Multi-poly-Bernoulli numbers and related zeta functions

Kaneko, Masanobu
Faculty of Mathematics, Kyushu University : Professor

Tsumura, Hirofumi
Department of Mathematics and Information Sciences, Tokyo Metropolitan University

出版情報：Nagoya Mathematical Journal. 232, pp.19-54, 2017-05-08. 名古屋大学理学部数学教室
バージョン：

Multi-poly-Bernoulli numbers and related zeta functions

Masanobu Kaneko* and Hirofumi Tsumura

Abstract

We construct and study a certain zeta function which interpolates multi-poly-Bernoulli numbers at non-positive integers and whose values at positive integers are linear combinations of multiple zeta values. This function can be regarded as the one to be paired up with the ξ-function defined by Arakawa and the first-named author. We show that both are closely related to the multiple zeta functions. Further we define multi-indexed poly-Bernoulli numbers, and generalize the duality formulas for poly-Bernoulli numbers by introducing more general zeta functions.

1. Introduction

In this paper, we investigate the function defined by

$$\eta(k_1,\ldots,k_r;s)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{\mathrm{Li}_{k_1,\ldots,k_r}(1-e^t)}{1-e^t}\,dt \tag{1.1}$$

and its generalizations, in connection with multi-poly-Bernoulli numbers and multiple zeta values (we shall give the precise definitions later in §2). This function can be viewed as a twin sibling of the function $\xi(k_1,\ldots,k_r;s)$,

$$\xi(k_1,\ldots,k_r;s)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{\mathrm{Li}_{k_1,\ldots,k_r}(1-e^{-t})}{e^t-1}\,dt, \tag{1.2}$$

which was introduced and studied in [4]. The present paper may constitute a natural continuation of the work [4].

To explain our results in some detail, we first give an overview of the necessary background. For an integer $k\in\mathbb{Z}$, two types of poly-Bernoulli numbers $\{B_n^{(k)}\}$ and $\{C_n^{(k)}\}$ are defined as follows (see Kaneko [20] and Arakawa-Kaneko [4], also Arakawa-Ibukiyama-Kaneko [3]):

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}=\sum_{n=0}^\infty B_n^{(k)}\frac{t^n}{n!}, \tag{1.3}$$

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}=\sum_{n=0}^\infty C_n^{(k)}\frac{t^n}{n!}, \tag{1.4}$$

*This work was supported by Japan Society for the Promotion of Science, Grant-in-Aid for Scientific Research (B) 23340010.

2010 Mathematics Subject Classification: Primary 11B68; Secondary 11M32

Key words: multi-poly-Bernoulli number, multiple zeta value, Arakawa-Kaneko zeta function

where $\mathrm{Li}_k(z)$ is the polylogarithm function defined by

$$\mathrm{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k} \quad (|z|<1). \tag{1.5}$$

Since $\mathrm{Li}_1(z) = -\log(1-z)$, we see that $B_n^{(1)}$ (resp. $C_n^{(1)}$) coincides with the ordinary Bernoulli number B_n defined by

$$\frac{te^t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \qquad \left(\text{resp. } \frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}\right).$$

A number of formulas, including closed formulas of $B_n^{(k)}$ and $C_n^{(k)}$ in terms of the Stirling numbers of the second kind as well as the duality formulas

$$B_n^{(-k)} = B_k^{(-n)}, \tag{1.6}$$

$$C_n^{(-k-1)} = C_k^{(-n-1)} \tag{1.7}$$

that hold for $k, n \in \mathbb{Z}_{\geq 0}$, have been established (see [20, Theorems 1 and 2] and [21, § 2]). We also mention that Brewbaker [9] gave a purely combinatorial interpretation of the number $B_n^{(-k)}$ of negative upper index as the number of 'Lonesum-matrices' with n rows and k columns.

A multiple version of $B_n^{(k)}$ is defined in [4, p. 202, Remarks (ii)] by

$$\frac{\mathrm{Li}_{k_1,\dots,k_r}(1-e^{-t})}{(1-e^{-t})^r} = \sum_{n=0}^{\infty} \mathbb{B}_n^{(k_1,\dots,k_r)} \frac{t^n}{n!} \quad (k_1,\dots,k_r \in \mathbb{Z}), \tag{1.8}$$

where

$$\mathrm{Li}_{k_1,\dots,k_r}(z) = \sum_{1\leq m_1<\cdots<m_r} \frac{z^{m_r}}{m_1^{k_1} m_2^{k_2} \cdots m_r^{k_r}} \tag{1.9}$$

is the multiple polylogarithm. Hamahata and Masubuchi [14, 15] investigated some properties of $\mathbb{B}_n^{(k_1,\dots,k_r)}$, and gave several generalizations of the known results in the single-index case. Based on this research, Bayad and Hamahata [8] further studied these numbers. Furusho [12, p. 269] also refers to (1.8).

More recently, Imatomi, Takeda and the first-named author [18] defined and studied another type of multi-poly-Bernoulli numbers given by

$$\frac{\mathrm{Li}_{k_1,\dots,k_r}(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k_1,\dots,k_r)} \frac{t^n}{n!}, \tag{1.10}$$

$$\frac{\mathrm{Li}_{k_1,\dots,k_r}(1-e^{-t})}{e^t-1} = \sum_{n=0}^{\infty} C_n^{(k_1,\dots,k_r)} \frac{t^n}{n!} \tag{1.11}$$

for $k_1,\dots,k_r \in \mathbb{Z}$. They proved several formulas for $B_n^{(k_1,\dots,k_r)}$ and $C_n^{(k_1,\dots,k_r)}$, and further gave an important relation between $C_{p-2}^{(k_1,\dots,k_r)}$ and the 'finite multiple zeta value', that is,

$$\sum_{1\leq m_1<\cdots<m_r<p} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}} \equiv -C_{p-2}^{(k_1,\dots,k_{r-1},k_r-1)} \mod p \tag{1.12}$$

for any prime number p.

The function (1.2) for $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 1}$ can be analytically continued to an entire function of the complex variable $s \in \mathbb{C}$ ([4, Sections 3 and 4]). The particular case $r = k = 1$ gives $\xi(1; s) = s\zeta(s+1)$. Hence $\xi(k_1, \ldots, k_r; s)$ can be regarded as a multi-indexed zeta function. It is shown in [4] that the values at non-positive integers of $\xi(k; s)$ interpolate poly-Bernoulli numbers $C_m^{(k)}$,

$$\xi(k; -m) = (-1)^m C_m^{(k)} \tag{1.13}$$

for $k \in \mathbb{Z}_{\geq 1}$ and $m \in \mathbb{Z}_{\geq 0}$. And also by investigating $\xi(k_1, \ldots, k_r; s)$ and its values at positive integer arguments, one produces many relations among multiple zeta values defined by

$$\zeta(l_1, \ldots, l_r) = \sum_{1 \leq m_1 < \cdots < m_r} \frac{1}{m_1^{l_1} \cdots m_r^{l_r}} \quad (= \mathrm{Li}_{l_1, \ldots, l_r}(1)) \tag{1.14}$$

for $l_1, \ldots, l_r \in \mathbb{Z}_{\geq 1}$ with $l_r \geq 2$ ([4, Corollary 11]).

Recently, further properties of $\xi(k_1, \ldots, k_r; s)$ and related results have been given by several authors (see, for example, Bayad-Hamahata [6, 7], Coppo-Candelpergher [10], Sasaki [28], and Young [31]).

In this paper, we conduct a basic study of the function (1.1) and relate it to the multi-poly-Bernoulli numbers $B_n^{(k_1, \ldots, k_r)}$ as well as multiple zeta (or 'zeta-star') values. Note that the only difference in both definitions (1.1) and (1.2) is, up to sign, the arguments $1 - e^t$ and $1 - e^{-t}$ of $\mathrm{Li}_{k_1, \ldots, k_r}(z)$ in the integrands. One sees in the main body of the paper a remarkable contrast between 'B-type' poly-Bernoulli numbers and those of 'C-type', and between multiple zeta and zeta-star values. We further investigate the case of non-positive indices k_i in connection with a yet more generalized 'multi-indexed' poly-Bernoulli number.

The paper is organized as follows. In §2, we give the analytic continuation of $\eta(k_1, \ldots, k_r; s)$ in the case of positive indices, and formulas for values at integer arguments (Theorems 2.3 and 2.5). In §3, we study relations between two functions $\eta(k_1, \ldots, k_r; s)$ and $\xi(k_1, \ldots, k_r; s)$ (Proposition 3.2), as well as relations with the single variable multiple zeta function (Definition 3.1 and Theorem 3.6). We turn in §4 to the study of $\eta(k_1, \ldots, k_r; s)$ in the negative index case and give a certain duality formula for $B_m^{(-k_1, \ldots, -k_r)}$ (Definition 4.3 and Theorems 4.4 and 4.7). We carry forward the study of negative index case in §5 and define the 'multi-indexed' poly-Bernoulli numbers $\{B_{m_1, \ldots, m_r}^{(k_1, \ldots, k_r), (d)}\}$ for $(k_1, \ldots, k_r) \in \mathbb{Z}^r$, $(m_1, \ldots, m_r) \in \mathbb{Z}_{\geq 0}^r$ and $d \in \{1, \ldots, r\}$ (Definition 5.1), which include (1.8) and (1.10) as special cases. We prove the 'multi-indexed' duality formula for them in the case $d = r$ (Theorem 5.4).

2. The function $\eta(k_1, \ldots, k_r; s)$ for positive indices and its values at integers

2.1 Analytic continuation and the values at non-positive integers

We start with the definition in the case of positive indices.

Definition 2.1. For positive integers $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 1}$, let

$$\eta(k_1, \ldots, k_r; s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\mathrm{Li}_{k_1, \ldots, k_r}(1 - e^t)}{1 - e^t} dt$$

for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1 - r$, where $\Gamma(s)$ is the gamma function. When $r = 1$, we often denote $\eta(k; s)$ by $\eta_k(s)$.

The integral on the right-hand side converges absolutely in the domain $\mathrm{Re}(s) > 1 - r$, as is seen from the following lemma.

Lemma 2.2. (i) *For $k_1, \dots, k_r \in \mathbb{Z}_{\geq 1}$, the function $\mathrm{Li}_{k_1,\dots,k_r}(1 - e^t)$ is holomorphic for $t \in \mathbb{C}$ with $|\mathrm{Im}(t)| < \pi$.*
(ii) *For $k_1, \dots, k_r \in \mathbb{Z}_{\geq 1}$ and $t \in \mathbb{R}_{>0}$, we have the estimates*

$$\mathrm{Li}_{k_1,\dots,k_r}(1 - e^t) = O\left(t^r\right) \qquad (t \to 0) \tag{2.1}$$

and

$$\mathrm{Li}_{k_1,\dots,k_r}(1 - e^t) = O\left(t^{k_1+\cdots+k_r}\right) \qquad (t \to \infty). \tag{2.2}$$

Proof. As is well-known, we can regard the function $\mathrm{Li}_{k_1,\dots,k_r}(z)$ as a single-valued holomorphic function in the simply connected domain $\mathbb{C} \smallsetminus [1, \infty)$, via the process of iterated integration starting with $\mathrm{Li}_1(z) = \int_0^z dz/(1-z)$. Noting that $1-e^t \in [1, \infty)$ is equivalent to $\mathrm{Im}(t) = (2j+1)\pi$ for some $j \in \mathbb{Z}$, we have the assertion (i).

The estimate (2.1) is clear from the definition of $\mathrm{Li}_{k_1,\dots,k_r}(z)$, because its Taylor series at $z = 0$ starts with the term $z^r/1^{k_1} \cdots r^{k_r}$. As for (2.2), we proceed by induction on the 'weight' $k_1 + \cdots + k_r$ as follows by using the formula

$$\frac{d}{dz}\mathrm{Li}_{k_1,\dots,k_r}(z) = \begin{cases} \frac{1}{z}\mathrm{Li}_{k_1,\dots,k_{r-1},k_r-1}(z) & (k_r > 1) \\ \frac{1}{1-z}\mathrm{Li}_{k_1,\dots,k_{r-1}}(z) & (k_r = 1), \end{cases} \tag{2.3}$$

which is easy to derive and is the basis of the analytic continuation of $\mathrm{Li}_{k_1,\dots,k_r}(z)$ mentioned above. If $r = k_1 = 1$, then we have $\mathrm{Li}_1(1 - e^t) = -t$ and the desired estimate holds. Suppose the weight k is larger than 1 and the assertion holds for any weight less than k. If $k_r > 1$, then by (2.3) we have

$$\begin{aligned} |\mathrm{Li}_{k_1,\dots,k_r}(1 - e^t)| &= \left| \int_0^{1-e^t} \frac{\mathrm{Li}_{k_1,\dots,k_r-1}(u)}{u} du \right| \\ &= \left| \int_0^t \frac{1}{1 - e^v} \mathrm{Li}_{k_1,\dots,k_r-1}(1 - e^v)(-e^v) dv \right| \qquad (u := 1 - e^v) \\ &\leq \int_0^\varepsilon \left| e^v \frac{\mathrm{Li}_{k_1,\dots,k_r-1}(1 - e^v)}{e^v - 1} \right| dv + \int_\varepsilon^t \left| \frac{e^v}{e^v - 1} \mathrm{Li}_{k_1,\dots,k_r-1}(1 - e^v) \right| dv \end{aligned}$$

for small $\varepsilon > 0$. The former integral is $O(1)$ because the integrand is continuous on $[0, \varepsilon]$. On the other hand, by induction hypothesis, the integrand of the latter integral is $O\left(v^{k_1+\cdots+k_r-1}\right)$ as $v \to \infty$. Therefore the latter integral is $O\left(t^{k_1+\cdots+k_r}\right)$ as $t \to \infty$. The case of $k_r = 1$ is similarly proved also by using (2.3), and is omitted here. $\square$

We now show that the function $\eta(k_1, \dots, k_r; s)$ can be analytically continued to an entire function, and interpolates multi-poly-Bernoulli numbers $B_m^{(k_1,\dots,k_r)}$ at non-positive integer arguments.

