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Partial Isometries, Duality, and Determinantal Point Processes

Dedicated to Professor Hirofumi Osada on the occasion of his 60th birthday

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Abstract

A determinantal point process (DPP) is an ensemble of random nonnegative-integer-valued Radon measures Ξ on a space S with measure λ, whose correlation functions are all given by determinants specified by an integral kernel K called the correlation kernel. We consider a pair of Hilbert spaces, $H_\ell, \ell = 1, 2$, which are assumed to be realized as L^2-spaces, $L^2(S_\ell, \lambda_\ell)$, $\ell = 1, 2$, and introduce a bounded linear operator $\mathcal{W} : H_1 \to H_2$ and its adjoint $\mathcal{W}^* : H_2 \to H_1$. We show that if $\mathcal{W}$ is a partial isometry of locally Hilbert–Schmidt class, then we have a unique DPP (Ξ_1, K_1, λ_1) associated with $\mathcal{W}^*\mathcal{W}$. In addition, if $\mathcal{W}^*$ is also of locally Hilbert–Schmidt class, then we have a unique pair of DPPs, $(\Xi_\ell, K_\ell, \lambda_\ell)$, $\ell = 1, 2$. We also give a practical framework which makes $\mathcal{W}$ and $\mathcal{W}^*$ satisfy the above conditions. Our framework to construct pairs of DPPs implies useful duality relations between DPPs making pairs. For a correlation kernel of a given DPP our formula can provide plural different expressions, which reveal different aspects of the DPP. In order to demonstrate these advantages of our framework as well as to show that the class of DPPs obtained by this method is large enough to study universal structures in a variety of DPPs, we report plenty of examples of DPPs in one-, two-, and higher-dimensional spaces S, where several types of weak convergence from finite DPPs to infinite DPPs are given. One-parameter ($d \in \mathbb{N}$) series of infinite DPPs on $S = \mathbb{R}^d$ and $\mathbb{C}^d$ are discussed, which we call the Euclidean and the Heisenberg families of DPPs, respectively, following the terminologies of Zelditch.

Keywords: Determinantal point processes; correlation kernels; partial isometry; locally Hilbert–Schmidt operators; duality; reproducing kernels; random matrix theory

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1 Introduction

Let S be a base space, which is a locally compact Hausdorff space with countable base, and λ be a Radon measure on S. The configuration space over S is given by the set of nonnegative-integer-valued Radon measures;

$$\mathrm{Conf}(S) = \left\{ \xi = \sum_j \delta_{x_j} : x_j \in S, \, \xi(\Lambda) < \infty \text{ for all bounded set } \Lambda \subset S \right\}.$$

$\mathrm{Conf}(S)$ is equipped with the topological Borel σ-fields with respect to the vague topology; we say $\xi_n, n \in \mathbb{N} := \{1, 2, \dots\}$ converges to ξ in the vague topology, if $\int_S f(x)\xi_n(dx) \to \int_S f(x)\xi(dx)$, $\forall f \in \mathcal{C}_\mathrm{c}(S)$, where $\mathcal{C}_\mathrm{c}(S)$ is the set of all continuous real-valued functions with compact support. A *point process* on S is a $\mathrm{Conf}(S)$-valued random variable $\Xi = \Xi(\cdot, \omega)$ on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. If $\Xi(\{x\}) \in \{0, 1\}$ for any point $x \in S$, then the point process is said to be *simple*.

Assume that $\Lambda_j, j = 1, \dots, m$, $m \in \mathbb{N}$ are disjoint bounded sets in S and $k_j \in \mathbb{N}_0 := \{0, 1, \dots\}, j = 1, \dots, m$ satisfy $\sum_{j=1}^m k_j = n \in \mathbb{N}_0$. A symmetric measure λ^n on S^n is called the n-th *correlation measure*, if it satisfies

$$\mathbf{E}\left[\prod_{j=1}^m \frac{\Xi(\Lambda_j)!}{(\Xi(\Lambda_j) - k_j)!} \right] = \lambda^n(\Lambda_1^{k_1} \times \cdots \times \Lambda_m^{k_m}),$$

where if $\Xi(\Lambda_j) - k_j < 0$, we interpret $\Xi(\Lambda_j)!/(\Xi(\Lambda_j) - k_j)! = 0$. If λ^n is absolutely continuous with respect to the n-product measure $\lambda^{\otimes n}$, the Radon–Nikodym derivative $\rho^n(x_1, \dots, x_n)$ is called the *n-point correlation function* with respect to the background measure λ;

$$\lambda^n(dx_1 \cdots dx_n) = \rho^n(x_1, \dots, x_n)\lambda^{\otimes n}(dx_1 \cdots dx_n).$$

Determinantal point process (DPP) is defined as follows [40, 53, 58, 54, 55, 28, 29].

Definition 1.1 *A simple point process Ξ on (S, λ) is said to be a determinantal point process (DPP) with correlation kernel $K : S \times S \to \mathbb{C}$ if it has correlation functions $\{\rho^n\}_{n \in \mathbb{N}}$, and they are given by*

$$\rho^n(x_1, \dots, x_n) = \det_{1 \le j,k \le n} [K(x_j, x_k)] \quad \text{for every } n \in \mathbb{N}, \text{ and } x_1, \dots, x_n \in S. \tag{1.1}$$

The triplet $(\Xi, K, \lambda(dx))$ denotes the DPP; $\Xi \in \mathrm{Conf}(S)$, specified by the correlation kernel K with respect to the measure $\lambda(dx)$.

If the integral projection operator $\mathcal{K}$ on $L^2(S, \lambda)$ with a kernel K is of rank $N \in \mathbb{N}$, then the number of points is N a.s. If $N < \infty$ (resp. $N = \infty$), we call the system a *finite DPP* (resp. an *infinite DPP*). The density of points with respect to the background measure $\lambda(dx)$ is given by

$$\rho(x) := \rho^1(x) = K(x, x).$$

The DPP is negatively correlated as shown by

$$\begin{aligned} \rho^2(x, x') &= \det \begin{bmatrix} K(x, x) & K(x, x') \\ K(x', x) & K(x', x') \end{bmatrix} \\ &= K(x, x)K(x', x') - |K(x, x')|^2 \le \rho(x)\rho(x'), \quad x, x' \in S, \end{aligned} \tag{1.2}$$

provided that K is Hermitian.

Let H be a separable Hilbert space. For linear operators $\mathcal{A}, \mathcal{B}$ on H, we say that $\mathcal{A}$ is positive definite and write $\mathcal{A} \geq O$ if $\langle \mathcal{A} f, f \rangle_H \geq 0$ for any $f \in H$, and write $\mathcal{A} \geq \mathcal{B}$ if $\mathcal{A} - \mathcal{B} \geq O$. The operator $\mathcal{A}^* \mathcal{A}$ is positive definite and it admits a unique positive-definite square-root $\sqrt{\mathcal{A}^* \mathcal{A}}$ which is denoted by $|\mathcal{A}|$. Let $\{\phi_n\}_{n \geq 1}$ be an orthonormal basis of H. For $\mathcal{A} \geq O$, we define the trace of $\mathcal{A}$ by

$$\operatorname{Tr} \mathcal{A} := \sum_{n=1}^{\infty} \langle \mathcal{A} \phi_n, \phi_n \rangle_H,$$

which does not depend on the choice of an orthonormal basis. A bounded linear operator $\mathcal{A}$ is said to be *of trace class* or a *trace class operator* if the trace norm $\|\mathcal{A}\|_1 := \operatorname{Tr} |\mathcal{A}|$ is finite. The trace $\operatorname{Tr} \mathcal{A}$ is defined whenever $\|\mathcal{A}\|_1 < \infty$.

Now, we consider the case $H = L^2(S, \lambda)$. For a compact set $\Lambda \subset S$, the projection from $L^2(S, \lambda)$ to the space of all functions vanishing outside Λ λ-a.e. is denoted by $\mathcal{P}_\Lambda$. $\mathcal{P}_\Lambda$ is the operation of multiplication of the indicator function $\mathbf{1}_\Lambda$ of the set Λ; $\mathbf{1}_\Lambda(x) = 1$ if $x \in \Lambda$, and $\mathbf{1}_\Lambda(x) = 0$ otherwise. We say that a bounded linear operator $\mathcal{A}$ on $L^2(S, \lambda)$ is *of locally trace class* or a *locally trace class operator*, if the restriction of $\mathcal{A}$ to each compact subset Λ is of trace class; that is,

$$\|\mathcal{A}_\Lambda\|_1 < \infty \quad \text{with} \quad \mathcal{A}_\Lambda := \mathcal{P}_\Lambda \mathcal{A} \mathcal{P}_\Lambda \quad \text{for any compact set } \Lambda \subset S.$$

The totality of locally trace class operators on $L^2(S, \lambda)$ is denoted by $\mathcal{I}_{1,\mathrm{loc}}(S, \lambda)$. It is known that [53, 58, 54, 55], if $\mathcal{K} \in \mathcal{I}_{1,\mathrm{loc}}(S, \lambda)$ and $O \leq \mathcal{K} \leq I$, where I is the identity operator, then we have a unique DPP on S with the determinantal correlation functions (1.1) with respect to λ and the correlation kernel K is given by the Hermitian integral kernel for $\mathcal{K}$ (see Section 2.1 below).

In the present paper, we consider the case in which

$$\mathcal{K} f = f \quad \text{for all } f \in (\ker \mathcal{K})^\perp \subset L^2(S, \lambda).$$

Here $(\ker \mathcal{K})^\perp$ denotes the *orthogonal complement of the kernel space* of $\mathcal{K}$. That is, $\mathcal{K}$ is an *orthogonal projection*. By definition, it is obvious that the condition $O \leq \mathcal{K} \leq I$ is satisfied. The purpose of the present paper is to propose a useful method to provide orthogonal projections $\mathcal{K}$ and DPPs whose correlation kernels are given by the Hermitian integral kernels of $\mathcal{K}$, $K(x, x')$, $x, x' \in S$.

We consider a pair of Hilbert spaces, $H_\ell, \ell = 1, 2$, which are assumed to be realized as L^2-spaces, $L^2(S_\ell, \lambda_\ell)$, $\ell = 1, 2$. We introduce a bounded linear operator $\mathcal{W}$ and its adjoint $\mathcal{W}^*$,

$$\mathcal{W} : H_1 \to H_2, \quad \mathcal{W}^* : H_2 \to H_1. \tag{1.3}$$

Then, we have the following basic existence theorem of DPP via a partial isometry $\mathcal{W}$ of locally Hilbert–Schmidt class.

Theorem 1.2 *Assume that $\mathcal{W} : L^2(S_1, \lambda_1) \to L^2(S_2, \lambda_2)$ is a partial isometry of locally Hilbert–Schmidt class. Then, there exists a unique DPP $(\Xi_1, K_{S_1}, \lambda_1)$ on S_1 with*

$$K_{S_1}(x, x') = \int_{S_2} \overline{W(y, x)} W(y, x') \lambda_2(dy), \tag{1.4}$$

where $\mathcal{W}$ admits a measurable kernel $W : S_2 \times S_1 \to \mathbb{C}$ such that $\Psi_1 \in L^2_{\mathrm{loc}}(S_1, \lambda_1)$ with $\Psi_1(x) := \|W(\cdot, x)\|_{L^2(S_2, \lambda_2)}$ $(x \in S_1)$.

The definitions of partial isometries, locally Hilbert–Schmidt operators, and $L^2_{\rm loc}(S,\lambda)$ will be given in Section 2.2. There we will show basic properties of them and Theorem 1.2 will be concluded from the well-known existence theorem of DPP [53, 58, 54, 55] (Theorem 2.1).

We assume that

(i) $\mathcal{W}$ is a *partial isometry*,

(ii) both $\mathcal{W}$ and $\mathcal{W}^*$ are *locally Hilbert–Schmidt operators.*

Under the assumption (i), the adjoint $\mathcal{W}^*$ is also a partial isometry. If two conditions (i) and (ii) are satisfied, by Theorem 1.2, we have a unique pair of DPPs, $(\Xi_\ell, K_\ell, \lambda_\ell)$, $\ell = 1, 2$, where the correlation kernel K_1 (resp. K_2) is given by the integral kernel of $\mathcal{W}^*\mathcal{W}$ (resp. $\mathcal{W}\mathcal{W}^*$) (Theorem 2.4) as in (1.4). We give a practical framework which makes $\mathcal{W}$ and $\mathcal{W}^*$ satisfy the above two assumptions (Corollaries 2.9 and 2.10).

One of the advantages of our framework is that the obtained pairs of DPPs satisfy useful duality relations, which will be reported in Sections 2.4, 3.1.2, 4.1.4. As mentioned above, one of a pair of DPPs discussed here is associated with a Hilbert space H_1 having an orthogonal projection $\mathcal{K}_1$, and $\mathcal{K}_1$ is given in the form $\mathcal{K}_1 = \mathcal{W}^*\mathcal{W}$. This equality can be regarded as a *decomposition formula* of $\mathcal{K}_1$ by a product of an operator $\mathcal{W}$ and its dual $\mathcal{W}^*$ acting as (1.3) provided that another Hilbert space H_2 is chosen. We note the fact that for a given DPP associated with H_1 and $\mathcal{K}_1$, choice of H_2 is not unique. As demonstrated in Sections 4.1.1–4.1.3 using the Ginibre DPPs on $\mathbb{C}$, such multivalency in our framework can give plural different expressions for one correlation kernel K_1 and they will help us to study different aspects of the DPP which we consider.

In order to demonstrate the class of DPPs obtained by our framework is large enough to study a variety of DPPs and universal structures behind them, we show plenty of examples of DPPs in one- and two-dimensional spaces. In particular, we use the symbols of classical and affine roots systems (e.g., $A_{N-1}, B_N, C_N, D_N, N \in \mathbb{N}$) to classify finite DPPs. Several types of weak convergence theorems of finite DPPs to infinite DPPs are given. We will show that in the one-dimensional space, there are three universal DPPs with an infinite number of points specified by the correlation kernels,

$$
\begin{aligned}
K_{\rm sinc}(x,x') &:= \frac{\sin(x-x')}{\pi(x-x')} = \frac{1}{2\pi}\int_{-1}^{1} e^{i\gamma(x-x')}d\gamma, \quad x,x' \in \mathbb{R}, \\
K^{(1/2)}_{\rm Bessel}(x,x') &:= \frac{\sin(x-x')}{\pi(x-x')} - \frac{\sin(x+x')}{\pi(x+x')} = \frac{1}{\pi}\int_{-1}^{1} \sin(\gamma x)\sin(\gamma x')d\gamma, \quad x,x' \in [0,\infty), \\
K^{(-1/2)}_{\rm Bessel}(x,x') &:= \frac{\sin(x-x')}{\pi(x-x')} + \frac{\sin(x+x')}{\pi(x+x')} = \frac{1}{\pi}\int_{-1}^{1} \cos(\gamma x)\cos(\gamma x')d\gamma, \quad x,x' \in [0,\infty),
\end{aligned}
$$

where $i := \sqrt{-1}$. $K_{\rm sinc}$ is usually called the sine kernel in random matrix theory [43], but it shall be called the *sinc kernel.* $K^{(1/2)}_{\rm Bessel}$ and $K^{(-1/2)}_{\rm Bessel}$ are special cases of the *Bessel kernels* $K^{(\nu)}_{\rm Bessel}$, $\nu \in (-1,\infty)$ with indices $\nu = 1/2$ and $-1/2$, respectively [21]. Note that $K_{\rm sinc}(x,x') = \{K^{(1/2)}_{\rm Bessel}(x,x') + K^{(-1/2)}_{\rm Bessel}(x,x')\}/2$, $x,x' \in [0,\infty)$. Corresponding to the threefold of DPPs with the correlation kernels, $K_{\rm sinc}$, $K^{(1/2)}_{\rm Bessel}$, $K^{(-1/2)}_{\rm Bessel}$, we also show the three universal DPPs on $\mathbb{C}$, whose correlation kernels are given by

$$\begin{aligned}
K^A_{\text{Ginibre}}(x,x') &:= e^{x\overline{x'}} = \sum_{n=0}^{\infty} \frac{(x\overline{x'})^n}{n!}, \\
K^C_{\text{Ginibre}}(x,x') &:= \sinh(x\overline{x'}) = \sum_{n=0}^{\infty} \frac{(x\overline{x'})^{2n+1}}{(2n+1)!}, \\
K^D_{\text{Ginibre}}(x,x') &:= \cosh(x\overline{x'}) = \sum_{n=0}^{\infty} \frac{(x\overline{x'})^{2n}}{(2n)!}, \quad x, x' \in \mathbb{C},
\end{aligned}$$

where $\overline{x'}$ denotes the complex conjugate of x'. K^A_{Ginibre} is known as the correlation kernel of the *Ginibre ensemble* in random matrix theory [23, 21], and K^C_{Ginibre} and K^D_{Ginibre} were studied in [34]. Note that $K^A_{\text{Ginibre}}(x,x') = K^C_{\text{Ginibre}}(x,x') + K^D_{\text{Ginibre}}(x,x')$, $x, x' \in \mathbb{C}$.

Our method to generate DPPs is also valid in higher dimensional spaces. We will state that the DPP with the sinc kernel K_{sinc} is the lowest-dimensional ($d = 1$) example of the one-parameter ($d \in \mathbb{N}$) family of DPPs on $\mathbb{R}^d$, whose correlation kernels are given by

$$\begin{aligned}
K^{(d)}_{\text{Euclid}}(x,x') &:= \frac{1}{(2\pi)^{d/2}} \frac{J_{d/2}(\|x-x'\|_{\mathbb{R}^d})}{\|x-x'\|^{d/2}_{\mathbb{R}^d}} \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{B}^d} e^{i\gamma\cdot(x-x')} d\gamma, \quad x, x' \in \mathbb{R}^d,
\end{aligned}$$

with respect to the Lebesgue measures of $\mathbb{R}^d$, $\lambda(dx) = dx$, where J_ν is the Bessel function of the first kind, $\|x-x'\|_{\mathbb{R}^d}$ is the Euclidean distance between x and x' in $\mathbb{R}^d$, and $\mathbb{B}^d$ is the unit ball in $\mathbb{R}^d$ centered at the origin. We also claim that the Ginibre ensemble is the lowest-dimensional example ($d = 1$) of another one-parameter ($d \in \mathbb{N}$) family of DPPs on $\mathbb{C}^d$, whose correlation kernel is given by

$$K^{(d)}_{\text{Heisenberg}}(x,x') := e^{x\cdot\overline{x'}}, \quad x, x' \in \mathbb{C}^d,$$

where the background measure λ is assumed to be the d-dimensional complex normal distribution. We call these two families of DPPs the *Euclidean family* of DPPs and the *Heisenberg family* of DPPs, respectively, following the terminologies by Zelditch [66]. See also [14, 57, 67, 18].

The paper is organized as follows. In Section 2 we give main theorems which enable us to generate DPPs. Sections 3 and 4 are devoted to a variety of examples of DPPs obtained by our framework for the one-dimensional and the two-dimensional spaces, respectively. Examples in spaces with arbitrary dimensions $d \in \mathbb{N}$ are given in Section 5. We list out open problems in Section 6. Appendices A and C are used to explain useful multivariate functions and determinantal formulas associated with the classical and the affine root systems, respectively. The definitions and basic properties of the Jacobi theta functions are summarized in Appendix B.

2 Main Theorems

2.1 Existence theorem of DPPs

We recall the existence theorem for DPPs. Let (S, λ) be a σ-finite measure space. We assume that $\mathcal{K} \in \mathcal{I}_{1,\text{loc}}(S, \lambda)$. If, in addition, $\mathcal{K} \geq O$, then it admits a Hermitian integral kernel $K(x, x')$ such that (cf. [22])

(i) $\det_{1 \le j,k \le n} [K(x_j, x_k)] \ge 0$ for $\lambda^{\otimes n}$-a.e. $(x_1, \dots, x_n)$ with every $n \in \mathbb{N}$,

(ii) $K_{x'} := K(\cdot, x') \in L^2(S, \lambda)$ for λ-a.e. x',

(iii) $\operatorname{Tr} \mathcal{K}_\Lambda = \int_\Lambda K(x,x)\lambda(dx)$, $\Lambda \subset S$ and

$$\operatorname{Tr}(\mathcal{P}_\Lambda \mathcal{K}^n \mathcal{P}_\Lambda) = \int_\Lambda \langle K_{x'}, \mathcal{K}^{n-2} K_{x'} \rangle_{L^2(S,\lambda)} \lambda(dx'), \quad \forall n \in \{2, 3, \dots\}.$$

This is based on the fact that every positive definite trace class operator has the form $\mathcal{B}^*\mathcal{B}$ of a Hilbert–Schmidt operator $\mathcal{B}$ together with a similar idea of the proof of Proposition 2.3 mentioned below.

Theorem 2.1 ([53, 58, 54, 55]) *Assume that $\mathcal{K} \in \mathcal{I}_{1,\mathrm{loc}}(S, \lambda)$ and $O \le \mathcal{K} \le I$. Then there exists a unique DPP (Ξ, K, λ) on S.*

If $\mathcal{K} \in \mathcal{I}_{1,\mathrm{loc}}(S, \lambda)$ is a projection onto a closed subspace $H \subset L^2(S, \lambda)$, one has *the DPP associated with K and λ*, or one may say *the DPP associated with the subspace H*. This situation often appears in the setting of *reproducing kernel Hilbert space* [9]. Let $\mathcal{F} = \mathcal{F}(S)$ be a Hilbert space of complex functions on S with inner product $\langle \cdot, \cdot \rangle_{\mathcal{F}}$. A function $K(x, x')$ on $S \times S$ is said to be a *reproducing kernel* of $\mathcal{F}$ if

1. For every $x' \in S$, the function $K(\cdot, x')$ belongs to $\mathcal{F}$.
2. The function $K(x, x')$ has reproducing kernel property; that is, for any $f \in \mathcal{F}$,

$$f(x') = \langle f(\cdot), K(\cdot, x') \rangle_{\mathcal{F}}.$$

A reproducing kernel of $\mathcal{F}$ is unique if exists, and it exists if and only if the point evaluation map $\mathcal{F} \ni f \mapsto f(x) \in \mathbb{C}$ is bounded for every $x \in S$. The Moore–Aronszajn theorem states that if a kernel $K(\cdot, \cdot)$ on $S \times S$ is positive definite in the sense that for any $n \ge 1$, $x_1, \dots, x_n \in S$, the matrix $(K(x_j, x_k))_{j,k \in \{1,\dots,n\}}$ is positive definite, then there exists a unique Hilbert space H_K of functions with inner product in which $K(x, x')$ is a reproducing kernel [9]. If H_K is realized in $L^2(S, \lambda)$ for some measure λ, the kernel $K(x, x')$ defines a projection onto H_K.

2.2 Partial isometries, locally Hilbert–Schmidt operators, and DPPs

First we recall the notion of partial isometries between Hilbert spaces [25, 26]. Let $H_\ell, \ell = 1, 2$ be separable Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_{H_\ell}$. For a bounded linear operator $\mathcal{W} : H_1 \to H_2$, the adjoint of $\mathcal{W}$ is defined as the operator $\mathcal{W}^* : H_2 \to H_1$, such that

$$\langle \mathcal{W} f, g \rangle_{H_2} = \langle f, \mathcal{W}^* g \rangle_{H_1} \quad \text{for all } f \in H_1 \text{ and } g \in H_2. \tag{2.1}$$

A linear operator $\mathcal{W}$ is called an *isometry* if

$$\|\mathcal{W} f\|_{H_2} = \|f\|_{H_1} \quad \text{for all } f \in H_1.$$

The kernel space of $\mathcal{W}$ is denoted as $\ker \mathcal{W}$ and its orthogonal complement is written as $(\ker \mathcal{W})^\perp$. A linear operator $\mathcal{W}$ is called a *partial isometry*, if

$$\|\mathcal{W} f\|_{H_2} = \|f\|_{H_1} \quad \text{for all } f \in (\ker \mathcal{W})^\perp.$$

For the partial isometry $\mathcal{W}$, $(\ker \mathcal{W})^{\perp}$ is called the *initial space* and the range of $\mathcal{W}$, ran$\mathcal{W}$, is called the *final space*. By the definition (2.1), $\|\mathcal{W}f\|_{H_2}^2 = \langle \mathcal{W}f, \mathcal{W}f\rangle_{H_2} = \langle f, \mathcal{W}^*\mathcal{W}f\rangle_{H_1}$. As is suggested from this equality, we have the following fact for partial isometries. Although this might be known, we give a proof below.

Lemma 2.2 *Let H_1 and H_2 be separable Hilbert spaces and $\mathcal{W} : H_1 \to H_2$ be a bounded operator. Then, the following are equivalent.*

(i) *$\mathcal{W}$ is a partial isometry.*

(ii) *$\mathcal{W}^*\mathcal{W}$ is a projection on H_1, which acts as the identity on $(\ker \mathcal{W})^{\perp}$.*

(iii) *$\mathcal{W} = \mathcal{W}\mathcal{W}^*\mathcal{W}$.*

Moreover, $\mathcal{W}$ is a partial isometry if and only if so is $\mathcal{W}^*$.

Proof When $H_1 = H_2$, this fact is well-known (cf. [25]). If we apply it to $H = H_1 \oplus H_2$ and $\widetilde{\mathcal{W}} : H \to H$ defined by

$$\widetilde{\mathcal{W}} := \begin{pmatrix} O & O \\ \mathcal{W} & O \end{pmatrix},$$

the assertion is followed by verifying $\widetilde{\mathcal{W}^*}\widetilde{\mathcal{W}} = \mathcal{W}^*\mathcal{W} \oplus O$, $\widetilde{\mathcal{W}}\widetilde{\mathcal{W}^*} = O \oplus \mathcal{W}\mathcal{W}^*$ and

$$\widetilde{\mathcal{W}}\widetilde{\mathcal{W}}^*\widetilde{\mathcal{W}} = \begin{pmatrix} O & O \\ \mathcal{W}\mathcal{W}^*\mathcal{W} & O \end{pmatrix}.$$

∎

We note that the conditions (i), (ii) and (iii) above, and the conditions (i)', (ii)' and (iii)' obtained by applying Lemma 2.2 to the adjoint $\mathcal{W}^* : H_2 \to H_1$ are all equivalent.

Assumption 1 $\mathcal{W}$ is a partial isometry.

By Lemma 2.2, under Assumption 1, $\mathcal{W}^*$ is also a partial isometry and hence the operator $\mathcal{W}^*\mathcal{W}$ (resp. $\mathcal{W}\mathcal{W}^*$) is the projection onto the initial space of $\mathcal{W}$ (resp. the final space of $\mathcal{W}$).

Now we assume that H_1 and H_2 are realized as L^2-spaces, $L^2(S_1, \lambda_1)$ and $L^2(S_2, \lambda_2)$, respectively.

A bounded linear operator $\mathcal{A} : L^2(S_1, \lambda_1) \to L^2(S_2, \lambda_2)$ is a *Hilbert–Schmidt operator* if Hilbert–Schmidt norm is finite; $\|\mathcal{A}\|_{\mathrm{HS}}^2 := \mathrm{Tr}\,(\mathcal{A}^*\mathcal{A}) < \infty$. We say that $\mathcal{A}$ is a *locally Hilbert–Schmidt operator* or *of locally Hilbert–Schmidt class*, if $\mathcal{A}\mathcal{P}_{\Lambda}$ is a Hilbert–Schmidt operator for any compact set $\Lambda \subset S$. It is known as the *kernel theorem* that every Hilbert–Schmidt operator $\mathcal{A} : L^2(S_1, \lambda_1) \to L^2(S_2, \lambda_2)$ is defined as an integral operator with kernel $A \in L^2(S_1 \times S_2, \lambda_1 \otimes \lambda_2)$ (cf. Theorem 12.6.2 [10]). In Proposition 2.3, we prove a local version of the kernel theorem.

We put the second assumption.

Assumption 2 (i) $\mathcal{W}$ is a locally Hilbert–Schmidt operator, and (ii) $\mathcal{W}^*$ is a locally Hilbert–Schmidt operator.

We note that for any compact set $\Lambda_1 \subset S_1$, the operator $\mathcal{W}\mathcal{P}_{\Lambda_1}$ is of Hilbert–Schmidt class if and only if the operator $\mathcal{P}_{\Lambda_1}\mathcal{W}^*\mathcal{W}\mathcal{P}_{\Lambda_1}$ is of trace class since

$$\|\mathcal{W}\mathcal{P}_{\Lambda_1}\|_{\mathrm{HS}}^2 = \mathrm{Tr}\left((\mathcal{W}\mathcal{P}_{\Lambda_1})^*\mathcal{W}\mathcal{P}_{\Lambda_1}\right) = \mathrm{Tr}\left(\mathcal{P}_{\Lambda_1}\mathcal{W}^*\mathcal{W}\mathcal{P}_{\Lambda_1}\right) < \infty.$$

Therefore, Assumption 2 (i) (resp. Assumption 2 (ii)) is equivalent to the following Assumption 2' (i) (resp. Assumption 2' (ii)), which guarantees the existence of DPP associated with $\mathcal{W}^*\mathcal{W}$ (resp. $\mathcal{W}\mathcal{W}^*$).

Assumption 2' (i) $\mathcal{W}^*\mathcal{W} \in \mathcal{I}_{1,\mathrm{loc}}(S_1, \lambda_1)$ and (ii) $\mathcal{W}\mathcal{W}^* \in \mathcal{I}_{1,\mathrm{loc}}(S_2, \lambda_2)$.

Given a measure space (S, λ), if $f \in L^2(\Lambda, \lambda)$ for all compact subsets Λ of S, then f is said to be locally L^2-integrable. The set of all such functions is denoted by $L^2_{\mathrm{loc}}(S, \lambda)$. By this definition if $\mathcal{P}_\Lambda f \in L^2(S, \lambda)$ for any compact set $\Lambda \subset S$, then $f \in L^2_{\mathrm{loc}}(S, \lambda)$. The following proposition is a *local* version of the kernel theorem for Hilbert–Schmidt operators.

Proposition 2.3 *Suppose Assumption* 2 (i) *holds. Then,* $\mathcal{W}$ *is regarded as an integral operator associated with a kernel* $W : S_2 \times S_1 \to \mathbb{C}$*;*

$$(\mathcal{W}f)(y) = \int_{S_1} W(y, x) f(x) \lambda_1(dx), \quad f \in L^2(S_1, \lambda_1), \tag{2.2}$$

such that $\Psi_1 \in L^2_{\mathrm{loc}}(S_1, \lambda_1)$*, where* $\Psi_1(x) := \|W(\cdot, x)\|_{L^2(S_2, \lambda_2)}, x \in S_1$.

