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出版情報：Hokkaido mathematical journal. 50 (2), pp.187-205, 2021-06. 北海道大学理学部数学科
バージョン：

Hokkaido Mathematical Journal Vol. 50 (2021) p. 187–205 doi:10.14492/hokmj/2018-938

Asymptotic behavior in time of solutions to complex-valued nonlinear Klein-Gordon equation in one space dimension

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(Received November 29, 2018; Revised February 25, 2020)

Abstract. We consider the long time behavior of solutions to the initial value problem for the "complex-valued" cubic nonlinear Klein-Gordon equation (NLKG) in one space dimension. In [12], Sunagawa derived the L^∞ decay estimate of solutions to (NLKG). In this note, we obtain the large time asymptotic profile of solutions to (NLKG).

Key words: nonlinear Klein-Gordon equation, scattering problem.

1. Introduction

In this note, we consider the long time behavior of solutions to the initial value problem for the "complex-valued" cubic nonlinear Klein-Gordon equation in one space dimension:

$$\begin{cases} (\Box + 1)u = \lambda |u|^2 u & t \in \mathbb{R}, x \in \mathbb{R}, \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x) & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $\Box = \partial_t^2 - \partial_x^2$ is d'Alembertian, $u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is an unknown function, $u_0, u_1 : \mathbb{R} \to \mathbb{C}$ are given functions, and λ is a non-zero real constant. The complex-valued nonlinear Klein-Gordon equation/system arise in various fields of physics. For example, the nonlinear Dirac equation, which is an important model in the relativistic quantum fields [4], [14], can be reduced to a system of complex valued nonlinear Klein-Gordon equations. Therefore we believe that our study will aid in understanding the long time behavior of solution to such physical models.

Since L^∞ decay rate of solution to the one dimensional linear Klein-Gordon equation is $O(|t|^{-1/2})$ as $|t| \to \infty$, the linear scattering theory in-

2020 Mathematics Subject Classification : Primary 35L71; Secondary 35B40, 81Q05.

dicates that the cubic nonlinear term is the long range type. In fact, it is well-known that the non-trivial solutions to (1.1) do not scatter to the free solution, see [5], [6], [11]. Therefore the asymptotic behavior in time of solution to (1.1) is different from that of the linear equation. For the real-valued case, the asymptotic behavior in time of solution to (1.1) is studied by the several authors. Delort [2] obtained an asymptotic profile of a time global solution to the general quasilinear Klein-Gordon equation including (1.1) for the small initial data with compact support. See also Lindblad and Soffer [9] for an alternative proof of his result for (1.1). Note that the compact support assumption in [2] is removed by Hayashi and Naumkin [7] for (1.1). Recently, Stingo [13] extended Delort's result for the general quasilinear Klein-Gordon equation to mildly decaying initial data.

For the complex-valued case, Sunagawa [12] derived the L^∞ decay estimate of solutions to (1.1). The main purpose of this note is to obtain the large time asymptotic profile of solutions to the initial value problem (1.1). We consider the case $t \geqslant 0$ only since the case $t \leqslant 0$ can be treated in a similar way.

Our main result is as follows.

Theorem 1.1 *Let $m \geqslant 11$ be an integer. Then, there exists $\varepsilon_0 > 0$ with the following properties: If u_0 and u_1 are compactly supported and satisfy $\varepsilon := \|u_0\|_{H^m} + \|u_1\|_{H^{m-1}} \leqslant \varepsilon_0$, then, there exists a unique global solution $u \in C([0,\infty); H^m(\mathbb{R})) \cap C^1([0,\infty); H^{m-1}(\mathbb{R}))$ to (1.1) which satisfies*

$$\|u(t)\|_{L^\infty_x} \leqslant C\varepsilon(1+t)^{-1/2} \tag{1.2}$$

for any $t \geqslant 0$. Furthermore, there exist $\Phi_\pm \in L^\infty(\mathbb{R})$ such that

$$\begin{aligned} u(t,x) = {} & \frac{1}{t^{1/2}}\Phi_+\left(\frac{x}{t}\right)\exp\left(i\sqrt{t^2-|x|^2} + i\Psi_+\left(\frac{x}{t}\right)\log t\right) \\ & + \frac{1}{t^{1/2}}\Phi_-\left(\frac{x}{t}\right)\exp\left(-i\sqrt{t^2-|x|^2} + i\Psi_-\left(\frac{x}{t}\right)\log t\right) \\ & + O\left(\varepsilon t^{-3/2+C\varepsilon}\right) \qquad \text{as } t \to \infty, \end{aligned} \tag{1.3}$$

where $\Psi_\pm$ are given by

$$\Psi_+(y) = -\frac{1}{2}\lambda\sqrt{1-|y|^2}\left(|\Phi_+(y)|^2 + 2\,|\Phi_-(y)|^2\right),$$
$$\Psi_-(y) = \frac{1}{2}\lambda\sqrt{1-|y|^2}\left(2\,|\Phi_+(y)|^2 + |\Phi_-(y)|^2\right)$$

and C is a positive constant independent of ε.

From (1.3), we see that an asymptotic profile of time global solution to (1.1) is given by solution to the linear Klein-Gordon equation with a logarithmic phase correction. It is known that $\Phi_- = \overline{\Phi}_+$ for the real-valued case (see [2]). Note that in [10] we constructed a solution to (1.1) which converge to "prescribed" final states in the sense of (1.3).

Remark 1.2 We mention the paper by Candy and Lindblad [1] who obtained the large time asymptotics of solutions for the one dimensional cubic nonlinear Dirac equation which is called the Thirring model [14] and can be reduced to a system of complex-valued Klein-Gordon equations with derivative interactions. Here, the asymptotic behavior of the solution is given by the solution to the linear Dirac equation with a logarithmic phase correction as in Theorem 1.1.

