Quasi-derivation relations for multiple zeta values revisited

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QUASI-DERIVATION RELATIONS FOR MULTIPLE ZETA VALUES REVISITED

MASANOBU KANEKO, HIDEKI MURAHARA, AND TAKUYA MURAKAMI

Abstract. We take another look at the so-called quasi-derivation relations in the theory of multiple zeta values, by giving a certain formula for the quasi-derivation operator. In doing so, we are not only able to prove the quasi-derivation relations in a simpler manner but also give an analog of the quasi-derivation relations for finite multiple zeta values.

1. Introduction

The *quasi-derivation relations* in the theory of multiple zeta values is a generalization, proposed by the first-named author and established by T. Tanaka, of a set of linear relations known as *derivation relations*, which we are first going to recall.

We use Hoffman's algebraic setup ([5]) with a slightly different convention. Let $\mathfrak{H} := \mathbb{Q}\langle x, y\rangle$ be the noncommutative polynomial algebra in two indeterminates x and y. This was introduced in order to encode multiple zeta values in the way the monomial $yx^{k_1-1}yx^{k_2-1}\cdots yx^{k_r-1}$ corresponds to the multiple zeta value

$$\zeta(k_1,k_2,\ldots,k_r) := \sum_{0<n_1<\cdots<n_r} \frac{1}{n_1^{k_1}n_2^{k_2}\cdots n_r^{k_r}}$$

when $k_r > 1$, which is a real number as the limiting value of a convergent series. If we denote by Z the $\mathbb{Q}$-linear map from $y\mathfrak{H}x$ to $\mathbb{R}$ assigning each monomial $yx^{k_1-1}yx^{k_2-1}\cdots yx^{k_r-1}$ to $\zeta(k_1,\ldots,k_r)$, the derivation relations state that

$$Z(\partial_n(w)) = 0$$

for all $n \geq 1$ and $w \in y\mathfrak{H}x$. Here the operator ∂_n is a $\mathbb{Q}$-linear derivation on $\mathfrak{H}$ determined uniquely by $\partial_n(x) = y(x+y)^{n-1}x$ and $\partial_n(y) = -y(x+y)^{n-1}x$. Set $z = x+y$, so that $\partial_n(z) = 0$. We use this repeatedly in the sequel.

In order to introduce the quasi-derivation relations, we first define a $\mathbb{Q}$-linear map $\theta := \theta^{(c)} : \mathfrak{H} \to \mathfrak{H}$ with a parameter $c \in \mathbb{Q}$ (we often drop c from the notation) by setting

$$\theta(u) = uz = u(x+y) \quad \text{for} \quad u = x,\, y$$

and requiring

$$\theta(ww') = \theta(w)w' + w\theta(w') + cH(w)\partial_1(w')$$

for $w, w' \in \mathfrak{H}$, where H is the $\mathbb{Q}$-linear map from $\mathfrak{H}$ to itself defined by $H(w) = \deg(w)\cdot w$ for any monomial $w \in \mathfrak{H}$ ($\deg(w)$ is the degree of w). This is well defined because H is a derivation on $\mathfrak{H}$. Now we define the quasi-derivation map $\partial_n^{(c)}$. Write $\mathrm{ad}(\theta)$ the adjoint operator by θ, i.e., $\mathrm{ad}(\theta)(\partial) := [\theta, \partial] = \theta\partial - \partial\theta$.

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Definition 1.1. For each positive integer n and any rational number c, we define a $\mathbb{Q}$-linear map $\partial_n^{(c)} \colon \mathfrak{H} \to \mathfrak{H}$ by

$$\partial_n^{(c)} := \frac{1}{(n-1)!}\mathrm{ad}(\theta)^{n-1}(\partial_1).$$

Then the quasi-derivation relations of Tanaka [13] is stated as

$$Z(\partial_n^{(c)}(w)) = 0$$

for all $n \geq 1$, $c \in \mathbb{Q}$, and $w \in y\mathfrak{H}x$. Our aim in this paper is to take another look at this relation, or rather at the operator $\partial_n^{(c)}$.

Remark 1.2. 1) We have changed the definition of $\theta = \theta^{(c)}$ by shifting the original ([8, 13]) by the derivation $w \to [z, w]/2 = (zw - wz)/2$. However, we can check that this does not change $\partial_n^{(c)}(w)$. Note also that the convention of the order of the product in $\mathfrak{H}$ there is opposite from ours.