Theorem 2.3. *For positive integers $k_1,\dots,k_r \in \mathbb{Z}_{\geq 1}$, the function $\eta(k_1,\dots,k_r;s)$ can be analytically continued to an entire function on the whole complex plane. And the values of $\eta(k_1,\dots,k_r;s)$ at non-positive integers are given by*

$$\eta(k_1,\dots,k_r;-m) = B_m^{(k_1,\dots,k_r)} \quad (m \in \mathbb{Z}_{\geq 0}). \tag{2.4}$$

In particular, $\eta_k(-m) = B_m^{(k)}$ for $k \in \mathbb{Z}_{\geq 1}$ and $m \in \mathbb{Z}_{\geq 0}$.

Proof. In order to prove this theorem, we adopt here the method of contour integral representation (see, for example, [30, Theorem 4.2]). Let $\mathcal{C}$ be the standard contour, namely the path consisting of the positive real axis from the infinity to (sufficiently small) ε ('top side'), a counter clockwise circle C_ε around the origin of radius ε, and the positive real axis from ε to the infinity ('bottom side'). Let

$$\begin{aligned} H(k_1,\dots,k_r;s) &= \int_{\mathcal{C}} t^{s-1}\frac{\mathrm{Li}_{k_1,\dots,k_r}(1-e^t)}{1-e^t}dt \\ &= (e^{2\pi i s}-1)\int_\varepsilon^\infty t^{s-1}\frac{\mathrm{Li}_{k_1,\dots,k_r}(1-e^t)}{1-e^t}dt + \int_{C_\varepsilon} t^{s-1}\frac{\mathrm{Li}_{k_1,\dots,k_r}(1-e^t)}{1-e^t}dt. \end{aligned}$$

It follows from Lemma 2.2 that $H(k_1,\dots,k_r;s)$ is entire, because the integrand has no singularity on $\mathcal{C}$ and the contour integral is absolutely convergent for all $s \in \mathbb{C}$. Suppose $\mathrm{Re}(s) > 1-r$. The last integral tends to 0 as $\varepsilon \to 0$. Hence

$$\eta(k_1,\dots,k_r;s) = \frac{1}{(e^{2\pi i s}-1)\Gamma(s)}H(k_1,\dots,k_r;s),$$

which can be analytically continued to $\mathbb{C}$, and is entire. In fact $\eta(k_1,\dots,k_r;s)$ is holomorphic for $\mathrm{Re}(s) > 0$, hence has no singularity at any positive integer. Set $s = -m \in \mathbb{Z}_{\leq 0}$. Then, by (1.10),

$$\begin{aligned} \eta(k_1,\dots,k_r;-m) &= \frac{(-1)^m m!}{2\pi i}H(k_1,\dots,k_r;-m) \\ &= \frac{(-1)^m m!}{2\pi i}\int_{C_\varepsilon} t^{-m-1}\sum_{n=0}^\infty B_n^{(k_1,\dots,k_r)}\frac{(-t)^n}{n!}dt = B_m^{(k_1,\dots,k_r)}. \end{aligned}$$

This completes the proof. □

Remark 2.4. Using the same method as above or the method used in [4], we can establish the analytic continuation of $\xi(k_1,\dots,k_r;s)$ to an entire function, and see that

$$\xi(k_1,\dots,k_r;-m) = (-1)^m C_m^{(k_1,\dots,k_r)} \quad (m \in \mathbb{Z}_{\geq 0}) \tag{2.5}$$

for $k_1,\dots,k_r \in \mathbb{Z}_{\geq 1}$, which is a multiple version of (1.13).

2.2 Values at positive integers

About the values at positive integer arguments, we prove formulas for both $\xi(k_1,\dots,k_r;s)$ and $\eta(k_1,\dots,k_r;s)$, for general index $(k_1,\dots,k_r)$. These formulas generalize [4, Theorem 9 (i)], and have remarkable similarity in that one obtains the formula for $\eta(k_1,\dots,k_r;s)$ just by replacing multiple zeta values in the one for $\xi(k_1,\ldots,k_r;s)$ with multiple 'zeta-star' values. Recall the multiple zeta-star value is a real number defined by

$$\zeta^\star(l_1,\ldots,l_r)=\sum_{1\le m_1\le\cdots\le m_r}\frac{1}{m_1^{l_1}\cdots m_r^{l_r}} \tag{2.6}$$

for $l_1,\ldots,l_r\in\mathbb{Z}_{\ge1}$ with $l_r\ge2$. This was first studied (for general r) by Hoffman in [16].

To state our theorem, we further introduce some notation. For an index set $\mathbf{k}=(k_1,\ldots,k_r)\in\mathbb{Z}_{\ge1}^r$, put $\mathbf{k}_+=(k_1,\ldots,k_{r-1},k_r+1)$. The usual dual index of an *admissible* index (i.e. the one that the last entry is greater than one) $\mathbf{k}$ is denoted by $\mathbf{k}^*$. For $\mathbf{j}=(j_1,\ldots,j_r)\in\mathbb{Z}_{\ge0}^r$, we write $|\mathbf{j}|=j_1+\cdots+j_r$ and call it the weight of $\mathbf{j}$, and $d(\mathbf{j})=r$, the depth of $\mathbf{j}$. For two such indices $\mathbf{k}$ and $\mathbf{j}$ of the same depth, we denote by $\mathbf{k}+\mathbf{j}$ the index obtained by the component-wise addition, $\mathbf{k}+\mathbf{j}=(k_1+j_1,\ldots,k_r+j_r)$, and by $b(\mathbf{k};\mathbf{j})$ the quantity given by

$$b(\mathbf{k};\mathbf{j}):=\prod_{i=1}^{r}\binom{k_i+j_i-1}{j_i}.$$

Theorem 2.5. *For any index set $\mathbf{k}=(k_1,\ldots,k_r)\in\mathbb{Z}_{\ge1}^r$ and any $m\in\mathbb{Z}_{\ge1}$, we have*

$$\xi(k_1,\ldots,k_r;m)=\sum_{|\mathbf{j}|=m-1,\,d(\mathbf{j})=n}b((\mathbf{k}_+)^*;\mathbf{j})\,\zeta((\mathbf{k}_+)^*+\mathbf{j}) \tag{2.7}$$

and

$$\eta(k_1,\ldots,k_r;m)=(-1)^{r-1}\sum_{|\mathbf{j}|=m-1,\,d(\mathbf{j})=n}b((\mathbf{k}_+)^*;\mathbf{j})\,\zeta^\star((\mathbf{k}_+)^*+\mathbf{j}), \tag{2.8}$$

where both sums are over all $\mathbf{j}\in\mathbb{Z}_{\ge0}^r$ of weight $m-1$ and depth $n:=d(\mathbf{k}_+^*)$ $(=|\mathbf{k}|+1-d(\mathbf{k}))$.

In particular, we have

$$\xi(k_1,\ldots,k_r;1)=\zeta((\mathbf{k}_+)^*)\quad(=\zeta(\mathbf{k}_+),\ \textit{by the duality of multiple zeta values})$$

and

$$\eta(k_1,\ldots,k_r;1)=(-1)^{r-1}\zeta^\star((\mathbf{k}_+)^*).$$

In order to prove the theorem, we give certain multiple integral expressions of the functions $\xi(k_1,\ldots,k_r;s)$ and $\eta(k_1,\ldots,k_r;s)$.

Proposition 2.6. *Notations being as above, write $(\mathbf{k}_+)^*=(l_1,\ldots,l_n)$. Then we have, for* $\mathrm{Re}(s)>1-r$,

(i)
$$\xi(k_1,\ldots,k_r;s)=\frac{1}{\prod_{i=1}^n\Gamma(l_i)\cdot\Gamma(s)}\int_0^\infty\cdots\int_0^\infty(x_1+\cdots+x_n)^{s-1}x_1^{l_1-1}\cdots x_n^{l_n-1}\\ \times\frac{1}{e^{x_1+\cdots+x_n}-1}\cdot\frac{1}{e^{x_2+\cdots+x_n}-1}\cdot\cdots\cdot\frac{1}{e^{x_n}-1}dx_1\cdots dx_n.$$

(ii)
$$\eta(k_1,\ldots,k_r;s)=\frac{(-1)^{r-1}}{\prod_{i=1}^n\Gamma(l_i)\cdot\Gamma(s)}\int_0^\infty\cdots\int_0^\infty(x_1+\cdots+x_n)^{s-1}x_1^{l_1-1}\cdots x_n^{l_n-1}\\ \times\frac{1}{e^{x_1+\cdots+x_n}-1}\cdot\frac{e^{x_2+\cdots+x_n}}{e^{x_2+\cdots+x_n}-1}\cdot\cdots\cdot\frac{e^{x_n}}{e^{x_n}-1}dx_1\cdots dx_n.$$

Proof. First write the index $(k_1,\ldots,k_r)$ as

$$(k_1,\ldots,k_r)=(\underbrace{1,\ldots,1}_{a_1-1},b_1+1,\ldots,\underbrace{1,\ldots,1}_{a_h-1},b_h+1),$$

with (uniquely determined) integers $h\ge 1$, $a_i\ge 1$ $(1\le i\le h)$, $b_i\ge 1$ $(1\le i\le h-1)$, and $b_h\ge 0$. Then, by performing the intermediate integrals of repeated $dz/(1-z)$ in the standard iterated integral coming from (2.3), we obtain the following iterated integral expression of the multiple polylogarithm $\mathrm{Li}_{k_1,\ldots,k_r}(z)$:

$$\begin{aligned}
\mathrm{Li}_{k_1,\ldots,k_r}(z)=&\underbrace{\int_0^z\frac{dx_h}{x_h}\int_0^{x_h}\cdots\cdots\int_0^{x_h}\frac{dx_h}{x_h}}_{b_h}\int_0^{x_h}\frac{1}{a_h!}\log\Bigl(\frac{1-x_{h-1}}{1-x_h}\Bigr)^{a_h}\frac{dx_{h-1}}{x_{h-1}}\\
&\cdot\underbrace{\int_0^{x_{h-1}}\frac{dx_{h-1}}{x_{h-1}}\cdots\cdots\int_0^{x_{h-1}}\frac{dx_{h-1}}{x_{h-1}}}_{b_{h-1}-1}\int_0^{x_{h-1}}\frac{1}{a_{h-1}!}\log\Bigl(\frac{1-x_{h-2}}{1-x_{h-1}}\Bigr)^{a_{h-1}}\frac{dx_{h-2}}{x_{h-2}}\cdots\cdots\\
&\cdots\cdots\underbrace{\int_0^{x_3}\frac{dx_3}{x_3}\cdots\cdots\int_0^{x_3}\frac{dx_3}{x_3}}_{b_3-1}\int_0^{x_3}\frac{1}{a_3!}\log\Bigl(\frac{1-x_2}{1-x_3}\Bigr)^{a_3}\frac{dx_2}{x_2}\underbrace{\int_0^{x_2}\frac{dx_2}{x_2}\cdots\int_0^{x_2}\frac{dx_2}{x_2}}_{b_2-1}\\
&\cdot\int_0^{x_2}\frac{1}{a_2!}\log\Bigl(\frac{1-x_1}{1-x_2}\Bigr)^{a_2}\underbrace{\frac{dx_1}{x_1}\int_0^{x_1}\cdots\cdots\int_0^{x_1}\frac{dx_1}{x_1}\int_0^{x_1}}_{b_1-1}\frac{\bigl(-\log(1-x)\bigr)^{a_1}}{a_1!}\frac{dx}{x}.
\end{aligned}$$

Here, to ease notation, we used the same variable in the repetitions of integrals $\int_0^x dx/x$, and we understand $x_h=z$ if $b_h=0$. The paths of integrations are in the domain $\mathbb{C}\setminus[1,\infty)$, and the formula is valid for $z\in\mathbb{C}\setminus[1,\infty)$. We may check this formula by differentiating both sides repeatedly and using (2.3). Putting $z=1-e^{-t}$ and $1-e^t$, changing variables accordingly, and suitably labeling the variables, we obtain

(2.9)
$$\begin{aligned}
\mathrm{Li}_{k_1,\ldots,k_r}(1-e^{-t})=&\int_0^t\int_0^{t_{b_1+\cdots+b_h}}\cdots\int_0^{t_2}\underbrace{\frac{1}{e^{t_{b_1+\cdots+b_h}}-1}\cdots\cdots\frac{1}{e^{t_{b_1+\cdots+b_{h-1}+2}}-1}}_{b_h-1}\\
&\times\frac{1}{a_h!}\frac{(t_{b_1+\cdots+b_{h-1}+1}-t_{b_1+\cdots+b_{h-1}})^{a_h}}{e^{t_{b_1+\cdots+b_{h-1}+1}}-1}\cdot\underbrace{\frac{1}{e^{t_{b_1+\cdots+b_{h-1}}}-1}\cdots\cdots\frac{1}{e^{t_{b_1+\cdots+b_{h-2}+2}}-1}}_{b_{h-1}-1}\\
&\times\cdots\cdots\\
&\times\frac{1}{a_3!}\frac{(t_{b_1+b_2+1}-t_{b_1+b_2})^{a_3}}{e^{t_{b_1+b_2+1}}-1}\cdot\underbrace{\frac{1}{e^{t_{b_1+b_2}}-1}\cdots\cdots\frac{1}{e^{t_{b_1+2}}-1}}_{b_2-1}\\
&\times\frac{1}{a_2!}\frac{(t_{b_1+1}-t_{b_1})^{a_2}}{e^{t_{b_1+1}}-1}\cdot\underbrace{\frac{1}{e^{t_{b_1}}-1}\cdots\cdots\frac{1}{e^{t_2}-1}}_{b_1-1}\cdot\frac{1}{a_1!}\frac{t_1^{a_1}}{e^{t_1}-1}\,dt_1\,dt_2\cdots dt_{b_1+\cdots+b_h},
\end{aligned}$$

and

(2.10)
$$\mathrm{Li}_{k_1,\ldots,k_r}(1-e^{t})=(-1)^r\int_0^t\int_0^{t_{b_1+\cdots+b_h}}\cdots\int_0^{t_2}\underbrace{\frac{e^{t_{b_1+\cdots+b_h}}}{e^{t_{b_1+\cdots+b_h}}-1}\cdots\cdots\frac{e^{t_{b_1+\cdots+b_{h-1}+2}}}{e^{t_{b_1+\cdots+b_{h-1}+2}}-1}}_{b_h-1}$$

$$\times \frac{1}{a_h!} \frac{(t_{b_1+\cdots+b_{h-1}+1} - t_{b_1+\cdots+b_{h-1}})^{a_h} e^{t_{b_1+\cdots+b_{h-1}+1}}}{e^{t_{b_1+\cdots+b_{h-1}+1}} - 1} \cdot \underbrace{\frac{e^{t_{b_1+\cdots+b_{h-1}}}}{e^{t_{b_1+\cdots+b_{h-1}}} - 1} \cdots\cdots \frac{e^{t_{b_1+\cdots+b_{h-2}+2}}}{e^{t_{b_1+\cdots+b_{h-2}+2}} - 1}}_{b_{h-1}-1}$$

$$\times \cdots\cdots$$

$$\times \frac{1}{a_3!} \frac{(t_{b_1+b_2+1} - t_{b_1+b_2})^{a_3} e^{t_{b_1+b_2+1}}}{e^{t_{b_1+b_2+1}} - 1} \cdot \underbrace{\frac{e^{t_{b_1+b_2}}}{e^{t_{b_1+b_2}} - 1} \cdots\cdots \frac{e^{t_{b_1+2}}}{e^{t_{b_1+2}} - 1}}_{b_2-1}$$

$$\times \frac{1}{a_2!} \frac{(t_{b_1+1} - t_{b_1})^{a_2} e^{t_{b_1+1}}}{e^{t_{b_1+1}} - 1} \cdot \underbrace{\frac{e^{t_{b_1}}}{e^{t_{b_1}} - 1} \cdots\cdots \frac{e^{t_2}}{e^{t_2} - 1}}_{b_1-1} \cdot \frac{1}{a_1!} \frac{t_1^{a_1} e^{t_1}}{e^{t_1} - 1} \, dt_1 \, dt_2 \cdots dt_{b_1+\cdots+b_h}.$$

