

Multi-poly-Bernoulli numbers and related zeta functions

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Masanobu Kaneko* and Hirofumi Tsumura

Abstract

We construct and study a certain zeta function which interpolates multi-poly-Bernoulli numbers at non-positive integers and whose values at positive integers are linear combinations of multiple zeta values. This function can be regarded as the one to be paired up with the ξ -function defined by Arakawa and the first-named author. We show that both are closely related to the multiple zeta functions. Further we define multi-indexed poly-Bernoulli numbers, and generalize the duality formulas for poly-Bernoulli numbers by introducing more general zeta functions.

1. Introduction

In this paper, we investigate the function defined by

$$(1.1) \quad \eta(k_1, \dots, k_r; s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\text{Li}_{k_1, \dots, k_r}(1 - e^{-t})}{1 - e^{-t}} dt$$

and its generalizations, in connection with multi-poly-Bernoulli numbers and multiple zeta values (we shall give the precise definitions later in §2). This function can be viewed as a twin sibling of the function $\xi(k_1, \dots, k_r; s)$,

$$(1.2) \quad \xi(k_1, \dots, k_r; s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\text{Li}_{k_1, \dots, k_r}(1 - e^{-t})}{e^t - 1} dt,$$

which was introduced and studied in [4]. The present paper may constitute a natural continuation of the work [4].

To explain our results in some detail, we first give an overview of the necessary background. For an integer $k \in \mathbb{Z}$, two types of poly-Bernoulli numbers $\{B_n^{(k)}\}$ and $\{C_n^{(k)}\}$ are defined as follows (see Kaneko [20] and Arakawa-Kaneko [4], also Arakawa-Ibukiyama-Kaneko [3]):

$$(1.3) \quad \frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} = \sum_{n=0}^\infty B_n^{(k)} \frac{t^n}{n!},$$

$$(1.4) \quad \frac{\text{Li}_k(1 - e^{-t})}{e^t - 1} = \sum_{n=0}^\infty C_n^{(k)} \frac{t^n}{n!},$$

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where $\text{Li}_k(z)$ is the polylogarithm function defined by

$$(1.5) \quad \text{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k} \quad (|z| < 1).$$

Since $\text{Li}_1(z) = -\log(1-z)$, we see that $B_n^{(1)}$ (resp. $C_n^{(1)}$) coincides with the ordinary Bernoulli number B_n defined by

$$\frac{te^t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad \left(\text{resp. } \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \right).$$

A number of formulas, including closed formulas of $B_n^{(k)}$ and $C_n^{(k)}$ in terms of the Stirling numbers of the second kind as well as the duality formulas

$$(1.6) \quad B_n^{(-k)} = B_k^{(-n)},$$

$$(1.7) \quad C_n^{(-k-1)} = C_k^{(-n-1)}$$

that hold for $k, n \in \mathbb{Z}_{\geq 0}$, have been established (see [20, Theorems 1 and 2] and [21, § 2]). We also mention that Brewbaker [9] gave a purely combinatorial interpretation of the number $B_n^{(-k)}$ of negative upper index as the number of ‘Lonesum-matrices’ with n rows and k columns.

A multiple version of $B_n^{(k)}$ is defined in [4, p. 202, Remarks (ii)] by

$$(1.8) \quad \frac{\text{Li}_{k_1, \dots, k_r}(1 - e^{-t})}{(1 - e^{-t})^r} = \sum_{n=0}^{\infty} \mathbb{B}_n^{(k_1, \dots, k_r)} \frac{t^n}{n!} \quad (k_1, \dots, k_r \in \mathbb{Z}),$$

where

$$(1.9) \quad \text{Li}_{k_1, \dots, k_r}(z) = \sum_{1 \leq m_1 < \dots < m_r} \frac{z^{m_r}}{m_1^{k_1} m_2^{k_2} \dots m_r^{k_r}}$$

is the multiple polylogarithm. Hamahata and Masubuchi [14, 15] investigated some properties of $\mathbb{B}_n^{(k_1, \dots, k_r)}$, and gave several generalizations of the known results in the single-index case. Based on this research, Bayad and Hamahata [8] further studied these numbers. Furusho [12, p. 269] also refers to (1.8).

More recently, Imatomi, Takeda and the first-named author [18] defined and studied another type of multi-poly-Bernoulli numbers given by

$$(1.10) \quad \frac{\text{Li}_{k_1, \dots, k_r}(1 - e^{-t})}{1 - e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k_1, \dots, k_r)} \frac{t^n}{n!},$$

$$(1.11) \quad \frac{\text{Li}_{k_1, \dots, k_r}(1 - e^{-t})}{e^t - 1} = \sum_{n=0}^{\infty} C_n^{(k_1, \dots, k_r)} \frac{t^n}{n!}$$

for $k_1, \dots, k_r \in \mathbb{Z}$. They proved several formulas for $B_n^{(k_1, \dots, k_r)}$ and $C_n^{(k_1, \dots, k_r)}$, and further gave an important relation between $C_{p-2}^{(k_1, \dots, k_r)}$ and the ‘finite multiple zeta value’, that is,

$$(1.12) \quad \sum_{1 \leq m_1 < \dots < m_r < p} \frac{1}{m_1^{k_1} \dots m_r^{k_r}} \equiv -C_{p-2}^{(k_1, \dots, k_{r-1}, k_r-1)} \pmod{p}$$

for any prime number p .

The function (1.2) for $k_1, \dots, k_r \in \mathbb{Z}_{\geq 1}$ can be analytically continued to an entire function of the complex variable $s \in \mathbb{C}$ ([4, Sections 3 and 4]). The particular case $r = k = 1$ gives $\xi(1; s) = s\zeta(s+1)$. Hence $\xi(k_1, \dots, k_r; s)$ can be regarded as a multi-indexed zeta function. It is shown in [4] that the values at non-positive integers of $\xi(k; s)$ interpolate poly-Bernoulli numbers $C_m^{(k)}$,

$$(1.13) \quad \xi(k; -m) = (-1)^m C_m^{(k)}$$

for $k \in \mathbb{Z}_{\geq 1}$ and $m \in \mathbb{Z}_{\geq 0}$. And also by investigating $\xi(k_1, \dots, k_r; s)$ and its values at positive integer arguments, one produces many relations among multiple zeta values defined by

$$(1.14) \quad \zeta(l_1, \dots, l_r) = \sum_{1 \leq m_1 < \dots < m_r} \frac{1}{m_1^{l_1} \dots m_r^{l_r}} \quad (= \text{Li}_{l_1, \dots, l_r}(1))$$

for $l_1, \dots, l_r \in \mathbb{Z}_{\geq 1}$ with $l_r \geq 2$ ([4, Corollary 11]).

Recently, further properties of $\xi(k_1, \dots, k_r; s)$ and related results have been given by several authors (see, for example, Bayad-Hamahata [6, 7], Coppo-Candelpergher [10], Sasaki [28], and Young [31]).

In this paper, we conduct a basic study of the function (1.1) and relate it to the multi-poly-Bernoulli numbers $B_n^{(k_1, \dots, k_r)}$ as well as multiple zeta (or ‘zeta-star’) values. Note that the only difference in both definitions (1.1) and (1.2) is, up to sign, the arguments $1 - e^t$ and $1 - e^{-t}$ of $\text{Li}_{k_1, \dots, k_r}(z)$ in the integrands. One sees in the main body of the paper a remarkable contrast between ‘ B -type’ poly-Bernoulli numbers and those of ‘ C -type’, and between multiple zeta and zeta-star values. We further investigate the case of non-positive indices k_i in connection with a yet more generalized ‘multi-indexed’ poly-Bernoulli number.

The paper is organized as follows. In §2, we give the analytic continuation of $\eta(k_1, \dots, k_r; s)$ in the case of positive indices, and formulas for values at integer arguments (Theorems 2.3 and 2.5). In §3, we study relations between two functions $\eta(k_1, \dots, k_r; s)$ and $\xi(k_1, \dots, k_r; s)$ (Proposition 3.2), as well as relations with the single variable multiple zeta function (Definition 3.1 and Theorem 3.6). We turn in §4 to the study of $\eta(k_1, \dots, k_r; s)$ in the negative index case and give a certain duality formula for $B_m^{(-k_1, \dots, -k_r)}$ (Definition 4.3 and Theorems 4.4 and 4.7). We carry forward the study of negative index case in §5 and define the ‘multi-indexed’ poly-Bernoulli numbers $\{B_{m_1, \dots, m_r}^{(k_1, \dots, k_r), (d)}\}$ for $(k_1, \dots, k_r) \in \mathbb{Z}^r$, $(m_1, \dots, m_r) \in \mathbb{Z}_{\geq 0}^r$ and $d \in \{1, \dots, r\}$ (Definition 5.1), which include (1.8) and (1.10) as special cases. We prove the ‘multi-indexed’ duality formula for them in the case $d = r$ (Theorem 5.4).

2. The function $\eta(k_1, \dots, k_r; s)$ for positive indices and its values at integers

2.1 Analytic continuation and the values at non-positive integers

We start with the definition in the case of positive indices.

Definition 2.1. For positive integers $k_1, \dots, k_r \in \mathbb{Z}_{\geq 1}$, let

$$\eta(k_1, \dots, k_r; s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\text{Li}_{k_1, \dots, k_r}(1 - e^t)}{1 - e^t} dt$$

for $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1 - r$, where $\Gamma(s)$ is the gamma function. When $r = 1$, we often denote $\eta(k; s)$ by $\eta_k(s)$.

The integral on the right-hand side converges absolutely in the domain $\operatorname{Re}(s) > 1 - r$, as is seen from the following lemma.

Lemma 2.2. (i) For $k_1, \dots, k_r \in \mathbb{Z}_{\geq 1}$, the function $\operatorname{Li}_{k_1, \dots, k_r}(1 - e^t)$ is holomorphic for $t \in \mathbb{C}$ with $|\operatorname{Im}(t)| < \pi$.

(ii) For $k_1, \dots, k_r \in \mathbb{Z}_{\geq 1}$ and $t \in \mathbb{R}_{>0}$, we have the estimates

$$(2.1) \quad \operatorname{Li}_{k_1, \dots, k_r}(1 - e^t) = O(t^r) \quad (t \rightarrow 0)$$

and

$$(2.2) \quad \operatorname{Li}_{k_1, \dots, k_r}(1 - e^t) = O(t^{k_1 + \dots + k_r}) \quad (t \rightarrow \infty).$$

Proof. As is well-known, we can regard the function $\operatorname{Li}_{k_1, \dots, k_r}(z)$ as a single-valued holomorphic function in the simply connected domain $\mathbb{C} \setminus [1, \infty)$, via the process of iterated integration starting with $\operatorname{Li}_1(z) = \int_0^z dz/(1-z)$. Noting that $1 - e^t \in [1, \infty)$ is equivalent to $\operatorname{Im}(t) = (2j+1)\pi$ for some $j \in \mathbb{Z}$, we have the assertion (i).

The estimate (2.1) is clear from the definition of $\operatorname{Li}_{k_1, \dots, k_r}(z)$, because its Taylor series at $z = 0$ starts with the term $z^r/1^{k_1} \dots r^{k_r}$. As for (2.2), we proceed by induction on the ‘weight’ $k_1 + \dots + k_r$ as follows by using the formula

$$(2.3) \quad \frac{d}{dz} \operatorname{Li}_{k_1, \dots, k_r}(z) = \begin{cases} \frac{1}{z} \operatorname{Li}_{k_1, \dots, k_{r-1}, k_r-1}(z) & (k_r > 1) \\ \frac{1}{1-z} \operatorname{Li}_{k_1, \dots, k_{r-1}}(z) & (k_r = 1), \end{cases}$$

which is easy to derive and is the basis of the analytic continuation of $\operatorname{Li}_{k_1, \dots, k_r}(z)$ mentioned above. If $r = k_1 = 1$, then we have $\operatorname{Li}_1(1 - e^t) = -t$ and the desired estimate holds. Suppose the weight k is larger than 1 and the assertion holds for any weight less than k . If $k_r > 1$, then by (2.3) we have

$$\begin{aligned} |\operatorname{Li}_{k_1, \dots, k_r}(1 - e^t)| &= \left| \int_0^{1-e^t} \frac{\operatorname{Li}_{k_1, \dots, k_{r-1}}(u)}{u} du \right| \\ &= \left| \int_0^t \frac{1}{1-e^v} \operatorname{Li}_{k_1, \dots, k_{r-1}}(1 - e^v) (-e^v) dv \right| \quad (u := 1 - e^v) \\ &\leq \int_0^\varepsilon \left| e^v \frac{\operatorname{Li}_{k_1, \dots, k_{r-1}}(1 - e^v)}{e^v - 1} \right| dv + \int_\varepsilon^t \left| \frac{e^v}{e^v - 1} \operatorname{Li}_{k_1, \dots, k_{r-1}}(1 - e^v) \right| dv \end{aligned}$$

for small $\varepsilon > 0$. The former integral is $O(1)$ because the integrand is continuous on $[0, \varepsilon]$. On the other hand, by induction hypothesis, the integrand of the latter integral is $O(v^{k_1 + \dots + k_{r-1} - 1})$ as $v \rightarrow \infty$. Therefore the latter integral is $O(t^{k_1 + \dots + k_r})$ as $t \rightarrow \infty$. The case of $k_r = 1$ is similarly proved also by using (2.3), and is omitted here. \square

We now show that the function $\eta(k_1, \dots, k_r; s)$ can be analytically continued to an entire function, and interpolates multi-poly-Bernoulli numbers $B_m^{(k_1, \dots, k_r)}$ at non-positive integer arguments.

