# First order semi-local invariants of stable maps of 3-manifolds into the plane

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# FIRST ORDER SEMI-LOCAL INVARIANTS OF STABLE MAPS OF 3-MANIFOLDS INTO THE PLANE

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ABSTRACT. In the late 1980's, Vassiliev introduced new graded numerical invariants of knots, which are now called Vassiliev invariants or finite-type invariants. Since his definition, many people have been trying to construct Vassiliev type invariants for various mapping spaces. In the early 1990's, Arnold and Goryunov introduced the notion of first order (local) invariants of *stable maps*.

In this paper, we define and study *first order semi-local invariants* of stable maps and those of *stable fold maps* of a closed orientable 3-dimensional manifold into the plane. Here, a stable fold map is a stable map with only fold singular points and a first order semi-local invariant is an isotopy invariant which is constructed by looking at the singular value set locally and the singular fibers semi-locally. We show that there are essentially seven first order semi-local invariants. For a stable map, six of them count the number of *singular fibers* of a given type which appear discretely (there are exactly six types of such singular fibers), and the other one is the *rotation number* of the singular value set. Besides these invariants, for stable fold maps, the *Bennequin invariant* of the singular value set corresponding to definite fold points is also a first order semi-local invariant. Our study of codimension 1 unstable fold maps provides invariants for the connected components of the set of all fold maps.

CONTENTS



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## 1. Introduction

1.1. **History.** Vassiliev [50] introduced a wonderful method to define graded numerical invariants of knots, which are now called *Vassiliev invariants* or *finite-type invariants*. He constructed these invariants by carefully studying a certain stratification of the mapping space $C^\infty(S^1, \mathbf{R}^3)$. Since his definition, many people have been trying to construct Vassiliev type invariants for various mapping spaces. Arnold [4] introduced "basic invariants" (we call them *Arnold invariants*) for stable immersions of $S^1$ into $\mathbf{R}^2$, which brought a new insight to the classical subject of the topology/geometry of plane curves. Arnold invariants are regarded as a special kind of first order Vassiliev type invariants, which are objects of great interest by themselves. Arnold invariants of plane curves (and those of wavefronts) were studied by many authors, for example, Aicardi [2], Goryunov [17], Tchernov [46, 47], etc. The construction of this kind of order one invariants may work for stable maps of manifolds whose dimensions are greater than 1. In fact, as a kind of a generalization of the $J^{\pm}$-invariants (not involving the strangeness invariant $St$), Goryunov [16] introduced and studied first order *local* invariants of stable maps of an oriented closed surface into $\mathbf{R}^3$. Aicardi and Ohmoto [3] worked on first order local invariants of stable maps of a closed surface into $\mathbf{R}^2$ (see also [39]). It should be remarked that in both cases, these first order "local" invariants are determined by numerical invariants of discrete critical sets and a certain Bennequin invariant of the critical value set (note that this is related to the $J^+$-theory of plane curves). See Remark 6.5 for the other results about Vassiliev (finite) type invariants. In these works, almost all invariants are essentially reduced to order one invariants.

1.2. **Purpose.** In this paper, we consider the case where the source manifolds are closed orientable 3-dimensional manifolds and the target manifold is the plane. In all the cases mentioned in the previous subsection, the dimensions of the target manifolds are greater than or equal to those of the source manifolds. Thus for any point in the target manifold, the inverse image of this point consists of a finite number of points, provided that the map is proper and generic enough. Hence, in order to study first order (local) invariants of such stable maps, we have only to consider multi-germs along zero dimensional sets. However, if the dimension of the source manifold is strictly greater than that of the target manifold, then the inverse image of a point (or the fiber over a point) is no longer a discrete set. In general, this forms a complex of positive dimension. Hence, if we consider multi-germs only along singular points in a fiber to study first order invariants of stable maps, then it is expected that little information about stable maps appears in these invariants. Thus, to get much information about stable maps from first order invariants, we need to study map germs along a whole fiber of positive dimension.

We define and study *first order semi-local invariants* of such stable maps. A first order semi-local invariant is a special kind of a first order invariant: when a homotopy in the mapping space crosses a codimension 1 stratum transversely at a codimension 1 unstable map, the jump of the invariant is determined by the homeomorphism type of the local deformation of the singular value set near the codimension 1 singular value and by the diffeomorphism types of associated singular fibers. Note that the notion of a diffeomorphism of singular fibers modulo regular components was first used implicitly by Kushner, Levine and Porto [26, 29]. After that Saeki [45] gave a precise definition of this notion (including regular components).

This is the first study of first order invariants when the dimension of the source manifold is strictly greater than that of the target manifold, as long as the author knows.

1.3. **Organization of the paper.** The paper is organized as follows.

In Section 2, we review the classification of multi-germs $(\mathbf{R}^3, S) \to (\mathbf{R}^2, y)$ up to $\mathcal{A}$-equivalence (i.e., $C^\infty$ right-left equivalence), where $S$ is a set of finitely many isolated points. We list the $\mathcal{A}$-equivalence classes of miniversal unfoldings of such multi-germs whose parameter spaces have dimensions 0, 1 or 2. They were studied by Rieger-Ruas [41], Gibson-Hobbs [14], Nabarro [35] and Rieger [40], and we use their results in this paper. We have to add two exceptional $\mathcal{A}$-equivalence classes of miniversal unfoldings of multi-germs which correspond to $D_4^\pm$ or a quadruplefold. The reason is as follows. Miniversal unfoldings of these multi-germs have parameter spaces of dimension 3. However, their $\mathcal{A}$-modalities are all equal to 1. On the parameter space $\mathbf{R}^3$ of their miniversal unfoldings, one coordinate $t$ of the coordinates $(a, b, t) \in \mathbf{R}^3$ corresponds to the $\mathcal{A}$-modality. Thus $D_4^\pm$ and a quadruplefold are considered to be 1-parameter families of $\mathcal{A}$-equivalence classes. To obtain first order invariants, we have to consider each such 1-parameter family to constitute a stratum, and the codimension of each such stratum is equal to 2. For details, see [40] and Subsection 2.2.

In Section 3, we define stable maps and unstable maps of codimensions 1 and 2. By using the classification of miniversal unfoldings of Section 2, we study local deformations of singular value sets and the associated local singular fibers near singular points.

In Section 4, we first define the notion of the *weak equivalence* for *singular fibers* (preimages of singular values). This equivalence relation reflects homeomorphism types of (local deformations of) singular value sets and diffeomorphism types of the associated semi-local singular fibers. This equivalence relation is related to the $C^\infty$ equivalence for map germs along singular fibers. See [45] for the definition of the $C^\infty$ equivalence. Since our equivalence is weaker than the $C^\infty$ equivalence, we use the term "weak" for our equivalence relation. We classify singular fibers of stable maps and unstable maps of codimensions 1 and 2 up to this equivalence relation (see Theorems 4.4, 4.7 and 4.8). For a stable map, there are exactly six weak equivalence classes of singular fibers which appear discretely. Then, we define the equivalence relation, which we also call weak equivalence for simplicity, for unstable maps of codimensions 1 and 2 and classify them up to this equivalence relation. We then define the coorientation of each weak equivalence class of codimension 1 unstable maps by looking at the local deformation of its singular value set and the associated (semi-local) singular fibers.

In Section 5, we construct the *Vassiliev cochain complex* for the weak equivalence classes of unstable maps of codimensions 1 and 2. (As general references about the Vassiliev cochain complex, see [23, 49].)

In Section 6, we define first order semi-local invariants of stable maps. These invariants are constructed from the cocycles of the Vassiliev cochain complex mentioned above and they are isotopy invariants of stable maps.

In Section 7, we determine the first order semi-local invariants of stable maps by using the Vassiliev cochain complex constructed in Section 5 (see Theorem 7.2). More precisely, we show that there are essentially seven first order semi-local invariants of stable maps. By a careful study of homotopies which intersect codimension 1 strata (the hypersurface in the mapping space which consists of the codimension 1 unstable maps) transversely, we give geometric interpretations of all the invariants. It turns out that for a stable map, six of them count the number of singular fibers of a given weak equivalence class which appear discretely (there are exactly six such weak equivalence classes by Theorem 4.4) and the other one is the "rotation number" of the singular value set (see Theorem 7.3). Then we construct several explicit examples of stable maps $f : S^3 \to \mathbf{R}^2$. By using these examples, we show that the above seven first order semi-local invariants together with a (non-zero) constant invariant are linearly independent for stable maps of an arbitrary closed orientable 3-dimensional manifold.

Note that these types of results would be impossible if we used the multi-germ of a given map only along the singular points in a fiber instead of considering the map germ along a whole singular fiber.

In Section 8, we subdivide the weak equivalence classes of unstable maps of codimensions 1 and 2 by using a global property of such maps. To subdivide them, we look at their singular value sets globally. By using such a finer classification, we give an additional first order invariant of stable maps, which is a non-local invariant, and give a geometric interpretation of this invariant (see Propositions 8.1 and 8.2).

In Section 9, we consider the space of all *fold maps*. A fold map is a smooth map with only fold singular points. By using the Vassiliev cochain complex for the weak equivalence classes of unstable fold maps of codimensions 1 and 2, we determine the first order semi-local invariants of *stable fold maps* and give geometric interpretations of all the invariants (see Theorems 9.3 and 9.4).

By combining our results with other results about first order invariants which are already known, we may conjecture that first order invariants can give information only about the 0-dimensional strata of the critical value set, endowed with the topology of the associated fibers, or the topology of the critical value set (e.g. the rotation number or the Bennequin invariant).

In Section 10, we give several invariants for the connected components of the space of all fold maps. These invariants are obtained by a careful study of codimension 1 unstable fold maps carried out in the previous sections.

Throughout the paper, all manifolds and maps are differentiable of class $C^\infty$.

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## 2. Classification of multi-germs

In this section, we quickly review the classification of multi-germs by $\mathcal{A}$-equivalence (that is, $C^\infty$ right-left equivalence).

2.1. **$\mathcal{A}$-equivalence of multi-germs.** In this subsection, we review some fundamental concepts and results from singularity theory. For details, see [5, 10, 39, 51].

Let $f : (M, S) \to (\mathbf{R}^2, y)$ be a multi-germ at finitely many isolated points $S$ of $f^{-1}(y)$, where $M$ is a 3-dimensional manifold. When $S$ consists exactly of one point, we also say that $f$ is a *mono-germ*. An *unfolding* of such a multi-germ $f : (M, S) \to (\mathbf{R}^2, y)$ with parameter space $\mathbf{R}^s$ centered at $t_0 \in \mathbf{R}^s$ means a multi-germ $F : (M \times \mathbf{R}^s, S \times \{t_0\}) \to (\mathbf{R}^2, y)$ such that $F(x, t_0) = f(x)$.