Proof From Assumption 2 (i) and the kernel theorem for Hilbert–Schmidt operators (cf. Theorem 12.6.2 [10]), for each compact set $\Lambda \subset S_1$, there exists a kernel $W_\Lambda \in L^2(S_2 \times S_1, \lambda_2 \otimes \lambda_1)$ such that

$$\mathcal{W}\mathcal{P}_\Lambda f(y) = \int_{S_1} W_\Lambda(y, x) f(x) \lambda_1(dx).$$

Since $\mathcal{W}\mathcal{P}_\Lambda f(y) = 0$ for all $f \in L^2(S_1, \lambda_1)$ whose support is contained in Λ^{c}, $W_\Lambda(y, x) = 0$ on $S_2 \times \Lambda^{\mathrm{c}}$ for $\lambda_2 \otimes \lambda_1$-a.e.$(y, x)$. We take two compact sets Λ and Λ' with $\Lambda \subset \Lambda' \subset S_1$. For any $f \in L^2(S_1, \lambda_1)$ whose support is contained in Λ, we see that $\mathcal{W}\mathcal{P}_{\Lambda'} f = \mathcal{W}\mathcal{P}_{\Lambda'}\mathcal{P}_\Lambda f = \mathcal{W}\mathcal{P}_\Lambda f$. Hence, for any $\Lambda \subset \Lambda' \subset S_1$,

$$W_{\Lambda'}(y, x)\mathbf{1}_\Lambda(x) = W_\Lambda(y, x) \quad \text{on } S_2 \times S_1 \text{ for } \lambda_2 \otimes \lambda_1\text{-a.e.}(y, x).$$

From this consistency, we can define $W(y, x) \in L^2_{\mathrm{loc}}(S_2 \times S_1, \lambda_2 \otimes \lambda_1)$ so that for any compact set $\Lambda \subset S_1$,

$$W_\Lambda(y, x) = W(y, x)\mathbf{1}_\Lambda(x) \quad \text{on } S_2 \times S_1 \text{ for } \lambda_2 \otimes \lambda_1\text{-a.e.}(y, x). \tag{2.3}$$

Since $W_\Lambda \in L^2(S_2 \times S_1, \lambda_2 \otimes \lambda_1)$, by (2.3), $\|W(\cdot, x)\mathbf{1}_\Lambda(x)\|_{L^2(S_2, \lambda_2)} = \Psi_1(x)\mathbf{1}_\Lambda(x)$ is finite λ_1-a.e.x, and also $\Psi_1 \mathbf{1}_\Lambda \in L^2(S_1, \lambda_1)$. This means that $\Psi_1 \in L^2_{\mathrm{loc}}(S_1, \lambda_1)$. This completes the proof. ∎

From Proposition 2.3, under Assumption 2 (ii), the dual operator $\mathcal{W}^*$ also admits an integral kernel $W^* : S_1 \times S_2 \to \mathbb{C}$ such that $\Psi_2 \in L^2_{\mathrm{loc}}(S_2, \lambda_2)$, where $\Psi_2(y) := \|W^*(\cdot, y)\|_{L^2(S_1, \lambda_1)}, y \in S_2$. It is easy to see that $W^*(x, y) = \overline{W(y, x)}$ for $\lambda_1 \otimes \lambda_2$-a.e.$(x, y)$. Then

$$(\mathcal{W}^* g)(x) = \int_{S_2} \overline{W(y, x)} g(y) \lambda_2(dy), \quad g \in L^2(S_2, \lambda_2). \tag{2.4}$$

Following (2.2) and (2.4), we have

$$(\mathcal{W}^*\mathcal{W}f)(x) = \int_{S_1} K_{S_1}(x, x') f(x') \lambda_1(dx'), \quad f \in L^2(S_1, \lambda_1),$$
$$(\mathcal{W}\mathcal{W}^* g)(y) = \int_{S_2} K_{S_2}(y, y') g(y') \lambda_2(dy'), \quad g \in L^2(S_2, \lambda_2),$$

with the integral kernels,

$$K_{S_1}(x,x') = \int_{S_2} \overline{W(y,x)} W(y,x') \lambda_2(dy) = \langle W(\cdot,x'), W(\cdot,x) \rangle_{L^2(S_2,\lambda_2)},$$
$$K_{S_2}(y,y') = \int_{S_1} W(y,x) \overline{W(y',x)} \lambda_1(dx) = \langle W(y,\cdot), W(y',\cdot) \rangle_{L^2(S_1,\lambda_1)}. \tag{2.5}$$

We see that $\overline{K_{S_1}(x',x)} = K_{S_1}(x,x')$ and $\overline{K_{S_2}(y',y)} = K_{S_2}(y,y')$.

Under Assumptions 1 and 2 (i), we obtain Theorem 1.2 in Introduction as an immediate consequence of the well-known existence theorem of DPP (Theorem 2.1). We also state the following theorem to emphasize duality of DPPs, and it is a starting-point for our discussion in the present paper.

Theorem 2.4 *Under Assumptions* 1 *and* 2*, associated with* $\mathcal{W}^* \mathcal{W}$ *and* $\mathcal{W}\mathcal{W}^*$*, there exists a unique pair of DPPs;* $(\Xi_1, K_{S_1}, \lambda_1(dx))$ *on* S_1 *and* $(\Xi_2, K_{S_2}, \lambda_2(dy))$ *on* S_2*. The correlation kernels* $K_{S_\ell}, \ell = 1, 2$ *are Hermitian and given by* (2.5)*.*

Note that the densities of the DPPs, $(\Xi_1, K_{S_1}, \lambda_1(dx))$ and $(\Xi_2, K_{S_2}, \lambda_2(dy))$, are given by

$$\rho_1(x) = K_{S_1}(x,x) = \int_{S_2} |W(y,x)|^2 \lambda_2(dy) = \|W(\cdot,x)\|^2_{L^2(S_2,\lambda_2)}, \quad x \in S_1,$$
$$\rho_2(y) = K_{S_2}(y,y) = \int_{S_1} |W(y,x)|^2 \lambda_1(dx) = \|W(y,\cdot)\|^2_{L^2(S_1,\lambda_1)}, \quad y \in S_2,$$

with respect to the background measures $\lambda_1(dx)$ and $\lambda_2(dy)$, respectively.

We say that a pair of DPPs $(\Xi_1, K_{S_1}, \lambda_1(dx))$ on S_1 and $(\Xi_2, K_{S_2}, \lambda_2(dy))$ on S_2 is associated with $\mathcal{W}$. One of the advantages of our framework is that the obtained pairs of DPPs satisfy useful duality relations, which will be reported in Sections 2.4, 3.1.2, and 4.1.4. Now we concentrate on one of a pair of DPPs constructed in our framework, $(\Xi_1, K_{S_1}, \lambda_1)$. The correlation kernel K_{S_1} is given by the first equation of (2.5), which is an integral kernel for $f \in L^2(S_1, \lambda_1)$. We can regard this equation as a *decomposition formula* of K_{S_1} by a product of W and $\overline{W}$. Since W is an integral kernel for an isometry $L^2(S_1, \lambda_1) \to L^2(S_2, \lambda_2)$, as a matter of course, it depends on a choice of another Hilbert space $L^2(S_2, \lambda_2)$. We note that a given DPP, $(\Xi_1, K_{S_1}, \lambda_1)$, choice of $L^2(S_2, \lambda_2)$ is not unique. Such multivalency gives plural different expressions for one correlation kernel K_{S_1} and they reveal different aspects of the DPP as demonstrated in Sections 4.1.1–4.1.3.

2.3 Basic properties of DPPs

For $v = (v^{(1)}, \dots, v^{(d)}) \in \mathbb{R}^d$, $y = (y^{(1)}, \dots, y^{(d)}) \in \mathbb{R}^d$, $d \in \mathbb{N}$, the inner product of them is given by $v \cdot y = y \cdot v := \sum_{a=1}^d v^{(a)} y^{(a)}$, and $|v|^2 := v \cdot v$. When $S \subset \mathbb{C}^d, d \in \mathbb{N}$, $x \in S$ has d complex components; $x = (x^{(1)}, \dots, x^{(d)})$ with $x^{(a)} = \Re x^{(a)} + i \Im x^{(a)}$, $a = 1, \dots, d$. In order to describe clearly such a complex structure, we set $x_{\mathrm{R}} = (\Re x^{(1)}, \dots, \Re x^{(d)}) \in \mathbb{R}^d$, $x_{\mathrm{I}} = (\Im x^{(1)}, \dots, \Im x^{(d)}) \in \mathbb{R}^d$, and write $x = x_{\mathrm{R}} + i x_{\mathrm{I}}$ in this paper. The Lebesgue measure is written as $dx = dx_{\mathrm{R}} dx_{\mathrm{I}} := \prod_{a=1}^d d\Re x^{(a)} d\Im x^{(a)}$. The complex conjugate of $x = x_{\mathrm{R}} + i x_{\mathrm{I}}$ is defined as $\overline{x} = x_{\mathrm{R}} - i x_{\mathrm{I}}$. For $x = x_{\mathrm{R}} + i x_{\mathrm{I}}$, $x' = x'_{\mathrm{R}} + i x'_{\mathrm{I}} \in \mathbb{C}^d$, we use the *Hermitian inner product*;

$$x \cdot \overline{x'} := (x_{\mathrm{R}} + i x_{\mathrm{I}}) \cdot (x'_{\mathrm{R}} - i x'_{\mathrm{I}}) = (x_{\mathrm{R}} \cdot x'_{\mathrm{R}} + x_{\mathrm{I}} \cdot x'_{\mathrm{I}}) - i (x_{\mathrm{R}} \cdot x'_{\mathrm{I}} - x_{\mathrm{I}} \cdot x'_{\mathrm{R}})$$

and define

$$|x|^2 := x \cdot \overline{x} = |x_{\mathrm{R}}|^2 + |x_{\mathrm{I}}|^2, \quad x \in \mathbb{C}^d.$$

For $(\Xi, K, \lambda(dx))$ defined on $S = \mathbb{R}^d$, $S = \mathbb{C}^d$, or on a space having appropriate periodicities or symmetries, we write $\Xi = \sum_j \delta_{X_j}$ and introduce the following operations.

(Shift) For $u \in S$, $\mathcal{S}_u \Xi := \sum_j \delta_{X_j - u}$,

$$\mathcal{S}_u K(x, x') = K(x+u, x'+u),$$

and $\mathcal{S}_u \lambda(dx) = \lambda(u + dx)$. We write $(\mathcal{S}_u\Xi, \mathcal{S}_u K, \mathcal{S}_u \lambda(dx))$ simply as $\mathcal{S}_u(\Xi, K, \lambda(dx))$.

(Dilatation) For $c > 0$, we set $c \circ \Xi := \sum_j \delta_{cX_j}$

$$c \circ K(x, x') := K\left(\frac{x}{c}, \frac{x'}{c}\right), \quad x, x' \in cS := \{cx : x \in S\},$$

and $c \circ \lambda(dx) := \lambda(dx/c)$. We define $c \circ (\Xi, K, \lambda(dx)) := (c \circ \Xi, c \circ K, c \circ \lambda(dx))$.

Moreover, we also consider the following operations.

(Square root) For $(\Xi, K, \lambda(dx))$ on $S = [0, \infty)$, we put $\Xi^{\langle 1/2 \rangle} := \sum_j \delta_{\sqrt{X_j}}$, $K^{\langle 1/2 \rangle}(x, x') := K(x^2, x'^2)$, and $\lambda^{\langle 1/2 \rangle}(dx) := (\lambda \circ v^{-1})(dx)$, where $v(x) = \sqrt{x}$. We define $(\Xi, K, \lambda(dx))^{\langle 1/2 \rangle} := (\Xi^{\langle 1/2 \rangle}, K^{\langle 1/2 \rangle}, \lambda^{\langle 1/2 \rangle}(dx))$ on $[0, \infty)$.

(Gauge transformation) For non-vanishing $u : S \to \mathbb{C}$, a gauge transformation of K by u is defined as

$$K(x, x') \mapsto \widetilde{K}_u(x, x') := u(x) K(x, x') u(x')^{-1}.$$

In particular, when $u : S \to \mathrm{U}(1)$, the $\mathrm{U}(1)$-gauge transformation of K is given by

$$K(x, x') \mapsto \widetilde{K}_u(x, x') = u(x) K(x, x') \overline{u(x')}.$$

We will use the following basic properties of DPP.

[**Gauge invariance**] For any $u : S \to \mathbb{C}$, a gauge transformation does not change the probability law of DPP;

$$(\Xi, K, \lambda(dx)) \stackrel{\text{(law)}}{=} (\Xi, \widetilde{K}_u, \lambda(dx)).$$

[**Measure change**] For a measurable function $g : S \to [0, \infty)$,

$$(\Xi, K(x, x'), g(x)\lambda(dx)) \stackrel{\text{(law)}}{=} (\Xi, \sqrt{g(x)} K(x, x') \sqrt{g(x')}, \lambda(dx)). \tag{2.6}$$

[**Mapping and scaling**] For a one-to-one measurable mapping $h : S \to \widehat{S}$, if we set

$$\widehat{\Xi} := \sum_j \delta_{h(X_j)}, \quad \widehat{K}(x, x') := K(h^{-1}(x), h^{-1}(y)), \quad \widehat{\lambda}(dx) := (\lambda \circ h^{-1})(dx),$$

then $(\widehat{\Xi}, \widehat{K}, \widehat{\lambda}(dx))$ is a DPP on $\widehat{S}$. In particular, when $h(x) = x - u, u \in S$, $(\widehat{\Xi}, \widehat{K}, \widehat{\lambda}(dx)) = \mathcal{S}_u(\Xi, K, \lambda(dx))$, when $h(x) = cx, c > 0$, $(\widehat{\Xi}, \widehat{K}, \widehat{\lambda}(dx)) = c \circ (\Xi, K, \lambda(dx))$, and when $h(x) =$

$\sqrt{x}$ for $S = [0, \infty)$, $(\widehat{\Xi}, \widehat{K}, \widehat{\lambda}(dx)) = (\Xi, K, \lambda(dx))^{\langle 1/2 \rangle}$. If $c \circ \lambda(dx) = c^{-d}\lambda(dx)$, then (2.6) with $g(x) \equiv c > 0$ gives

$$c \circ (\Xi, K, \lambda(dx)) \stackrel{\text{(law)}}{=} (c \circ \Xi, K_c, \lambda(dx)), \quad c > 0,$$

with

$$K_c(x, x') := \frac{1}{c^d} K\left(\frac{x}{c}, \frac{x'}{c}\right),$$

where the base space is given by cS.

We will give some limit theorems for DPPs in this paper. Consider a DPP which depends on a continuous parameter, or a series of DPPs labeled by a discrete parameter (e.g., the number of points $N \in \mathbb{N}$), and describe the system by $(\Xi, K_p, \lambda_p(dx))$ with the continuous or discrete parameter p. If $(\Xi, K_p, \lambda_p(dx))$ converges to a DPP, $(\Xi, K, \lambda(dx))$, as $p \to \infty$, weakly in the vague topology, we write this limit theorem as $(\Xi, K_p, \lambda_p(dx)) \stackrel{p\to\infty}{\Longrightarrow} (\Xi, K, \lambda(dx))$. The weak convergence of DPPs is verified by the uniform convergence of the kernel $K_p \to K$ on each compact set $C \subset S \times S$ [54].

2.4 Duality relations

For $f \in \mathcal{C}_c(S)$, the Laplace transform of the probability measure $\mathbf{P}$ for a point process Ξ is defined as

$$\Psi[f] := \mathbf{E}\left[\exp\left(\int_S f(x)\Xi(dx)\right)\right]. \tag{2.7}$$

For the DPP, $(\Xi, K, \lambda(dx))$, this is given by the Fredholm determinant on $L^2(S, \lambda)$ [56],

$$\operatorname*{Det}_{L^2(S,\lambda)}[I - (1 - e^f)\mathcal{K}] := 1 + \sum_{n\in\mathbb{N}} \frac{(-1)^n}{n!} \int_{S^n} \det_{1\le j,k\le n}[K(x_j, x_k)] \prod_{\ell=1}^n (1 - e^{f(x_\ell)}) \lambda^{\otimes n}(d\boldsymbol{x}).$$

Lemma 2.5 *Between two DPPs,* $(\Xi_1, K_{S_1}, \lambda_1(dx))$ *on* S_1 *and* $(\Xi_2, K_{S_2}, \lambda_2(dy))$ *on* S_2*, given by Theorem* 2.4*, the following equality holds with an arbitrary parameter* $\alpha \in \mathbb{C}$*,*

$$\operatorname*{Det}_{L^2(S_1,\lambda_1)}[I + \alpha\mathcal{K}_{S_1}] = \operatorname*{Det}_{L^2(S_2,\lambda_2)}[I + \alpha\mathcal{K}_{S_2}]. \tag{2.8}$$

Proof We recall that if $\mathcal{AB}$ and $\mathcal{BA}$ are trace class operators on a Hilbert space H then [56]

$$\operatorname*{Det}_H[I + \mathcal{BA}] = \operatorname*{Det}_H[I + \mathcal{AB}]. \tag{2.9}$$

Now we have $\mathcal{A} : H_1 \to H_2$ and $\mathcal{B} : H_2 \to H_1$ between two Hilbert spaces H_1 and H_2. Let $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{B}}$ be two operators on $H_1 \oplus H_2$ defined by

$$\widetilde{\mathcal{A}} := \begin{pmatrix} O & O \\ \mathcal{A} & O \end{pmatrix}, \quad \widetilde{\mathcal{B}} := \begin{pmatrix} O & \mathcal{B} \\ O & O \end{pmatrix}$$

Then, $\widetilde{\mathcal{A}}\widetilde{\mathcal{B}}$ and $\widetilde{\mathcal{B}}\widetilde{\mathcal{A}}$ are diagonal operators $O \oplus \mathcal{AB}$ and $\mathcal{BA} \oplus O$, respectively, and hence also they are trace class operators. By applying (2.9) to $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{B}}$ with $H := H_1 \oplus H_2$, we obtain

$$\operatorname*{Det}_{H_1}[I + \mathcal{BA}] = \operatorname*{Det}_{H_2}[I + \mathcal{AB}].$$

Consequently, taking $\mathcal{A} = \sqrt{\alpha}\mathcal{W}$, $\mathcal{B} = \sqrt{\alpha}\mathcal{W}^*$, $H_1 = L^2(S_1, \lambda_1)$, and $H_2 = L^2(S_2, \lambda_2)$ yields (2.8). ∎

For $\Lambda_\ell \subset S_\ell, \ell = 1, 2$, let

$$\widetilde{\mathcal{W}} := \mathcal{P}_{\Lambda_2}\mathcal{W}\mathcal{P}_{\Lambda_1}, \quad \mathcal{K}_{S_1}^{(\Lambda_2)} := \mathcal{W}^*\mathcal{P}_{\Lambda_2}\mathcal{W}, \quad \mathcal{K}_{S_2}^{(\Lambda_1)} := \mathcal{W}\mathcal{P}_{\Lambda_1}\mathcal{W}^*. \tag{2.10}$$

They admit the following integral kernels,

$$\begin{aligned} \widetilde{W}(y, x) &= \mathbf{1}_{\Lambda_2}(y) W(y, x) \mathbf{1}_{\Lambda_1}(x), \\ K_{S_1}^{(\Lambda_2)}(x, x') &= \int_{\Lambda_2} \overline{W(y, x)} W(y, x') \lambda_2(dy), \\ K_{S_2}^{(\Lambda_1)}(y, y') &= \int_{\Lambda_1} W(y, x) \overline{W(y', x)} \lambda_1(dx). \end{aligned} \tag{2.11}$$

Using Lemma 2.5, the following theorem is proved.

Theorem 2.6 *Let* $(\Xi_1^{(\Lambda_2)}, K_{S_1}^{(\Lambda_2)}, \lambda_1(dx))$ *and* $(\Xi_2^{(\Lambda_1)}, K_{S_2}^{(\Lambda_1)}, \lambda_2(dy))$ *be DPPs associated with the kernels* $K_{S_1}^{(\Lambda_2)}$ *and* $K_{S_2}^{(\Lambda_1)}$ *given by* (2.11)*, respectively. Then,* $\Xi_1^{(\Lambda_2)}(\Lambda_1) \stackrel{\text{(law)}}{=} \Xi_2^{(\Lambda_1)}(\Lambda_2)$*, i.e.,*

$$\mathbf{P}(\Xi_1^{(\Lambda_2)}(\Lambda_1) = m) = \mathbf{P}(\Xi_2^{(\Lambda_1)}(\Lambda_2) = m), \quad \forall m \in \mathbb{N}_0.$$

Proof As a special case of (2.7) with $f(x) = \mathbf{1}_{\Lambda_1}(x) \log z$ for $\Xi = \Xi_1^{(\Lambda_2)}$, $z \in \mathbb{C}$, we have the equality,

$$\mathbf{E}\left[z^{\Xi_1^{(\Lambda_2)}(\Lambda_1)}\right] = \underset{L^2(S_1,\lambda_1)}{\mathrm{Det}}[I - (1 - z)\mathcal{P}_{\Lambda_1}\mathcal{K}_{S_1}^{(\Lambda_2)}\mathcal{P}_{\Lambda_1}], \tag{2.12}$$

where $\mathcal{K}_{S_1}^{(\Lambda_2)}$ is defined by (2.10). Here LHS is the moment generating function of $\Xi_1^{(\Lambda_2)}(\Lambda_1)$ and RHS gives its Fredholm determinantal expression. By replacing $\mathcal{W}$ by $\widetilde{\mathcal{W}}$ and letting $\alpha = -(1 - z)$ in the proof of Lemma 2.5, we obtain the equality,

$$\underset{L^2(S_1,\lambda_1)}{\mathrm{Det}}[I - (1 - z)\mathcal{P}_{\Lambda_1}\mathcal{K}_{S_1}^{(\Lambda_2)}\mathcal{P}_{\Lambda_1}] = \underset{L^2(S_2,\lambda_2)}{\mathrm{Det}}[I - (1 - z)\mathcal{P}_{\Lambda_2}\mathcal{K}_{S_2}^{(\Lambda_1)}\mathcal{P}_{\Lambda_2}].$$

Through (2.12) and the similar equality for $\mathbf{E}\left[z^{\Xi_2^{(\Lambda_1)}(\Lambda_2)}\right]$, we obtain the corresponding equivalence between the moment generating functions of $\Xi_1^{(\Lambda_2)}(\Lambda_1)$ and $\Xi_2^{(\Lambda_1)}(\Lambda_2)$, and hence the statement of the proposition is proved. ∎

Examples of duality relations will be given in Sections 3.1.2 and 4.1.4. Theorem 2.6 was used to analyze hyperuniformity [62] of the Heisenberg family of DPPs in [42].

2.5 Orthonormal functions and correlation kernels

In addition to $L^2(S_\ell, \lambda_\ell)$, $\ell = 1, 2$, we introduce $L^2(\Gamma, \nu)$ as a parameter space for functions in $L^2(S_\ell, \lambda_\ell), \ell = 1, 2$. Assume that there are two families of measurable functions $\{\psi_1(x, \gamma) : x \in S_1, \gamma \in \Gamma\}$ and $\{\psi_2(y, \gamma) : y \in S_2, \gamma \in \Gamma\}$ such that two bounded operators $\mathcal{U}_\ell : L^2(S_\ell, \lambda_\ell) \to L^2(\Gamma, \nu)$ given by

$$(\mathcal{U}_\ell f)(\gamma) := \int_{S_\ell} \overline{\psi_\ell(x, \gamma)} f(x) \lambda_\ell(dx), \quad \ell = 1, 2,$$

are well-defined. Then, their adjoints $\mathcal{U}_\ell^* : L^2(\Gamma,\nu) \to L^2(S_\ell,\lambda_\ell), \ell = 1,2$ are given by

$$(\mathcal{U}_\ell^* F)(\cdot) = \int_\Gamma \psi_\ell(\cdot,\gamma)F(\gamma)\nu(d\gamma).$$

A typical example of $\mathcal{U}_1$ is the Fourier transform, i.e., $\psi_1(x,\gamma) = e^{ix\gamma}$. In this case, for any γ, the function $\psi_1(\cdot,\gamma)$ is *not* in $L^2(\mathbb{R},dx)$. Now we define $\mathcal{W} : L^2(S_1,\lambda_1) \to L^2(S_2,\lambda_2)$ by $\mathcal{W} = \mathcal{U}_2^*\mathcal{U}_1$, i.e.,

$$(\mathcal{W}f)(y) = \int_\Gamma \psi_2(y,\gamma)(\mathcal{U}_1 f)(\gamma)\nu(d\gamma). \tag{2.13}$$

Let I_Γ be an identity in $L^2(\Gamma,\nu)$. We can see the following.

Lemma 2.7 *If $\mathcal{U}_\ell\mathcal{U}_\ell^* = I_\Gamma$ for $\ell = 1,2$, then both $\mathcal{W}$ and $\mathcal{W}^*$ are partial isometries.*

Proof By the assumption, we see that

$$\mathcal{W}\mathcal{W}^*\mathcal{W} = (\mathcal{U}_2^*\mathcal{U}_1)(\mathcal{U}_1^*\mathcal{U}_2)(\mathcal{U}_2^*\mathcal{U}_1) = \mathcal{U}_2^*\mathcal{U}_1 = \mathcal{W}.$$

From Lemma 2.2, $\mathcal{W}$ is a partial isometry. By symmetry, the assertion for $\mathcal{W}^*$ also follows. ∎

We note that $\mathcal{W}^*\mathcal{W} = \mathcal{U}_1^*\mathcal{U}_1$ and $\mathcal{W}\mathcal{W}^* = \mathcal{U}_2^*\mathcal{U}_2$. Hence, $\mathcal{U}_\ell, \ell = 1,2$ are also partial isometries. In addition, $\mathcal{W}^*\mathcal{W}$ is a locally trace class operator if and only if so is $\mathcal{U}_1^*\mathcal{U}_1$. Therefore, $\mathcal{W}$ is of locally Hilbert–Schmidt class if and only if so is $\mathcal{U}_1$.

Now we rewrite the condition for $\mathcal{U}_1$ to be of locally Hilbert–Schmidt class in terms of the function $\psi_1(x,\gamma), x \in S_1, \gamma \in \Gamma$.

Lemma 2.8 *Let $\Psi_1(x) := \|\psi_1(x,\cdot)\|_{L^2(\Gamma,\nu)}$, $x \in S_1$ and assume that $\Psi_1 \in L^2_{\mathrm{loc}}(S_1,\lambda_1)$. Then, the operator $\mathcal{U}_1$ is of locally Hilbert–Schmidt class.*

Proof For a compact set $\Lambda \subset S_1$, we see that

$$\begin{aligned}
|\mathcal{P}_\Lambda\mathcal{U}_1^*\mathcal{U}_1\mathcal{P}_\Lambda f(x)| &= \left|\mathbf{1}_\Lambda(x)\int_\Gamma \nu(d\gamma)\psi_1(x,\gamma)\int_{S_1}\overline{\psi_1(x',\gamma)}\mathbf{1}_\Lambda(x')f(x')\lambda_1(dx')\right| \\
&\le \mathbf{1}_\Lambda(x)\Psi_1(x)\int_{S_1}\mathbf{1}_\Lambda(x')\Psi_1(x')|f(x')|\lambda_1(dx') \\
&\le \mathcal{P}_\Lambda\Psi_1(x)\|\mathcal{P}_\Lambda\Psi_1\|_{L^2(S_1,\lambda_1)}\|\mathcal{P}_\Lambda f\|_{L^2(S_1,\lambda_1)}.
\end{aligned}$$

By Fubini's theorem, we have

$$\mathcal{P}_\Lambda\mathcal{U}_1^*\mathcal{U}_1\mathcal{P}_\Lambda f(x) = \int_{S_1}\lambda_1(dx')f(x')\left(\int_\Gamma \mathbf{1}_\Lambda(x)\psi_1(x,\gamma)\overline{\mathbf{1}_\Lambda(x')\psi_1(x',\gamma)}\nu(d\gamma)\right)$$

and hence

$$\|\mathcal{U}_1\mathcal{P}_\Lambda\|_{\mathrm{HS}}^2 = \mathrm{Tr}\,(\mathcal{P}_\Lambda\mathcal{U}_1^*\mathcal{U}_1\mathcal{P}_\Lambda) = \int_{S_1}\lambda_1(dx)\mathbf{1}_\Lambda(x)\left(\int_\Gamma |\psi_1(x,\gamma)|^2\nu(d\gamma)\right) = \|\mathcal{P}_\Lambda\Psi_1\|_{L^2(S_1,\lambda_1)}^2 < \infty.$$

This completes the proof. ∎

Now we put the following.

Assumption 3 For $\ell = 1,2$,

(i) $\mathcal{U}_\ell \mathcal{U}_\ell^* = I_\Gamma$,

(ii) $\Psi_\ell \in L^2_{\rm loc}(S_\ell, \lambda_\ell)$, where $\Psi_\ell(x) := \|\psi_\ell(x, \cdot)\|_{L^2(\Gamma,\nu)}$, $x \in S_\ell$.

Assumption 3(i) can be rephrased as the following orthonormality relations:

$$\langle \psi_\ell(\cdot, \gamma), \psi_\ell(\cdot, \gamma')\rangle_{L^2(S_\ell,\lambda_\ell)} \nu(d\gamma) = \delta(\gamma - \gamma') d\gamma, \quad \gamma, \gamma' \in \Gamma, \quad \ell = 1, 2.$$

We often use these relations below.

The following is immediately obtained as a corollary of Theorem 2.4.

Corollary 2.9 *Let $\mathcal{W} = \mathcal{U}_2^* \mathcal{U}_1$ as in the above. We assume Assumption 3. Then, there exists a unique pair of DPPs; $(\Xi_1, K_{S_1}, \lambda_1(dx))$ on S_1 and $(\Xi_2, K_{S_2}, \lambda_2(dy))$ on S_2. Here the correlation kernels $K_{S_\ell}, \ell = 1, 2$ are given by*

$$\begin{aligned} K_{S_1}(x, x') &= \int_\Gamma \psi_1(x, \gamma)\overline{\psi_1(x', \gamma)}\nu(d\gamma) = \langle \psi_1(x, \cdot), \psi_1(x', \cdot)\rangle_{L^2(\Gamma,\nu)}, \\ K_{S_2}(y, y') &= \int_\Gamma \psi_2(y, \gamma)\overline{\psi_2(y', \gamma)}\nu(d\gamma) = \langle \psi_2(y, \cdot), \psi_2(y', \cdot)\rangle_{L^2(\Gamma,\nu)}. \end{aligned} \tag{2.14}$$

In particular, the densities of the DPPs are given by $\rho_1(x) = K_{S_1}(x, x) = \Psi_1(x)^2, x \in S_1$ and $\rho_2(y) = K_{S_2}(y, y) = \Psi_2(y)^2, y \in S_2$ with respect to the background measures $\lambda_1(dx)$ and $\lambda_2(dy)$, respectively.

Remark 1 Consider the symmetric case such that $L^2(S_1, \lambda_1) = L^2(S_2, \lambda_2) =: L^2(S, \lambda)$, $\psi_1 = \psi_2 =: \psi$, $\nu = \lambda|_\Gamma$, $\Gamma \subseteq S$. In this case, $\mathcal{W} = \mathcal{U}^*\mathcal{U}$ with

$$(\mathcal{U}f)(\gamma) = \int_S \overline{\psi(x, \gamma)} f(x)\lambda(dx).$$

Then $K_{S_1} = K_{S_2} = W =: K$ is given by

$$K(x, x') = \int_\Gamma \psi(x, \gamma)\overline{\psi(x', \gamma)}\lambda(d\gamma). \tag{2.15}$$

This is Hermitian; $\overline{K(x', x)} = K(x, x')$, and satisfies the reproducing property

$$K(x, x') = \int_S K(x, \zeta)K(\zeta, x')\lambda(d\zeta).$$

Now we consider a simplified version of the preceding setting. Let $\Gamma \subseteq S_2$ and $\nu = \lambda_2|_\Gamma$. We define $\mathcal{U}_2 : L^2(S_2, \lambda_2) \to L^2(\Gamma, \nu)$ as the restriction onto Γ, and then its adjoint $\mathcal{U}_2^*$ is given by $(\mathcal{U}_2^* F)(y) = F(y)$ for $y \in \Gamma$, and by 0 for $y \in S_2 \setminus \Gamma$. We write the extension $\widetilde{F} = \mathcal{U}_2^* F$ for $F \in L^2(\Gamma, \nu)$. It is obvious that $\mathcal{U}_2\mathcal{U}_2^* = I_\Gamma$ and hence $\mathcal{U}_2$ is a partial isometry.

For $\Gamma \subseteq S_2$, we assume that there is a family of measurable functions $\{\psi_1(x, y) : x \in S_1, y \in \Gamma\}$ such that a bounded operator $\mathcal{U}_1 : L^2(S_1, \lambda_1) \to L^2(\Gamma, \nu)$ given by

$$(\mathcal{U}_1 f)(\gamma) := \int_{S_1} \overline{\psi_1(x, \gamma)} f(x)\lambda_1(dx) \quad (\gamma \in \Gamma)$$

is well-defined.

Assumption 3'

(i) $\mathcal{U}_1\mathcal{U}_1^* = I_\Gamma$,

(ii) $\Psi_1 \in L^2_{\mathrm{loc}}(S_1, \lambda_1)$, where $\Psi_1(x) := \|\psi_1(x, \cdot)\|_{L^2(\Gamma,\nu)}$, $x \in S_1$.

Assumption 3'(i) can be rephrased as the following orthonormality relation:

$$\langle \psi_1(\cdot, y), \psi_1(\cdot, y')\rangle_{L^2(S_1,\lambda_1)}\lambda_2(dy) = \delta(y - y')dy, \quad y, y' \in \Gamma.$$

Now we define $\mathcal{W} : L^2(S_1, \lambda_1) \to L^2(S_2, \lambda_2)$ by $\mathcal{W} = \mathcal{U}_2^*\mathcal{U}_1$ as before. In this case, we have

$$(\mathcal{W}f)(y) = \mathbf{1}_\Gamma(y) \int_{S_1} \overline{\tilde{\psi}_1(x, y)} f(x)\lambda_1(dx),$$

and hence

$$W(y, x) = \overline{\tilde{\psi}_1(x, y)}\mathbf{1}_\Gamma(y). \tag{2.16}$$

It follows from Assumption 3' that $\mathcal{W}$ is a partial isometry. Corollary 2.9 is reduced to the following.