Theorem 2.3. For positive integers $k_1, \dots, k_r \in \mathbb{Z}_{\geq 1}$, the function $\eta(k_1, \dots, k_r; s)$ can be analytically continued to an entire function on the whole complex plane. And the values of $\eta(k_1, \dots, k_r; s)$ at non-positive integers are given by

$$(2.4) \quad \eta(k_1, \dots, k_r; -m) = B_m^{(k_1, \dots, k_r)} \quad (m \in \mathbb{Z}_{\geq 0}).$$

In particular, $\eta_k(-m) = B_m^{(k)}$ for $k \in \mathbb{Z}_{\geq 1}$ and $m \in \mathbb{Z}_{\geq 0}$.

Proof. In order to prove this theorem, we adopt here the method of contour integral representation (see, for example, [30, Theorem 4.2]). Let \mathcal{C} be the standard contour, namely the path consisting of the positive real axis from the infinity to (sufficiently small) ε ('top side'), a counter clockwise circle C_ε around the origin of radius ε , and the positive real axis from ε to the infinity ('bottom side'). Let

$$\begin{aligned} H(k_1, \dots, k_r; s) &= \int_{\mathcal{C}} t^{s-1} \frac{\text{Li}_{k_1, \dots, k_r}(1 - e^t)}{1 - e^t} dt \\ &= (e^{2\pi i s} - 1) \int_{\varepsilon}^{\infty} t^{s-1} \frac{\text{Li}_{k_1, \dots, k_r}(1 - e^t)}{1 - e^t} dt + \int_{C_\varepsilon} t^{s-1} \frac{\text{Li}_{k_1, \dots, k_r}(1 - e^t)}{1 - e^t} dt. \end{aligned}$$

It follows from Lemma 2.2 that $H(k_1, \dots, k_r; s)$ is entire, because the integrand has no singularity on \mathcal{C} and the contour integral is absolutely convergent for all $s \in \mathbb{C}$. Suppose $\text{Re}(s) > 1 - r$. The last integral tends to 0 as $\varepsilon \rightarrow 0$. Hence

$$\eta(k_1, \dots, k_r; s) = \frac{1}{(e^{2\pi i s} - 1)\Gamma(s)} H(k_1, \dots, k_r; s),$$

which can be analytically continued to \mathbb{C} , and is entire. In fact $\eta(k_1, \dots, k_r; s)$ is holomorphic for $\text{Re}(s) > 0$, hence has no singularity at any positive integer. Set $s = -m \in \mathbb{Z}_{\leq 0}$. Then, by (1.10),

$$\begin{aligned} \eta(k_1, \dots, k_r; -m) &= \frac{(-1)^m m!}{2\pi i} H(k_1, \dots, k_r; -m) \\ &= \frac{(-1)^m m!}{2\pi i} \int_{C_\varepsilon} t^{-m-1} \sum_{n=0}^{\infty} B_n^{(k_1, \dots, k_r)} \frac{(-t)^n}{n!} dt = B_m^{(k_1, \dots, k_r)}. \end{aligned}$$

This completes the proof. □

Remark 2.4. Using the same method as above or the method used in [4], we can establish the analytic continuation of $\xi(k_1, \dots, k_r; s)$ to an entire function, and see that

$$(2.5) \quad \xi(k_1, \dots, k_r; -m) = (-1)^m C_m^{(k_1, \dots, k_r)} \quad (m \in \mathbb{Z}_{\geq 0})$$

for $k_1, \dots, k_r \in \mathbb{Z}_{\geq 1}$, which is a multiple version of (1.13).

2.2 Values at positive integers

About the values at positive integer arguments, we prove formulas for both $\xi(k_1, \dots, k_r; s)$ and $\eta(k_1, \dots, k_r; s)$, for general index (k_1, \dots, k_r) . These formulas generalize [4, Theorem 9 (i)], and have remarkable similarity in that one obtains the formula for $\eta(k_1, \dots, k_r; s)$ just by replacing

multiple zeta values in the one for $\xi(k_1, \dots, k_r; s)$ with multiple ‘zeta-star’ values. Recall the multiple zeta-star value is a real number defined by

$$(2.6) \quad \zeta^*(l_1, \dots, l_r) = \sum_{1 \leq m_1 \leq \dots \leq m_r} \frac{1}{m_1^{l_1} \dots m_r^{l_r}}$$

for $l_1, \dots, l_r \in \mathbb{Z}_{\geq 1}$ with $l_r \geq 2$. This was first studied (for general r) by Hoffman in [16].

To state our theorem, we further introduce some notation. For an index set $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}_{\geq 1}^r$, put $\mathbf{k}_+ = (k_1, \dots, k_{r-1}, k_r + 1)$. The usual dual index of an *admissible* index (i.e. the one that the last entry is greater than one) \mathbf{k} is denoted by \mathbf{k}^* . For $\mathbf{j} = (j_1, \dots, j_r) \in \mathbb{Z}_{\geq 0}^r$, we write $|\mathbf{j}| = j_1 + \dots + j_r$ and call it the weight of \mathbf{j} , and $d(\mathbf{j}) = r$, the depth of \mathbf{j} . For two such indices \mathbf{k} and \mathbf{j} of the same depth, we denote by $\mathbf{k} + \mathbf{j}$ the index obtained by the component-wise addition, $\mathbf{k} + \mathbf{j} = (k_1 + j_1, \dots, k_r + j_r)$, and by $b(\mathbf{k}; \mathbf{j})$ the quantity given by

$$b(\mathbf{k}; \mathbf{j}) := \prod_{i=1}^r \binom{k_i + j_i - 1}{j_i}.$$

Theorem 2.5. *For any index set $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}_{\geq 1}^r$ and any $m \in \mathbb{Z}_{\geq 1}$, we have*

$$(2.7) \quad \xi(k_1, \dots, k_r; m) = \sum_{|\mathbf{j}|=m-1, d(\mathbf{j})=n} b((\mathbf{k}_+)^*; \mathbf{j}) \zeta((\mathbf{k}_+)^* + \mathbf{j})$$

and

$$(2.8) \quad \eta(k_1, \dots, k_r; m) = (-1)^{r-1} \sum_{|\mathbf{j}|=m-1, d(\mathbf{j})=n} b((\mathbf{k}_+)^*; \mathbf{j}) \zeta^*((\mathbf{k}_+)^* + \mathbf{j}),$$

where both sums are over all $\mathbf{j} \in \mathbb{Z}_{\geq 0}^r$ of weight $m - 1$ and depth $n := d(\mathbf{k}_+^*) (= |\mathbf{k}| + 1 - d(\mathbf{k}))$.

In particular, we have

$$\xi(k_1, \dots, k_r; 1) = \zeta((\mathbf{k}_+)^*) \quad (= \zeta(\mathbf{k}_+), \text{ by the duality of multiple zeta values})$$

and

$$\eta(k_1, \dots, k_r; 1) = (-1)^{r-1} \zeta^*((\mathbf{k}_+)^*).$$

In order to prove the theorem, we give certain multiple integral expressions of the functions $\xi(k_1, \dots, k_r; s)$ and $\eta(k_1, \dots, k_r; s)$.

Proposition 2.6. *Notations being as above, write $(\mathbf{k}_+)^* = (l_1, \dots, l_n)$. Then we have, for $\operatorname{Re}(s) > 1 - r$,*

$$(i) \quad \begin{aligned} \xi(k_1, \dots, k_r; s) &= \frac{1}{\prod_{i=1}^n \Gamma(l_i) \cdot \Gamma(s)} \int_0^\infty \dots \int_0^\infty (x_1 + \dots + x_n)^{s-1} x_1^{l_1-1} \dots x_n^{l_n-1} \\ &\quad \times \frac{1}{e^{x_1+\dots+x_n} - 1} \cdot \frac{1}{e^{x_2+\dots+x_n} - 1} \dots \frac{1}{e^{x_n} - 1} dx_1 \dots dx_n. \end{aligned}$$

$$(ii) \quad \begin{aligned} \eta(k_1, \dots, k_r; s) &= \frac{(-1)^{r-1}}{\prod_{i=1}^n \Gamma(l_i) \cdot \Gamma(s)} \int_0^\infty \dots \int_0^\infty (x_1 + \dots + x_n)^{s-1} x_1^{l_1-1} \dots x_n^{l_n-1} \\ &\quad \times \frac{1}{e^{x_1+\dots+x_n} - 1} \cdot \frac{e^{x_2+\dots+x_n}}{e^{x_2+\dots+x_n} - 1} \dots \frac{e^{x_n}}{e^{x_n} - 1} dx_1 \dots dx_n. \end{aligned}$$

Proof. First write the index (k_1, \dots, k_r) as

$$(k_1, \dots, k_r) = (\underbrace{1, \dots, 1}_{a_1-1}, b_1 + 1, \dots, \underbrace{1, \dots, 1}_{a_h-1}, b_h + 1),$$

with (uniquely determined) integers $h \geq 1$, $a_i \geq 1$ ($1 \leq i \leq h$), $b_i \geq 1$ ($1 \leq i \leq h-1$), and $b_h \geq 0$. Then, by performing the intermediate integrals of repeated $dz/(1-z)$ in the standard iterated integral coming from (2.3), we obtain the following iterated integral expression of the multiple polylogarithm $\text{Li}_{k_1, \dots, k_r}(z)$:

$$\begin{aligned} \text{Li}_{k_1, \dots, k_r}(z) &= \underbrace{\int_0^z \frac{dx_h}{x_h} \int_0^{x_h} \dots \int_0^{x_h} \frac{dx_h}{x_h} \int_0^{x_h} \frac{1}{a_h!} \log\left(\frac{1-x_{h-1}}{1-x_h}\right)^{a_h} \frac{dx_{h-1}}{x_{h-1}}}_{b_h} \\ &\cdot \underbrace{\int_0^{x_{h-1}} \frac{dx_{h-1}}{x_{h-1}} \dots \int_0^{x_{h-1}} \frac{dx_{h-1}}{x_{h-1}} \int_0^{x_{h-1}} \frac{1}{a_{h-1}!} \log\left(\frac{1-x_{h-2}}{1-x_{h-1}}\right)^{a_{h-1}} \frac{dx_{h-2}}{x_{h-2}} \dots}_{b_{h-1}-1} \\ &\dots \underbrace{\int_0^{x_3} \frac{dx_3}{x_3} \dots \int_0^{x_3} \frac{dx_3}{x_3} \int_0^{x_3} \frac{1}{a_3!} \log\left(\frac{1-x_2}{1-x_3}\right)^{a_3} \frac{dx_2}{x_2} \int_0^{x_2} \frac{dx_2}{x_2} \dots \int_0^{x_2} \frac{dx_2}{x_2}}_{b_3-1} \\ &\cdot \underbrace{\int_0^{x_2} \frac{1}{a_2!} \log\left(\frac{1-x_1}{1-x_2}\right)^{a_2} \frac{dx_1}{x_1} \int_0^{x_1} \dots \int_0^{x_1} \frac{dx_1}{x_1} \int_0^{x_1} \frac{(-\log(1-x))^{a_1}}{a_1!} \frac{dx}{x}}_{b_1-1}. \end{aligned}$$