Let $M_i$ be 3-dimensional manifolds ($i = 1, 2$). Let $f_i : (M_i, S_i) \to (\mathbf{R}^2, y_i)$ be multi-germs and $F_i : (M_i \times \mathbf{R}^s, S_i \times \{t_i\}) \to (\mathbf{R}^2, y_i)$ unfoldings of $f_i$ with parameter space $\mathbf{R}^s$ centered at $t_i$ ($i = 1, 2$). We say that $F_1$ and $F_2$ are *$\mathcal{A}$-equivalent* if there exist a diffeomorphism germ $\varphi : (\mathbf{R}^s, t_1) \to (\mathbf{R}^s, t_2)$, and unfoldings $\widetilde{R} : (M_1 \times \mathbf{R}^s, S_1 \times \{t_1\}) \to (M_2, S_2)$ and $\widetilde{L} : (\mathbf{R}^2 \times \mathbf{R}^s, (y_1, t_1)) \to (\mathbf{R}^2, y_2)$ of diffeomorphism germs $R : (M_1, S_1) \to (M_2, S_2)$ and $L : (\mathbf{R}^2, y_1) \to (\mathbf{R}^2, y_2)$ respectively, such that the following diagram is commutative:

$$\begin{array}{ccccc}
(M_1 \times \mathbf{R}^s, S_1 \times \{t_1\}) & \xrightarrow{\widetilde{F_1}} & (\mathbf{R}^2 \times \mathbf{R}^s, (y_1, t_1)) & \xrightarrow{\pi} & (\mathbf{R}^s, t_1) \\
{\scriptstyle (\widetilde{R}, \varphi)} \downarrow & & {\scriptstyle (\widetilde{L}, \varphi)} \downarrow & & {\scriptstyle \varphi} \downarrow \\
(M_2 \times \mathbf{R}^s, S_2 \times \{t_2\}) & \xrightarrow{\widetilde{F_1}} & (\mathbf{R}^2 \times \mathbf{R}^s, (y_2, t_2)) & \xrightarrow{\pi} & (\mathbf{R}^s, t_2).
\end{array}$$

Here, $\pi$ is the projection to the second factor, $(\widetilde{R}, \varphi)$ (or $(\widetilde{L}, \varphi)$) is defined by $(\widetilde{R}, \varphi)(x, t) = (\widetilde{R}(x, t), \varphi(t))$ (resp. by $(\widetilde{L}, \varphi)(y, t) = (\widetilde{L}(y, t), \varphi(t))$), and the map $\widetilde{F_i}$, $i = 1, 2$, is defined by $\widetilde{F_i}(x, t) = (F_i(x, t), t)$.

Two unfoldings $F_1$ and $F_2$ of $f : (M, S) \to (\mathbf{R}^2, y)$ with the same parameter space $\mathbf{R}^s$ are said to be *$f$-isomorphic* if $F_1$ and $F_2$ are $\mathcal{A}$-equivalent with $R$ and $L$ being the identity multi-germs $\mathrm{id}_M$ and $\mathrm{id}_{\mathbf{R}^2}$ respectively.

Let $F : (M \times \mathbf{R}^{s_1}, S \times \{t_f\}) \to (\mathbf{R}^2, y)$ be an unfolding of $f : (M, S) \to (\mathbf{R}^2, y)$ and $g : (\mathbf{R}^{s_2}, t_g) \to (\mathbf{R}^{s_1}, t_f)$ a smooth map germ. We define the induced unfolding $g^*F : (M \times \mathbf{R}^{s_2}, S \times \{t_g\}) \to (\mathbf{R}^2, y)$ by $g^*F(x, w) = F(x, g(w))$, which is also an unfolding of $f$. An unfolding $F$ of $f$ is called a *universal unfolding* if any unfolding $G$ of $f$ is $f$-isomorphic to an unfolding induced from $F$. A universal unfolding of a multi-germ $f$ is called a *miniversal unfolding* if the parameter space has the minimal dimension among all universal unfoldings of $f$.

For a multi-germ $f : (M, S) \to (\mathbf{R}^2, y)$, let $\theta(f)_S$ denote the set of $C^\infty$ vector fields along $f$. That is, it is the set of multi-germs $\zeta : (M, S) \to T\mathbf{R}^2$ such that $\zeta(x) \in T\mathbf{R}^2_{f(x)}$ $(x \in M)$. We set $\theta(M)_S = \theta(\mathrm{id}_M)_S$ and $\theta(\mathbf{R}^2)_y = \theta(\mathrm{id}_{\mathbf{R}^2})_y$. The two maps $tf : \theta(M)_S \to \theta(f)_S$ and $\omega f : \theta(\mathbf{R}^2)_y \to \theta(f)_S$ are defined by $tf(\xi) = df \circ \xi$ and $\omega f(\eta) = \eta \circ f$ respectively. The *extended tangent space* $T\mathcal{A}_e f$ is defined by

$$T\mathcal{A}_e f = tf(\theta(M)_S) + \omega f(\theta(\mathbf{R}^2)_y) \subset \theta(f)_S$$

and the dimension of the quotient vector space $\theta(f)_S/T\mathcal{A}_e f$ is called the $\mathcal{A}_e$*-codimension* of $f$.

Note that if the $\mathcal{A}_e$-codimension of $f$ is finite, then it admits a universal unfolding and the $\mathcal{A}_e$-codimension coincides with the dimension of the parameter space of a miniversal unfolding of $f$ (see [39, 51]). A multi-germ $f : (M, S) \to (\mathbf{R}^2, y)$ is said to be $\mathcal{A}_e$*-finite* if the $\mathcal{A}_e$-codimension of $f$ is finite. It should be noted that every $\mathcal{A}_e$-finite multi-germ is finitely determined. That is, its $\mathcal{A}$-equivalence class is determined by its jet of finite order, and hence it is represented by a polynomial multi-germ (see [51]).

2.2. **Classification of multi-germs.** Let us consider the classification of those $\mathcal{A}_e$-finite multi-germs whose $\mathcal{A}_e$-codimension minus $\mathcal{A}$-modality is strictly less than three. In the following, let $m_A(f) \in \mathbf{Z}$ denote the value of ($\mathcal{A}_e$-codimension) $-$ ($\mathcal{A}$-modality) for $f$. The *modality* is defined as follows. Suppose that a Lie group $G$ acts on a variety $V$, then a theorem of Rosenlicht [42] implies that $V$ has a uniquely determined finite stratification $\mathcal{S}$ such that the action of $G$ on each stratum $S$ defines a fibration $S \to S/G$. If a point $p \in V$ is in a stratum $S$ such that dim $S/G = m$, then we say that the *modality* of $p \in V$ is equal to $m \in \mathbf{Z}_{\geq 0}$, where $\mathbf{Z}_{\geq 0}$ is the set of non-negative integers. The $\mathcal{A}$-modality of an $\mathcal{A}_e$-finite mono-germ $f : (\mathbf{R}^3, x) \to (\mathbf{R}^2, y)$ is the modality of an $\mathcal{A}$-sufficient jet $j^k f$ in $J^k(3,2)_{x,y}$ under the action of the Lie group $\mathcal{A}^k$ of $k$-jets of elements of $\mathcal{A}$. If $f$ is not a mono-germ, then we can define the $\mathcal{A}$-modality similarly. For details, see [40, 52].

Let $f : (\mathbf{R}^3, S) \to (\mathbf{R}^2, 0)$ be an $\mathcal{A}_e$-finite multi-germ, where $S$ is a set of finitely many isolated points of $f^{-1}(0)$. To determine first order invariants of stable maps, we may assume that $m_A(f)$ is equal to 0, 1 or 2.

For $f$ with $m_A(f) = 0, 1$ or 2, the $\mathcal{A}$-equivalence classification of mono-germs and their miniversal unfoldings has been obtained by Nabarro [35], Rieger [40], and Rieger-Ruas [41]. The $\mathcal{A}$-equivalence classification of multi-germs and their miniversal unfoldings has been studied by Gibson-Hobbs [14]. In fact, they considered multi-germs $f : (\mathbf{R}^2, S) \to$

$(\mathbf{R}^2, 0)$ and classified them by $\mathcal{A}$-equivalence. We can use similar arguments for multi-germs $f : (\mathbf{R}^3, S) \to (\mathbf{R}^2, 0)$ and obtain the required $\mathcal{A}$-equivalence classification.

Let $f : (\mathbf{R}^3, S) \to (\mathbf{R}^2, 0)$ be a multi-germ. We put $S(f) = \{q \in \mathbf{R}^3 \mid \text{rank } df_q < 2\}$ and call it the *singular set germ* of $f$. Furthermore, we call $f(S(f))$ the *singular value set germ* of $f$.

Let $f : (\mathbf{R}^3, S) \to (\mathbf{R}^2, 0)$ be a multi-germ such that $S \subset S(f)$ and $m_A(f) \leq 2$. Then we have $S = \{q_1, \ldots, q_k\}$, $1 \leq k \leq 4$, and there exist local coordinates $(x_i, y_i, z_i)$ and $(X, Y)$ around $q_i \in \mathbf{R}^3$, $1 \leq i \leq k$, and $f(q_i) = 0 \in \mathbf{R}^2$ respectively such that a miniversal unfolding of $f$ is expressed by one of the polynomials listed in Tables 1–5 with respect to the local coordinates.

Table 1 Table 2 Table 3 Table 4 Table 5

We remark that by [14, 35, 40, 41], Tables 1–5 give the complete list of $\mathcal{A}$-equivalence classes of multi-germs $f : (\mathbf{R}^3, S) \to (\mathbf{R}^2, 0)$ such that $S \subset S(f)$ and $m_A(f) \leq 2$. In our situation, the $\mathcal{A}_e$-codimensions of $D_4^{\pm}$ and a quadruplefold are equal to 3 and their $\mathcal{A}$-modalities are equal to 1. For the miniversal unfoldings for these cases $t$ is the parameter of modality in Tables 3 and 5. For the other classes, their $\mathcal{A}$-modalities are equal to 0.

Let $f : (\mathbf{R}^3, S) \to (\mathbf{R}^2, 0)$ be a stable germ in Table 1. Then using the local normal forms in Table 1, we see that the singular value set germ $f(S(f))$ around 0 is as depicted in Figure 1, where (1) corresponds to a fold point, (2) corresponds to a cusp point, and (3) corresponds to a nodefold.

Figure 1

Let $G : (\mathbf{R}^3 \times \mathbf{R}, S \times \{0\}) \to (\mathbf{R}^2, 0)$ be a 1-parameter unfolding in Table 2. We define $g_t : \mathbf{R}^3 \to \mathbf{R}^2$ by $g_t(q) = G(q, t)$. Suppose that $0 \in g_0(S(g_0))$ and $S \subset S(g_0)$. Then using the local normal forms in Table 2, we see that the deformations of the singular value set germ $g_t(S(g_t))$ around 0 are as depicted in Figure 2, where (1) corresponds to lips, (2) corresponds to beaks, (3) corresponds to a swallowtail, (4) corresponds to a cusp-plus-fold, (5) corresponds to a tacnodefold and (6) corresponds to a triplefold.

Figure 2

Let $H : (\mathbf{R}^3 \times \mathbf{R}^2, S \times \{(0,0)\}) \to (\mathbf{R}^2, 0)$ be a 2-parameter unfolding in Tables 3 or 4 other than $D_4^{\pm}$. We define $h_{a,b} : \mathbf{R}^3 \to \mathbf{R}^2$ by $h_{a,b}(q) = H(q, a, b)$. Suppose that $0 \in h_{0,0}(S(h_{0,0}))$ and $S \subset S(h_{0,0})$. Then using the local normal forms in Tables 3 or 4, we see that the deformations of the singular value set germ $h_{a,b}(S(h_{a,b}))$ around 0 are as depicted in Figure 3. Let $H : (\mathbf{R}^3 \times \mathbf{R}^3, S \times \{(0,0,0)\}) \to (\mathbf{R}^2, 0)$ be a 3-parameter unfolding in Tables 3 or 5 which corresponds to $D_4^{\pm}$ or a quadruplefold. We fix $t = t_0 \in \mathbf{R}$ in the corresponding local normal form and define $h_{a,b} : \mathbf{R}^3 \to \mathbf{R}^2$ by $h_{a,b}(q) = H(q, a, b, t_0)$. Suppose that $0 \in h_{0,0}(S(h_{0,0}))$ and $S \subset S(h_{0,0})$. Then using the local normal forms, we see that the deformations of the singular value set germ $h_{a,b}(S(h_{a,b}))$ around 0 are as depicted in Figure 3. In Figure 3, (1) corresponds to a goose, (2) corresponds to a butterfly, (3) corresponds to gulls, (4) corresponds to $D_4^+$, (5) corresponds to $D_4^-$, (6) corresponds to a lips-plus-fold, (7) corresponds to a beaks-plus-fold, (8) corresponds to a swallowtail-plus-fold, (9) corresponds to a cusp-plus-cusp, (10) corresponds to a cusp-plus-fold tangency, (11) corresponds to a flecnodefold, (12) corresponds to a nodefold-plus-cusp, (13) corresponds to a tacnodefold-plus-fold, and (14) corresponds to a quadruplefold.