Corollary 2.10 *Let $\mathcal{W} = \mathcal{U}_2^*\mathcal{U}_1$ as in the above. We assume Assumption 3'. Then there exists a unique DPP, (Ξ, K, λ_1) on S_1 with the correlation kernel*

$$K_{S_1}(x, x') = \int_\Gamma \psi_1(x, y)\overline{\psi_1(x', y)}\lambda_2(dy) = \langle \tilde{\psi}_1(x, \cdot), \tilde{\psi}_1(x', \cdot)\rangle_{L^2(\Gamma,\lambda_2)}. \tag{2.17}$$

In particular, the density of the DPP is given by $\rho_1(x) = K_{S_1}(x, x) = \Psi_1(x)^2, x \in S_1$ with respect to the background measures $\lambda_1(dx)$.

Proof The proof is the same as before. ∎

Remark 2 The correlation kernel (2.17) is the same as the correlation kernel (2.15) shown in Remark 1 in the symmetric case, $L^2(S_1, \lambda_1) = L^2(S_2, \lambda_2)$ and $\psi_1 = \psi_2$, of the pair of DPPs given by Corollary 2.9.

2.6 Weyl–Heisenberg ensembles of DPPs

The family of DPPs given by Corollary 2.10 is a generalization of the class of DPPs called the *Weyl–Heisenberg ensembles* studied by Abreu *et al.* [1, 3, 2]. For $d \in \mathbb{N}$, let

$$S_1 = \mathbb{C}^d, \quad S_2 = \Gamma = \mathbb{R}^d,$$

with the Lebesgue measures $\lambda_1(dx) = dx_\mathrm{R}dx_\mathrm{I}$, $\lambda_2(dy) = dy$, where $x = x_\mathrm{R} + ix_\mathrm{I}$ with $x_\mathrm{R}, x_\mathrm{I} \in \mathbb{R}^d$. We consider the case that ψ_1 in the setting (2.16) of W is given of the form

$$\psi_1(x, y) = \psi_1(x_\mathrm{R} + ix_\mathrm{I}, y) = G(y - x_\mathrm{R})e^{2\pi i y\cdot x_\mathrm{I}} \quad \text{with } G \in L^2(\mathbb{R}^d, dx_\mathrm{R}), \tag{2.18}$$

where $y \cdot x_\mathrm{I}$ denotes the inner product in $\mathbb{R}^d$. In this setting, since $\Psi_1(x) \equiv \|G\|_{L^2(\mathbb{R}^d,dx_\mathrm{R})}$, $x \in \mathbb{C}^d$, Assumption 3'(ii) is satisfied. Also, we have

$$\langle \psi_1(\cdot, y), \psi_1(\cdot, y')\rangle_{L^2(S_1,\lambda_1)} = \int_{\mathbb{R}^d} dx_\mathrm{R}\, G(y - x_\mathrm{R})\overline{G(y' - x_\mathrm{R})} \int_{\mathbb{R}^d} dx_\mathrm{I}\, e^{2\pi i(y-y')\cdot x_\mathrm{I}}$$

Since $\int_{\mathbb{R}^d} e^{2\pi i y\cdot x} dx = \delta(y), y \in \mathbb{R}^d$, the above is equal to $\|G\|^2_{L^2(\mathbb{R}^d, dx_{\rm R})} \delta(y-y')$. Hence, the norm $\|G\|_{L^2(\mathbb{R}^d, dx_{\rm R})}$ must be 1 for Assumption 3'(i). Therefore, in this setting (2.18), Assumption 3' will be reduced to the following.

Assumption 4 The function G in (2.18) has norm 1 in $L^2(\mathbb{R}^d, dx_{\rm R})$.

Under the setting (2.16) with $\Gamma = \mathbb{R}^d$ and (2.18), if Assumption 4 is satisfied, then the operator $\mathcal{W}$ and the correlation kernel K_{S_1} are written as

$$
\begin{aligned}
(\mathcal{W}_{\rm WH} f)(y) &= \int_{\mathbb{C}^d} \overline{G(y-x_{\rm R})} e^{-2\pi i y\cdot x_{\rm I}} f(x_{\rm R} + i x_{\rm I}) dx_{\rm R} dx_{\rm I}, \quad f \in L^2(\mathbb{C}^d, dx_{\rm R} dx_{\rm I}), \\
(\mathcal{W}^*_{\rm WH} g)(x) &= \int_{\mathbb{R}^d} G(y-x_{\rm R}) e^{2\pi i y\cdot x_{\rm I}} g(y) dy, \quad g \in L^2(\mathbb{R}^d, dy), \\
K_{\rm WH}(x, x') &= \int_{\mathbb{R}^d} G(y-x_{\rm R}) \overline{G(y-x'_{\rm R})} e^{2\pi i y\cdot(x_{\rm I} - x'_{\rm I})} dy,
\end{aligned}
\tag{2.19}
$$

for $(x, x') = (x_{\rm R} + i x_{\rm I}, x'_{\rm R} + i x'_{\rm I}) \in \mathbb{C}^d \times \mathbb{C}^d$. The second formula in (2.19) is regarded as the *short-time Fourier transform* of $g \in L^2(\mathbb{R}^d, dy)$ with respect to a *window function* $G \in L^2(\mathbb{R}^d, dx_{\rm R})$ [24]. The formulas (2.19) define the Weyl–Heisenberg ensemble of DPP, $(\Xi, K_{\rm WH}, dx_{\rm R} dx_{\rm I})$, studied in [1, 3, 2].

Proposition 2.11 *Under Assumption* 4*, the Weyl–Heisenberg class of DPPs specified by the window function* $G \in L^2(\mathbb{R}^d, dx_{\rm R})$ *is a special case of the family of DPPs given by Corollary* 2.10*, in which* $\Gamma = \mathbb{R}^d$*,* $S_1 = \mathbb{C}^d$*,* $\lambda_1(dx) = dx_{\rm R} dx_{\rm I}$*,* $\lambda_2(dy) = dy$*, and* ψ_1 *in* (2.16) *is given of the form* (2.18)*.*

3 Examples in One-dimensional Spaces

3.1 Finite DPPs in $\mathbb{R}$ associated with classical orthonormal polynomials

3.1.1 Classical orthonormal polynomials and DPPs

Let $S_1 = S_2 = \mathbb{R}$. Assume that we have two sets of orthonormal functions $\{\varphi_n\}_{n\in\mathbb{N}_0}$ and $\{\phi_n\}_{n\in\mathbb{N}_0}$ with respect to the measures λ_1 and λ_2, respectively,

$$
\begin{aligned}
\langle \varphi_n, \varphi_m \rangle_{L^2(\mathbb{R}, \lambda_1)} &= \int_{\mathbb{R}} \varphi_n(x) \overline{\varphi_m(x)} \lambda_1(dx) = \delta_{nm}, \\
\langle \phi_n, \phi_m \rangle_{L^2(\mathbb{R}, \lambda_2)} &= \int_{\mathbb{R}} \phi_n(y) \overline{\phi_m(y)} \lambda_2(dy) = \delta_{nm}, \quad n, m \in \mathbb{N}_0.
\end{aligned}
\tag{3.1}
$$

Then for an arbitrary but fixed $N \in \mathbb{N}$, we set $\Gamma = \{0, 1, \dots, N-1\} \subsetneq \mathbb{N}_0$, $\psi_1(\cdot, \gamma) = \varphi_\gamma(\cdot)$, $\psi_2(\cdot, \gamma) = \phi_\gamma(\cdot)$, $\gamma \in \Gamma$, and consider $\ell^2(\Gamma)$ as $L^2(\Gamma, \nu)$ in the setting of Section 2.5. We see that $\int_{\mathbb{R}} \|\varphi_\cdot(x)\|^2_{\ell^2(\Gamma)} \lambda_1(dx) = \sum_{n=0}^{N-1} \|\varphi_n\|^2_{L^2(\mathbb{R}, \lambda_1)} = N$ and $\int_{\mathbb{R}} \|\phi_\cdot(y)\|^2_{\ell^2(\Gamma)} \lambda_2(dy) = \sum_{n=0}^{N-1} \|\phi_n\|^2_{L^2(\mathbb{R}, \lambda_2)} = N$. Hence Assumption 3 is satisfied for any $N \in \mathbb{N}$. Then the integral kernel for $\mathcal{W}$ defined by (2.13) is given by

$$
W(y, x) = \sum_{n=0}^{N-1} \overline{\varphi_n(x)} \phi_n(y).
$$

By Corollary 2.9, we have a pair of DPPs on $\mathbb{R}$, $(\Xi_1, K_\varphi^{(N)}, \lambda_1(dx))$ and $(\Xi_2, K_\phi^{(N)}, \lambda_2(dy))$, where the correlation kernels are given by

$$K_\varphi^{(N)}(x,x') = \sum_{n=0}^{N-1} \varphi_n(x)\overline{\varphi_n(x')}, \quad K_\phi^{(N)}(y,y') = \sum_{n=0}^{N-1} \phi_n(y)\overline{\phi_n(y')}, \tag{3.2}$$

respectively. Here N gives the number of points for each DPPs. If we can use the three-term relations in $\{\varphi_n\}_{n\in\mathbb{N}_0}$ or $\{\phi_n\}_{n\in\mathbb{N}_0}$, (3.2) can be written in the Christoffel–Darboux form (see, for instance, Proposition 5.1.3 in [21]). As a matter of course, if we have three or more than three, say M distinct sets of orthonormal functions satisfying Assumption 3 with a common Γ, then by applying Corollary 2.9 to every pair of them, we will obtain M distinct finite DPPs. See examples given in Sections 3.2, 3.3, 3.5, 4.3, 4.4.

Even if we have only one set of orthonormal functions, for example, only the first one $\{\varphi_n\}_{n\in\mathbb{N}_0}$ in (3.1), we can obtain a DPP (labeled by the number of particles $N \in \mathbb{N}$) following Corollary 2.10. In such a case, we set

$$W(n,x) = \overline{\varphi_n(x)}\mathbf{1}_\Gamma(n) \tag{3.3}$$

with $\Gamma = \{0,1,\dots,N-1\}$ for (2.16). Then we have a DPP, $(\Xi, K_\varphi^{(N)}, \lambda_1(dx))$. See examples given in Sections 4.2, 5.1, and 5.2.

Remark 3 If Γ is a finite set, $|\Gamma| < \infty$, and the parameter space is given by $\ell^2(\Gamma)$, Assumption 3(ii) (resp. Assumption 3'(ii)) is concluded from 3(i) (resp. 3'(i)) as shown below. Since $\Psi(x)^2 := \|\varphi_\cdot(x)\|_{\ell^2(\Gamma)}^2 = \sum_{n\in\Gamma} |\varphi_n(x)|^2$, $x \in S$, we have $\int_S \Psi(x)^2\lambda(dx) = \sum_{n\in\Gamma} \|\varphi_n\|_{L^2(S,\lambda)}^2$. Then, if $\{\varphi_n\}_{n\in\Lambda}$ are normalized, the above integral is equal to $|\Gamma| < \infty$. This implies $\Psi \in L^2(S,\lambda) \subset L^2_{\mathrm{loc}}(S,\lambda)$. See finite DPPs given in Sections 3.2, 3.3, 4.2, 4.3, 4.4, 5.2.

Now we give classical examples of DPPs associated with real-valued orthonormal polynomials. Let $\lambda_{\mathrm{N}(m,\sigma^2)}(dx)$ denote the *normal distribution*,

$$\lambda_{\mathrm{N}(m,\sigma^2)}(dx) := \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-m)^2/(2\sigma^2)} dx, \quad m \in \mathbb{R}, \quad \sigma > 0,$$

and $\lambda_{\Gamma(a,b)}(dy)$ do the *Gamma distribution*,

$$\lambda_{\Gamma(a,b)}(dy) := \frac{b^a}{\Gamma(a)} y^{a-1} e^{-by} \mathbf{1}_{[0,\infty)}(y) dy, \quad a > 0, \quad b > 0,$$

with the Gamma function $\Gamma(z) := \int_0^\infty u^{z-1}e^{-u}du, \Re z > 0$. We set

$$\lambda_1(dx) = \lambda_{\mathrm{N}(0,1/2)}(dx) = \frac{1}{\sqrt{\pi}} e^{-x^2} dx,$$
$$\varphi_n(x) = \frac{1}{\sqrt{2^n n!}} H_n(x), \quad n \in \mathbb{N}_0, \tag{3.4}$$

and

$$\lambda_2(dy) = \lambda_{\Gamma(\nu+1,1)}(dy) = \frac{1}{\Gamma(\nu+1)} y^\nu e^{-y} \mathbf{1}_{[0,\infty)}(y) dy,$$
$$\phi_n(y) = \phi_n^{(\nu)}(y) = \sqrt{\frac{\Gamma(n+1)\Gamma(\nu+1)}{\Gamma(n+\nu+1)}}\, L_n^{(\nu)}(y), \quad n \in \mathbb{N}_0, \tag{3.5}$$

with parameter $\nu \in (-1, \infty)$. Here $\{H_n(x)\}_{n\in\mathbb{N}_0}$ are the *Hermite polynomials*,

$$\begin{aligned} H_n(x) &:= (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \\ &= n! \sum_{k=0}^{[n/2]} \frac{(-1)^k (2x)^{n-2k}}{k!(n-2k)!}, \quad n \in \mathbb{N}_0, \end{aligned} \tag{3.6}$$

where $[a]$ denotes the largest integer not greater than $a \in \mathbb{R}$, and $\{L_n^{(\nu)}(x)\}_{n\in\mathbb{N}_0}$ are the *Laguerre polynomials*,

$$\begin{aligned} L_n^{(\nu)}(x) &:= \frac{1}{n!} x^{-\nu} e^x \frac{d^n}{dx^n} \Big(x^{n+\nu} e^{-x} \Big) \\ &= \sum_{k=0}^{n} \frac{(\nu+k+1)_{n-k}}{(n-k)!k!} (-x)^k, \quad n \in \mathbb{N}_0, \quad \nu \in (-1, \infty), \end{aligned} \tag{3.7}$$

where $(\alpha)_n := \alpha(\alpha+1)\cdots(\alpha+n-1) = \Gamma(\alpha+n)/\Gamma(\alpha)$, $n \in \mathbb{N}$, $(\alpha)_0 := 1$. The correlation kernels (3.2) are written in the *Christoffel–Darboux form* as,

$$K_{\varphi}^{(N)}(x, x') = K_{\text{Hermite}}^{(N)}(x, x') = \sqrt{\frac{N}{2}} \frac{\varphi_N(x)\varphi_{N-1}(x') - \varphi_N(x')\varphi_{N-1}(x)}{x - x'}, \quad x, x' \in \mathbb{R},$$

and

$$\begin{aligned} K_{\phi}^{(N)}(y, y') &= K_{\text{Laguerre}}^{(\nu,N)}(y, y') \\ &= -\sqrt{N(N+\nu)} \frac{\phi_N^{(\nu)}(y)\phi_{N-1}^{(\nu)}(y') - \phi_N^{(\nu)}(y')\phi_{N-1}^{(\nu)}(y)}{y - y'}, \quad y, y' \in [0, \infty). \end{aligned}$$

When $x = x'$ or $y = y'$, we make sense of the above formulas by using L'Hôpital's rule. The former is called the *Hermite kernel* and the latter is the *Laguerre kernel*.

By definition, for a finite DPP $(\Xi, K, \lambda(dx))$ with N points in S, the probability density with respect to $\lambda^{\otimes N}(dx_1 \cdots dx_N)$ is given by $\rho^N(x_1, \dots, x_N) = \det_{1\le j,k\le N}[K(x_j, x_k)], \boldsymbol{x} = (x_1, \dots, x_N) \in S^N$. Using the *Vandermonde determinantal formula*, $\det_{1\le j,k\le N}(z_k^{j-1}) = \prod_{1\le j<k\le N}(z_k - z_j)$, which is also given as the type A_{N-1} of Weyl denominator formula (A.1) in Appendix A, we can verify that the probability densities of the DPPs $(\Xi, K_{\text{Hermite}}^{(N)}, \lambda_{\text{N}(0,1/2)}(dx))$ and $(\Xi, K_{\text{Laguerre}}^{(N)}, \lambda_{\Gamma(\nu+1,1)}(dy))$ with respect to the Lebesgue measures $d\boldsymbol{x} = \prod_{j=1}^N dx_j$ and $d\boldsymbol{y} = \prod_{j=1}^N dy_j$ are given as

$$\begin{aligned} \mathbf{p}_{\text{Hermite}}^{(N)}(\boldsymbol{x}) &= \frac{1}{Z_{\text{Hermite}}^{(N)}} \prod_{1\le j<k\le N} (x_k - x_j)^2 \prod_{\ell=1}^N e^{-x_\ell^2}, \quad \boldsymbol{x} \in \mathbb{R}^N, \\ \mathbf{p}_{\text{Laguerre}}^{(\nu,N)}(\boldsymbol{y}) &= \frac{1}{Z_{\text{Laguerre}}^{(\nu,N)}} \prod_{1\le j<k\le N} (y_k - y_j)^2 \prod_{\ell=1}^N y_\ell^{\nu} e^{-y_\ell}, \quad \nu > -1, \quad \boldsymbol{y} \in [0, \infty)^N, \end{aligned} \tag{3.8}$$

with the normalization constants $Z_{\text{Hermite}}^{(N)}$ and $Z_{\text{Laguerre}}^{(\nu,N)}$.

The DPP $(\Xi, K_{\text{Hermite}}^{(N)}, \lambda_{\text{N}(0,1/2)}(dy))$ describes the eigenvalue distribution of $N \times N$ Hermitian random matrices in the *Gaussian unitary ensemble* (GUE). When $\nu \in \mathbb{N}_0$, the DPP $(\Xi, K_{\text{Laguerre}}^{(N)},$

$\lambda_{\Gamma(\nu+1,1)}(dx)$) describes the distribution of the nonnegative square roots of eigenvalues of $M^\dagger M$, where M is $(N+\nu)\times N$ complex random matrix in the *chiral Gaussian ensemble* (chGUE) and $M^\dagger$ is its Hermitian conjugate. The probability density (3.8) can be extended to any $\nu \in (-1,\infty)$ and it is called the *complex Laguerre ensemble* or the *complex Wishart ensemble*. Many other examples of one-dimensional DPPs are given as eigenvalue ensembles of Hermitian random matrices in the literature of random matrix theory (see, for instance, [43, 21, 36]).

3.1.2 Duality relations between DPPs in continuous and discrete spaces

We consider the simplified setting (3.3) of W with $\Gamma = \mathbb{N}_0$. If we set $\Lambda_1 = [r,\infty) \subset S_1 = \mathbb{R}, r \in \mathbb{R}$ and $\Lambda_2 = \{0,1,\dots,N-1\} \subset S_2 = \Gamma = \mathbb{N}_0, N \in \mathbb{N}$ in (2.11), we obtain

$$\begin{aligned} K_{\mathbb{R}}^{\{0,1,\dots,N-1\}}(x,x') &= \sum_{n=0}^{N-1} \varphi_n(x)\overline{\varphi_n(x')}, \quad x,x' \in \mathbb{R}, \\ K_{\mathbb{N}_0}^{[r,\infty)}(n,n') &= \int_r^\infty \overline{\varphi_n(x)}\varphi_{n'}(x)\lambda_1(dx), \quad n,n' \in \mathbb{N}_0. \end{aligned} \tag{3.9}$$

When $\lambda_1(dx)$ and $\{\varphi_n\}_{n\in\mathbb{N}_0}$ are given by (3.4) or by (3.5), the kernels (3.9) are given by

$$\begin{aligned} K_{\text{DHermite}^+(r)}(n,n') &= (\pi 2^{n+n'} n! n'!)^{-1/2} \int_r^\infty H_n(x)H_{n'}(x)e^{-x^2}dx \\ &= -(\pi n! n'! 2^{n+n'+2})^{-1/2} e^{-r^2} \frac{H_{n+1}(r)H_{n'}(r) - H_n(r)H_{n'+1}(r)}{n-n'}, \end{aligned}$$

and, provided $r > 0$,

$$\begin{aligned} K_{\text{DLaguerre}^+(r,\nu+1)}(n,n') &= \left(\frac{n!n'!}{\Gamma(n+\nu+1)\Gamma(n'+\nu+1)}\right)^{1/2} \int_r^\infty L_n^{(\nu)}(x)L_{n'}^{(\nu)}(x)x^\nu e^{-x}dx \\ &= \left(\frac{n!n'!}{\Gamma(n+\nu+1)\Gamma(n'+\nu+1)}\right)^{1/2} r^{\nu+1}e^{-r}\frac{L_{n-1}^{(\nu+1)}(r)L_{n'}^{(\nu)}(r) - L_n^{(\nu)}(r)L_{n'-1}^{(\nu+1)}(r)}{n-n'}, \end{aligned}$$

with the convention that $L_{-1}^{(\nu)}(r) = 0$, respectively (see Propositions 3.3 and 3.4 in [15]). Borodin and Olshanski called the correlation kernels $K_{\text{DHermite}^+(r)}$ and $K_{\text{DLaguerre}^+(r,\nu+1)}$ the *discrete Hermite kernel* and the *discrete Laguerre kernel*, respectively [15]. Theorem 2.6 gives

$$\mathbf{P}(\Xi_1^{\{0,1,\dots,N-1\}}([r,\infty)) = m) = \mathbf{P}(\Xi_2^{[r,\infty)}(\{0,1,\dots,N-1\}) = m), \quad \forall m \in \mathbb{N}_0, \tag{3.10}$$

where LHS denotes the probability that the number of points in the interval $[r,\infty)$ is m for the N-point continuous DPP on $\mathbb{R}$ such as $(\Xi_1, K_{\text{Hermite}}^{(N)}, \lambda_{N(0,1/2)}(dx))$ or $(\Xi_1, K_{\text{Laguerre}}^{(N)}, \lambda_{\Gamma(\nu+1,1)}(dx))$, $\nu \in (-1,\infty)$, while RHS does the probability that the number of points in $\{0,1,\dots,N-1\}$ is m for the discrete DPP on $\mathbb{N}_0$ such as $(\Xi_2, K_{\text{DHermite}^+(r)})$ or $(\Xi_2, K_{\text{DLaguerre}^+(r,\nu+1)})$, $\nu \in (-1,\infty)$. The *duality between continuous and discrete ensembles* of Borodin and Olshanski (Theorem 3.7 in [15]) is a special case with $m = 0$ of the equality (3.10).

3.2 Finite DPPs in intervals related with classical root systems

Let $N \in \mathbb{N}$ and consider the four types of *classical root systems* denoted by A_{N-1}, B_N, C_N, and D_N (see Appendix A). We set $S^{A_{N-1}} = \mathbb{S}^1 = [0,2\pi)$, the unit circle, with a uniform measure

$\lambda^{A_{N-1}}(dx) = \lambda_{[0,2\pi)}(dx) := dx/(2\pi)$, and $S^{R_N} = [0,\pi]$, the upper half-circle, with $\lambda^{R_N}(dx) = \lambda_{[0,\pi]}(dx) := dx/\pi$ for $R_N = B_N, C_N, D_N$.

For a fixed $N \in \mathbb{N}$, we introduce the four sets of functions $\{\varphi_n^{R_N}\}_{n=1}^N$ on S^{R_N} defined as

$$\varphi_n^{R_N}(x) := \begin{cases} e^{-i(\mathcal{N}^{A_{N-1}}-2J^{A_{N-1}}(n))x/2}, & R_N = A_{N-1}, \\ \sin\left[(\mathcal{N}^{R_N} - 2J^{R_N}(n))x/2\right], & R_N = B_N, C_N, \\ \cos\left[(\mathcal{N}^{D_N} - 2J^{D_N}(n))x/2\right], & R_N = D_N, \end{cases}$$

where

$$\mathcal{N}^{R_N} := \begin{cases} N, & R_N = A_{N-1}, \\ 2N-1, & R_N = B_N, \\ 2(N+1), & R_N = C_N, \\ 2(N-1), & R_N = D_N. \end{cases} \tag{3.11}$$

and

$$J^{R_N}(n) := \begin{cases} n - 1/2, & R_N = A_{N-1}, \\ n-1, & R_N = B_N, D_N, \\ n, & R_N = C_N. \end{cases} \tag{3.12}$$

It is easy to verify that they satisfy the following orthonormality relations,

$$\begin{aligned} &\langle \varphi_n^{A_{N-1}}, \varphi_m^{A_{N-1}} \rangle_{L^2(\mathbb{S}^1, \lambda_{[0,2\pi)})} = \delta_{nm}, \\ &\quad \langle \varphi_n^{R_N}, \varphi_m^{R_N} \rangle_{L^2([0,\pi], \lambda_{[0,\pi]})} = \delta_{nm}, \quad R_N = B_N, C_N, D_N, \quad \text{if } n, m \in \{1, \dots, N\}. \end{aligned}$$

We put $\Gamma = \{1, \dots, N\}, N \in \mathbb{N}$ and $L^2(\Gamma, \nu) = \ell^2(\Gamma)$. By the argument given in Remark 3 in Section 3.1.1, Assumption 3 is verified, and hence Corollary 2.9 gives the four types of DPPs; $(\Xi, K^{A_{N-1}}, \lambda_{[0,2\pi)}(dx))$ on $\mathbb{S}^1$, and $(\Xi, K^{R_N}, \lambda_{[0,\pi]}(dx))$ on $[0,\pi]$, $R_N = B_N, C_N, D_N$, with the correlation kernels,

$$\begin{aligned} K^{R_N}(x,x') &= \sum_{n=1}^N \varphi_n^{R_N}(x) \overline{\varphi_n^{R_N}(x')} \\ &= \begin{cases} \dfrac{\sin\{N(x-x')/2\}}{\sin\{(x-x')/2\}}, & R_N = A_{N-1}, \\ \dfrac{1}{2}\left[\dfrac{\sin\{N(x-x')\}}{\sin\{(x-x')/2\}} - \dfrac{\sin\{N(x+x')\}}{\sin\{(x+x')/2\}}\right], & R_N = B_N, \\ \dfrac{1}{2}\left[\dfrac{\sin\{(2N+1)(x-x')/2\}}{\sin\{(x-x')/2\}} - \dfrac{\sin\{(2N+1)(x+x')/2\}}{\sin\{(x+x')/2\}}\right], & R_N = C_N, \\ \dfrac{1}{2}\left[\dfrac{\sin\{(2N-1)(x-x')/2\}}{\sin\{(x-x')/2\}} + \dfrac{\sin\{(2N-1)(x+x')/2\}}{\sin\{(x+x')/2\}}\right], & R_N = D_N. \end{cases} \end{aligned}$$

By Lemma A.1 in Appendix A, the probability densities for these finite DPPs with respect to the Lebesgue measures, $d\boldsymbol{x} = \prod_{j=1}^N dx_j$ are given as

$$\mathbf{p}^{A_{N-1}}(\boldsymbol{x}) = \frac{1}{Z^{A_{N-1}}} \prod_{1 \le j < k \le N} \sin^2 \frac{x_k - x_j}{2}, \quad \boldsymbol{x} \in [0, 2\pi)^N,$$

$$\mathbf{p}^{B_N}(\boldsymbol{x}) = \frac{1}{Z^{B_N}} \prod_{1 \le j < k \le N} \left(\sin^2 \frac{x_k - x_j}{2} \sin^2 \frac{x_k + x_j}{2} \right) \prod_{\ell=1}^N \sin^2 \frac{x_\ell}{2}, \quad \boldsymbol{x} \in [0, \pi]^N,$$

$$\mathbf{p}^{C_N}(\boldsymbol{x}) = \frac{1}{Z^{C_N}} \prod_{1 \le j < k \le N} \left(\sin^2 \frac{x_k - x_j}{2} \sin^2 \frac{x_k + x_j}{2} \right) \prod_{\ell=1}^N \sin^2 x_\ell, \quad \boldsymbol{x} \in [0, \pi]^N,$$

$$\mathbf{p}^{D_N}(\boldsymbol{x}) = \frac{1}{Z^{D_N}} \prod_{1 \le j < k \le N} \left(\sin^2 \frac{x_k - x_j}{2} \sin^2 \frac{x_k + x_j}{2} \right), \quad \boldsymbol{x} \in [0, \pi]^N,$$

with the normalization constants Z^{R_N}.

The DPP, $(\Xi, K^{A_{N-1}}, \lambda_{[0,2\pi)}(dx))$ is known as the *circular unitary ensemble* (CUE) in random matrix theory (see Section 11.8 in [43]). These four types of DPPs, $(\Xi, K^{A_{N-1}}, \lambda_{[0,2\pi)}(dx))$, $(\Xi, K^{R_N}, \lambda_{[0,\pi]}(dx))$, $R_N = B_N, C_N, D_N$ are realized as the eigenvalue distributions of random matrices in the *classical groups*, U(N), SO$(2N+1)$, Sp(N), and SO$(2N)$, respectively. (See Section 2.3 c) in [58] and Section 5.5 in [21].)

3.3 Finite DPPs in intervals related with affine root systems

We define the following four types of functions;

$$\begin{aligned}
\Theta^A(\sigma, z, \tau) &:= e^{2\pi i \sigma z} \vartheta_2(\sigma\tau + z; \tau), \\
\Theta^B(\sigma, z, \tau) &:= e^{2\pi i \sigma z} \vartheta_1(\sigma\tau + z; \tau) - e^{-2\pi i \sigma z} \vartheta_1(\sigma\tau - z; \tau), \\
\Theta^C(\sigma, z, \tau) &:= e^{2\pi i \sigma z} \vartheta_2(\sigma\tau + z; \tau) - e^{-2\pi i \sigma z} \vartheta_2(\sigma\tau - z; \tau), \\
\Theta^D(\sigma, z, \tau) &:= e^{2\pi i \sigma z} \vartheta_2(\sigma\tau + z; \tau) + e^{-2\pi i \sigma z} \vartheta_2(\sigma\tau - z; \tau),
\end{aligned} \tag{3.13}$$

for $\sigma \in \mathbb{R}, z \in \mathbb{C}, \tau \in \mathbb{H} := \{z \in \mathbb{C} : \Im z > 0\}$, where $\vartheta_\mu(v; \tau), \mu = 1, 2$ are the *Jacobi theta functions*. See Appendix B for definitions and the basic properties of the Jacobi theta functions.