Remark 1.3 In [10], we studied large time behavior of complex-valued solutions to the Klein-Gordon equation with a gauge invariant quadratic nonlinearity in two space dimensions:

$$(\Box + 1)u = \lambda|u|u \qquad t \in \mathbb{R}, x \in \mathbb{R}^2, \tag{1.4}$$

where $\Box = \partial_t^2 - \partial_{x_1}^2 - \partial_{x_2}^2$ is d'Alembertian, $u : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{C}$ is an unknown function, and λ is a non-zero real constant. As in the one dimensional cubic case, (1.4) is long range type. We constructed a solution to (1.4) which converges to prescribed final states, where the final state is given by the free solution with a logarithmic phase correction. Note that the logarithmic phase correction given by [10] has one more parameter which is characterized by the final data. It is an interesting open question whether all small global solutions to (1.4) behave like such an asymptotic profile.

We give an outline of the proof of Theorem 1.1. As in [2], [3], [12], we first reduce the Cauchy problem for (1.1) on $[0,\infty)\times\mathbb{R}$ into the initial value problem in the interior of hyperbola:

$$D = \{(t,x) \in \mathbb{R} \times \mathbb{R} \mid t \geqslant 0,\ (t+2B)^2 - |x|^2 \geqslant \tau_0^2\}, \tag{1.5}$$

where B and τ_0 are positive constants which are determined by the initial data (see Section 2). Then we analyze the solution to (1.1) in D by using the hyperbolic coordinate:

$$t + 2B = \tau \cosh z, \qquad x = \tau \sinh z, \qquad \tau \geqslant \tau_0,\ z \in \mathbb{R}.$$

For a solution u to (1.1), we introduce a new unknown function $v = v(\tau, z)$ via the identity

$$u(t,x) = \frac{1}{\tau^{1/2} \cosh \kappa z} v(\tau, z), \tag{1.6}$$

where κ is a positive constant which is fixed later. Indeed, we shall choose $\kappa > 5/2$ to derive a large time asymptotics of u from an asymptotics of v in τ, see Section 4 for the detail.

Furthermore, we define functions $\phi_\pm = \phi_\pm(\tau, z)$ by

$$v(\tau, z) = \phi_+(\tau, z) e^{i\tau} + \phi_-(\tau, z) e^{-i\tau}.$$

For the real-valued case, we are able to take $\phi_- = \overline{\phi}_+$. However, for the complex-valued case, we cannot expect such a relation. Hence, to determine $\phi_\pm$ uniquely, we impose

$$\partial_\tau v(\tau, z) = i\phi_+(\tau, z) e^{i\tau} - i\phi_-(\tau, z) e^{-i\tau},$$

that is,

$$\phi_\pm(\tau, z) = \frac{1}{2}(v(\tau, z) \mp i\partial_\tau v(\tau, z)) e^{\mp i\tau}.$$

Then the evolution equations for $\phi_\pm$ are given by

$$\begin{aligned} \partial_\tau \phi_\pm = &\mp \frac{i}{2} \frac{\lambda}{\tau (\cosh \kappa z)^2} |\phi_+ e^{i\tau} + \phi_- e^{-i\tau}|^2 (\phi_+ e^{i\tau} + \phi_- e^{-i\tau}) e^{\mp i\tau} \\ &+ O(\tau^{-2}), \end{aligned} \tag{1.7}$$

as $\tau \to \infty$ (see (3.5) for the derivation), where the first term on the right hand side of (1.7) is contribution from the nonlinear term. We rewrite (1.7) as follows:

$$\begin{aligned}\partial_\tau \phi_+ = &-\frac{i}{2}\frac{\lambda}{\tau(\cosh\kappa z)^2}(|\phi_+|^2+2|\phi_-|^2)\phi_+ \\ &-\frac{i}{2}\frac{\lambda}{\tau(\cosh\kappa z)^2}\{\phi_+^2\overline{\phi}_- e^{2i\tau}+(2|\phi_+|^2+|\phi_-|^2)\phi_- e^{-2i\tau}+\overline{\phi}_+\phi_-^2 e^{-4i\tau}\} \\ &+O(\tau^{-2}),\end{aligned}$$

We have a similar equation for ϕ_-. For the second term (non-resonant term), we shall see that the oscillation factors $e^{i\omega\tau}$ enable us to replace the non-resonant term by a term which has better decay by a normal forms procedure. To treat the first term (resonant term), we consider the large time behavior of $|\phi_\pm|^2$. Since the resonant terms for the evolution equations of $|\phi_\pm|^2$ are purely imaginary, we see that $\{|\phi_\pm(\tau,\cdot)|^2\}_{\tau\geqslant\tau_0}$ are Cauchy sequences in L^∞_z as $\tau\to\infty$ and $|\phi_\pm(\tau,\cdot)|^2$ converge to some non-negative functions $a_\pm\in L^\infty_z$ as $\tau\to\infty$. By using the function $a_\pm$ and the gauge transform, we obtain the large time asymptotic profiles of $\phi_\pm$ which yields (1.3).

The rest of the paper is organized as follows. In Section 2, we reduce the Cauchy problem for (1.1) on $[0,\infty)\times\mathbb{R}$ into the initial value problem on the domain D given by (1.5). Then, in Section 3, we derive the L^∞ estimate of the solution to (1.1) in D. In Section 4, we obtain the asymptotic profile of solution to (1.1) by analyzing the large time behaviors of $\phi_\pm$.

2. Reduction of the problem

In this section, we reduce the Cauchy problem for (1.1) on $[0,\infty)\times\mathbb{R}$ into the initial value problem on the interior of the hyperbola D by using the argument by [2], [3], [12].