2) As noted in [6], the special case $c = 0$ gives the original derivation ∂_n: $\partial_n = \partial_n^{(0)}$. This together with works of Connes-Moscovicci [1, 2] motivated us to define $\partial_n^{(c)}(w)$ in [8].

3) From $\theta(z^r) = rz^{r+1}$ $(r \geq 1)$ and $\partial_n(z) = 0$, we see that $\partial_n^{(c)}(wz) = \partial_n^{(c)}(w)z$ and $\partial_n^{(c)}(zw) = z\partial_n^{(c)}(w)$. We need to use this at several points later.

2. Main Theorem

We present a formula for $\partial_n^{(c)}(w)$ when w is in $\mathfrak{H}x$. To describe the formula, we define a product $\diamond$ on $\mathfrak{H}$ introduced in Hirose-Murahara-Onozuka [3] by

$$w_1 \diamond w_2 := \phi\big(\phi(w_1) * \phi(w_2)\big) \quad (w_1, w_2 \in \mathfrak{H}), \tag{1}$$

where ϕ is an involutive automorphism of $\mathfrak{H}$ determined by

$$\phi(x) = z = x + y \quad \text{and} \quad \phi(y) = -y,$$

and $*$ is the harmonic product on $\mathfrak{H}$ (see [5, 4] for the precise definition of $*$). This is an associative and commutative binary operation with $1 \diamond w = w \diamond 1 = w$ for any $w \in \mathfrak{H}$. In [3], the definition of $\diamond$ is given in an inductive manner like the definition of $*$ in [4]. Later we only use the shuffle-type equality

$$xw_1 \diamond yw_2 = x(w_1 \diamond yw_2) + y(xw_1 \diamond w_2), \tag{2}$$

which holds for any $w_1, w_2 \in \mathfrak{H}$.

We define a specific element $q_n = q_n^{(c)}$ in $\mathfrak{H}$ for each $n \geq 1$ as follows.

Definition 2.1. Let $\tilde{\theta} = \tilde{\theta}^{(c)}$ be the map from $\mathfrak{H}$ to itself given by

$$\tilde{\theta}(w) := \theta(w) + cH(w)y \quad (w \in \mathfrak{H}).$$

For each positive integer n, we define

$$q_n := \frac{1}{(n-1)!}\tilde{\theta}^{n-1}(y).$$

We thus have $q_1 = y$ and $q_n = \tilde{\theta}(q_{n-1})/(n-1)$ for $n \geq 2$.

Note that $q_n = q_n^{(c)}$ is in $y\mathfrak{H}$, as can be seen inductively by the definition. We shall give an explicit formula for q_n in the next section. Here is our main theorem.

Theorem 2.2. *For all $n \geq 1$ and $c \in \mathbb{Q}$, we have*

$$\partial_n^{(c)}(wx) = (w \diamond q_n)x \quad (w \in \mathfrak{H}).$$

Assuming the theorem, it is straightforward to deduce the quasi-derivation relations from Kawashima's relations (strictly speaking, its "linear part"). Recall the linear part of Kawashima's relations [11] asserts that

$$Z(\phi(w_1 * w_2)x) = 0$$

for any $w_1, w_2 \in y\mathfrak{H}$. Using this and the definition (1) of $\diamond$, we see that

$$Z\big(\partial_n^{(c)}(ywx)\big) = Z\big((yw \diamond q_n)x\big) = Z\big(\phi\big(\phi(yw) * \phi(q_n)\big)x\big) = 0$$

because both $\phi(yw)$ and $\phi(q_n)$ are in $y\mathfrak{H}$. This is the quasi-derivation relations.

Another immediate corollary to the theorem is the commutativity of the operators $\partial_n^{(c)}$, that is, $\partial_{n_1}^{(c_1)}$ and $\partial_{n_2}^{(c_2)}$ commute with each other for any $n_1, n_2 \geq 1$ and $c_1, c_2 \in \mathbb{Q}$. This was proved in [13] but the argument was quite involved. Here we may show

$$[\partial_{n_1}^{(c_1)}, \partial_{n_2}^{(c_2)}](w) = 0$$

first for $w \in \mathfrak{H}x$ as

$$\begin{aligned}[\partial_{n_1}^{(c_1)}, \partial_{n_2}^{(c_2)}](wx) &= (\partial_{n_1}^{(c_1)}\partial_{n_2}^{(c_2)} - \partial_{n_2}^{(c_2)}\partial_{n_1}^{(c_1)})(wx)\\ &= ((w \diamond q_{n_2}) \diamond q_{n_1})x - ((w \diamond q_{n_1}) \diamond q_{n_2})x\\ &= 0\end{aligned}$$

because the product $\diamond$ is associative and commutative, and then for the general case by induction on the degree of w by noting $\partial_n^{(c)}(wz) = \partial_n^{(c)}(w)z$ as remarked before.