The factor $(-1)^r$ on the right of (2.10) comes from $(-1)^{a_1+\cdots+a_h} = (-1)^r$. Plugging (2.9) and (2.10) into the definitions (1.2) and (1.1) respectively and making the change of variables

$$t = x_1+\cdots+x_n,\ t_{b_1+\cdots+b_h} = x_2+\cdots+x_n,\ t_{b_1+\cdots+b_h-1} = x_3+\cdots+x_n, \ldots, t_2 = x_{n-1}+x_n,\ t_1 = x_n,$$

we obtain the proposition. One should note that the dual index $(\mathbf{k}_+)^* = (l_1, \ldots, l_n)$ is given by

$$(\mathbf{k}_+)^* = (\underbrace{1, \ldots, 1}_{b_h}, a_h + 1, \underbrace{1, \ldots, 1}_{b_{h-1}-1}, a_{h-1} + 1, \ldots, \underbrace{1, \ldots, 1}_{b_1-1}, a_1 + 1)$$

and the depth n is equal to $b_1 + \cdots + b_h + 1$, and that (the trivial) $x_i^{l_i-1} = 1$ when $l_i = 1$. □

Proof of Theorem 2.5. Set $s = m$ in the integral expressions in the proposition, and expand $(x_1 + \cdots + x_k)^{m-1}$ by the multinomial theorem. Then the formula in the theorem follows from the lemma below. □

Lemma 2.7. *For $l_1, \ldots, l_r \in \mathbb{Z}_{\geq 1}$ with $l_r \geq 2$, we have*

$$\zeta(l_1, \ldots, l_r) = \frac{1}{\prod_{j=1}^r \Gamma(l_j)} \int_0^\infty \cdots \int_0^\infty \frac{x_1^{l_1-1} \cdots x_r^{l_r-1}}{e^{x_1+\cdots+x_r} - 1} \cdot \frac{1}{e^{x_2+\cdots+x_r} - 1} \cdots\cdots \frac{1}{e^{x_r} - 1} dx_1 \cdots dx_r$$

and

$$\zeta^\star(l_1, \ldots, l_r) = \frac{1}{\prod_{j=1}^r \Gamma(l_j)} \int_0^\infty \cdots \int_0^\infty \frac{x_1^{l_1-1} \cdots x_r^{l_r-1}}{e^{x_1+\cdots+x_r} - 1} \cdot \frac{e^{x_2+\cdots+x_r}}{e^{x_2+\cdots+x_r} - 1} \cdots\cdots \frac{e^{x_r}}{e^{x_r} - 1} dx_1 \cdots dx_r.$$

Proof. The first formula is given in [4, Theorem 3 (i)]. As for the second, we may proceed similarly by using $n^{-s} = \Gamma(s)^{-1} \int_0^\infty t^{s-1} e^{-nt} \, dt$ to have

$$\begin{aligned}
\zeta^\star(l_1, \ldots, l_r) &= \sum_{m_1=1}^\infty \sum_{m_2,\ldots,m_r=0}^\infty \frac{1}{m_1^{l_1} (m_1 + m_2)^{l_2} \cdots\cdots (m_1 + \cdots + m_r)^{l_r}} \\
&= \frac{1}{\prod_{j=1}^r \Gamma(l_j)} \sum_{m_1=1}^\infty \sum_{m_2,\ldots,m_r=0}^\infty \int_0^\infty \cdots \int_0^\infty x_1^{l_1-1} e^{-m_1 x_1} \cdot x_2^{l_2-1} e^{-(m_1+m_2)x_2} \cdots \\
&\qquad\qquad \cdots x_r^{l_r-1} e^{-(m_1+\cdots+m_r)x_r} \, dx_1 \cdots dx_r \\
&= \frac{1}{\prod_{j=1}^r \Gamma(l_j)} \sum_{m_1=1}^\infty \sum_{m_2,\ldots,m_r=0}^\infty \int_0^\infty \cdots \int_0^\infty x_1^{l_1-1} \cdots x_r^{l_r-1} e^{-m_1(x_1+\cdots+x_r)} \cdot e^{-m_2(x_2+\cdots+x_r)} \cdots
\end{aligned}$$

$$\cdots e^{-m_r x_r}\, dx_1 \cdots dx_r$$

$$= \frac{1}{\prod_{j=1}^{r}\Gamma(l_j)} \int_0^\infty \cdots \int_0^\infty \frac{x_1^{l_1-1}\cdots x_r^{l_r-1}}{e^{x_1+\cdots+x_r}-1} \cdot \frac{e^{x_2+\cdots+x_r}}{e^{x_2+\cdots+x_r}-1} \cdot \cdots \cdot \frac{e^{x_r}}{e^{x_r}-1} dx_1 \cdots dx_r.$$

□

We record here one corollary to the theorem in the case of $\eta_k(m)$ (compare with the similar formula in [4, Theorem 9 (i)]). Noting $(k+1)^* = (\underbrace{1,\ldots,1}_{k-1},2)$, we have

Corollary 2.8. *For $k, m \geq 1$, we have*

$$\eta_k(m) = \sum_{\substack{j_1,\ldots,j_{k-1}\geq 1, j_k\geq 2\\ j_1+\cdots+j_k=k+m}} (j_k-1)\zeta^\star(j_1,\ldots,j_{k-1},j_k). \tag{2.11}$$

3. Relations among the functions ξ, η and ζ, and their consequences to multiple zeta values and multi-poly-Bernoulli numbers

In this section, we first deduce that each of the functions η and ξ can be written as a linear combination of the other by the *same* formula. This is a consequence of the so-called Landen-type connection formula for the multiple polylogarithm $\mathrm{Li}_{k_1,\ldots,k_r}(z)$. We then establish a formula for $\xi(k_1,\ldots,k_r;s)$ in terms of the single-variable multiple zeta function

$$\zeta(l_1,\ldots,l_r;s) = \sum_{1\leq m_1<\cdots<m_r<m} \frac{1}{m_1^{l_1}\cdots m_r^{l_r} m^s} \tag{3.1}$$

defined for positive integers $l_1,\ldots,l_r$, the analytic continuation of which has been given in [4] (the analytic continuation of a more general multi-variable multiple zeta function is established in [1]). This answers to the question posed in §5 of [4]. As a result, the function $\eta(k_1,\ldots,k_r;s)$ can also be written by the multiple zeta functions of the type above. We then give a formula for values at positive integers of $\xi(k_1,\ldots,k_r;s)$, and hence of $\eta(k_1,\ldots,k_r;s)$, in terms of the 'shuffle regularized values' of multiple zeta values, and thereby derive some consequences on the values of $\eta_k(s)$.

Let $\mathbf{k} = (k_1,\ldots,k_r) \in \mathbb{Z}_{\geq 1}^r$ be an index set. Recall that $\mathbf{k}$ is said to be admissible if the last entry k_r is greater than 1, the weight of $\mathbf{k}$ is the sum $k_1+\cdots+k_r$, and the depth is the length r of the index. For two indices $\mathbf{k}$ and $\mathbf{k}'$ of the same weight, we say $\mathbf{k}'$ refines $\mathbf{k}$, denoted $\mathbf{k} \preceq \mathbf{k}'$, if $\mathbf{k}$ is obtained from $\mathbf{k}'$ by replacing some commas by +'s. For example, $(5) = (2+3) \preceq (2,3)$, $(2,3) = (1+1,2+1) \preceq (1,1,2,1)$, etc. The standard expression of a multiple zeta-star value as a sum of multiple zeta values is written as

$$\zeta^\star(\mathbf{k}) = \sum_{\substack{\mathbf{k}'\preceq\mathbf{k}\\ \text{admissible}}} \zeta(\mathbf{k}'),$$

where the sum on the right runs over the admissible indices $\mathbf{k}'$ such that $\mathbf{k}$ refines $\mathbf{k}'$.

The following formula is known as the Landen connection formula for the multiple polylogarithm ([26, Proposition 9]).

Lemma 3.1. *For any index* $\mathbf{k}$ *of depth* r*, we have*

$$\mathrm{Li}_{\mathbf{k}}\Big(\frac{z}{z-1}\Big) = (-1)^r \sum_{\mathbf{k}\preceq\mathbf{k}'} \mathrm{Li}_{\mathbf{k}'}(z). \tag{3.2}$$

We can prove this by induction on weight and by using (2.3), see [26].

By using this and noting $z/(z-1) = 1-e^t$ (resp. $1-e^{-t}$) if $z = 1-e^{-t}$ (resp. $1-e^t$), we immediately obtain the following proposition.

Proposition 3.2. *Let* $\mathbf{k}$ *be any index set and* r *its depth. We have the relations*

$$\eta(\mathbf{k};s) = (-1)^{r-1} \sum_{\mathbf{k}\preceq\mathbf{k}'} \xi(\mathbf{k}';s) \tag{3.3}$$

and

$$\xi(\mathbf{k};s) = (-1)^{r-1} \sum_{\mathbf{k}\preceq\mathbf{k}'} \eta(\mathbf{k}';s). \tag{3.4}$$

Corollary 3.3. *Let* k *be a positive integer. Then we have*

$$\eta_k(s) = \sum_{\mathbf{k}:\textit{weight } k} \xi(\mathbf{k};s) \tag{3.5}$$

and

$$\xi_k(s) = \sum_{\mathbf{k}:\textit{weight } k} \eta(\mathbf{k};s), \tag{3.6}$$

where the sums run over all indices of weight k*. Here we have written* $\xi_k(s)$ *for* $\xi(k;s)$*.*

Proof. The index (k) is of depth 1 and all indices of weight k (admissible or non-admissible) refine (k). □

We mention here that, also by taking $\mathbf{k} = (k)$ in Lemma 3.1 and setting $z = 1-e^t$ or $1-e^{-t}$, one immediately obtains a kind of sum formulas for multi-poly-Bernoulli numbers as follows (compare with similar formulas in [17, Theorem 3.1]).

Corollary 3.4. *For* $k \geq 1$ *and* $m \geq 0$*, we have*

$$B_m^{(k)} = (-1)^m \sum_{\substack{k_1+\cdots+k_r=k \\ k_i, r\geq 1}} C_m^{(k_1,\ldots,k_r)} \tag{3.7}$$

and

$$C_m^{(k)} = (-1)^m \sum_{\substack{k_1+\cdots+k_r=k \\ k_i, r\geq 1}} B_m^{(k_1,\ldots,k_r)}. \tag{3.8}$$

Next, we prove an Euler-type connection formula for the multiple polylogarithm. If an index $\mathbf{k}$ is of weight $|\mathbf{k}|$, we also say the multiple zeta value $\zeta(\mathbf{k})$ is of weight $|\mathbf{k}|$.

Lemma 3.5. *Let* $\mathbf{k}$ *be any index. Then we have*

$$\mathrm{Li}_{\mathbf{k}}(1-z)=\sum_{\mathbf{k}',\,j\geq 0}c_{\mathbf{k}}(\mathbf{k}';j)\mathrm{Li}_{\underbrace{1,\ldots,1}_{j}}(1-z)\mathrm{Li}_{\mathbf{k}'}(z), \tag{3.9}$$

where the sum on the right runs over indices $\mathbf{k}'$ *and integers* $j\geq 0$ *that satisfy* $|\mathbf{k}'|+j\leq|\mathbf{k}|$*, and* $c_{\mathbf{k}}(\mathbf{k}';j)$ *is a* $\mathbb{Q}$*-linear combination of multiple zeta values of weight* $|\mathbf{k}|-|\mathbf{k}'|-j$*. We understand* $\mathrm{Li}_{\emptyset}(z)=1$ *and* $|\emptyset|=0$ *for the empty index* $\emptyset$*, and the constant* 1 *is regarded as a multiple zeta value of weight* 0*.*

Proof. We proceed by induction on the weight of $\mathbf{k}$. When $\mathbf{k}=(1)$, the trivial identity $\mathrm{Li}_1(1-z)=\mathrm{Li}_1(1-z)$ is the one asserted. Suppose the weight $|\mathbf{k}|$ of $\mathbf{k}$ is greater than 1 and assume the statement holds for any index of weight less than $|\mathbf{k}|$. For $\mathbf{k}=(k_1,\ldots,k_r)$, set $\mathbf{k}_-=(k_1,\ldots,k_{r-1},k_r-1)$ and $\mathbf{k}_+=(k_1,\ldots,k_{r-1},k_r+1)$.