Figure 3

In Figure 3, on each 2-dimensional region $R$ of the parameter space, we have depicted $h_{a,b}(S(h_{a,b})) \subset \mathbf{R}^2$ for $(a,b) \in R$. Some parameter spaces in Figure 3 may not strictly coincide with the corresponding $(a,b)$-plane for $H$ in Tables 3–5. For each of these cases, we need to compose an orientation preserving homeomorphism on the $(a,b)$-plane to obtain the corresponding parameter space in Figure 3.

## 3. Stable maps and unstable maps of codimensions one and two

In this section, we define *stable maps* and *unstable maps of codimensions* 1 and 2 by using the $\mathcal{A}$-equivalence classification of multi-germs as in Tables 1–5. We also study local behaviors of their singular value sets and their singular fibers.

Let $f : M \to \mathbf{R}^2$ be a smooth map. We set $S(f) = \{q \in M \mid \text{rank } df_q < 2\}$ and call it the *singular set* of $f$. Furthermore, we call $f(S(f))$ the *singular value set* of $f$. When $y \in \mathbf{R}^2$ is in the singular value set of $f : M \to \mathbf{R}^2$, we call $f^{-1}(y)$ a *singular fiber*; otherwise, a *regular fiber*.

3.1. **Stable maps.** Let $M$ be a closed 3-dimensional manifold and $f : M \to \mathbf{R}^2$ a smooth map. We denote the set of such maps by $C^\infty(M, \mathbf{R}^2)$ which is equipped with the Whitney $C^\infty$-topology. A smooth map $f$ is said to be *stable* if in $C^\infty(M, \mathbf{R}^2)$, there exists an open neighborhood $U$ of $f$ such that for any $g \in U$, $g$ is $C^\infty$ *right-left equivalent* to $f$, that is, there exist two diffeomorphisms $\Phi : M \to M$ and $\varphi : \mathbf{R}^2 \to \mathbf{R}^2$ such that the following diagram is commutative:

$$\begin{array}{ccc} M & \xrightarrow{\Phi} & M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \mathbf{R}^2 & \xrightarrow{\varphi} & \mathbf{R}^2. \end{array}$$

It is known that the set of stable maps is open and dense in $C^\infty(M, \mathbf{R}^2)$ (see [32]). Note that for a stable map $f : M \to \mathbf{R}^2$, $S(f)$ is a compact 1-dimensional submanifold of $M$. The following characterization of stable maps is well-known (see [15, 26, 29, 53] and Table 1).

**Proposition 3.1.** *A smooth map $f : M \to \mathbf{R}^2$ of a closed 3-dimensional manifold into the plane is stable if and only if the following conditions are satisfied.*

(i) *For every $q \in M$, there exist local coordinates $(x, y, z)$ and $(X, Y)$ around $q \in M$ and $f(q) \in \mathbf{R}^2$ respectively such that one of the following holds:*

$$(X \circ f, Y \circ f) = \begin{cases} (x, y), & q\text{:regular point}, \\ (x, y^2 + z^2), & q\text{:definite fold point}, \\ (x, y^2 - z^2), & q\text{:indefinite fold point}, \\ (x, y^3 + xy + z^2), & q\text{:cusp point}. \end{cases}$$

(ii) *For every $y \in f(S(f))$, $f^{-1}(y) \cap S(f)$ consists of at most two points and the multi-germ $(f|S(f), f^{-1}(y) \cap S(f))$ is right-left equivalent to one of the three multi-germs as depicted in Figure* 1: (1) *represents a single immersion germ which corresponds to a fold point,* (2) *corresponds to a cusp point, and* (3) *represents a normal crossing of two immersion germs each of which corresponds to a fold point.*

Suppose that for a stable map $f : M \to \mathbf{R}^2$, there are distinct singular points $q_1$ and $q_2$ in $S(f)$ such that $y = f(q_1) = f(q_2) \in \mathbf{R}^2$ holds. In this case, we call $y$ a *nodefold* of $f$ or a *node* of $f(S(f))$. Note that $S(f)$ is a closed 1-dimensional submanifold of $M$, that the number of nodefolds of $f$ is finite and that the number of cusps on each component of $S(f)$ is even (see [27]).

**Definition 3.2** ([45]). Let $M_i$ be manifolds and $A_i \subset M_i$ subsets, $i = 0, 1$. A continuous map $g : A_0 \to A_1$ is said to be *smooth* if for every point $q \in A_0$, there exists a smooth map $\widetilde{g} : V \to M_1$ defined on a neighborhood $V$ of $q$ in $M_0$ such that $\widetilde{g}|V \cap A_0 = g|V \cap A_0$. Furthermore, a smooth map $g : A_0 \to A_1$ is a *diffeomorphism* if it is a homeomorphism and its inverse is also smooth.

Let $q$ be a singular point of a stable map $f : M \to \mathbf{R}^2$. Then, using the local normal forms in Table 1, we can easily describe the diffeomorphism type of a neighborhood of $q$ in $f^{-1}(f(q))$. That is, we easily get the following local characterization of singular fibers.

**Lemma 3.3.** *Let $f : M \to \mathbf{R}^2$ be a stable map of a closed 3-dimensional manifold into the plane. Every singular point $q$ of $f$ has one of the following neighborhoods in its corresponding singular fiber (see Figure* 4):

(1) *isolated point diffeomorphic to $\{(y, z) \in \mathbf{R}^2 \mid y^2 + z^2 = 0\}$, if $q$ is a definite fold point,*
(2) *union of two transverse arcs diffeomorphic to $\{(y, z) \in \mathbf{R}^2 \mid y^2 - z^2 = 0\}$, if $q$ is an indefinite fold point,*
(3) *3/2-cuspidal arc diffeomorphic to $\{(y, z) \in \mathbf{R}^2 \mid y^3 + z^2 = 0\}$, if $q$ is a cusp point.*

Figure 4

For the local nearby fibers, we have the following.

**Lemma 3.4.** *Let $f : M \to \mathbf{R}^2$ be a stable map. Suppose that $q \in S(f)$ is a singular point such that $f^{-1}(f(q)) \cap S(f) = \{q\}$ or that $q_1$ and $q_2 \in S(f)$ are distinct singular points with $f(q_1) = f(q_2)$ and $f^{-1}(f(q_1)) \cap S(f) = \{q_1, q_2\}$. Then the local fibers near $q$ or the pair $q_1, q_2$ are as described in Figure 5, where* D *means "definite" and* I *means "indefinite". In Figure 5, each 0- or 1-dimensional object except $f(S(f)) \subset \mathbf{R}^2$ represents a portion of the fiber over the corresponding point in the plane. They are drawn with thin lines and $f(S(f))$ is drawn with thick lines.*

Figure 5

In Figure 5, some of the edges of $f(S(f))$ are oriented. For the definition of the orientation on $f(S(f))$, see Remark 4.6.

3.2. **Unstable maps of codimensions one and two.** Let $M$ be a closed 3-dimensional manifold. In this subsection we define and study unstable maps of codimensions 1 and 2.

**Definition 3.5.** If for a singular value $y \in \mathbf{R}^2$ of a smooth map $f : M \to \mathbf{R}^2$, the multi-germ $f : (M, f^{-1}(y) \cap S(f)) \to (\mathbf{R}^2, y)$ is $\mathcal{A}$-equivalent to a stable multi-germ in Table 1, then we call $y$ a *stable singular value* of $f$ and $f^{-1}(y)$ a *stable singular fiber* of $f$.

**Definition 3.6.** Let $f : M \to \mathbf{R}^2$ be a smooth map. Suppose that for a singular value $y \in \mathbf{R}^2$, $f^{-1}(y) \cap S(f)$ is a finite set and that the multi-germ $f : (M, f^{-1}(y) \cap S(f)) \to (\mathbf{R}^2, y)$ is not $\mathcal{A}$-equivalent to any stable multi-germs in Table 1.

(1) If there exists a 1-parameter unfolding $G : (M \times \mathbf{R}, (f^{-1}(y) \cap S(f)) \times \{0\}) \to (\mathbf{R}^2, y)$ of $f : (M, f^{-1}(y) \cap S(f)) \to (\mathbf{R}^2, y)$ which is $\mathcal{A}$-equivalent to one of the 1-parameter unfoldings in Table 2, then we call $y$ a *codimension* 1 *singular value* of $f$ and $f^{-1}(y)$ a *codimension* 1 *singular fiber* of $f$.

(2) If there exists a 2-parameter unfolding $H : (M \times \mathbf{R}^2, (f^{-1}(y) \cap S(f)) \times \{0\}) \to (\mathbf{R}^2, y)$ of $f : (M, f^{-1}(y) \cap S(f)) \to (\mathbf{R}^2, y)$ which is $\mathcal{A}$-equivalent to one of the 2-parameter unfoldings in Tables 3–5, other than those for $D_4^{\pm}$ or a quadruplefold, or if there exists a 3-parameter unfolding $H : (M \times \mathbf{R}^3, (f^{-1}(y) \cap S(f)) \times \{0\}) \to (\mathbf{R}^2, y)$ of $f : (M, f^{-1}(y) \cap S(f)) \to (\mathbf{R}^2, y)$ which is $\mathcal{A}$-equivalent to the 3-parameter unfolding of $D_4^{\pm}$ or a quadruplefold in Tables 3 or 5 around the parameter $(0, 0, t_0)$ for some $t_0$, then we call $y$ a *codimension* 2 *singular value* of $f$ and $f^{-1}(y)$ a *codimension* 2 *singular fiber* of $f$.

Note that we use the term "codimension $i$ singular fiber of a smooth map" in a sense different from the term "codimension $i$ singular fiber of a stable map" used in [45] ($i = 1, 2$).

**Definition 3.7.** A smooth map $f : M \to \mathbf{R}^2$ is said to be a *codimension* 1 (resp. 2) *unstable map* if there exists a unique codimension 1 (resp. 2) singular value of $f$ and the other singular values are all stable singular values of $f$.

*Remark* 3.8. Let $f : M \to \mathbf{R}^2$ be a smooth map. Suppose that $f$ has exactly two codimension 1 singular values and that the other singular values are all stable singular values of $f$. We can regard such an $f$ as a codimension 2 unstable map in $C^\infty(M, \mathbf{R}^2)$ in a natural sense. However, for the study of first order semi-local invariants of stable maps, we can ignore such kind of maps. We will explain the reason in Remark 5.2. For the study of first order non-local invariants of stable maps, we have to consider such kind of maps (see Section 8).

**Definition 3.9.** Let $f$ and $g : M \to \mathbf{R}^2$ be two stable maps of a closed 3-dimensional manifold $M$ into the plane and $I \subset \mathbf{R}$ a closed interval such that $\partial I = \{a, b\}$ and $a < b$. Let $\tau : I \to C^\infty(M, \mathbf{R}^2)$ be a continuous map which connects $f$ and $g$, i.e., $\tau(a) = f$ and $\tau(b) = g$. We call $\tau$ a *continuous path* between $f$ and $g$.

For a continuous path $\tau$, we define the associated continuous map $F : M \times I \to \mathbf{R}^2$ by $F(x,t) = \tau(t)(x)$ $(x \in M, t \in I)$. Note that $f_t$ is a smooth map for each $t \in I$, $f_a = f$ and $f_b = g$, where $f_t : M \to \mathbf{R}^2$ is defined by $f_t(x) = F(x,t)$. By an approximation theorem, there exists a smooth map $G : M \times I \to \mathbf{R}^2$ which is an approximation of $F$ such that $g_a = f$ and $g_b = g$, where $g_t$ is defined by $g_t(x) = G(x,t)$ (see [34]). We call $G$ a *smooth homotopy* between $f$ and $g$. To choose a suitable smooth homotopy, we use the following parameterized multi-transversality theorem.

