Positive Definiteness in Linear Matrix Inequality Problem for H-infinity Output Feedback Control Problem

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Abstract—We focus on the positive definiteness in the linear matrix inequality (LMI) problem obtained from H_∞ output feedback control problem. In particular, we provide a necessary condition for an inequality constraint of the LMI problem, which is called the coupling constraint, to strictly hold at optimal solutions. It is reasonable to expect that the condition does not hold for most generalized plants. In other words, the strict inequality at the optimal solutions rarely happens in H_∞ output feedback control for most generalized plants. Thus, we must apply remedies for constructing a controller after solving the LMI problem.

Index Terms—LMIs, linear systems, optimization.

I. Introduction

THE importance of H_∞ control was proposed by Zames [1], and it has been studied from the viewpoints of theory, application, and computation since then (See e.g., [2] and references therein). Further, the two approaches to deal with H_∞ control problem, the algebraic Riccati equations/inequalities approach (e.g., [3], [4]) and linear matrix inequality (LMI) problem (e.g., [5], [6], [7]), were proposed and investigated thoroughly.

For H_∞ state feedback control, [8], [9] provided characterizations of the optimality via the algebraic Riccati equation. These characterizations were obtained under only the so-called standard assumption, i.e., the stabilizability of the plant, and therefore, a rather satisfactory theory for H_∞ state feedback control was established. For instance, the optimization was formulated as the infimum problem and [9] provided characterizations for when the optimal value is attained. On the other hand, although some characterizations of the optimality and suboptimality for H_∞ output feedback control were discussed under the stabilizability and some additional assumptions on zeros, they are less satisfactory in comparison to H_∞ state feedback control.

In the LMI approach, the optimization of H_∞ control is formulated as an LMI problem, which is a convex optimization problem. Thus, one can apply an SDP solver to the resulting LMI problem obtained from H_∞ control and obtain the optimal value of the problem. If an inequality constraint strictly holds at a computed solution, one can construct a controller from the solution. Otherwise, the solution may be available to construct a reduced-order controller under some restrictive conditions. See e.g., [7, Section IV].

The motivation of this study is to provide a necessary condition for the inequality constraint of the LMI problem, which is called the coupling constraint, to strictly hold in H_∞ output feedback control. The condition is that the optimal value is equal to one of the lower bounds simply obtained from the definition of H_∞ output feedback control optimization. It is reasonable to expect that, in general, the equality in a generalized plant is rather unusual. Thus, the strict inequality at the optimal solutions rarely happens in H_∞ output feedback control for most generalized plants. This means that a remedy to construct a controller, which is proposed in e.g., [7, Section IV.B], is necessary after solving the LMI problem for most generalized plants.

The following notation and symbols are used in this letter. Let $\mathbb{R}$, $\mathbb{C}$, $\mathbb{S}^n$, $\mathbb{S}^n_+$ and $\mathbb{S}^n_{++}$ be the sets of real numbers, complex numbers, $n\times n$ symmetry matrices, $n\times n$ positive semidefinite matrices and $n\times n$ positive definite matrices, respectively. j denotes the imaginary unit. $\bar{a}$ denotes the conjugate of $a\in\mathbb{C}$. For $b\in\mathbb{C}^n$, b^H denotes its conjugate transpose. $\sigma_{\max}(M)$ is the largest singular value of a matrix M. We define $\mathrm{He}(M) = M+M^T$ for any square matrix M.

II. Problem setting

We deal with the following generalized plant.

$$\begin{bmatrix} G_{zw} & G_{zu} \\ G_{yw} & G_{yu} \end{bmatrix} : \begin{cases} \dot{x} = Ax + B_1 w + B_2 u, \\ z = C_1 x + D_{11} w + D_{12} u, \\ y = C_2 x + D_{21} w, \end{cases} \quad (1)$$

where $x(t)\in\mathbb{R}^n$, $w(t)\in\mathbb{R}^{m_1}$, $u(t)\in\mathbb{R}^{m_2}$, $z(t)\in\mathbb{R}^{p_1}$ and $y(t)\in\mathbb{R}^{p_2}$. Here, the coefficient matrices in (1) have appropriate sizes. We design the following dynamical controller for (1).

$$K : \begin{cases} \dot{x}_K &= A_K x_K + B_K y, \\ u &= C_K x_K + D_K y, \end{cases} \quad (2)$$

where $x_K(t)\in\mathbb{R}^n$. We impose the following assumptions to (1) throughout this letter.

Assumption 1: (A, B_2) is stabilizable and (A, C_2) is detectable.

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Assumption 2: A has no purely imaginary eigenvalues.

We provide a formulation of the optimization of H_∞ output feedback control problem based on transfer functions of (1) and (2) as follows:

$$\gamma^* = \inf_{K\in\mathcal{K}} \sup_{s\in j\mathbb{R}} \sigma_{\max}(G_{\rm cl}(K,s)), \tag{3}$$

where $G_{\rm cl}(K,s) = G_{zw}(s) + G_{zu}(s)K(s)(I_n - G_{yu}(s)K(s))^{-1}G_{yw}(s)$ and $\mathcal{K}$ is the set of a rational function on s that makes the transfer function $G_{\rm cl}(K,s)$ internally stable.

We provide another formulation of H_∞ output feedback control problem, which is an LMI problem. For this, we introduce some notation and symbols. The state space representation of the closed loop of (1) with (2) is formulated as follows:

$$\begin{cases} \begin{bmatrix} \dot{x} \\ \dot{x}_K \end{bmatrix} = \begin{bmatrix} \tilde{A} & B_2C_K \\ B_KC_2 & A_K \end{bmatrix}\begin{bmatrix} x \\ x_K \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ B_KD_{21} \end{bmatrix} w, \\ z = \begin{bmatrix} \tilde{C}_1 & D_{12}C_K \end{bmatrix}\begin{bmatrix} x \\ x_K \end{bmatrix} + \tilde{D}w \end{cases}$$

where $\tilde{A}$, $\tilde{B}_1$, $\tilde{C}_1$, and $\tilde{D}$ are defined by $\tilde{A} = A + B_2D_KC_2$, $\tilde{B}_1 = B_1 + B_2D_KD_{21}$, $\tilde{C}_1 = C_1 + D_{12}D_KC_2$, and $\tilde{D} = D_{11} + D_{12}D_KD_{21}$, respectively.