First assume that $\mathbf{k}$ is admissible. Then, by (2.3) and induction hypothesis, we have

$$\frac{d}{dz}\mathrm{Li}_{\mathbf{k}}(1-z)=-\frac{\mathrm{Li}_{\mathbf{k}_-}(1-z)}{1-z}=-\frac{1}{1-z}\sum_{\mathbf{l},\,j\geq 0}c_{\mathbf{k}_-}(\mathbf{l};j)\mathrm{Li}_{\underbrace{1,\ldots,1}_{j}}(1-z)\mathrm{Li}_{\mathbf{l}}(z),$$

the right-hand side being of a desired form. Here, again by (2.3), we see that

$$\frac{1}{1-z}\mathrm{Li}_{\underbrace{1,\ldots,1}_{j}}(1-z)\mathrm{Li}_{\mathbf{l}}(z)=\frac{d}{dz}\left(\sum_{i=0}^{j}\mathrm{Li}_{\underbrace{1,\ldots,1}_{j-i}}(1-z)\mathrm{Li}_{\mathbf{l},1+i}(z)\right).$$

We therefore conclude

$$\mathrm{Li}_{\mathbf{k}}(1-z)=-\sum_{\mathbf{l},\,j\geq 0}c_{\mathbf{k}_-}(\mathbf{l};j)\sum_{i=0}^{j}\mathrm{Li}_{\underbrace{1,\ldots,1}_{j-i}}(1-z)\mathrm{Li}_{\mathbf{l},1+i}(z)+C$$

with some constant C. Since $\lim_{z\to 0}\mathrm{Li}_{\underbrace{1,\ldots,1}_{j-i}}(1-z)\mathrm{Li}_{\mathbf{l},1+i}(z)=0$, we find $C=\zeta(\mathbf{k})$ by setting $z\to 0$, and obtain the desired expression for $\mathrm{Li}_{\mathbf{k}}(1-z)$.

When $\mathbf{k}$ is not necessarily admissible, write $\mathbf{k}=(\mathbf{k}_0,\underbrace{1,\ldots,1}_{q})$ with an admissible $\mathbf{k}_0$ and $q\geq 0$. We prove the identity by induction on q. The case $q=0$ ($\mathbf{k}$ is admissible) is already done. Suppose $q\geq 1$ and assume the claim is true for smaller q. Then by assumption we have the expression

$$\mathrm{Li}_{\mathbf{k}_0,\underbrace{1,\ldots,1}_{q-1}}(1-z)=\sum_{\mathbf{m},\,j\geq 0}c_{\mathbf{k}'}(\mathbf{m};j)\mathrm{Li}_{\underbrace{1,\ldots,1}_{j}}(1-z)\mathrm{Li}_{\mathbf{m}}(z),$$

where we have put $\mathbf{k}'=(\mathbf{k}_0,\underbrace{1,\ldots,1}_{q-1})$. We multiply $\mathrm{Li}_1(1-z)$ on both sides. Then, by the shuffle product, the left-hand side becomes the sum of the form

$$q\,\mathrm{Li}_{\mathbf{k}}(1-z)+\sum_{\mathbf{k}_0':\text{admissible}}\mathrm{Li}_{\mathbf{k}_0',\underbrace{1,\ldots,1}_{q-1}}(1-z),$$

and each term in the sum is written in the claimed form by induction hypothesis. On the other hand, the right-hand side becomes also of the form desired because

$$\mathrm{Li}_1(1-z)\mathrm{Li}_{\underbrace{1,\ldots,1}_{j}}(1-z)=(j+1)\mathrm{Li}_{\underbrace{1,\ldots,1}_{j+1}}(1-z).$$

Hence $\mathrm{Li}_{\mathbf{k}}(1-z)$ is of the form as claimed. □

With the lemma, we are now able to establish the following (see [4, §5, Problem (i)]).

Theorem 3.6. *Let* $\mathbf{k}$ *be any index set. The function* $\xi(\mathbf{k};s)$ *can be written in terms of multiple zeta functions as*

$$\xi(\mathbf{k};s)=\sum_{\mathbf{k}',\,j\geq 0}c_{\mathbf{k}}(\mathbf{k}';j)\binom{s+j-1}{j}\zeta(\mathbf{k}';s+j).$$

Here, the sum is over indices $\mathbf{k}'$ *and integers* $j\geq 0$ *satisfying* $|\mathbf{k}'|+j\leq|\mathbf{k}|$*, and* $c_{\mathbf{k}}(\mathbf{k}';j)$ *is a* $\mathbb{Q}$*-linear combination of multiple zeta values of weight* $|\mathbf{k}|-|\mathbf{k}'|-j$*. The index* $\mathbf{k}'$ *may be* $\emptyset$ *and for this we set* $\zeta(\emptyset;s+j)=\zeta(s+j)$*.*

Proof. By setting $z=e^{-t}$ in the lemma and using

$$\mathrm{Li}_{\underbrace{1,\ldots,1}_{j}}(z)=\frac{(-\log(1-z))^j}{j!},\tag{3.10}$$

we have

$$\mathrm{Li}_{\mathbf{k}}(1-e^{-t})=\sum_{\mathbf{k}',\,j\geq 0}c_{\mathbf{k}}(\mathbf{k}';j)\frac{t^j}{j!}\mathrm{Li}_{\mathbf{k}'}(e^{-t}).$$

Substituting this into the definition (1.2) of $\xi(\mathbf{k};s)$ and using the formula ([4, Proposition 2, (i)])

$$\zeta(\mathbf{k};s)=\frac{1}{\Gamma(s)}\int_0^\infty\frac{t^{s-1}}{e^t-1}\mathrm{Li}_{\mathbf{k}}(e^{-t})\,dt,$$

we immediately obtain the theorem. □

Remark 3.7. This theorem generalizes [4, Theorem 8], where the corresponding formula for $\mathrm{Li}_{\underbrace{1,\ldots,1}_{r-1},k}(1-z)$ is

$$\begin{aligned}\mathrm{Li}_{\underbrace{1,\ldots,1}_{r-1},k}(1-z)=(-1)^{k-1}&\sum_{\substack{j_1+\cdots+j_k=r+k\\ \forall j_i\geq 1}}\mathrm{Li}_{\underbrace{1,\ldots,1}_{j_k-1}}(1-z)\mathrm{Li}_{j_1,\ldots,j_{k-1}}(z)\\ &+\sum_{j=0}^{k-2}(-1)^j\zeta(\underbrace{1,\ldots,1}_{r-1},k-j)\mathrm{Li}_{\underbrace{1,\ldots,1}_{j}}(z).\end{aligned}$$

As pointed out by Shu Oi, one can deduce Lemma 3.5 by induction using [27, Prop. 5]. However, to describe the right-hand side of the lemma explicitly is a different problem and neither proof gives such a formula in general. See also [25] for a related topic.

Example 3.8. Apart from the trivial case $(1,\ldots,1)$, examples of the identity in Lemma 3.5 up to weight 4 are:

$$\begin{aligned}\mathrm{Li}_2(1-z)&=-\mathrm{Li}_2(z)-\mathrm{Li}_1(1-z)\mathrm{Li}_1(z)+\zeta(2),\\ \mathrm{Li}_3(1-z)&=\mathrm{Li}_{1,2}(z)+\mathrm{Li}_{2,1}(z)+\mathrm{Li}_1(1-z)\mathrm{Li}_{1,1}(z)-\zeta(2)\mathrm{Li}_1(z)+\zeta(3),\\ \mathrm{Li}_{1,2}(1-z)&=-\mathrm{Li}_3(z)-\mathrm{Li}_1(1-z)\mathrm{Li}_2(z)-\mathrm{Li}_{1,1}(1-z)\mathrm{Li}_1(z)+\zeta(3),\\ \mathrm{Li}_{2,1}(1-z)&=2\mathrm{Li}_3(z)+\mathrm{Li}_1(1-z)\mathrm{Li}_2(z)+\zeta(2)\mathrm{Li}_1(1-z)-2\zeta(3),\\ \mathrm{Li}_4(1-z)&=-\mathrm{Li}_{1,1,2}(z)-\mathrm{Li}_{1,2,1}(z)-\mathrm{Li}_{2,1,1}(z)-\mathrm{Li}_1(1-z)\mathrm{Li}_{1,1,1}(z)\\ &\quad+\zeta(2)\mathrm{Li}_{1,1}(z)-\zeta(3)\mathrm{Li}_1(z)+\zeta(4),\end{aligned}$$

$$\begin{aligned}
\mathrm{Li}_{1,3}(1-z) &= \mathrm{Li}_{1,3}(z)+\mathrm{Li}_{2,2}(z)+\mathrm{Li}_{3,1}(z)+\mathrm{Li}_1(1-z)\mathrm{Li}_{1,2}(z)+\mathrm{Li}_1(1-z)\mathrm{Li}_{2,1}(z)\\
&\quad+\mathrm{Li}_{1,1}(1-z)\mathrm{Li}_{1,1}(z)-\zeta(3)\mathrm{Li}_1(z)+\frac{1}{4}\zeta(4),\\
\mathrm{Li}_{2,2}(1-z) &= -\mathrm{Li}_{2,2}(z)-2\mathrm{Li}_{3,1}(z)-\mathrm{Li}_1(1-z)\mathrm{Li}_{2,1}(z)-\zeta(2)\mathrm{Li}_1(1-z)\mathrm{Li}_1(z)\\
&\quad-\zeta(2)\mathrm{Li}_2(z)+2\zeta(3)\mathrm{Li}_1(z)+\frac{3}{4}\zeta(4),\\
\mathrm{Li}_{3,1}(1-z) &= -2\mathrm{Li}_{1,3}(z)-\mathrm{Li}_{2,2}(z)-\mathrm{Li}_1(1-z)\mathrm{Li}_{1,2}(z)+\zeta(2)\mathrm{Li}_2(z)\\
&\quad+\zeta(3)\mathrm{Li}_1(1-z)-\frac{5}{4}\zeta(4),\\
\mathrm{Li}_{1,1,2}(1-z) &= -\mathrm{Li}_4(z)-\mathrm{Li}_1(1-z)\mathrm{Li}_3(z)-\mathrm{Li}_{1,1}(1-z)\mathrm{Li}_2(z)\\
&\quad-\mathrm{Li}_{1,1,1}(1-z)\mathrm{Li}_1(z)+\zeta(4),\\
\mathrm{Li}_{1,2,1}(1-z) &= 3\mathrm{Li}_4(z)+2\mathrm{Li}_1(1-z)\mathrm{Li}_3(z)+\mathrm{Li}_{1,1}(1-z)\mathrm{Li}_2(z)+\zeta(3)\mathrm{Li}_1(1-z)\\
&\quad-3\zeta(4),\\
\mathrm{Li}_{2,1,1}(1-z) &= -3\mathrm{Li}_4(z)-\mathrm{Li}_1(1-z)\mathrm{Li}_3(z)+\zeta(2)\mathrm{Li}_{1,1}(1-z)-2\zeta(3)\mathrm{Li}_1(1-z)\\
&\quad+3\zeta(4).
\end{aligned}$$

Accordingly, we have

$$\begin{aligned}
\xi(2;s) &= -\zeta(2;s)-s\zeta(1;s+1)+\zeta(2)\zeta(s),\\
\xi(3;s) &= \zeta(1,2;s)+\zeta(2,1;s)+s\zeta(1,1;s+1)-\zeta(2)\zeta(1;s)+\zeta(3)\zeta(s),\\
\xi(1,2;s) &= -\zeta(3;s)-s\zeta(2;s+1)-\frac{s(s+1)}{2}\zeta(1;s+2)+\zeta(3)\zeta(s),\\
\xi(2,1;s) &= 2\zeta(3;s)+s\zeta(2;s+1)+\zeta(2)s\zeta(s+1)-2\zeta(3)\zeta(s),\\
\xi(4;s) &= -\zeta(1,1,2;s)-\zeta(1,2,1;s)-\zeta(2,1,1;s)-s\zeta(1,1,1;s+1)\\
&\quad+\zeta(2)\zeta(1,1;s)-\zeta(3)\zeta(1;s)+\zeta(4)\zeta(s),\\
\xi(1,3;s) &= \zeta(1,3;s)+\zeta(2,2;s)+\zeta(3,1;s)+s\zeta(1,2;s+1)+s\zeta(2,1;s+1)\\
&\quad+\frac{s(s+1)}{2}\zeta(1,1;s+2)-\zeta(3)\zeta(1;s)+\frac{1}{4}\zeta(4)\zeta(s),\\
\xi(2,2;s) &= -\zeta(2,2;s)-2\zeta(3,1;s)-s\zeta(2,1;s+1)-\zeta(2)s\zeta(1;s+1)\\
&\quad-\zeta(2)\zeta(2;s)+2\zeta(3)\zeta(1;s)+\frac{3}{4}\zeta(4)\zeta(s),\\
\xi(3,1;s) &= -2\zeta(1,3;s)-\zeta(2,2;s)-s\zeta(1,2;s+1)+\zeta(2)\zeta(2;s)\\
&\quad+\zeta(3)s\zeta(s+1)-\frac{5}{4}\zeta(4)\zeta(s),\\
\xi(1,1,2;s) &= -\zeta(4;s)-s\zeta(3;s+1)-\frac{s(s+1)}{2}\zeta(2;s+2)\\
&\quad-\frac{s(s+1)(s+2)}{6}\zeta(1;s+3)+\zeta(4)\zeta(s),\\
\xi(1,2,1;s) &= 3\zeta(4;s)+2s\zeta(3;s+1)+\frac{s(s+1)}{2}\zeta(2;s+2)+\zeta(3)s\zeta(s+1)\\
&\quad-3\zeta(4)\zeta(s),\\
\xi(2,1,1;s) &= -3\zeta(4;s)-s\zeta(3;s+1)+\zeta(2)\frac{s(s+1)}{2}\zeta(s+2)-2\zeta(3)s\zeta(s+1)\\
&\quad+3\zeta(4)\zeta(s).
\end{aligned}$$