Let $N$, $Q$ and $P$ be manifolds and $F : N \times Q \to P$ a smooth map. For each $q \in Q$, the smooth map $F_q : N \to P$ is defined by $F_q(x) = F(x,q)$. We denote by $N^{(k)}$ the set of all $(x_1, \dots, x_k) \in N^k$ such that $x_1, \dots, x_k$ are distinct points in $N$. Let $J^r(N,P)$ be the $r$-jet space and $j^r F_q(x)$ the $r$-jet of $F_q$ at $x \in N$. We define ${}_kJ^r(N,P)$ by ${}_kJ^r(N,P) = (\pi_N^k)^{-1} N^{(k)}$, where $\pi_N : J^r(N,P) \to N$ is the projection.

We define the parameterized jet extension ${}_kj^r F : N^{(k)} \times Q^k \to {}_kJ^r(N,P) \times Q^k$ by

$$ {}_kj^r F(x_1, \dots, x_k, q_1, \dots, q_k) = (j^r F_{q_1}(x_1), \dots, j^r F_{q_k}(x_k), q_1, \dots, q_k). $$

Then we have the following proposition.

**Proposition 3.10** (Parameterized multi-transversality theorem). *Let $W_1, W_2, \dots$ be countably many submanifolds in ${}_kJ^r(N,P) \times Q^k$. Then, the set*

$$ T = \{F \in C^\infty(N \times Q, P) \mid {}_kj^r F \text{ is transverse to every } W_1, W_2, \dots\} $$

*is a residual subset and is dense in $C^\infty(N \times Q, P)$.*

The above proposition follows from the (ordinary) multi-transversality theorem proved in [30, 31] (see [20] for details). For the case of $k = 1$, see [1, 9].

By the above proposition, we may assume that the smooth homotopy $G : M \times I \to \mathbf{R}^2$ approximating the continuous map $F : M \times I \to \mathbf{R}^2$ associated with a continuous path $\tau : I \to C^\infty(M, \mathbf{R}^2)$ satisfies the following:

(1) there is a finite set of parameter values $a < t_1 < t_2 < \cdots < t_l < b$ (possibly empty) in the open interval $\text{Int}I = (a, b)$ such that the following holds.

(1-1) For any $t \in I \setminus \{t_1, \ldots, t_l\}$, the map $g_t : M \to \mathbf{R}^2$ is stable, where $g_t$ is defined by $g_t(x) = G(x, t)$.

(1-2) For each $t_i$ $(i = 1, \ldots, l)$, $g_{t_i}$ is a codimension 1 unstable map.

(1-3) Let $y_i \in \mathbf{R}^2$ be the codimension 1 singular value of $g_{t_i}$ $(i = 1, \ldots, l)$. Then

$$G : (M \times (t_i - \varepsilon, t_i + \varepsilon), (g_{t_i}^{-1}(y_i) \cap S(g_{t_i})) \times \{t_i\}) \to (\mathbf{R}^2, y_i) \qquad (3.1)$$

is $\mathcal{A}$-equivalent to one of the 1-parameter unfoldings in Table 2, where $\varepsilon$ is a sufficiently small positive real number.

We call such a $G$ a *generic homotopy* between $f$ and $g$ and call each $t_i$ $(1 \leq i \leq l)$ a *codimension* 1 *bifurcation value* of $G$. If there is no codimension 1 bifurcation value of $G$ in $I$, then we call $G$ an *isotopy* between $f$ and $g$, and if there exists an isotopy between $f$ and $g$, then we say that $f$ and $g$ are *isotopic*. We say that $f$ is the *initial stable map* of $G$ and $g$ is the *terminal stable map* of $G$. For a generic homotopy, if the initial stable map and the terminal one are the same, then we call it a *generic loop*.

Let $\rho : W \to C^\infty(M, \mathbf{R}^2)$ be a continuous map such that for the associated continuous map $F : M \times W \to \mathbf{R}^2$, the restriction $F|M \times \partial W$ is a generic loop. Here, $W \subset \mathbf{R}^2$ is a closed disk and $F$ is defined by $F(x, w) = \rho(w)(x)$. By an approximation theorem, there exists a smooth map $G : M \times W \to \mathbf{R}^2$ which is an approximation of $F$ such that $F|M \times \partial W = G|M \times \partial W$. By Proposition 3.10, we may assume that the smooth map $G$ satisfies the following conditions.

(2) The closed disk $W$ is stratified into finitely many 2-, 1- and 0-dimensional strata and $\partial W$ is a union of finitely many 1- and 0-dimensional strata. They satisfy the following.

(2-1) For any point $w$ in each 2-dimensional stratum, $g_w$ is a stable map, where $g_w$ is defined by $g_w(x) = G(x, w)$.

(2-2) For any point $w$ in each 1-dimensional stratum contained in $\text{Int}W$, $g_w$ is a codimension 1 unstable map. Let $y \in \mathbf{R}^2$ be the codimension 1 singular value of $g_w$ and $I_w \subset \text{Int}W$ a small open arc passing through $w$ which is transverse to the 1-dimensional stratum of $w$. Then

$$G|M \times I_w : (M \times I_w, (g_w^{-1}(y) \cap S(g_w)) \times \{w\}) \to (\mathbf{R}^2, y)$$

is $\mathcal{A}$-equivalent to one of the 1-parameter unfoldings in Table 2.

(2-3) For each 0-dimensional stratum $w_j$ in Int$W$ ($j = 1, \ldots, m$), $g_{w_j}$ is a codimension 2 unstable map. Let $y_j \in \mathbf{R}^2$ be the codimension 2 singular value of $g_{w_j}$. Then, either

(2-3a)

$$G : (M \times W, (g_{w_j}^{-1}(y_j) \cap S(g_{w_j})) \times \{w_j\}) \to (\mathbf{R}^2, y_j) \tag{3.2}$$

is $\mathcal{A}$-equivalent to one of the 2-parameter unfoldings in Tables 3 or 4 other than those for $D_4^{\pm}$, or

(2-3b) there exists a $t_0 \in \mathbf{R}$ such that (3.2) is $\mathcal{A}$-equivalent to the unfolding of $D_4^{\pm}$ or a quadruplefold in Tables 3 or 5 with $t = t_0$.

We call such a $G$ a *generic* 2*-parameter family* and we call each $w_j$ ($1 \leq j \leq m$) a *codimension* 2 *bifurcation value* of $G$ in $W$.

Let $F : M \times I \to \mathbf{R}^2$ be a generic homotopy such that a closed interval $I$ contains 0 and 0 is the unique codimension 1 bifurcation value in $I$. Then, the open interval Int$I = (a, b) \subset \mathbf{R}$ is stratified into two 1-dimensional strata and one 0-dimensional stratum (i.e., the origin). We call such a stratified open interval Int$I$ a *codimension* 1 *bifurcation diagram* of $f_0$. Here, $f_t : M \to \mathbf{R}^2$ is defined by $f_t(x) = F(x, t)$.

Let $G : M \times W \to \mathbf{R}^2$ be a generic 2-parameter family such that 0 is the unique codimension 2 bifurcation value in the closed disk $W$. Then, the open disk Int$W \subset \mathbf{R}^2$ is naturally stratified into several 2-dimensional strata, several 1-dimensional strata and one 0-dimensional stratum (i.e., the origin). We call such a stratified open disk Int$W$ a *codimension* 2 *bifurcation diagram* of $g_0$. Here, $g_w : M \to \mathbf{R}^2$ is defined by $g_w(x) = G(x, w)$ (see Figure 3).

For a bifurcation diagram, we usually consider that each stratum contains some extra information on the stable (or codimension 1 or 2 unstable) maps corresponding to the stratum, such as their singular value sets, their singular fibers, etc. (see Figures 2 and 3).

*Remark* 3.11. Let $G_i : M \times W_i \to \mathbf{R}^2$ be two generic 2-parameter families such that for each $G_i$, $0 \in W_i$ is the unique codimension 2 bifurcation value ($i = 1, 2$). Suppose that both $G_i$ are $\mathcal{A}$-equivalent to the unfolding of $D_4^{\pm}$ or a quadruplefold in Tables 3 or 5 with $t = t_i$ ($t_1 \neq t_2$) (see (2-3b) above). By using the normal form of $D_4^{\pm}$ or a quadruplefold in Tables 3 or 5, we see that there exists a homeomorphism $\varphi : \mathrm{Int}W_1 \to \mathrm{Int}W_2$ which preserves the codimension 2 bifurcation diagrams of Int$W_1$ and Int$W_2$ (see Figure 3).

3.3. **Bifurcation diagrams.** In this subsection, we study bifurcation diagrams of unstable maps of codimensions 1 and 2, and clarify the deformations of their singular value sets and their singular fibers locally.

Let $M$ be a closed 3-dimensional manifold and $F : M \times I \to \mathbf{R}^2$ a generic homotopy such that $I$ is a closed interval and $0 \in \mathrm{Int} I$ is the unique codimension 1 bifurcation value of $F$. Suppose that $y \in \mathbf{R}^2$ is the codimension 1 singular value of $f_0$, where $f_t$ is defined by $f_t(x) = F(x,t)$. We call such an $F$ a *generic homotopy around* $f_0$. The deformation of the singular value set $f_t(S(f_t))$ around $y$ is as depicted in Figure 2.

Let $f : M \to \mathbf{R}^2$ be a codimension 1 unstable map and $y \in \mathbf{R}^2$ the codimension 1 singular value of $f$. Suppose that $q \in f^{-1}(y) \cap S(f)$ is a singular point in $f^{-1}(y)$. Using the local normal forms in Table 2, we can easily describe the diffeomorphism type of a neighborhood of $q$ in $f^{-1}(y)$. If $f^{-1}(y) \cap S(f)$ has two or more points, then $q$ has one of the neighborhoods as listed in Lemma 3.3 in its corresponding singular fiber. In the following lemma, we describe the local characterization of codimension 1 singular fibers when $\{q\} = f^{-1}(y) \cap S(f)$ holds.

**Lemma 3.12.** *Let $f : M \to \mathbf{R}^2$ be a codimension* 1 *unstable map of a closed* 3*-dimensional manifold into the plane and $y \in \mathbf{R}^2$ the codimension* 1 *singular value of $f$. Suppose that $f^{-1}(y) \cap S(f)$ consists of a single point, say $q$. Then $q$ has one of the following neighborhoods in its corresponding singular fiber (see Figure* 6*):*

(1) 3/2*-cuspidal arc diffeomorphic to $\{(y,z) \in \mathbf{R}^2 \mid y^3 + z^2 = 0\}$, if $q$ corresponds to lips or beaks,*

(2) *isolated point diffeomorphic to $\{(y,z) \in \mathbf{R}^2 \mid y^4 + z^2 = 0\}$, if $q$ is a definite swallowtail,*

(3) *union of two tangent arcs diffeomorphic to $\{(y,z) \in \mathbf{R}^2 \mid y^4 - z^2 = 0\}$, if $q$ is an indefinite swallowtail.*

Figure 6

Note that in Figure 6, the black square (2) represents an isolated point. However, we do not use the black dot as in Figure 4 (1) in order to distinguish the fiber corresponding to a definite fold from that corresponding to a definite swallowtail.

For the local nearby fibers of stable maps appearing in a generic homotopy around a codimension 1 unstable map, we have the following.

**Lemma 3.13.** *Let $f : M \to \mathbf{R}^2$ be a codimension* 1 *unstable map of a closed* 3*-dimensional manifold into the plane and $y_0 \in \mathbf{R}^2$ the codimension* 1 *singular value of $f$. Suppose that $F : M \times [-1,1] \to \mathbf{R}^2$ is a generic homotopy around $f$, and define $f_t$ by $f_t(x) = F(x,t)$. Then the local fibers of $f_t$ near $f_0^{-1}(y_0) \cap S(f_0)$ are as depicted in Figure* 7*, where we replace $t$ by $-t$ for $f_t$ if necessary. In Figure* 7*, each* 0*- or* 1*-dimensional object except $f_t(S(f_t)) \subset \mathbf{R}^2$ represents a portion of the fiber over the corresponding point in the plane. They are drawn with thin lines and $f_t(S(f_t))$ is drawn with thick lines.*

Figure 7

In Figure 7, some of the edges of $f_t(S(f_t))$ are oriented. For the definition of the orientation on $f_t(S(f_t))$, see Remark 4.6. Note that the figures in Figure 7 are in one-to-one correspondence with the normal forms in Table 2 and these figures do not depend on the choice of a generic homotopy $F$ up to diffeomorphism.

