.322	1.647	2.177	2.694
a_{21}	30	50	70	90	100
r	1.068	1.045	1.028	1.015	1.010

As is seen from the results, similar characteristics is found in tables like optimal productions in previous section. The value of r increases depending on the size of jumps and also the occurrence

Table 2: Change of ratio r depending on θ_{11}(Upper:upward jump) and θ_{21}(Lower:downward jump)

θ_{11}	0.01	0.02	0.03	0.04	0.05
r	1.647	2.295	2.943	3.591	4.239
a_{21}	0.01	0.02	0.03	0.04	0.05
r	1.028	1.057	1.085	1.114	1.142

Table 3: Change of ratio r depending on θ_{12}(Upper:upward jump) and θ_{22}(Lower:downward jump)

θ_{12}	0.3	0.4	0.5	0.6	0.7	0.8
r	4.394	3.295	2.636	2.197	1.883	1.647
θ_{22}	0.3	0.4	0.5	0.6	0.7	0.8
r	2.705	2.112	1.67	1.382	1.175	1.021

probability of jumps. We also see remarkable effects of low probability used for the returning process ('back' process) after the occurrence of jumps. Distinct advantage of portfolio using forecast is found over ordinary investment without changing the weight among securities.

6 Conclusion

We have shown the estimation of occurrence of jumps in time series models including jump diffusion processes and their applications. For future works we will extend the model to make more applicable to optimal portfolio of securities.

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Appendix A. Reduction of optimal production

For simplicity, we suppose at first the time series includes only upward jump. By dividing the range of integration $t \sim T$ in equation (3) into two parts as $t \sim t + dt$ and $t + dt \sim T$, then we have following transfomed equation,

$$V = E[\int_{t}^{t+dt} Pe^{-\rho(\tau-t)}d\tau + \int_{t+dt}^{T} Pe^{-\rho(\tau-t)}d\tau]$$

Sine the second term on the right hand side of equation can be expressed in the same form as the first term by shifting the time scale with dt, then we have following expression.

$$V = E[\int_{t}^{t+dt} Pe^{-\rho(\tau-t)}d\tau + \int_{t+dt}^{T} Pe^{-\rho(\tau-t)}V(P+dt)d\tau]$$

Next, we transform the term $V(t + dt)$ by using the Ito's Lemma then we have the expression.

$$0 = P + L(V)dt + \sum_{k=1}^{2} E[V_k^{(+,P)} - V]\lambda_k dt$$

$$L(V) = V_t + \frac{1}{2}\sigma^2 V_{PP} - \rho V$$

Here, the subject k correspond to go and back process of the occurrence of jump, and $V_k^{(+,P)}$ is the value of V if jumps occurred. The evaluation is the value of function when the jumps occurred, and we take expectations. By removing terms decreasing fasted than dt, and also dividing terms by dt, then we have finally the equation above.

Since we have the relation $P + dP = P + (J_k - P) = J_k + ...$, we calculate $E[V_k^{(+,P)} - P]$ approximately as

$$\int_{-\infty}^{\infty} V(P = J_k) r_k dJ_k$$

where J_k is the variable for normal distribution $N_k(.,.)$, and r_k is the variable for the standard normal probability distribution corresponds to $N_k(.,.)$. We replace the value of V at t by the value at $t + dt$ (known value) based on the solution process for the PDF using finite difference done in the backward manner from $t = T$ to $t = 0$.

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