From these and (3.5) of Corollary 3.3, we have for instance

$$\eta_2(s) = \xi(2;s)+\xi(1,1;s)$$

$$
\begin{aligned}
&= -\zeta(2;s) - s\zeta(1;s+1) + \zeta(2)\zeta(s) + \frac{s(s+1)}{2}\zeta(s+2),\\
\eta_3(s) &= \xi(3;s) + \xi(1,2;s) + \xi(2,1;s) + \xi(1,1,1;s)\\
&= \zeta(3;s) + \zeta(1,2;s) + \zeta(2,1;s) + s\zeta(1,1;s+1) - \frac{s(s+1)}{2}\zeta(1;s+2)\\
&\quad -\zeta(2)\zeta(1;s) + \zeta(2)s\zeta(s+1) + \frac{s(s+1)(s+2)}{6}\zeta(s+3),\\
\eta_4(s) &= \xi(4;s) + \xi(1,3;s) + \xi(2,2;s) + \xi(3,1;s) + \xi(1,1,2;s) + \xi(1,2,1;s)\\
&\quad +\xi(2,1,1;s) + \xi(1,1,1,1;s)\\
&= -\zeta(4;s) - \zeta(1,3;s) - \zeta(2,2;s) - \zeta(3,1;s) - \zeta(1,1,2;s) - \zeta(1,2,1;s)\\
&\quad -\zeta(2,1,1;s) - s\zeta(1,1,1;s+1) + \zeta(2)\zeta(1,1;s) + \frac{s(s+1)}{2}\zeta(1,1;s+2)\\
&\quad -\zeta(2)s\zeta(1;s+1) + \zeta(2)\frac{s(s+1)}{2}\zeta(s+2) - \frac{s(s+1)(s+2)}{6}\zeta(1;s+3)\\
&\quad +\frac{7}{4}\zeta(4)\zeta(s) + \frac{s(s+1)(s+2)(s+3)}{24}\zeta(s+4).
\end{aligned}
$$

Before closing this section, we present a curious observation. Recall the formula

$$\xi_k(m) = \zeta^\star(\underbrace{1,\ldots,1}_{m-1},k+1)$$

discovered by Ohno [24]. Comparing this with the two formulas (2.11) and [4, Corollary 10], one may expect

$$\eta_k(m) \overset{?}{=} \zeta(\underbrace{1,\ldots,1}_{m-1},k+1).$$

This is not true in fact. However, we found experimentally the identities

$$\eta_k(m) = \eta_m(k) \tag{3.11}$$

and

$$\sum_{j=1}^{k-1}(-1)^{j-1}\eta_{k-j}(j) = \begin{cases} 2(1-2^{1-k})\zeta(k) & (k\text{: even}),\\ 0 & (k\text{: odd}).\end{cases} \tag{3.12}$$

These are respectively analogous to the duality relation

$$\zeta(\underbrace{1,\ldots,1}_{m-1},k+1) = \zeta(\underbrace{1,\ldots,1}_{k-1},m+1)$$

and the relation

$$\sum_{j=1}^{k-1}(-1)^{j-1}\zeta(\underbrace{1,\ldots,1}_{j-1},k-j+1) = \begin{cases} 2(1-2^{1-k})\zeta(k) & (k\text{: even}),\\ 0 & (k\text{: odd}),\end{cases}$$

which is a special case of the Le-Murakami relation [23] (or one can derive this from the well-known generating series identity [2], [11]

$$1 - \sum_{k>j\ge 1}\zeta(\underbrace{1,\ldots,1}_{j-1},k-j+1)X^{k-j}Y^j = \frac{\Gamma(1-X)\Gamma(1-Y)}{\Gamma(1-X-Y)}$$

by setting $Y=-X$ and using the reflection formula for the gamma function.)

We are still not able to prove (3.11)*, but could prove (3.12) by using the following general

*Quite recently, Shuji Yamamoto communicated to the authors that he found a proof.

formula for the value $\xi(\mathbf{k};m)$ and the relation (3.3) in Proposition 3.2. We shall discuss this and other aspects of 'height one' multiple zeta values in more detail in a subsequent paper [22].

Proposition 3.9. *Let* $\mathbf{k}$ *be any index and* $m \geq 1$ *an integer. Then we have*

$$\xi(\mathbf{k};m) = (-1)^{m-1}\zeta^{ш}(\mathbf{k}_+, \underbrace{1,\ldots,1}_{m-1}), \tag{3.13}$$

where $\zeta^{ш}$ *stands for the 'shuffle regularized' value, which is the constant term of the shuffle regularized polynomial defined in [19].*

Proof. By making the change of variable $x = 1 - e^{-t}$ in the definition (1.2), we have

$$\xi(\mathbf{k};s) = \frac{1}{\Gamma(s)}\int_0^1 \big(-\log(1-x)\big)^{s-1}\mathrm{Li}_{\mathbf{k}}(x)\,\frac{dx}{x}.$$

Put $s = m$ and use (3.10) to obtain

$$\xi(\mathbf{k};m) = \int_0^1 \mathrm{Li}_{\underbrace{1,\ldots,1}_{m-1}}(x)\mathrm{Li}_{\mathbf{k}}(x)\,\frac{dx}{x}.$$

The regularization formula [19, Eq. (5.2)], together with the shuffle product of $\mathrm{Li}_{\underbrace{1,\ldots,1}_{m-1}}(x)\mathrm{Li}_{\mathbf{k}}(x)$, immediately gives (3.13). $\square$

By using (3.13) and (3.5), we can write $\eta_k(m)$ in terms of shuffle regularized values. The following expression seems to follow from that formula by taking the dual, but we have not yet worked it out in detail.

$$\eta_k(m) \stackrel{?}{=} \binom{m+k}{k}\zeta(m+k) - \sum_{\substack{2\leq r\leq k+1\\ j_1+\cdots+j_r=m+k-r-1}} \binom{j_1+\cdots+j_{r-1}}{k-r+1}\zeta(j_1+1,\cdots,j_{r-1}+1,j_r+2).$$

4. The function $\eta(k_1,\ldots,k_r;s)$ for non-positive indices

In this section, as in the case of positive indices, we construct η-functions with non-positive indices. It is known that $\mathrm{Li}_{-k}(z)$ can be expressed as

$$\mathrm{Li}_{-k}(z) = \frac{P(z;k)}{(1-z)^k}$$

for $k \in \mathbb{Z}_{\geq 0}$, where $P(x;k) \in \mathbb{Z}[x]$ is a monic polynomial satisfying

$$\deg P(x;k) = \begin{cases} 1 & (k=0) \\ k & (k\geq 1), \end{cases}$$
$$x \mid P(x;k)$$

(see, for example, Shimura [29, Equations (2.17), (4.2) and (4.6)]; Note that the above $P(x;k)$ coincides with $xP_{k+1}(x)$ in [29]). We first extend this fact to multiple polylogarithms with non-positive indices as follows.

Lemma 4.1. *For $k_1, \dots, k_r \in \mathbb{Z}_{\geq 0}$, there exists a polynomial $P(x; k_1, \dots, k_r) \in \mathbb{Z}[x]$ such that*

$$\mathrm{Li}_{-k_1,\dots,-k_r}(z) = \frac{P(z; k_1, \dots, k_r)}{(1-z)^{k_1+\cdots+k_r+r}}, \tag{4.1}$$

$$\deg P(x; k_1, \dots, k_r) = \begin{cases} r & (k_1 = \cdots = k_r = 0) \\ k_1 + \cdots + k_r + r - 1 & (\text{otherwise}), \end{cases} \tag{4.2}$$

$$x^r \mid P(x; k_1, \dots, k_r). \tag{4.3}$$

More explicitly, $P(x; \underbrace{0, 0, \dots, 0}_{r}) = x^r$.

Proof. We prove this lemma by the double induction on $r \geq 1$ and $K = k_1 + \cdots + k_r \geq 0$. The case $r = 1$ is as mentioned above. For $r \geq 2$, we assume the case of $r-1$ holds and consider the case of r. When $K = k_1 + \cdots + k_r = 0$, namely $k_1 = \cdots = k_r = 0$, we have

$$\begin{aligned} \mathrm{Li}_{0,\dots,0}(z) &= \sum_{m_1 < \cdots < m_r} x^{m_r} = \sum_{m_1 < \cdots < m_{r-1}} \sum_{m_r = m_{r-1}+1}^{\infty} z^{m_r} \\ &= \frac{z}{1-z} \sum_{m_1 < \cdots < m_{r-1}} z^{m_{r-1}} = \cdots = \frac{z^r}{(1-z)^r}, \end{aligned}$$

which implies (4.1)–(4.3) hold, and also $P(x; 0, \dots, 0) = x^r$. Hence we assume the case $K = k_1 + \cdots + k_r - 1$ holds and consider the case $K = k_1 + \cdots + k_r (\geq 1)$. We consider the two cases $k_r = 0$ and $k_r \geq 1$ separately. First we assume $k_r = 0$. Then, by induction hypothesis, we have

$$\begin{aligned} \mathrm{Li}_{-k_1,\dots,-k_{r-1},0}(z) &= \sum_{m_1 < \cdots < m_{r-1}} m_1^{k_1} \cdots m_{r-1}^{k_{r-1}} \sum_{m_r = m_{r-1}+1}^{\infty} z^{m_r} \\ &= \frac{z}{1-z} \sum_{m_1 < \cdots < m_{r-1}} m_1^{k_1} \cdots m_{r-1}^{k_{r-1}} z^{m_{r-1}} \\ &= \frac{z}{1-z} \frac{P(z : k_1, \dots, k_{r-1})}{(1-z)^{k_1+\cdots+k_{r-1}+r-1}}. \end{aligned}$$

Let $P(z; k_1, \dots, k_{r-1}, 0) = zP(z; k_1, \dots, k_{r-1})$. Then (4.1)–(4.3) hold.

Next we assume $k_r \geq 1$. Then, using the same formula as in (2.3) and the induction hypothesis, we have

$$\begin{aligned} &\mathrm{Li}_{-k_1,\dots,-k_{r-1},-k_r}(z) = z\frac{d}{dz}\mathrm{Li}_{-k_1,\dots,-k_r+1}(z) \\ &= z\frac{d}{dz}\left(\frac{P(z : k_1, \dots, k_r - 1)}{(1-z)^{k_1+\cdots+k_r-1+r}}\right) \\ &= \frac{z\left\{P'(z : k_1, \dots, k_r - 1)(1-z) + (k_1 + \cdots + k_r - 1 + r)P(z; k_1, \dots, k_r - 1)\right\}}{(1-z)^{k_1+\cdots+k_r+r}}. \end{aligned}$$

If $k_1 = \cdots = k_{r-1} = 0$ and $k_r = 1$, then the numerator, that is, $P(0, \dots, 0, -1)$ equals rz^r, using the above results. If not, the degree of the numerator equals $k_1 + \cdots + k_r + r - 1$ by induction hypothesis. The both cases satisfy (4.1)–(4.3). This completes the proof of the lemma. □

Remark 4.2. In the case $r \geq 2$, $P(x; k_1, \dots, k_r)$ is not necessarily a monic polynomial. For example, we have $\mathrm{Li}_{0,-1}(z) = 2z^2/(1-z)^3$, so $P(x; 0, 1) = 2x^2$.

We obtain from (4.1) and (4.2) that

$$\mathrm{Li}_{-k_1,\ldots,-k_r}(1-e^t)=\frac{P(1-e^t;k_1,\ldots,k_r)}{e^{(k_1+\cdots+k_r+r)t}}=\begin{cases}O(1) & (k_1=\cdots=k_r=0)\\ O(e^{-t}) & (\text{otherwise})\end{cases} \tag{4.4}$$

as $t\to\infty$, and from (4.3) that

$$\mathrm{Li}_{-k_1,\ldots,-k_r}(1-e^t)=O(t^r)\quad(t\to 0). \tag{4.5}$$

Therefore we can define the following.

Definition 4.3. For $k_1,\ldots,k_r\in\mathbb{Z}_{\geq 0}$, define

$$\eta(-k_1,\ldots,-k_r;s)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{\mathrm{Li}_{-k_1,\ldots,-k_r}(1-e^t)}{1-e^t}dt \tag{4.6}$$

for $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1-r$. In the case $r=1$, denote $\eta(-k;s)$ by $\eta_{-k}(s)$.

We see that the integral on the right-hand side of (4.6) is absolutely convergent for $\mathrm{Re}(s)>1-r$. Hence $\eta(-k_1,\ldots,-k_r;s)$ is holomorphic for $\mathrm{Re}(s)>1-r$. By the same method as in the proof of Theorem 2.3 for $\eta(k_1,\ldots,k_r;s)$, we can similarly obtain the following.

Theorem 4.4. *For $k_1,\ldots,k_r\in\mathbb{Z}_{\geq 0}$, $\eta(-k_1,\ldots,-k_r;s)$ can be analytically continued to an entire function on the whole complex plane, and satisfies*

$$\eta(-k_1,\ldots,-k_r;-m)=B_m^{(-k_1,\ldots,-k_r)}\quad(m\in\mathbb{Z}_{\geq 0}). \tag{4.7}$$

In particular, $\eta_{-k}(-m)=B_m^{(-k)}$ $(k\in\mathbb{Z}_{\geq 0},\ m\in\mathbb{Z}_{\geq 0})$.

It should be noted that $\xi(-k_1,\ldots,-k_r;s)$ cannot be defined by replacing $\{k_j\}$ by $\{-k_j\}$ in (1.2). In fact, even if $r=1$ and $k=0$ in (1.2), we see that

$$\xi_0(s)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{\mathrm{Li}_0(1-e^{-t})}{e^t-1}dt=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}dt,$$

which is not convergent for any $s\in\mathbb{C}$. Therefore we modify the definition (1.2) as follows.