The LMI problem of H_∞ output feedback control problem via the change in variables is formulated in (4). Here, $*$ indicates the transpose of the lower triangular matrix.

Remark 1: From [7], the optimal value of (4) is equal to the optimal value γ^* of (3). On the other hand, (4) may not have optimal solutions even if γ^* is finite. It was proved in [10] that (4) has an optimal solution if the following conditions 1) and 2) hold for (1).

1) For all $\lambda \in \mathbb{C}$ with $\Re(\lambda) \le 0$, rank $\left[\begin{smallmatrix} A-\lambda I_n & B_2 \\ C_1 & D_{12}\end{smallmatrix}\right] = n + m_2$ and rank $\left[\begin{smallmatrix} A^T-\lambda I_n & C_2^T \\ B_1^T & D_{21}^T\end{smallmatrix}\right] = n + p_2$.
2) Both D_{12} and D_{21}^T have full column ranks.

From the viewpoint of the algebraic Riccati equations/inequalities approach, [3, Section V.G] discussed the optimality and suboptimality of controllers under a weaker condition than 1) above.

An optimal solution of (4) is said to be positive definite if the solution satisfies

$$\begin{bmatrix} X & * \\ -I_n & Y \end{bmatrix} \in \mathbb{S}^{2n}_{++}. \tag{5}$$

If an optimal solution of (4) is positive definite, then one can compute controller parameters in (2) by using the solution. See [7] for the detail.

III. Lower bounds

Define $\sigma(M,x,y) = \left[\begin{smallmatrix} x^H & y^H \end{smallmatrix}\right]\left[\begin{smallmatrix} O & M \\ M^H & O\end{smallmatrix}\right]\left[\begin{smallmatrix} x \\ y\end{smallmatrix}\right]$. To obtain the lower bounds of γ^*, we use a characterization of the largest singular value.

Lemma 1: For any $M \in \mathbb{C}^{m\times n}$, we can formulate $\sigma_{\max}(M)$ as follows.

$$\sigma_{\max}(M) = \sup\left\{\sigma(M,x,y) : \begin{array}{l} \|x\|_2^2 + \|y\|_2^2 = 1, \\ x\in\mathbb{C}^m, y\in\mathbb{C}^n \end{array}\right\},$$

Proof: It follows from the singular value decomposition (SVD) of M that there exist $\Sigma \in \mathbb{R}^{m\times n}$ and unitary matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ such that $M = U\Sigma V$ and all the off-diagonals of Σ are zeros. Moreover, we have $\sigma_{\max}(M) = \Sigma_{11} \ge \Sigma_{22} \ge \cdots \ge \Sigma_{pp} \ge 0$ and $p = \min\{m,n\}$.

For any (x,y), define $z = U^Hx$ and $w = Vy$. Then

$$\begin{aligned} \sigma(M,x,y) = \sigma(\Sigma,z,w) &= \sum_{k=1}^p \Sigma_{kk}(\bar{z}_kw_k + z_k\bar{w}_k) \\ &\le \Sigma_{11}(z^Hw + w^Hz) \le \Sigma_{11}(\|z\|_2^2 + \|w\|_2^2). \end{aligned}$$

Furthermore, the equalities hold if we choose $\sqrt{2}x$ and $\sqrt{2}y$ as the first column vector of U and V, respectively. Since U and V are unitary, the desired result follows from these discussions. ■

Using Lemma 1, we provide a lower bound of γ^*. For any $K \in \mathcal{K}$ and $s \in j\mathbb{R}$, we have

$$\begin{aligned} &\sigma_{\max}(G_{\rm cl}(K,s)) \\ &= \sup\left\{\sigma(G_{\rm cl}(K,s),v_2,v_3) : \begin{array}{l} \|v_2\|_2^2 + \|v_3\|_2^2 = 1, \\ v_2\in\mathbb{C}^{p_1}, v_3\in\mathbb{C}^{m_1} \end{array}\right\} \\ &\ge \sup\left\{\sigma(G_{\rm cl}(K,s),v_2,v_3) : \begin{array}{l} \|v_2\|_2^2 + \|v_3\|_2^2 = 1, \\ v_2^HG_{zu}(s) = 0, \\ v_2\in\mathbb{C}^{p_1}, v_3\in\mathbb{C}^{m_1} \end{array}\right\} \\ &= \sup\left\{\sigma(G_{zw}(s),v_2,v_3) : \begin{array}{l} \|v_2\|_2^2 + \|v_3\|_2^2 = 1, \\ v_2^HG_{zu}(s) = 0, \\ v_2\in\mathbb{C}^{p_1}, v_3\in\mathbb{C}^{m_1} \end{array}\right\}. \end{aligned}$$

$\gamma_1^*(s)$ denotes the optimal value of the last maximization problem. Then, we obtain $\gamma^* \ge \gamma_1^*(s)$ for all $s \in j\mathbb{R}$. Similarly, we have

$$\begin{aligned} &\sigma_{\max}(G_{\rm cl}(K,s)) \\ &\ge \sup\left\{\sigma(G_{\rm cl}(K,s),v_2,v_3) : \begin{array}{l} \|v_2\|_2^2 + \|v_3\|_2^2 = 1, \\ G_{yw}(s)v_3 = 0, \\ v_2\in\mathbb{C}^{p_1}, v_3\in\mathbb{C}^{m_1} \end{array}\right\} \\ &= \sup\left\{\sigma(G_{zw}(s),v_2,v_3) : \begin{array}{l} \|v_2\|_2^2 + \|v_3\|_2^2 = 1, \\ G_{yw}(s)v_3 = 0, \\ v_2\in\mathbb{C}^{p_1}, v_3\in\mathbb{C}^{m_1} \end{array}\right\}. \end{aligned}$$

We denote the last maximization problem by $\gamma_2^*(s)$. Then, we obtain $\gamma^* \ge \gamma_2^*(s)$ for all $s \in j\mathbb{R}$. It should be noted that these lower bounds are obtained by removing the controller $K(s)$ from the objective function of (3). This idea can be also seen at the proof of necessity in [11, Theorem 2.6].