Let B be a positive constant which satisfies

$$\text{supp } u_0\cup \text{ supp } u_1\subset\{x\in\mathbb{R}\mid |x|\leqslant B\}.$$

We fix a positive number $\tau_0>\max\{1,2B\}$. We employ the following result by Delort [2, Proposition 1.4].

Proposition 2.1 *Let $m\geqslant 3$ be an integer. Then there exists $\varepsilon_0>0$*

with the following properties: *If* $(u_0, u_1) \in H^m(\mathbb{R}) \times H^{m-1}(\mathbb{R})$ *satisfies* $\varepsilon := \|u_0\|_{H^m} + \|u_1\|_{H^{m-1}} \leqslant \varepsilon_0$, *then, there exist* $T \geqslant (\tau_0^2 - 3B^3)/(2B)$ *and unique solution* $u \in C([0,T]; H^m(\mathbb{R})) \cap C^1([0,T]; H^{m-1}(\mathbb{R}))$ *to* (1.1)*. Especially,* u *is defined on* $\{(t,x) \in \mathbb{R}^2 | |x| \leqslant (\tau_0^2 - 3B^2)/(2B), (t+2B)^2 - |x|^2 = \tau_0^2\}$*. Furthermore,* u *satisfies*

$$\sum_{0 \leqslant j+k \leqslant m} \int_{|y| \leqslant \frac{\tau_0^2 - 3B^2}{2B}} \left| \partial_t^j \partial_x^k u \big|_{(t,x) = (\sqrt{\tau_0^2 + y^2} - 2B, y)} \right|^2 dy \leqslant C\varepsilon^2.$$

Proof of Proposition 2.1. By the standard local existence theorem for the nonlinear Klein-Gordon equation (see [8] for instance), there exist $T = O(1/\varepsilon)$ and unique solution $u \in C([0,T]; H^m(\mathbb{R})) \cap C^1([0,T]; H^{m-1}(\mathbb{R}))$ to (1.1) such that

$$\sup_{0 \leqslant t \leqslant T} (\|u(t)\|_{H^m} + \|\partial_t u(t)\|_{H^{m-1}}) \leqslant C\varepsilon.$$

Furthermore, by the property of finite speed of propagation for the solution to (1.1), we have $\operatorname{supp} u(t) \subset \{x \in \mathbb{R} \mid |x| \leqslant t + B\}$ for $0 \leqslant t \leqslant T$. Choosing $\varepsilon > 0$ sufficiently small, we can take $T \geqslant (\tau_0^2 - 3B^2)/(2B)$. Then we have the conclusion. □

By Proposition 2.1 and the property of finite speed of propagation for the solution to (1.1), it suffices to consider the solution u to (1.1) on the domain D.

3. L^∞ estimate of solution

In this section, we derive the L^∞ decay estimate of solution to (1.1) in the region D given by (1.5). Although the L^∞ decay estimate of solution has already been proven by [12], we give the detail of the proof because we use the several estimates in the proof to derive the asymptotic behavior of the solution.

As in the papers [2], [3], [9], [12], we use the hyperbolic coordinate

$$t + 2B = \tau \cosh z, \qquad x = \tau \sinh z, \qquad \tau \geqslant \tau_0, \ z \in \mathbb{R}.$$

Let u be a solution to (1.1). We introduce a new unknown function $v = v(\tau, z)$ via the identity

$$u(t,x) = \frac{1}{\tau^{1/2}\cosh\kappa z} v(\tau, z), \tag{3.1}$$

where $\kappa > 5/2$ is fixed later (see Section 4). Furthermore, we define functions $\phi_\pm = \phi_\pm(\tau, z)$ by

$$v(\tau,z) = \phi_+(\tau,z)e^{i\tau} + \phi_-(\tau,z)e^{-i\tau}, \tag{3.2}$$

$$\partial_\tau v(\tau,z) = i\phi_+(\tau,z)e^{i\tau} - i\phi_-(\tau,z)e^{-i\tau}. \tag{3.3}$$

By a simple calculation, we see that v satisfies

$$\partial_\tau^2 v + v = \frac{\lambda}{\tau(\cosh\kappa z)^2}|v|^2 v + \frac{1}{\tau^2} P v, \tag{3.4}$$

where P is a differential operator given by

$$P = \partial_z^2 - 2\kappa(\tanh\kappa z)\partial_z + \kappa^2 - \frac{1}{4} - \frac{2\kappa^2}{(\cosh\kappa z)^2}.$$

Hence we obtain

$$\partial_\tau \phi_\pm = \mp\frac{i}{2}e^{\mp i\tau}(\partial_\tau^2 v + v) = \mp\frac{i}{2}\frac{\lambda}{\tau(\cosh\kappa z)^2}|v|^2 v e^{\mp i\tau} \mp \frac{i}{2}\frac{1}{\tau^2} P v e^{\mp i\tau}. \tag{3.5}$$

We derive the L^∞ estimates for $\phi_\pm$. For $T \geqslant \tau_0$, let

$$M_T := \sup_{(\tau,z)\in[\tau_0,T]\times\mathbb{R}} \left(|\phi_+(\tau,z)|^2 + |\phi_-(\tau,z)|^2\right)^{1/2}.$$

Lemma 3.1 *Let $m = \lfloor 2\kappa \rfloor + 6$[1] and $\kappa > 0$. Assume that $\|u_0\|_{H^m} + \|u_1\|_{H^{m-1}} \leqslant \varepsilon_0$, where ε_0 is given by Proposition 2.1. Then, there exists $\varepsilon_1 > 0$ such that if $M_T \leqslant \varepsilon^{1/2}$ holds for some $0 < \varepsilon \leqslant \varepsilon_1$ and $T \geqslant \tau_0$, then we have $M_T \leqslant C\varepsilon$, where C is a positive constant independent of ε and T.*

If $\kappa > 5/2$ and $\|u_0\|_{H^{\lfloor 2\kappa\rfloor+6}} + \|u_1\|_{H^{\lfloor 2\kappa\rfloor+5}}$ is sufficiently small, then Proposition 2.1 ensures that $M_{\tau_0} \leqslant \varepsilon \leqslant \varepsilon_1$. Therefore we need $m \geqslant 11$ in Theorem 1.1. Combining this with Lemma 3.1 and the standard continuity

[1] $\lfloor \cdot \rfloor$ denotes the usual floor function.

argument, we obtain $M_T \leqslant C\varepsilon$ for any $T \geqslant \tau_0$. Hence we obtain the decay estimate (1.2) for the solution to (1.1).