Let us now study codimension 2 unstable maps. Let $F : M \times W \to \mathbf{R}^2$ be a generic 2-parameter family such that 0 is the unique codimension 2 bifurcation value in the closed disk $W$. Suppose that $y \in \mathbf{R}^2$ is the codimension 2 singular value of $f_0$, where $f_w$ is defined by $f_w(x) = F(x, w)$. We call such an $F$ a *generic 2-parameter family around* $f_0$. Then the deformation of the singular value set $f_w(S(f_w))$ around $y$ is as depicted in Figure 3.

Let $f : M \to \mathbf{R}^2$ be a codimension 2 unstable map and $y \in \mathbf{R}^2$ the codimension 2 singular value of $f$. Let $q \in f^{-1}(y) \cap S(f)$ be a singular point in $f^{-1}(y)$. Using the normal forms in Tables 3–5, we can easily describe the diffeomorphism type of a neighborhood of $q$ in $f^{-1}(y)$. If $f^{-1}(y) \cap S(f)$ has two or more points, then $q$ has one of the neighborhoods as listed in Lemmas 3.3 or 3.12 in its corresponding singular fiber. In the following lemma, we describe the local characterization of codimension 2 singular fibers when $\{q\} = f^{-1}(y) \cap S(f)$ holds.

**Lemma 3.14.** *Let $f : M \to \mathbf{R}^2$ be a codimension 2 unstable map of a closed 3-dimensional manifold into the plane and $y \in \mathbf{R}^2$ the codimension 2 singular value of $f$. Suppose that $f^{-1}(y) \cap S(f)$ consists of a single point, say $q$. Then $q$ has one of the following neighborhoods in its corresponding singular fiber (see Figure 8):*

(1) *3/2-cuspidal arc diffeomorphic to $\{(y, z) \in \mathbf{R}^2 \mid y^3 + z^2 = 0\}$, if $q$ is a goose,*
(2) *5/2-cuspidal arc diffeomorphic to $\{(y, z) \in \mathbf{R}^2 \mid y^5+y^7+z^2 = 0\}$, if $q$ is a butterfly,*
(3) *isolated point diffeomorphic to $\{(y, z) \in \mathbf{R}^2 \mid y^4 + y^5 + z^2 = 0\}$, if $q$ corresponds to definite gulls,*
(4) *union of two tangent arcs diffeomorphic to $\{(y, z) \in \mathbf{R}^2 \mid y^4 + y^5 - z^2 = 0\}$, if $q$ corresponds to indefinite gulls,*
(5) *union of an arc and an isolated point diffeomorphic to $\{(y, z) \in \mathbf{R}^2 \mid y^3 + y^2z + z^3 + z^5 = 0\}$, if $q$ is a $D_4^+$ point,*
(6) *union of three arcs meeting at a point with distinct tangents diffeomorphic to $\{(y, z) \in \mathbf{R}^2 \mid y^3 - 2y^2z + z^3 + z^5 = 0\}$, if $q$ is a $D_4^-$ point.*

Figure 8

By [35, 40], if $q$ is a $D_4^+$ point (resp. $D_4^-$ point), the normal form $H(x, y, z, 0; 0, t)$ in Table 3 is $C^\infty\mathcal{K}$-equivalent to the map of the form $\widetilde{h}(x, y, z) = (x, xy + y^2z + z^3)$ (resp. $\widetilde{h}(x, y, z) = (x, xy - y^2z + z^3)$). Because of the definition of $C^\infty\mathcal{K}$-equivalence, a neighborhood of $q$ in its corresponding singular fiber is diffeomorphic to a neighborhood of

$(0,0,0) \in \mathbf{R}^3$ in $\widetilde{h}^{-1}(0,0)$. Therefore, $q$ has a neighborhood in its corresponding singular fiber diffeomorphic to $\{(y,z) \in \mathbf{R}^2 \mid \pm y^2 z + z^3 = 0\}$ and we have the figures depicted as in Figure 8 (5) and (6).

Note that in Figure 8 (2), the "shape of Y", (3) the black square, and (5) the union of an arc and an isolated point represent a 5/2-cuspidal arc, an isolated point and a line respectively. We use these symbols to distinguish a 3/2-cuspidal arc and (2), a black dot and (3) (see the paragraph just after Lemma 3.12), and a regular arc (without singular points) and (5).

For the local nearby fibers of a codimension 2 unstable map, we have the following.

**Lemma 3.15.** *Let $f : M \to \mathbf{R}^2$ be a codimension 2 unstable map of a closed 3-dimensional manifold into the plane and $y_0 \in \mathbf{R}^2$ the codimension 2 singular value of $f$. Then the local fibers of $f$ near $f^{-1}(y_0) \cap S(f)$ are as depicted in Figure 9. In Figure 9, each 0- or 1-dimensional object except $f(S(f)) \subset \mathbf{R}^2$ represents a portion of the fiber over the corresponding point in the plane. They are drawn with thin lines and $f(S(f))$ are drawn with thick lines.*

Figure 9

In Figure 9, some of the edges of $f(S(f))$ are oriented. For the definition of the orientation on $f(S(f))$, see Remark 4.6. Note that Figure 9 has one-to-one correspondence with Tables 3–5.

Note that for a generic 2-parameter family $F$ around a codimension 2 unstable map $f$, we can depict figures similar to those given in Lemma 3.13. But the statement and the figures would be so complicated that we do not write them down here.

## 4. Classification of singular fibers

In this section, we first give a precise definition of the *weak equivalence* for singular fibers. We classify singular fibers of stable maps and unstable maps of codimensions 1 and 2 up to this equivalence relation. Then we define an equivalence relation for unstable maps of codimensions 1 and 2, which is based on the weak equivalence of singular fibers of codimensions 1 and 2. We also call this equivalence the weak equivalence for simplicity, for unstable maps of codimensions 1 and 2. We classify unstable maps of codimensions 1 and 2 up to this equivalence relation.

4.1. **Definition of an equivalence of fibers.** Let $f$ and $g : M \to \mathbf{R}^2$ be two smooth maps of a closed manifold $M$ into the plane. For $y_f$ and $y_g \in \mathbf{R}^2$, we say that the fiber of $f$ over $y_f$ and that of $g$ over $y_g$ are *diffeomorphic* to each other if $f^{-1}(y_f)$ and $g^{-1}(y_g)$ are diffeomorphic in the sense of Definition 3.2. Let $\pi : S^1 \times \mathrm{Int}D^2 \to \mathrm{Int}D^2$ be the projection to the second factor. Since $\pi$ is a submersion, the fiber $\pi^{-1}(0)$ is a regular fiber and is diffeomorphic to $S^1$. We denote by $\cup_{i=1}^{\alpha} S_i^1$ the disjoint union of $\alpha$ copies of $S^1$ ($\alpha \geq 1$) and by $\pi_\alpha : (\cup_{i=1}^{\alpha} S_i^1) \times \mathrm{Int}D^2 \to \mathrm{Int}D^2$ the projection to the second factor.

**Definition 4.1.** Let $f$ and $g : M \to \mathbf{R}^2$ be two smooth maps of a closed 3-dimensional manifold $M$, and $f^{-1}(y_f)$ and $g^{-1}(y_g)$ two singular fibers of $f$ and $g$ over $y_f$ and $y_g \in \mathbf{R}^2$ respectively. Suppose that $U_f$ (resp. $U_g$) is a small open disk neighborhood of $y_f$ (resp. $y_g$).

(1) Suppose that both $f$ and $g$ are stable maps. We say that the two singular fibers $f^{-1}(y_f)$ and $g^{-1}(y_g)$ are *weakly equivalent* if there exist a homeomorphism $\varphi : (U_f, y_f) \to (U_g, y_g)$ and non-negative integers $\alpha$ and $\beta$ such that $\varphi(U_f \cap f(S(f))) = U_g \cap g(S(g))$ and for each $y \in U_f$, the disjoint union $f^{-1}(y) \cup (\pi_\alpha^{-1}(0))$ is diffeomorphic to the disjoint union $g^{-1}(\varphi(y)) \cup (\pi_\beta^{-1}(0))$.

(2) Suppose that both $f$ and $g$ are codimension 1 unstable maps and that $y_f \in \mathbf{R}^2$ (resp. $y_g$) is the unique codimension 1 singular value of $f$ (resp. $g$). Let $F : M \times I_f \to \mathbf{R}^2$ (resp. $G : M \times I_g \to \mathbf{R}^2$) be a generic homotopy around $f$ (resp. $g$) such that $f_0 = f$ (resp. $g_0 = g$) holds, where $I_f$ and $I_g \subset \mathbf{R}$ are small closed intervals containing 0 and $f_t$ (resp. $g_t$) is defined by $f_t(x) = F(x,t)$ (resp. $g_t(x) = G(x,t)$). The two singular fibers $f^{-1}(y_f)$ and $g^{-1}(y_g)$ are *weakly equivalent* if their exist a homeomorphism $\varphi : I_f \times U_f \to I_g \times U_g$ of the form $\varphi(t,y) = (\psi(t), \varphi_t(y))$ and non-negative integers $\alpha$ and $\beta$ such that $\psi(0) = 0$, $\varphi_0(y_f) = y_g$ and $\varphi_t(U_f \cap f_t(S(f_t))) = U_g \cap g_{\psi(t)}(S(g_{\psi(t)}))$ for all $t \in I_f$ and that for each $y \in U_f$, the disjoint union $f_t^{-1}(y) \cup (\pi_\alpha^{-1}(0))$ is diffeomorphic to the disjoint union $g_{\psi(t)}^{-1}(\varphi_t(y)) \cup (\pi_\beta^{-1}(0))$ for all $t \in I_f$.

(3) Suppose that both $f$ and $g$ are codimension 2 unstable maps and that $y_f \in \mathbf{R}^2$ (resp. $y_g$) is the unique codimension 2 singular value of $f$ (resp. $g$). Let $F : M \times W_f \to \mathbf{R}^2$ (resp. $G : M \times W_g \to \mathbf{R}^2$) be a generic 2-parameter family around $f$ (resp. $g$) such that $f_0 = f$ (resp. $g_0 = g$) holds, where $W_f$ and $W_g \subset \mathbf{R}^2$ are sufficiently small closed disks containing the origin and $f_w$ (resp. $g_w$) is defined by $f_w(x) = F(x,w)$ (resp. $g_w(x) = G(x,w)$). The two singular fibers $f^{-1}(y_f)$ and $g^{-1}(y_g)$ are *weakly equivalent* if there exist a homeomorphism $\varphi : W_f \times U_f \to W_g \times U_g$ of the form $\varphi(w,y) = (\psi(w), \varphi_w(y))$ and non-negative integers $\alpha$ and $\beta$ such that $\psi(0) = 0$, $\varphi_0(y_f) = y_g$ and $\varphi_w(U_f \cap f_w(S(f_w))) = U_g \cap g_{\psi(w)}(S(g_{\psi(w)}))$ for all $w \in W_f$ and that for each $y \in U_f$, the disjoint union $f_w^{-1}(y) \cup (\pi_\alpha^{-1}(0))$ is diffeomorphic to the disjoint union $g_{\psi(w)}^{-1}(\varphi_w(y)) \cup (\pi_\beta^{-1}(0))$.

Note that the non-negative integers $\alpha$ and $\beta$ should not depend on $y \in U_f$ or $t$ or $w$.

We remark that Definition 4.1 (2) and (3) do not depend on the choices of $F$ and $G$ respectively. We can prove this by using an argument similar to that in the proof of Theorem 4.7 in Subsection 4.3.