We can also obtain a lower bound of γ^* by setting $s = \infty$. In fact, for all $K \in \mathcal{K}$, because $G_{zw}(\infty) = D_{11}$, $G_{zu}(\infty) = D_{12}$, $G_{yw}(\infty) = D_{21}$ and $K(\infty) = D_K$, we have

$$\begin{aligned} &\sigma_{\max}(G_{\rm cl}(K,\infty)) = \sigma_{\max}(D_{11} + D_{12}D_KD_{21}) \\ &= \inf\left\{\gamma : \begin{array}{l} \begin{bmatrix} \gamma I_{p_1} & * \\ -D_{11}^T - (D_{12}D_KD_{21})^T & \gamma I_{m_1} \end{bmatrix} \in \mathbb{S}_+^{p_1+m_1}, \\ \gamma\in\mathbb{R} \end{array}\right\}. \end{aligned}$$

The last equality holds due to a characterization of the maximum singular values in e.g., [2, Section 2.1]. We

$$\inf\left\{\begin{array}{l}\gamma : \quad -\begin{bmatrix} \mathrm{He}(AX+B_2\hat{C}_K) & * & * & * \\ C_1X+D_{12}\hat{C}_K & -\gamma I_{p_1} & * & * \\ (B_1+B_2\hat{D}_KD_{21})^T & (D_{11}+D_{12}\hat{D}_KD_{21})^T & -\gamma I_{m_1} & * \\ (\hat{A}_K+A+B_2\hat{D}_KC_2)^T & (C_1+D_{12}\hat{D}_KC_2)^T & YB_1+\hat{B}_KD_{21} & \mathrm{He}(YA+\hat{B}_KC_2)\end{bmatrix} \in \mathbb{S}_+^{2n+m_1+p_1}, \\ \begin{bmatrix} X & * \\ -I_n & Y\end{bmatrix} \in \mathbb{S}_+^{2n}, \left[\begin{array}{c|c} \hat{A}_K & \hat{B}_K \\ \hline \hat{C}_K & \hat{D}_K\end{array}\right] \in \mathbb{R}^{(n+p_2)\times(n+m_2)}, \gamma \in \mathbb{R}\end{array}\right\} \tag{4}$$

denote the last minimization by $\gamma^*(D_K)$. Then, we obtain the following result.

Proposition 1: We have $\gamma^* \geq \gamma_1^*(s), \gamma_2^*(s)$ for all $s \in j\mathbb{R}$ and $\gamma^* \geq \gamma_\infty^* := \inf_{D_K} \gamma^*(D_K)$.

Remark 2: If the normal rank of $G_{zu}(s)$, i.e., the maximum of the rank of $G_{zu}(s)$ over $s \in \mathbb{C}$, is p_1, then $s \in \mathbb{C}$ that satisfies $v_2^H G_{zu}(s) = 0$ for some $v_2 \in \mathbb{C}^{p_1}$ is a transmission zero of G_{zu}. See [12, Lemma 3.28]. In addition, if $p_1 = 1$, then $\gamma_1^*(j\omega)$ can be simplified as follows.

$$\gamma_1^*(j\omega) = \begin{cases} \sigma_{\max}(G_{zw}(j\omega)) & \text{if } G_{zu}(j\omega) = 0, \\ 0 & \text{otherwise}\end{cases}$$

These also hold for G_{yw} and $\gamma_2^*(s)$.

IV. Main theorem

Theorem 1: Under Assumptions 1 and 2, if (4) has a positive definite optimal solution, at least one of the following holds.

1) There exists $\omega_1^* \in \mathbb{R}$ such that $\gamma^* = \gamma_1^*(j\omega_1^*)$,
2) there exists $\omega_2^* \in \mathbb{R}$ such that $\gamma^* = \gamma_2^*(j\omega_2^*)$ or
3) $\gamma^* = \gamma_\infty^*$.

Theorem 1 implies that it is rare that (4) has a positive definite optimal solution. To emphasize it, we provide the following result from Theorem 1 in the case of the generalized plant with the single-input-single-output.

Corollary 1: Let $m_1 = m_2 = p_1 = p_2 = 1$. Under Assumptions 1 and 2, if (4) has a positive definite optimal solution, at least one of the following holds.

1) There exists $\omega_1^* \in \mathbb{R}$ such that $\gamma^* = |G_{zw}(j\omega_1^*)|$ and $G_{zu}(j\omega_1^*) = 0$,
2) there exists $\omega_2^* \in \mathbb{R}$ such that $\gamma^* = |G_{zw}(j\omega_2^*)|$ and $G_{yw}(j\omega_2^*) = 0$,
3) $\gamma^* = |D_{11}|$ or 0.

Proof: 1) and 2) follow from Theorem 1 and Remark 2. For 3), if both D_{12} and D_{21} are nonzero, we obtain $\gamma_\infty^* = 0$ by $D_K = -D_{11}/D_{12}D_{21}$. Otherwise, $\gamma_\infty^* = \sigma_{\max}(D_{11}) = |D_{11}|$. ■

If 1) (resp. 2)) in Corollary 1 holds, then it follows from the definition of $G_{\mathrm{cl}}(K, s)$ that $\sigma_{\max}(G_{\mathrm{cl}}(K, j\omega_1^*)) = \gamma^*$ (resp. $\sigma_{\max}(G_{\mathrm{cl}}(K, j\omega_2^*)) = \gamma^*$) for each admissible controller $K \in \mathcal{K}$. This is a restrictive property for generalized plants (1).