Proof of Lemma 3.1. We assume that v satisfies $M_T \leqslant \varepsilon^{1/2}$. Throughout the proof, we denote by C or C_j positive constants which are independent of ε and T.

We first prove

$$|Pv(\tau, z)| \leqslant C\varepsilon\tau^{C_1\varepsilon} \tag{3.6}$$

for any $\tau_0 \leqslant \tau \leqslant T$. To show (3.6), we employ the energy estimates used by Delort, Fang and Xue [3]. Let us define

$$E_m[v](\tau) := \frac{1}{2}\sum_{j=1}^{m}\int_{\mathbb{R}}\left(\left|\partial_\tau\partial_z^j v(\tau,z)\right|^2 + \frac{1}{\tau^2}\left|\partial_z^{1+j}v(\tau,z)\right|^2 + \left|\partial_z^j v(\tau,z)\right|^2\right) dz.$$

We shall show

$$E_3[v](\tau) \leqslant C\varepsilon^2\tau^{C\varepsilon}. \tag{3.7}$$

Then (3.6) follows from the Sobolev embedding and (3.7). Let us prove (3.7). By a similar argument as that in [3, Proposition 2.1.2], for any $m \in \mathbb{Z}_+$ and $\ell = 0, 1$, we have

$$\begin{aligned}\frac{d}{d\tau}E_m[v](\tau) \leqslant{}& \frac{2\kappa}{\tau^{1+\ell}}E_{m+\ell}[v](\tau) + CE_m[v]^{1/2}(\tau)\left\|\left(\partial_\tau^2 - \tau^{-2}P + 1\right)v(\tau)\right\|_{H_z^m}\\ &+ \frac{C}{\tau^2}E_m[v](\tau).\end{aligned}$$

Equation (3.4) and the assumption $M_T \leqslant \varepsilon^{1/2}$ yield

$$\begin{aligned}&\left\|\left(\partial_\tau^2 - \tau^{-2}P + 1\right)v(\tau)\right\|_{H_z^m}\\ &\quad\leqslant \frac{|\lambda|}{\tau}\||v|^2v\|_{H_z^m} \leqslant C\frac{|\lambda|}{\tau}\|v\|_{L_z^\infty}^2\|v\|_{H_z^m} \leqslant \frac{C}{\tau}\varepsilon E_m[v]^{1/2}(\tau).\end{aligned}$$

Hence

$$\frac{d}{d\tau}E_m[v](\tau) \leqslant \frac{2\kappa}{\tau^{1+\ell}}E_{m+\ell}[v](\tau) + \left(\frac{C\varepsilon}{\tau} + \frac{C}{\tau^2}\right)E_m[v](\tau). \tag{3.8}$$

By (3.8) with $m = \lfloor 2\kappa \rfloor + 5$ and $\ell = 0$, we obtain

$$\begin{aligned}\frac{d}{d\tau}E_{\lfloor 2\kappa\rfloor+5}[v](\tau) &\leqslant \left(\frac{2\kappa + C_2\varepsilon}{\tau} + \frac{C}{\tau^2}\right)E_{\lfloor 2\kappa\rfloor+5}[v](\tau)\\ &\leqslant \left(\frac{\lfloor 2\kappa\rfloor + 5/4}{\tau} + \frac{C}{\tau^2}\right)E_{\lfloor 2\kappa\rfloor+5}[v](\tau),\end{aligned}$$

provided $C_2\varepsilon \leqslant 1/4$. Then the Gronwall lemma and Proposition 2.1 yield

$$\begin{aligned}E_{\lfloor 2\kappa\rfloor+5}[v](\tau) &\leqslant E_{\lfloor 2\kappa\rfloor+5}[v](\tau_0)\exp\left(\int_{\tau_0}^{\tau}\left(\frac{\lfloor 2\kappa\rfloor + 5/4}{\sigma} + \frac{C}{\sigma^2}\right)d\sigma\right)\\ &\leqslant C\varepsilon^2\tau^{\lfloor 2\kappa\rfloor+5/4}.\end{aligned} \tag{3.9}$$

Combining (3.9) and (3.8) with $m = \lfloor 2\kappa\rfloor + 4$ and $\ell = 1$, we obtain

$$\begin{aligned}\frac{d}{d\tau}E_{\lfloor 2\kappa\rfloor+4}[v](\tau) &\leqslant \frac{2\kappa}{\tau^2}E_{\lfloor 2\kappa\rfloor+5}[v](\tau) + \left(\frac{C\varepsilon}{\tau} + \frac{C}{\tau^2}\right)E_{\lfloor 2\kappa\rfloor+4}[v](\tau)\\ &\leqslant C\varepsilon^2\tau^{\lfloor 2\kappa\rfloor-3/4} + \left(\frac{C\varepsilon}{\tau} + \frac{C}{\tau^2}\right)E_{\lfloor 2\kappa\rfloor+4}[v](\tau).\end{aligned}$$

Then by the Gronwall lemma and Proposition 2.1, we have

$$E_{\lfloor 2\kappa\rfloor+4}[v](\tau) \leqslant C\varepsilon^2\tau^{\lfloor 2\kappa\rfloor+1/4}.$$

Repeating this argument, we obtain

$$E_4[v](\tau) \leqslant C\varepsilon^2\tau^{1/4}. \tag{3.10}$$

Combining (3.10) and (3.8) with $m = 3$ and $\ell = 1$, we have

$$\begin{aligned}\frac{d}{d\tau}E_3[v](\tau) &\leqslant \frac{2\kappa}{\tau^2}E_4[v](\tau) + \left(\frac{C\varepsilon}{\tau} + \frac{C}{\tau^2}\right)E_3[v](\tau)\\ &\leqslant \frac{C\varepsilon^2}{\tau^{7/4}} + \left(\frac{C\varepsilon}{\tau} + \frac{C}{\tau^2}\right)E_3[v](\tau).\end{aligned}$$

Then the Gronwall lemma and Proposition 2.1 yield (3.7).