Here we consider the seven types of *irreducible reduced affine root systems* $R_N = A_{N-1}$, B_N, $B_N^\vee$, C_N, $C_N^\vee$, BC_N, D_N, $N \in \mathbb{N}$ [41, 51] (see Appendix C). We put $S^{A_{N-1}} = \mathbb{S}^1 = [0, 2\pi)$ with $\lambda^{A_{N-1}}(dx) = \lambda_{[0,2\pi)}(dx)$, and $S^{R_N} = [0, \pi]$ with $\lambda^{R_N}(dx) = \lambda_{[0,\pi]}(dx)$ for $R_N = B_N$, $B_N^\vee$, C_N, $C_N^\vee$, BC_N, D_N. We assume that $\tau \in \mathbb{H}$ is pure imaginary, that is,

$$\tau = i\Im\tau \in i(0, \infty).$$

For a fixed $N \in \mathbb{N}$, we define the seven sets of functions $\{\varphi_n^{R_N}(x; \tau)\}_{n=1}^N$ on S^{R_N} as

$$\varphi_n^{R_N}(x; \tau) := \frac{1}{\sqrt{m_n^{R_N}(\tau)}} \Theta^{\sharp(R_N)} \left(\frac{J^{R_N}(n)}{\mathcal{N}^{R_N}}, \mathcal{N}^{R_N} \frac{x}{2\pi}, \tau \right),$$

where

$$\sharp(R_N) := \begin{cases} A, & \text{if } R_N = A_{N-1}, \\ B, & \text{if } R_N = B_N, B_N^\vee, \\ C, & \text{if } R_N = C_N, C_N^\vee, BC_N, \\ D, & \text{if } R_N = D_N, \end{cases} \tag{3.14}$$

$$\mathcal{N}^{R_N} := \begin{cases} N, & R_N = A_{N-1}, \\ 2N-1, & R_N = B_N, \\ 2N, & R_N = B_N^\vee, C_N^\vee, \\ 2(N+1), & R_N = C_N, \\ 2N+1, & R_N = BC_N, \\ 2(N-1), & R_N = D_N, \end{cases} \tag{3.15}$$

$$J^{R_N}(n) := \begin{cases} n-1/2, & R_N = A_{N-1}, C_N^\vee, \\ n-1, & R_N = B_N, B_N^\vee, D_N, \\ n, & R_N = C_N, BC_N, \end{cases} \tag{3.16}$$

and we set

$$\begin{aligned} m_n^{A_{N-1}}(\tau) &:= \vartheta_2(2J^{A_{N-1}}(n)\tau/\mathcal{N}^{A_{N-1}}; 2\tau), \quad n \in \{1, \dots, N\}, \\ m_n^{R_N}(\tau) &:= 2\vartheta_2(2J^{R_N}(n)\tau/\mathcal{N}^{R_N}; 2\tau), \quad n \in \{1, \dots, N\}, \quad \text{for } R_N = C_N, C_N^\vee, BC_N, \\ m_n^{R_N}(\tau) &:= \begin{cases} 4\vartheta_2(0; 2\tau), & n = 1, \\ 2\vartheta_2(2J^{R_N}(j)\tau/\mathcal{N}^{R_N}; 2\tau), & n \in \{2, 3, \dots, N\}, \end{cases} \quad \text{for } R_N = B_N, B_N^\vee, \\ m_n^{D_N}(\tau) &:= \begin{cases} 4\vartheta_2(0; 2\tau), & n = 1, \\ 2\vartheta_2(2J^{D_N}(j)\tau/\mathcal{N}^{D_N}; 2\tau), & n \in \{2, 3, \dots, N-1\}, \\ 4\vartheta_2(2(N-1)\tau/\mathcal{N}^{D_N}; 2\tau), & n = N. \end{cases} \end{aligned}$$

For $N \in \mathbb{N}$, the following orthonormality relations can be proved as a special case of Lemma 2.1 in [33],

$$\begin{aligned} \langle \varphi_n^{A_{N-1}}(\cdot, \tau), \varphi_m^{A_{N-1}}(\cdot, \tau) \rangle_{L^2(\mathbb{S}^1, \lambda_{[0,2\pi)})} &= \delta_{nm}, \\ \langle \varphi_n^{R_N}(\cdot, \tau), \varphi_m^{R_N}(\cdot, \tau) \rangle_{L^2([0,\pi], \lambda_{[0,\pi]})} &= \delta_{nm}, \quad R_N = B_N, B_N^\vee, C_N, C_N^\vee, BC_N, D_N, \\ & n, m \in \Gamma := \{1, \dots, N\}. \end{aligned}$$

By the argument given in Remark 3 in Section 3.1.1, Assumption 3 is verified, and hence Corollary 2.9 gives the seven types of DPPs, $(\Xi, K_\tau^{R_N}, \lambda^{R_N}(dx))$ with the correlation kernels,

$$K_\tau^{R_N}(x, x') = \sum_{n=1}^N \varphi_n^{R_N}(x; \tau) \overline{\varphi_n^{R_N}(x'; \tau)}, \quad R_N = A_{N-1}, B_N, B_N^\vee, C_N, C_N^\vee, BC_N, R_N.$$

Thanks to the *Macdonald denominator formula* proved by Rosengren and Schlosser [51] (see (3.1) with (3.2) in [33] in the present notations), the probability densities for these finite DPPs with respect to the Lebesgue measure, $d\boldsymbol{x} = \prod_{j=1}^N dx_j$ are given as follows,

$$\mathbf{p}_\tau^{A_{N-1}}(\boldsymbol{x}) = \begin{cases} \dfrac{1}{Z^{A_{N-1}}(\tau)} \left| \vartheta_0 \left(\displaystyle\sum_{j=1}^N \frac{x_j}{2\pi}; \frac{\tau}{\mathcal{N}^{A_{N-1}}} \right) W^{A_{N-1}} \left(\frac{\boldsymbol{x}}{2\pi}; \frac{\tau}{\mathcal{N}^{A_{N-1}}} \right) \right|^2, & \text{if } N \text{ is even}, \\ \dfrac{1}{Z^{A_{N-1}}(\tau)} \left| \vartheta_3 \left(\displaystyle\sum_{j=1}^N \frac{x_j}{2\pi}; \frac{\tau}{\mathcal{N}^{A_{N-1}}} \right) W^{A_{N-1}} \left(\frac{\boldsymbol{x}}{2\pi}; \frac{\tau}{\mathcal{N}^{A_{N-1}}} \right) \right|^2, & \text{if } N \text{ is odd}, \\ & \boldsymbol{x} \in [0, 2\pi)^N, \end{cases}$$

$$\mathbf{p}_\tau^{R_N}(\boldsymbol{x}) = \frac{1}{Z^{R_N}(\tau)} \left| W^{R_N} \left(\frac{\boldsymbol{x}}{2\pi}; \frac{\tau}{\mathcal{N}^{R_N}} \right) \right|^2, \ \boldsymbol{x} \in [0, \pi]^N, \ R_N = B_N, B_N^\vee, C_N, C_N^\vee, BC_N, D_N,$$

where W^{R_N} are the Macdonald denominators given by (C.1) in Appendix C and $Z^{R_N}(\tau)$ are the normalization constants. By the properties (B.3) and (B.6) of the Jacobi theta functions, it is easy to verify the following,

$$\begin{aligned} &\mathcal{S}_{2\pi/N}(\Xi, K_\tau^{A_{N-1}}, \lambda_{[0,2\pi)}) \overset{\text{(law)}}{=} (\Xi, K_\tau^{A_{N-1}}, \lambda_{[0,2\pi)}), \\ &\rho_\tau^{R_N}(0) = 0, \quad R_N = B_N, C_N^\vee, BC_N, \\ &\rho_\tau^{R_N}(0) = \rho_\tau^{R_N}(\pi) = 0, \quad R_N = B_N^\vee, C_N. \end{aligned}$$

In [33], it was proved that these seven types of DPPs are realized as the particle configurations at the middle time $t = t_*/2$ of the *noncolliding Brownian bridges* in time duration $[0, t_*]$, provided $t_* = 4\pi \Im\tau > 0$, whose initial configurations at $t = 0$ and final configurations at $t = t_*$ are fixed to be specially chosen configurations depending on the types $R_N, N \in \mathbb{N}$.

As $\Im\tau \to \infty$, the temporal inhomogeneity in such noncolliding Brownian bridges vanishes. Associated with such limit transitions, the following degeneracies are observed in the weak convergence of DPPs from the seven types of affine root systems to the four types of classical root systems,

$$\begin{aligned} &(\Xi, K_\tau^{A_{N-1}}, \lambda_{[0,2\pi)}(dx)) \overset{\Im\tau\to\infty}{\Longrightarrow} (\Xi, K^{A_{N-1}}, \lambda_{[0,2\pi)}(dx)), \\ &\left. \begin{array}{l} (\Xi, K_\tau^{B_N}, \lambda_{[0,\pi]}(dx)) \\ (\Xi, K_\tau^{C_N^\vee}, \lambda_{[0,\pi]}(dx)) \\ (\Xi, K_\tau^{BC_N}, \lambda_{[0,\pi]}(dx)) \end{array} \right\} \overset{\Im\tau\to\infty}{\Longrightarrow} (\Xi, K^{B_N}, \lambda_{[0,\pi]}(dx)), \\ &\left. \begin{array}{l} (\Xi, K_\tau^{B_N^\vee}, \lambda_{[0,\pi]}(dx)) \\ (\Xi, K_\tau^{C_N}, \lambda_{[0,\pi]}(dx)) \end{array} \right\} \overset{\Im\tau\to\infty}{\Longrightarrow} (\Xi, K^{C_N}, \lambda_{[0,\pi]}(dx)), \\ &(\Xi, K_\tau^{D_N}, \lambda_{[0,\pi]}(dx)) \overset{\Im\tau\to\infty}{\Longrightarrow} (\Xi, K^{D_N}, \lambda_{[0,\pi]}(dx)), \end{aligned} \tag{3.17}$$

where the DPPs, $(\Xi, K^{A_{N-1}}, \lambda_{[0,2\pi)})$ and $(\Xi, K^{R_N}, \lambda_{[0,\pi]})$, $R_N = B_N, C_N, D_N$ were given in Section 3.2.

3.4 Infinite DPPs in $\mathbb{R}$ associated with classical orthonormal functions

Here we give examples of infinite DPPs obtained by Corollary 2.10.

(i) DPP with the *sinc kernel* : We set $S_1 = \mathbb{R}$, $\lambda_1(dx) = dx$, $\Gamma = (-1, 1)$, $\nu(dy) = \lambda_2(dy) = dy$, and put

$$\psi_1(x, y) = \frac{1}{\sqrt{2\pi}} e^{ixy}.$$

We see that $\Psi_1(x)^2 \equiv 1/\pi, x \in \mathbb{R}$ and thus Assumption 3'(ii) is satisfied. The correlation kernel K_{S_1} is given by

$$K_{\text{sinc}}(x, x') = \frac{1}{2\pi} \int_{-1}^{1} e^{iy(x-x')} dy = \frac{\sin(x - x')}{\pi(x - x')}, \quad x, x' \in \mathbb{R}.$$

(ii) DPP with the *Airy kernel* : We set $S_1 = \mathbb{R}$, $\lambda_1(dx) = dx$, $\Gamma = [0, \infty)$, $\nu(dy) = \lambda_2(dy) = dy$, and put

$$\psi_1(x, y) = \text{Ai}(x + y),$$

where $\text{Ai}(x)$ denotes the *Airy function* [45]

$$\text{Ai}(x) := \frac{1}{\pi} \int_0^\infty \cos\left(\frac{k^3}{3} + kx\right) dk.$$

We see that $\Psi_1(x)^2 = -x\text{Ai}(x)^2 + \text{Ai}'(x)^2, x \in \mathbb{R}$ and thus Assumption 3(ii) is satisfied. The correlation kernel K_{S_1} is given by

$$K_{\text{Airy}}(x, x') = \int_0^\infty \text{Ai}(x + y)\text{Ai}(x' + y) dy = \frac{\text{Ai}(x)\text{Ai}'(x') - \text{Ai}(x')\text{Ai}'(x)}{x - x'}, \quad x, x' \in \mathbb{R},$$

where $\text{Ai}'(x) := d\text{Ai}(x)/dx$.

(iii) DPP with the *Bessel kernel*: We set $S_1 = [0, \infty)$, $\lambda_1(dx) = dx$, $\Gamma = [0, 1]$, $\nu(dy) = \lambda_2(dy) = dy$. With parameter $\nu > -1$ we put

$$\psi_1(x, y) = \sqrt{xy} J_\nu(xy),$$

where J_ν is the *Bessel function of the first kind* defined by

$$J_\nu(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu + n + 1)} \left(\frac{x}{2}\right)^{2n+\nu}, \quad x \in \mathbb{C} \setminus (-\infty, 0). \tag{3.18}$$

We see that $\Psi_1(x)^2 = x\{J_\nu(x)^2 - J_{\nu-1}(x)J_{\nu+1}(x)\}/2$, $x \in [0, \infty)$ and thus Assumption 3'(ii) is satisfied. The correlation kernel K_{S_1} is given by

$$\begin{aligned} K_{\text{Bessel}}^{(\nu)}(x, x') &= \int_0^1 \sqrt{xy} J_\nu(xy) \sqrt{x'y} J_\nu(x'y) dy \\ &= \frac{\sqrt{xx'}}{x^2 - (x')^2} \Big\{ J_\nu(x) x' J_\nu'(x') - x J_\nu'(x) J_\nu(x') \Big\}, \quad x, x' \in [0, \infty), \end{aligned} \tag{3.19}$$

where $J_\nu'(x) := dJ_\nu(x)/dx$.

These three kinds of infinite DPPs, $(\Xi, K_{\text{sinc}}, dx)$, $(\Xi, K_{\text{Airy}}, dx)$, and $(\Xi, K_{\text{Bessel}}^{(\nu)}, \mathbf{1}_{[0,\infty)}(x)dx)$, are obtained as the scaling limits of the finite DPPs, $(\Xi, K_{\text{Hermite}}^{(N)}, \lambda_{\text{N}(0,1/2)}(dx))$ and $(\Xi, K_{\text{Laguerre}}^{(\nu,N)}, \lambda_{\Gamma(\nu+1,1)}(dx))$, given in Section 3.1 as follows.

(i) *Bulk scaling limit*,

$$\sqrt{2N} \circ (\Xi, K_{\rm Hermite}^{(N)}, \lambda_{{\rm N}(0,1/2)}(dx)) \stackrel{N\to\infty}{\Longrightarrow} (\Xi, K_{\rm sinc}, dx).$$

(ii) *Soft-edge scaling limit*,

$$\sqrt{2}N^{1/6} \circ \mathcal{S}_{\sqrt{2N}} (\Xi, K_{\rm Hermite}^{(N)}, \lambda_{{\rm N}(0,1/2)}(dx)) \stackrel{N\to\infty}{\Longrightarrow} (\Xi, K_{\rm Airy}, dx).$$

(iii) *Hard-edge scaling limit*, for $\nu > -1$,

$$4N \circ \Big((\Xi, K_{\rm Laguerre}^{(\nu,N)}, \lambda_{\Gamma(\nu+1,1)}(dx))^{\langle 1/2 \rangle} \Big) \stackrel{N\to\infty}{\Longrightarrow} (\Xi, K_{\rm Bessel}^{(\nu)}, \mathbf{1}_{[0,\infty)}(x)dx).$$

See, for instance, [43, 21, 8, 32], for more details.

The DPPs with the sinc kernel and the Bessel kernel with the special values of parameter ν can be obtained as the bulk scaling limits of the DPPs, $(\Xi, K^{R_N}, \lambda^{R_N}(dx))$, $R_N = A_{N-1}, B_N, C_N, D_N$ given in Section 3.2 as

$$\begin{aligned}
\frac{N}{2} \circ (\Xi, K^{A_{N-1}}, \lambda_{[0,2\pi)}(dx)) &\stackrel{N\to\infty}{\Longrightarrow} (\Xi, K_{\rm sinc}, dx), \\
\left.\begin{array}{l} N \circ (\Xi, K^{B_N}, \lambda_{[0,\pi]}(dx)) \\ N \circ (\Xi, K^{C_N}, \lambda_{[0,\pi]}(dx)) \end{array}\right\} &\stackrel{N\to\infty}{\Longrightarrow} (\Xi, K_{\rm Bessel}^{(1/2)}, \mathbf{1}_{[0,\infty)}(x)dx), \\
N \circ (\Xi, K^{D_N}, \lambda_{[0,\pi]}(dx)) &\stackrel{N\to\infty}{\Longrightarrow} (\Xi, K_{\rm Bessel}^{(-1/2)}, \mathbf{1}_{[0,\infty)}(x)dx),
\end{aligned} \tag{3.20}$$

where

$$\begin{aligned}
K_{\rm Bessel}^{(1/2)}(x,x') &= \frac{\sin(x-x')}{\pi(x-x')} - \frac{\sin(x+x')}{\pi(x+x')}, \quad x, x' \in [0,\infty), \\
K_{\rm Bessel}^{(-1/2)}(x,x') &= \frac{\sin(x-x')}{\pi(x-x')} + \frac{\sin(x+x')}{\pi(x+x')}, \quad x, x' \in [0,\infty).
\end{aligned}$$

Since $J_{1/2}(x) = \sqrt{2/(\pi x)} \sin x$ and $J_{-1/2}(x) = \sqrt{2/(\pi x)} \cos x$, the above correlation kernels are readily obtained from (3.19) by setting $\nu = 1/2$ and $-1/2$, respectively.

3.5 Infinite DPPs in $\mathbb{R}$ associated with orthonormal theta functions

Let $S = \mathbb{R}$ with $\lambda^A(dx) = dx$, and $\lambda^R(dx) = \mathbf{1}_{[0,\infty)}(x)dx$ for the types $R = B, C, D$. Here we assume that $\tau \in \mathbb{H}$ is pure imaginary. We put

$$\begin{aligned}
\psi^A(x, \gamma; \tau) &:= \frac{\Theta^A(\gamma, x/\pi, \tau)}{\sqrt{\pi \vartheta_2(2\tau\gamma; 2\tau)}}, \\
\psi^R(x, \gamma; \tau) &:= \frac{\Theta^R(\gamma/2, x/\pi, \tau)}{\sqrt{2\pi \vartheta_2(\tau\gamma; 2\tau)}}, \quad R = B, C, D,
\end{aligned}$$

and $\Gamma = (0,1)$ with $\nu(d\gamma) = d\gamma$, where $\Theta^R, R = A, B, C, D$ are given by (3.13).

Using the equalities

$$\frac{1}{\pi}\int_{\mathbb{R}} e^{2i\{(\gamma-\gamma')+(n-m)\}x}dx = \delta_{nm}\delta(\gamma-\gamma'),$$
$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\{(\gamma+\gamma')+2(n+m-1)\}x}dx = 0, \quad \text{for } n,m\in\mathbb{Z}, \quad \gamma,\gamma'\in\Gamma,$$

we can show the orthonormality relations [33];

$$\langle \psi^A(\cdot,\gamma;\tau), \psi^A(\cdot,\gamma';\tau)\rangle_{L^2(\mathbb{R},dx)} = \delta(\gamma-\gamma'),$$
$$\langle \psi^R(\cdot,\gamma;\tau), \psi^R(\cdot,\gamma';\tau)\rangle_{L^2(\mathbb{R},\mathbf{1}_{[0,\infty)}(x)dx)} = \delta(\gamma-\gamma'), \quad R=A,B,C,D, \quad \gamma,\gamma'\in\Gamma.$$

We can also evaluate the upper bounds for $\Psi^R(x;\tau)^2 := \|\psi^R(x,\cdot;\tau)\|^2_{L^2(\Gamma,\nu)}$, $x\in\mathbb{R}$, $R=A,B,C,D$, and confirm that Assumption 3(ii) is also satisfied. Hence by Corollary 2.9, we obtain the four types of infinite DPPs (Ξ, K_τ^A, dx), $(\Xi, K_\tau^R, \mathbf{1}_{[0,\infty)}(x)dx)$, $R=B,C,D$. The correlation kernels are written as follows,

$$\begin{aligned}
K_\tau^A(x,x') &= \frac{1}{\pi}\int_0^1 e^{2i(x-x')\gamma}\frac{\vartheta_2(x/\pi+\tau\gamma;\tau)\vartheta_2(x'/\pi-\tau\gamma;\tau)}{\vartheta_2(2\tau\gamma;2\tau)}d\gamma, \quad x,x'\in\mathbb{R},\\
K_\tau^B(x,x') &= \frac{1}{2\pi}\Bigg[\int_{-1}^1 e^{i(x-x')\gamma}\frac{\vartheta_1(x/\pi+\tau\gamma/2;\tau)\vartheta_1(x'/\pi-\tau\gamma/2;\tau)}{\vartheta_2(\tau\gamma;2\tau)}d\gamma\\
&\quad+\int_{-1}^1 e^{i(x+x')\gamma}\frac{\vartheta_1(x/\pi+\tau\gamma/2;\tau)\vartheta_1(x'/\pi+\tau\gamma/2);\tau)}{\vartheta_2(\tau\gamma;2\tau)}d\gamma\Bigg], \quad x,x'\in[0,\infty),\\
K_\tau^C(x,x') &= \frac{1}{2\pi}\Bigg[\int_{-1}^1 e^{i(x-x')\gamma}\frac{\vartheta_2(x/\pi+\tau\gamma/2;\tau)\vartheta_2(x'/\pi-\tau\gamma/2;\tau)}{\vartheta_2(\tau\gamma;2\tau)}d\gamma\\
&\quad-\int_{-1}^1 e^{i(x+x')\gamma}\frac{\vartheta_2(x/\pi+\tau\gamma/2;\tau)\vartheta_2(x'/\pi+\tau\gamma/2);\tau)}{\vartheta_2(\tau\gamma;2\tau)}d\gamma\Bigg], \quad x,x'\in[0,\infty),\\
K_\tau^D(x,x') &= \frac{1}{2\pi}\Bigg[\int_{-1}^1 e^{i(x-x')\gamma}\frac{\vartheta_2(x/\pi+\tau\gamma/2;\tau)\vartheta_2(x'/\pi-\tau\gamma/2;\tau)}{\vartheta_2(\tau\gamma;2\tau)}d\gamma\\
&\quad+\int_{-1}^1 e^{i(x+x')\gamma}\frac{\vartheta_2(x/\pi+\tau\gamma/2;\tau)\vartheta_2(x'/\pi+\tau\gamma/2);\tau)}{\vartheta_2(\tau\gamma;2\tau)}d\gamma\Bigg], \quad x,x'\in[0,\infty). \quad (3.21)
\end{aligned}$$

If we change the integral variables appropriately, the above become the correlation kernels $\mathcal{K}^R_{t_*/2}$, $R=A,B,C,D$, with $t_*=4\pi\Im\tau$, given in Lemma 3.5 in [33]. Using the quasi-periodicity (B.3) of the theta functions, we can show that the DPP (Ξ, K_τ^A, dx) has a periodicity of π; $\mathcal{S}_\pi K_\tau^A(x,x') = K_\tau^A(x,x')$, $x,x'\in\mathbb{R}$. By the symmetry (B.2) of the theta functions, we see that $\rho_\tau^R(0)=K_\tau^R(0,0)=0$, $R=B,C$.

The infinite DPPs associated with the above correlation kernels (3.21) are obtained as the bulk scaling limits of the finite DPPs in intervals studied in Section 3.3 [33];

$$\begin{aligned}
\frac{N}{2} \circ (\Xi, K_\tau^{A_{N-1}}, \lambda_{[0,2\pi)}(dx)) &\overset{N\to\infty}{\Longrightarrow} (\Xi, K_\tau^A, dx), \\
\left.\begin{array}{r} N \circ (\Xi, K_\tau^{B_N}, \lambda_{[0,\pi]}(dx)) \\ N \circ (\Xi, K_\tau^{B_N^\vee}, \lambda_{[0,\pi]}(dx)) \end{array}\right\} &\overset{N\to\infty}{\Longrightarrow} (\Xi, K_\tau^B, \mathbf{1}_{[0,\infty)}(x)dx), \\
\left.\begin{array}{r} N \circ (\Xi, K_\tau^{C_N}, \lambda_{[0,\pi]}(dx)) \\ N \circ (\Xi, K_\tau^{C_N^\vee}, \lambda_{[0,\pi]}(dx)) \\ N \circ (\Xi, K_\tau^{BC_N}, \lambda_{[0,\pi]}(dx)) \end{array}\right\} &\overset{N\to\infty}{\Longrightarrow} (\Xi, K_\tau^C, \mathbf{1}_{[0,\infty)}(x)dx), \\
N \circ (\Xi, K_\tau^{D_N}, \lambda_{[0,\pi]}(dx)) &\overset{N\to\infty}{\Longrightarrow} (\Xi, K_\tau^D, \mathbf{1}_{[0,\infty)}(x)dx).
\end{aligned} \tag{3.22}$$

If we take the limit $\Im\tau \to \infty$ in (3.21), we obtain the following three infinite DPPs,

$$\begin{aligned}
(\Xi, K_\tau^A, dx) &\overset{\Im\tau\to\infty}{\Longrightarrow} (\Xi, K_{\rm sinc}, dx), \\
\left.\begin{array}{r} (\Xi, K_\tau^B, \mathbf{1}_{[0,\infty)}(x)dx) \\ (\Xi, K_\tau^C, \mathbf{1}_{[0,\infty)}(x)dx) \end{array}\right\} &\overset{\Im\tau\to\infty}{\Longrightarrow} (\Xi, K_{\rm Bessel}^{(1/2)}, \mathbf{1}_{[0,\infty)}(x)dx), \\
(\Xi, K_\tau^D, \mathbf{1}_{[0,\infty)}(x)dx) &\overset{\Im\tau\to\infty}{\Longrightarrow} (\Xi, K_{\rm Bessel}^{(-1/2)}, \mathbf{1}_{[0,\infty)}(x)dx),
\end{aligned} \tag{3.23}$$

which are the same as the limiting DPPs given by (3.20).

Remark 4 The results (3.17), (3.20), (3.22), and (3.23) imply that, in the limit transitions from the finite DPPs $(\Xi, K_\tau^{R_N}, \lambda^{R_N})$, $R_N = A_{N-1}, B_N, B_N^\vee, C_N, C_N^\vee, BC_N, D_N$, to the infinite DPPs $(\Xi, K_{\rm sinc}, dx)$, $(\Xi, K_{\rm Bessel}^{(1/2)}, \mathbf{1}_{[0,\infty)}(x)dx)$, $(\Xi, K_{\rm Bessel}^{(-1/2)}, \mathbf{1}_{[0,\infty)}(x)dx)$, the scaling limits associated with $N \to \infty$ and the limit $\Im\tau \to \infty$ are commutable.

4 Examples in Two-dimensional Spaces

4.1 Infinite DPPs on $\mathbb{C}$: Ginibre and Ginibre-type DPPs

4.1.1 Three types of Ginibre DPPs

Let $\lambda_{{\rm N}(m,\sigma^2;\mathbb{C})}(dx)$ denote the *complex normal distribution*,

$$\begin{aligned}
\lambda_{{\rm N}(m,\sigma;\mathbb{C})}(dx) &:= \frac{1}{\pi\sigma^2} e^{-|x-m|^2/\sigma^2} dx \\
&= \frac{1}{\pi\sigma^2} e^{-(x_{\rm R}-m_{\rm R})^2/\sigma^2 - (x_{\rm I}-m_{\rm I})^2/\sigma^2} dx_{\rm R} dx_{\rm I},
\end{aligned}$$

$m \in \mathbb{C}, m_{\rm R} := \Re m, m_{\rm I} := \Im m, \sigma > 0$. We set $S = \mathbb{C}$,

$$\begin{aligned}
\lambda(dx) = \lambda_{{\rm N}(0,1;\mathbb{C})}(dx) &= \frac{1}{\pi} e^{-|x|^2} dx \\
&= \lambda_{{\rm N}(0,1/2)}(dx_{\rm R}) \lambda_{{\rm N}(0,1/2)}(dx_{\rm I}),
\end{aligned}$$

and

$$\begin{aligned}
\psi^A(x,\gamma) &:= e^{-(x_{\rm R}^2 - x_{\rm I}^2)/2 + 2x\gamma}, \\
\psi^C(x,\gamma) &:= \sqrt{2}\sinh(2x\gamma) e^{-(x_{\rm R}^2 - x_{\rm I}^2)/2}, \\
\psi^D(x,\gamma) &:= \sqrt{2}\cosh(2x\gamma) e^{-(x_{\rm R}^2 - x_{\rm I}^2)/2}.
\end{aligned}$$

It is easy to confirm that

$$\frac{1}{\pi}\int_{\mathbb{R}} \psi^A(x,\gamma)\overline{\psi^A(x,\gamma')}e^{-x_{\rm I}^2}dx_{\rm I} = e^{-(x_{\rm R}^2-4x_{\rm R}\gamma)}\delta(\gamma-\gamma'),$$

$$\frac{1}{\pi}\int_{\mathbb{R}} \psi^R(x,\gamma)\overline{\psi^R(x,\gamma')}e^{-x_{\rm I}^2}dx_{\rm I} = e^{-x_{\rm R}^2}\cosh(4x_{\rm R}\gamma)\times\begin{cases}\delta(\gamma-\gamma')-\delta(\gamma+\gamma'), & R=C,\\ \delta(\gamma-\gamma')+\delta(\gamma+\gamma'), & R=D.\end{cases}$$

Therefore, we have

$$\langle \psi^A(\cdot,\gamma),\psi^A(\cdot,\gamma')\rangle_{L^2(\mathbb{C},\lambda_{\rm N(0,1;\mathbb{C})})}\nu(d\gamma) = \delta(\gamma-\gamma')d\gamma, \quad \gamma,\gamma'\in\Gamma^A := \mathbb{R},$$

$$\langle \psi^R(\cdot,\gamma),\psi^R(\cdot,\gamma')\rangle_{L^2(\mathbb{C},\lambda_{\rm N(0,1;\mathbb{C})})}\nu(d\gamma) = \delta(\gamma-\gamma')d\gamma, \quad \gamma,\gamma'\in\Gamma^R := (0,\infty), \quad R=C,D,$$

with $\nu(d\gamma)=\lambda_{\rm N(0,1/4)}(d\gamma)=\sqrt{2/\pi}e^{-2\gamma^2}d\gamma$. We also see that $\Psi^A(x)^2 := \|\psi^A(x,\cdot)\|^2_{L^2(\Gamma^A,\nu)} = e^{|x|^2}$, $\Psi^C(x)^2 := \|\psi^C(x,\cdot)\|^2_{L^2(\Gamma^C,\nu)} = \sinh|x|^2$, and $\Psi^D(x)^2 := \|\psi^D(x,\cdot)\|^2_{L^2(\Gamma^D,\nu)} = \cosh|x|^2$, $x\in\mathbb{C}$. Thus Assumption 3 is satisfied and we can apply Corollary 2.9. The kernels (2.14) of obtained DPPs are given as

$$K^A(x,x') = \sqrt{\frac{2}{\pi}}e^{-\{(x_{\rm R}^2-x_{\rm I}^2)+({x'_{\rm R}}^2-{x'_{\rm I}}^2)\}/2}\int_{-\infty}^{\infty} e^{-2\{\gamma^2-(x+\overline{x'})\gamma\}}d\gamma,$$

$$K^C(x,x') = 2\sqrt{\frac{2}{\pi}}e^{-\{(x_{\rm R}^2-x_{\rm I}^2)+({x'_{\rm R}}^2-{x'_{\rm I}}^2)\}/2}\int_0^{\infty} e^{-2\gamma^2}\sinh(2x\gamma)\sinh(2\overline{x'}\gamma)d\gamma,$$

$$K^D(x,x') = 2\sqrt{\frac{2}{\pi}}e^{-\{(x_{\rm R}^2-x_{\rm I}^2)+({x'_{\rm R}}^2-{x'_{\rm I}}^2)\}/2}\int_0^{\infty} e^{-2\gamma^2}\cosh(2x\gamma)\cosh(2\overline{x'}\gamma)d\gamma.$$

The integrals are performed and we obtain $K^R(x,x') = e^{ix_{\rm R}x_{\rm I}}K^R_{\rm Ginibre}(x,x')e^{-ix'_{\rm R}x'_{\rm I}}$, $R=A,C,D$, with

$$K^A_{\rm Ginibre}(x,x') = e^{x\overline{x'}}, \tag{4.1}$$

$$K^C_{\rm Ginibre}(x,x') = \sinh(x\overline{x'}), \tag{4.2}$$

$$K^D_{\rm Ginibre}(x,x') = \cosh(x\overline{x'}), \quad x,x'\in\mathbb{C}. \tag{4.3}$$

Due to the gauge invariance of DPP mentioned in Section 2.3, the obtained three types of infinite DPPs on $\mathbb{C}$ are written as $(\Xi, K^R_{\rm Ginibre}, \lambda_{\rm N(0,1;\mathbb{C})}(dx))$, $R=A,C,D$. The DPP, $(\Xi,\ K^A_{\rm Ginibre}, \lambda_{\rm N(0,1;\mathbb{C})}(dx))$ with (4.1) describes the eigenvalue distribution of the Gaussian random complex matrix in the bulk scaling limit, which is called the *complex Ginibre ensemble* [23, 43, 28, 29, 21, 52]. This density profile is uniform with the Lebesgue measure dx on $\mathbb{C}$ as

$$\rho_{\rm Ginibre}(x)dx = K^A_{\rm Ginibre}(x,x)\lambda_{\rm N(0,1;\mathbb{C})}(dx) = \frac{1}{\pi}dx_{\rm R}dx_{\rm I}, \quad x\in\mathbb{C}.$$

On the other hands, the Ginibre DPPs of types C and D with the correlation kernels (4.2) and (4.3) are rotationally symmetric around the origin, but non-uniform on $\mathbb{C}$. The density profiles with the Lebesgue measure dx on $\mathbb{C}$ are given by

$$\rho^C_{\rm Ginibre}(x)dx = K^C_{\rm Ginibre}(x,x)\lambda_{\rm N(0,1;\mathbb{C})}(dx) = \frac{1}{2\pi}(1-e^{-2|x|^2})dx_{\rm R}dx_{\rm I}, \quad x\in\mathbb{C},$$

$$\rho^D_{\rm Ginibre}(x)dx = K^D_{\rm Ginibre}(x,x)\lambda_{\rm N(0,1;\mathbb{C})}(dx) = \frac{1}{2\pi}(1+e^{-2|x|^2})dx_{\rm R}dx_{\rm I}, \quad x\in\mathbb{C}.$$

They were first obtained in [34] by taking the limit $W \to \infty$ keeping the density of points of the infinite DPPs in the strip on $\mathbb{C}$, $\{z \in \mathbb{C} : 0 \le \Im z \le W\}$. See (4.16) in Section 4.4, which represents the corresponding limit transitions. See also Remarks 5 and 8 in [34] in which the present Ginibre DPPs of types C and D are discussed as new examples of the *Mittag–Leffler fields* studied by [5, 6, 7].