Proof of Theorem 2.2. We need some lemmas. Recall $z = x + y$.

Lemma 2.3. *For $w_1, w_2 \in \mathfrak{H}$, we have*

$$zw_1 \diamond w_2 = w_1 \diamond zw_2 = z(w_1 \diamond w_2).$$

Proof. This follows from $\phi(z) = x$, $\phi(x) = z$ and $xw_1 * w_2 = w_1 * xw_2 = x(w_1 * w_2)$. See also [3]. □

Lemma 2.4. *For $w \in \mathfrak{H}$, we have $\partial_1(w) = w \diamond y - wy$.*

Proof. We proceed by induction on $\deg(w)$. The case $\deg(w) = 0$ is obvious because $\partial_1(1) = 0$. Suppose $\deg(w) \geq 1$. By linearity, it is enough to prove the equation when w is of the form zw' and xw'. If $w = zw'$, we have, by using the induction hypothesis and Lemma 2.3,

$$\partial_1(w) = \partial_1(zw') = z\partial_1(w') = z(w' \diamond y - w'y) = zw' \diamond y - zw'y = w \diamond y - wy.$$

When $w = xw'$, we similarly compute (using equation (2))

$$\begin{aligned}\partial_1(w) = \partial_1(xw') &= yxw' + x\partial_1(w') = yxw' + x(w' \diamond y - w'y)\\ &= y(xw' \diamond 1) + x(w' \diamond y) - xw'y = xw' \diamond y - xw'y\\ &= w \diamond y - wy.\end{aligned}$$

□

Lemma 2.5. *For $u \in \mathbb{Q}x + \mathbb{Q}y$, we have*

$$\tilde{\theta}(uw) = u\big(\tilde{\theta}(w) + zw + c(w \diamond y)\big).$$

Proof. We only need to show the equation for $u = x$ and y. By the definition of $\tilde{\theta}$, we have

$$\begin{aligned}\tilde{\theta}(uw) &= \theta(uw) + cH(uw)y \\ &= uzw + u\theta(w) + cu\partial_1(w) + cuwy + cuH(w)y \\ &= u\big(\tilde{\theta}(w) + zw + c(\partial_1(w) + wy)\big).\end{aligned}$$

From Lemma 2.4, we complete the proof. □

We need one more preparatory result, which may be of interest in its own right.

Proposition 2.6. *The $\mathbb{Q}$-linear map $\tilde{\theta}$ is a derivation on $\mathfrak{H}$ with respect to the product $\diamond$, i.e., the equation*

$$\tilde{\theta}(w_1 \diamond w_2) = \tilde{\theta}(w_1) \diamond w_2 + w_1 \diamond \tilde{\theta}(w_2) \tag{3}$$

holds for any $w_1, w_2 \in \mathfrak{H}$.

Proof. We prove this by induction on $\deg(w_1) + \deg(w_2)$. The case $\deg(w_1) + \deg(w_2) = 0$ holds trivially:

$$\tilde{\theta}(1 \diamond 1) = \tilde{\theta}(1) = 0 = \tilde{\theta}(1) \diamond 1 + 1 \diamond \tilde{\theta}(1).$$

When $\deg(w_1) + \deg(w_2) \geq 1$, we first prove when w_1 is of the form $w_1 = zw_1'$. By the definition of $\tilde{\theta}$ and Lemmas 2.3 and 2.5, we have

$$\tilde{\theta}(zw_1' \diamond w_2) = \tilde{\theta}(z(w_1' \diamond w_2)) = z\big(\tilde{\theta}(w_1' \diamond w_2) + z(w_1' \diamond w_2) + c(w_1' \diamond w_2 \diamond y)\big).$$