We turn to the estimates for $\phi_\pm$. From (3.5), (3.6) and the assumption $M_T \leqslant \varepsilon^{1/2}$, we see

$$|\partial_\tau \phi_\pm(\tau, z)| \leqslant \frac{C\varepsilon^{3/2}}{\tau} + \frac{C\varepsilon}{\tau^{2-C_1\varepsilon}} \leqslant \frac{C\varepsilon}{\tau}, \tag{3.11}$$

provided $C_1\varepsilon \leqslant 1$. This is not enough to obtain the boundedness of $\phi_\pm$. To derive the desired bound, we make use of the oscillation factors in the nonlinear term. A direct calculation shows

$$\begin{aligned} |v|^2 v = {} & \phi_+^2 \overline{\phi}_- e^{3i\tau} + (|\phi_+|^2 + 2|\phi_-|^2)\phi_+ e^{i\tau} \\ & + (2|\phi_+|^2 + |\phi_-|^2)\phi_- e^{-i\tau} + \overline{\phi}_+ \phi_-^2 e^{-3i\tau}. \end{aligned} \tag{3.12}$$

Substituting (3.12) into (3.5), we find

$$\partial_\tau \phi_+ = -\frac{i}{2}\frac{\lambda}{\tau(\cosh \kappa z)^2}(|\phi_+|^2 + 2|\phi_-|^2)\phi_+ + \frac{1}{\tau}N_+ - \frac{i}{2}\frac{1}{\tau^2}Pve^{-i\tau}, \tag{3.13}$$

$$\partial_\tau \phi_- = \frac{i}{2}\frac{\lambda}{\tau(\cosh \kappa z)^2}(2|\phi_+|^2 + |\phi_-|^2)\phi_- + \frac{1}{\tau}N_- + \frac{i}{2}\frac{1}{\tau^2}Pve^{i\tau}, \tag{3.14}$$

where $N_\pm = N_\pm(\tau, z)$ are given by

$$N_+ = -\frac{i}{2}\frac{\lambda}{(\cosh \kappa z)^2}\{\phi_+^2 \overline{\phi}_- e^{2i\tau} + (2|\phi_+|^2 + |\phi_-|^2)\phi_- e^{-2i\tau} + \overline{\phi}_+ \phi_-^2 e^{-4i\tau}\},$$

$$N_- = \frac{i}{2}\frac{\lambda}{(\cosh \kappa z)^2}\{\phi_+^2 \overline{\phi}_- e^{4i\tau} + (|\phi_+|^2 + 2|\phi_-|^2)\phi_+ e^{2i\tau} + \overline{\phi}_+ \phi_-^2 e^{-2i\tau}\}.$$

Since the first term on the right hand side of (3.13) is purely imaginary, (3.13) implies

$$\begin{aligned} |\phi_+(\tau, z)|^2 = {} & |\phi_+(\tau_0, z)|^2 + 2\int_{\tau_0}^{\tau} \frac{1}{\sigma}\Re(N_+\overline{\phi}_+)d\sigma \\ & + \int_{\tau_0}^{\tau} \frac{1}{\sigma^2}\Im(Pv\overline{\phi}_+ e^{-i\tau})d\sigma. \end{aligned} \tag{3.15}$$

By the Sobolev embedding and Proposition 2.1, we have

$$|\phi_+(\tau_0, z)|^2 \leqslant C\varepsilon^2. \tag{3.16}$$

To evaluate the second term on the right hand side of (3.15), we make use of the oscillation factor $e^{i\omega\tau}, \omega \neq 0$. Integrating by parts, we have

$$\begin{aligned}
&\int_{\tau_0}^{\tau} \frac{e^{2i\sigma}}{\sigma} |\phi_+|^2 \phi_+ \overline{\phi}_-(\sigma, z) d\sigma \\
&\quad = \frac{e^{2i\tau}}{2i\tau} |\phi_+|^2 \phi_+ \overline{\phi}_-(\tau, z) - \frac{e^{2i\tau}}{2i\tau_0} |\phi_+|^2 \phi_+ \overline{\phi}_-(\tau_0, z) \\
&\qquad + \frac{1}{2i} \int_{\tau_0}^{\tau} \frac{e^{2i\sigma}}{\sigma^2} |\phi_+|^2 \phi_+ \overline{\phi}_-(\sigma, z) d\sigma - \frac{1}{2i} \int_{\tau_0}^{\tau} \frac{e^{2i\sigma}}{\sigma} \partial_\sigma (|\phi_+|^2 \phi_+ \overline{\phi}_-)(\sigma, z) d\sigma.
\end{aligned}$$

Combining the above identity, $M_T \leqslant \varepsilon^{1/2}$ and (3.11), we find

$$\begin{aligned}
&\left| \int_{\tau_0}^{\tau} \frac{e^{2i\sigma}}{\sigma} |\phi_+|^2 \phi_+ \overline{\phi}_-(\sigma, z) d\sigma \right| \\
&\quad \leqslant \frac{C\varepsilon^2}{\tau} + C\varepsilon^2 + C\varepsilon^2 \int_{\tau_0}^{\tau} \frac{d\sigma}{\sigma^2} + C\varepsilon^2 \int_{\tau_0}^{\tau} \frac{d\sigma}{\sigma^2} \leqslant C\varepsilon^2.
\end{aligned}$$