Although the conditions in Theorem 1 cannot be simplified in the case of the multi-input-multi-output, the condition 1) (resp. 2)) is independent of the transfer function G_{yw} (resp. G_{zu}). The condition 3) uses only the feedthrough terms D_{ij}. Therefore it is reasonable to expect that (4) rarely has a positive definite optimal solution.

V. Proof of Theorem 1

The proof of Theorem 1 involves the following four steps: (i) simplify (4) to another LMI problem (6), (ii) obtain the dual problem (7) of (6), (iii) characterize the optimality of the dual solution, and (iv) reconstruct an optimal solution of the dual problem. We then obtain the conclusion of Theorem 1 from the solution. We note that we used a similar technique to [13], [14] in (iii).

A. Step (i): simplification of the LMI problem (4)

Consider another LMI problem (6). We can prove the equivalence between (4) and (6) by the so-called elimination lemma in [2, Section 2.6.2]. In fact, we first consider the interior of the feasible region of (4). It follows from Assumption 1 that the interior is nonempty. The interior can be reformulated by replacing positive semidefiniteness by positive definiteness. Then, we can apply the elimination lemma to one of the inequalities that express the interior of the feasible region. By taking the closure to the set obtained by the elimination lemma, we can observe the equivalence between (4) and (6).

B. Step (ii): formulation of dual of the LMI problem (6)

We can obtain the dual problem (7) of (6) by constructing the Lagrange dual problem of (6). Here, the blanks in the matrices denote the zero matrices with appropriate sizes and $S_1 \bullet S_2 = \mathrm{Tr}(S_1^T S_2)$ for any $S_1, S_2 \in \mathbb{R}^{m\times n}$.

It follows from Assumption 1 that (6) is strictly feasible, i.e., it has a solution at which all constraints strictly hold. Then, it follows from the strong duality theorem on the semidefinite program presented in e.g., [15, Theorem 2.3] that the optimal value of (6) is equal to that of (7) and (7) has an optimal solution.

C. Step (iii): characterization of the optimality in (7)

Assume that (6) has a positive definite optimal solution, i.e., (5) holds. Let (Z, V, W) be an optimal solution of (7). Then, it satisfies $W = O$ due to the complementarity condition on the semidefinite program in e.g., [15, Section 2.4]. Hence, the dual optimal solution (Z, V, O) satisfies

$$\begin{aligned}\gamma^* = \mathrm{Tr}\left(B_1^T Z_{31}^T + D_{11}^T Z_{32}^T\right) + \mathrm{Tr}\left(C_1 V_{31}^T + D_{11} V_{32}^T\right) \\ \mathrm{Tr}\left(B_1 Z_{31} + D_{11} Z_{32}\right) + \mathrm{Tr}\left(C_1^T V_{31} + D_{11}^T V_{32}\right),\end{aligned} \tag{8}$$

$$I_{p_1} \bullet Z_{22} + I_{m_1} \bullet Z_{33} + I_{m_1} \bullet V_{22} + I_{p_1} \bullet V_{33} = 1, \tag{9}$$

$$\mathrm{He}(A^T Z_{11} + C_1^T Z_{21}) = O, B_2^T Z_{11} + D_{12}^T Z_{21} = O, \tag{10}$$

$$\mathrm{He}(AV_{11} + B_1 V_{21}) = O, C_2 V_{11} + D_{21} V_{21} = O, \tag{11}$$

$$D_{21}\begin{bmatrix} Z_{31} & Z_{32}\end{bmatrix}\begin{bmatrix} B_2 \\ D_{12}\end{bmatrix} + \begin{bmatrix} C_2 & D_{21}\end{bmatrix}\begin{bmatrix} V_{31}^T \\ V_{32}^T\end{bmatrix} D_{12} = O, \tag{12}$$

$$
\inf\left\{\gamma : \begin{array}{l} -\begin{bmatrix} \mathrm{He}(AX+B_2\hat{C}_K) & * & * \\ C_1X+D_{12}\hat{C}_K & -\gamma I_{p_1} & * \\ (B_1+B_2\hat{D}_KD_{21})^T & (D_{11}+D_{12}\hat{D}_KD_{21})^T & -\gamma I_{m_1} \end{bmatrix} \in \mathbb{S}_+^{n+m_1+p_1}, \begin{bmatrix} X & * \\ -I_n & Y \end{bmatrix} \in \mathbb{S}_+^{2n}, \\ -\begin{bmatrix} \mathrm{He}(A^TY+C_2^T\hat{B}_K^T) & * & * \\ B_1^TY+D_{21}^T\hat{B}_K^T & -\gamma I_{m_1} & * \\ C_1+D_{12}\hat{D}_KC_2 & D_{11}+D_{12}\hat{D}_KD_{21} & -\gamma I_{p_1} \end{bmatrix} \in \mathbb{S}_+^{n+m_1+p_1}, \begin{array}{l} \hat{B}_K \in \mathbb{R}^{n\times p_2}, \hat{C}_K \in \mathbb{R}^{m_2\times n}, \\ \hat{D}_K \in \mathbb{R}^{m_2\times p_2}, \gamma \in \mathbb{R}. \end{array} \end{array}\right\} \tag{6}
$$