Definition 4.5. For $k_1,\ldots,k_r\in\mathbb{Z}_{\geq 0}$ with $(k_1,\ldots,k_r)\neq(0,\ldots,0)$, define

$$\widetilde{\xi}(-k_1,\ldots,-k_r;s)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{\mathrm{Li}_{-k_1,\ldots,-k_r}(1-e^t)}{e^{-t}-1}dt \tag{4.8}$$

for $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1-r$. In the case $r=1$, denote $\widetilde{\xi}(-k;s)$ by $\widetilde{\xi}_{-k}(s)$ for $k\geq 1$.

We see from (4.4) and (4.5) that (4.8) is well-defined. Also it is noted that $\widetilde{\xi}(k_1,\ldots,k_r;s)$ cannot be defined by replacing $\{-k_j\}$ by $\{k_j\}$ in (4.8) for $(k_j)\in\mathbb{Z}_{\geq 1}^r$.

In a way parallel to deriving Theorem 4.4, we can obtain the following.

Theorem 4.6. *For $k_1,\ldots,k_r\in\mathbb{Z}_{\geq 0}$ with $(k_1,\ldots,k_r)\neq(0,\ldots,0)$, $\widetilde{\xi}(-k_1,\ldots,-k_r;s)$ can be analytically continued to an entire function on the whole complex plane, and satisfies*

$$\widetilde{\xi}(-k_1,\ldots,-k_r;-m)=C_m^{(-k_1,\ldots,-k_r)}\quad(m\in\mathbb{Z}_{\geq 0}). \tag{4.9}$$

In particular, $\widetilde{\xi}_{-k}(-m)=C_m^{(-k)}$ $(k\in\mathbb{Z}_{\geq 1},\ m\in\mathbb{Z}_{\geq 0})$.

Next we give certain duality formulas for $B_n^{(k_1,\dots,k_r)}$ which is a generalization of (1.6). To state this, we define another type of multi-poly-Bernoulli numbers by

$$\begin{aligned}(4.10)\qquad &\sum_{a=0}^{r-1}(-1)^a\binom{r-1}{a}\sum_{l_1,\dots,l_r\geq 1}\frac{\prod_{j=1}^{r}\left(1-e^{-\sum_{\nu=j}^{r}x_\nu}\right)^{l_j-1}}{(l_1+\cdots+l_r-a)^s}\\ &\quad=\sum_{m_1,\dots,m_r\geq 0}\mathfrak{B}_{m_1,\dots,m_r}^{(s)}\frac{x_1^{m_1}\cdots x_r^{m_r}}{m_1!\cdots m_r!}\end{aligned}$$

for $s\in\mathbb{C}$. In the case $r=1$, we see that $\mathfrak{B}_m^{(k)}=B_m^{(k)}$ for $k\in\mathbb{Z}$. Then we obtain the following result which is a kind of the duality formula. In fact, this coincides with (1.6) in the case $r=1$.

Theorem 4.7. *For $k_1,\dots,k_r\in\mathbb{Z}_{\geq 0}$,*

$$(4.11)\qquad \eta(-k_1,\dots,-k_r;s)=\mathfrak{B}_{k_1,\dots,k_r}^{(s)}.$$

Therefore, for $m\in\mathbb{Z}_{\geq 0}$,

$$(4.12)\qquad B_m^{(-k_1,\dots,-k_r)}=\mathfrak{B}_{k_1,\dots,k_r}^{(-m)}.$$

Proof. We first prepare the following relation which will be proved in the next section (see Lemma 5.9):

$$(4.13)\qquad \prod_{j=1}^{r}\frac{e^{\sum_{\nu=j}^{r}x_\nu}(1-e^t)}{1-e^{\sum_{\nu=j}^{r}x_\nu}(1-e^t)}=\sum_{k_1,\dots,k_r\geq 0}\mathrm{Li}_{-k_1,\dots,-k_r}(1-e^t)\frac{x_1^{k_1}\cdots x_r^{k_r}}{k_1!\cdots k_r!}$$

holds around the origin. Let

$$\mathcal{F}(x_1,\dots,x_r;s)=\sum_{k_1,\dots,k_r\geq 0}\eta(-k_1,\dots,-k_r;s)\frac{x_1^{k_1}\cdots x_r^{k_r}}{k_1!\cdots k_r!}.$$

As a generalization of [18, Proposition 5], we have from (4.13) that

$$\begin{aligned}\mathcal{F}(x_1,\dots,x_r;s)&=\frac{1}{\Gamma(s)}\int_0^\infty\frac{t^{s-1}}{1-e^t}\prod_{j=1}^{r}\frac{e^{\sum_{\nu=j}^{r}x_\nu}(1-e^t)}{1-e^{\sum_{\nu=j}^{r}x_\nu}(1-e^t)}dt\\ &=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}(1-e^t)^{r-1}e^{-rt}\prod_{j=1}^{r}\frac{1}{1-e^{-t}\left(1-e^{-\sum_{\nu=j}^{r}x_\nu}\right)}dt\\ &=\frac{1}{\Gamma(s)}\sum_{a=0}^{r-1}(-1)^a\binom{r-1}{a}\sum_{m_1,\dots,m_r\geq 0}\prod_{j=1}^{r}\left(1-e^{-\sum_{\nu=j}^{r}x_\nu}\right)^{m_j}\\ &\qquad\times\int_0^\infty t^{s-1}e^{(a-r)t}\prod_{j=1}^{r}e^{-m_jt}\,dt\\ &=\sum_{a=0}^{r-1}(-1)^a\binom{r-1}{a}\sum_{m_1,\dots,m_r\geq 0}\frac{\prod_{j=1}^{r}\left(1-e^{-\sum_{\nu=j}^{r}x_\nu}\right)^{m_j}}{(m_1+\cdots+m_r+r-a)^s}.\end{aligned}$$

Therefore, by (4.10), we obtain (4.11). Further, setting $s=-m$ in (4.11) and using (4.7), we obtain (4.12). $\square$

Remark 4.8. In the case $r = 1$, (4.11) implies $\eta_{-k}(s) = B_k^{(s)}$. Thus, using Theorem 4.4, we obtain the duality formula (1.6), which is also written as

$$\eta_{-k}(-m) = \eta_{-m}(-k) \tag{4.14}$$

for $k, m \in \mathbb{Z}_{\geq 0}$. This is exactly contrasted with the positive index case (3.11). Furthermore, by the same method, we can show that $\widetilde{\xi}_{-k-1}(-m) = C_{k+1}^{(-m)}$ for $k, m \in \mathbb{Z}_{\geq 0}$. Hence, using Theorem 4.6 in the case $r = 1$, we obtain the duality formula (1.7).

Example 4.9. When $r = 2$, we can calculate directly from (4.10) that $\mathfrak{B}_{1,0}^{(s)} = 3^{-s} - 2^{-s}$. On the other hand, as mentioned in Lemma 4.1, we have $\mathrm{Li}_{-1,0}(z) = z^2/(1-z)^3$. Hence the left-hand side of (1.10) equals

$$\frac{\mathrm{Li}_{-1,0}(1-e^{-t})}{1-e^{-t}} = \frac{1-e^{-t}}{e^{-3t}} = e^{3t} - e^{2t},$$

hence $B_m^{(-1,0)} = 3^m - 2^m$. Thus we can verify $B_m^{(-1,0)} = \mathfrak{B}_{1,0}^{(-m)}$.

5. Multi-indexed poly-Bernoulli numbers and duality formulas

In this section, we define multi-indexed poly-Bernoulli numbers (see Definition 5.1) and prove the duality formula for them, namely a multi-indexed version of (1.6) (see Theorem 5.4).

For this aim, we first recall multiple polylogarithms of $*$-type and of ш-type in several variables defined by

$$\mathrm{Li}^{*}_{s_1,\ldots,s_r}(z_1,\ldots,z_r) = \sum_{1\leq m_1<\cdots<m_r} \frac{z_1^{m_1}\cdots z_r^{m_r}}{m_1^{s_1} m_2^{s_2}\cdots m_r^{s_r}}, \tag{5.1}$$

$$\begin{aligned}\mathrm{Li}^{\text{ш}}_{s_1,\ldots,s_r}(z_1,\ldots,z_r) &= \sum_{1\leq m_1<\cdots<m_r} \frac{z_1^{m_1} z_2^{m_2-m_1}\cdots z_r^{m_r-m_{r-1}}}{m_1^{s_1} m_2^{s_2}\cdots m_r^{s_r}} \\ &= \sum_{l_1,\ldots,l_r=1}^{\infty} \frac{z_1^{l_1} z_2^{l_2}\cdots z_r^{l_r}}{l_1^{s_1}(l_1+l_2)^{s_2}\cdots(l_1+\cdots+l_r)^{s_r}}\end{aligned} \tag{5.2}$$

for $s_1,\ldots,s_r \in \mathbb{C}$ and $z_1,\cdots,z_r \in \mathbb{C}$ with $|z_j| \leq 1$ $(1 \leq j \leq r)$ (see, for example, [13]). The symbols $*$ and ш are derived from the harmonic product and the shuffle product in the theory of multiple zeta values. In fact, Arakawa and the first-named author defined the two types of multiple L-values $L^{*}(k_1,\ldots,k_r; f_1,\ldots,f_r)$ of $*$-type and $L^{\text{ш}}(k_1,\ldots,k_r; f_1,\ldots,f_r)$ of ш-type associated to periodic functions $\{f_j\}$ (see [5]), defined by replacing $\{z_j^m\}$ by $\{f_j(m)\}$ and setting $(s_j) = (k_j) \in \mathbb{Z}_{\geq 1}^r$ on the right-hand sides of (5.1) and (5.2) for $(k_1,\ldots,k_r) \in \mathbb{Z}_{\geq 1}^r$. Note that

$$\mathrm{Li}^{*}_{s_1,\ldots,s_r}(z_1,\ldots,z_r) = \mathrm{Li}^{\text{ш}}_{s_1,\ldots,s_r}\Big(\prod_{j=1}^{r} z_j, \prod_{j=2}^{r} z_j, \ldots, z_{r-1}z_r, z_r\Big). \tag{5.3}$$

Definition 5.1 (Multi-indexed poly-Bernoulli numbers)**.** For $s_1,\ldots,s_r \in \mathbb{C}$ and $d \in \{1,2,\ldots,r\}$, the multi-indexed poly-Bernoulli numbers $\{B_{m_1,\ldots,m_r}^{(s_1,s_2,\ldots,s_r),(d)}\}$ are defined by

$$F(x_1,\ldots,x_r; s_1,\ldots,s_r; d) = \frac{\mathrm{Li}^{\text{ш}}_{s_1,\cdots,s_r}\left(1-e^{-\sum_{\nu=1}^{r} x_\nu},\ldots,1-e^{-x_{r-1}-x_r},1-e^{-x_r}\right)}{\prod_{j=1}^{d}\left(1-e^{-\sum_{\nu=j}^{r} x_\nu}\right)} \tag{5.4}$$

$$\left(= \sum_{l_1,\ldots,l_r=1}^{\infty} \frac{\prod_{j=1}^{r} \left(1 - e^{-\sum_{\nu=j}^{r} x_\nu}\right)^{l_j - \delta_j(d)}}{\prod_{j=1}^{r} \left(\sum_{\nu=1}^{j} l_\nu\right)^{s_j}} \right)$$
$$= \sum_{m_1,\ldots,m_r=0}^{\infty} B_{m_1,\ldots,m_r}^{(s_1,\ldots,s_r),(d)} \frac{x_1^{m_1} \cdots x_r^{m_r}}{m_1! \cdots m_r!},$$

where $\delta_j(d) = 1$ $(j \leq d)$, $= 0$ $(j > d)$.

Remark 5.2. Note that $\mathrm{Li}^{ш}_{k_1,\ldots,k_r}(z,\ldots,z) = \mathrm{Li}_{k_1,\ldots,k_r}(z)$ defined by (1.9). Suppose $x_1 = \cdots = x_{r-1} = 0$ and $(s_j) = (k_j) \in \mathbb{Z}^r$ in (5.4). We immediately see that if $d = 1$ then

$$B_{0,\ldots,0,m}^{(k_1,\ldots,k_r),(1)} = B_m^{(k_1,\ldots,k_r)} \qquad (m \in \mathbb{Z}_{\geq 0})$$

(see (1.6)), and if $d = r$ then

$$B_{0,\ldots,0,m}^{(k_1,\ldots,k_r),(r)} = \mathbb{B}_m^{(k_1,\ldots,k_r)} \qquad (m \in \mathbb{Z}_{\geq 0})$$

(see (1.8)).

Remark 5.3. Let

$$\Lambda_r = \{(x_1,\ldots,x_r) \in \mathbb{C}^r \mid |1 - e^{-\sum_{\nu=j}^{r} x_\nu}| < 1 \ (1 \leq j \leq r)\}. \tag{5.5}$$

Then we can see that

$$\mathrm{Li}^{ш}_{s_1,\cdots,s_r}\left(1 - e^{-\sum_{\nu=1}^{r} x_\nu}, \ldots, 1 - e^{-x_{r-1}-x_r}, 1 - e^{-x_r}\right) \quad (s_1,\ldots,s_r \in \mathbb{C})$$

is absolutely convergent for $(x_j) \in \Lambda_r$. Also $F(x_1,\ldots,x_r;s_1,\ldots,s_r;d)$ is absolutely convergent in the region $\Lambda_r \times \mathbb{C}^r$, so is holomorphic. Hence $B_{m_1,\ldots,m_r}^{(s_1,\ldots,s_r),(d)}$ is an entire function, because

$$B_{m_1,\ldots,m_r}^{(s_1,\ldots,s_r),(d)} = \left(\frac{\partial}{\partial x_1}\right)^{m_1} \cdots \left(\frac{\partial}{\partial x_r}\right)^{m_r} F(x_1,\ldots,x_r;s_1,\ldots,s_r;d)\bigg|_{(x_1,\ldots,x_r)=(0,\ldots,0)}$$

is holomorphic for all $(s_1,\ldots,s_r) \in \mathbb{C}^r$.