In a similar way, we obtain

$$\left| 2 \int_{\tau_0}^{\tau} \frac{1}{\sigma} \Re(N_+ \overline{\phi}_+) d\sigma \right| \leqslant C\varepsilon^2. \tag{3.17}$$

For the third term on the right hand side of (3.15), the Young inequality and (3.6) yield

$$\begin{aligned}
\left| \int_{\tau_0}^{\tau} \frac{1}{\sigma^2} \Im(Pv\overline{\phi}_+ e^{-i\tau}) d\sigma \right| &\leqslant \frac{1}{2} \int_{\tau_0}^{\tau} \frac{1}{\sigma^2} (|Pv|^2 + |\phi_+|^2) d\sigma \\
&\leqslant C\varepsilon^2 \int_{\tau_0}^{\tau} \frac{1}{\sigma^{2-2C_1\varepsilon}} d\sigma + \int_{\tau_0}^{\tau} \frac{1}{\sigma^2} |\phi_+|^2 d\sigma \\
&\leqslant C\varepsilon^2 + \int_{\tau_0}^{\tau} \frac{1}{\sigma^2} |\phi_+|^2 d\sigma,
\end{aligned} \tag{3.18}$$

provided that $C_1\varepsilon \leqslant 1/4$.

Let $\varepsilon_1 = \min(1, 1/(4C_1), 1/(4C_2))$. Then by (3.15), (3.16), (3.17) and (3.18), we see that if $0 < \varepsilon \leqslant \varepsilon_1$, then

$$|\phi_+(\tau, z)|^2 \leqslant C\varepsilon^2 + \int_{\tau_0}^{\tau} \frac{1}{\sigma^2}|\phi_+|^2(\sigma, z)d\sigma.$$

We have the similar estimate for ϕ_-. Then the Gronwall lemma yields $M_T \leqslant C\varepsilon$. This completes the proof of Lemma 3.1. □

4. Large time asymptotics of solution

In this section we derive the large time asymptotics of solution to (1.1). By Lemma 3.1, for any $\tau \geqslant \tau_0$ and $z \in \mathbb{R}$, we have

$$|\phi_\pm(\tau, z)| \leqslant C\varepsilon. \tag{4.1}$$

Combining (4.1) with the argument used in the previous section, for any $\tau \geqslant \tau_0$ and $z \in \mathbb{R}$, we have

$$|Pv(\tau, z)| \leqslant C\varepsilon\tau^{C\varepsilon}, \qquad |\partial_\tau\phi_\pm(\tau, z)| \leqslant \frac{C\varepsilon}{\tau}. \tag{4.2}$$

From (3.13), for $\tau_0 \leqslant \tau_1 < \tau_2$ we have

$$\begin{aligned} &|\phi_+(\tau_2, z)|^2 - |\phi_+(\tau_1, z)|^2 \\ &\quad = 2\int_{\tau_1}^{\tau_2} \frac{1}{\sigma}\Re(N_+\overline{\phi}_+)d\sigma + \int_{\tau_1}^{\tau_2} \frac{1}{\sigma^2}\Im(Pv\overline{\phi}_+e^{-i\tau})d\sigma. \end{aligned} \tag{4.3}$$

Combining the same argument as that in (3.17) with (4.1) and (4.2), we have

$$\left|2\int_{\tau_1}^{\tau_2} \frac{1}{\sigma}\Re(N_+\overline{\phi}_+)d\sigma\right| \leqslant \frac{C\varepsilon^4}{\tau_1}. \tag{4.4}$$

From (4.1) and (4.2), we see

$$\left|\int_{\tau_1}^{\tau_2} \frac{1}{\sigma^2}\Im(Pv\overline{\phi}_+e^{-i\tau})d\sigma\right| \leqslant \frac{C\varepsilon^2}{\tau_1^{1-C\varepsilon}}. \tag{4.5}$$

Collecting (4.3), (4.4) and (4.5), we obtain

$$\big||\phi_+(\tau_2,z)|^2 - |\phi_+(\tau_1,z)|^2\big| \leqslant \frac{C\varepsilon^2}{\tau_1^{1-C\varepsilon}}. \tag{4.6}$$

We have the similar estimate for ϕ_-. Therefore, we find that $\{|\phi_\pm(\tau,\cdot)|^2\}_{\tau\geqslant\tau_0}$ are Cauchy sequences in L^∞_z as $\tau\to\infty$. Hence there exist non-negative functions $a_\pm\in L^\infty_z$ such that

$$\||\phi_\pm(\tau)|^2 - a_\pm\|_{L^\infty_z} \leqslant \frac{C\varepsilon^2}{\tau^{1-C\varepsilon}} \tag{4.7}$$

for $\tau\geqslant\tau_0$. Note that a_+ satisfies

$$a_+(z) = |\phi_+(\tau_0,z)|^2 + 2\int_{\tau_0}^\infty \frac{1}{\sigma}\Re(N_+\overline{\phi}_+)d\sigma + \int_{\tau_0}^\infty \frac{1}{\sigma^2}\Im(Pv\overline{\phi}_+e^{-i\tau})d\sigma.$$

Hence (3.6) yields $a_+\in W^{1,\infty}_z$. Similarly, we have $a_-\in W^{1,\infty}_z$.