Here, to ease notation, we used the same variable in the repetitions of integrals $\int_0^x dx/x$, and we understand $x_h = z$ if $b_h = 0$. The paths of integrations are in the domain $\mathbb{C} \setminus [1, \infty)$, and the formula is valid for $z \in \mathbb{C} \setminus [1, \infty)$. We may check this formula by differentiating both sides repeatedly and using (2.3). Putting $z = 1 - e^{-t}$ and $1 - e^t$, changing variables accordingly, and suitably labeling the variables, we obtain

$$\begin{aligned} (2.9) \quad \text{Li}_{k_1, \dots, k_r}(1 - e^{-t}) &= \int_0^t \int_0^{t_{b_1+\dots+b_h}} \dots \int_0^{t_2} \underbrace{\frac{1}{e^{t_{b_1+\dots+b_h}-1}} \dots \frac{1}{e^{t_{b_1+\dots+b_{h-1}+2}-1}}}_{b_h-1} \\ &\times \frac{1}{a_h!} \frac{(t_{b_1+\dots+b_{h-1}+1} - t_{b_1+\dots+b_{h-1}})^{a_h}}{e^{t_{b_1+\dots+b_{h-1}+1}-1}} \cdot \underbrace{\frac{1}{e^{t_{b_1+\dots+b_{h-1}}-1}} \dots \frac{1}{e^{t_{b_1+\dots+b_{h-2}+2}-1}}}_{b_{h-1}-1} \\ &\times \dots \times \frac{1}{a_3!} \frac{(t_{b_1+b_2+1} - t_{b_1+b_2})^{a_3}}{e^{t_{b_1+b_2+1}-1}} \cdot \underbrace{\frac{1}{e^{t_{b_1+b_2}-1}} \dots \frac{1}{e^{t_{b_1+2}-1}}}_{b_2-1} \\ &\times \frac{1}{a_2!} \frac{(t_{b_1+1} - t_{b_1})^{a_2}}{e^{t_{b_1+1}-1}} \cdot \underbrace{\frac{1}{e^{t_{b_1}-1}} \dots \frac{1}{e^{t_2-1}}}_{b_1-1} \cdot \frac{1}{a_1!} \frac{t_1^{a_1}}{e^{t_1}-1} dt_1 dt_2 \dots dt_{b_1+\dots+b_h}, \end{aligned}$$

and

(2.10)

$$\text{Li}_{k_1, \dots, k_r}(1 - e^t) = (-1)^r \int_0^t \int_0^{t_{b_1+\dots+b_h}} \dots \int_0^{t_2} \underbrace{\frac{e^{t_{b_1+\dots+b_h}}}{e^{t_{b_1+\dots+b_h}-1}} \dots \frac{e^{t_{b_1+\dots+b_{h-1}+2}}}{e^{t_{b_1+\dots+b_{h-1}+2}-1}}}_{b_h-1}$$

$$\begin{aligned}
& \times \frac{1}{a_h!} \frac{(t_{b_1+\dots+b_{h-1}+1} - t_{b_1+\dots+b_{h-1}})^{a_h} e^{t_{b_1+\dots+b_{h-1}+1}}}{e^{t_{b_1+\dots+b_{h-1}+1}} - 1} \cdot \underbrace{\frac{e^{t_{b_1+\dots+b_{h-1}}}}{e^{t_{b_1+\dots+b_{h-1}}}-1} \dots \frac{e^{t_{b_1+\dots+b_{h-2}+2}}}{e^{t_{b_1+\dots+b_{h-2}+2}}-1}}_{b_{h-1}-1} \\
& \times \dots \times \frac{1}{a_3!} \frac{(t_{b_1+b_2+1} - t_{b_1+b_2})^{a_3} e^{t_{b_1+b_2+1}}}{e^{t_{b_1+b_2+1}} - 1} \cdot \underbrace{\frac{e^{t_{b_1+b_2}}}{e^{t_{b_1+b_2}}-1} \dots \frac{e^{t_{b_1+2}}}{e^{t_{b_1+2}}-1}}_{b_2-1} \\
& \times \frac{1}{a_2!} \frac{(t_{b_1+1} - t_{b_1})^{a_2} e^{t_{b_1+1}}}{e^{t_{b_1+1}} - 1} \cdot \underbrace{\frac{e^{t_{b_1}}}{e^{t_{b_1}}-1} \dots \frac{e^{t_2}}{e^{t_2}-1}}_{b_1-1} \cdot \frac{1}{a_1!} \frac{t_1^{a_1} e^{t_1}}{e^{t_1}-1} dt_1 dt_2 \dots dt_{b_1+\dots+b_h}.
\end{aligned}$$

The factor $(-1)^r$ on the right of (2.10) comes from $(-1)^{a_1+\dots+a_h} = (-1)^r$. Plugging (2.9) and (2.10) into the definitions (1.2) and (1.1) respectively and making the change of variables

$$t = x_1 + \dots + x_n, t_{b_1+\dots+b_h} = x_2 + \dots + x_n, t_{b_1+\dots+b_{h-1}} = x_3 + \dots + x_n, \dots, t_2 = x_{n-1} + x_n, t_1 = x_n,$$

we obtain the proposition. One should note that the dual index $(\mathbf{k}_+)^* = (l_1, \dots, l_n)$ is given by

$$(\mathbf{k}_+)^* = (\underbrace{1, \dots, 1}_{b_h}, a_h + 1, \underbrace{1, \dots, 1}_{b_{h-1}-1}, a_{h-1} + 1, \dots, \underbrace{1, \dots, 1}_{b_1-1}, a_1 + 1)$$

and the depth n is equal to $b_1 + \dots + b_h + 1$, and that (the trivial) $x_i^{l_i-1} = 1$ when $l_i = 1$. \square

Proof of Theorem 2.5. Set $s = m$ in the integral expressions in the proposition, and expand $(x_1 + \dots + x_k)^{m-1}$ by the multinomial theorem. Then the formula in the theorem follows from the lemma below. \square

Lemma 2.7. For $l_1, \dots, l_r \in \mathbb{Z}_{\geq 1}$ with $l_r \geq 2$, we have

$$\zeta(l_1, \dots, l_r) = \frac{1}{\prod_{j=1}^r \Gamma(l_j)} \int_0^\infty \dots \int_0^\infty \frac{x_1^{l_1-1} \dots x_r^{l_r-1}}{e^{x_1+\dots+x_r} - 1} \cdot \frac{1}{e^{x_2+\dots+x_r} - 1} \dots \frac{1}{e^{x_r} - 1} dx_1 \dots dx_r$$

and

$$\zeta^*(l_1, \dots, l_r) = \frac{1}{\prod_{j=1}^r \Gamma(l_j)} \int_0^\infty \dots \int_0^\infty \frac{x_1^{l_1-1} \dots x_r^{l_r-1}}{e^{x_1+\dots+x_r} - 1} \cdot \frac{e^{x_2+\dots+x_r}}{e^{x_2+\dots+x_r} - 1} \dots \frac{e^{x_r}}{e^{x_r} - 1} dx_1 \dots dx_r.$$

Proof. The first formula is given in [4, Theorem 3 (i)]. As for the second, we may proceed similarly by using $n^{-s} = \Gamma(s)^{-1} \int_0^\infty t^{s-1} e^{-nt} dt$ to have

$$\begin{aligned}
\zeta^*(l_1, \dots, l_r) &= \sum_{m_1=1}^\infty \sum_{m_2, \dots, m_r=0}^\infty \frac{1}{m_1^{l_1} (m_1 + m_2)^{l_2} \dots (m_1 + \dots + m_r)^{l_r}} \\
&= \frac{1}{\prod_{j=1}^r \Gamma(l_j)} \sum_{m_1=1}^\infty \sum_{m_2, \dots, m_r=0}^\infty \int_0^\infty \dots \int_0^\infty x_1^{l_1-1} e^{-m_1 x_1} \cdot x_2^{l_2-1} e^{-(m_1+m_2)x_2} \dots \\
&\quad \dots x_r^{l_r-1} e^{-(m_1+\dots+m_r)x_r} dx_1 \dots dx_r \\
&= \frac{1}{\prod_{j=1}^r \Gamma(l_j)} \sum_{m_1=1}^\infty \sum_{m_2, \dots, m_r=0}^\infty \int_0^\infty \dots \int_0^\infty x_1^{l_1-1} \dots x_r^{l_r-1} e^{-m_1(x_1+\dots+x_r)} \cdot e^{-m_2(x_2+\dots+x_r)} \dots
\end{aligned}$$

$$= \frac{1}{\prod_{j=1}^r \Gamma(l_j)} \int_0^\infty \cdots \int_0^\infty \frac{x_1^{l_1-1} \cdots x_r^{l_r-1}}{e^{x_1+\cdots+x_r} - 1} \cdot \frac{e^{x_2+\cdots+x_r}}{e^{x_2+\cdots+x_r} - 1} \cdots \frac{e^{x_r}}{e^{x_r} - 1} dx_1 \cdots dx_r.$$

□

We record here one corollary to the theorem in the case of $\eta_k(m)$ (compare with the similar formula in [4, Theorem 9 (i)]). Noting $(k+1)^* = \underbrace{(1, \dots, 1)}_{k-1}, 2$, we have

Corollary 2.8. *For $k, m \geq 1$, we have*

$$(2.11) \quad \eta_k(m) = \sum_{\substack{j_1, \dots, j_{k-1} \geq 1, j_k \geq 2 \\ j_1 + \cdots + j_k = k+m}} (j_k - 1) \zeta^*(j_1, \dots, j_{k-1}, j_k).$$

3. Relations among the functions ξ , η and ζ , and their consequences to multiple zeta values and multi-poly-Bernoulli numbers

In this section, we first deduce that each of the functions η and ξ can be written as a linear combination of the other by the *same* formula. This is a consequence of the so-called Landen-type connection formula for the multiple polylogarithm $\text{Li}_{k_1, \dots, k_r}(z)$. We then establish a formula for $\xi(k_1, \dots, k_r; s)$ in terms of the single-variable multiple zeta function

$$(3.1) \quad \zeta(l_1, \dots, l_r; s) = \sum_{1 \leq m_1 < \cdots < m_r < m} \frac{1}{m_1^{l_1} \cdots m_r^{l_r} m^s}$$

defined for positive integers l_1, \dots, l_r , the analytic continuation of which has been given in [4] (the analytic continuation of a more general multi-variable multiple zeta function is established in [1]). This answers to the question posed in §5 of [4]. As a result, the function $\eta(k_1, \dots, k_r; s)$ can also be written by the multiple zeta functions of the type above. We then give a formula for values at positive integers of $\xi(k_1, \dots, k_r; s)$, and hence of $\eta(k_1, \dots, k_r; s)$, in terms of the ‘shuffle regularized values’ of multiple zeta values, and thereby derive some consequences on the values of $\eta_k(s)$.

Let $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}_{\geq 1}^r$ be an index set. Recall that \mathbf{k} is said to be admissible if the last entry k_r is greater than 1, the weight of \mathbf{k} is the sum $k_1 + \cdots + k_r$, and the depth is the length r of the index. For two indices \mathbf{k} and \mathbf{k}' of the same weight, we say \mathbf{k}' refines \mathbf{k} , denoted $\mathbf{k} \preceq \mathbf{k}'$, if \mathbf{k} is obtained from \mathbf{k}' by replacing some commas by +’s. For example, $(5) = (2+3) \preceq (2, 3)$, $(2, 3) = (1+1, 2+1) \preceq (1, 1, 2, 1)$, etc. The standard expression of a multiple zeta-star value as a sum of multiple zeta values is written as

$$\zeta^*(\mathbf{k}) = \sum_{\substack{\mathbf{k}' \preceq \mathbf{k} \\ \text{admissible}}} \zeta(\mathbf{k}'),$$

where the sum on the right runs over the admissible indices \mathbf{k}' such that \mathbf{k} refines \mathbf{k}' .

The following formula is known as the Landen connection formula for the multiple polylogarithm ([26, Proposition 9]).

Lemma 3.1. *For any index \mathbf{k} of depth r , we have*

$$(3.2) \quad \text{Li}_{\mathbf{k}}\left(\frac{z}{z-1}\right) = (-1)^r \sum_{\mathbf{k} \preceq \mathbf{k}'} \text{Li}_{\mathbf{k}'}(z).$$

We can prove this by induction on weight and by using (2.3), see [26].

By using this and noting $z/(z-1) = 1 - e^t$ (resp. $1 - e^{-t}$) if $z = 1 - e^{-t}$ (resp. $1 - e^t$), we immediately obtain the following proposition.

Proposition 3.2. *Let \mathbf{k} be any index set and r its depth. We have the relations*

$$(3.3) \quad \eta(\mathbf{k}; s) = (-1)^{r-1} \sum_{\mathbf{k} \preceq \mathbf{k}'} \xi(\mathbf{k}'; s)$$

and

$$(3.4) \quad \xi(\mathbf{k}; s) = (-1)^{r-1} \sum_{\mathbf{k} \preceq \mathbf{k}'} \eta(\mathbf{k}'; s).$$

Corollary 3.3. *Let k be a positive integer. Then we have*

$$(3.5) \quad \eta_k(s) = \sum_{\mathbf{k}: \text{weight } k} \xi(\mathbf{k}; s)$$

and

$$(3.6) \quad \xi_k(s) = \sum_{\mathbf{k}: \text{weight } k} \eta(\mathbf{k}; s),$$

where the sums run over all indices of weight k . Here we have written $\xi_k(s)$ for $\xi(k; s)$.

Proof. The index (k) is of depth 1 and all indices of weight k (admissible or non-admissible) refine (k) . \square

We mention here that, also by taking $\mathbf{k} = (k)$ in Lemma 3.1 and setting $z = 1 - e^t$ or $1 - e^{-t}$, one immediately obtains a kind of sum formulas for multi-poly-Bernoulli numbers as follows (compare with similar formulas in [17, Theorem 3.1]).