$$
\left\{\begin{array}{ll} \sup & \begin{bmatrix} & & * \\ & & * \\ B_1^T & D_{11}^T & \end{bmatrix} \bullet Z + \begin{bmatrix} & & * \\ & & * \\ C_1 & D_{11} & \end{bmatrix} \bullet V + \begin{bmatrix} & * \\ I_n & \end{bmatrix} \bullet W \\ \text{s.t.} & I_{p_1} \bullet Z_{22} + I_{m_1} \bullet Z_{33} + I_{m_1} \bullet V_{22} + I_{p_1} \bullet V_{33} = 1, \mathrm{He}(A^TZ_{11}+C_1^TZ_{21}) = W_{11}, \\ & B_2^TZ_{11}+D_{12}^TZ_{21} = O, \mathrm{He}(AV_{11}+B_1V_{21}) = W_{22}, C_2V_{11}+D_{21}V_{21} = O, \\ & D_{21}\begin{bmatrix} Z_{31} & Z_{32} \end{bmatrix}\begin{bmatrix} B_2 \\ D_{12} \end{bmatrix} + \begin{bmatrix} C_2 & D_{21} \end{bmatrix}\begin{bmatrix} V_{31}^T \\ V_{32}^T \end{bmatrix} D_{12} = O, W = \begin{bmatrix} W_{11} & * \\ W_{21} & W_{22} \end{bmatrix} \in \mathbb{S}_+^{2n}, \\ & Z = \begin{bmatrix} Z_{11} & * & * \\ Z_{21} & Z_{22} & * \\ Z_{31} & Z_{32} & Z_{33} \end{bmatrix} \in \mathbb{S}_+^{n+p_1+m_1}, V = \begin{bmatrix} V_{11} & * & * \\ V_{21} & V_{22} & * \\ V_{31} & V_{32} & V_{33} \end{bmatrix} \in \mathbb{S}_+^{n+p_1+m_1}. \end{array}\right. \tag{7}
$$

and the positive semidefiniteness of Z and V.

Because Z and V are positive semidefinite, we decompose Z and V as follows.

$$
Z = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}^T + \begin{bmatrix} O_n & * & * \\ O & \tilde{Z}_{22} & * \\ O & \tilde{Z}_{32} & \tilde{Z}_{33} \end{bmatrix} =: Z_1^* + Z_2^*,
$$

$$
V = \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix}\begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix}^T + \begin{bmatrix} O_n & * & * \\ O & \tilde{V}_{22} & * \\ O & \tilde{V}_{32} & \tilde{V}_{33} \end{bmatrix} =: V_1^* + V_2^*,
$$

where both F_1 and G_1 are of full column rank or vanished, and the second terms in the above are positive semidefinite. If $Z_{11} = O_n$, then $F_j \ (j = 1, 2, 3)$ is eliminated from the above decomposition of Z, i.e. $Z = Z_2^*$. This also holds in the decomposition of V. Let r_F and r_G be the ranks of F_1 and G_1, respectively.

We can rewrite (10) and (11) by using the following lemma. The proof is provided in Appendix A.

Lemma 2: Assume that the matrices F_1 and F_2 satisfy $\mathrm{He}((A^TF_1 + C_1^TF_2)F_1^T) = O$. In addition, we assume that F_1 is of full column rank with rank r. Then there exists $\Omega \in \mathbb{R}^{r\times r}$ such that $\mathrm{He}(\Omega) = O_r$ and $A^TF_1 + C_1^TF_2 = F_1\Omega$.

It follows from Lemma 2 that there exists an $\Omega_F \in \mathbb{R}^{r_F\times r_F}$ such that $\mathrm{He}(\Omega_F) = O_{r_F}$ and

$$
A^TF_1 + C_1^TF_2 = F_1\Omega_F, B_2^TF_1 + D_{12}^TF_2 = O. \tag{13}
$$

Similarly, there exists an $\Omega_G \in \mathbb{R}^{r_G\times r_G}$ such that $\mathrm{He}(\Omega_G) = O_{r_G}$ and

$$
AG_1 + B_1G_2 = G_1\Omega_G, C_2G_1 + D_{21}G_2 = O. \tag{14}
$$

It is clear that $\begin{bmatrix} Z_{31} & Z_{32} \end{bmatrix} = \begin{bmatrix} F_3F_1^T & F_3F_2^T + \tilde{Z}_{32} \end{bmatrix}$, and $\begin{bmatrix} V_{31} & V_{32} \end{bmatrix} = \begin{bmatrix} G_3G_1^T & G_3G_2^T + \tilde{V}_{32} \end{bmatrix}$. Thus, it follows from (13) and (14) that (12) is equivalent to

$$
D_{21}\left(\tilde{Z}_{32} + \tilde{V}_{32}^T\right)D_{12} = O. \tag{15}
$$

Because all skew-symmetric matrices can be diagonalized using a unitary matrix, we have $\Lambda_\circ = U_\circ^H\Omega_\circ U_\circ \ (\circ = F, G)$, where $\Lambda_\circ \in \mathbb{C}^{r_\circ\times r_\circ}$ is diagonal and $U_\circ \in \mathbb{C}^{r_\circ\times r_\circ}$ is a unitary matrix. In particular, all diagonal elements $\lambda_{\circ,p}$ $(p = 1, \ldots, r_\circ)$ in $\Lambda_\circ$ are purely imaginary numbers. We set $F_\ell U_F = \begin{bmatrix} f_{\ell,1} & \ldots & f_{\ell,r_F} \end{bmatrix}$ and $G_\ell U_G = \begin{bmatrix} g_{\ell,1} & \ldots & g_{\ell,r_G} \end{bmatrix}$ for $\ell = 1, 2, 3$. (13) and (14) imply $A^Tf_{1,p} + C_1^Tf_{2,p} = \lambda_{F,p}f_{1,p}$, $B_2^Tf_{1,p} + D_{12}^Tf_{2,p} = 0$, $Ag_{1,p} + B_1g_{2,p} = \lambda_{G,p}g_{1,p}$ and $C_2g_{1,p} + D_{21}g_{2,p} = 0$ for all p. We note that each $f_{2,p}$ is not the zero vector for all p if F_1 is not vanished. Otherwise, we obtain $A^Tf_{1,p} = \lambda_{F,p}f_{1,p}$ and $B_2^Tf_{1,p} = 0$, and thus it contradicts Assumption 1. Similarly, no $g_{2,p}$ is the zero vector. In addition, from Assumption 2, we obtain for all p

$$
\begin{aligned} f_{1,p} &= (\lambda_{F,p}I_n - A^T)^{-1}C_1^Tf_{2,p}, f_{2,p}^HG_{zu}(\bar{\lambda}_{F,p}) = 0, \\ g_{1,p} &= (\lambda_{G,p}I_n - A)^{-1}B_1g_{2,p}, G_{yw}(\lambda_{G,p})g_{2,p} = 0. \end{aligned} \tag{16}
$$