4.1.2 Ginibre and Ginibre-type DPPs as examples of Weyl–Heisenberg ensembles

Let $d = 1$ and consider the following window function,

$$G(x_{\rm R}) = 2^{1/4} e^{-\pi x_{\rm R}^2}, \quad x_{\rm R} \in \mathbb{R}. \tag{4.4}$$

It is obvious that Assumption 4 is satisfied, $\|G\|^2_{L^2(\mathbb{R}, dx_{\rm R})} = 1$. In this case (2.19) becomes [1, 3, 2]

$$K_{\rm WH}(x, x') = \frac{e^{\pi i x_{\rm R} x_{\rm I}}}{e^{\pi i x'_{\rm R} x'_{\rm I}}} K^A_{\rm Ginibre}(\sqrt{\pi} x, \sqrt{\pi} x') e^{-\pi(|x|^2+|x'|^2)/2}.$$

By taking into account the direct decomposition

$$L^2(\mathbb{R}) = L^2_{\rm odd}(\mathbb{R}) \oplus L^2_{\rm even}(\mathbb{R}),$$

we have

$$\mathcal{W}^*_{\rm WH}(L^2(\mathbb{R})) = \mathcal{W}^*_{\rm WH}(L^2_{\rm odd}(\mathbb{R})) \oplus \mathcal{W}^*_{\rm WH}(L^2_{\rm even}(\mathbb{R})).$$

When $G(-y) = G(y)$, we have

$$\begin{aligned}
\mathcal{W}^*_{\rm WH}(L^2_{\rm odd}(\mathbb{R})) &\subset \{F \in L^2(\mathbb{C}) : F(-x) = -F(x), x \in \mathbb{C}\} =: L^2_{\rm odd}(\mathbb{C}), \\
\mathcal{W}^*_{\rm WH}(L^2_{\rm even}(\mathbb{R})) &\subset \{F \in L^2(\mathbb{C}) : F(-x) = F(x), x \in \mathbb{C}\} =: L^2_{\rm even}(\mathbb{C}).
\end{aligned}$$

We consider the restriction of operator

$$\mathcal{W}_{\rm WH}\Big|_{L^2_{\rm odd}(\mathbb{C})} : L^2_{\rm odd}(\mathbb{C}) \to L^2_{\rm odd}(\mathbb{R}).$$

and its adjoint

$$\mathcal{W}^*_{\rm WH}\Big|_{L^2_{\rm odd}(\mathbb{R})} : L^2_{\rm odd}(\mathbb{R}) \to L^2_{\rm odd}(\mathbb{C}).$$

Then, the kernel of the operator $\mathcal{W}^*_{\rm WH}\Big|_{L^2_{\rm odd}(\mathbb{R})} \mathcal{W}_{\rm WH}\Big|_{L^2_{\rm odd}(\mathbb{C})}$ is given by

$$K^{\rm odd}_{\rm WH}(x, x') = \frac{1}{2}(K_{\rm WH}(x, x') - K_{\rm WH}(x, -x')).$$

Similarly, we have the kernel,

$$K^{\rm even}_{\rm WH}(x, x') = \frac{1}{2}(K_{\rm WH}(x, x') + K_{\rm WH}(x, -x')).$$

When the window function G is given by (4.4), we obtain

$$\begin{aligned}
K^{\rm odd}_{\rm WH}(x, x') &= \frac{e^{\pi i x_{\rm R} x_{\rm I}}}{e^{\pi i x'_{\rm R} x'_{\rm I}}} K^C_{\rm Ginibre}(\sqrt{\pi} x, \sqrt{\pi} x') e^{-\pi(|x|^2+|x'|^2)/2}, \\
K^{\rm even}_{\rm WH}(x, x') &= \frac{e^{\pi i x_{\rm R} x_{\rm I}}}{e^{\pi i x'_{\rm R} x'_{\rm I}}} K^D_{\rm Ginibre}(\sqrt{\pi} x, \sqrt{\pi} x') e^{-\pi(|x|^2+|x'|^2)/2}, \quad x, x' \in \mathbb{C},
\end{aligned}$$

where K^C_{Ginibre} and K^D_{Ginibre} are given by (4.2) and (4.3), respectively.

The Ginibre DPP of type A is extended to *Ginibre-type DPPs* indexed by $q \in \mathbb{N}_0$, which are introduced in [52] and also known as the infinite pure *polyanalytic ensembles* (cf. [2]). Each Ginibre-type DPP with index $q \in \mathbb{N}_0$ is associated with the correlation kernel

$$K^{(q)}_{\text{Ginibre-type}}(x, x') := L^{(0)}_q(|x - x'|^2) K^A_{\text{Ginibre}}(x, x'), \quad x, x' \in \mathbb{C}, \tag{4.5}$$

where $L^{(0)}_q$ is the q-th Laguerre polynomial (3.7) with parameter $\nu = 0$ and K^A_{Ginibre} is defined by (4.1). This DPP can be viewed as the Weyl–Heisenberg ensemble, $(\Xi, K^{h_q}_{\text{WH}}, dx_{\text{R}} dx_{\text{I}})$, with the window function $G(x) = h_q(x)$, $x \in \mathbb{R}$, which is defined using the q-th Hermite polynomial (3.6) as

$$h_q(x) := \frac{2^{-q/2+1/4}}{\sqrt{q!}} e^{-\pi x^2} H_q(\sqrt{2\pi} x), \quad x \in \mathbb{R}, \quad q \in \mathbb{N}_0.$$

Indeed, we see that

$$K^{h_q}_{\text{WH}}(x, x') = \frac{e^{\pi i x_{\text{R}} x_{\text{I}}}}{e^{\pi i x'_{\text{R}} x'_{\text{I}}}} K^{(q)}_{\text{Ginibre-type}}(\sqrt{\pi} x, \sqrt{\pi} x') e^{-\pi(|x|^2 + |x'|^2)/2}, \quad x, x' \in \mathbb{C}, \quad q \in \mathbb{N}_0.$$

See [2] for more details about the Weyl–Heisenberg aspect of finite polyanalytic ensembles. Other examples of the Weyl–Heisenberg ensembles are given in [1, 3, 2].

4.1.3 Representations of Ginibre and Ginibre-type kernels in the Bargmann–Fock space and the eigenspaces of Landau levels

We consider an application of Corollary 2.10. Let $S_1 = \mathbb{C}$ and $S_2 = \mathbb{N}_0$ with $\lambda_1(dx) = \lambda_{\text{N}(0,1;\mathbb{C})}(dx)$. We put

$$\varphi_n(x) := \frac{x^n}{\sqrt{n!}}, \quad n \in \mathbb{N}_0. \tag{4.6}$$

Note that $\{\varphi_n(x)\}_{n \in \mathbb{N}_0}$ forms a complete orthonormal system of the *Bargmann–Fock space*, which is the space of square-integrable analytic functions on $\mathbb{C}$ with respect to the complex Gaussian measure;

$$\langle \varphi_n, \varphi_m \rangle_{L^2(\mathbb{C}, \lambda_{\text{N}(0,1;\mathbb{C})})} = \delta_{nm}, \quad n, m \in \mathbb{N}_0.$$

We assume that $\Gamma = S_2 = \mathbb{N}_0$. We can see that $\|\varphi_\cdot(x)\|_{\ell^2(\Gamma)} = \sum_{n \in \mathbb{N}_0} |x|^{2n}/n! = e^{|x|^2}$, $x \in \mathbb{C}$. Hence Assumption 3' is satisfied. By Corollary 2.10, we obtain the DPP on $\mathbb{C}$ in which the correlation kernel with respect to $\lambda_{\text{N}(0,1:\mathbb{C})}$ is given by

$$\begin{aligned} K_{\text{BF}}(x, x') &= \sum_{n \in \mathbb{N}_0} \varphi_n(x) \overline{\varphi_n(x')} = \sum_{n=0}^{\infty} \frac{(x \overline{x'})^n}{n!} \\ &= e^{x \overline{x'}}, \quad x, x' \in \mathbb{C}. \end{aligned}$$

This is the reproducing kernel in the Bargmann–Fock space and obtained DPP is identified with $(\Xi, K^A_{\text{Ginibre}}, \lambda_{\text{N}(0,1;\mathbb{C})}(dx))$. See [52, 16, 2].

If we set $\Gamma = 2\mathbb{N}_0 + 1 = \{1, 3, 5, \dots\}$ or $\Gamma = 2\mathbb{N}_0 = \{0, 2, 4, \dots\}$, we will obtain the DPPs with the following kernels

$$K_{\mathrm{BF}}^{\mathrm{odd}}(x, x') = \sum_{k=0}^{\infty} \frac{(x\overline{x'})^{2k+1}}{(2k+1)!} = \sinh(x\overline{x'}),$$
$$K_{\mathrm{BF}}^{\mathrm{even}}(x, x') = \sum_{k=0}^{\infty} \frac{(x\overline{x'})^{2k}}{(2k)!} = \cosh(x\overline{x'}), \quad x, x' \in \mathbb{C}.$$

The obtained DPPs are identified with $(\Xi, K_{\mathrm{Ginibre}}^{C}, \lambda_{\mathrm{N}(0,1;\mathbb{C})}(dx))$ and $(\Xi, K_{\mathrm{Ginibre}}^{D}, \lambda_{\mathrm{N}(0,1;\mathbb{C})}(dx))$, respectively.

The correlation kernel of Ginibre-type DPP (4.5) admits the similar representation in terms of the *complex Hermite polynomials* defined by

$$H_{p,q}(\zeta, \overline{\zeta}) := (-1)^{p+q} e^{\zeta\overline{\zeta}} \frac{\partial^p}{\partial \overline{\zeta}^p} \frac{\partial^q}{\partial \zeta^q} e^{-\zeta\overline{\zeta}}, \quad \zeta \in \mathbb{C}, \quad p, q \in \mathbb{N}_0,$$

which were introduced by Itô [30]. We note that their generating function is given by

$$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} H_{p,q}(\zeta, \overline{\zeta}) \frac{s^p t^q}{p! q!} = \exp(\zeta s + \overline{\zeta} t - st)$$

and the set $\{H_{p,q}(\zeta, \overline{\zeta})/\sqrt{p!q!} : p, q \in \mathbb{N}_0\}$ forms a complete orthonormal system of $L^2(\mathbb{C}, \lambda_{\mathrm{N}(0,1;\mathbb{C})}(d\zeta))$. Let $S_1 = \mathbb{C}$ and $S_2 = \mathbb{N}_0$ with $\lambda_1(dx) = \lambda_{\mathrm{N}(0,1;\mathbb{C})}(dx)$, and for fixed $q \in \mathbb{N}_0$, define

$$\varphi_n^{(q)}(x) := \frac{1}{\sqrt{n!q!}} H_{n,q}(x, \overline{x}), \quad x \in \mathbb{C}, \quad n \in \mathbb{N}_0.$$

Then $\{\varphi_n^{(q)}(x)\}_{n \in \mathbb{N}_0}$ forms a complete orthonormal system of the eigenspace corresponding to the q-th *Landau level*, which coincides with the Bargmann–Fock space when $q = 0$. Since the following formula is known

$$L_q^{(0)}(|\zeta - \eta|^2) e^{\zeta\overline{\eta}} = \sum_{p=0}^{\infty} \frac{1}{p!q!} H_{p,q}(\zeta, \overline{\zeta}) \overline{H_{p,q}(\eta, \overline{\eta})}, \quad \zeta, \eta \in \mathbb{C}, \quad q \in \mathbb{N}_0,$$

we obtain the following expansion formula for (4.5),

$$K_{\text{Ginibre-type}}^{(q)}(x, x') = \sum_{n=0}^{\infty} \varphi_n^{(q)}(x) \overline{\varphi_n^{(q)}(x')}, \quad x, x' \in \mathbb{C}, \quad q \in \mathbb{N}_0.$$

The obtained DPPs are identified with $(\Xi, K_{\text{Ginibre-type}}^{(q)}, \lambda_{\mathrm{N}(0,1;\mathbb{C})}(dx))$, $q \in \mathbb{N}_0$ constructed as Weyl–Heisenberg ensembles in Section 4.1.2.

4.1.4 Application of duality relations

We consider the simplified setting (3.3) of W with (4.6) and $\Gamma = \mathbb{N}_0$. If we set $\lambda_1(dx) = \lambda_{\mathrm{N}(0,1;\mathbb{C})}(dx)$, Λ_1 be a disk (i.e., two-dimensional ball) $\mathbb{B}_r^2$ with radius $r \in (0, \infty)$ centered at the origin in $S_1 = \mathbb{C} \simeq \mathbb{R}^2$ and $\Lambda_2 = S_2 = \mathbb{N}_0$ in (2.11), we obtain

$$\begin{aligned} K_{\mathbb{C}}^{(\mathbb{N}_0)}(x, x') &= \sum_{n=0}^{\infty} \varphi_n(x)\overline{\varphi_n(x')} = e^{x\overline{x'}} \\ &= K^A_{\text{Ginibre}}(x, x'), \quad x, x' \in \mathbb{C}, \end{aligned}$$

where K^A_{Ginibre} denotes the correlation kernel of the Ginibre DPP of type A, and

$$\begin{aligned} K_{\mathbb{N}_0}^{(\mathbb{B}_r^2)}(n, n') &= \int_{\mathbb{B}_r^2} \overline{\varphi_n(x)}\varphi_{n'}(x)\lambda_{\mathrm{N}(0,1;\mathbb{C})}(dx) = \frac{1}{\pi\sqrt{n!n'!}} \int_0^r ds\, e^{-s^2} s^{n+n'+1} \int_0^{2\pi} d\theta\, e^{i\theta(n'-n)} \\ &= 2\delta_{nn'} \int_0^r \frac{s^{2n+1}e^{-s^2}}{n!} ds = \delta_{nn'} \int_0^{r^2} \lambda_{\Gamma(n+1,1)}(du), \quad n, n' \in \mathbb{N}_0. \end{aligned}$$

Define

$$\lambda_n(r) := \int_0^{r^2} \frac{u^n e^{-u}}{n!} du = \sum_{k=n+1}^{\infty} \frac{r^{2k}e^{-r^2}}{k!}, \quad n \in \mathbb{N}_0, \quad r \in (0, \infty),$$

where the second equality is due to Eq.(4.1) in [52]. That is, if we write the Gamma distribution with parameters (a, b) as $\Gamma(a, b)$ (see Section 3.1.1) and the Poisson distribution with parameter c as $\mathrm{Po}(c)$,

$$\lambda_n(r) := \mathbf{P}(R_n \le r^2) = \mathbf{P}(Y_{r^2} \ge n + 1),$$

provided $R_n \sim \Gamma(n+1, 1)$ and $Y_{r^2} \sim \mathrm{Po}(r^2)$. Then DPP $(\Xi_2^{(\mathbb{B}_r^2)}, K_{\mathbb{N}_0}^{(\mathbb{B}_r^2)})$ on $\mathbb{N}_0$ is the product measure $\bigotimes_{n\in\mathbb{N}_0} \mu^{\text{Bernoulli}}_{\lambda_n(r)}$ under the natural identification between $\{0, 1\}^{\mathbb{N}_0}$ and the power set of $\mathbb{N}_0$, where $\mu_p^{\text{Bernoulli}}$ denotes the Bernoulli measure of probability $p \in [0, 1]$. Theorem 2.6 gives the duality relation

$$\mathbf{P}(\Xi^A_{\text{Ginibre}}(\mathbb{B}_r^2) = m) = \mathbf{P}(\Xi_2^{(\mathbb{B}_r^2)}(\mathbb{N}_0) = m), \quad \forall m \in \mathbb{N}_0,$$

where we have identified the DPP, $(\Xi_1^{(\mathbb{N}_0)}, K_{\mathbb{C}}^{(\mathbb{N}_0)}, \lambda_1(dx))$ with the Ginibre DPP of type A, $(\Xi^A_{\text{Ginibre}}, K^A_{\text{Ginibre}}, \lambda_{\mathrm{N}(0,1;\mathbb{C})})$. If we introduce a series of random variables $X_n^{(r)} \in \{0, 1\}, n \in \mathbb{N}_0$, which are mutually independent and $X_n^{(r)} \sim \mu^{\text{Bernoulli}}_{\lambda_n(r)}$, $n \in \mathbb{N}_0$, then the above implies the equivalence in probability law

$$\Xi^A_{\text{Ginibre}}(\mathbb{B}_r^2) \stackrel{\text{(law)}}{=} \Xi_2^{(\mathbb{B}_r^2)}(\mathbb{N}_0) \stackrel{\text{(law)}}{=} \sum_{n\in\mathbb{N}_0} X_n^{(r)}, \quad r \in (0, \infty).$$

Similarly, we have the following equalities by the results in Section 4.1.3 and Theorem 2.6,

$$\Xi^C_{\text{Ginibre}}(\mathbb{B}_r^2) \stackrel{\text{(law)}}{=} \sum_{n\in 2\mathbb{N}_0+1} X_n^{(r)}, \quad \Xi^D_{\text{Ginibre}}(\mathbb{B}_r^2) \stackrel{\text{(law)}}{=} \sum_{n\in 2\mathbb{N}_0} X_n^{(r)}, \quad r \in (0, \infty).$$

The argument above is valid for general *radially symmetric DPPs* associated with radially symmetric finite measure $\lambda_1(dx) = p(|x|)dx$ on $\mathbb{C}$. Let $\varphi_n(x) = a_n x^n, n \in \mathbb{N}_0$ be an orthonormal system in $L^2(\mathbb{C}, \lambda_1)$ where $a_n > 0, n \in \mathbb{N}_0$ are the normalization constants, and we set

$$\begin{aligned} K_{\mathbb{C}}^{(\mathbb{N}_0)}(x, x') &= \sum_{n=0}^{\infty} \varphi_n(x)\overline{\varphi_n(x')} = \sum_{n=0}^{\infty} a_n^2 (x\overline{x'})^n \quad x, x' \in \mathbb{C}, \\ K_{\mathbb{N}_0}^{(\mathbb{B}_r^2)}(n, n') &= \int_{\mathbb{B}_r^2} \overline{\varphi_n(x)}\varphi_{n'}(x)\lambda_1(dx) = \delta_{nn'}\lambda_n(r) \quad n, n' \in \mathbb{N}_0, \end{aligned}$$

where

$$\lambda_n(r) := \frac{1}{Z_n}\int_0^{r^2} u^n p(\sqrt{u})du$$

with $Z_n = \int_0^\infty u^n p(\sqrt{u})du$. Then DPP $(\Xi_1^{(\mathbb{N}_0)}, K_{\mathbb{C}}^{(\mathbb{N}_0)}, p(|x|)dx)$ on $\mathbb{C}$ is radially symmetric and DPP $(\Xi_2^{(\mathbb{B}_r^2)}, K_{\mathbb{N}_0}^{(\mathbb{B}_r^2)})$ on $\mathbb{N}_0$ is again identified with the product measure $\bigotimes_{n\in\mathbb{N}_0}\mu_{\lambda_n(r)}^{\mathrm{Bernoulli}}$. For example, if $p(s) = \pi^{-1}e^{-s^2}$ and $a_n = 1/\sqrt{n!}$, then $(\Xi_1^{(\mathbb{N}_0)}, K_{\mathbb{C}}^{(\mathbb{N}_0)}, p(|x|)dx)$ is the Ginibre DPP of type A. The function $\lambda_n(r)$ is considered as a probability distribution function on $[0,\infty)$ and hence there exist independent random variables $R_n, n \in \mathbb{N}_0$ such that

$$\lambda_n(r) = \mathbf{P}(R_n \le r^2).$$

If we define $X_n^{(r)} = \mathbf{1}_{\{R_n \le r^2\}}$ for each $n \in \mathbb{N}_0$, then Theorem 2.6 gives the duality relation

$$\Xi_1^{(\mathbb{N}_0)}(\mathbb{B}_r^2) \stackrel{\text{(law)}}{=} \Xi_2^{(\mathbb{B}_r^2)}(\mathbb{N}_0) \stackrel{\text{(law)}}{=} \sum_{n\in\mathbb{N}_0} X_n^{(r)}, \quad r \in (0,\infty).$$

Indeed, $\{X_n^{(r)}, n \in \mathbb{N}_0\}$ are mutually independent $\{0,1\}$-valued random variables whose laws are given by $\{\mu_{\lambda_n(r)}^{\mathrm{Bernoulli}}, n \in \mathbb{N}_0\}$. If we take a set $\Lambda_2 \subset \mathbb{N}_0$, then DPP $(\Xi_1^{(\Lambda_2)}, K_{\mathbb{C}}^{(\Lambda_2)}, p(|x|)dx)$ satisfies

$$\Xi_1^{(\Lambda_2)}(\mathbb{B}_r^2) \stackrel{\text{(law)}}{=} \Xi_2^{(\mathbb{B}_r^2)}(\Lambda_2) \stackrel{\text{(law)}}{=} \sum_{n\in\Lambda_2} X_n^{(r)}, \quad r \in (0,\infty).$$

We note that if we write $\Xi_1^{(\mathbb{N}_0)} = \sum_j \delta_{X_j}$, then $\sum_j \delta_{|X_j|^2}$ is equal to $\sum_{n\in\mathbb{N}_0}\delta_{R_n}$ in law, which was discussed in Theorem 4.7.1 in [29] by constructing $\{R_n\}_{n\in\mathbb{N}_0}$ in terms of *size-biased sampling*.

4.2 Finite DPPs on sphere $\mathbb{S}^2$

Let $\mathbb{S}^2 := \{x \in \mathbb{R}^3 : \|x\|_{\mathbb{R}^3} = 1\}$ be the two-dimensional unit sphere centered at the origin in the three-dimensional Euclidean space $\mathbb{R}^3$, where $\|\cdot\|_{\mathbb{R}^3}$ denotes the Euclidean distance in $\mathbb{R}^3$. We will use the following coordinates for $x = (x^{(1)}, x^{(2)}, x^{(3)})$ on $\mathbb{S}^2$,

$$x^{(1)} = \sin\theta\cos\varphi, \quad x^{(2)} = \sin\theta\sin\varphi, \quad x^{(3)} = \cos\theta, \quad \theta \in [0,\pi], \quad \varphi \in [0, 2\pi). \tag{4.7}$$

We consider the case that $S_1 = \mathbb{S}^2$ and $S_2 = \mathbb{N}_0$, in which we assume that $\lambda_1(dx)$ is given by the *Lebesgue surface area measure* $d\sigma_2(x)$ on $\mathbb{S}^2$ such that

$$\lambda_1(dx) = d\sigma_2(x) = d\sigma_2(\theta,\varphi) := \sin\theta d\theta d\varphi, \quad \lambda_1(\mathbb{S}^2) = \sigma_2(\mathbb{S}^2) = 4\pi.$$

For $n \in \{0, 1, \dots, N-1\}, N \in \mathbb{N}$, put

$$\varphi_n^{\mathbb{S}^2}(x) = \varphi_n^{\mathbb{S}^2}(\theta,\varphi) := \frac{1}{\sqrt{h_n}}e^{-in\varphi}\sin^n\frac{\theta}{2}\cos^{N-1-n}\frac{\theta}{2}, \quad \theta \in [0,\pi], \quad \varphi \in [0,2\pi), \tag{4.8}$$

with

$$h_n = h_n^{(N)} := \frac{4\pi}{N}\binom{N-1}{n}^{-1}.$$

It is easy to confirm the following orthonormality relations on $\mathbb{S}^2$,

$$\langle \varphi_n^{\mathbb{S}^2}(\cdot), \varphi_m^{\mathbb{S}^2}(\cdot)\rangle_{L^2(\mathbb{S}^2; d\sigma_2)} = \int_0^\pi d\theta \int_0^{2\pi} d\varphi\, \varphi_n^{\mathbb{S}^2}(\theta,\varphi)\overline{\varphi_m^{\mathbb{S}^2}(\theta,\varphi)} d\sigma_2(\theta,\varphi) = \delta_{nm}, \quad n, m \in \mathbb{N}_0.$$

We set $\psi_1(\cdot, n) = \varphi_n^{\mathbb{S}^2}(\cdot)$, $n \in \Gamma := \{0, 1, \dots, N-1\}$, $N \in \mathbb{N}_0$, By the argument given in Remark 3 in Section 3.1.1, we see Assumption 3' is satisfied. Then Corollary 2.10 gives the DPP with N points on $\mathbb{S}^2$, $(\Xi, K_{\mathbb{S}^2}^{(N)}, d\sigma_2(x))$, whose correlation kernel is given by

$$\begin{aligned} K_{\mathbb{S}^2}^{(N)}(x,x') &= K_{\mathbb{S}^2}^{(N)}((\theta,\varphi),(\theta',\varphi')) \\ &= \frac{N}{4\pi}\sum_{n=0}^{N-1}\binom{N-1}{n}\left(e^{-i(\varphi-\varphi')}\sin\frac{\theta}{2}\sin\frac{\theta'}{2}\right)^n \left(\cos\frac{\theta}{2}\cos\frac{\theta'}{2}\right)^{N-1-n} \\ &= \frac{N}{4\pi}\left(e^{-i(\varphi-\varphi')}\sin\frac{\theta}{2}\sin\frac{\theta'}{2} + \cos\frac{\theta}{2}\cos\frac{\theta'}{2}\right)^{N-1}. \end{aligned} \tag{4.9}$$

The density of points with respect to $d\sigma_2(x)$ is given by

$$\rho(x) = K_{\mathbb{S}^2}^{(N)}(x,x) = \frac{N}{4\pi} = \text{constant}, \quad x \in \mathbb{S}^2.$$

For two points $x = (\theta, \varphi)$ and $x' = (\theta', \varphi')$ on $\mathbb{S}^2$,

$$\begin{aligned} \|x - x'\|_{\mathbb{R}^3}^2 &= (\sin\theta\cos\varphi - \sin\theta'\cos\varphi')^2 + (\sin\theta\sin\varphi - \sin\theta'\sin\varphi')^2 + (\cos\theta - \cos\theta')^2 \\ &= |\Phi(x-x')|^2, \end{aligned}$$

with

$$\begin{aligned} \Phi(x-x') &:= 2\left[\sin\frac{\theta-\theta'}{2}\cos\frac{\varphi-\varphi'}{2} - i\sin\frac{\theta+\theta'}{2}\sin\frac{\varphi-\varphi'}{2}\right] \\ &= 2\cos\frac{\theta}{2}\cos\frac{\theta'}{2}e^{i(\varphi+\varphi')/2}\left[e^{-i\varphi}\tan\frac{\theta}{2} - e^{-i\varphi'}\tan\frac{\theta'}{2}\right]. \end{aligned}$$

Then we can show that the probability density of this DPP with respect to $d\sigma_2(\boldsymbol{x}) = \prod_{j=1}^N d\sigma_2(x_j)$ is given as

$$\mathbf{p}_{\mathbb{S}^2}^{(N)}(\boldsymbol{x}) = \frac{1}{Z_{\mathbb{S}^2}^{(N)}}\prod_{1\le j<k\le N}\|x_k - x_j\|_{\mathbb{R}^3}^2,$$

with

$$Z_{\mathbb{S}^2}^{(N)} = \frac{2^{N(N+1)}\pi^N}{(N!)^{N-1}}\left(\prod_{j=1}^N (j-1)!\right)^2.$$

Since $\|x - x'\|_{\mathbb{R}^3}^2 = 2 - 2x\cdot x'$ for $x, x' \in \mathbb{S}^2$, we have the equality

$$\frac{1}{2}(1 + x\cdot x') = \left|e^{-i(\varphi-\varphi')}\sin\frac{\theta}{2}\sin\frac{\theta'}{2} + \cos\frac{\theta}{2}\cos\frac{\theta'}{2}\right|^2.$$

Hence the absolute value of (4.9) is written as

$$\left|K_{\mathbb{S}^2}^{(N)}(x,x')\right| = \frac{N}{4\pi}\left(\frac{1+x\cdot x'}{2}\right)^{(N-1)/2},$$

and hence the two-point correlation function (1.2) with respect to $d\sigma_2(x)$ is given by

$$\rho^2(x,x') = \left(\frac{N}{4\pi}\right)^2 \left[1 - \left(\frac{1+x\cdot x'}{2}\right)^{N-1}\right], \quad x,x' \in \mathbb{S}^2.$$

The system $(\Xi, K_{\mathbb{S}^2}^{(N)}, d\sigma_2(x))$ is uniform and isotropic on $\mathbb{S}^2$, which is called the *spherical ensemble* [39, 21, 4, 11, 12].

Remark 5 Let G_1 and G_2 be $N \times N$ independent random matrices, whose entries are i.i.d. following $\mathrm{N}(0,1;\mathbb{C})$. Krishnapur [39] studied the statistical ensemble of the eigenvalues $\boldsymbol{z} = (z_1, \dots, z_N)$ on $\mathbb{C}$ of $G_1^{-1}G_2$ and proved that it gives the DPP $(\sum_j \delta_{Z_j}, K_{G_1^{-1}G_2}^{(N)}, \lambda(dz))$ with

$$K_{G_1^{-1}G_2}^{(N)}(z,z') = (1+z\overline{z'})^{N-1}, \quad \lambda(dz) = \frac{N}{\pi}\frac{dz}{(1+|z|^2)^{N+1}},$$

which implies that the probability density of $\boldsymbol{z}$ with respect to the Lebesgue measure $d\boldsymbol{z} = \prod_{j=1}^N dz_j$ on $\mathbb{C}$ is given by

$$\mathbf{p}_{G_1^{-1}G_2}^{(N)}(\boldsymbol{z}) = \frac{1}{Z_{G_1^{-1}G_2}^{(N)}} \prod_{1\le j<k\le N} |z_k - z_j|^2 \prod_{\ell=1}^N \frac{1}{(1+|z_\ell|^2)^{N+1}}$$

with a normalization constant $Z_{G_1^{-1}G_2}^{(N)}$. Krishnapur claimed that if we consider the stereographic projection from $\mathbb{S}^2$ to $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ which makes an equatorial plane of $\mathbb{S}^2$, then the DPP, $(\sum_j \delta_{Z_j}, K_{G_1^{-1}G_2}^{(N)}, \lambda(dz))$ is realized as the image of the DPP in the spherical ensemble, $(\Xi, K_{\mathbb{S}^2}^{(N)}, d\sigma_2(x))$ [39]. Actually if we consider the stereographic projection such that the north pole of $\mathbb{S}^2$ $(\theta = 0)$ is mapped to the origin of $\widehat{\mathbb{C}}$ and the south pole of $\mathbb{S}^2$ $(\theta = \pi)$ is to ∞, the image of $x = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \in \mathbb{S}^2$ is given by

$$z = e^{i\varphi}\tan\frac{\theta}{2} \in \widehat{\mathbb{C}}, \quad \theta \in [0,\pi], \quad \varphi \in [0, 2\pi).$$

We see that

$$\|x - x'\|_{\mathbb{R}^3}^2 = \frac{4}{(1+|z|^2)(1+|z'|^2)}|z-z'|^2$$

and

$$d\sigma_2(x) = \frac{4}{(1+|z|^2)^2}dz.$$

Hence we can verify the statement of Krishnapur [39].

The equivalent system with the spherical ensemble of DPP was studied by Caillol [17] as a *two-dimensional one-component plasma model* in physics. It is interesting to see that he used the *Cayley–Klein parameters* defined by

$$\alpha := e^{i\varphi/2}\cos\frac{\theta}{2}, \quad \beta := -ie^{-i\varphi/2}\sin\frac{\theta}{2}, \quad \varphi \in [0, 2\pi), \quad \theta \in [0,\pi].$$

The orthonormal functions (4.8) can be identified with the following up to irrelevant factors,

$$\widetilde{\varphi}_n^{\mathbb{S}^2}(\alpha,\beta)=\frac{1}{\sqrt{h_n}}\alpha^{N-1-n}\beta^n, \quad n\in\{0,1,\dots,N-1\}.$$

If we define

$$\langle(\alpha,\beta),(\alpha',\beta')\rangle_{\mathrm{CK}}:=\alpha\overline{\alpha'}+\beta\overline{\beta'},$$

the correlation kernel (4.9) is written as

$$K_{\mathbb{S}^2}^{(N)}(x,x')=K_{\mathbb{S}^2}^{(N)}((\alpha,\beta),(\alpha',\beta'))=\frac{N}{4\pi}\left(\langle(\alpha,\beta),(\alpha',\beta')\rangle_{\mathrm{CK}}\right)^{N-1}.$$

Following the claim given in [17] (see also Section 15.6.2 in [21]), we consider the vicinity of the north pole, $x_{\mathrm{np}}=(0,0,1)\in\mathbb{R}^3$, that is $\theta\sim 0$. We put

$$\theta=\frac{2r}{\sqrt{N}}, \quad \theta'=\frac{2r'}{\sqrt{N}},$$

and take the limit $N\to\infty$ keeping r and r' be constants. Then in (4.9), we see that

$$\begin{aligned}
\sin\frac{\theta}{2}\sin\frac{\theta'}{2}&\sim\frac{1}{4}\theta\theta'=\frac{rr'}{N},\\
\cos\frac{\theta}{2}\cos\frac{\theta'}{2}&\sim 1-\frac{\theta^2+\theta'^2}{8}=1-\frac{r^2+r'^2}{2N}.
\end{aligned}$$

We set $re^{i\varphi}=z, r'e^{i\varphi'}=z'\in\mathbb{C}$ with $rdrd\varphi=dz$. Then the kernel given by (4.9) multiplied by $d\sigma_2$ has the following limit,

$$\begin{aligned}
&\lim_{N\to\infty}K_{\mathbb{S}^2}^{(N)}((\theta,\varphi),(\theta',\varphi'))d\sigma_2(\theta,\varphi)\Big|_{\theta=2|z|/\sqrt{N},\theta'=2|z'|/\sqrt{N}}\\
&=\lim_{N\to\infty}\frac{N}{4\pi}\left(1+\frac{1}{N}\left\{z\overline{z'}-\frac{|z|^2+|z'|^2}{2}\right\}\right)^N\frac{4}{N}dz\\
&=\frac{1}{\pi}e^{z\overline{z'}-(|z|^2+|z'|^2)/2}dz.
\end{aligned}$$

Since the spherical ensemble is uniform and isotropic on $\mathbb{S}^2$, we obtain the same limiting DPP in the vicinity of any point on $\mathbb{S}^2$. This implies the following limit theorem [17, 35].