On the other hand, we have

$$\begin{aligned}&\tilde{\theta}(zw_1') \diamond w_2 + zw_1' \diamond \tilde{\theta}(w_2) \\ &= z\big(\tilde{\theta}(w_1') + zw_1' + c(w_1' \diamond y)\big) \diamond w_2 + z\big(w_1' \diamond \tilde{\theta}(w_2)\big) \\ &= z\big(\tilde{\theta}(w_1') \diamond w_2 + w_1' \diamond \tilde{\theta}(w_2) + z(w_1' \diamond w_2) + c(w_1' \diamond w_2 \diamond y)\big).\end{aligned}$$

Hence by the induction hypothesis we obtain

$$\tilde{\theta}(zw_1' \diamond w_2) = \tilde{\theta}(zw_1') \diamond w_2 + zw_1' \diamond \tilde{\theta}(w_2).$$

Since the binary operator $\diamond$ is commutative and bilinear, it suffices then to prove equation (3) only in the case where $w_1 = xw_1'$ and $w_2 = yw_2'$. By using equation (2) and Lemma 2.5, we have

$$\begin{aligned}&\tilde{\theta}(xw_1' \diamond yw_2') \\ &= \tilde{\theta}\left(x(w_1' \diamond yw_2') + y(xw_1' \diamond w_2')\right) \\ &= x\big(\tilde{\theta}(w_1' \diamond yw_2') + z(w_1' \diamond yw_2') + c(w_1' \diamond yw_2' \diamond y)\big) \\ &\quad + y\big(\tilde{\theta}(xw_1' \diamond w_2') + z(xw_1' \diamond w_2') + c(xw_1' \diamond w_2' \diamond y)\big)\end{aligned}$$

and

$$\begin{aligned}&\tilde{\theta}(xw_1') \diamond yw_2' + xw_1' \diamond \tilde{\theta}(yw_2') \\ &= x\big(\big(\tilde{\theta}(w_1') + zw_1' + c(w_1' \diamond y)\big) \diamond yw_2'\big) + y\big(\tilde{\theta}(xw_1') \diamond w_2'\big) \\ &\quad + x\big(w_1' \diamond \tilde{\theta}(yw_2')\big) + y\big(xw_1' \diamond \big(\tilde{\theta}(w_2') + zw_2' + c(w_2' \diamond y)\big)\big) \\ &= x\big(\tilde{\theta}(w_1') \diamond yw_2' + w_1' \diamond \tilde{\theta}(yw_2') + z(w_1' \diamond yw_2') + c(w_1' \diamond yw_2' \diamond y)\big) \\ &\quad + y\big(\tilde{\theta}(xw_1') \diamond w_2' + xw_1' \diamond \tilde{\theta}(w_2') + z(xw_1' \diamond w_2') + c(xw_1' \diamond w_2' \diamond y)\big).\end{aligned}$$

From these, we see by the induction hypothesis that

$$\tilde{\theta}(xw_1' \diamond yw_2') = \tilde{\theta}(xw_1') \diamond yw_2' + xw_1' \diamond \tilde{\theta}(yw_2')$$

holds. □

Now we prove Theorem 2.2 by induction on n. When $n = 1$, we have

$$\partial_1^{(c)}(wx) = \partial_1(wx) = \partial_1(w)x + wyx = (\partial_1(w) + wy)x = (w \diamond y)x = (w \diamond q_1)x$$

by Lemma 2.4. When $n \geq 2$, we have

$$\begin{aligned}\partial_n^{(c)}(wx) &= \frac{1}{n-1} ad(\theta)(\partial_{n-1}^{(c)})(wx) \\ &= \frac{1}{n-1}\left(\theta\partial_{n-1}^{(c)}(wx) - \partial_{n-1}^{(c)}\theta(wx)\right).\end{aligned}$$

By the induction hypothesis, we have

$$\begin{aligned}\theta\partial_{n-1}^{(c)}(wx) &= \theta((w \diamond q_{n-1})x) \\ &= \theta(w \diamond q_{n-1})x + (w \diamond q_{n-1})xz + cH(w \diamond q_{n-1})yx \\ &= \tilde{\theta}(w \diamond q_{n-1})x + (w \diamond q_{n-1})xz\end{aligned}$$

and

$$\begin{aligned}\partial_{n-1}^{(c)}\theta(wx) &= \partial_{n-1}^{(c)}\left(\theta(w)x + wxz + cH(w)yx\right) \\ &= (\theta(w) \diamond q_{n-1})x + (w \diamond q_{n-1})xz + c(H(w)y \diamond q_{n-1})x \\ &= (\tilde{\theta}(w) \diamond q_{n-1})x + (w \diamond q_{n-1})xz.\end{aligned}$$