We can rewrite the equations (8) and (9) by using $f_{\ell,p}$ and $g_{\ell,p}$. In fact, because U_F is unitary, Z has the form of $Z = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} U_F \left(\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} U_F\right)^H + Z_2^*$. Thus, we can rewrite the first terms in (8) and (9) as follows.

$$
\begin{aligned} \mathrm{Tr}\left(B_1^TZ_{31}^T + D_{11}^TZ_{32}^T\right) &= \sum_{p=1}^{r_F} f_{3,p}^HG_{zw}(\bar{\lambda}_{F,p})f_{2,p} \\ &\quad + \begin{bmatrix} & * \\ D_{11}^T & \end{bmatrix} \bullet \begin{bmatrix} \tilde{Z}_{22} & * \\ \tilde{Z}_{32} & \tilde{Z}_{33} \end{bmatrix} \\ I_{p_1} \bullet Z_{22} + I_{m_1} \bullet Z_{33} &= \sum_{p=1}^{r_F} \left(\|f_{2,p}\|_2^2 + \|f_{3,p}\|_2^2\right) \end{aligned}
$$

Further, V has a similar form, and we can rewrite the second terms in (8) and (9). Hence, the equations (8) and

(9) are rewritten by substituting $f_{1,p}$ and $g_{1,p}$ as follows.

$$\begin{aligned}\gamma^* &= \sum_{p=1}^{r_F} \sigma\left(G_{zw}(\bar{\lambda}_{F,p}), f_{2,p}, f_{3,p}\right)\\ &\quad + \sum_{p=1}^{r_G} \sigma\left(G_{zw}(\lambda_{G,p}), g_{3,p}, g_{2,p}\right)\\ &\quad + \begin{bmatrix} & * \\ D_{11}^T & \end{bmatrix} \bullet \begin{bmatrix}\tilde{Z}_{22}+\tilde{V}_{33} & * \\ \tilde{Z}_{32}+\tilde{V}_{32}^T & \tilde{Z}_{33}+\tilde{V}_{22}\end{bmatrix}, \end{aligned} \quad (17)$$

$$\begin{aligned}1 &= \sum_{p=1}^{r_F}\left(\|f_{2,p}\|_2^2 + \|f_{3,p}\|_2^2\right) + \sum_{p=1}^{r_G}\left(\|g_{2,p}\|_2^2 + \|g_{3,p}\|_2^2\right)\\ &\quad + \begin{bmatrix} I_{p_1} & \\ & I_{m_1}\end{bmatrix} \bullet \begin{bmatrix}\tilde{Z}_{22}+\tilde{V}_{33} & * \\ \tilde{Z}_{32}+\tilde{V}_{32}^T & \tilde{Z}_{33}+\tilde{V}_{22}\end{bmatrix}. \end{aligned} \quad (18)$$

D. Step (iv): reconstruction of the optimal solution

We construct another optimal solution with a simpler expression from the optimal solution (Z, V, O) of (7). We assume that $f_{2,p} \neq 0$ and $g_{2,p} \neq 0$ for all p and that the matrix $\begin{bmatrix}\tilde{Z}_{22}+\tilde{V}_{33} & * \\ \tilde{Z}_{32}+\tilde{V}_{32}^T & \tilde{Z}_{33}+\tilde{V}_{22}\end{bmatrix}$ is nonzero. Otherwise, we can ignore the terms of $f_{\ell,p}, g_{\ell,p}$ or the matrix in (17) and (18). From the assumption, we have for all p

$$\begin{aligned}&\alpha_p := \|f_{2,p}\|_2^2 + \|f_{3,p}\|_2^2 < 1, \beta_p := \|g_{2,p}\|_2^2 + \|g_{3,p}\|_2^2 < 1,\\ &\delta := \begin{bmatrix} I_{p_1} & \\ & I_{m_1}\end{bmatrix} \bullet \begin{bmatrix}\tilde{Z}_{22}+\tilde{V}_{33} & * \\ \tilde{Z}_{32}+\tilde{V}_{32}^T & \tilde{Z}_{33}+\tilde{V}_{22}\end{bmatrix} < 1.\end{aligned}$$

Then, it follows from (18) that $\sum_p \alpha_p + \sum_p \beta_p + \delta = 1$. Moreover, because all α_p, β_p and δ are nonzeros, we can define $\hat{Z}_p$, $\hat{V}_p$ and $\hat{S}$ by

$$\begin{aligned}\hat{Z}_p &= \frac{1}{2\alpha_p}\,\mathrm{He}\left(\begin{bmatrix}f_{1,p}\\ f_{2,p}\\ f_{3,p}\end{bmatrix}\begin{bmatrix}f_{1,p}\\ f_{2,p}\\ f_{3,p}\end{bmatrix}^H\right),\\ \hat{V}_p &= \frac{1}{2\beta_p}\,\mathrm{He}\left(\begin{bmatrix}g_{1,p}\\ g_{2,p}\\ g_{3,p}\end{bmatrix}\begin{bmatrix}g_{1,p}\\ g_{2,p}\\ g_{3,p}\end{bmatrix}^H\right),\\ \hat{S} &= \frac{1}{\delta}\begin{bmatrix}\tilde{Z}_{22}+\tilde{V}_{33} & * \\ \tilde{Z}_{32}+\tilde{V}_{32}^T & \tilde{Z}_{33}+\tilde{V}_{22}\end{bmatrix}.\end{aligned}$$