By (3.13) and (3.14),

$$\partial_\tau\phi_+(z) = -\frac{i}{2}\frac{\lambda}{\tau(\cosh\kappa z)^2}(a_+(z)+2a_-(z))\phi_+(z) + R_+(\tau,z), \tag{4.8}$$

$$\partial_\tau\phi_-(z) = \frac{i}{2}\frac{\lambda}{\tau(\cosh\kappa z)^2}(2a_+(z)+a_-(z))\phi_-(z) + R_-(\tau,z), \tag{4.9}$$

where $R_\pm = R_\pm(\tau,z)$ are given by

$$\begin{aligned} R_+ &= -\frac{i}{2}\frac{\lambda}{\tau(\cosh\kappa z)^2}(|\phi_+|^2+2|\phi_-|^2)\phi_+ + \frac{i}{2}\frac{\lambda}{\tau(\cosh\kappa z)^2}(a_++2a_-)\phi_+ \\ &\quad + \frac{1}{\tau}N_+ - \frac{i}{2}\frac{1}{\tau^2}Pve^{-i\tau}, \\ R_- &= \frac{i}{2}\frac{\lambda}{\tau(\cosh\kappa z)^2}(2|\phi_+|^2+|\phi_-|^2)\phi_- - \frac{i}{2}\frac{\lambda}{\tau(\cosh\kappa z)^2}(2a_++a_-)\phi_- \\ &\quad + \frac{1}{\tau}N_- + \frac{i}{2}\frac{1}{\tau^2}Pve^{i\tau}. \end{aligned}$$

It follows from (4.8) and (4.9) that

$$\partial_\tau \left\{ \phi_+(z) \exp\left(\frac{i}{2} \frac{\lambda}{(\cosh \kappa z)^2} (a_+(z) + 2a_-(z)) \log \tau \right) \right\}$$
$$= R_+(\tau, z) \exp\left(\frac{i}{2} \frac{\lambda}{(\cosh \kappa z)^2} (a_+(z) + 2a_-(z)) \log \tau \right),$$
$$\partial_\tau \left\{ \phi_-(z) \exp\left(-\frac{i}{2} \frac{\lambda}{(\cosh \kappa z)^2} (2a_+(z) + a_-(z)) \log \tau \right) \right\}$$
$$= R_-(\tau, z) \exp\left(-\frac{i}{2} \frac{\lambda}{(\cosh \kappa z)^2} (2a_+(z) + a_-(z)) \log \tau \right).$$

By (4.1), (4.2), (4.7) and the argument used in the proof of (3.17), we have

$$\left| \int_{\tau_1}^{\tau_2} R_+(\sigma, z) \exp\left(\frac{i}{2} \frac{\lambda}{(\cosh \kappa z)^2} (a_+(z) + 2a_-(z)) \log \sigma \right) d\sigma \right| \leqslant C\varepsilon \tau_1^{-1+C\varepsilon},$$
$$\left| \int_{\tau_1}^{\tau_2} R_1(\sigma, z) \exp\left(-\frac{i}{2} \frac{\lambda}{(\cosh \kappa z)^2} (2a_+(z) + a_-(z)) \log \sigma \right) d\sigma \right| \leqslant C\varepsilon \tau_1^{-1+C\varepsilon}$$

for any $0 < \tau_1 < \tau_2$. By the above estimates, we see that $\phi_+ \exp(\frac{i}{2} \frac{\lambda}{(\cosh \kappa z)^2} (a_+ + 2a_-) \log \tau)$ and $\phi_- \exp(-\frac{i}{2} \frac{\lambda}{(\cosh \kappa z)^2} (2a_+ + a_-) \log \tau)$ are Cauchy sequences in L_z^∞ as $\tau \to \infty$. Therefore we find that there exist functions $b_\pm \in L_z^\infty$ such that

$$\left\| \phi_+(\tau) \exp\left(\frac{i}{2} \frac{\lambda}{(\cosh \kappa z)^2} (a_+ + 2a_-) \log \tau \right) - b_+ \right\|_{L_z^\infty} \leqslant \frac{C\varepsilon}{\tau^{1-C\varepsilon}},$$
$$\left\| \phi_-(\tau) \exp\left(-\frac{i}{2} \frac{\lambda}{(\cosh \kappa z)^2} (2a_+ + a_-) \log \tau \right) - b_- \right\|_{L_z^\infty} \leqslant \frac{C\varepsilon}{\tau^{1-C\varepsilon}}$$

for $\tau \geqslant \tau_0$. Especially, we have $a_\pm = |b_\pm|^2$ and $b_\pm \in W_z^{1,\infty}$. Hence we have

$$\phi_+(\tau, z) = b_+(z) \exp\left(-\frac{i}{2} \frac{\lambda}{(\cosh \kappa z)^2} (|b_+(z)|^2 + 2|b_-(z)|^2) \log \tau \right) + O\left(\frac{\varepsilon}{\tau^{1-C\varepsilon}} \right), \tag{4.10}$$

$$\phi_-(\tau,z) = b_-(z)\exp\left(\frac{i}{2}\frac{\lambda}{(\cosh\kappa z)^2}(2|b_+(z)|^2+|b_-(z)|^2)\log\tau\right)$$
$$+\,O\left(\frac{\varepsilon}{\tau^{1-C\varepsilon}}\right) \tag{4.11}$$

as $\tau\to\infty$. Let

$$c_\pm(z) := \frac{(\cosh z)^{1/2}}{\cosh\kappa z}b_\pm(z), \qquad d_\pm(y) := \begin{cases} c_\pm(\tanh^{-1}y) & \text{if } |y|<1, \\ 0 & \text{if } |y|\geqslant 1. \end{cases}$$

Then by (3.1), (3.2), (4.10) and (4.11), we have

$$\begin{aligned} u(t,x) = {} & \frac{1}{(t+2B)^{1/2}}d_+(y)\exp\Biggl(i(t+2B)\sqrt{1-|y|^2} \\ & -\frac{i}{2}\lambda\sqrt{1-|y|^2}\left(|d_+(y)|^2+2|d_-(y)|^2\right) \\ & \times\log\left((t+2B)\sqrt{1-|y|^2}\right)\Biggr)\Bigg|_{y=\frac{x}{t+2B}} \\ & +\frac{1}{(t+2B)^{1/2}}d_-(y)\exp\Biggl(-i(t+2B)\sqrt{1-|y|^2} \\ & +\frac{i}{2}\lambda\sqrt{1-|y|^2}\left(2|d_+(y)|^2+|d_-(y)|^2\right) \\ & \times\log\left((t+2B)\sqrt{1-|y|^2}\right)\Biggr)\Bigg|_{y=\frac{x}{t+2B}} \\ & +O\left(\varepsilon t^{-\frac{3}{2}+C\varepsilon}\right), \end{aligned} \tag{4.12}$$

as $t\to\infty$.