Corollary 3.4. *For $k \geq 1$ and $m \geq 0$, we have*

$$(3.7) \quad B_m^{(k)} = (-1)^m \sum_{\substack{k_1 + \dots + k_r = k \\ k_i, r \geq 1}} C_m^{(k_1, \dots, k_r)}$$

and

$$(3.8) \quad C_m^{(k)} = (-1)^m \sum_{\substack{k_1 + \dots + k_r = k \\ k_i, r \geq 1}} B_m^{(k_1, \dots, k_r)}.$$

Next, we prove an Euler-type connection formula for the multiple polylogarithm. If an index \mathbf{k} is of weight $|\mathbf{k}|$, we also say the multiple zeta value $\zeta(\mathbf{k})$ is of weight $|\mathbf{k}|$.

Lemma 3.5. *Let \mathbf{k} be any index. Then we have*

$$(3.9) \quad \text{Li}_{\mathbf{k}}(1-z) = \sum_{\mathbf{k}', j \geq 0} c_{\mathbf{k}}(\mathbf{k}'; j) \underbrace{\text{Li}_{1, \dots, 1}}_j(1-z) \text{Li}_{\mathbf{k}'}(z),$$

where the sum on the right runs over indices \mathbf{k}' and integers $j \geq 0$ that satisfy $|\mathbf{k}'| + j \leq |\mathbf{k}|$, and $c_{\mathbf{k}}(\mathbf{k}'; j)$ is a \mathbb{Q} -linear combination of multiple zeta values of weight $|\mathbf{k}| - |\mathbf{k}'| - j$. We understand $\text{Li}_{\emptyset}(z) = 1$ and $|\emptyset| = 0$ for the empty index \emptyset , and the constant 1 is regarded as a multiple zeta value of weight 0.

Proof. We proceed by induction on the weight of \mathbf{k} . When $\mathbf{k} = (1)$, the trivial identity $\text{Li}_1(1-z) = \text{Li}_1(1-z)$ is the one asserted. Suppose the weight $|\mathbf{k}|$ of \mathbf{k} is greater than 1 and assume the statement holds for any index of weight less than $|\mathbf{k}|$. For $\mathbf{k} = (k_1, \dots, k_r)$, set $\mathbf{k}_- = (k_1, \dots, k_{r-1}, k_r - 1)$ and $\mathbf{k}_+ = (k_1, \dots, k_{r-1}, k_r + 1)$.

First assume that \mathbf{k} is admissible. Then, by (2.3) and induction hypothesis, we have

$$\frac{d}{dz} \text{Li}_{\mathbf{k}}(1-z) = -\frac{\text{Li}_{\mathbf{k}_-}(1-z)}{1-z} = -\frac{1}{1-z} \sum_{\mathbf{l}, j \geq 0} c_{\mathbf{k}_-}(\mathbf{l}; j) \underbrace{\text{Li}_{1, \dots, 1}}_j(1-z) \text{Li}_{\mathbf{l}}(z),$$

the right-hand side being of a desired form. Here, again by (2.3), we see that

$$\frac{1}{1-z} \underbrace{\text{Li}_{1, \dots, 1}}_j(1-z) \text{Li}_{\mathbf{l}}(z) = \frac{d}{dz} \left(\sum_{i=0}^j \underbrace{\text{Li}_{1, \dots, 1}}_{j-i}(1-z) \text{Li}_{\mathbf{l}, 1+i}(z) \right).$$

We therefore conclude

$$\text{Li}_{\mathbf{k}}(1-z) = - \sum_{\mathbf{l}, j \geq 0} c_{\mathbf{k}_-}(\mathbf{l}; j) \sum_{i=0}^j \underbrace{\text{Li}_{1, \dots, 1}}_{j-i}(1-z) \text{Li}_{\mathbf{l}, 1+i}(z) + C$$

with some constant C . Since $\lim_{z \rightarrow 0} \underbrace{\text{Li}_{1, \dots, 1}}_{j-i}(1-z) \text{Li}_{\mathbf{l}, 1+i}(z) = 0$, we find $C = \zeta(\mathbf{k})$ by setting $z \rightarrow 0$, and obtain the desired expression for $\text{Li}_{\mathbf{k}}(1-z)$.

When \mathbf{k} is not necessarily admissible, write $\mathbf{k} = (\mathbf{k}_0, \underbrace{1, \dots, 1}_q)$ with an admissible \mathbf{k}_0 and $q \geq 0$. We prove the identity by induction on q . The case $q = 0$ (\mathbf{k} is admissible) is already done. Suppose $q \geq 1$ and assume the claim is true for smaller q . Then by assumption we have the expression

$$\text{Li}_{\mathbf{k}_0, \underbrace{1, \dots, 1}_{q-1}}(1-z) = \sum_{\mathbf{m}, j \geq 0} c_{\mathbf{k}'}(\mathbf{m}; j) \underbrace{\text{Li}_{1, \dots, 1}}_j(1-z) \text{Li}_{\mathbf{m}}(z),$$

where we have put $\mathbf{k}' = (\mathbf{k}_0, \underbrace{1, \dots, 1}_{q-1})$. We multiply $\text{Li}_1(1-z)$ on both sides. Then, by the shuffle product, the left-hand side becomes the sum of the form

$$q \text{Li}_{\mathbf{k}}(1-z) + \sum_{\mathbf{k}_0': \text{admissible}} \underbrace{\text{Li}_{\mathbf{k}_0', 1, \dots, 1}}_{q-1}(1-z),$$

and each term in the sum is written in the claimed form by induction hypothesis. On the other hand, the right-hand side becomes also of the form desired because

$$\text{Li}_1(1-z) \underbrace{\text{Li}_{1, \dots, 1}}_j(1-z) = (j+1) \underbrace{\text{Li}_{1, \dots, 1}}_{j+1}(1-z).$$

Hence $\text{Li}_{\mathbf{k}}(1-z)$ is of the form as claimed. \square

With the lemma, we are now able to establish the following (see [4, §5, Problem (i)]).

Theorem 3.6. *Let \mathbf{k} be any index set. The function $\xi(\mathbf{k}; s)$ can be written in terms of multiple zeta functions as*

$$\xi(\mathbf{k}; s) = \sum_{\mathbf{k}', j \geq 0} c_{\mathbf{k}}(\mathbf{k}'; j) \binom{s+j-1}{j} \zeta(\mathbf{k}'; s+j).$$

Here, the sum is over indices \mathbf{k}' and integers $j \geq 0$ satisfying $|\mathbf{k}'| + j \leq |\mathbf{k}|$, and $c_{\mathbf{k}}(\mathbf{k}'; j)$ is a \mathbb{Q} -linear combination of multiple zeta values of weight $|\mathbf{k}| - |\mathbf{k}'| - j$. The index \mathbf{k}' may be \emptyset and for this we set $\zeta(\emptyset; s+j) = \zeta(s+j)$.

Proof. By setting $z = e^{-t}$ in the lemma and using

$$(3.10) \quad \text{Li}_{\underbrace{1, \dots, 1}_j}(z) = \frac{(-\log(1-z))^j}{j!},$$

we have

$$\text{Li}_{\mathbf{k}}(1 - e^{-t}) = \sum_{\mathbf{k}', j \geq 0} c_{\mathbf{k}}(\mathbf{k}'; j) \frac{t^j}{j!} \text{Li}_{\mathbf{k}'}(e^{-t}).$$

Substituting this into the definition (1.2) of $\xi(\mathbf{k}; s)$ and using the formula ([4, Proposition 2, (i)])

$$\zeta(\mathbf{k}; s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} \text{Li}_{\mathbf{k}}(e^{-t}) dt,$$

we immediately obtain the theorem. \square

Remark 3.7. This theorem generalizes [4, Theorem 8], where the corresponding formula for $\text{Li}_{\underbrace{1, \dots, 1}_{r-1}, k}(1-z)$ is

$$\begin{aligned} \text{Li}_{\underbrace{1, \dots, 1}_{r-1}, k}(1-z) &= (-1)^{k-1} \sum_{\substack{j_1 + \dots + j_k = r+k \\ \forall j_i \geq 1}} \text{Li}_{\underbrace{1, \dots, 1}_{j_k-1}}(1-z) \text{Li}_{j_1, \dots, j_{k-1}}(z) \\ &\quad + \sum_{j=0}^{k-2} (-1)^j \zeta(\underbrace{1, \dots, 1}_{r-1}, k-j) \text{Li}_{\underbrace{1, \dots, 1}_j}(z). \end{aligned}$$

As pointed out by Shu Oi, one can deduce Lemma 3.5 by induction using [27, Prop. 5]. However, to describe the right-hand side of the lemma explicitly is a different problem and neither proof gives such a formula in general. See also [25] for a related topic.

Example 3.8. Apart from the trivial case $(1, \dots, 1)$, examples of the identity in Lemma 3.5 up to weight 4 are:

$$\begin{aligned} \text{Li}_2(1-z) &= -\text{Li}_2(z) - \text{Li}_1(1-z)\text{Li}_1(z) + \zeta(2), \\ \text{Li}_3(1-z) &= \text{Li}_{1,2}(z) + \text{Li}_{2,1}(z) + \text{Li}_1(1-z)\text{Li}_{1,1}(z) - \zeta(2)\text{Li}_1(z) + \zeta(3), \\ \text{Li}_{1,2}(1-z) &= -\text{Li}_3(z) - \text{Li}_1(1-z)\text{Li}_2(z) - \text{Li}_{1,1}(1-z)\text{Li}_1(z) + \zeta(3), \\ \text{Li}_{2,1}(1-z) &= 2\text{Li}_3(z) + \text{Li}_1(1-z)\text{Li}_2(z) + \zeta(2)\text{Li}_1(1-z) - 2\zeta(3), \\ \text{Li}_4(1-z) &= -\text{Li}_{1,1,2}(z) - \text{Li}_{1,2,1}(z) - \text{Li}_{2,1,1}(z) - \text{Li}_1(1-z)\text{Li}_{1,1,1}(z) \\ &\quad + \zeta(2)\text{Li}_{1,1}(z) - \zeta(3)\text{Li}_1(z) + \zeta(4), \end{aligned}$$

$$\begin{aligned}
\text{Li}_{1,3}(1-z) &= \text{Li}_{1,3}(z) + \text{Li}_{2,2}(z) + \text{Li}_{3,1}(z) + \text{Li}_1(1-z)\text{Li}_{1,2}(z) + \text{Li}_1(1-z)\text{Li}_{2,1}(z) \\
&\quad + \text{Li}_{1,1}(1-z)\text{Li}_{1,1}(z) - \zeta(3)\text{Li}_1(z) + \frac{1}{4}\zeta(4), \\
\text{Li}_{2,2}(1-z) &= -\text{Li}_{2,2}(z) - 2\text{Li}_{3,1}(z) - \text{Li}_1(1-z)\text{Li}_{2,1}(z) - \zeta(2)\text{Li}_1(1-z)\text{Li}_1(z) \\
&\quad - \zeta(2)\text{Li}_2(z) + 2\zeta(3)\text{Li}_1(z) + \frac{3}{4}\zeta(4), \\
\text{Li}_{3,1}(1-z) &= -2\text{Li}_{1,3}(z) - \text{Li}_{2,2}(z) - \text{Li}_1(1-z)\text{Li}_{1,2}(z) + \zeta(2)\text{Li}_2(z) \\
&\quad + \zeta(3)\text{Li}_1(1-z) - \frac{5}{4}\zeta(4), \\
\text{Li}_{1,1,2}(1-z) &= -\text{Li}_4(z) - \text{Li}_1(1-z)\text{Li}_3(z) - \text{Li}_{1,1}(1-z)\text{Li}_2(z) \\
&\quad - \text{Li}_{1,1,1}(1-z)\text{Li}_1(z) + \zeta(4), \\
\text{Li}_{1,2,1}(1-z) &= 3\text{Li}_4(z) + 2\text{Li}_1(1-z)\text{Li}_3(z) + \text{Li}_{1,1}(1-z)\text{Li}_2(z) + \zeta(3)\text{Li}_1(1-z) \\
&\quad - 3\zeta(4), \\
\text{Li}_{2,1,1}(1-z) &= -3\text{Li}_4(z) - \text{Li}_1(1-z)\text{Li}_3(z) + \zeta(2)\text{Li}_{1,1}(1-z) - 2\zeta(3)\text{Li}_1(1-z) \\
&\quad + 3\zeta(4).
\end{aligned}$$