We therefore obtain by Proposition 2.6

$$\begin{aligned}\partial_n^{(c)}(wx) &= \frac{1}{n-1}\big(\tilde{\theta}(w \diamond q_{n-1}) - (\tilde{\theta}(w) \diamond q_{n-1})\big)x = \frac{1}{n-1}\big(w \diamond \tilde{\theta}(q_{n-1})\big)x \\ &= (w \diamond q_n)x,\end{aligned}$$

which completes the proof. □

3. Explicit formula for q_n

We now describe the element $q_n = q_n^{(c)}$ in an explicit manner. For any index $\boldsymbol{l} = (l_1, \ldots, l_s) \in \mathbb{N}^s$, we define $a(\boldsymbol{l}) = a(l_1, \ldots, l_s) \in \mathbb{Q}$ (or $\in \mathbb{Z}[c]$ if we view c as a variable) inductively by $a(1) := 1$ and

$$a(\boldsymbol{l}) := \sum_{i=1}^{s}\big(l_i - 1 - (l_1 + \cdots + l_{i-1})c\big)\, a(\boldsymbol{l}^{(i)}),$$

where

$$\boldsymbol{l}^{(i)} = \begin{cases} (l_1, \ldots, l_{i-1}, l_{i+1}, \ldots, l_s) & \text{if } l_i = 1, \\ (l_1, \ldots, l_{i-1}, l_i - 1, l_{i+1}, \ldots, l_s) & \text{if } l_i > 1. \end{cases}$$

Proposition 3.1. *For $n \geq 1$, we have*

$$q_n = -\frac{1}{(n-1)!}\sum_{|\boldsymbol{l}|=n} a(\boldsymbol{l})w(\boldsymbol{l}), \tag{4}$$

where the sum runs over all indices $\boldsymbol{l} = (l_1, \ldots, l_s) \in \mathbb{N}^s$ of any length s and of weight $|\boldsymbol{l}| := l_1 + \cdots + l_s = n$, and $w(\boldsymbol{l}) = \phi(yx^{l_1-1} \cdots yx^{l_s-1}) = (-1)^s yz^{l_1-1} \cdots yz^{l_s-1}$.

Proof. Let q'_n denote the right-hand side of (4). We prove (4) by induction on n. When $n=1$, we easily see $q'_1 = y$.

Suppose $n \geq 2$. We want to show that $q'_n = \tilde{\theta}(q'_{n-1})/(n-1)$. Since $\theta(z^m) = mz^{m+1}$ and $\partial_1(z) = 0$, we have

$$\theta(yz^{k-1}) = yz^k + (k-1)yz^k = kyz^k,$$

and so

$$\begin{aligned}
&\theta(yz^{k_1-1}\cdots yz^{k_r-1})\\
&= \sum_{j=1}^{r} yz^{k_1-1}\cdots yz^{k_{j-1}-1}\cdot k_j yz^{k_j}\cdot yz^{k_{j+1}-1}\cdots yz^{k_r-1}\\
&\quad + c\sum_{1\leq i<j\leq r} yz^{k_1-1}\cdots H(yz^{k_i-1})\cdots \partial_1(yz^{k_j-1})\cdots yz^{k_r-1}\\
&= \sum_{j=1}^{r} k_j\, yz^{k_1-1}\cdots yz^{k_{j-1}-1}yz^{k_j}yz^{k_{j+1}-1}\cdots yz^{k_r-1}\\
&\quad - c\sum_{1\leq i<j\leq r} yz^{k_1-1}\cdots (k_i yz^{k_i-1})\cdots y(z-y)z^{k_j-1}yz^{k_{j+1}-1}\cdots yz^{k_r-1}\\
&= \sum_{j=1}^{r} k_j\, yz^{k_1-1}\cdots yz^{k_{j-1}-1}yz^{k_j}yz^{k_{j+1}-1}\cdots yz^{k_r-1}\\
&\quad - c\sum_{j=2}^{r}(k_1+\cdots+k_{j-1})yz^{k_1-1}\cdots yz^{k_{j-1}-1}y(z-y)z^{k_j-1}yz^{k_{j+1}-1}\cdots yz^{k_r-1}.
\end{aligned}$$