Furthermore, introducing

$$\begin{aligned} e_+(y) := {} & d_+(y)\exp\left(2iB\sqrt{1-|y|^2}\right) \\ & \times\exp\left(-\frac{i}{2}\lambda\sqrt{1-|y|^2}(|d_+(y)|^2+2|d_-(y)|^2)\log\sqrt{1-|y|^2}\right), \end{aligned}$$

$$
\begin{aligned}
e_-(y) :=& \, d_-(y) \exp\left(-2iB\sqrt{1-|y|^2}\right) \\
&\times \exp\left(\frac{i}{2}\lambda\sqrt{1-|y|^2}(2|d_+(y)|^2 + |d_-(y)|^2)\log\sqrt{1-|y|^2}\right),
\end{aligned}
$$

we see that the first term on the right hand side of (4.12) is equal to

$$
\begin{aligned}
&\frac{1}{(t+2B)^{1/2}} e_+(y)\exp\left(it\sqrt{1-|y|^2}\right) \\
&\times \exp\left(-\frac{i}{2}\lambda\sqrt{1-|y|^2}\left(|e_+(y)|^2 + 2|e_-(y)|^2\right)\log t\right) \\
&\times \exp\left(i\Theta_+(t,y)\right)\Big|_{y=\frac{x}{t+2B}} + O\left(\varepsilon t^{-3/2+C\varepsilon}\right),
\end{aligned} \tag{4.13}
$$

as $t \to \infty$, where Θ_+ is defined by

$$
\Theta_+(t,y) = -\frac{i}{2}\lambda\sqrt{1-|y|^2}(|d_+(y)|^2 + 2|d_-(y)|^2)\log\left(1+\frac{2B}{t}\right).
$$

A direct calculation shows (c.f. [2, page 59])

$$
\left|\frac{1}{(t+2B)^{1/2}} - \frac{1}{t^{1/2}}\right| \leqslant Ct^{-3/2}, \tag{4.14}
$$

$$
\begin{aligned}
&\left|\sqrt{1-\left|\frac{x}{t+2B}\right|^2} - \sqrt{1-\left|\frac{x}{t}\right|^2} - \frac{2B\,(x/t)^2}{t\sqrt{1-|x/t|^2}}\right| \\
&\qquad \leqslant Ct^{-2}\left(1-\left|\frac{x}{t}\right|^2\right)^{-3/2}.
\end{aligned} \tag{4.15}
$$

Since $b_\pm \in W_z^{1,\infty}$, we see that the function

$$
g_+(t,y) = e_+(y)\exp\left(-\frac{i}{2}\lambda\sqrt{1-|y|^2}(|e_+(y)|^2 + 2|e_-(y)|^2)\log t\right)
$$

satisfies

$$|\partial_y^j g_+(t,y)| \leqslant C_j \varepsilon (\log t)^j (1-|y|)^{\kappa/2-1/4-j}$$

for $j = 0, 1$. Hence

$$\begin{aligned}
&\left| g_+\left(t, \frac{x}{t+2B}\right) - g_+\left(t, \frac{x}{t}\right)\right| \\
&\leqslant C_1 \varepsilon \frac{\log t}{t} \left\{ \left(1 - \frac{|x|}{t}\right)_+ + \frac{1}{t} \right\}^{\kappa/2-5/4} \leqslant C_1 \varepsilon \frac{\log t}{t} \qquad (4.16)
\end{aligned}$$

for $\kappa > 5/2$. We also note

$$\exp\left(i\Theta_+\left(t, \frac{x}{t+2B}\right)\right) = 1 + O\left(\frac{\varepsilon^2}{t}\right). \qquad (4.17)$$

Combining (4.13), (4.14), (4.15), (4.16) and (4.17), we have that the first term on the right hand side of (4.12) is equal to

$$\begin{aligned}
&\frac{1}{t^{1/2}} e_+\left(\frac{x}{t}\right) \exp\left(it\sqrt{1 - \left|\frac{x}{t}\right|^2} + i\frac{2B\,(x/t)^2}{\sqrt{1-|x/t|^2}} \right) \\
&\quad \times \exp\left(-\frac{i}{2}\lambda \sqrt{1 - \left|\frac{x}{t}\right|^2} \left(\left| e_+\left(\frac{x}{t}\right)\right|^2 + 2\left| e_-\left(\frac{x}{t}\right)\right|^2 \right) \log t \right) \\
&+ O\left(\varepsilon t^{-3/2+C\varepsilon}\right),
\end{aligned}$$

as $t \to \infty$. We have a similar asymptotic formula for the second term on the right hand side of (4.12). Hence choosing $\Phi_\pm(y) = e_\pm(y)\exp\left(\pm i \frac{2By^2}{\sqrt{1-|y|^2}}\right)$, we obtain the asymptotic formula (1.3). This completes the proof of Theorem 1.1.

Acknowledgments We thank the referee for careful reading of the manuscript and for fruitful suggestions. J.S was supported by JSPS KAKENHI Grant Number JP17H02851.

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