Proposition 4.1 *The following weak convergence is established,*

$$\frac{\sqrt{N}}{2}\circ\left(\Xi,K_{\mathbb{S}^2}^{(N)},d\sigma_2(x)\right)\stackrel{N\to\infty}{\Longrightarrow}\left(\Xi,K_{\mathrm{Ginibre}}^{A},\lambda_{\mathrm{N}(0,1;\mathbb{C})}(dx)\right),$$

where the limit point process is the Ginibre DPP of type A given in Section 4.1.

4.3 Finite DPPs on torus $\mathbb{T}^2$

We will consider the finite DPPs on a surface of torus with double periodicity of $2\omega_1:=2\pi$ and $2\omega_3:=2\tau\pi$, where we assume that $\tau=i\Im\tau\in i(0,\infty)$. The surface of such a torus $\mathbb{T}^2=\mathbb{T}^2(2\pi,2\tau\pi):=\mathbb{S}^1(2\pi)\times\mathbb{S}^1(2\pi\Im\tau)$ can be identified with a rectangular domain in $\mathbb{C}$,

$$D_{(2\pi,2\tau\pi)}:=\{z\in\mathbb{C}:0\le\Re z<2\pi,0\le\Im z<2\pi\Im\tau\}\subset\mathbb{C} \quad \text{with double periodicity of } (2\pi,2\tau\pi).$$

So we first consider the systems on $D_{(2\pi,2\tau\pi)}$.

Let $S = \mathbb{C}$ with $\lambda(dx) = \mathbf{1}_{D_{(2\pi,2\tau\pi)}}(x) dx_{\mathrm{R}} dx_{\mathrm{I}}$. For $N \in \mathbb{N}$, put

$$\varphi_n^{R_N,(2\pi,2\tau\pi)}(x) := \frac{e^{-\mathcal{N}^{R_N} i x_{\mathrm{I}}^2/(4\tau\pi)}}{\sqrt{h_n^{R_N}(\tau)}} \Theta^{\sharp(R_N)} \left(\frac{J^{R_N}(n)}{\mathcal{N}^{R_N}}, \mathcal{N}^{R_N} \frac{x}{2\pi}, \mathcal{N}^{R_N}\tau \right), \quad n \in \{1, \dots, N\}.$$

where Θ^R, $\sharp(R_N)$, $\mathcal{N}^{R_N}$, and $J^{R_N}(n)$ are given by (3.13), (3.14), (3.15), and (3.16), respectively, and

$$h_n^{A_{N-1}}(\tau) := 4\pi^2 \sqrt{\frac{\Im\tau}{2\mathcal{N}^{A_{N-1}}}} e^{-2\tau\pi i J^{A_{N-1}}(n)^2/\mathcal{N}^{A_{N-1}}}, \quad n \in \{1, \dots, N\},$$

$$h_n^{R_N}(\tau) := 8\pi^2 \sqrt{\frac{\Im\tau}{2\mathcal{N}^{R_N}}} e^{-2\tau\pi i J^{R_N}(n)^2/\mathcal{N}^{R_N}}, \quad n \in \{1, \dots, N\}, \quad \text{for } R_N = C_N, C_N^\vee, BC_N,$$

$$h_n^{R_N}(\tau) := \begin{cases} 16\pi^2 \sqrt{\dfrac{\Im\tau}{2\mathcal{N}^{R_N}}}, & n = 1, \\[2ex] 8\pi^2 \sqrt{\dfrac{\Im\tau}{2\mathcal{N}^{R_N}}} e^{-2\tau\pi i J^{R_N}(n)^2/\mathcal{N}^{R_N}}, & n \in \{2, 3, \dots, N\}, \end{cases} \quad \text{for } R_N = B_N, B_N^\vee,$$

$$h_n^{D_N}(\tau) := \begin{cases} 16\pi^2 \sqrt{\dfrac{\Im\tau}{2\mathcal{N}^{R_N}}} e^{-2\tau\pi i J^{D_N}(n)^2/\mathcal{N}^{D_N}}, & n \in \{1, N\}, \\[2ex] 8\pi^2 \sqrt{\dfrac{\Im\tau}{2\mathcal{N}^{R_N}}} e^{-2\tau\pi i J^{D_N}(n)^2/\mathcal{N}^{D_N}}, & n \in \{2, 3, \dots, N-1\}. \end{cases}$$

The following orthonormality relations were proved in [34],

$$\langle \varphi_n^{R_N,(2\pi,2\tau\pi)}, \varphi_m^{R_N,(2\pi,2\tau\pi)} \rangle_{L^2(\mathbb{C}, \mathbf{1}_{D_{(2\pi,2\tau\pi)}}(x)dx)} = \delta_{nm}, \quad n, m \in \Gamma := \{1, \dots, N\},$$

$R_N = A_{N-1}, B_N, B_N^\vee, C_N, C_N^\vee, BC_N, D_N$. By the argument given in Remark 3 in Section 3.1.1, we see Assumption 3 is satisfied. Then Corollary 2.9 gives the seven types of DPPs with the correlation kernels,

$$K^{R_N,(2\pi,2\tau\pi)}(x, x') = \sum_{n=1}^{N} \varphi_n^{R_N,(2\pi,2\tau\pi)}(x) \overline{\varphi_n^{R_N,(2\pi,2\tau\pi)}(x')}, \tag{4.10}$$

with respect to the measure $\lambda(dx) = \mathbf{1}_{D_{(2\pi,2\tau\pi)}} dx$ on $\mathbb{C}$ for $R_N = A_{N-1}, B_N, B_N^\vee, C_N, C_N^\vee, BC_N, D_N$.

Using the quasi-periodicity of the Jacobi theta functions (B.3) and (B.4), we can show that the correlation kernels are quasi-double-periodic as [34],

$$\begin{aligned} K^{R_N,(2\pi,2\tau\pi)}(x + 2\pi, x') &= K^{R_N,(2\pi,2\tau\pi)}(x, x' + 2\pi) \\ &= \begin{cases} (-1)^{\mathcal{N}^{A_{N-1}}} K^{R_N,(2\pi,2\tau\pi)}(x, x'), & R_N = A_{N-1}, \\ -K^{R_N,(2\pi,2\tau\pi)}(x, x'), & R_N = B_N, C_N^\vee, BC_N, \\ K^{R_N,(2\pi,2\tau\pi)}(x, x'), & R_N = B_N^\vee, C_N, D_N, \end{cases} \end{aligned}$$

$$K^{R_N,(2\pi,2\tau\pi)}(x + 2\tau\pi, x') = \begin{cases} e^{-\mathcal{N}^{R_N} i x_{\mathrm{R}}} K^{R_N,(2\pi,2\tau\pi)}(x, x'), & R_N = A_{N-1}, C_N, C_N^\vee, BC_N, D_N, \\ -e^{-\mathcal{N}^{R_N} i x_{\mathrm{R}}} K^{R_N,(2\pi,2\tau\pi)}(x, x'), & R_N = B_N, B_N^\vee, \end{cases}$$

$$K^{R_N,(2\pi,2\tau\pi)}(x, x' + 2\tau\pi) = \begin{cases} e^{\mathcal{N}^{R_N} i x'_{\mathrm{R}}} K^{R_N,(2\pi,2\tau\pi)}(x, x'), & R_N = A_{N-1}, C_N, C_N^\vee, BC_N, D_N, \\ -e^{\mathcal{N}^{R_N} i x'_{\mathrm{R}}} K^{R_N,(2\pi,2\tau\pi)}(x, x'), & R_N = B_N, B_N^\vee. \end{cases}$$

The above implies the following double periodicity (up to an irrelevant gauge transformation),

$$\begin{aligned}\mathcal{S}_{2\pi}K^{R_N,(2\pi,2\tau\pi)}(x,x') &= \frac{e^{\mathcal{N}^{R_N} i x_{\mathrm{R}}}}{e^{\mathcal{N}^{R_N} i x'_{\mathrm{R}}}}\mathcal{S}_{2\tau\pi}K^{R_N,(2\pi,2\tau\pi)}(x,x') \\ &= K^{R_N,(2\pi,2\tau\pi)}(x,x'), \quad x,x' \in D_{(2\pi,2\tau\pi)}.\end{aligned}$$

In other words, we have obtained the seven types of DPPs with a finite number of points N on a surface of torus $\mathbb{T}^2(2\pi, 2\tau\pi)$. Hence here we write them as $\left(\Xi, K^{R_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}, dx\right)$, $R_N = A_{N-1}$, B_N, $B_N^\vee$, C_N, $C_N^\vee$, BC_N, D_N. Using the *Macdonald denominator formula* given by Rosengren and Schlosser [51] (see (2.6) in [34] in the present notations), the probability densities for these finite DPPs with respect to the Lebesgue measures, $d\boldsymbol{x} = \prod_{j=1}^N dx_j$ are given as follows;

$$\begin{aligned}\mathbf{p}^{A_{N-1}}_{\mathbb{T}^2(2\pi,2\tau\pi)}(\boldsymbol{x}) &= \frac{1}{Z^{A_{N-1}}_{\mathbb{T}^2(2\pi,2\tau\pi)}} \exp\left(-\frac{\mathcal{N}^{A_{N-1}}}{2\pi\Im\tau}\sum_{j=1}^N (x_j)_{\mathrm{I}}^2\right) \\ &\quad\times\begin{cases}\left|\vartheta_0\left(\displaystyle\sum_{k=1}^N \frac{x_k}{2\pi};\tau\right) W^{A_{N-1}}\left(\frac{\boldsymbol{x}}{2\pi};\tau\right)\right|^2, & \text{if } N \text{ is even},\\ \left|\vartheta_3\left(\displaystyle\sum_{k=1}^N \frac{x_k}{2\pi};\tau\right) W^{A_{N-1}}\left(\frac{\boldsymbol{x}}{2\pi};\tau\right)\right|^2, & \text{if } N \text{ is odd},\end{cases} \\ \mathbf{p}^{R_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}(x) &= \frac{1}{Z^{R_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}} \exp\left(-\frac{\mathcal{N}^{R_N}}{2\pi\Im\tau}\sum_{j=1}^N (x_j)_{\mathrm{I}}^2\right)\left|W^{R_N}\left(\frac{\boldsymbol{x}}{2\pi};\tau\right)\right|^2, \\ &\quad R_N = B_N, B_N^\vee, C_N, C_N^\vee, BC_N, D_N,\end{aligned} \tag{4.11}$$

for $\boldsymbol{x} \in (\mathbb{T}^2(2\pi, 2\tau\pi))^N$, where W^{R_N} are the Macdonald denominators given by (C.1) in Appendix C and $Z^{R_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}$ are normalization constants [34].

We can prove the following symmetry properties for the present DPPs on $\mathbb{T}^2(2\pi, 2\tau\pi)$.

Proposition 4.2 (i) *The finite DPPs* $\left(\Xi, K^{R_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}, dx\right)$ *with* $\tau = i\Im\tau \in i(0,\infty)$ *have the following shift invariance,*

$$\begin{aligned}&\mathcal{S}_{2\pi/N}(\Xi, K^{A_{N-1}}_{\mathbb{T}^2(2\pi,2\tau\pi)}, dx) \stackrel{\text{(law)}}{=} (\Xi, K^{A_{N-1}}_{\mathbb{T}^2(2\pi,2\tau\pi)}, dx),\\ &\mathcal{S}_{2\tau\pi/N}(\Xi, K^{A_{N-1}}_{\mathbb{T}^2(2\pi,2\tau\pi)}, dx) \stackrel{\text{(law)}}{=} (\Xi, K^{A_{N-1}}_{\mathbb{T}^2(2\pi,2\tau\pi)}, dx),\\ &\mathcal{S}_{\pi}(\Xi, K^{R_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}, dx) \stackrel{\text{(law)}}{=} (\Xi, K^{R_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}, dx), \quad R_N = B_N^\vee, C_N, D_N,\\ &\mathcal{S}_{\tau\pi}(\Xi, K^{R_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}, dx) \stackrel{\text{(law)}}{=} (\Xi, K^{R_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}, dx), \quad R_N = C_N, C_N^\vee, BC_N, D_N.\end{aligned}$$

(ii) *The densities of points* $\rho^{R_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}(x)$ *given by* $K^{R_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}(x,x)$ *have the following zeros,*

$$
\begin{aligned}
&\rho^{B_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}(0) = 0,\\
&\rho^{B_N^\vee}_{\mathbb{T}^2(2\pi,2\tau\pi)}(0) = \rho^{B_N^\vee}_{\mathbb{T}^2(2\pi,2\tau\pi)}(\pi) = 0,\\
&\rho^{R_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}(0) = \rho^{R_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}(\tau\pi) = 0, \quad R_N = C_N^\vee, BC_N,\\
&\rho^{C_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}(0) = \rho^{C_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}(\pi) = \rho^{C_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}(\tau\pi) = 0.
\end{aligned}
$$

Proof (i) It is easy to verify the statements if we use the formulas (4.11) with (C.1) for the probability densities. Use the formulas (B.3)–(B.5), which show the change of values of $\vartheta_\mu(v;\tau)$, $\mu = 0,1,3$, due to the shift of variable $v \to v+1, v \to v+\tau$ and $v \to v+\tau/2$, respectively. For the shift $\mathcal{S}_{\tau\pi}$, note the fact that

$$
\mathcal{S}_{\tau\pi} \exp\left(-\frac{\mathcal{N}^{R_N}}{2\pi\Im\tau}\sum_{j=1}^N (x_j)_{\rm I}^2\right) = \exp\left(-\frac{\mathcal{N}^{R_N}}{2\pi\Im\tau}\sum_{j=1}^N (x_j)_{\rm I}^2\right)\prod_{\ell=1}^N e^{2\pi i\mathcal{N}^{R_N}(ix_\ell/2\pi+\tau/4)}.
$$

As a matter of course, the statements can be proved also by showing the shift invariance of the correlation kernels (4.10) up to irrelevant gauge transformations. (ii) By the properties (B.6) of the Jacobi theta functions, the zeros of densities $\rho^{R_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}(x)$ are determined as above. Then the proof is complete. ∎

We note that the periods $2\pi/N \in (0,\infty)$ and $2\tau\pi/N \in i(0,\infty)$ of $(\Xi, K^{A_{N-1}}_{\mathbb{T}^2(2\pi,2\tau\pi)}, dx)$ shown by Proposition 4.2 (i) become zeros as $N \to \infty$. Hence, as the $N \to \infty$ limit of $(\Xi, K^{A_{N-1}}_{\mathbb{T}^2(2\pi,2\tau\pi)}, dx)$, it is expected to obtain a uniform system of infinite number of points on $\mathbb{C}$. Actually we can prove the following limit theorems.

Proposition 4.3 *The following weak convergence is established,*

$$
\begin{aligned}
&\frac{1}{2}\sqrt{\frac{N}{\pi\Im\tau}} \circ \left(\Xi, K^{A_{N-1}}_{\mathbb{T}^2(2\pi,2\tau\pi)}, dx\right) \stackrel{N\to\infty}{\Longrightarrow} \left(\Xi, K^{A}_{\rm Ginibre}, \lambda_{{\rm N}(0,1;\mathbb{C})}(dx)\right),\\
&\sqrt{\frac{N}{2\pi\Im\tau}} \circ \left(\Xi, K^{R_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}, dx\right) \stackrel{N\to\infty}{\Longrightarrow} \left(\Xi, K^{C}_{\rm Ginibre}, \lambda_{{\rm N}(0,1;\mathbb{C})}(dx)\right), \quad R_N = B_N, B_N^\vee, C_N, C_N^\vee, BC_N,\\
&\sqrt{\frac{N}{2\pi\Im\tau}} \circ \left(\Xi, K^{D_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}, dx\right) \stackrel{N\to\infty}{\Longrightarrow} \left(\Xi, K^{D}_{\rm Ginibre}, \lambda_{{\rm N}(0,1;\mathbb{C})}(dx)\right),
\end{aligned}
$$

where the limit point processes are the three types of Ginibre DPPs given in Section 4.1.

Proof By (B.7), we see that

$$
\begin{aligned}
\varphi_n^{A_{N-1},(2\pi,2\tau\pi)}\left(2\sqrt{\frac{\pi\Im\tau}{N}}x\right) &\sim N^{1/4}\frac{1}{2\pi}\left(\frac{2}{\Im\tau}\right)^{1/4} e^{-N\pi\Im\tau/4 - i\sqrt{N\pi\Im\tau}x - x_{\rm I}^2}\\
&\quad\times \exp\left[-\pi\Im\tau\left(\frac{n-1/2}{\sqrt{N}}\right)^2 + (2i\sqrt{\pi\Im\tau}x + \sqrt{N}\pi\Im\tau)\frac{n-1/2}{\sqrt{N}}\right],
\end{aligned}
$$

as $N \to \infty$ with $(n-1/2)/\sqrt{N} = \mathrm{O}(1) > 0$. Then

$$
\begin{aligned}
&K^{A_{N-1}}_{\mathbb{T}^2(2\pi,2\tau\pi)}\left(2\sqrt{\frac{\pi\Im\tau}{N}}x, 2\sqrt{\frac{\pi\Im\tau}{N}}x'\right)\left(2\sqrt{\frac{\pi\Im\tau}{N}}\right)^2 \\
&\quad\sim \frac{1}{\pi}\sqrt{2\Im\tau}e^{-N\pi\Im\tau/2-i\sqrt{N\pi\Im\tau}(x-\overline{x'})}e^{-(x_{\mathrm{I}}^2+x_{\mathrm{I}}'^2)}\int_0^\infty e^{-2\pi\Im\tau u^2+\{2i\sqrt{\pi\Im\tau}(x-\overline{x'})+2\sqrt{N}\pi\Im\tau\}u}du \\
&\quad= \frac{1}{\pi}\sqrt{2\Im\tau}e^{-(x_{\mathrm{I}}^2+x_{\mathrm{I}}'^2)-(x-\overline{x'})^2/2}\int_{-\sqrt{N}/2-i(x-\overline{x'})/\{2\sqrt{\pi\Im\tau}\}}^\infty e^{-2\pi\Im\tau v^2}dv \\
&\quad\to \frac{1}{\pi}e^{-(x_{\mathrm{I}}^2+x_{\mathrm{I}}'^2)-(x-\overline{x'})^2/2} \quad \text{as } N \to \infty.
\end{aligned}
$$

It implies that

$$
\lim_{N\to\infty} K^{A_{N-1}}_{\mathbb{T}^2(2\pi,2\tau\pi)}\left(2\sqrt{\frac{\pi\Im\tau}{N}}x, 2\sqrt{\frac{\pi\Im\tau}{N}}x'\right)\left(2\sqrt{\frac{\pi\Im\tau}{N}}\right)^2 = \frac{e^{-ix_{\mathrm{R}}x_{\mathrm{I}}}}{e^{-ix'_{\mathrm{R}}x'_{\mathrm{I}}}}K^{A}_{\mathrm{Ginibre}}(x,x')\frac{1}{\pi}e^{-(|x|^2+|x'|^2)/2}.
$$

Similarly, we can show that

$$
\lim_{N\to\infty} K^{R_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}\left(\sqrt{\frac{2\pi\Im\tau}{N}}x, \sqrt{\frac{2\pi\Im\tau}{N}}x'\right)\left(\sqrt{\frac{2\pi\Im\tau}{N}}\right)^2 = \frac{e^{-ix_{\mathrm{R}}x_{\mathrm{I}}}}{e^{-ix'_{\mathrm{R}}x'_{\mathrm{I}}}}K^{C}_{\mathrm{Ginibre}}(x,x')\frac{1}{\pi}e^{-(|x|^2+|x'|^2)/2},
$$

for $R_N = B_N, B_N^\vee, C_N, C_N^\vee, BC_N$, and

$$
\lim_{N\to\infty} K^{D_N}_{\mathbb{T}^2(2\pi,2\tau\pi)}\left(\sqrt{\frac{2\pi\Im\tau}{N}}x, \sqrt{\frac{2\pi\Im\tau}{N}}x'\right)\left(\sqrt{\frac{2\pi\Im\tau}{N}}\right)^2 = \frac{e^{-ix_{\mathrm{R}}x_{\mathrm{I}}}}{e^{-ix'_{\mathrm{R}}x'_{\mathrm{I}}}}K^{D}_{\mathrm{Ginibre}}(x,x')\frac{1}{\pi}e^{-(|x|^2+|x'|^2)/2}.
$$

Then the statement will be proved. ∎

4.4 Finite and infinite DPPs on cylinder $\mathbb{R} \times \mathbb{S}^1$

Here we consider the finite DPPs on a surface of cylinder with infinite length having periodicity of $2\pi\alpha$ in the circumference direction, $\alpha \in (0,\infty)$, which we write here as $\mathbb{R} \times \mathbb{S}^1(2\pi\alpha)$. The surface of $\mathbb{R} \times \mathbb{S}^1(2\pi\alpha)$ can be identified with a strip with width $2\pi\alpha$ in $\mathbb{C}$,

$$
D_{2\pi\alpha} := \{z \in \mathbb{C} : 0 \le \Im z < 2\pi\alpha\} \subset \mathbb{C} \quad \text{with periodicity of } 2\pi i\alpha.
$$

So we first consider the systems on $D_{2\pi\alpha}$. Let $S = \mathbb{C}$. For $N \in \mathbb{N}$, we set

$$
\lambda(dx) = \lambda_{\mathrm{N}(0,1/4)}(dx_{\mathrm{R}})\lambda_{[0,2\pi\alpha)}(dx_{\mathrm{I}}) = \frac{1}{\sqrt{2}\pi^{3/2}\alpha}e^{-2x_{\mathrm{R}}^2}\mathbf{1}_{D_{2\pi\alpha}}(x)dx_{\mathrm{R}}dx_{\mathrm{I}}.
$$

Define

$$
\begin{aligned}
\varphi_n^{A_{N-1},2\pi\alpha}(x) &:= e^{-[(\mathcal{N}^{A_{N-1}}-2J^{A_{N-1}}(n))^2/(16\alpha^2)+(\mathcal{N}^{A_{N-1}}-2J^{A_{N-1}}(n))x/(2\alpha)]}, \quad n \in \{1,\dots,N\}, \\
\varphi_n^{R_N,2\pi\alpha}(x) &:= \sqrt{2}e^{-(\mathcal{N}^{R_N}-2J^{R_N}(n))^2/(16\alpha^2)}\sinh\left[(\mathcal{N}^{R_N}-2J^{R_N}(n))\frac{x}{2\alpha}\right], \quad n \in \{1,\dots,N\}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } R_N = B_N, C_N, \\
\varphi_n^{D_N,2\pi\alpha}(x) &:= \begin{cases} \sqrt{2}e^{-(\mathcal{N}^{D_N}-2J^{D_N}(n))^2/(16\alpha^2)}\cosh\left[(\mathcal{N}^{D_N}-2J^{D_N}(n))\frac{x}{2\alpha}\right], & n \in \{1,\dots,N-1\}, \\ 1, & n = N, \end{cases}
\end{aligned}
$$

where $\mathcal{N}^{R_N}$ and $J^{R_N}(n)$, $R_N = A_{N-1}, B_N, C_N, D_N$ are given by (3.11) and (3.12), respectively. They have periodicity or quasi-periodicity of $2\pi i\alpha$,

$$\varphi_n^{R_N,2\pi\alpha}(x+2\pi i\alpha) = \begin{cases} (-1)^{N+1}\varphi_n^{R_N,2\pi\alpha}(x), & R_N = A_{N-1}, \\ -\varphi_n^{R_N,2\pi\alpha}(x), & R_N = B_N, \\ \varphi_n^{R_N,2\pi\alpha}(x), & R_N = C_N, D_N. \end{cases} \tag{4.12}$$

It is easy to verify the following orthonormality relations; for $R_N = A_{N-1}, B_N, C_N, D_N$,

$$\langle \varphi_n^{R_N,2\pi\alpha}, \varphi_m^{R_N,w\pi\alpha} \rangle_{L^2(\mathbb{C}, \lambda_{\mathrm{N}(0,1/4)}(dx_{\mathrm{R}})\lambda_{[0,2\pi\alpha)}(dx_{\mathrm{I}}))} = \delta_{nm}, \quad n, m \in \Gamma := \{1, \dots, N\}.$$

By the argument given in Remark 3 in Section 3.1.1, we see Assumption 3' is satisfied. Then Corollary 2.9 gives the following four types of DPPs with the correlation kernels,

$$K^{R_N,2\pi\alpha}(x,x') = \sum_{n=1}^{N} \varphi_n^{R_N,2\pi\alpha}(x)\overline{\varphi_n^{R_N,2\pi\alpha}(x')}, \quad R_N = A_{N-1}, B_N, C_N, D_N.$$

From (4.12), we can see the periodicity or quasi-periodicity of $2\pi i\alpha$ in the correlation kernels,

$$K^{R_N,2\pi\alpha}(x+2\pi i\alpha, x') = K^{R_N,2\pi\alpha}(x, x'+2\pi i\alpha) = \begin{cases} (-1)^{N+1}K^{R_N,2\pi\alpha}(x,x'), & R_N = A_{N-1}, \\ -K^{R_N,2\pi\alpha}(x,x'), & R_N = B_N, \\ K^{R_N,2\pi\alpha}(x,x'), & R_N = C_N, D_N, \end{cases}$$

which implies

$$\mathcal{S}_{2\pi i\alpha} K^{R_N,2\pi\alpha}(x,x') = K^{R_N,2\pi\alpha}(x,x'), \quad x, x' \in D_{2\pi\alpha}.$$

That is, we have obtained the four types of DPPs with N points on a surface of cylinder, $\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)$. Hence here we write them as $\left(\Xi, K^{R_N}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}, \lambda_{\mathrm{N}(0,1/4)}(dx_{\mathrm{R}})\lambda_{[0,2\pi\alpha)}(dx_{\mathrm{I}})\right)$, $R_N = A_{N-1}$, B_N, C_N, D_N. (See [19] for related systems in two dimensions.)

By Lemma A.1 in Appendix A, the probability densities for these finite DPPs with respect to the Lebesgue measures, $d\boldsymbol{x} = \prod_{j=1}^N dx_j$ are given as follows; for $\boldsymbol{x} \in (\mathbb{R}\times\mathbb{S}^1(2\pi\alpha))^N$, $\alpha \in (0,\infty)$,

$$\begin{aligned}
\mathbf{p}_\alpha^{A_{N-1}}(\boldsymbol{x}) &= \frac{1}{Z_\alpha^{A_{N-1}}} \prod_{1\le j<k\le N} \sinh^2 \frac{x_k - x_j}{2\alpha} \prod_{\ell=1}^{N} e^{-2(x_\ell)_{\mathrm{R}}^2} \\
\mathbf{p}_\alpha^{B_N}(\boldsymbol{x}) &= \frac{1}{Z_\alpha^{B_N}} \prod_{1\le j<k\le N} \left(\sinh^2 \frac{x_k - x_j}{2\alpha} \sinh^2 \frac{x_k + x_j}{2\alpha}\right) \prod_{\ell=1}^{N} \left(\sinh^2 \frac{x_\ell}{2\alpha} e^{-2(x_\ell)_{\mathrm{R}}^2}\right), \\
\mathbf{p}_\alpha^{C_N}(\boldsymbol{x}) &= \frac{1}{Z_\alpha^{C_N}} \prod_{1\le j<k\le N} \left(\sinh^2 \frac{x_k - x_j}{2\alpha} \sinh^2 \frac{x_k + x_j}{2\alpha}\right) \prod_{\ell=1}^{N} \left(\sinh^2 \frac{x_\ell}{\alpha} e^{-2(x_\ell)_{\mathrm{R}}^2}\right), \\
\mathbf{p}_\alpha^{D_N}(\boldsymbol{x}) &= \frac{1}{Z_\alpha^{D_N}} \prod_{1\le j<k\le N} \left(\sinh^2 \frac{x_k - x_j}{2\alpha} \sinh^2 \frac{x_k + x_j}{2\alpha}\right) \prod_{\ell=1}^{N} e^{-2(x_\ell)_{\mathrm{R}}^2},
\end{aligned} \tag{4.13}$$

with normalization constants $Z_\alpha^{R_N}$.

If we use the formulas (4.13), it is easy to verify the following symmetry properties.

Proposition 4.4 (i) *The infinite DPPs* $\Big(\Xi, K^{R_N}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}, \lambda_{\mathrm{N}(0,1/4)}(dx_{\mathrm{R}})\lambda_{[0,2\pi\alpha)}(dx_{\mathrm{I}})\Big)$ *with* $\alpha \in (0,\infty)$ *have the following shift invariance,*

$$\begin{aligned}
&\mathcal{S}_{i\theta}\Big(\Xi, K^{A_{N-1}}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}, \lambda_{\mathrm{N}(0,1/4)}(dx_{\mathrm{R}})\lambda_{[0,2\pi\alpha)}(dx_{\mathrm{I}})\Big) \\
&\qquad \stackrel{\text{(law)}}{=} \Big(\Xi, K^{A_{N-1}}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}, \lambda_{\mathrm{N}(0,1/4)}(dx_{\mathrm{R}})\lambda_{[0,2\pi\alpha)}(dx_{\mathrm{I}})\Big), \quad \forall\theta \in [0, 2\pi\alpha), \\
&\mathcal{S}_{i\pi\alpha}\Big(\Xi, K^{R_N}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}, \lambda_{\mathrm{N}(0,1/4)}(dx_{\mathrm{R}})\lambda_{[0,2\pi\alpha)}(dx_{\mathrm{I}})\Big) \\
&\qquad \stackrel{\text{(law)}}{=} \Big(\Xi, K^{R_N}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}, \lambda_{\mathrm{N}(0,1/4)}(dx_{\mathrm{R}})\lambda_{[0,2\pi\alpha)}(dx_{\mathrm{I}})\Big), \quad R_N = C_N, D_N.
\end{aligned}$$

(ii) *The densities of points* $\rho^{R_N}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}(x)$ *given by* $K^{R_N}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}(x,x)$ *have the following zeros,*

$$\rho^{B_N}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}(0) = 0, \quad \rho^{C_N}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}(0) = \rho^{C_N}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}(i\pi\alpha) = 0.$$

Using the Jacobi theta functions (B.1), the limits of the correlation kernels in $N \to \infty$ can be expressed as follows,

$$\begin{aligned}
\lim_{\ell\to\infty} K^{A_{2\ell-1}}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}(x,x') &= \vartheta_2\left(\frac{i(x+\overline{x'})}{2\pi\alpha}; \frac{i}{2\pi\alpha^2}\right) =: K^{A,\mathrm{even}}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}(x,x'), \\
\lim_{\ell\to\infty} K^{A_{2\ell}}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}(x,x') &= \vartheta_3\left(\frac{i(x+\overline{x'})}{2\pi\alpha}; \frac{i}{2\pi\alpha^2}\right) =: K^{A,\mathrm{odd}}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}(x,x'), \\
\lim_{N\to\infty} K^{B_N}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}(x,x') &= \frac{1}{2}\left\{\vartheta_2\left(\frac{i(x+\overline{x'})}{2\pi\alpha}; \frac{i}{2\pi\alpha^2}\right) - \vartheta_2\left(\frac{i(x-\overline{x'})}{2\pi\alpha}; \frac{i}{2\pi\alpha^2}\right)\right\}, \\
&=: K^{B}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}(x,x') \\
\lim_{N\to\infty} K^{C_N}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}(x,x') &= \frac{1}{2}\left\{\vartheta_3\left(\frac{i(x+\overline{x'})}{2\pi\alpha}; \frac{i}{2\pi\alpha^2}\right) - \vartheta_3\left(\frac{i(x-\overline{x'})}{2\pi\alpha}; \frac{i}{2\pi\alpha^2}\right)\right\} \\
&=: K^{C}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}(x,x') \\
\lim_{N\to\infty} K^{D_N}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}(x,x') &= \frac{1}{2}\left\{\vartheta_3\left(\frac{i(x+\overline{x'})}{2\pi\alpha}; \frac{i}{2\pi\alpha^2}\right) + \vartheta_3\left(\frac{i(x-\overline{x'})}{2\pi\alpha}; \frac{i}{2\pi\alpha^2}\right)\right\} \\
&=: K^{D}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}(x,x'),
\end{aligned} \tag{4.14}$$

$x, x' \in \mathbb{R}\times\mathbb{S}^1(2\pi\alpha)$.