Accordingly, we have

$$\begin{aligned}
\xi(2; s) &= -\zeta(2; s) - s\zeta(1; s+1) + \zeta(2)\zeta(s), \\
\xi(3; s) &= \zeta(1, 2; s) + \zeta(2, 1; s) + s\zeta(1, 1; s+1) - \zeta(2)\zeta(1; s) + \zeta(3)\zeta(s), \\
\xi(1, 2; s) &= -\zeta(3; s) - s\zeta(2; s+1) - \frac{s(s+1)}{2}\zeta(1; s+2) + \zeta(3)\zeta(s), \\
\xi(2, 1; s) &= 2\zeta(3; s) + s\zeta(2; s+1) + \zeta(2)s\zeta(s+1) - 2\zeta(3)\zeta(s), \\
\xi(4; s) &= -\zeta(1, 1, 2; s) - \zeta(1, 2, 1; s) - \zeta(2, 1, 1; s) - s\zeta(1, 1, 1; s+1) \\
&\quad + \zeta(2)\zeta(1, 1; s) - \zeta(3)\zeta(1; s) + \zeta(4)\zeta(s), \\
\xi(1, 3; s) &= \zeta(1, 3; s) + \zeta(2, 2; s) + \zeta(3, 1; s) + s\zeta(1, 2; s+1) + s\zeta(2, 1; s+1) \\
&\quad + \frac{s(s+1)}{2}\zeta(1, 1; s+2) - \zeta(3)\zeta(1; s) + \frac{1}{4}\zeta(4)\zeta(s), \\
\xi(2, 2; s) &= -\zeta(2, 2; s) - 2\zeta(3, 1; s) - s\zeta(2, 1; s+1) - \zeta(2)s\zeta(1; s+1) \\
&\quad - \zeta(2)\zeta(2; s) + 2\zeta(3)\zeta(1; s) + \frac{3}{4}\zeta(4)\zeta(s), \\
\xi(3, 1; s) &= -2\zeta(1, 3; s) - \zeta(2, 2; s) - s\zeta(1, 2; s+1) + \zeta(2)\zeta(2; s) \\
&\quad + \zeta(3)s\zeta(s+1) - \frac{5}{4}\zeta(4)\zeta(s), \\
\xi(1, 1, 2; s) &= -\zeta(4; s) - s\zeta(3; s+1) - \frac{s(s+1)}{2}\zeta(2; s+2) \\
&\quad - \frac{s(s+1)(s+2)}{6}\zeta(1; s+3) + \zeta(4)\zeta(s), \\
\xi(1, 2, 1; s) &= 3\zeta(4; s) + 2s\zeta(3; s+1) + \frac{s(s+1)}{2}\zeta(2; s+2) + \zeta(3)s\zeta(s+1) \\
&\quad - 3\zeta(4)\zeta(s), \\
\xi(2, 1, 1; s) &= -3\zeta(4; s) - s\zeta(3; s+1) + \zeta(2)\frac{s(s+1)}{2}\zeta(s+2) - 2\zeta(3)s\zeta(s+1) \\
&\quad + 3\zeta(4)\zeta(s).
\end{aligned}$$

From these and (3.5) of Corollary 3.3, we have for instance

$$\eta_2(s) = \xi(2; s) + \xi(1, 1; s)$$

$$\begin{aligned}
&= -\zeta(2; s) - s\zeta(1; s+1) + \zeta(2)\zeta(s) + \frac{s(s+1)}{2}\zeta(s+2), \\
\eta_3(s) &= \xi(3; s) + \xi(1, 2; s) + \xi(2, 1; s) + \xi(1, 1, 1; s) \\
&= \zeta(3; s) + \zeta(1, 2; s) + \zeta(2, 1; s) + s\zeta(1, 1; s+1) - \frac{s(s+1)}{2}\zeta(1; s+2) \\
&\quad - \zeta(2)\zeta(1; s) + \zeta(2)s\zeta(s+1) + \frac{s(s+1)(s+2)}{6}\zeta(s+3), \\
\eta_4(s) &= \xi(4; s) + \xi(1, 3; s) + \xi(2, 2; s) + \xi(3, 1; s) + \xi(1, 1, 2; s) + \xi(1, 2, 1; s) \\
&\quad + \xi(2, 1, 1; s) + \xi(1, 1, 1, 1; s) \\
&= -\zeta(4; s) - \zeta(1, 3; s) - \zeta(2, 2; s) - \zeta(3, 1; s) - \zeta(1, 1, 2; s) - \zeta(1, 2, 1; s) \\
&\quad - \zeta(2, 1, 1; s) - s\zeta(1, 1, 1; s+1) + \zeta(2)\zeta(1, 1; s) + \frac{s(s+1)}{2}\zeta(1, 1; s+2) \\
&\quad - \zeta(2)s\zeta(1; s+1) + \zeta(2)\frac{s(s+1)}{2}\zeta(s+2) - \frac{s(s+1)(s+2)}{6}\zeta(1; s+3) \\
&\quad + \frac{7}{4}\zeta(4)\zeta(s) + \frac{s(s+1)(s+2)(s+3)}{24}\zeta(s+4).
\end{aligned}$$

Before closing this section, we present a curious observation. Recall the formula

$$\xi_k(m) = \zeta^*(\underbrace{1, \dots, 1}_{m-1}, k+1)$$

discovered by Ohno [24]. Comparing this with the two formulas (2.11) and [4, Corollary 10], one may expect

$$\eta_k(m) \stackrel{?}{=} \zeta(\underbrace{1, \dots, 1}_{m-1}, k+1).$$

This is not true in fact. However, we found experimentally the identities

$$(3.11) \quad \eta_k(m) = \eta_m(k)$$

and

$$(3.12) \quad \sum_{j=1}^{k-1} (-1)^{j-1} \eta_{k-j}(j) = \begin{cases} 2(1 - 2^{1-k})\zeta(k) & (k: \text{even}), \\ 0 & (k: \text{odd}). \end{cases}$$

These are respectively analogous to the duality relation

$$\zeta(\underbrace{1, \dots, 1}_{m-1}, k+1) = \zeta(\underbrace{1, \dots, 1}_{k-1}, m+1)$$

and the relation

$$\sum_{j=1}^{k-1} (-1)^{j-1} \zeta(\underbrace{1, \dots, 1}_{j-1}, k-j+1) = \begin{cases} 2(1 - 2^{1-k})\zeta(k) & (k: \text{even}), \\ 0 & (k: \text{odd}), \end{cases}$$

which is a special case of the Le-Murakami relation [23] (or one can derive this from the well-known generating series identity [2], [11])

$$1 - \sum_{k>j\geq 1} \zeta(\underbrace{1, \dots, 1}_{j-1}, k-j+1) X^{k-j} Y^j = \frac{\Gamma(1-X)\Gamma(1-Y)}{\Gamma(1-X-Y)}$$

by setting $Y = -X$ and using the reflection formula for the gamma function.)

We are still not able to prove (3.11)*, but could prove (3.12) by using the following general

*Quite recently, Shuji Yamamoto communicated to the authors that he found a proof.

formula for the value $\xi(\mathbf{k}; m)$ and the relation (3.3) in Proposition 3.2. We shall discuss this and other aspects of ‘height one’ multiple zeta values in more detail in a subsequent paper [22].

Proposition 3.9. *Let \mathbf{k} be any index and $m \geq 1$ an integer. Then we have*

$$(3.13) \quad \xi(\mathbf{k}; m) = (-1)^{m-1} \zeta^{\text{III}}(\mathbf{k}_+, \underbrace{1, \dots, 1}_{m-1}),$$

where ζ^{III} stands for the ‘shuffle regularized’ value, which is the constant term of the shuffle regularized polynomial defined in [19].

Proof. By making the change of variable $x = 1 - e^{-t}$ in the definition (1.2), we have

$$\xi(\mathbf{k}; s) = \frac{1}{\Gamma(s)} \int_0^1 (-\log(1-x))^{s-1} \text{Li}_{\mathbf{k}}(x) \frac{dx}{x}.$$

Put $s = m$ and use (3.10) to obtain

$$\xi(\mathbf{k}; m) = \int_0^1 \underbrace{\text{Li}_{1, \dots, 1}(x)}_{m-1} \text{Li}_{\mathbf{k}}(x) \frac{dx}{x}.$$

The regularization formula [19, Eq. (5.2)], together with the shuffle product of $\underbrace{\text{Li}_{1, \dots, 1}(x)}_{m-1} \text{Li}_{\mathbf{k}}(x)$, immediately gives (3.13). \square

By using (3.13) and (3.5), we can write $\eta_{\mathbf{k}}(m)$ in terms of shuffle regularized values. The following expression seems to follow from that formula by taking the dual, but we have not yet worked it out in detail.

$$\eta_{\mathbf{k}}(m) \stackrel{?}{=} \binom{m+k}{k} \zeta(m+k) - \sum_{\substack{2 \leq r \leq k+1 \\ j_1 + \dots + j_r = m+k-r-1}} \binom{j_1 + \dots + j_{r-1}}{k-r+1} \zeta(j_1+1, \dots, j_{r-1}+1, j_r+2).$$

4. The function $\eta(k_1, \dots, k_r; s)$ for non-positive indices

In this section, as in the case of positive indices, we construct η -functions with non-positive indices. It is known that $\text{Li}_{-k}(z)$ can be expressed as

$$\text{Li}_{-k}(z) = \frac{P(z; k)}{(1-z)^k}$$

for $k \in \mathbb{Z}_{\geq 0}$, where $P(x; k) \in \mathbb{Z}[x]$ is a monic polynomial satisfying

$$\deg P(x; k) = \begin{cases} 1 & (k=0) \\ k & (k \geq 1), \end{cases}$$

$$x \mid P(x; k)$$

(see, for example, Shimura [29, Equations (2.17), (4.2) and (4.6)]; Note that the above $P(x; k)$ coincides with $xP_{k+1}(x)$ in [29]). We first extend this fact to multiple polylogarithms with non-positive indices as follows.

Lemma 4.1. For $k_1, \dots, k_r \in \mathbb{Z}_{\geq 0}$, there exists a polynomial $P(x; k_1, \dots, k_r) \in \mathbb{Z}[x]$ such that

$$(4.1) \quad \text{Li}_{-k_1, \dots, -k_r}(z) = \frac{P(z; k_1, \dots, k_r)}{(1-z)^{k_1 + \dots + k_r + r}},$$

$$(4.2) \quad \deg P(x; k_1, \dots, k_r) = \begin{cases} r & (k_1 = \dots = k_r = 0) \\ k_1 + \dots + k_r + r - 1 & (\text{otherwise}), \end{cases}$$

$$(4.3) \quad x^r \mid P(x; k_1, \dots, k_r).$$

More explicitly, $P(x; \underbrace{0, 0, \dots, 0}_r) = x^r$.

Proof. We prove this lemma by the double induction on $r \geq 1$ and $K = k_1 + \dots + k_r \geq 0$. The case $r = 1$ is as mentioned above. For $r \geq 2$, we assume the case of $r - 1$ holds and consider the case of r . When $K = k_1 + \dots + k_r = 0$, namely $k_1 = \dots = k_r = 0$, we have

$$\begin{aligned} \text{Li}_{0, \dots, 0}(z) &= \sum_{m_1 < \dots < m_r} x^{m_r} = \sum_{m_1 < \dots < m_{r-1}} \sum_{m_r = m_{r-1} + 1}^{\infty} z^{m_r} \\ &= \frac{z}{1-z} \sum_{m_1 < \dots < m_{r-1}} z^{m_{r-1}} = \dots = \frac{z^r}{(1-z)^r}, \end{aligned}$$

which implies (4.1)–(4.3) hold, and also $P(x; 0, \dots, 0) = x^r$. Hence we assume the case $K = k_1 + \dots + k_r - 1$ holds and consider the case $K = k_1 + \dots + k_r (\geq 1)$. We consider the two cases $k_r = 0$ and $k_r \geq 1$ separately. First we assume $k_r = 0$. Then, by induction hypothesis, we have

$$\begin{aligned} \text{Li}_{-k_1, \dots, -k_{r-1}, 0}(z) &= \sum_{m_1 < \dots < m_{r-1}} m_1^{k_1} \dots m_{r-1}^{k_{r-1}} \sum_{m_r = m_{r-1} + 1}^{\infty} z^{m_r} \\ &= \frac{z}{1-z} \sum_{m_1 < \dots < m_{r-1}} m_1^{k_1} \dots m_{r-1}^{k_{r-1}} z^{m_{r-1}} \\ &= \frac{z}{1-z} \frac{P(z; k_1, \dots, k_{r-1})}{(1-z)^{k_1 + \dots + k_{r-1} + r - 1}}. \end{aligned}$$

Let $P(z; k_1, \dots, k_{r-1}, 0) = zP(z; k_1, \dots, k_{r-1})$. Then (4.1)–(4.3) hold.

Next we assume $k_r \geq 1$. Then, using the same formula as in (2.3) and the induction hypothesis, we have

$$\begin{aligned} \text{Li}_{-k_1, \dots, -k_{r-1}, -k_r}(z) &= z \frac{d}{dz} \text{Li}_{-k_1, \dots, -k_r + 1}(z) \\ &= z \frac{d}{dz} \left(\frac{P(z; k_1, \dots, k_r - 1)}{(1-z)^{k_1 + \dots + k_r - 1 + r}} \right) \\ &= \frac{z \{P'(z; k_1, \dots, k_r - 1)(1-z) + (k_1 + \dots + k_r - 1 + r)P(z; k_1, \dots, k_r - 1)\}}{(1-z)^{k_1 + \dots + k_r + r}}. \end{aligned}$$

If $k_1 = \dots = k_{r-1} = 0$ and $k_r = 1$, then the numerator, that is, $P(0, \dots, 0, -1)$ equals rz^r , using the above results. If not, the degree of the numerator equals $k_1 + \dots + k_r + r - 1$ by induction hypothesis. The both cases satisfy (4.1)–(4.3). This completes the proof of the lemma. \square

Remark 4.2. In the case $r \geq 2$, $P(x; k_1, \dots, k_r)$ is not necessarily a monic polynomial. For example, we have $\text{Li}_{0, -1}(z) = 2z^2/(1-z)^3$, so $P(x; 0, 1) = 2x^2$.