Since $cH(yz^{k_1-1}\cdots yz^{k_r-1})y = c(k_1+\cdots+k_r)yz^{k_1-1}\cdots yz^{k_r-1}y$, we finally obtain for $\boldsymbol{k} = (k_1,\ldots,k_r)$

$$\begin{aligned}
&\tilde{\theta}(w(\boldsymbol{k}))\\
&= (-1)^r\tilde{\theta}(yz^{k_1-1}\cdots yz^{k_r-1})\\
&= (-1)^r\sum_{j=1}^{r}\big(k_j - c(k_1+\cdots+k_{j-1})\big)yz^{k_1-1}\cdots yz^{k_{j-1}-1}yz^{k_j}yz^{k_{j+1}-1}\cdots yz^{k_r-1}\\
&\quad - (-1)^{r+1}c\sum_{j=1}^{r}(k_1+\cdots+k_j)yz^{k_1-1}\cdots yz^{k_j-1}\cdot y\cdot yz^{k_{j+1}-1}\cdots yz^{k_r-1}.
\end{aligned}$$

If we write

$$\tilde{\theta}(q'_{n-1}) = -\frac{1}{(n-2)!}\sum_{|\boldsymbol{l}|=n} a'(\boldsymbol{l})w(\boldsymbol{l}),$$

we see from this that the coefficient $a'(\boldsymbol{l})$ of $w(\boldsymbol{l}) = (-1)^s yz^{l_1-1}\cdots yz^{l_s-1}$ is given exactly by $a(\boldsymbol{l})$ as defined recursively. □

4. Quasi-derivation relations for finite multiple zeta values

In this section, we briefly discuss how the quasi-derivation relations look like for "finite" multiple zeta values. There are two versions, denoted $\zeta_{\mathcal{A}}(k_1,\ldots,k_r)$ and $\zeta_{\mathcal{S}}(k_1,\ldots,k_r)$, of "finite" analogues of multiple zeta values. The former lives in the $\mathbb{Q}$-algebra $\mathcal{A} := \prod_p \mathbb{F}_p/\bigoplus_p \mathbb{F}_p$ and the latter the quotient $\mathbb{Q}$-algebra of classical multiple zeta values modulo the ideal generated by $\zeta(2)$. It is conjectured that the two versions satisfy completely the same relations, and there is a conjectural isomorphism between two $\mathbb{Q}$-algebras generated by those two versions. For more on finite multiple zeta values, see for instance [9].

Denote by $Z_{\mathcal{F}}$ the $\mathbb{Q}$-linear map from $y\mathfrak{H}$ to either algebra assigning the monomial $yx^{k_1-1}\cdots yx^{k_r-1}$ to $\zeta_{\mathcal{A}}(k_1,\ldots,k_r)$ or $\zeta_{\mathcal{S}}(k_1,\ldots,k_r)$. Then the derivation relations for finite multiple zeta values established by the second-named author [12] is the relation

$$Z_{\mathcal{F}}(\partial_n(w)x^{-1})=0 \tag{5}$$

that holds for all $w\in y\mathfrak{H}x$.

As a consequence of our Theorem 2.2, we have the following.

Theorem 4.1 (Quasi-derivation relations for finite multiple zeta values)**.** *For all $n\geq 1$ and $c\in\mathbb{Q}$, we have*

$$Z_{\mathcal{F}}(\partial_n^{(c)}(w)x^{-1})=Z_{\mathcal{F}}(wx^{-1})Z_{\mathcal{F}}(q_n^{(c)})\quad(w\in y\mathfrak{H}x).$$

Proof. This is almost immediate from Theorem 2.2 if one notes $Z_{\mathcal{F}}\circ\phi=Z_{\mathcal{F}}$ and $Z_{\mathcal{F}}$ is a $*$-homomorphism (for these, see [7, 9, 10]). □

Remark 4.2. When $c=0$, we can easily compute that $q_n^{(0)}=yz^{n-1}$. Since $Z_{\mathcal{F}}(yz^{n-1})=Z_{\mathcal{F}}\big(\phi(yz^{n-1})\big)=-Z_{\mathcal{F}}(yx^{n-1})=-\zeta_{\mathcal{F}}(n)=0$ for $\mathcal{F}=\mathcal{A}$ or S, we see that Theorem 4.1 generalizes the derivation relations (5).

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