Proposition 4.5 *The five limit kernels* (4.14) *define the five kinds of infinite DPPs on the cylinder,* $(\Xi, K^{A,\sharp}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}, \lambda_{\mathrm{N}(0,1/4)}(dx_{\mathrm{R}})\lambda_{[0,2\pi\alpha)}(dx_{\mathrm{I}}))$, $\sharp =$*even, odd, and* $(\Xi, K^{R}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}, \lambda_{\mathrm{N}(0,1/4)}(dx_{\mathrm{R}}) \lambda_{[0,2\pi\alpha)}(dx_{\mathrm{I}}))$, $R = B, C, D$.

The particle densities at $x \in D_{2\pi\alpha} \simeq \mathbb{R}\times\mathbb{S}^1(2\pi\alpha)$ are obtained from the limit kernels (4.14) by setting $x' = x$. Since $x + \overline{x} = 2x_{\mathrm{R}}$, $x - \overline{x} = 2ix_{\mathrm{I}}$, the definitions of the Jacobi theta functions (B.1) with parity (B.2) give explicit expressions for them; for instance,

$$\begin{aligned}
\rho^B_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}(x) &= \frac{1}{2}\left\{\vartheta_2\left(\frac{ix_{\rm R}}{\pi\alpha};\frac{i}{2\pi\alpha^2}\right)-\vartheta_2\left(\frac{x_{\rm I}}{2\pi\alpha};\frac{i}{2\pi\alpha^2}\right)\right\} \\
&= \sum_{n=1}^{\infty} e^{-(n-1/2)^2/2\alpha^2}\Big[\cosh\{(2n-1)x_{\rm R}/\alpha\}-\cos\{(2n-1)x_{\rm I}/\alpha\}\Big], \\
\rho^C_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}(x) &= \frac{1}{2}\left\{\vartheta_3\left(\frac{ix_{\rm R}}{\pi\alpha};\frac{i}{2\pi\alpha^2}\right)-\vartheta_3\left(\frac{x_{\rm I}}{2\pi\alpha};\frac{i}{2\pi\alpha^2}\right)\right\} \\
&= \sum_{n=1}^{\infty} e^{-n^2/2\alpha^2}\Big[\cosh(2nx_{\rm R}/\alpha)-\cos(2nx_{\rm I}/\alpha)\Big].
\end{aligned}$$

They show that the obtained particle densities are indeed nonnegative, and that

$$\rho^B_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}(0)=0, \quad \rho^C_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}(0)=\rho^C_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}(i\pi\alpha)=0.$$

Remark 6 In [34], the infinite DPPs on a strip in $\mathbb{C}$ were introduced by taking an anisotropic scaling limit associated with $N\to\infty$ of the doubly periodic DPPs (Theorem 3.4 in [34]). There the limiting correlation kernels are expressed by the integrals of products of Jacobi's theta functions. We have found that, if we correctly perform Jacobi's imaginary transformations (B.8) of the integrands, the integrals can be calculated and the results are identified with the correlation kernels simply given by (4.14).

Using the quasi-periodicity of the Jacobi theta functions (B.4), we can show that, for $\sharp=$ even, odd,

$$\mathcal{S}_{1/(2\alpha)}K^{A,\sharp}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}(x,x')\sqrt{\frac{2}{\pi}}e^{-(x_{\rm R}^2+{x'_{\rm R}}^2)}\frac{1}{2\pi\alpha}=\frac{e^{ix_{\rm I}/\alpha}}{e^{ix'_{\rm I}/\alpha}}K^{A,\sharp}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}(x,x')\sqrt{\frac{2}{\pi}}e^{-(x_{\rm R}^2+{x'_{\rm R}}^2)}\frac{1}{2\pi\alpha}.$$

By the gauge invariance, this implies the shift invariance,

$$\begin{aligned}
&\mathcal{S}_{1/(2\alpha)}\left(\Xi, K^{A,\sharp}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}, \lambda_{{\rm N}(0,1/4)}(dx_{\rm R})\lambda_{[0,2\pi\alpha)}(dx_{\rm I})\right) \\
&\qquad \stackrel{\rm (law)}{=} \left(\Xi, K^{A,\sharp}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}, \lambda_{{\rm N}(0,1/4)}(dx_{\rm R})\lambda_{[0,2\pi\alpha)}(dx_{\rm I})\right),
\end{aligned} \tag{4.15}$$

$\sharp=$ even, odd. Moreover, by the properties (B.5) of the Jacobi theta functions, we obtain the equality

$$\mathcal{S}_{1/(4\alpha)}K^{A,{\rm even}}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}(x,x')\sqrt{\frac{2}{\pi}}e^{-(x_{\rm R}^2+{x'_{\rm R}}^2)}\frac{1}{2\pi\alpha}=\frac{e^{ix_{\rm I}/(2\alpha)}}{e^{ix'_{\rm I}/(2\alpha)}}K^{A,{\rm odd}}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}(x,x')\sqrt{\frac{2}{\pi}}e^{-(x_{\rm R}^2+{x'_{\rm R}}^2)}\frac{1}{2\pi\alpha}.$$

Hence again by the gauge invariance,

$$\mathcal{S}_{1/(4\alpha)}\left(\Xi, K^{A,{\rm even}}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}, \lambda_{{\rm N}(0,1/4)}(dx_{\rm R})\lambda_{[0,2\pi\alpha)}(dx_{\rm I})\right) \stackrel{\rm (law)}{=} \left(\Xi, K^{A,{\rm odd}}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}, \lambda_{{\rm N}(0,1/4)}(dx_{\rm R})\lambda_{[0,2\pi\alpha)}(dx_{\rm I})\right),$$

that is, the even-limit and the odd-limit of type A are equivalent up to the shift by $1/(4\alpha)$ in the real-axis direction.

We note that the period $1/(2\alpha)\in(0,\infty)$ of $\left(\Xi, K^{A,\sharp}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}, \lambda_{{\rm N}(0,1/4)}(dx_{\rm R})\lambda_{[0,2\pi\alpha)}(dx_{\rm I})\right)$, $\sharp=$ even, odd, shown by (4.15) becomes zero as $\alpha\to\infty$. Hence, as the $\alpha\to\infty$ limits of these DPPs, a uniform system of infinite number of points on $\mathbb{C}$ will be obtained. In order to see such limit transitions, first we perform Jacobi's imaginary transformations (B.8).

Lemma 4.6 *The following equalities hold,*

$$
\begin{aligned}
&K^{A,\mathrm{even}}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}(x,x')\sqrt{\frac{2}{\pi}}e^{-(x_\mathrm{R}^2+{x'_\mathrm{R}}^2)}\frac{1}{2\pi\alpha}=\frac{e^{ix_\mathrm{R}x_\mathrm{I}}}{e^{ix'_\mathrm{R}x'_\mathrm{I}}}e^{x\overline{x'}}\frac{1}{\pi}e^{-(|x|^2+|x'|^2)/2}\vartheta_0((x+\overline{x'})\alpha;2\pi i\alpha^2),\\
&K^{A,\mathrm{odd}}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}(x,x')\sqrt{\frac{2}{\pi}}e^{-(x_\mathrm{R}^2+{x'_\mathrm{R}}^2)}\frac{1}{2\pi\alpha}=\frac{e^{ix_\mathrm{R}x_\mathrm{I}}}{e^{ix'_\mathrm{R}x'_\mathrm{I}}}e^{x\overline{x'}}\frac{1}{\pi}e^{-(|x|^2+|x'|^2)/2}\vartheta_3((x+\overline{x'})\alpha;2\pi i\alpha^2),\\
&K^{B}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}(x,x')\sqrt{\frac{2}{\pi}}e^{-(x_\mathrm{R}^2+{x'_\mathrm{R}}^2)}\frac{1}{2\pi\alpha}\\
&\quad=\frac{e^{ix_\mathrm{R}x_\mathrm{I}}}{e^{ix'_\mathrm{R}x'_\mathrm{I}}}\Big\{e^{x\overline{x'}}\vartheta_0((x+\overline{x'})\alpha;2\pi i\alpha^2)-e^{-x\overline{x'}}\vartheta_0((x-\overline{x'})\alpha;2\pi i\alpha^2)\Big\}\frac{1}{2\pi}e^{-(|x|^2+|x'|^2)/2},\\
&K^{C}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}(x,x')\sqrt{\frac{2}{\pi}}e^{-(x_\mathrm{R}^2+{x'_\mathrm{R}}^2)}\frac{1}{2\pi\alpha}\\
&\quad=\frac{e^{ix_\mathrm{R}x_\mathrm{I}}}{e^{ix'_\mathrm{R}x'_\mathrm{I}}}\Big\{e^{x\overline{x'}}\vartheta_3((x+\overline{x'})\alpha;2\pi i\alpha^2)-e^{-x\overline{x'}}\vartheta_3((x-\overline{x'})\alpha;2\pi i\alpha^2)\Big\}\frac{1}{2\pi}e^{-(|x|^2+|x'|^2)/2},\\
&K^{D}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}(x,x')\sqrt{\frac{2}{\pi}}e^{-(x_\mathrm{R}^2+{x'_\mathrm{R}}^2)}\frac{1}{2\pi\alpha}\\
&\quad=\frac{e^{ix_\mathrm{R}x_\mathrm{I}}}{e^{ix'_\mathrm{R}x'_\mathrm{I}}}\Big\{e^{x\overline{x'}}\vartheta_3((x+\overline{x'})\alpha;2\pi i\alpha^2)+e^{-x\overline{x'}}\vartheta_3((x-\overline{x'})\alpha;2\pi i\alpha^2)\Big\}\frac{1}{2\pi}e^{-(|x|^2+|x'|^2)/2}.
\end{aligned}
$$

By the asymptotics of the Jacobi theta functions (B.7), the following limit transitions are immediately concluded from the expressions in Lemma 4.6.

Proposition 4.7 *The following limit transitions from the four types of infinite DPPs on* $\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)$ *to the three types of Ginibre DPPs on* $\mathbb{C}$ *are established,*

$$
\begin{aligned}
&\Big(\Xi, K^{A,\sharp}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}, \lambda_{\mathrm{N}(0,1/4)}(dx_\mathrm{R})\lambda_{[0,2\pi\alpha)}(dx_\mathrm{I})\Big) \overset{\alpha\to\infty}{\Longrightarrow} \Big(\Xi, K^{A}_{\mathrm{Ginibre}}, \lambda_{\mathrm{N}(0,1;\mathbb{C})}(dx)\Big), \quad \sharp=\mathrm{even},\mathrm{odd},\\
&\left.\begin{array}{l}\Big(\Xi, K^{B}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}, \lambda_{\mathrm{N}(0,1/4)}(dx_\mathrm{R})\lambda_{[0,2\pi\alpha)}(dx_\mathrm{I})\Big)\\ \Big(\Xi, K^{C}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}, \lambda_{\mathrm{N}(0,1/4)}(dx_\mathrm{R})\lambda_{[0,2\pi\alpha)}(dx_\mathrm{I})\Big)\end{array}\right\} \overset{\alpha\to\infty}{\Longrightarrow} \Big(\Xi, K^{C}_{\mathrm{Ginibre}}, \lambda_{\mathrm{N}(0,1;\mathbb{C})}(dx)\Big),\\
&\Big(\Xi, K^{D}_{\mathbb{R}\times\mathbb{S}^1(2\pi\alpha)}, \lambda_{\mathrm{N}(0,1/4)}(dx_\mathrm{R})\lambda_{[0,2\pi\alpha)}(dx_\mathrm{I})\Big) \overset{\alpha\to\infty}{\Longrightarrow} \Big(\Xi, K^{D}_{\mathrm{Ginibre}}, \lambda_{\mathrm{N}(0,1;\mathbb{C})}(dx)\Big).
\end{aligned} \tag{4.16}
$$

5 Examples in Spaces with Arbitrary Dimensions

5.1 Heisenberg family of infinite DPPs on $\mathbb{C}^d$

The Ginibre DPP of type A on $\mathbb{C}$ given in Section 4.1 can be generalized to the DPPs on $\mathbb{C}^d$ for $d \geq 2$. This generalization was done by [1, 3, 2] as the Weyl–Heisenberg ensembles of DPP, but here we derive the DPPs on $\mathbb{C}^d$, $d \in \mathbb{N}$, following Corollary 2.10 given in Section 2.5.

Let $S_1 = \mathbb{C}^d$, $S_2 = \Gamma = \mathbb{R}^d$,

$$\begin{aligned}
\lambda_1(dx) &= \prod_{a=1}^{d} \lambda_{\mathrm{N}(0,1;\mathbb{C})}(dx^{(a)}) = \frac{1}{\pi^d} e^{-|x|^2} = \frac{1}{\pi^d} e^{-(|x_{\mathrm{R}}|^2+|x_{\mathrm{I}}|^2)} \\
&=: \lambda_{\mathrm{N}(0,1;\mathbb{C}^d)}(dx), \\
\lambda_2(dy) = \nu(dy) &= \prod_{a=1}^{d} \lambda_{\mathrm{N}(0,1/4)}(dy^{(a)}) = \left(\frac{2}{\pi}\right)^{d/2} e^{-2|y|^2},
\end{aligned}$$

and

$$\psi_1(x, y) = e^{-(|x_{\mathrm{R}}|^2-|x_{\mathrm{I}}|^2)/2+2(x_{\mathrm{R}}\cdot y+ix_{\mathrm{I}}\cdot y)}, \quad x = x_{\mathrm{R}} + ix_{\mathrm{I}} \in \mathbb{C}^d, \quad y \in \mathbb{R}^d.$$

We see that $\Psi_1(x)^2 := \|\psi_1(x, \cdot)\|^2_{L^2(\mathbb{R}^d,\nu)} = e^{|x|^2}$, $x \in \mathbb{C}^d$. Hence Assumption 3' is satisfied and then, by Corollary 2.10, we obtain the DPP on $\mathbb{C}^d$ with the correlation kernel,

$$\begin{aligned}
K^{(d)}(x, x') &= \left(\frac{2}{\pi}\right)^{d/2} e^{-\{(|x_{\mathrm{R}}|^2-|x_{\mathrm{I}}|^2)+(|x'_{\mathrm{R}}|^2-|x'_{\mathrm{I}}|^2)\}/2} \int_{\mathbb{R}^d} e^{-2[|y|^2-\{(x_{\mathrm{R}}+ix_{\mathrm{I}})+(x'_{\mathrm{R}}-ix'_{\mathrm{I}})\}\cdot y]} dy \\
&= \frac{e^{ix_{\mathrm{R}}\cdot x_{\mathrm{I}}}}{e^{ix'_{\mathrm{R}}\cdot x'_{\mathrm{I}}}} K^{(d)}_{\mathrm{Heisenberg}}(x, x')
\end{aligned}$$

with

$$K^{(d)}_{\mathrm{Heisenberg}}(x, x') = e^{x\cdot\overline{x'}}, \quad x, x' \in \mathbb{C}^d.$$

The kernels in this form on $\mathbb{C}^d, d \in \mathbb{N}$ have been studied by Zelditch and his coworkers (see [66, 14] and references therein), who identified them with the Szegö kernels for the reduced Heisenberg group [20, 59, 24]. Here we call the DPPs associated with the correlation kernels in this form the *Heisenberg family of DPPs* on $\mathbb{C}^d, d \in \mathbb{N}$. This family includes the Ginibre DPP of type A as the lowest dimensional case with $d = 1$.

Definition 5.1 *The Heisenberg family of DPPs is a one-parameter ($d \in \mathbb{N}$) family of* $\left(\Xi, K^{(d)}_{\mathrm{Heisenberg}}, \lambda_{\mathrm{N}(0,1;\mathbb{C}^d)}(dx)\right)$ *with*

$$K^{(d)}_{\mathrm{Heisenberg}}(x, x') := e^{x\cdot\overline{x'}}, \quad x, x' \in \mathbb{C}^d.$$

Since

$$K^{(d)}_{\mathrm{Heisenberg}}(x, x)\lambda_{\mathrm{N}(0,1;\mathbb{C}^d)}(dx) = \frac{1}{\pi^d} dx, \quad x \in \mathbb{C}^d,$$

every DPP in the Heisenberg family is uniform on $\mathbb{C}^d$ and the density with respect to the Lebesgue measure dx is given by $1/\pi^d$. Hyperuniformity [62] of the Heisenberg family of DPPs has been studied in [3, 42].

5.2 Finite DPPs on $\mathbb{S}^d$

First we recall basic properties of spherical harmonics on $\mathbb{S}^d$ [44]. For $d \in \mathbb{N}$, let $\mathcal{P} = \mathcal{P}(\mathbb{R}^{d+1})$ be a vector space of all complex-valued polynomials on $\mathbb{R}^{d+1}$, and $\mathcal{P}_k, k \in \mathbb{N}_0$, be its subspaces consisting of homogeneous polynomials of degree k; $p(x) = \sum_{|\alpha|=k} c_\alpha x^\alpha$, $c_\alpha \in \mathbb{C}, x = (x^{(1)}, \dots, x^{(d+1)}) \in \mathbb{R}^{d+1}$, where we have used the notations $x^\alpha := \prod_{a=1}^{d+1}(x^{(a)})^{\alpha_a}$ with $\alpha := (\alpha_1, \ldots, \alpha_{d+1}) \in \mathbb{N}_0^{d+1}$, $|\alpha| := \sum_{a=1}^{d+1} \alpha_a$. The vector space of all harmonic functions in $\mathcal{P}$ is denoted by $\mathcal{H} = \{p \in \mathcal{P} : \Delta p = 0\}$ and let $\mathcal{H}_k = \mathcal{H} \cap \mathcal{P}_k$, $k \in \mathbb{N}_0$.

Now we consider a unit sphere in $\mathbb{R}^{d+1}$ denoted by $\mathbb{S}^d$, in which we use the polar coordinates for $x = (x^{(1)}, \ldots, x^{(d+1)}) \in \mathbb{S}^d$,

$$\begin{aligned} x^{(1)} &= \sin\theta_d \cdots \sin\theta_2 \sin\theta_1, \\ x^{(a)} &= \sin\theta_d \cdots \sin\theta_a \cos\theta_{a-1}, \quad a = 2, \ldots, d, \\ x^{(d+1)} &= \cos\theta_d, \quad \text{with } \theta_1 \in [0, 2\pi), \quad \theta_a \in [0, \pi], \quad a = 2, \ldots, d. \end{aligned} \tag{5.1}$$

Note that $\|x\|_{\mathbb{R}^{d+1}}^2 := \sum_{a=1}^{d+1} {x^{(a)}}^2 = 1$. For $d = 2$, if we put $\theta_1 = \pi/2 - \varphi$ and $\theta_2 = \theta$, the polar coordinates (4.7) used in Section 4.2 are obtained. The standard measure on $\mathbb{S}^d$ is given by the *Lebesgue area measure* expressed as

$$d\sigma_d(x) = \sin^{d-1}\theta_d \sin^{d-2}\theta_{d-1} \cdots \sin\theta_2 d\theta_1 \cdots d\theta_d, \quad x \in \mathbb{S}^d. \tag{5.2}$$

The total measure of $\mathbb{S}^d$ is calculated as

$$\omega_d := \sigma_d(\mathbb{S}^d) = \frac{2\pi^{(d+1)/2}}{\Gamma((d+1)/2)}. \tag{5.3}$$

We write the space of harmonic polynomials in $\mathcal{H}_k$ restricted on $\mathbb{S}^d$ as

$$\mathcal{Y}_{(d,k)} = \left\{ h\Big|_{\mathbb{S}^d} : h \in \mathcal{H}_k \right\}, \quad k \in \mathbb{N}_0.$$

We can see that

$$D(d,k) := \dim \mathcal{Y}_{(d,k)} = \frac{(d+2k-1)(d+k-2)!}{(d-1)!k!} = \frac{2}{(d-1)!}k^{d-1} + \mathrm{o}(k^{d-1}). \tag{5.4}$$

Consider an orthonormal basis $\{Y_j^{(d,k)}\}_{j=1}^{D(d,k)}$ of $\mathcal{Y}_{(d,k)}$ with respect to $d\sigma_d$;

$$\langle Y_n^{(d,k)}, Y_m^{(d,k)} \rangle_{L^2(\mathbb{S}^d, d\sigma_d)} = \int_{\mathbb{S}^d} Y_n^{(d,k)}(x) \overline{Y_m^{(d,k)}(x)} d\sigma_d(x) = \delta_{nm}, \quad n, m \in \mathbb{N}_0. \tag{5.5}$$

Then, if we put

$$K^{\mathcal{Y}_{(d,k)}}(x, x') = \sum_{j=1}^{D(d,k)} Y_j^{(d,k)}(x) \overline{Y_j^{(d,k)}(x')}, \quad x' \in \mathbb{S}^d,$$

then $\{K^{\mathcal{Y}_{(d,k)}}(x, x')\}_{x, x' \in \mathbb{S}^d}$ give the reproducing kernel in $\mathcal{Y}^{(d,k)}$ in the sense that

$$Y(x') = \int_{\mathbb{S}^d} Y(x) \overline{K^{\mathcal{Y}_{(d,k)}}(x, x')} d\sigma_d(x), \quad \forall Y \in \mathcal{Y}_{(d,k)}.$$

For $\lambda > -1/2$, we define

$$P_k^\lambda(x) := {}_2F_1\left(-k, k+2\lambda; \lambda + \frac{1}{2}; \frac{1-x}{2}\right),$$

where ${}_2F_1$ denotes the *Gauss hypergeometric function*, ${}_2F_1(\alpha,\beta,\gamma;z) := \sum_{n=0}^{\infty}\{(\alpha)_n(\beta)_n/\gamma)_n\}z^n/n!$, with $(\alpha)_n := \alpha(\alpha+1)\cdots(\alpha+n-1) = \Gamma(\alpha+n)/\Gamma(\alpha)$, $n \in \mathbb{N}$, $(\alpha)_0 := 1$. Then the following equality is established,

$$K^{\mathcal{Y}_{(d,k)}}(x,x') = \frac{D(d,k)}{\omega_d} P_k^{(d-1)/2}(x \cdot x'), \quad x, x' \in \mathbb{S}^d,$$

where ω_d and $D(d,k)$ are given by (5.3) and (5.4), respectively, and $x \cdot x' := \sum_{a=1}^{d+1} x^{(a)} x'^{(a)}$.

We see that $K^{\mathcal{Y}_{(d,k)}}(x,x')$ is $\mathrm{O}(d+1,\mathbb{R})$-invariant in the sense

$$K^{\mathcal{Y}_{(d,k)}}(gx,gx') = K^{\mathcal{Y}_{(d,k)}}(x,x'), \quad \forall g \in \mathrm{O}(d+1,\mathbb{R}), \quad \forall x, x' \in \mathbb{S}^d.$$

Let $\{e_1,\dots,e_{d+1}\}$ be the standard basis of $\mathbb{R}^{d+1}$ and L_0 be the stabilizer subgroup of $\mathrm{SO}(d+1,\mathbb{R})$ at e_{d+1} represented as

$$L_0 = \left\{ \begin{pmatrix} A & \mathbf{0} \\ {}^t\mathbf{0} & 1 \end{pmatrix} : A \in \mathrm{SO}(d,\mathbb{R}) \right\}.$$

We define

$$\mathcal{Y}_{(d,k)}^{L_0} = \{Y \in \mathcal{Y}_{(d,k)} : Y(\ell x) = Y(x), \quad \forall \ell \in L_0, \quad \forall x \in \mathbb{S}^d\},$$

Then $K^{\mathcal{Y}_{(d,k)}} \in \mathcal{Y}_{(d,k)}^{L_0}$. The function $P_k^{\lambda}(s)$ is called the *ultraspherical polynomial* [45]. The space $\mathcal{Y}_{(d,k)}^{L_0}$ is a one-dimensional vector space generated by $P_k^{(d-1)/2}(x \cdot e_{d+1})$. In general, any L_0-invariant function is a constant for each L_0-orbit, $\mathcal{O} = \{x \in \mathbb{S}^d : x \cdot e_{d+1} = a\}$, $-1 \le a \le 1$, and hence functions in $\mathcal{Y}_{(d,k)}^{L_0}$ is called the *zonal harmonics* of degree k. Note that, when we set

$$C_k^{\lambda}(x) := \binom{k+2\lambda-1}{k} P_k^{\lambda}(x),$$

we call $C_k^{\lambda}(s)$ the *Gegenbauer polynomial* of degree k [45].

Fix $d \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then, if we consider the case that $S_1 = \mathbb{S}^d$, $S_2 = \mathbb{N}$ with $\lambda_1(dx) = d\sigma_d(x)$, $L^2(\Gamma,\nu) = \ell^2(\{1,\dots,D(d,k)\}) \subset S_2$, and $\psi_1(x,n) = Y_n^{(d,k)}(x)$, then (5.5) with Remark 3 in Section 3.1.1 guarantees Assumption 3'. Hence Corollary 2.10 determines a unique DPP on $\mathbb{S}^d$, in which the correlation kernel is given by [13]

$$\begin{aligned} K^{\mathcal{Y}_{(d,k)}}(x,x') &= \frac{D(d,k)}{\omega_d} P_k^{(d-1)/2}(x \cdot x') \\ &= \frac{d-1+2k}{(d-1)\omega_d} C_k^{(d-1)/2}(x \cdot x'). \end{aligned}$$

It is obvious that the obtained DPP is rotationally invariant on $\mathbb{S}^d$, since the kernel $K^{\mathcal{Y}_{(d,k)}}(x,x')$ depend only on the inner product $x \cdot x'$. The density of points is uniform on $\mathbb{S}^d$ and is given with respect to $\sigma_d(dx)$ by

$$\begin{aligned} \rho^{\mathcal{Y}_{(d,k)}} &= K^{\mathcal{Y}_{(d,k)}}(x,x) = \frac{D(d,k)}{\omega_d} P_k^{(d-1)/2}(1) \\ &= \frac{D(d,k)}{\omega_d} = \frac{2k^{d-1}}{(d-1)!\omega_d} + \mathrm{o}(k^{d-1}), \end{aligned}$$

where we have used the fact that $P_k^{\lambda}(1) = {}_2F_1(-k, k+2\lambda, \lambda+1/2; 0) = 1, \lambda > -1/2$ [45].

Next we consider the DPP on $\mathbb{S}^d$ for fixed $d \in \mathbb{N}$ and $L \in \mathbb{N}$ such that the correlation kernel is given by the following finite sum [13],

$$
\begin{aligned}
K^{(N(d,L))}_{\mathrm{harmonic}(\mathbb{S}^d)}(x, x') &:= \sum_{k=0}^{L-1} K^{\mathcal{Y}_{(d,k)}}(x, x') = \frac{1}{\omega_d} \sum_{k=0}^{L-1} D(d,k) P_k^{(d-1)/2}(x \cdot x') \\
&= \frac{1}{\omega_d} \sum_{k=0}^{L-1} \frac{d-1+2k}{d-1} C_k^{(d-1)/2}(x \cdot x'),
\end{aligned} \tag{5.6}
$$

where the total number of points on $\mathbb{S}^d$ is given by

$$
\begin{aligned}
N(d, L) &= \sum_{k=0}^{L-1} D(d,k) = \frac{2L+d-2}{d} \binom{d+L-2}{L-1} \\
&= \frac{2}{d!} L^d + \mathrm{o}(L^d).
\end{aligned} \tag{5.7}
$$

The DPP $(\Xi, K^{(N)}_{\mathrm{harmonic}(\mathbb{S}^d)}, d\sigma_d(x))$ is rotationally invariant in $\mathbb{S}^d$ and is called the *harmonic ensemble* in $\mathbb{S}^d$ with N points by Beltrán *et al.* [13]. We note the recurrence relation of the Gegenbauer polynomials (see, Eq.(18.9.7) in [45]),

$$
(n+\lambda) C_n^{\lambda}(x) = \lambda (C_n^{\lambda+1}(x) - C_{n-2}^{\lambda+1}(x)).
$$

This implies that

$$
\frac{d-1+2k}{d-1} C_k^{(d-1)/2}(x) = C_k^{(d+1)/2}(x) - C_{k-2}^{(d+1)/2}(x), \quad k \geq 2.
$$

Since $C_0^{\lambda}(x) = 1, C_1^{\lambda}(x) = 2\lambda x$, we obtain the following expression for the correlation kernel,

$$
K^{(N(d,L))}_{\mathrm{harmonic}(\mathbb{S}^d)}(x, x') = \frac{1}{\omega_d} \left[C_{L-1}^{(d+1)/2}(x \cdot x') + C_{L-2}^{(d+1)/2}(x \cdot x') \right].
$$

If we introduce the *Jacobi polynomials* defined as [45]

$$
P_n^{(\alpha,\beta)}(x) := \frac{(\alpha+1)_n}{n!} {}_2F_1 \left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2} \right),
$$

and use the contiguous relation, $(b-a)\, {}_2F_1(a,b;c;z) + a\, {}_2F_1(a+1,b;c;z) - b\, {}_2F_1(a,b+1;c;z) = 0$, the above is written as follows [13],

$$
K^{(N(d,L))}_{\mathrm{harmonic}(\mathbb{S}^d)}(x, x') = \frac{1}{\omega_d} \frac{N(d,L)}{\binom{L+d/2-1}{L-1}} P_{L-1}^{(d/2,(d-2)/2)}(x \cdot x'), \tag{5.8}
$$

where $\binom{L+d/2-1}{L-1} := \Gamma(L+d/2)/\{(L-1)!\Gamma(d/2+1)\} = P_{L-1}^{(d/2,(d-2)/2)}(1)$.

In particular, when $d=1$, for $x = (x^{(1)}, x^{(2)}) = (\sin\theta, \cos\theta)$, $x' = (x'^{(1)}, x'^{(2)}) = (\sin\theta', \cos\theta') \in \mathbb{S}^1 \subset \mathbb{R}^2$, $\theta, \theta' \in [0, 2\pi)$, we have $x \cdot x' = \cos(\theta - \theta')$ and

$$
\begin{aligned}
K^{(N(1,L))}_{\mathrm{harmonic}(\mathbb{S}^1)}(x, x') d\sigma_1(x) &= \frac{1}{2\pi} {}_2F_1 \left(\frac{1-(2L-1)}{2}, \frac{1+(2L-1)}{2}; \frac{3}{2}; \sin^2 \frac{\theta - \theta'}{2} \right) d\theta \\
&= \frac{\sin\{(2L-1)(\theta-\theta')/2\}}{\sin\{(\theta-\theta')/2\}} \frac{d\theta}{2\pi} \\
&= \frac{\sin\{N(\theta-\theta')/2\}}{\sin\{(\theta-\theta')/2\}} \frac{d\theta}{2\pi},
\end{aligned} \tag{5.9}
$$

where we have used the fact that $N(1,L) = 2L-1$ given by (5.7). This verifies the identification of the 1-sphere case of the present DPP with the CUE, $(\Xi, K^{A_{N-1}}, \lambda_{[0,2\pi)}(d\theta))$, given in Section 3.2.

On the other hand, when $d=2$, (5.7) gives $N(2,L) = L^2$ and

$$\begin{aligned} K^{(N(2,L))}_{\mathrm{harmonic}(\mathbb{S}^2)}(x,x') &= \frac{L^2}{4\pi} {}_2F_1\left(-L+1, L+1; 2; \frac{1-x\cdot x'}{2}\right) \\ &= \frac{N}{4\pi} {}_2F_1\left(-\sqrt{N}+1, \sqrt{N}+1; 2; \frac{\|x-x'\|^2_{\mathbb{R}^3}}{4}\right), \end{aligned}$$

which is different from $K^{(N)}_{\mathbb{S}^2}(x,x')$ given by (4.9) in Section 4.2.