*Remark* 4.2. We can define another equivalence relation for singular fibers of unstable maps of codimensions 1 and 2 by ignoring the unfoldings as follows. Suppose that both $f$ and $g : M \to \mathbf{R}^2$ are codimension 1 (or 2) unstable maps of a closed manifold and that $y_f$ and $y_g \in \mathbf{R}^2$ are codimension 1 (resp. 2) singular values of $f$ and $g$ respectively. The two singular fibers $f^{-1}(y_f)$ and $g^{-1}(y_g)$ are *coarsely equivalent* if there exist a homeomorphism $\varphi : (U_f, y_f) \to (U_g, y_g)$ and non-negative integers $\alpha$ and $\beta$ such that $\varphi(U_f \cap f(S(f))) = U_g \cap g(S(g))$ and for each $y \in U_f$, the disjoint union $f^{-1}(y) \cup (\pi_\alpha^{-1}(0))$ is diffeomorphic to the disjoint union $g^{-1}(\varphi(y)) \cup (\pi_\beta^{-1}(0))$. We will see later that this equivalence is strictly weaker than the weak equivalence. For example, in Theorem 4.7 of Subsection 4.3, $\mathrm{III}_2^4$ and $\mathrm{III}_3^4$ are not weakly equivalent but are coarsely equivalent (see Figure 12 in Section 4.3), and in Theorem 4.8 of Subsection 4.3, $\mathrm{IV}_1^9, \mathrm{IV}_2^9$ and $\mathrm{IV}_3^9$ are not weakly equivalent to each other but are all coarsely equivalent (see Figure 13 (p) in Section 4.3).

*Remark* 4.3. We have several equivalence relations for stable, codimension 1 or codimension 2 singular fibers. They are "diffeomorphism up to regular $S^1$-components" (considered implicitly in [26]), "coarse equivalence", "weak equivalence" and "$C^\infty$ equivalence up to regular $S^1$-components" (originally defined in [45]). Let us write "$A$" $\Rightarrow$ "$B$" by using an arrow if the equivalence $A$ implies the equivalence $B$. Then, we see easily that

$$\text{“}C^\infty\text{ equivalence up to regular }S^1\text{-components”} \underset{(a)}{\Rightarrow} \text{“weak equivalence”}$$

$$\underset{(b)}{\Rightarrow} \text{“coarse equivalence”} \underset{(c)}{\Rightarrow} \text{“diffeomorphism up to regular }S^1\text{-components”}.$$

The converse of $(a)$ does not hold. This follows from the fact that $D_4^\pm$ and the quadruplefold have positive $\mathcal{A}$-modalities. That is, for $D_4^\pm$ or the quadruplefold, we have infinitely many $C^\infty$ equivalence classes of multi-germs. By Remark 4.2, the converse of $(b)$ does not hold. Figures 4 (1) and 6 (2) show that the converse of $(c)$ does not hold (see Lemma 3.12 in Subsection 3.3).

4.2. **Classification of singular fibers of stable maps.** In what follows, let $M$ be a closed *orientable* 3-dimensional manifold. In this subsection, we get a classification of stable singular fibers up to weak equivalence.

We have the following theorem which was implicitly proved in [26, 29].

**Theorem 4.4.** *Let $f : M \to \mathbf{R}^2$ be a stable map of a closed orientable 3-dimensional manifold into the plane. Then, every singular fiber of $f$ is weakly equivalent to one of the fibers as in Figure* 10, *and no two fibers in Figure* 10 *are weakly equivalent. In Figure* 10, *we have described the deformation of singular fibers around each stable singular fiber to clarify the weak equivalence class.*

Figure 10

In Figure 10, I* and II* mean the names of the corresponding weak equivalence classes of singular fibers. Note that we have named the fibers so that each connected singular fiber has its own digit or letter and a disconnected fiber has the name consisting of the digits or letters of its connected components containing singular points. Hence, the number of digits in the superscript coincides with the number of connected components which contain singular points. In this figure, singular value sets are drawn with thick lines and the orientations on the singular value sets correspond to those defined in Remark 4.6. Note that Figure 5 (1) can be regarded as the deformation of the singular fiber of type $\mathrm{I}^0$.

*Proof of Theorem* 4.4. The proof is simpler than that of [45, Theorems 3.1, 4.5 and 4.15]. For completeness, we give a detailed proof here.

Let us take a point $r \in f(S(f))$. If $r$ corresponds to Figure 1 (1), then $f^{-1}(r)$ contains exactly one singular point $q \in M$, which is a fold point. If $q$ is a definite fold point, then the component of the singular fiber containing $q$ is diffeomorphic to one point (the singular fiber of type $\mathrm{I}^0$ in Figure 10) by Lemma 3.3.

Suppose that $q$ is an indefinite fold point and $f(q) = r \in \mathbf{R}^2$. By Proposition 3.1, the germ of $f$ at $q$ is $\mathcal{A}$-equivalent to the germ of $f_1(x,y,z) = (x, y^2 - z^2)$ $(= (u,v))$ at the origin: i.e., there exist diffeomorphisms $\widetilde{\varphi}_1 : V \to V_1$ and $\varphi_1 : (\mathbf{R}^2, f(q)) \to (\mathbf{R}^2, (0,0))$ such that $\widetilde{\varphi}_1(q) = (0,0,0)$, $\varphi_1(f(q)) = (0,0)$ and $\varphi_1^{-1} \circ f_1 \circ \widetilde{\varphi}_1 = f$ on $V$, where $V$ is a sufficiently small open neighborhood of $q$ in $M$ and $V_1$ is an open neighborhood of the origin in $\mathbf{R}^3$ of the form

$$V_1 = \{(x,y,z) \in \mathbf{R}^3 \mid x^2 + y^2 + z^2 \leq \varepsilon, \|f_1(x,y,z)\| < \delta\}$$

for $1 \gg \varepsilon \gg \delta > 0$. Let $U_0 \subset \mathbf{R}^2$ be a small open interval defined by $u = 0$ and $|v| < \delta$ with respect to the above coordinates $(u,v)$. Then $V_1 \cap f_1^{-1}(U_0)$ is a surface as depicted in Figure 11 (1) (see also Figure 5 (2)). Note that the map

$$f_1|\partial(V_1 \cap f_1^{-1}(U_0)) : \partial(V_1 \cap f_1^{-1}(U_0)) \to U_0$$

is a proper submersion. Since the map $f_1|V_1 \cap f_1^{-1}(U_0) : V_1 \cap f_1^{-1}(U_0) \to U_0$ is a Morse function and a Morse function is a submersion outside of the critical points, $f^{-1}(r) \setminus V$ in $M$ is a compact 1-dimensional smooth manifold which is diffeomorphic to the disjoint union of two arcs and some circles. Therefore, $f^{-1}(r)$ is diffeomorphic to the disjoint union of a "figure eight" type singular fiber (a singular fiber of type $\mathrm{I}^1$ in Figure 10) or the complex as depicted in Figure 11 (2), and some circles. If a fiber as in Figure 11 (2) appears, then $M$ must contain a punctured Möbius band times $[-1,1]$, and hence is non-orientable. Since we have assumed that $M$ is orientable, this does not occur. Hence, we see that the singular fiber $f^{-1}(r)$ is diffeomorphic to the disjoint union of a "figure eight" type singular fiber and a finite number of non-singular circles.

Figure 11

If $r$ corresponds to Figure 1 (3), then $f^{-1}(r)$ contains exactly two singular points, say $q_1$ and $q_2$, which are fold points. Since they have neighborhoods as in Lemma 3.3 (1) or (2) in $f^{-1}(r)$, and since $f$ is a submersion outside of the singular points, we see that there are only a finite number of possibilities for the diffeomorphism type of the union of the components of $f^{-1}(r)$ containing $q_1$ and $q_2$: for example, if both $q_1$ and $q_2$ are indefinite folds, then it is obtained from two copies of the figure as in Figure 4 (2) by connecting their end points by four arcs. Then we can use Lemma 3.4 to obtain the nearby fibers of each possible singular fiber: for example, see Figure 10, $\mathrm{II}^{1,1}, \mathrm{II}^2$ and $\mathrm{II}^3$. Excluding the possibilities such that a singular fiber as in Figure 11 (2) appears as a nearby fiber, we get the singular fibers and corresponding nearby fibers as depicted in Figure 10, $\mathrm{II}^{0,0}, \mathrm{II}^{0,1}, \mathrm{II}^{1,1}, \mathrm{II}^2$ and $\mathrm{II}^3$.

By a similar argument, we see that if $r$ corresponds to Figure 1 (2), then we obtain the singular fiber of type $\mathrm{II}^a$.

Thus we have proved that every singular fiber is diffeomorphic to the union of one of the fibers listed in the theorem and a finite number of non-singular circles.

In order to complete the proof, we have only to show that if two singular fibers in the list are diffeomorphic after omitting all non-singular circles, then they are weakly equivalent.

Let $f$ and $g : M \to \mathbf{R}^2$ be stable maps of a closed orientable 3-dimensional manifold $M$ into the plane. Let $\pi_\alpha : (\cup_{i=1}^{\alpha} S_i^1) \times \mathrm{Int}D^2 \to \mathrm{Int}D^2$ and $\pi_\beta : (\cup_{j=1}^{\beta} S_j^1) \times \mathrm{Int}D^2 \to \mathrm{Int}D^2$ be the projections to the second factors, where $\alpha$ and $\beta$ are non-negative integers, and $S_i^1$ and $S_j^1$ are copies of $S^1$. Let us take $r_f \in f(S(f)) \subset \mathbf{R}^2$ and $r_g \in g(S(g)) \subset \mathbf{R}^2$. Suppose that the disjoint union $f^{-1}(r_f) \cup (\pi_\alpha^{-1}(0))$ and the disjoint union $g^{-1}(r_g) \cup (\pi_\beta^{-1}(0))$ are diffeomorphic to each other.

If the singular fibers over $r_f$ and $r_g$ are both of type $\mathrm{I}^0$, then there exist neighborhoods $U_f$ of $r_f$ and $U_g$ of $r_g$ such that the sets $U_f \cap f(S(f))$ and $U_g \cap g(S(g))$ are as depicted in Figure 1 (1). In particular, there exists a homeomorphism $\varphi : (U_f, r_f) \to (U_g, r_g)$ such that $\varphi(U_f \cap f(S(f))) = U_g \cap g(S(g))$. Note that we can describe the degeneration of fibers of $f$ over $U_f$ and that of $g$ over $U_g$ using Lemma 3.4 and Figure 5. Then we see that the homeomorphism $\varphi$ can be chosen so that $f^{-1}(r) \cup (\pi_\alpha^{-1}(0))$ is diffeomorphic to $g^{-1}(\varphi(r)) \cup (\pi_\beta^{-1}(0))$ for all $r \in U_f$. Thus, the two fibers $f^{-1}(r_f)$ and $g^{-1}(r_g)$ are weakly equivalent.

Similar arguments work also in the cases $\mathrm{I}^1$, $\mathrm{II}^a$, $\mathrm{II}^{0,0}$, $\mathrm{II}^{0,1}$, $\mathrm{II}^{1,1}$, $\mathrm{II}^2$ and $\mathrm{II}^3$.

This completes the proof of Theorem 4.4. □

*Remark* 4.5. The list of fibers given in Figure 10 was already obtained by Kushner, Levine and Porto [26, 29], although they did not describe explicitly the equivalence relation for their classification.

*Remark* 4.6 ([29]). Let $f : M \to \mathbf{R}^2$ be a stable map or an unstable map of codimension 1 or 2, where $M$ is a closed orientable 3-dimensional manifold. By the singular value set $f(S(f)) \subset \mathbf{R}^2$, $\mathbf{R}^2$ is naturally stratified into 2-, 1- and 0-dimensional strata. Note that the union of 1- and 0-dimensional strata forms $f(S(f))$. On each 1-dimensional stratum of $f(S(f))$, we can define an orientation as follows. We fix the canonical orientation on $\mathbf{R}^2$. Let $\Omega$ be a connected component of $\mathbf{R}^2 \setminus f(S(f))$. We associate to $\Omega$ a non-negative integer $n_f(\Omega)$, which is the number of connected components of the fiber of $f$ over any point of $\Omega$. Every 1-dimensional stratum in $f(S(f))$ is adjacent to exactly two connected components of $\mathbf{R}^2 \setminus f(S(f))$. Since these two components have distinct $n_f(\Omega)$-values (see Figure 10), we can orient each 1-dimensional stratum in $f(S(f))$ so that the region with the larger $n_f(\Omega)$-value is on its left. In Figures 5, 7, 9 and 10, we have oriented some edges of $f(S(f))$ by this rule.