We obtain from (4.1) and (4.2) that

$$(4.4) \quad \text{Li}_{-k_1, \dots, -k_r}(1 - e^t) = \frac{P(1 - e^t; k_1, \dots, k_r)}{e^{(k_1 + \dots + k_r + r)t}} = \begin{cases} O(1) & (k_1 = \dots = k_r = 0) \\ O(e^{-t}) & (\text{otherwise}) \end{cases}$$

as $t \rightarrow \infty$, and from (4.3) that

$$(4.5) \quad \text{Li}_{-k_1, \dots, -k_r}(1 - e^t) = O(t^r) \quad (t \rightarrow 0).$$

Therefore we can define the following.

Definition 4.3. For $k_1, \dots, k_r \in \mathbb{Z}_{\geq 0}$, define

$$(4.6) \quad \eta(-k_1, \dots, -k_r; s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\text{Li}_{-k_1, \dots, -k_r}(1 - e^{-t})}{1 - e^{-t}} dt$$

for $s \in \mathbb{C}$ with $\text{Re}(s) > 1 - r$. In the case $r = 1$, denote $\eta(-k; s)$ by $\eta_{-k}(s)$.

We see that the integral on the right-hand side of (4.6) is absolutely convergent for $\text{Re}(s) > 1 - r$. Hence $\eta(-k_1, \dots, -k_r; s)$ is holomorphic for $\text{Re}(s) > 1 - r$. By the same method as in the proof of Theorem 2.3 for $\eta(k_1, \dots, k_r; s)$, we can similarly obtain the following.

Theorem 4.4. For $k_1, \dots, k_r \in \mathbb{Z}_{\geq 0}$, $\eta(-k_1, \dots, -k_r; s)$ can be analytically continued to an entire function on the whole complex plane, and satisfies

$$(4.7) \quad \eta(-k_1, \dots, -k_r; -m) = B_m^{(-k_1, \dots, -k_r)} \quad (m \in \mathbb{Z}_{\geq 0}).$$

In particular, $\eta_{-k}(-m) = B_m^{(-k)}$ ($k \in \mathbb{Z}_{\geq 0}$, $m \in \mathbb{Z}_{\geq 0}$).

It should be noted that $\xi(-k_1, \dots, -k_r; s)$ cannot be defined by replacing $\{k_j\}$ by $\{-k_j\}$ in (1.2). In fact, even if $r = 1$ and $k = 0$ in (1.2), we see that

$$\xi_0(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\text{Li}_0(1 - e^{-t})}{e^{-t} - 1} dt = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} dt,$$

which is not convergent for any $s \in \mathbb{C}$. Therefore we modify the definition (1.2) as follows.

Definition 4.5. For $k_1, \dots, k_r \in \mathbb{Z}_{\geq 0}$ with $(k_1, \dots, k_r) \neq (0, \dots, 0)$, define

$$(4.8) \quad \tilde{\xi}(-k_1, \dots, -k_r; s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\text{Li}_{-k_1, \dots, -k_r}(1 - e^{-t})}{e^{-t} - 1} dt$$

for $s \in \mathbb{C}$ with $\text{Re}(s) > 1 - r$. In the case $r = 1$, denote $\tilde{\xi}(-k; s)$ by $\tilde{\xi}_{-k}(s)$ for $k \geq 1$.

We see from (4.4) and (4.5) that (4.8) is well-defined. Also it is noted that $\tilde{\xi}(k_1, \dots, k_r; s)$ cannot be defined by replacing $\{-k_j\}$ by $\{k_j\}$ in (4.8) for $(k_j) \in \mathbb{Z}_{\geq 1}^r$.

In a way parallel to deriving Theorem 4.4, we can obtain the following.

Theorem 4.6. For $k_1, \dots, k_r \in \mathbb{Z}_{\geq 0}$ with $(k_1, \dots, k_r) \neq (0, \dots, 0)$, $\tilde{\xi}(-k_1, \dots, -k_r; s)$ can be analytically continued to an entire function on the whole complex plane, and satisfies

$$(4.9) \quad \tilde{\xi}(-k_1, \dots, -k_r; -m) = C_m^{(-k_1, \dots, -k_r)} \quad (m \in \mathbb{Z}_{\geq 0}).$$

In particular, $\tilde{\xi}_{-k}(-m) = C_m^{(-k)}$ ($k \in \mathbb{Z}_{\geq 1}$, $m \in \mathbb{Z}_{\geq 0}$).

Next we give certain duality formulas for $B_n^{(k_1, \dots, k_r)}$ which is a generalization of (1.6). To state this, we define another type of multi-poly-Bernoulli numbers by

$$(4.10) \quad \sum_{a=0}^{r-1} (-1)^a \binom{r-1}{a} \sum_{l_1, \dots, l_r \geq 1} \frac{\prod_{j=1}^r (1 - e^{-\sum_{\nu=j}^r x_\nu})^{l_j-1}}{(l_1 + \dots + l_r - a)^s} \\ = \sum_{m_1, \dots, m_r \geq 0} \mathfrak{B}_{m_1, \dots, m_r}^{(s)} \frac{x_1^{m_1} \dots x_r^{m_r}}{m_1! \dots m_r!}$$

for $s \in \mathbb{C}$. In the case $r = 1$, we see that $\mathfrak{B}_m^{(k)} = B_m^{(k)}$ for $k \in \mathbb{Z}$. Then we obtain the following result which is a kind of the duality formula. In fact, this coincides with (1.6) in the case $r = 1$.

Theorem 4.7. For $k_1, \dots, k_r \in \mathbb{Z}_{\geq 0}$,

$$(4.11) \quad \eta(-k_1, \dots, -k_r; s) = \mathfrak{B}_{k_1, \dots, k_r}^{(s)}.$$

Therefore, for $m \in \mathbb{Z}_{\geq 0}$,

$$(4.12) \quad B_m^{(-k_1, \dots, -k_r)} = \mathfrak{B}_{k_1, \dots, k_r}^{(-m)}.$$

Proof. We first prepare the following relation which will be proved in the next section (see Lemma 5.9):

$$(4.13) \quad \prod_{j=1}^r \frac{e^{\sum_{\nu=j}^r x_\nu} (1 - e^t)}{1 - e^{\sum_{\nu=j}^r x_\nu} (1 - e^t)} = \sum_{k_1, \dots, k_r \geq 0} \text{Li}_{-k_1, \dots, -k_r}(1 - e^t) \frac{x_1^{k_1} \dots x_r^{k_r}}{k_1! \dots k_r!}$$

holds around the origin. Let

$$\mathcal{F}(x_1, \dots, x_r; s) = \sum_{k_1, \dots, k_r \geq 0} \eta(-k_1, \dots, -k_r; s) \frac{x_1^{k_1} \dots x_r^{k_r}}{k_1! \dots k_r!}.$$

As a generalization of [18, Proposition 5], we have from (4.13) that

$$\begin{aligned} \mathcal{F}(x_1, \dots, x_r; s) &= \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{1 - e^t} \prod_{j=1}^r \frac{e^{\sum_{\nu=j}^r x_\nu} (1 - e^t)}{1 - e^{\sum_{\nu=j}^r x_\nu} (1 - e^t)} dt \\ &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} (1 - e^t)^{r-1} e^{-rt} \prod_{j=1}^r \frac{1}{1 - e^{-t} (1 - e^{-\sum_{\nu=j}^r x_\nu})} dt \\ &= \frac{1}{\Gamma(s)} \sum_{a=0}^{r-1} (-1)^a \binom{r-1}{a} \sum_{m_1, \dots, m_r \geq 0} \prod_{j=1}^r (1 - e^{-\sum_{\nu=j}^r x_\nu})^{m_j} \\ &\quad \times \int_0^\infty t^{s-1} e^{(a-r)t} \prod_{j=1}^r e^{-m_j t} dt \\ &= \sum_{a=0}^{r-1} (-1)^a \binom{r-1}{a} \sum_{m_1, \dots, m_r \geq 0} \frac{\prod_{j=1}^r (1 - e^{-\sum_{\nu=j}^r x_\nu})^{m_j}}{(m_1 + \dots + m_r + r - a)^s}. \end{aligned}$$

Therefore, by (4.10), we obtain (4.11). Further, setting $s = -m$ in (4.11) and using (4.7), we obtain (4.12). \square

Remark 4.8. In the case $r = 1$, (4.11) implies $\eta_{-k}(s) = B_k^{(s)}$. Thus, using Theorem 4.4, we obtain the duality formula (1.6), which is also written as

$$(4.14) \quad \eta_{-k}(-m) = \eta_{-m}(-k)$$

for $k, m \in \mathbb{Z}_{\geq 0}$. This is exactly contrasted with the positive index case (3.11). Furthermore, by the same method, we can show that $\tilde{\xi}_{-k-1}(-m) = C_{k+1}^{(-m)}$ for $k, m \in \mathbb{Z}_{\geq 0}$. Hence, using Theorem 4.6 in the case $r = 1$, we obtain the duality formula (1.7).

Example 4.9. When $r = 2$, we can calculate directly from (4.10) that $\mathfrak{B}_{1,0}^{(s)} = 3^{-s} - 2^{-s}$. On the other hand, as mentioned in Lemma 4.1, we have $\text{Li}_{-1,0}(z) = z^2/(1-z)^3$. Hence the left-hand side of (1.10) equals

$$\frac{\text{Li}_{-1,0}(1 - e^{-t})}{1 - e^{-t}} = \frac{1 - e^{-t}}{e^{-3t}} = e^{3t} - e^{2t},$$

hence $B_m^{(-1,0)} = 3^m - 2^m$. Thus we can verify $B_m^{(-1,0)} = \mathfrak{B}_{1,0}^{(-m)}$.

5. Multi-indexed poly-Bernoulli numbers and duality formulas

In this section, we define multi-indexed poly-Bernoulli numbers (see Definition 5.1) and prove the duality formula for them, namely a multi-indexed version of (1.6) (see Theorem 5.4).

For this aim, we first recall multiple polylogarithms of $*$ -type and of \mathfrak{w} -type in several variables defined by

$$(5.1) \quad \text{Li}_{s_1, \dots, s_r}^*(z_1, \dots, z_r) = \sum_{1 \leq m_1 < \dots < m_r} \frac{z_1^{m_1} \dots z_r^{m_r}}{m_1^{s_1} m_2^{s_2} \dots m_r^{s_r}},$$

$$(5.2) \quad \begin{aligned} \text{Li}_{s_1, \dots, s_r}^{\mathfrak{w}}(z_1, \dots, z_r) &= \sum_{1 \leq m_1 < \dots < m_r} \frac{z_1^{m_1} z_2^{m_2 - m_1} \dots z_r^{m_r - m_{r-1}}}{m_1^{s_1} m_2^{s_2} \dots m_r^{s_r}} \\ &= \sum_{l_1, \dots, l_r=1}^{\infty} \frac{z_1^{l_1} z_2^{l_2} \dots z_r^{l_r}}{l_1^{s_1} (l_1 + l_2)^{s_2} \dots (l_1 + \dots + l_r)^{s_r}} \end{aligned}$$

for $s_1, \dots, s_r \in \mathbb{C}$ and $z_1, \dots, z_r \in \mathbb{C}$ with $|z_j| \leq 1$ ($1 \leq j \leq r$) (see, for example, [13]). The symbols $*$ and \mathfrak{w} are derived from the harmonic product and the shuffle product in the theory of multiple zeta values. In fact, Arakawa and the first-named author defined the two types of multiple L -values $L^*(k_1, \dots, k_r; f_1, \dots, f_r)$ of $*$ -type and $L^{\mathfrak{w}}(k_1, \dots, k_r; f_1, \dots, f_r)$ of \mathfrak{w} -type associated to periodic functions $\{f_j\}$ (see [5]), defined by replacing $\{z_j^m\}$ by $\{f_j(m)\}$ and setting $(s_j) = (k_j) \in \mathbb{Z}_{\geq 1}^r$ on the right-hand sides of (5.1) and (5.2) for $(k_1, \dots, k_r) \in \mathbb{Z}_{\geq 1}^r$. Note that

$$(5.3) \quad \text{Li}_{s_1, \dots, s_r}^*(z_1, \dots, z_r) = \text{Li}_{s_1, \dots, s_r}^{\mathfrak{w}}\left(\prod_{j=1}^r z_j, \prod_{j=2}^r z_j, \dots, z_{r-1} z_r, z_r\right).$$

Definition 5.1 (Multi-indexed poly-Bernoulli numbers). For $s_1, \dots, s_r \in \mathbb{C}$ and $d \in \{1, 2, \dots, r\}$, the multi-indexed poly-Bernoulli numbers $\{B_{m_1, \dots, m_r}^{(s_1, s_2, \dots, s_r), (d)}\}$ are defined by

$$(5.4) \quad F(x_1, \dots, x_r; s_1, \dots, s_r; d) = \frac{\text{Li}_{s_1, \dots, s_r}^{\mathfrak{w}}\left(1 - e^{-\sum_{\nu=1}^r x_\nu}, \dots, 1 - e^{-x_{r-1} - x_r}, 1 - e^{-x_r}\right)}{\prod_{j=1}^d \left(1 - e^{-\sum_{\nu=j}^r x_\nu}\right)}$$

$$\begin{aligned}
& \left(= \sum_{l_1, \dots, l_r=1}^{\infty} \frac{\prod_{j=1}^r \left(1 - e^{-\sum_{\nu=j}^r x_{\nu}}\right)^{l_j - \delta_j(d)}}{\prod_{j=1}^r \left(\sum_{\nu=1}^j l_{\nu}\right)^{s_j}} \right) \\
& = \sum_{m_1, \dots, m_r=0}^{\infty} B_{m_1, \dots, m_r}^{(s_1, \dots, s_r), (d)} \frac{x_1^{m_1} \cdots x_r^{m_r}}{m_1! \cdots m_r!},
\end{aligned}$$

where $\delta_j(d) = 1$ ($j \leq d$), $= 0$ ($j > d$).