In the preceding section, we gave a certain duality formula for $B_m^{(k_1,\ldots,k_r)}$ (see Theorem 4.7). By the similar method, we can prove certain duality formulas for $B_{m_1,\ldots,m_r}^{(k_1,\ldots,k_r),(d)}$, though they may be complicated. Hence, in the rest of this section, we will consider the case $d = r$. For emphasis, we denote $B_{m_1,\ldots,m_r}^{(s_1,\ldots,s_r),(r)}$ by $\mathbb{B}_{m_1,\ldots,m_r}^{(s_1,\ldots,s_r)}$. Note that $\delta_j(r) = 1$ for any j. With this notation, we prove the following duality formulas.

Theorem 5.4. *For $m_1,\ldots,m_r$, $k_1,\ldots,k_r \in \mathbb{Z}_{\geq 0}$,*

$$\mathbb{B}_{m_1,\ldots,m_r}^{(-k_1,\ldots,-k_r)} = \mathbb{B}_{k_1,\ldots,k_r}^{(-m_1,\ldots,-m_r)}. \tag{5.6}$$

Now we aim to prove this theorem. First we generalize Lemma 4.1 as follows.

Lemma 5.5. *For $k_1,\ldots,k_r \in \mathbb{Z}_{\geq 0}$, there exists a polynomial $\widetilde{P}(x_1,\ldots,x_r;k_1,\ldots,k_r) \in \mathbb{Z}[x_1,\ldots,x_r]$ such that*

$$\mathrm{Li}^{*}_{-k_1,\ldots,-k_r}(z_1,\ldots,z_r) = \frac{\widetilde{P}(\prod_{j=1}^{r} z_j, \prod_{j=2}^{r} z_j, \ldots, z_{r-1}z_r, z_r; k_1,\ldots,k_r)}{\prod_{j=1}^{r} \left(1 - \prod_{\nu=j}^{r} z_\nu\right)^{\sum_{\nu=j}^{r} k_\nu + 1}}, \tag{5.7}$$

$$\deg_{x_j} \widetilde{P}(x_1, \ldots, x_r; k_1, \ldots, k_r) \le \sum_{\nu=j}^{r} k_\nu + 1, \tag{5.8}$$

$$(x_1 \cdots x_r) \mid \widetilde{P}(x_1, \ldots, x_r; k_1, \ldots, k_r). \tag{5.9}$$

Set $y_j = \prod_{\nu=j}^{r} z_\nu$ $(1 \le j \le r)$*. Then* (5.7) *implies*

$$\mathrm{Li}^{ш}_{-k_1,\ldots,-k_r}(y_1, \ldots, y_r) = \frac{\widetilde{P}(y_1, \cdots, y_r; k_1, \ldots, k_r)}{\prod_{j=1}^{r} (1 - y_j)^{\sum_{\nu=j}^{r} k_\nu + 1}}. \tag{5.10}$$

Proof. In order to prove this lemma, we have only to use the same method as in Lemma 4.1 by induction on r. Since the case of $r = 1$ is proven, we consider the case of $r \ge 2$. Further, when $K = k_1 + \cdots + k_r = 0$, it is easy to have the assertion. Hence we think about a general case $K = k_1 + \cdots + k_r (\ge 1)$. When $k_r = 0$, we have

$$\begin{aligned}
\mathrm{Li}^{*}_{-k_1,\ldots,-k_r}(z_1, \ldots, z_r) &= \frac{z_r}{1 - z_r} \mathrm{Li}^{*}_{-k_1,\ldots,-k_{r-1}}(z_1, \ldots, z_{r-2}, z_{r-1} z_r) \\
&= \frac{z_r}{1 - z_r} \frac{\widetilde{P}(\prod_{j=1}^{r} z_j, \ldots, z_{r-1} z_r; k_1, \ldots, k_{r-1})}{\prod_{j=1}^{r-1} \left(1 - \prod_{\nu=j}^{r} z_j\right)^{\sum_{\nu=j}^{r} k_\nu + 1}}.
\end{aligned}$$

Therefore, setting $\widetilde{P}(x_1, \cdots, x_r; k_1, \ldots, k_{r-1}, 0) = x_r \widetilde{P}(x_1, \ldots, x_{r-1}; k_1, \ldots, k_{r-1})$, we can verify (5.7)–(5.9).

Next we consider the case $k_r \ge 1$. For $k \in \mathbb{Z}_{\ge 0}$, we inductively define a subset $\{c^{(k)}_{j,\nu}\}_{0 \le j,\nu \le k+1}$ of $\mathbb{Z}$ by

$$\frac{d}{dz}\left(\sum_{m>l} m^k z^m\right) = \frac{1}{(1 - z)^{k+2}} \sum_{j=0}^{k+1} \sum_{\nu=0}^{k+1} c^{(k)}_{j,\nu} l^\nu z^{l+j}. \tag{5.11}$$

In fact, by

$$\frac{d}{dz}\left(\sum_{m>l} z^m\right) = \frac{1}{(1 - z)^2}\left(z^l + l z^l - l z^{l+1}\right),$$

and

$$\sum_{m>l} m^k z^m = z \frac{d}{dz}\left(\sum_{m>l} m^{k-1} z^m\right) \qquad (k \ge 1),$$

we can determine $\{c^{(k)}_{j,\nu}\}$ by (5.11). Using this notation, we have

$$\begin{aligned}
&\mathrm{Li}^{*}_{-k_1,\ldots,-k_r}(z_1, \ldots, z_r) = z_r \frac{d}{dz_r} \mathrm{Li}^{*}_{-k_1,\ldots,-k_r+1}(z_1, \ldots, z_r) \\
&= z_r \sum_{m_1 < \cdots < m_{r-1}} m_1^{k_1} \cdots m_{r-1}^{k_{r-1}} z_1^{m_1} \cdots z_{r-1}^{m_{r-1}} \frac{\sum_{j=0}^{k_r} \sum_{\nu=0}^{k_r} c^{(k_r-1)}_{j,\nu} m_{r-1}^{\nu} z_r^{m_{r-1}+j}}{(1 - z_r)^{k_r+1}} \\
&= \frac{1}{(1 - z_r)^{k_r+1}} \sum_{j=0}^{k_r} \sum_{\nu=0}^{k_r} c^{(k_r-1)}_{j,\nu} z_r^{j+1} \sum_{m_1 < \cdots < m_{r-1}} m_1^{k_1} \cdots m_{r-1}^{k_{r-1}+\nu} z_1^{m_1} \cdots (z_{r-1} z_r)^{m_{r-1}}.
\end{aligned}$$

By the induction hypothesis in the case $r - 1$, this is equal to

$$\frac{1}{(1 - z_r)^{k_r+1}} \sum_{j=0}^{k_r} \sum_{\nu=0}^{k_r} c^{(k_r-1)}_{j,\nu} z_r^{j+1} \frac{\widetilde{P}(\prod_{j=1}^{r-1} z_j, \ldots, z_{r-1} z_r; k_1, \ldots, k_{r-2}, k_{r-1} + \nu)}{\prod_{j=1}^{r-2} \left(1 - \prod_{\nu=j}^{r} z_j\right)^{\sum_{\nu=j}^{r} k_\nu + 1} (1 - z_{r-1} z_r)^{k_{r-1}+\nu+1}}.$$

Therefore we set

$$
\begin{aligned}
&\widetilde{P}(x_1,\cdots,x_r;k_1,\ldots,k_r)\\
&=\sum_{j=0}^{k_r}\sum_{\nu=0}^{k_r} c_{j,\nu}^{(k_r-1)} x_r^{j+1}(1-x_{r-1})^{k_r-\nu}\widetilde{P}(x_1,\cdots,x_{r-1};k_1,\ldots,k_{r-2},k_{r-1}+\nu).
\end{aligned}
$$

Then this satisfies (5.7)–(5.9). This completes the proof. □

From this result, we can reach the following definition.

Definition 5.6. For $k_1,\ldots,k_r\in\mathbb{Z}_{\geq 0}$, define

$$
\begin{aligned}
\eta(-k_1,\ldots,-k_r;s_1,\ldots,s_r)&=\frac{1}{\prod_{j=1}^r\Gamma(s_j)}\int_0^\infty\cdots\int_0^\infty\prod_{j=1}^r t_j^{s_j-1}\\
&\times\frac{\mathrm{Li}^{ш}_{-k_1,\ldots,-k_r}(1-e^{\sum_{\nu=1}^r t_\nu},\ldots,1-e^{t_{r-1}+t_r},1-e^{t_r})}{\prod_{j=1}^r(1-e^{\sum_{\nu=j}^r t_\nu})}\prod_{j=1}^r dt_j
\end{aligned}
\tag{5.12}
$$

for $s_1,\ldots,s_r\in\mathbb{C}$ with $\mathrm{Re}(s_j)>0$ $(1\leq j\leq r)$.

Lemma 5.5 ensures that the integral on the right-hand side of (5.12) is absolutely convergent for $\mathrm{Re}(s_j)>0$. By the same method as in the proof of Theorem 2.3 for $\eta(k_1,\ldots,k_r;s)$, we can similarly obtain the following.

Theorem 5.7. *For $k_1,\ldots,k_r\in\mathbb{Z}_{\geq 0}$, $\eta(-k_1,\ldots,-k_r;s_1,\ldots,s_r)$ can be analytically continued to an entire function on the whole complex space, and satisfies*

$$
\eta(-k_1,\ldots,-k_r;-m_1,\ldots,-m_r)=\mathbb{B}_{m_1,\ldots,m_r}^{(-k_1,\ldots,-k_r)}\quad(m_1,\ldots,m_r\in\mathbb{Z}_{\geq 0}).
\tag{5.13}
$$

Proof. As in the proof of Theorem 2.3, let

$$
\begin{aligned}
&H(-k_1,\ldots,-k_r;s_1,\ldots,s_r)\\
&=\int_{\mathcal{C}^r}\prod_{j=1}^r t_j^{s_j-1}\frac{\mathrm{Li}^{ш}_{-k_1,\ldots,-k_r}(1-e^{\sum_{\nu=1}^r t_\nu},\ldots,1-e^{t_r})}{\prod_{j=1}^r(1-e^{\sum_{\nu=j}^r t_\nu})}\prod_{j=1}^r dt_j\\
&=\prod_{j=1}^r(e^{2\pi i s_j}-1)\int_\varepsilon^\infty\cdots\int_\varepsilon^\infty\prod_{j=1}^r t_j^{s_j-1}\frac{\mathrm{Li}^{ш}_{-k_1,\ldots,-k_r}(1-e^{\sum_{\nu=1}^r t_\nu},\ldots,1-e^{t_r})}{\prod_{j=1}^r(1-e^{\sum_{\nu=j}^r t_\nu})}\prod_{j=1}^r dt_j\\
&\qquad+\int_{C_\varepsilon^r}\prod_{j=1}^r t_j^{s_j-1}\frac{\mathrm{Li}^{ш}_{-k_1,\ldots,-k_r}(1-e^{\sum_{\nu=1}^r t_\nu},\ldots,1-e^{t_r})}{\prod_{j=1}^r(1-e^{\sum_{\nu=j}^r t_\nu})}\prod_{j=1}^r dt_j,
\end{aligned}
\tag{5.14}
$$

where $\mathcal{C}^r$ is the direct product of the contour $\mathcal{C}$ defined before. Note that the integrand on the second member has no singularity on $\mathcal{C}^r$. It follows from Lemma 5.5 that $H(-k_1,\ldots,-k_r;s_1,\ldots,s_r)$ is absolutely convergent for any $(s_j)\in\mathbb{C}^r$, namely is entire. Suppose $\mathrm{Re}(s_j)>0$ for each j, the second integral tends to 0 as $\varepsilon\to 0$. Hence

$$
\eta(-k_1,\ldots,-k_r;s_1,\ldots,s_r)=\frac{1}{\prod_{j=1}^r(e^{2\pi i s_j}-1)\Gamma(s_j)}H(-k_1,\ldots,-k_r;s_1,\ldots,s_r),
$$

which can be analytically continued to $\mathbb{C}^r$. Also, setting $(s_1,\ldots,s_r)=(-m_1,\ldots,-m_r)\in\mathbb{Z}_{\leq 0}^r$ in (5.14), we obtain (5.13) from (5.4). This completes the proof. □

Next we directly construct the generating function of $\eta(-k_1,\ldots,-k_r;s_1,\ldots,s_r)$. We prepare the following two lemmas which we consider when (x_j) is in Λ_r defined by (5.5).