4.3. **Classification of unstable maps of codimensions one and two.** In this subsection, we will classify codimension $i$ singular fibers by the weak equivalence. Then we will classify unstable maps of codimensions 1 and 2 by using the weak equivalence of singular fibers.

We get the following classification of codimension 1 singular fibers.

**Theorem 4.7.** *Let $f : M \to \mathbf{R}^2$ be a codimension 1 unstable map of a closed orientable 3-dimensional manifold into the plane and $y \in \mathbf{R}^2$ the codimension 1 singular value of $f$. Then, the codimension 1 singular fiber $f^{-1}(y)$ of $f$ is weakly equivalent to one of the codimension 1 singular fibers as depicted in Figure 12, and no two fibers in Figure 12 are weakly equivalent. (In Figure 12, we have described the deformation of singular fibers on $f(S(f))$ around each codimension 1 singular fiber. We can describe the regular fibers on each 2-dimensional region around $y \in \mathbf{R}^2$ as well, although we have not included them in the figures. For the deformations of singular fibers of stable maps appearing in a generic homotopy around $f$, we can describe them by combining Figures 7 and 12.)*

Figure 12

In Figure 12, $\mathrm{III}^*_\star$ means the name of the weak equivalence class of the corresponding codimension 1 singular fiber. Each symbol $\mathrm{III}^*$ represents one diffeomorphism class (up to regular $S^1$-components) of a codimension 1 singular fiber. Note that $\mathrm{III}^a(l)$ and $\mathrm{III}^a(b)$ correspond to lips and beaks respectively. The subscript $\star$ in $\mathrm{III}^*_\star$ or the letter $\star$ in $\mathrm{III}^a(\star)$

means that we have two or more weak equivalence classes of codimension 1 singular fibers in the corresponding diffeomorphism class. In this figure, singular value sets are drawn with thick lines and the orientations on the singular value sets correspond to those defined in Remark 4.6.

Note that in Figure 12 (b), $\mathrm{III}^3_1$ and $\mathrm{III}^3_2$ are not weakly equivalent. We can distinguish them by looking at their generic homotopies and the deformations of singular fibers of stable maps appearing in these generic homotopies. By the same reason, $\mathrm{III}^4_2$ and $\mathrm{III}^4_3$ are not weakly equivalent, either.

Let $f : M \to \mathbf{R}^2$ be an unstable map of codimension 1 and $y \in \mathbf{R}^2$ the codimension 1 singular value of $f$. If the codimension 1 singular fiber $f^{-1}(y)$ belongs to $\mathrm{III}^*_\star$, then we call it a codimension 1 singular fiber of type $\mathrm{III}^*_\star$.

*Proof of Theorem* 4.7 (*sketch*). To prove the theorem, we can use almost the same argument as in the proof of Theorem 4.4 (see also [45]).

Let us take a point $y \in f(S(f))$. By an argument similar to that in the proof of Theorem 4.4, we can show that the union of the components of $f^{-1}(y)$ containing singular points is diffeomorphic to one of the fibers of type $\mathrm{III}^*$ listed in Figure 12.

In order to complete the proof, we have only to classify the singular fibers in each diffeomorphism class by the weak equivalence.

Let $f : M \to \mathbf{R}^2$ be a codimension 1 unstable map such that $y \in \mathbf{R}^2$ is the codimension 1 singular value of $f$ and that $f^{-1}(y)$ is diffeomorphic to a singular fiber of type $\mathrm{III}^*$. Let $F : M \times [-1, 1] \to \mathbf{R}^2$ be a generic homotopy around $f$. We define $f_t : M \to \mathbf{R}^2$ by $f_t(x) = F(x, t)$. The singular value set $f_t(S(f_t))$ around $y$ is as depicted in one of the figures of Figure 2 that corresponds to the type $\mathrm{III}^*$. Then we can describe all the possibilities for the degeneration of fibers of $f_t$ around $y$ by using Lemma 3.13 and Figure 7. We classify these degenerations of fibers by the "weak equivalence" in a sense as in Definition 4.1 and obtain all the weak equivalence classes. To prove that no two fibers in Figure 12 are weakly equivalent, we can easily show that there does not exist a homeomorphism $\varphi$ in a sense as in Definition 4.1.

This completes the proof of Theorem 4.7. □

Similarly, we get the following classification of codimension 2 singular fibers.

**Theorem 4.8.** *Let $f : M \to \mathbf{R}^2$ be a codimension 2 unstable map of a closed orientable 3-dimensional manifold into the plane and $y \in \mathbf{R}^2$ the codimension 2 singular value of $f$. Then, the codimension 2 singular fiber $f^{-1}(y)$ of $f$ is weakly equivalent to one of the codimension 2 singular fibers as depicted in Figure 13, and no two fibers in Figure 13 are weakly equivalent. (In Figure 13, we have described the deformation of singular fibers on $f(S(f))$ around each codimension 2 singular fiber. We can describe the regular fibers* *on each 2-dimensional region around $y \in \mathbf{R}^2$ as well, although we have not included them in the figures. For the deformations of singular fibers of stable and codimension* 1 *unstable maps appearing in a generic* 2*-parameter family around $f$, we can describe them by combining Figures* 3 *and* 9*, Tables* 3– 5 *and Theorems* 4.4 *and* 4.7.)

Figure 13

The proof of this theorem is similar to that of Theorems 4.4 and 4.7 (see also [45]), and is left to the reader.

In Figure 13, $\mathrm{IV}^*_\star$ means the name of the weak equivalence class of the corresponding codimension 2 singular fiber. Each symbol $\mathrm{IV}^*$ represents one diffeomorphism class (up to regular $S^1$-components) of a codimension 2 singular fiber. Note that the subscript $\star$ in $\mathrm{IV}^*_\star$ or the letter $\star$ in $\mathrm{IV}^*(\star)$ means that we have two or more weak equivalence classes of codimension 2 singular fibers in the corresponding diffeomorphism class. Furthermore, we use the convention as in Figure 8 for drawing the singular fibers in Figure 13. In this figure, singular value sets are drawn with thick lines and the orientations on the singular value sets correspond to those defined in Remark 4.6.

We remark that for example, $\mathrm{IV}^9_1$, $\mathrm{IV}^9_2$ and $\mathrm{IV}^9_3$ are not weakly equivalent. We can distinguish them by looking at their generic 2-parameter families and the deformations of singular fibers of stable maps appearing in these generic 2-parameter families.

**Definition 4.9.** Let $M$ be a closed orientable 3-dimensional manifold. Let $f_i : M \to \mathbf{R}^2$, $i = 1, 2$, be unstable maps of the same codimension $k$ $(= 1 \text{ or } 2)$, and $y_i \in \mathbf{R}^2$ the codimension $k$ singular value of $f_i$. We say that $f_1$ and $f_2$ are *weakly equivalent* if $f_1^{-1}(y_1)$ and $f_2^{-1}(y_2)$ are weakly equivalent in the sense of Definition 4.1.

We use the same expression "weakly equivalent" for the equivalence relation of singular fibers and that of unstable maps for simplicity.

We can classify codimension 1 (resp. 2) unstable maps by the weak equivalence by using Theorem 4.7 (resp. Theorem 4.8). By abuse of notation we use the symbol $\mathrm{III}^*_\star$ (resp. $\mathrm{IV}^*_\star$) for the weak equivalence class of maps in $\Gamma_1$ (resp. $\Gamma_2$) which have exactly one singular fiber of type $\mathrm{III}^*_\star$ (resp. $\mathrm{IV}^*_\star$). Note that these classes $\mathrm{III}^*_\star$ and $\mathrm{IV}^*_\star$ are considered to be strata of the set of all unstable maps $\Gamma$ in $C^\infty(M, \mathbf{R}^2)$. Note that each stratum $\mathrm{III}^*_\star$ or $\mathrm{IV}^*_\star$ may not necessarily be connected.

4.4. **Coorientations of codimension one strata.** Let $M$ be a closed orientable 3-dimensional manifold and $\Gamma_1$ the set of all codimension 1 unstable maps in $C^\infty(M, \mathbf{R}^2)$. In this subsection, we define a coorientation for each weak equivalence class of $\Gamma_1$.

Let $F : M \times [-1, 1] \to \mathbf{R}^2$ be a generic homotopy around a codimension 1 unstable map $f_0$, where $f_t : M \to \mathbf{R}^2$ is defined by $F(x,t) = f_t(x)$ $(x \in M, t \in [-1,1])$. Then we say that $F$ *crosses* $\Gamma_1$ *positively* at $f_0$ if one of the following holds.

(1) When $f_0 \in \mathrm{III}^a(l), \mathrm{III}^a(b), \mathrm{III}^b, \mathrm{III}^c$ or $\mathrm{III}^d$, the number of cusps of $f_1$ is greater than that of $f_{-1}$.
(2) When $f_0 \in \mathrm{III}^{0,a}_\star, \mathrm{III}^{1,a}_\star, \mathrm{III}^{e}_\star, \mathrm{III}^{0,0}_\star, \mathrm{III}^{0,1}_\star, \mathrm{III}^{1,1}_\star, \mathrm{III}^{2}_\star$ or $\mathrm{III}^{3}_\star$, the number of nodefolds of $f_1$ is greater than that of $f_{-1}$.
(3) When $f_0 \in \mathrm{III}^{0,0,0}_\star, \mathrm{III}^{0,0,1}_\star, \mathrm{III}^{0,1,1}_\star, \mathrm{III}^{1,1,1}_\star, \mathrm{III}^{0,2}_\star, \mathrm{III}^{1,2}_\star, \mathrm{III}^{0,3}_\star, \mathrm{III}^{1,3}_\star, \mathrm{III}^{4}_\star, \mathrm{III}^{5}_\star, \mathrm{III}^{6}_\star, \mathrm{III}^{7}_\star$ or $\mathrm{III}^8$, the number of connected components of the regular fiber over a point in the new-born triangle of $f_1$ is greater than that over a point in the vanishing triangle of $f_{-1}$.

If a generic homotopy $F$ does not satisfy the above property, then we say that $F$ crosses $\Gamma_1$ *negatively* at $f_0$. By the above definition, all weak equivalence classes of maps in $\Gamma_1 \subset C^\infty(M, \mathbf{R}^2)$ are cooriented.

## 5. The Vassiliev cochain complex for the weak equivalence

In this section we construct the *Vassiliev cochain complex* for the weak equivalence classes of unstable maps of codimensions 1 and 2.

Let $M$ be a closed orientable 3-dimensional manifold. In the following, we set $\mathcal{M} = C^\infty(M, \mathbf{R}^2)$. Let $\Gamma_i$ be the set of all codimension $i$ unstable maps in $\mathcal{M}$ $(i = 1, 2)$.

Let $f : M \to \mathbf{R}^2$ be a codimension 1 unstable map and $g : M \to \mathbf{R}^2$ a codimension 2 unstable map. Suppose that $F : M \times [-1, 1] \to \mathbf{R}^2$ is a generic homotopy around $f_0 = f$, where $F$ has the form $F(x, t) = f_t(x)$ $(x \in M, t \in [-1, 1])$. Let $W \subset \mathbf{R}^2$ be a small closed disk neighborhood of the origin in $\mathbf{R}^2$ and $G : M \times W \to \mathbf{R}^2$ a generic 2-parameter family around $g = g_0$, where $G$ has the form $G(x, w) = g_w(x)$ $(x \in M, w \in W)$. We fix an orientation of the parameter space $W$ of $G$ (for details, see below), while we assume that $F$ crosses $\Gamma_1$ positively at $f_0$ in the sense of Subsection 4.4.