Then, it follows from (13), (14) and (15) that $(\hat{Z}_p, O, O)$, $(O, \hat{V}_p, O)$, for all p and $(O, O, \hat{S})$ are feasible in (7) with the objective values being $d_1(\hat{Z}_p)$, $d_2(\hat{V}_p)$, and $d_\infty(\hat{S})$, respectively. Here, we define the functions $d_1(Z)$ and $d_2(V)$ for any $Z, V \in \mathbb{S}^{n+p_1+m_1}$ and $d_\infty(S)$ for any $S \in \mathbb{S}^{m_1+p_1}$.

$$d_1(Z) = \begin{bmatrix} & & * \\ & & * \\ B_1^T & D_{11}^T & \end{bmatrix} \bullet Z, d_2(V) = \begin{bmatrix} & & * \\ & & * \\ C_1 & D_{11} & \end{bmatrix} \bullet V,$$
$$d_\infty(S) = \begin{bmatrix} & * \\ D_{11}^T & \end{bmatrix} \bullet S.$$

We prove that at least one of these feasible solutions is optimal in (7). In fact, we note that

$$d_1(\hat{Z}_p) = \sigma\left(G_{zw}(\bar{\lambda}_{F,p}), f_{2,p}, f_{3,p}\right), \quad (19)$$
$$d_2(\hat{V}_p) = \sigma\left(G_{zw}(\lambda_{G,p}), g_{3,p}, g_{2,p}\right). \quad (20)$$

for all p. From (17), (18) and Proposition 1, we have

$$\begin{aligned}\gamma^* &= \sum_{p=1}^{r_F}\alpha_p d_1(\hat{Z}_p) + \sum_{p=1}^{r_G}\beta_p d_2(\hat{V}_p) + \delta d_\infty(\hat{S})\\ &\le \max\Big\{d_1(\hat{Z}_p)\ (p=1,\dots,r_F),\\ &\qquad d_2(\hat{V}_p)\ (p=1,\dots,r_G), d_\infty(\hat{S})\Big\} \le \gamma^*.\end{aligned}$$

The last inequality implies that at least one of $(\hat{Z}_p, O, O)$, $(O, \hat{V}_p, O)$, for all p and $(O, O, \hat{S})$ is optimal in (7).

If $(\hat{Z}_p, O, O)$ is optimal, then $\gamma^* = d_1(\hat{Z}_p) = \gamma_1^*(\bar{\lambda}_{F,p})$. In fact, it follows from the definition of α_p and (16) that the vector $(f_{2,p}/\sqrt{\alpha_p}, f_{3,p}/\sqrt{\alpha_p})$ satisfies

$$\left\|\frac{f_{2,p}}{\sqrt{\alpha_p}}\right\|_2^2 + \left\|\frac{f_{3,p}}{\sqrt{\alpha_p}}\right\|_2^2 = 1 \text{ and } \left(\frac{f_{2,p}}{\sqrt{\alpha_p}}\right)^H G_{zu}(\bar{\lambda}_{F,p}) = 0,$$

and thus, the objective value $\sigma\left(G_{zw}(\bar{\lambda}_{F,p}), \frac{f_{2,p}}{\sqrt{\alpha_p}}, \frac{f_{3,p}}{\sqrt{\alpha_p}}\right)$ is less than or equal to $\gamma_1^*(\bar{\lambda}_{F,p})$. This can be seen from the definition of $\gamma_1^*(s)$. Hence, it follows from Proposition 1 that $\gamma^* = d_1(\hat{Z}_p) \le \gamma_1^*(\bar{\lambda}_{F,p}) \le \gamma^*$. A similar result holds for the case in which $(O, \hat{V}_p, O)$ is optimal.

If $(O, O, \hat{S})$ is optimal, then $d_\infty(\hat{S}) = \gamma_\infty^*$. In fact, the dual of the minimization of $\gamma^*(D_K)$ over all $D_K \in \mathbb{R}^{m_2 \times p_2}$ is formulated as follows.

$$\sup\left\{d_\infty(S) :\ \begin{aligned}&S = \begin{bmatrix}S_{22} & * \\ S_{32} & S_{33}\end{bmatrix} \in \mathbb{S}_+^{p_1+m_1},\\ &I_{p_1}\bullet S_{22} + I_{m_1}\bullet S_{33} = 1,\\ &D_{21}S_{32}D_{12} = O\end{aligned}\right\}.$$

The optimal value of the above maximization is γ_∞^* because of the strong duality on the semidefinite program. It follows from (15) and the definition of δ that $\hat{S}$ is feasible in the above maximization problem. Hence, it follows from Proposition 1 that $\gamma^* = d_\infty(\hat{S}) \le \gamma_\infty^* \le \gamma^*$. Therefore, we obtain the desired result.

VI. H_∞ state feedback problem

We can obtain similar results to Proposition 1 and Theorem 1 for H_∞ state feedback problem. In this case, we deal with $C_2 = I_n$ and $D_{21} = O$ for (1) and consider a static feedback law $u = Kx$. Then, the LMI problem via the change in variables can be formulated as follows.

$$\inf\left\{\gamma :\ \begin{aligned}&-\begin{bmatrix}\mathrm{He}(AX+B_2Y) & * & * \\ C_1X + D_{12}Y & -\gamma I_{p_1} & * \\ B_1^T & D_{11}^T & -\gamma I_{m_1}\end{bmatrix} \in \mathbb{S}_+^N,\\ &X \in \mathbb{S}_+^n, Y \in \mathbb{R}^{m_2\times n}, \gamma \in \mathbb{R}\end{aligned}\right\}, \quad (21)$$

where $N = n + p_1 + m_1$. Let γ^* be the optimal value of (21).