Lemma 5.8. *For $(s_j)\in\mathbb{C}^r$ with $\mathrm{Re}(s_j)>0$ $(1\le j\le r)$,*

$$\begin{aligned}(5.15)\qquad & F(x_1,\ldots,x_r;s_1,\ldots,s_r;r)\\ &=\frac{1}{\prod_{j=1}^r\Gamma(s_j)}\int_0^\infty\cdots\int_0^\infty\prod_{j=1}^r\left\{t_j^{s_j-1}\frac{e^{\sum_{\nu=j}^r x_\nu}}{1-e^{\sum_{\nu=j}^r x_\nu}(1-e^{\sum_{\nu=j}^r t_\nu})}\right\}\prod_{j=1}^r dt_j.\end{aligned}$$

Proof. Substituting $n^{-s}=(1/\Gamma(s))\int_0^\infty t^{s-1}e^{-nt}dt$ into the second member of (5.4), we have

$$\begin{aligned}F(\{x_j\};\{s_j\};r)=&\sum_{l_1,\ldots,l_r=1}^\infty\prod_{j=1}^r\left(1-e^{-\sum_{\nu=j}^r x_\nu}\right)^{l_j-1}\frac{1}{\prod_{j=1}^r\Gamma(s_j)}\\ &\times\int_0^\infty\cdots\int_0^\infty\prod_{j=1}^r\left\{t_j^{s_j-1}\exp\left(-(\sum_{\nu=1}^j l_\nu)t_j\right)\right\}\prod_{j=1}^r dt_j.\end{aligned}$$

We see that the integrand on the right-hand side can be rewritten as

$$\prod_{j=1}^r t_j^{s_j-1}\prod_{j=1}^r\exp\left(-l_j(\sum_{\nu=j}^r t_\nu)\right).$$

Hence we have

$$\begin{aligned}F(\{x_j\};\{s_j\};r)=&\frac{1}{\prod_{j=1}^r\Gamma(s_j)\left(1-e^{-\sum_{\nu=j}^r x_\nu}\right)}\int_0^\infty\cdots\int_0^\infty\prod_{j=1}^r t_j^{s_j-1}\\ &\times\sum_{l_1,\ldots,l_r=1}^\infty\prod_{j=1}^r\left(1-e^{-\sum_{\nu=j}^r x_\nu}\right)^{l_j}e^{-l_j(\sum_{\nu=j}^r t_\nu)}\prod_{j=1}^r dt_j\\ =&\frac{1}{\prod_{j=1}^r\Gamma(s_j)}\int_0^\infty\cdots\int_0^\infty\prod_{j=1}^r t_j^{s_j-1}\frac{e^{-\sum_{\nu=j}^r t_\nu}}{1-(1-e^{-\sum_{\nu=j}^r x_\nu})e^{-\sum_{\nu=j}^r t_\nu}}\prod_{j=1}^r dt_j\\ =&\frac{1}{\prod_{j=1}^r\Gamma(s_j)}\int_0^\infty\cdots\int_0^\infty\prod_{j=1}^r t_j^{s_j-1}\frac{e^{\sum_{\nu=j}^r x_\nu}}{1-e^{\sum_{\nu=j}^r x_\nu}(1-e^{\sum_{\nu=j}^r t_\nu})}\prod_{j=1}^r dt_j.\end{aligned}$$

This completes the proof. $\square$

Lemma 5.9. *Let $z_1,\ldots,z_r\in\mathbb{C}$ and assume that $|z_j|$ $(1\le j\le r)$ are sufficiently small. Then*

$$(5.16)\qquad \prod_{j=1}^r\frac{z_je^{\sum_{\nu=j}^r x_\nu}}{1-z_je^{\sum_{\nu=j}^r x_\nu}}=\sum_{k_1,\ldots,k_r=0}^\infty\mathrm{Li}^{ш}_{-k_1,\ldots,-k_r}(z_1,\ldots,z_r)\frac{x_1^{k_1}\cdots x_r^{k_r}}{k_1!\cdots k_r!}.$$

Set $z_j=1-e^{\sum_{\nu=j}^r t_\nu}$ $(1\le j\le r)$ for $(t_j)\in\Lambda_r$. Then

$$\begin{aligned}(5.17)\qquad & \prod_{j=1}^r\frac{e^{\sum_{\nu=j}^r x_\nu}\left(1-e^{\sum_{\nu=j}^r t_\nu}\right)}{1-e^{\sum_{\nu=j}^r x_\nu}\left(1-e^{\sum_{\nu=j}^r t_\nu}\right)}\\ &=\sum_{k_1,\ldots,k_r=0}^\infty\mathrm{Li}^{ш}_{-k_1,\ldots,-k_r}(1-e^{\sum_{\nu=1}^r t_\nu},\ldots,1-e^{t_r})\frac{x_1^{k_1}\cdots x_r^{k_r}}{k_1!\cdots k_r!}.\end{aligned}$$

In particular, the case $t_1=\cdots=t_{r-1}=0$ and $t_r=t$ implies (4.13).

Proof. We have only to prove (5.16). Actually we have

$$\begin{aligned}
&\sum_{k_1,\ldots,k_r=0}^{\infty} \mathrm{Li}^{ш}_{-k_1,\ldots,-k_r}(z_1,\ldots,z_r)\frac{x_1^{k_1}\cdots x_r^{k_r}}{k_1!\cdots k_r!}\\
&=\sum_{k_1,\ldots,k_r=0}^{\infty}\sum_{m_1,\ldots,m_r=1}^{\infty}\prod_{j=1}^{r}\frac{((\sum_{\mu=1}^{j}m_\mu)x_j)^{k_j}}{k_j!}z^{m_j}\\
&=\sum_{m_1,\ldots,m_r=1}^{\infty}\prod_{j=1}^{r}z_j^{m_j}\prod_{\mu=1}^{j}e^{m_\mu x_j}\\
&=\sum_{m_1,\ldots,m_r=1}^{\infty}\prod_{j=1}^{r}\left(z_j e^{\sum_{\nu=j}^{r}x_\nu}\right)^{m_j}=\prod_{j=1}^{r}\frac{z_j e^{\sum_{\nu=j}^{r}x_\nu}}{1-z_j e^{\sum_{\nu=j}^{r}x_\nu}}.
\end{aligned}$$

Thus we have the assertion. □

Using these lemmas, we obtain the following.

Theorem 5.10. *For $k_1,\ldots,k_r\in\mathbb{Z}_{\geq 0}$,*

$$\eta(-k_1,\ldots,-k_r;s_1,\ldots,s_r)=\mathbb{B}_{k_1,\ldots,k_r}^{(s_1,\ldots,s_r)}. \tag{5.18}$$

Proof. By Lemmas 5.8 and 5.9, we have

$$\begin{aligned}
&F(x_1,\ldots,x_r;s_1,\ldots,s_r;r)\\
&=\frac{1}{\prod_{j=1}^{r}\Gamma(s_j)}\int_0^\infty\cdots\int_0^\infty\prod_{j=1}^{r}\left\{t_j^{s_j-1}\frac{e^{\sum_{\nu=j}^{r}x_\nu}}{1-e^{\sum_{\nu=j}^{r}x_\nu}(1-e^{\sum_{\nu=j}^{r}t_\nu})}\right\}\prod_{j=1}^{r}dt_j\\
&=\frac{1}{\prod_{j=1}^{r}\Gamma(s_j)}\sum_{k_1,\ldots,k_r=0}^{\infty}\left\{\int_0^\infty\cdots\int_0^\infty\prod_{j=1}^{r}t_j^{s_j-1}\right.\\
&\qquad\times\left.\frac{\mathrm{Li}^{ш}_{-k_1,\ldots,-k_r}(1-e^{\sum_{\nu=1}^{r}t_\nu},\ldots,1-e^{t_r})}{\prod_{j=1}^{r}\left(1-e^{\sum_{\nu=j}^{r}t_\nu}\right)}\prod_{j=1}^{r}dt_j\right\}\frac{x_1^{k_1}\cdots x_r^{k_r}}{k_1!\cdots k_r!}
\end{aligned}\tag{5.19}$$

for $\mathrm{Re}(s_j)>0$ $(1\leq j\leq r)$. Combining (5.4), (5.12) and (5.19), we obtain (5.18) for $\mathrm{Re}(s_j)>0$ $(1\leq j\leq r)$, hence for all $(s_j)\in\mathbb{C}$, because both sides of (5.18) are entire functions (see Remark 5.3). □

Proof of Theorem 5.4. Setting $(s_1,\ldots,s_r)=(-m_1,\ldots,-m_r)$ in (5.18), we obtain (5.6) from (5.13). This completes the proof of Theorem 5.4. □

Example 5.11. We can easily see that

$$\mathrm{Li}^{ш}_{-1,0}(z_1,z_2)=\frac{z_1z_2}{(1-z_1)^2(1-z_2)},\quad \mathrm{Li}^{ш}_{0,-1}(z_1,z_2)=\frac{z_1z_2(2-z_1-z_2)}{(1-z_1)^2(1-z_2)^2}.$$

Hence we have

$$\mathbb{B}_{m,n}^{(-1,0)}=2^m3^n,\quad \mathbb{B}_{m,n}^{(0,-1)}=(2^m+1)3^n\quad (m,n\in\mathbb{Z}_{\geq 0}).$$

Therefore $\mathbb{B}_{0,1}^{(-1,0)}=\mathbb{B}_{1,0}^{(0,-1)}=3$. Similarly we obtain, for example,

$$\mathbb{B}_{1,0}^{(-1,-2)}=\mathbb{B}_{1,2}^{(-1,0)}=18,\ \mathbb{B}_{1,2}^{(-3,-1)}=\mathbb{B}_{3,1}^{(-1,-2)}=1820,\ \mathbb{B}_{2,2}^{(-2,-1)}=\mathbb{B}_{2,1}^{(-2,-2)}=1958.$$

Remark 5.12. Hamahata and Masubuchi [14, Corollary 10] showed the special case of (5.6), namely

$$\mathbb{B}_{0,\ldots,0,m}^{(0,\ldots,0,-k)} = \mathbb{B}_{0,\ldots,0,k}^{(0,\ldots,0,-m)} \quad (m, k \in \mathbb{Z}_{\geq 0})$$

(see Remark 5.2). On the other hand, Theorem 4.7 corresponds to the case $d = 1 \neq r$ except for $r = 1$ (see Remark 5.2), hence is located in the outside of Theorem 5.4. Therefore, in (4.12), another type of multi-poly-Bernoulli numbers appear.

References

[1] S. Akiyama, S. Egami and Y. Tanigawa, Analytic continuation of multiple zeta functions and their values at non-positive integers, Acta Arith., **98** (2001), 107–116.

[2] K. Aomoto, Special values of hyperlogarithms and linear difference schemes, Illinois J. Math., **34-2** (1990), 191–216.

[3] T. Arakawa, T. Ibukiyama and M. Kaneko, *Bernoulli Numbers and Zeta Functions*, Springer, Tokyo, 2014.

[4] T. Arakawa and M. Kaneko, Multiple zeta values, poly-Bernoulli numbers, and related zeta functions, Nagoya Math. J., **153** (1999), 189–209.

[5] T. Arakawa and M. Kaneko, On multiple L-values, J. Math. Soc. Japan, **56** (2004), 967–991.

[6] A. Bayad and Y. Hamahata, Polylogarithms and poly-Bernoulli polynomials, Kyushu J. Math., **65** (2011), 15–24.

[7] A. Bayad and Y. Hamahata, Arakawa-Kaneko L-functions and generalized poly-Bernoulli polynomials, J. Number Theory, **131** (2011), 1020–1036.

[8] A. Bayad and Y. Hamahata, Multiple polylogarithms and multi-poly-Bernoulli polynomials, Funct. Approx. Comment. Math., **46** (2012), 45–61.

[9] C. Brewbaker, A combinatorial interpretation of the Poly-Bernoulli numbers and two Fermat analogues, Integers, **8** (2008), ♯A02.

[10] M. -A. Coppo and B. Candelpergher, The Arakawa-Kaneko zeta function, Ramanujan J., **22** (2010), 153–162.

[11] V. G. Drinfel'd, On quasitriangular quasi-Hopf algebras and a group closely connected with $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, Leningrad Math. J. **2** (1991), 829–860.

[12] H. Furusho, p-adic multiple zeta values. I. p-adic multiple polylogarithms and the p-adic KZ equation, Invent. Math., **155** (2004), 253–286.

[13] A. B. Goncharov, Multiple polylogarithms, cyclotomy, and modular complexes, Math. Res. Lett., **5** (1998), 497–516.

[14] Y. Hamahata and H. Masubuchi, Special multi-poly-Bernoulli numbers, J. Integer Seq., **10** (2007), Article 07.4.1.

[15] Y. Hamahata and H. Masubuchi, Recurrence formulae for multi-poly-Bernoulli numbers, Integers, **7** (2007), ♯ A46.11.

[16] M. Hoffman, Multiple harmonic series, Pacific J. Math., **152** (1992), 275–290.

[17] K. Imatomi, Multi-poly-Bernoulli-star numbers and finite multiple zeta-star values, Integers, **14** (2014), A51.

[18] K. Imatomi, M. Kaneko and E. Takeda, Multi-poly-Bernoulli numbers and finite multiple zeta values, J. Integer Sequences, **17** (2014), Article 14.4.5.

[19] K. Ihara, M. Kaneko and D. Zagier, Derivation and double shuffle relations for multiple zeta values, Compositio Math., **142** (2006), 307–338.

[20] M. Kaneko, Poly-Bernoulli numbers, J. Théor. Nombres Bordeaux, **9** (1997), 199–206.

[21] M. Kaneko, Poly-Bernoulli numbers and related zeta functions, *Algebraic and Analytic Aspects of Zeta Functions and L-functions*, MSJ Mem., **21**, pp. 73–85, Math. Soc. Japan, Tokyo, 2010.

[22] M. Kaneko and M. Sakata, Notes on height one multiple zeta values, in preparation.

[23] T. Q. T. Le and J. Murakami, Kontsevich's integral for the Homfly polynomial and relations between values of multiple zeta functions, Topology Appl., **62** (1995), 193–206.

[24] Y. Ohno, A generalization of the duality and sum formulas on the multiple zeta values, J. Number Theory, **74** (1999), 39–43.

[25] S. Oi, Gauss hypergeometric functions, multiple polylogarithms, and multiple zeta values, Publ. Res. Inst. Mat. Sci. **45** (2009), 981–1009.

[26] J. Okuda and K. Ueno, Relations for multiple zeta values and Mellin transforms of multiple polylogarithms, Publ. Res. Inst. Math. Sci., **40** (2004), 537–564.

[27] J. Okuda and K. Ueno, The sum formula for multiple zeta values and connection problem of the formal Knizhnik-Zamolodchikov equation, *Zeta Functions, Topology and Quantum Physics*, Dev. Math., **14**, Springer, New York, (2005), 145–70.

[28] Y. Sasaki, On generalized poly-Bernoulli numbers and related L-functions, J. Number Theory, **132** (2012), 156–170.

[29] G. Shimura, *Elementary Dirichlet Series and Modular Forms*, Springer Monographs in Mathematics, Springer, New York, 2007.

[30] L. C. Washington, *Introduction to Cyclotomic Fields*, Second edition, Graduate Texts in Mathematics **83**, Springer-Verlag, New York, 1997.

[31] P. T. Young, Symmetries of Bernoulli polynomial series and Arakawa-Kaneko zeta functions, J. Number Theory, **143** (2014), 142–161.

M. Kaneko: Faculty of Mathematics, Kyushu University, Motooka 744, Nishi-ku Fukuoka 819-0395, Japan
Tel & Fax: 092-802-4443
e-mail: `mkaneko@math.kyushu-u.ac.jp`

H. Tsumura: Department of Mathematics and Information Sciences, Tokyo Metropolitan University, 1-1, Minami-Ohsawa, Hachioji, Tokyo 192-0397 Japan
e-mail: `tsumura@tmu.ac.jp`