Remark 5.2. Note that $\text{Li}_{k_1, \dots, k_r}^{\mathbb{W}}(z, \dots, z) = \text{Li}_{k_1, \dots, k_r}(z)$ defined by (1.9). Suppose $x_1 = \cdots = x_{r-1} = 0$ and $(s_j) = (k_j) \in \mathbb{Z}^r$ in (5.4). We immediately see that if $d = 1$ then

$$B_{0, \dots, 0, m}^{(k_1, \dots, k_r), (1)} = B_m^{(k_1, \dots, k_r)} \quad (m \in \mathbb{Z}_{\geq 0})$$

(see (1.6)), and if $d = r$ then

$$B_{0, \dots, 0, m}^{(k_1, \dots, k_r), (r)} = \mathbb{B}_m^{(k_1, \dots, k_r)} \quad (m \in \mathbb{Z}_{\geq 0})$$

(see (1.8)).

Remark 5.3. Let

$$(5.5) \quad \Lambda_r = \{(x_1, \dots, x_r) \in \mathbb{C}^r \mid |1 - e^{-\sum_{\nu=j}^r x_{\nu}}| < 1 \ (1 \leq j \leq r)\}.$$

Then we can see that

$$\text{Li}_{s_1, \dots, s_r}^{\mathbb{W}} \left(1 - e^{-\sum_{\nu=1}^r x_{\nu}}, \dots, 1 - e^{-x_{r-1} - x_r}, 1 - e^{-x_r} \right) \quad (s_1, \dots, s_r \in \mathbb{C})$$

is absolutely convergent for $(x_j) \in \Lambda_r$. Also $F(x_1, \dots, x_r; s_1, \dots, s_r; d)$ is absolutely convergent in the region $\Lambda_r \times \mathbb{C}^r$, so is holomorphic. Hence $B_{m_1, \dots, m_r}^{(s_1, \dots, s_r), (d)}$ is an entire function, because

$$B_{m_1, \dots, m_r}^{(s_1, \dots, s_r), (d)} = \left(\frac{\partial}{\partial x_1} \right)^{m_1} \cdots \left(\frac{\partial}{\partial x_r} \right)^{m_r} F(x_1, \dots, x_r; s_1, \dots, s_r; d) \Big|_{(x_1, \dots, x_r) = (0, \dots, 0)}$$

is holomorphic for all $(s_1, \dots, s_r) \in \mathbb{C}^r$.

In the preceding section, we gave a certain duality formula for $B_m^{(k_1, \dots, k_r)}$ (see Theorem 4.7). By the similar method, we can prove certain duality formulas for $B_{m_1, \dots, m_r}^{(k_1, \dots, k_r), (d)}$, though they may be complicated. Hence, in the rest of this section, we will consider the case $d = r$. For emphasis, we denote $B_{m_1, \dots, m_r}^{(s_1, \dots, s_r), (r)}$ by $\mathbb{B}_{m_1, \dots, m_r}^{(s_1, \dots, s_r)}$. Note that $\delta_j(r) = 1$ for any j . With this notation, we prove the following duality formulas.

Theorem 5.4. For $m_1, \dots, m_r, k_1, \dots, k_r \in \mathbb{Z}_{\geq 0}$,

$$(5.6) \quad \mathbb{B}_{m_1, \dots, m_r}^{(-k_1, \dots, -k_r)} = \mathbb{B}_{k_1, \dots, k_r}^{(-m_1, \dots, -m_r)}.$$

Now we aim to prove this theorem. First we generalize Lemma 4.1 as follows.

Lemma 5.5. For $k_1, \dots, k_r \in \mathbb{Z}_{\geq 0}$, there exists a polynomial $\tilde{P}(x_1, \dots, x_r; k_1, \dots, k_r) \in \mathbb{Z}[x_1, \dots, x_r]$ such that

$$(5.7) \quad \text{Li}_{-k_1, \dots, -k_r}^*(z_1, \dots, z_r) = \frac{\tilde{P}(\prod_{j=1}^r z_j, \prod_{j=2}^r z_j, \dots, z_{r-1} z_r; k_1, \dots, k_r)}{\prod_{j=1}^r \left(1 - \prod_{\nu=j}^r z_{\nu}\right)^{\sum_{\nu=j}^r k_{\nu} + 1}},$$

$$(5.8) \quad \deg_{x_j} \tilde{P}(x_1, \dots, x_r; k_1, \dots, k_r) \leq \sum_{\nu=j}^r k_\nu + 1,$$

$$(5.9) \quad (x_1 \cdots x_r) \mid \tilde{P}(x_1, \dots, x_r; k_1, \dots, k_r).$$

Set $y_j = \prod_{\nu=j}^r z_\nu$ ($1 \leq j \leq r$). Then (5.7) implies

$$(5.10) \quad \text{Li}_{-k_1, \dots, -k_r}^{\square}(y_1, \dots, y_r) = \frac{\tilde{P}(y_1, \dots, y_r; k_1, \dots, k_r)}{\prod_{j=1}^r (1 - y_j)^{\sum_{\nu=j}^r k_\nu + 1}}.$$

Proof. In order to prove this lemma, we have only to use the same method as in Lemma 4.1 by induction on r . Since the case of $r = 1$ is proven, we consider the case of $r \geq 2$. Further, when $K = k_1 + \cdots + k_r = 0$, it is easy to have the assertion. Hence we think about a general case $K = k_1 + \cdots + k_r (\geq 1)$. When $k_r = 0$, we have

$$\begin{aligned} \text{Li}_{-k_1, \dots, -k_r}^*(z_1, \dots, z_r) &= \frac{z_r}{1 - z_r} \text{Li}_{-k_1, \dots, -k_{r-1}}^*(z_1, \dots, z_{r-2}, z_{r-1} z_r) \\ &= \frac{z_r}{1 - z_r} \frac{\tilde{P}(\prod_{j=1}^r z_j, \dots, z_{r-1} z_r; k_1, \dots, k_{r-1})}{\prod_{j=1}^{r-1} \left(1 - \prod_{\nu=j}^r z_\nu\right)^{\sum_{\nu=j}^r k_\nu + 1}}. \end{aligned}$$

Therefore, setting $\tilde{P}(x_1, \dots, x_r; k_1, \dots, k_{r-1}, 0) = x_r \tilde{P}(x_1, \dots, x_{r-1}; k_1, \dots, k_{r-1})$, we can verify (5.7)–(5.9).

Next we consider the case $k_r \geq 1$. For $k \in \mathbb{Z}_{\geq 0}$, we inductively define a subset $\{c_{j,\nu}^{(k)}\}_{0 \leq j, \nu \leq k+1}$ of \mathbb{Z} by

$$(5.11) \quad \frac{d}{dz} \left(\sum_{m>l} m^k z^m \right) = \frac{1}{(1-z)^{k+2}} \sum_{j=0}^{k+1} \sum_{\nu=0}^{k+1} c_{j,\nu}^{(k)} l^\nu z^{l+j}.$$

In fact, by

$$\frac{d}{dz} \left(\sum_{m>l} z^m \right) = \frac{1}{(1-z)^2} (z^l + l z^l - l z^{l+1}),$$

and

$$\sum_{m>l} m^k z^m = z \frac{d}{dz} \left(\sum_{m>l} m^{k-1} z^m \right) \quad (k \geq 1),$$

we can determine $\{c_{j,\nu}^{(k)}\}$ by (5.11). Using this notation, we have

$$\begin{aligned} \text{Li}_{-k_1, \dots, -k_r}^*(z_1, \dots, z_r) &= z_r \frac{d}{dz_r} \text{Li}_{-k_1, \dots, -k_{r-1}}^*(z_1, \dots, z_r) \\ &= z_r \sum_{m_1 < \cdots < m_{r-1}} m_1^{k_1} \cdots m_{r-1}^{k_{r-1}} z_1^{m_1} \cdots z_{r-1}^{m_{r-1}} \frac{\sum_{j=0}^{k_r} \sum_{\nu=0}^{k_r} c_{j,\nu}^{(k_r-1)} m_{r-1}^\nu z_r^{m_{r-1}+j}}{(1-z_r)^{k_r+1}} \\ &= \frac{1}{(1-z_r)^{k_r+1}} \sum_{j=0}^{k_r} \sum_{\nu=0}^{k_r} c_{j,\nu}^{(k_r-1)} z_r^{j+1} \sum_{m_1 < \cdots < m_{r-1}} m_1^{k_1} \cdots m_{r-1}^{k_{r-1}+\nu} z_1^{m_1} \cdots (z_{r-1} z_r)^{m_{r-1}}. \end{aligned}$$

By the induction hypothesis in the case $r-1$, this is equal to

$$\frac{1}{(1-z_r)^{k_r+1}} \sum_{j=0}^{k_r} \sum_{\nu=0}^{k_r} c_{j,\nu}^{(k_r-1)} z_r^{j+1} \frac{\tilde{P}(\prod_{j=1}^{r-1} z_j, \dots, z_{r-1} z_r; k_1, \dots, k_{r-2}, k_{r-1} + \nu)}{\prod_{j=1}^{r-2} \left(1 - \prod_{\nu=j}^r z_\nu\right)^{\sum_{\nu=j}^r k_\nu + 1} (1 - z_{r-1} z_r)^{k_{r-1} + \nu + 1}}.$$

Therefore we set

$$\begin{aligned} & \tilde{P}(x_1, \dots, x_r; k_1, \dots, k_r) \\ &= \sum_{j=0}^{k_r} \sum_{\nu=0}^{k_r} c_{j,\nu}^{(k_r-1)} x_r^{j+1} (1-x_{r-1})^{k_r-\nu} \tilde{P}(x_1, \dots, x_{r-1}; k_1, \dots, k_{r-2}, k_{r-1} + \nu). \end{aligned}$$

Then this satisfies (5.7)–(5.9). This completes the proof. \square

From this result, we can reach the following definition.

Definition 5.6. For $k_1, \dots, k_r \in \mathbb{Z}_{\geq 0}$, define

$$(5.12) \quad \begin{aligned} \eta(-k_1, \dots, -k_r; s_1, \dots, s_r) &= \frac{1}{\prod_{j=1}^r \Gamma(s_j)} \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^r t_j^{s_j-1} \\ &\times \frac{\text{Li}_{-k_1, \dots, -k_r}^\omega(1 - e^{\sum_{\nu=1}^r t_\nu}, \dots, 1 - e^{t_{r-1}+t_r}, 1 - e^{t_r})}{\prod_{j=1}^r (1 - e^{\sum_{\nu=j}^r t_\nu})} \prod_{j=1}^r dt_j \end{aligned}$$

for $s_1, \dots, s_r \in \mathbb{C}$ with $\text{Re}(s_j) > 0$ ($1 \leq j \leq r$).

Lemma 5.5 ensures that the integral on the right-hand side of (5.12) is absolutely convergent for $\text{Re}(s_j) > 0$. By the same method as in the proof of Theorem 2.3 for $\eta(k_1, \dots, k_r; s)$, we can similarly obtain the following.