5.3 Euclidean family of infinite DPPs on $\mathbb{R}^d$

We consider the vicinity of the north pole $e_{d+1} = (0,\dots,0,1)$ on $\mathbb{S}^d$ and put $\theta_d = R/L$, $R \in [0,\infty)$. Then, as $L \to \infty$, the polar coordinates (5.1) behave as

$$\begin{aligned} x^{(1)} &\sim \frac{R}{L} \sin\theta_{d-1} \cdots \sin\theta_2 \sin\theta_1 =: \frac{1}{L} \widetilde{x}^{(1)}, \\ x^{(a)} &\sim \frac{R}{L} \sin\theta_{d-1} \cdots \sin\theta_a \cos\theta_{a-1} =: \frac{1}{L} \widetilde{x}^{(a)}, \quad a = 2,\dots,d, \\ x^{(d+1)} &\sim 1 - \frac{1}{2}\left(\frac{R}{L}\right)^2. \end{aligned}$$

In this case, for $x, x' \in \mathbb{S}^d$,

$$x \cdot x' = \sum_{a=1}^{d+1} x^{(a)} x'^{(a)} = 1 - \frac{1}{2L^2} \|\widetilde{x} - \widetilde{x}'\|^2_{\mathbb{R}^d} + \mathrm{o}\left(\frac{1}{L^2}\right), \quad \text{as } L \to \infty,$$

where $\widetilde{x}, \widetilde{x}' \in \mathbb{R}^d$ and $\|\cdot\|_{\mathbb{R}^d}$ denotes the Euclidean norm in $\mathbb{R}^d$. Hence we can conclude that

$$x \cdot x' = \cos\left(\frac{r}{L}\right) + \mathrm{o}\left(\frac{1}{L^2}\right), \quad \text{with } r := \|\widetilde{x} - \widetilde{x}'\|_{\mathbb{R}^d}, \quad \text{as } L \to \infty. \tag{5.10}$$

In this limit, the measure on $\mathbb{S}^d$ given by (5.2) behaves as

$$\begin{aligned} d\sigma_d(x) &\sim \left(\frac{R}{L}\right)^{d-1} \sin^{d-2}\theta_{d-1} \cdots \sin\theta_2 \, d\theta_1 \cdots d\theta_{d-1} \frac{dR}{L} \\ &= \frac{1}{L^d} d\sigma_{d-1}(\widehat{x}) R^{d-1} dR = \frac{1}{L^d} d\widetilde{x}, \quad \widehat{x} \in \mathbb{S}^{d-1}, \quad \widetilde{x} \in \mathbb{R}^d. \end{aligned}$$

The following limit is proved for the correlation kernel $K^{(N(d,L))}_{\mathrm{harmonic}(\mathbb{S}^d)}$ given by (5.6).

Lemma 5.2 *When* (5.10) *holds, the limit*

$$k^{(d)}(r) = \lim_{L\to\infty} \frac{1}{L^d} K^{(N(d,L))}_{\mathrm{harmonic}(\mathbb{S}^d)}(x,x')$$

exists and has the following expressions,

$$k^{(d)}(r) = \frac{J_{d/2}(r)}{(2\pi r)^{d/2}} \tag{5.11}$$

$$= \frac{1}{(2\pi)^{d/2} r^{(d-2)/2}} \int_0^1 s^{d/2} J_{(d-2)/2}(rs)ds, \tag{5.12}$$

where $J_\nu(z)$ *is the Bessel function of the first kind with index* ν *defined by* (3.18).

Proof For $d \in \mathbb{N}$ and $r \in (0, \infty)$, the following formula of Mehler–Heine type is known (Theorem 8.1.1 in [61]); for $\alpha, \beta \in \mathbb{R}$,

$$\lim_{n\to\infty} n^{-\alpha} P_n^{(\alpha,\beta)} \left(\cos \frac{r}{n} \right) = \left(\frac{r}{2} \right)^{-\alpha} J_\alpha(r),$$

where the limit is uniform on compact subset of $\mathbb{C}$. Then under (5.10), (5.8) gives

$$\lim_{L\to\infty} \frac{1}{L^d} K_{\mathrm{harmonic}(\mathbb{S}^d)}^{(N(d,L))}(x, x') = \frac{2}{\omega_d d!} \Gamma(d/2+1) \left(\frac{2}{r} \right)^{d/2} J_{d/2}(r).$$

By (5.3) and the equality $\Gamma(2z) = \{2^{2z}/(2\sqrt{\pi})\}\Gamma(z)\Gamma(z+1/2)$, we can confirm that $2\Gamma(d/2 + 1)/(\omega_d d!) = 1/\{(2\sqrt{\pi})^d\}$. Hence (5.11) is proved. By the integral formula (see, for instance, Eq.(10.22.1) in [45]),

$$\int z^{\nu+1} J_\nu(z)dz = z^{\nu+1} J_{\nu+1}(z),$$

we can derive (5.12) from (5.11). The proof is complete. ∎

This result implies that for each $d \in \mathbb{N}$ we obtain an infinite-dimensional DPP on $\mathbb{R}^d$ such that it is uniform and isotropic on $\mathbb{R}^d$ and the correlation kernel is given by

$$K^{(d)}(x, x') = k^{(d)}(\|x - x'\|_{\mathbb{R}^d}), \quad x, x' \in \mathbb{R}^d, \tag{5.13}$$

where $k^{(d)}(r)$ is given by (5.11) and (5.12).

We can give the following alternative expression for $K^{(d)}$.

Lemma 5.3 *For* $d \in \mathbb{N}$, *the correlation kernel* $K^{(d)}$ *given by* (5.13) *with* (5.12) *is written as*

$$K^{(d)}(x, x') = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathbf{1}_{\mathbb{B}^d}(y) e^{i(x-x')\cdot y} dy = \frac{1}{(2\pi)^d} \int_{\mathbb{B}^d} e^{i(x-x')\cdot y} dy, \tag{5.14}$$

where $\mathbb{B}^d$ *denotes the unit ball centered at the origin;* $\mathbb{B}^d := \{y \in \mathbb{R}^d : |y| \le 1\}$.

Proof The statement is proved for $d = 1$ and 2 by direct calculation as follows. For $d = 1$, (5.12) gives

$$k^{(1)}(r) = \sqrt{\frac{r}{2\pi}} \int_0^1 s^{1/2} J_{-1/2}(rs)ds.$$

Here we use the equality $J_{-1/2}(z) = \sqrt{2/(\pi z)} \cos z$. Then

$$k^{(1)}(r) = \frac{1}{\pi} \int_0^1 \cos(rs)ds = \frac{1}{2\pi} \int_{-1}^1 e^{iry} dy,$$

which gives (5.14) with $d=1$, if we put $r=x-x'$ and regard an interval $[-1,1]\subset\mathbb{R}$ as $\mathbb{B}^1$. For $d=2$, (5.12) gives

$$k^{(2)}(r)=\frac{1}{2\pi}\int_0^1 sJ_0(rs)ds. \tag{5.15}$$

We use the following integral representation for J_0 given as Eq.(10.9.1) in [45],

$$J_0(z)=\frac{1}{\pi}\int_0^{\pi}\cos(z\cos\varphi)d\varphi=\frac{1}{2\pi}\int_0^{2\pi}e^{iz\cos\varphi}d\varphi.$$

Hence (5.15) is written as

$$k^{(2)}(r)=\frac{1}{(2\pi)^2}\int_0^1 ds\, s\int_0^{2\pi}d\theta\, e^{irs\cos\theta}. \tag{5.16}$$

We can identify the integral variables (s,θ) in (5.16) with the polar coordinates in $\mathbb{R}^2$ and (5.14) with $d=2$ is obtained, if we recognize $r=\|x-x'\|_{\mathbb{R}^2}$, $s=\|y\|_{\mathbb{R}^2}$, and $(x-x')\cdot y=rs\cos\theta$. Now we assume $d\geq 3$. In this case RHS of (5.14) is given by

$$\begin{aligned}I:=&\frac{1}{(2\pi)^d}\int_0^1 ds\, s^{d-1}\int_0^{2\pi}d\theta_1\int_0^{\pi}d\theta_2\,\sin\theta_2\\ &\times\cdots\times\int_0^{\pi}d\theta_{d-2}\sin^{d-3}\theta_{d-2}\int_0^{\pi}d\theta_{d-1}\sin^{d-2}\theta_{d-1}e^{irs\cos\theta_{d-1}}.\end{aligned}$$

Since

$$\int_0^{2\pi}d\theta_1\int_0^{\pi}d\theta_2\,\sin\theta_2\cdots\int_0^{\pi}d\theta_{d-2}\sin^{d-3}\theta_{d-2}=\sigma_{d-2}(\mathbb{S}^{d-2})=\omega_{d-2},$$

we have

$$I=\frac{w_{d-2}}{(2\pi)^d}\int_0^1 ds\, s^{d-1}\int_0^{\pi}d\theta_{d-1}\,\sin^{d-2}\theta_{d-1}e^{irs\cos\theta_{d-1}}.$$

If we use the following integral representation of the Bessel function of the first kind,

$$J_m(z)=\frac{1}{\sqrt{\pi}\Gamma(m+1/2)}\left(\frac{z}{2}\right)^m\int_0^{\pi}\sin^{2m}(\theta)e^{iz\cos\theta}d\theta,\quad m\in\frac{1}{2}\mathbb{N},$$

which is obtained from Eq.(10.9.4) in [45], the equivalence between (5.14) and (5.12) is verified. Hence the proof is complete. ∎

The kernel (5.14) is obtained as the correlation kernel K_{S_1} given by (2.17) in Corollary 2.10, if we consider the case such that $S_1=S_2=\mathbb{R}^d$, $\lambda_1(dx)=dx$, $\lambda_2(dy)=\nu(dy)=dy$, $\psi_1(x,y)=e^{ix\cdot y}/(2\pi)^{d/2}$, and $\Gamma=\mathbb{B}^d\subsetneq\mathbb{R}^d$. We see $\Psi_1(x)^2:=\|\psi_1(x,\cdot)\|^2_{L^2(\Gamma,d\nu)}\equiv|\mathbb{B}^d|/(2\pi)^d$, $x\in\mathbb{R}^d$, where the volume of $\mathbb{B}^d$ is denoted by $|\mathbb{B}^d|=\pi^{d/2}/\Gamma((d+2)/2)$.

The kernels $K^{(d)}$ on $\mathbb{R}^d, d\in\mathbb{N}$ have been studied by Zelditch and others (see [66, 57, 67, 18] and references therein), who regarded them as the Szegö kernels for the reduced Euclidean motion group [60, 65]. Here we call the DPPs associated with the correlation kernels in this form the *Euclidean family of DPPs* on $\mathbb{R}^d, d\in\mathbb{N}$.

Definition 5.4 *The Euclidean family of DPPs is a one-parameter* ($d \in \mathbb{N}$) *family of* $\left(\Xi, K_{\text{Euclid}}^{(d)}, dx\right)$ *with*

$$
\begin{aligned}
K_{\text{Euclid}}^{(d)}(x, x') &:= \frac{1}{(2\pi)^{d/2}} \frac{J_{d/2}(\|x-x'\|_{\mathbb{R}^d})}{\|x-x'\|_{\mathbb{R}^d}^{d/2}} \\
&= \frac{1}{(2\pi)^{d/2}} \frac{1}{\|x-x'\|_{\mathbb{R}^d}^{(d-2)/2}} \int_0^1 s^{d/2} J_{(d-2)/2}(\|x-x'\|_{\mathbb{R}^d} s) ds \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathbf{1}_{\mathbb{B}^d}(y) e^{i(x-x')\cdot y} dy = \frac{1}{(2\pi)^d} \int_{\mathbb{B}^d} e^{i(x-x')\cdot y} dy, \quad x, x' \in \mathbb{R}^d.
\end{aligned}
$$

The above result is summarized as follows [35].

Proposition 5.5 *The following is established for* $d \in \mathbb{N}$,

$$
\left(\frac{d!}{2}\right)^{1/d} N^{1/d} \circ \left(\Xi, K_{\text{harmonic}(\mathbb{S}^d)}^{(N)}, d\sigma_d(x)\right) \stackrel{N\to\infty}{\Longrightarrow} \left(\Xi, K_{\text{Euclid}}^{(d)}, dx\right).
$$

We see that

$$
K_{\text{Euclid}}^{(d)}(x, x) = \lim_{r\to 0} \frac{1}{(2\pi)^{d/2}} \frac{J_{d/2}(r)}{r^{d/2}} = \frac{1}{2^d \pi^{d/2} \Gamma((d+2)/2)}.
$$

Then the Euclidean family of DPPs are uniform on $\mathbb{R}^d$ with densities

$$
\rho_{\text{Euclid}}^{(d)} = \frac{1}{2^d \pi^{d/2} \Gamma((d+2)/2)}
$$

with respect to the Lebesgue measures dx of $\mathbb{R}^d$.

For lower dimensions, the correlation kernels and the densities are given as follows,

$$
\begin{aligned}
K_{\text{Euclid}}^{(1)}(x, x') &= \frac{\sin(x-x')}{\pi(x-x')} = K_{\text{sinc}}(x, x') \quad \text{with} \quad \rho_{\text{Euclid}}^{(1)} = \frac{1}{\pi}, \\
K_{\text{Euclid}}^{(2)}(x, x') &= \frac{J_1(\|x-x'\|_{\mathbb{R}^2})}{2\pi\|x-x'\|_{\mathbb{R}^2}} \quad \text{with} \quad \rho_{\text{Euclid}}^{(2)} = \frac{1}{4\pi}, \\
K_{\text{Euclid}}^{(3)}(x, x') &= \frac{1}{2\pi^2 \|x-x'\|_{\mathbb{R}^3}^2} \left(\frac{\sin\|x-x'\|_{\mathbb{R}^3}}{\|x-x'\|_{\mathbb{R}^3}} - \cos\|x-x'\|_{\mathbb{R}^3} \right) \quad \text{with} \quad \rho_{\text{Euclid}}^{(3)} = \frac{1}{6\pi^2}.
\end{aligned}
$$

This family of DPPs includes the DPP with the sinc kernel K_{sinc} as the lowest dimensional case with $d = 1$. Since $(\Xi, K_{\text{harmonic}(\mathbb{S}^1)}^{(N)}, d\sigma_1)$ has been identified with the CUE, $(\Xi, K^{A_{N-1}}, \lambda_{[0,2\pi)})$, by (5.9), Proposition 5.5 can be regarded as the multidimensional extension of the limit theorem from the CUE to the DPP with K_{sinc} given by the first line of (3.20). Note that, if d is odd,

$$
k^{(d)}(r) = \left(-\frac{1}{2\pi r} \frac{d}{dr}\right)^{(d-1)/2} \frac{\sin r}{\pi r}.
$$

This is proved by Rayleigh's formula for the spherical Bessel function of the first kind (Eq. (10.49.14) in [45]);

$$
j_m(x) := \sqrt{\frac{\pi}{2x}} J_{m+1/2}(x) = x^m \left(-\frac{1}{x}\frac{d}{dx}\right)^m \frac{\sin x}{x}, \quad m \in \mathbb{N}.
$$

6 Concluding Remarks

In Section 5.2, we studied the finite DPPs $(\Xi, K^{(N)}_{\text{harmonic}(\mathbb{S}^d)}, d\sigma_d(x))$, $N \in \mathbb{N}$ called the harmonic ensembles on $\mathbb{S}^d$, $d \in \mathbb{N}$ [13]. Then we proved as Proposition 5.5 in Section 5.3 that their bulk scaling limits are given by $(\Xi, K^{(d)}_{\text{Euclid}}, dx)$, which we call the Euclidean family of DPPs. On $\mathbb{S}^2$, there are two distinct types of uniform and isotropic DPPs, one of which is the harmonic ensemble $(\Xi, K^{(N)}_{\text{harmonic}(\mathbb{S}^2)}, d\sigma_2(x))$ studied in Section 5.2 [13] , and other of which is the DPP called the spherical ensemble $(\Xi, K^{(N)}_{\mathbb{S}^2}, d\sigma_2(x))$ studied in Section 4.2 [39, 4]. As mentioned above, the scaling limit of the former is given by $(\Xi, K^{(2)}_{\text{Euclid}}, dx)$, while as given by Proposition 4.1 the bulk scaling limit of the latter is $(\Xi, K^{A}_{\text{Ginibre}}, \lambda_{\text{N}(0,1;\mathbb{C})}(dx))$, which is equivalent with $(\Xi, K^{(1)}_{\text{Heisenberg}}, \lambda_{\text{N}(0,1;\mathbb{C})}(dx))$. The spherical ensemble on $\mathbb{S}^2$ shall be generalized to DPPs on the higher dimensional spheres $\mathbb{S}^{2d} \simeq \mathbb{C}^d$, $d \geq 2$ so that they are uniform and isotropic and their bulk scaling limits are given by DPPs in the Heisenberg family. The papers [11, 12] will be useful.

With $L^2(S, \lambda)$ and $L^2(\Gamma, \nu)$, we can consider the system of *biorthonormal functions*, which consists of a pair of distinct families of measurable functions $\{\psi(x, \gamma) : x \in S, \gamma \in \Gamma\}$ and $\{\varphi(x, \gamma) : x \in S, \gamma \in \Gamma\}$ satisfying the biorthonormality relations

$$\langle \psi(\cdot, \gamma), \varphi(\cdot, \gamma')\rangle_{L^2(S,\lambda)}\nu(d\gamma) = \delta(\gamma - \gamma')d\gamma, \quad \gamma, \gamma' \in \Gamma. \tag{6.1}$$

If the integral kernel defined by

$$K^{\text{bi}}(x, x') = \int_\Gamma \psi(x, \gamma)\overline{\varphi(x', \gamma)}\nu(d\gamma), \quad x, x' \in S, \tag{6.2}$$

is of finite rank, we can construct a finite DPP on S whose correlation kernel is given by (6.2) following a standard method of random matrix theory (see, for instance, Appendix C in [33]). By the biorthonormality (6.1), it is easy to verify that K^{bi} is a projection kernel, but it is not necessarily an orthogonal projection. This observation means that such a DPP is not constructed by the method reported in this paper. Generalization of the present framework in order to cover such DPPs associated with biorthonormal systems is required. Moreover, the dynamical extensions of DPPs called *determinantal processes* (see, for instance, [32]) shall be studied in the context of the present paper.

For finite DPPs, we can readily derive the systems of *stochastic interacting particle systems* whose stationary states are given by the DPPs. For example, with $N \in \mathbb{N}$, the system of stochastic differential equations (SDEs) on $\mathbb{S}^1$ [27, 31],

$$dX_j(t) = dB_j(t) + \frac{1}{2} \sum_{\substack{1 \leq k \leq N, \\ k \neq j}} \cot \frac{X_j(t) - X_k(t)}{2} dt, \quad j = 1, \dots, N, \quad t \geq 0, \tag{6.3}$$

driven by independent one-dimensional standard Brownian motions $B_j(t), j = 1, \dots, N, t \geq 0$ has the DPP $(\Xi, K^{A_{N-1}}, \lambda_{[0,2\pi)}(dx))$ given in Section 3.2 as a stationary probability measure. Another example is given by the system of SDEs on $\mathbb{C}$,

$$dZ_j(t) = dB^{\mathbb{C}}_j(t) - \frac{(N+1)Z_j(t)}{1 + |Z_j(t)|^2} dt + \sum_{\substack{1 \leq k \leq N, \\ k \neq j}} \frac{Z_j(t) - Z_k(t)}{|Z_j(t) - Z_k(t)|^2} dt, \quad j = 1, \dots, N, \quad t \geq 0, \tag{6.4}$$

$N \in \mathbb{N}$, driven by independent complex Brownian motions $B_j^{\mathbb{C}}(t) := B_j^{\mathrm{R}}(t) + iB_j^{\mathrm{I}}(t)$, where $B_j^{\mathrm{R}}(t), B_j^{\mathrm{I}}(t)$ are independent one-dimensional standard Brownian motions, $j = 1, \dots, N, t \geq 0$, does the DPP $(\sum_j \delta_{Z_j}, K_{G_1^{-1}G_2}^{(N)}, \lambda(dz))$ of Krishnapur [39] as a stationary probability measure, which is obtained as the stereographic projection of $(\Xi, K_{\mathbb{S}^2}^{(N)}, d\sigma_2(x))$ as explained in Remark 5 in Section 4.2. A general theory has been developed by Osada *et al.* for *infinite-dimensional stochastic differential equations* (ISDEs), some of which have infinite DPPs as invariant probability measures [46, 47, 48, 49, 37, 50]. We expect to obtain the *universal* ISDEs along the limit theorems given in Propositions 5.5 and 4.1 taking account of the fact that (6.3) and (6.4) might give useful approximations to characterize Osada's Dyson/Ginibre ISDEs.

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A Weyl Denominator Formulas

The *Weyl denominator formulas* for classical root systems play a fundamental role in Lie theory and related area. For reduced root systems they are given in the form,

$$\sum_{w \in W} \det(w) e^{w(\rho)-\rho} = \prod_{\alpha \in R_+} (1 - e^{-\alpha}),$$

where W is the Weyl group, R_+ the set of positive roots and $\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha$.

For classical root systems A_{N-1}, B_N, C_N and D_N, $N \in \mathbb{N}$, the explicit forms are given as follows,

$$\begin{aligned}
&(\text{type } A_{N-1}) && \det_{1 \leq j,k \leq N} \left(z_k^{j-1} \right) = \prod_{1 \leq j < k \leq N} (z_k - z_j), \\
&(\text{type } B_N) && \det_{1 \leq, j,k \leq N} \left(z_k^{j-N} - z_k^{N+1-j} \right) = \prod_{\ell=1}^{N} z_\ell^{1-N} (1 - z_\ell) \prod_{1 \leq j < k \leq N} (z_k - z_j)(1 - z_j z_k), \\
&(\text{type } C_N) && \det_{1 \leq, j,k \leq N} \left(z_k^{j-N-1} - z_k^{N+1-j} \right) = \prod_{\ell=1}^{N} z_\ell^{-N} (1 - z_\ell^2) \prod_{1 \leq j < k \leq N} (z_k - z_j)(1 - z_j z_k), \\
&(\text{type } D_N) && \det_{1 \leq, j,k \leq N} \left(z_k^{j-N} + z_k^{N-j} \right) = 2 \prod_{\ell=1}^{N} z_\ell^{1-N} \prod_{1 \leq j < k \leq N} (z_k - z_j)(1 - z_j z_k),
\end{aligned} \tag{A.1}$$

respectively. See, for instance, [51].

If we change the variables as

$$z_k = e^{-2i\zeta_k}, \quad \zeta_k \in \mathbb{C}, \quad k = 1, \dots, N, \tag{A.2}$$

then, the following equalities are derived from the above.

Lemma A.1 *For $\zeta_k \in \mathbb{C}, k = 1, \dots, N$, the following equalities are established.*

$$(\text{type}\, A_{N-1}) \quad \det_{1 \le j,k \le N} \left[e^{-i(\mathcal{N}^{A_{N-1}} - 2J^{A_{N-1}}(j))\zeta_k} \right] = (2i)^{N(N-1)/2} \prod_{1 \le j < k \le N} \sin(\zeta_k - \zeta_j).$$

$$\begin{aligned}(\text{type}\, B_N) \quad & \det_{1 \le j,k \le N} \left[\sin\{(\mathcal{N}^{B_N} - 2J^{B_N}(j))\zeta_k\} \right] \\ & = 2^{N(N-1)} \prod_{\ell=1}^{N} \sin \zeta_\ell \prod_{1 \le j < k \le N} \sin(\zeta_k - \zeta_j) \sin(\zeta_k + \zeta_j),\end{aligned}$$

$$\begin{aligned}(\text{type}\, C_N) \quad & \det_{1 \le j,k \le N} \left[\sin\{(\mathcal{N}^{C_N} - 2J^{C_N}(j))\zeta_k\} \right] \\ & = 2^{N(N-1)} \prod_{\ell=1}^{N} \sin(2z_\ell) \prod_{1 \le j < k \le N} \sin(\zeta_k - \zeta_j) \sin(\zeta_k + \zeta_j),\end{aligned}$$

$$\begin{aligned}(\text{type}\, D_N) \quad & \det_{1 \le j,k \le N} \left[\cos\{(\mathcal{N}^{D_N} - 2J^{D_N}(j))\zeta_k\} \right] \\ & = 2^{(N-1)^2} \prod_{1 \le j < k \le N} \sin(\zeta_k - \zeta_j) \sin(\zeta_k + \zeta_j),\end{aligned}$$

where $\mathcal{N}^{R_N}$ and $J^{R_N}(j)$, $R_N = A_{N-1}, B_N, C_N, D_N$, are given by (3.11) *and* (3.12).

B Jacobi Theta Functions

Let

$$z = e^{v\pi i}, \quad q = e^{\tau\pi i},$$

for $v \in \mathbb{C}$ and $\tau \in \mathbb{H}$. The Jacobi theta functions are defined as follows [64, 45],

$$\begin{aligned}\vartheta_0(v;\tau) &= \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2} z^{2n} = 1 + 2\sum_{n=1}^{\infty} (-1)^n e^{\tau\pi i n^2} \cos(2n\pi v), \\ \vartheta_1(v;\tau) &= i\sum_{n\in\mathbb{Z}} (-1)^n q^{(n-1/2)^2} z^{2n-1} = 2\sum_{n=1}^{\infty} (-1)^{n-1} e^{\tau\pi i(n-1/2)^2} \sin\{(2n-1)\pi v\}, \\ \vartheta_2(v;\tau) &= \sum_{n\in\mathbb{Z}} q^{(n-1/2)^2} z^{2n-1} = 2\sum_{n=1}^{\infty} e^{\tau\pi i(n-1/2)^2} \cos\{(2n-1)\pi v\}, \\ \vartheta_3(v;\tau) &= \sum_{n\in\mathbb{Z}} q^{n^2} z^{2n} = 1 + 2\sum_{n=1}^{\infty} e^{\tau\pi i n^2} \cos(2n\pi v). \end{aligned} \tag{B.1}$$

(Note that the present functions $\vartheta_\mu(v;\tau), \mu = 1, 2, 3$ are denoted by $\vartheta_\mu(\pi v, q)$, and $\vartheta_0(v;\tau)$ by $\vartheta_4(\pi v, q)$ in [64].) For $\Im\tau > 0$, $\vartheta_\mu(v;\tau)$, $\mu = 0, 1, 2, 3$ are holomorphic for $|v| < \infty$. The parity with respect to v is given by

$$\vartheta_1(-v;\tau) = -\vartheta_1(v;\tau), \quad \vartheta_\mu(-v;\tau) = \vartheta_\mu(v;\tau), \quad \mu = 0, 2, 3, \tag{B.2}$$

and they have the quasi-double-periodicity;

$$\vartheta_\mu(v+1;\tau) = \begin{cases} \vartheta_\mu(v;\tau), & \mu = 0, 3, \\ -\vartheta_\mu(v;\tau), & \mu = 1, 2, \end{cases} \tag{B.3}$$

$$\vartheta_\mu(v+\tau;\tau) = \begin{cases} -e^{-(2v+\tau)\pi i}\vartheta_\mu(v;\tau), & \mu = 0, 1, \\ e^{-(2v+\tau)\pi i}\vartheta_\mu(v;\tau), & \mu = 2, 3. \end{cases} \tag{B.4}$$

The following relations are derived by (B.1),

$$\begin{aligned}
\vartheta_0\left(v + \frac{\tau}{2};\tau\right) &= ie^{-(v+\tau/4)\pi i}\vartheta_1(v;\tau), \\
\vartheta_1\left(v + \frac{\tau}{2};\tau\right) &= ie^{-(v+\tau/4)\pi i}\vartheta_0(v;\tau), \\
\vartheta_2\left(v + \frac{\tau}{2};\tau\right) &= e^{-(v+\tau/4)\pi i}\vartheta_3(v;\tau), \\
\vartheta_3\left(v + \frac{\tau}{2};\tau\right) &= e^{-(v+\tau/4)\pi i}\vartheta_2(v;\tau).
\end{aligned} \tag{B.5}$$

By the definition (B.1), when $\tau \in \mathbb{H}$,

$$\begin{aligned}
&\vartheta_1(0;\tau) = \vartheta_1(1;\tau) = 0, \qquad \vartheta_1(x;\tau) > 0, \quad x \in (0,1), \\
&\vartheta_2(-1/2;\tau) = \vartheta_2(1/2;\tau) = 0, \qquad \vartheta_2(x;\tau) > 0, \quad x \in (-1/2, 1/2), \\
&\vartheta_0(x;\tau) > 0, \quad \vartheta_3(x;\tau) > 0, \quad x \in \mathbb{R}.
\end{aligned} \tag{B.6}$$

The asymptotics

$$\begin{aligned}
&\vartheta_0(v;\tau) \sim 1, \quad \vartheta_1(v;\tau) \sim 2e^{\tau\pi i/4}\sin(\pi v), \quad \vartheta_2(v;\tau) \sim 2e^{\tau\pi i/4}\cos(\pi v), \quad \vartheta_3(v;\tau) \sim 1, \\
&\qquad \text{in} \quad \Im\tau \to +\infty \quad (\textit{i.e.}, \quad q = e^{\tau\pi i} \to 0)
\end{aligned} \tag{B.7}$$

are known. We will use the following functional equations known as *Jacobi's imaginary transformation* [64, 45],

$$\begin{aligned}
\vartheta_0(v;\tau) &= e^{\pi i/4}\tau^{-1/2}e^{-\pi i v^2/\tau}\vartheta_2\left(\frac{v}{\tau}; -\frac{1}{\tau}\right), \\
\vartheta_1(v;\tau) &= e^{3\pi i/4}\tau^{-1/2}e^{-\pi i v^2/\tau}\vartheta_1\left(\frac{v}{\tau}; -\frac{1}{\tau}\right), \\
\vartheta_2(v;\tau) &= e^{\pi i/4}\tau^{-1/2}e^{-\pi i v^2/\tau}\vartheta_0\left(\frac{v}{\tau}; -\frac{1}{\tau}\right), \\
\vartheta_3(v;\tau) &= e^{\pi i/4}\tau^{-1/2}e^{-\pi i v^2/\tau}\vartheta_3\left(\frac{v}{\tau}; -\frac{1}{\tau}\right).
\end{aligned} \tag{B.8}$$

C Macdonald Denominators

Assume that $N \in \mathbb{N}$. As extensions of the Weyl denominators for classical root systems, Rosengren and Schlosser [51] studied the *Macdonald denominators* for the seven types of irreducible reduced affine root systems [41], $W^{R_N}(\boldsymbol{z};\tau)$, $\boldsymbol{z} = (z_1, \dots, z_N) \in \mathbb{C}^N$, $\tau \in \mathbb{H}$, $R_N = A_{N-1}$,

$B_N, B_N^\vee, C_N, C_N^\vee, BC_N, D_N$, $N \in \mathbb{N}$. (See also [63, 38].) Up to trivial factors they are written using the Jacobi theta functions as follows.

$$\begin{aligned}
W^{A_{N-1}}(\boldsymbol{z};\tau) &= \prod_{1\le j<k\le N} \vartheta_1(z_k - z_j;\tau), \\
W^{B_N}(\boldsymbol{z};\tau) &= \prod_{\ell=1}^{N} \vartheta_1(z_\ell;\tau) \prod_{1\le j<k\le N} \Big\{\vartheta_1(z_k - z_j;\tau)\vartheta_1(z_k + z_j;\tau)\Big\}, \\
W^{B_N^\vee}(\boldsymbol{z};\tau) &= \prod_{\ell=1}^{N} \vartheta_1(2z_\ell;2\tau) \prod_{1\le j<k\le N} \Big\{\vartheta_1(z_k - z_j;\tau)\vartheta_1(z_k + z_j;\tau)\Big\}, \\
W^{C_N}(\boldsymbol{z};\tau) &= \prod_{\ell=1}^{N} \vartheta_1(2z_\ell;\tau) \prod_{1\le j<k\le N} \Big\{\vartheta_1(z_k - z_j;\tau)\vartheta_1(z_k + z_j;\tau)\Big\}, \\
W^{C_N^\vee}(\boldsymbol{z};\tau) &= \prod_{\ell=1}^{N} \vartheta_1\left(z_\ell;\frac{\tau}{2}\right) \prod_{1\le j<k\le N} \Big\{\vartheta_1(z_k - z_j;\tau)\vartheta_1(z_k + z_j;\tau)\Big\}, \\
W^{BC_N}(\boldsymbol{z};\tau) &= \prod_{\ell=1}^{N} \Big\{\vartheta_1(z_\ell;\tau)\vartheta_0(2z_\ell;2\tau)\Big\} \prod_{1\le j<k\le N} \Big\{\vartheta_1(z_k - z_j;\tau)\vartheta_1(z_k + z_j;\tau)\Big\}, \\
W^{D_N}(\boldsymbol{z};\tau) &= \prod_{1\le j<k\le N} \Big\{\vartheta_1(z_k - z_j;\tau)\vartheta_1(z_k + z_j;\tau)\Big\}.
\end{aligned} \tag{C.1}$$

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