Proposition 2: $\gamma^* \ge \gamma_1^*(s)$ for all $s \in j\mathbb{R}$ and $\gamma^* \ge \sigma_{\max}(D_{11})$.

If (21) has an optimal solution (γ^*, X^*, Y^*) that satisfies $X^* \in \mathbb{S}_{++}^n$, then we can provide a static feedback law $u = K^*x$ by $K^* = Y^*(X^*)^{-1}$. We call such an optimal solution of (21) a positive definite optimal solution of (21).

Theorem 2: Under Assumptions 1 and 2, if (21) has a positive definite optimal solution, at least one of the following holds.

1) There exists $\omega^* \in \mathbb{R}$ such that $\gamma^* = \gamma_1^*(j\omega^*)$ or
2) $\gamma^* = \sigma_{\max}(D_{11})$.

Theorem 2 is proven in a similar manner to Theorem 1.

Remark 3: In [9, Theorem 2], the existence of an optimal controller for H_∞ state feedback control problem was characterized by using an algebraic Riccati equation/inequality under Assumption 1. For instance, in a more restrictive situation than Assumption 1, it was introduced in [9, a corollary of Theorem 2] that an algebraic Riccati equation (ARE) defined by γ^* has a positive definite solution if and only if there exists an optimal controller. Moreover, it was presented in [16] that the Hamiltonian matrix related to the ARE has a purely imaginary eigenvalue because of the optimality of γ^*. The investigation of the relationship between the eigenvalue and $j\omega^*$ of 1) in Theorem 2 is future work.

We can discuss the result for an H_∞ state feedback control in [17] from the viewpoint of Theorem 2. The authors in [17] considered the following generalized plant.

$$\left\{\begin{array}{rcl} \dot{x} &=& Ax + B_1 w + B_2 u, \\ z &=& \begin{bmatrix} I_n \\ O_{q\times n}\end{bmatrix} x + \begin{bmatrix} O_{n\times m} \\ D \end{bmatrix} u, \\ y &=& x, \end{array}\right. \tag{22}$$

where A is symmetric and Hurwitz stable, and $D \in \mathbb{R}^{q\times m}$ is of full column rank. We present a result discussed in [17].

Theorem 3: ([17, Theorem 1 and Remark 1]) Define $R = D^T D$. Then, an optimal solution (γ^*, X^*, Y^*) of the H_∞ state feedback control problem for (22) is $\gamma^* = \sqrt{\|B_1^T(A^2 + B_2 R^{-1} B_2^T)^{-1} B_1\|}$, $X^* = -A$ and $Y^* = -R^{-1}B_2^T$. Here, $\|M\|$ denotes the largest eigenvalue of a square matrix M. Consequently, an optimal state feedback gain K^* is provided by $K^* = R^{-1} B_2^T A^{-1}$.

For (22), it follows from Theorems 2 and 3 that the condition 1) in Theorem 2 holds because $D_{11} = O$ in (22). In addition, we can specify ω^* as follows.

Proposition 3: $\gamma^* = \gamma_1^*(0)$.

Proof: The transfer functions $G_{zw}(s)$ and $G_{zu}(s)$ are formulated as

$$G_{zw}(s) = \begin{bmatrix} (sI_n - A)^{-1} B_1 \\ O_{m\times n} \end{bmatrix}, G_{zu}(s) = \begin{bmatrix} (sI_n - A)^{-1} B_2 \\ D \end{bmatrix}.$$

Then, $\gamma_1^*(0)$ can be formulated by

$$\sup \left\{ \begin{array}{c} -v_1^H A^{-1} B_1 v_3 \\ -v_3^H B_1^T A^{-T} v_1 \end{array} : \begin{array}{c} B_2^T(-A^{-T}) v_1 + D^T v_2 = 0, \\ \|v_1\|_2^2 + \|v_2\|_2^2 + \|v_3\|_2^2 = 1, \\ v_1 \in \mathbb{C}^n, v_2 \in \mathbb{C}^q, v_3 \in \mathbb{C}^m \end{array} \right\}.$$

It is easy to verify that for each optimal solution $v^* = (v_1^*, v_2^*, v_3^*)$, v_2^* is spanned by some columns of D. Otherwise, it contradicts the optimality of v^*.

We define $\tilde{v}_1 = (-A)^{-T} v_1$ and $v_2 = D\tilde{v}_2$. Then, $\gamma_1^*(0)$ can be reformulated by

$$\sup \left\{ \begin{array}{c} \tilde{v}_1^H B_1 v_3 \\ +\tilde{v}_3^H B_1^T v_1 \end{array} : \begin{array}{c} \|A^T \tilde{v}_1\|_2^2 + \|DR^{-1}B_2^T \tilde{v}_1\|_2^2 \\ +\|v_3\|_2^2 = 1, \\ \tilde{v}_1 \in \mathbb{C}^n, v_3 \in \mathbb{C}^m \end{array} \right\}.$$

We define $x = (A^2 + B_2 R^{-1} B_2^T)^{1/2} \tilde{v}_1$, $y = v_3$ and $M = (A^2 + B_2 R^{-1} B_2^T)^{1/2} B_1$. Then, it follows from Lemma 1 that $\gamma_1^*(0)$ is the square root of the largest eigenvalue of $B_1^T(A^2 + B_2 R^{-1} B_2^T)^{-1} B_1$. ■

Appendix A
Proof of Lemma 2

Because F_1 is of full column rank, this statement follows from Lemma 3. Rantzer [14, (iii) of Lemma 3] provided a proof for an extension of Lemma 3.

Lemma 3: Let $F, G \in \mathbb{R}^{m\times n}$. Assume that F is of full column rank. Then, $\mathrm{He}(FG^T) = O_m$ if and only if there exists an $\Omega \in \mathbb{R}^{n\times n}$ such that $G = F\Omega$ and $\mathrm{He}(\Omega) = O_n$.

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