Theorem 5.7. For $k_1, \dots, k_r \in \mathbb{Z}_{\geq 0}$, $\eta(-k_1, \dots, -k_r; s_1, \dots, s_r)$ can be analytically continued to an entire function on the whole complex space, and satisfies

$$(5.13) \quad \eta(-k_1, \dots, -k_r; -m_1, \dots, -m_r) = \mathbb{B}_{m_1, \dots, m_r}^{(-k_1, \dots, -k_r)} \quad (m_1, \dots, m_r \in \mathbb{Z}_{\geq 0}).$$

Proof. As in the proof of Theorem 2.3, let

$$(5.14) \quad \begin{aligned} & H(-k_1, \dots, -k_r; s_1, \dots, s_r) \\ &= \int_{\mathcal{C}^r} \prod_{j=1}^r t_j^{s_j-1} \frac{\text{Li}_{-k_1, \dots, -k_r}^\omega(1 - e^{\sum_{\nu=1}^r t_\nu}, \dots, 1 - e^{t_r})}{\prod_{j=1}^r (1 - e^{\sum_{\nu=j}^r t_\nu})} \prod_{j=1}^r dt_j \\ &= \prod_{j=1}^r (e^{2\pi i s_j} - 1) \int_\varepsilon^\infty \cdots \int_\varepsilon^\infty \prod_{j=1}^r t_j^{s_j-1} \frac{\text{Li}_{-k_1, \dots, -k_r}^\omega(1 - e^{\sum_{\nu=1}^r t_\nu}, \dots, 1 - e^{t_r})}{\prod_{j=1}^r (1 - e^{\sum_{\nu=j}^r t_\nu})} \prod_{j=1}^r dt_j \\ &\quad + \int_{C_\varepsilon^r} \prod_{j=1}^r t_j^{s_j-1} \frac{\text{Li}_{-k_1, \dots, -k_r}^\omega(1 - e^{\sum_{\nu=1}^r t_\nu}, \dots, 1 - e^{t_r})}{\prod_{j=1}^r (1 - e^{\sum_{\nu=j}^r t_\nu})} \prod_{j=1}^r dt_j, \end{aligned}$$

where \mathcal{C}^r is the direct product of the contour \mathcal{C} defined before. Note that the integrand on the second member has no singularity on \mathcal{C}^r . It follows from Lemma 5.5 that $H(-k_1, \dots, -k_r; s_1, \dots, s_r)$ is absolutely convergent for any $(s_j) \in \mathbb{C}^r$, namely is entire. Suppose $\text{Re}(s_j) > 0$ for each j , the second integral tends to 0 as $\varepsilon \rightarrow 0$. Hence

$$\eta(-k_1, \dots, -k_r; s_1, \dots, s_r) = \frac{1}{\prod_{j=1}^r (e^{2\pi i s_j} - 1) \Gamma(s_j)} H(-k_1, \dots, -k_r; s_1, \dots, s_r),$$

which can be analytically continued to \mathbb{C}^r . Also, setting $(s_1, \dots, s_r) = (-m_1, \dots, -m_r) \in \mathbb{Z}_{\leq 0}^r$ in (5.14), we obtain (5.13) from (5.4). This completes the proof. \square

Next we directly construct the generating function of $\eta(-k_1, \dots, -k_r; s_1, \dots, s_r)$. We prepare the following two lemmas which we consider when (x_j) is in Λ_r defined by (5.5).

Lemma 5.8. *For $(s_j) \in \mathbb{C}^r$ with $\operatorname{Re}(s_j) > 0$ ($1 \leq j \leq r$),*

$$(5.15) \quad F(x_1, \dots, x_r; s_1, \dots, s_r; r) = \frac{1}{\prod_{j=1}^r \Gamma(s_j)} \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^r \left\{ t_j^{s_j-1} \frac{e^{\sum_{\nu=j}^r x_\nu}}{1 - e^{\sum_{\nu=j}^r x_\nu} (1 - e^{\sum_{\nu=j}^r t_\nu})} \right\} \prod_{j=1}^r dt_j.$$

Proof. Substituting $n^{-s} = (1/\Gamma(s)) \int_0^\infty t^{s-1} e^{-nt} dt$ into the second member of (5.4), we have

$$\begin{aligned} F(\{x_j\}; \{s_j\}; r) &= \sum_{l_1, \dots, l_r=1}^\infty \prod_{j=1}^r \left(1 - e^{-\sum_{\nu=j}^r x_\nu} \right)^{l_j-1} \frac{1}{\prod_{j=1}^r \Gamma(s_j)} \\ &\quad \times \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^r \left\{ t_j^{s_j-1} \exp \left(- \left(\sum_{\nu=1}^j l_\nu \right) t_j \right) \right\} \prod_{j=1}^r dt_j. \end{aligned}$$

We see that the integrand on the right-hand side can be rewritten as

$$\prod_{j=1}^r t_j^{s_j-1} \prod_{j=1}^r \exp \left(- l_j \left(\sum_{\nu=j}^r t_\nu \right) \right).$$

Hence we have

$$\begin{aligned} F(\{x_j\}; \{s_j\}; r) &= \frac{1}{\prod_{j=1}^r \Gamma(s_j) \left(1 - e^{-\sum_{\nu=j}^r x_\nu} \right)} \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^r t_j^{s_j-1} \\ &\quad \times \sum_{l_1, \dots, l_r=1}^\infty \prod_{j=1}^r \left(1 - e^{-\sum_{\nu=j}^r x_\nu} \right)^{l_j} e^{-l_j \left(\sum_{\nu=j}^r t_\nu \right)} \prod_{j=1}^r dt_j \\ &= \frac{1}{\prod_{j=1}^r \Gamma(s_j)} \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^r t_j^{s_j-1} \frac{e^{-\sum_{\nu=j}^r t_\nu}}{1 - (1 - e^{-\sum_{\nu=j}^r x_\nu}) e^{-\sum_{\nu=j}^r t_\nu}} \prod_{j=1}^r dt_j \\ &= \frac{1}{\prod_{j=1}^r \Gamma(s_j)} \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^r t_j^{s_j-1} \frac{e^{\sum_{\nu=j}^r x_\nu}}{1 - e^{\sum_{\nu=j}^r x_\nu} (1 - e^{\sum_{\nu=j}^r t_\nu})} \prod_{j=1}^r dt_j. \end{aligned}$$

This completes the proof. \square

Lemma 5.9. *Let $z_1, \dots, z_r \in \mathbb{C}$ and assume that $|z_j|$ ($1 \leq j \leq r$) are sufficiently small. Then*

$$(5.16) \quad \prod_{j=1}^r \frac{z_j e^{\sum_{\nu=j}^r x_\nu}}{1 - z_j e^{\sum_{\nu=j}^r x_\nu}} = \sum_{k_1, \dots, k_r=0}^\infty \operatorname{Li}_{-k_1, \dots, -k_r}^\mathbb{W}(z_1, \dots, z_r) \frac{x_1^{k_1} \cdots x_r^{k_r}}{k_1! \cdots k_r!}.$$

Set $z_j = 1 - e^{\sum_{\nu=j}^r t_\nu}$ ($1 \leq j \leq r$) for $(t_j) \in \Lambda_r$. Then

$$(5.17) \quad \begin{aligned} &\prod_{j=1}^r \frac{e^{\sum_{\nu=j}^r x_\nu} \left(1 - e^{\sum_{\nu=j}^r t_\nu} \right)}{1 - e^{\sum_{\nu=j}^r x_\nu} \left(1 - e^{\sum_{\nu=j}^r t_\nu} \right)} \\ &= \sum_{k_1, \dots, k_r=0}^\infty \operatorname{Li}_{-k_1, \dots, -k_r}^\mathbb{W}(1 - e^{\sum_{\nu=1}^r t_\nu}, \dots, 1 - e^{t_r}) \frac{x_1^{k_1} \cdots x_r^{k_r}}{k_1! \cdots k_r!}. \end{aligned}$$

In particular, the case $t_1 = \cdots = t_{r-1} = 0$ and $t_r = t$ implies (4.13).

Proof. We have only to prove (5.16). Actually we have

$$\begin{aligned}
& \sum_{k_1, \dots, k_r=0}^{\infty} \text{Li}_{-k_1, \dots, -k_r}^{\mathfrak{W}}(z_1, \dots, z_r) \frac{x_1^{k_1} \cdots x_r^{k_r}}{k_1! \cdots k_r!} \\
&= \sum_{k_1, \dots, k_r=0}^{\infty} \sum_{m_1, \dots, m_r=1}^{\infty} \prod_{j=1}^r \frac{((\sum_{\mu=1}^j m_{\mu}) x_j)^{k_j}}{k_j!} z_j^{m_j} \\
&= \sum_{m_1, \dots, m_r=1}^{\infty} \prod_{j=1}^r z_j^{m_j} \prod_{\mu=1}^j e^{m_{\mu} x_j} \\
&= \sum_{m_1, \dots, m_r=1}^{\infty} \prod_{j=1}^r \left(z_j e^{\sum_{\nu=j}^r x_{\nu}} \right)^{m_j} = \prod_{j=1}^r \frac{z_j e^{\sum_{\nu=j}^r x_{\nu}}}{1 - z_j e^{\sum_{\nu=j}^r x_{\nu}}}.
\end{aligned}$$

Thus we have the assertion. \square

Using these lemmas, we obtain the following.

Theorem 5.10. For $k_1, \dots, k_r \in \mathbb{Z}_{\geq 0}$,

$$(5.18) \quad \eta(-k_1, \dots, -k_r; s_1, \dots, s_r) = \mathbb{B}_{k_1, \dots, k_r}^{(s_1, \dots, s_r)}.$$

Proof. By Lemmas 5.8 and 5.9, we have

$$\begin{aligned}
(5.19) \quad & F(x_1, \dots, x_r; s_1, \dots, s_r; r) \\
&= \frac{1}{\prod_{j=1}^r \Gamma(s_j)} \int_0^{\infty} \cdots \int_0^{\infty} \prod_{j=1}^r \left\{ t_j^{s_j-1} \frac{e^{\sum_{\nu=j}^r x_{\nu}}}{1 - e^{\sum_{\nu=j}^r x_{\nu}} (1 - e^{\sum_{\nu=j}^r t_{\nu}})} \right\} \prod_{j=1}^r dt_j \\
&= \frac{1}{\prod_{j=1}^r \Gamma(s_j)} \sum_{k_1, \dots, k_r=0}^{\infty} \left\{ \int_0^{\infty} \cdots \int_0^{\infty} \prod_{j=1}^r t_j^{s_j-1} \right. \\
&\quad \times \left. \frac{\text{Li}_{-k_1, \dots, -k_r}^{\mathfrak{W}}(1 - e^{\sum_{\nu=1}^r t_{\nu}}, \dots, 1 - e^{t_r})}{\prod_{j=1}^r (1 - e^{\sum_{\nu=j}^r t_{\nu}})} \prod_{j=1}^r dt_j \right\} \frac{x_1^{k_1} \cdots x_r^{k_r}}{k_1! \cdots k_r!}
\end{aligned}$$

for $\text{Re}(s_j) > 0$ ($1 \leq j \leq r$). Combining (5.4), (5.12) and (5.19), we obtain (5.18) for $\text{Re}(s_j) > 0$ ($1 \leq j \leq r$), hence for all $(s_j) \in \mathbb{C}$, because both sides of (5.18) are entire functions (see Remark 5.3). \square

Proof of Theorem 5.4. Setting $(s_1, \dots, s_r) = (-m_1, \dots, -m_r)$ in (5.18), we obtain (5.6) from (5.13). This completes the proof of Theorem 5.4. \square

Example 5.11. We can easily see that

$$\text{Li}_{-1,0}^{\mathfrak{W}}(z_1, z_2) = \frac{z_1 z_2}{(1 - z_1)^2 (1 - z_2)}, \quad \text{Li}_{0,-1}^{\mathfrak{W}}(z_1, z_2) = \frac{z_1 z_2 (2 - z_1 - z_2)}{(1 - z_1)^2 (1 - z_2)^2}.$$

Hence we have

$$\mathbb{B}_{m,n}^{(-1,0)} = 2^m 3^n, \quad \mathbb{B}_{m,n}^{(0,-1)} = (2^m + 1) 3^n \quad (m, n \in \mathbb{Z}_{\geq 0}).$$

Therefore $\mathbb{B}_{0,1}^{(-1,0)} = \mathbb{B}_{1,0}^{(0,-1)} = 3$. Similarly we obtain, for example,

$$\mathbb{B}_{1,0}^{(-1,-2)} = \mathbb{B}_{1,2}^{(-1,0)} = 18, \quad \mathbb{B}_{1,2}^{(-3,-1)} = \mathbb{B}_{3,1}^{(-1,-2)} = 1820, \quad \mathbb{B}_{2,2}^{(-2,-1)} = \mathbb{B}_{2,1}^{(-2,-2)} = 1958.$$

Remark 5.12. Hamahata and Masubuchi [14, Corollary 10] showed the special case of (5.6), namely

$$\mathbb{B}_{0,\dots,0,m}^{(0,\dots,0,-k)} = \mathbb{B}_{0,\dots,0,k}^{(0,\dots,0,-m)} \quad (m, k \in \mathbb{Z}_{\geq 0})$$

(see Remark 5.2). On the other hand, Theorem 4.7 corresponds to the case $d = 1 \neq r$ except for $r = 1$ (see Remark 5.2), hence is located in the outside of Theorem 5.4. Therefore, in (4.12), another type of multi-poly-Bernoulli numbers appear.

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