Let $[f]_\mathcal{W}$ (resp. $[g]_\mathcal{W}$) be the cooriented weak equivalence class in $\Gamma_1$ (resp. $\Gamma_2$) of $f$ (resp. $g$), where the coorientation of $[f]_\mathcal{W}$ (resp. $[g]_\mathcal{W}$) is defined by the orientation of the parameter space of $F$ (resp. $G$). If we change the orientation of the parameter space of $F$ (resp. $G$), then the corresponding weak equivalence class is considered to be $-[f]_\mathcal{W}$ (resp. $-[g]_\mathcal{W}$).

Let us denote by $C^i(\mathcal{W}(\mathcal{M}); \mathbf{Z})$ the (finitely generated) free $\mathbf{Z}$-module generated by the cooriented weak equivalence classes of $\Gamma_i$, $i = 1, 2$. The rank of $C^1(\mathcal{W}(\mathcal{M}); \mathbf{Z})$ is equal to 56, while the rank of $C^2(\mathcal{W}(\mathcal{M}); \mathbf{Z})$ is equal to 389 (see Figures 12 and 13). Furthermore, we set $C^0(\mathcal{W}(\mathcal{M}); \mathbf{Z}) = C^3(\mathcal{W}(\mathcal{M}); \mathbf{Z}) = 0$. By Subsection 4.4, any weak equivalence class $\mathrm{III}^*_\star$ in $\Gamma_1$ is considered to be an element of $C^1(\mathcal{W}(\mathcal{M}); \mathbf{Z})$, and any weak equivalence class $\mathrm{IV}^*_\star$ in $\Gamma_2$ with a fixed coorientation is considered to be an element of $C^2(\mathcal{W}(\mathcal{M}); \mathbf{Z})$. We adopt the convention that the coorientation of each $\mathrm{IV}^*_\star$ is defined by the canonical orientation of $\mathbf{R}^2$ in Figure 3.

To define the coboundary operator $\delta : C^1(\mathcal{W}(\mathcal{M});\mathbf{Z}) \to C^2(\mathcal{W}(\mathcal{M});\mathbf{Z})$, we define the *incidence coefficient* $[\Theta : \Xi] \in \mathbf{Z}$ for every pair of generators $\Theta$ of $C^1(\mathcal{W}(\mathcal{M});\mathbf{Z})$ and $\Xi$ of $C^2(\mathcal{W}(\mathcal{M});\mathbf{Z})$ as follows.

Let $W(\Xi) \subset \mathbf{R}^2$ be a small closed disk neighborhood of the origin in $\mathbf{R}^2$ and $G : M \times W(\Xi) \to \mathbf{R}^2$ a generic 2-parameter family around $g = g_0$, where $G$ has the form $G(x, w) = g_w(x)$ $(x \in M, w \in W(\Xi))$ and $[g]_{\mathcal{W}} = \Xi \in C^2(\mathcal{W}(\mathcal{M});\mathbf{Z})$. We denote by $\Theta(\Xi)$ the subset of $W(\Xi)$ consisting of all points $w$ such that $g_w$ is an element of $\Theta$. If $\Theta(\Xi)$ is empty, then we define $[\Theta : \Xi]$ to be zero. Otherwise, near the origin of $W(\Xi)$, the closure of $\Theta(\Xi)$ is a union of curves in $W(\Xi)$ containing the origin as an end point. Take a sufficiently small circle around the origin which is transverse to $\Theta(\Xi)$. The circle is oriented by the orientation of $W(\Xi)$, while the orientation of the normal bundle of $\Theta(\Xi)$ is induced from the coorientation of $\Theta$. Thus the intersection number of the small circle and $\Theta(\Xi)$ is well-defined, and we define the incidence coefficient $[\Theta : \Xi] \in \mathbf{Z}$ as this intersection number.

The oriented open neighborhood $\mathrm{Int}W(\Xi) \subset \mathbf{R}^2$ of the origin in the parameter space of a generic 2-parameter family around a representative of $\Xi$ is stratified by the subsets $\Theta(\Xi)$ for various $\Theta$. By the definition of the weak equivalence relation (Definition 4.1 (2) and (3)), this stratification of $\mathrm{Int}W(\Xi)$ by $\Theta(\Xi)$ does not depend on the choice of a generic 2-parameter family $G$ around $g$ or a representative $g$ of $\Xi$ up to orientation preserving homeomorphisms. Therefore, the incidence coefficient $[\Theta : \Xi]$ is well-defined. This open disk $\mathrm{Int}W(\Xi)$ is stratified into several 2-dimensional strata, several 1-dimensional strata $\Theta(\Xi)$ and one 0-dimensional stratum $\Xi$ (i.e., the origin). We call this stratified parameter space $\mathrm{Int}W(\Xi)$ with the local stable singular value set $g_w(S(g_w)) \cap U \subset \mathbf{R}^2$ on each 2-dimensional stratum the *bifurcation diagram* of $\Xi$, where $w$ is an element of each 2-dimensional stratum of $\mathrm{Int}W(\Xi)$ (see the definition of a codimension 2 bifurcation diagram in Subsection 3.2 and the paragraph just before Remark 3.11 as well). Here, $U$ is a small open disk neighborhood around the codimension 2 singular value of $g_0$. For each cooriented codimension 2 weak equivalence class $\Xi$, the bifurcation diagram $\mathrm{Int}W(\Xi)$ of $\Xi$ is as depicted in Figure 14. We adopt the convention that the coorientation of each $\mathrm{IV}^*_\star$ is defined by the canonical orientation of $\mathbf{R}^2$ in Figure 14.

Figure 14

Note that for each weak equivalence class of codimension 2, there exists an orientation preserving homeomorphism between the corresponding parameter space of Figure 14 and the canonical parameter space given by Tables 3–5 such that the horizontal axis corresponds to $a$ and the vertical one corresponds to $b$.

By using the incidence coefficients defined above, we define the homomorphism

$$\delta : C^1(\mathcal{W}(\mathcal{M}); \mathbf{Z}) \to C^2(\mathcal{W}(\mathcal{M}); \mathbf{Z}) \text{ by } \delta(\Theta) = \sum_{\Xi} [\Theta : \Xi]\Xi,$$

where $\Theta$ is an arbitrary generator of $C^1(\mathcal{W}(\mathcal{M}); \mathbf{Z})$ and the summation runs over all the generators $\Xi$ of $C^2(\mathcal{W}(\mathcal{M}); \mathbf{Z})$. We call the cochain complex

$$0 \longrightarrow C^1(\mathcal{W}(\mathcal{M}); \mathbf{Z}) \xrightarrow{\ \delta\ } C^2(\mathcal{W}(\mathcal{M}); \mathbf{Z}) \longrightarrow 0$$

the *Vassiliev cochain complex* of the weak equivalence classes.

**Definition 5.1.** We call a non-trivial element of $\mathrm{Ker}(\delta)$ a *Vassiliev* 1-*cocycle* for the Vassiliev cochain complex of the weak equivalence classes.

*Remark* 5.2. Let $g : M \to \mathbf{R}^2$ be a smooth map with exactly two codimension 1 singular values such that the other points in $\mathbf{R}^2$ are either a regular value or a stable singular value. We can naturally define a generic 2-parameter family $G$ around $g$ and a weak equivalence relation for such maps by using a definition similar to that for codimension 2 unstable maps. We could consider the weak equivalence class $\Xi = [g]_{\mathcal{W}}$ of $g$ as an element of $C^2(\mathcal{W}(\mathcal{M}); \mathbf{Z})$ by fixing a coorientation (see Remark 3.8). By the definition of the incidence coefficient, we see that for such a $\Xi$, $[\Theta : \Xi]$ is always zero for any cooriented weak equivalence class $\Theta \in C^1(\mathcal{W}(\mathcal{M}); \mathbf{Z})$. Therefore, in order to determine the Vassiliev 1-cocycles and the associated first order semi-local invariants of stable maps, which will be defined in Subsection 6.2, we can omit such a $\Xi$ from $C^2(\mathcal{W}(\mathcal{M}); \mathbf{Z})$.

## 6. First order semi-local invariants

In this section, we define first order (semi-local) invariants of stable maps. These invariants are constructed from the cocycles of the Vassiliev cochain complex which has been constructed in the previous section. These invariants are isotopy invariants of stable maps.

**6.1. Semi-local invariants.** Let $M$ be a closed orientable 3-dimensional manifold and $\Gamma$ the set of all unstable maps in the mapping space $\mathcal{M} = C^\infty(M, \mathbf{R}^2)$. The set $\mathcal{M} \setminus \Gamma$ consists of all stable maps and we are interested in finding numerical invariants of path-connected components of $\mathcal{M} \setminus \Gamma$.

Let $\Gamma_1$ be the set of all codimension 1 unstable maps and $\Theta$ one of the cooriented weak equivalence classes in $\Gamma_1$. For any generic homotopy $F : M \times [-1, 1] \to \mathbf{R}^2$, we can count the algebraic intersection number of $F$ with $\Theta$. We denote by $\Theta(F) \in \mathbf{Z}$ this intersection number. That is, let $\{t_1, \ldots, t_k\}$ be the set of all codimension 1 bifurcation values $t_i$ of $F$ such that $f_{t_i} : M \to \mathbf{R}^2$ is in the weak equivalence class $\Theta$, where $f_{t_i}$ is defined by $F(x, t_i) = f_{t_i}(x)$. If $F$ crosses $\Gamma_1$ positively at $f_{t_i}$, then we define the sign of $t_i$ to be

$+1$; otherwise $-1$. Then we define the integer $\Theta(F)$ to be the sum of signs $\pm 1$ over all codimension 1 bifurcation values $t_i \in (-1,1)$ of type $\Theta$ $(1 \leq i \leq k)$.

Any formal sum $c = \sum_j a_j \Theta_j \in C^1(\mathcal{W}(\mathcal{M}); \mathbf{Z})$ such that $a_j \in \mathbf{Z}$ and each $\Theta_j$ is a cooriented weak equivalence class in $\Gamma_1$, defines an integer valued function on the set of all generic homotopies as follows:

$$c : \{\text{generic homotopies}\} \to \mathbf{Z}, \quad c(F) = \sum_j a_j \Theta_j(F).$$

**Definition 6.1.** We say that a formal sum $c = \sum a_j \Theta_j$ defines a *semi-local invariant* of stable maps if for any generic homotopy $F$, the value $c(F)$ depends only on the isotopy classes of the stable maps $f_{-1}$ and $f_1$, where $f_{-1}$ is the initial stable map of $F$ and $f_1$ is the terminal one.

In fact, given such a $c$, taking a distinguished map $g \in \mathcal{M} \setminus \Gamma$ and a constant $\alpha_0 \in \mathbf{Z}$, we can define the semi-local invariant $L_c : \mathcal{M} \setminus \Gamma \to \mathbf{Z}$ of stable maps by $L_c(f) = c(F) + \alpha_0$. Here, $F$ is any generic homotopy between $g$ and $f$. Note that $L_c$ is an isotopy invariant of stable maps and $L_c(g) = \alpha_0$.

*Remark* 6.2. "*Semi-local*" means the following. The increment of the value $c(F)$ at $\Theta_j$ is determined by the coorientation of $\Theta_j$. The weak equivalence class $\Theta_j$ and its coorientation can be recognized only by looking at the homeomorphism type of the local deformation of the singular value set and the diffeomorphism types of the associated semi-local singular fibers. For these fibers, we consider the diffeomorphism type of a whole fiber instead of the multi-germ at the singular points contained in a fiber. Therefore, we say that $L_c$ is a "*semi-local*" invariant.

6.2. **First order semi-local invariants.** Let $M$ be a closed orientable 3-dimensional manifold and $\mathcal{M} = C^\infty(M, \mathbf{R}^2)$ the mapping space. Since $\mathcal{M}$ is contractible, we have the following.

**Proposition 6.3.** *Let $c$ be a non-zero element in $C^1(\mathcal{W}(\mathcal{M}); \mathbf{Z})$. Then $c$ is a Vassiliev 1-cocycle if and only if $c$ induces a semi-local invariant $L_c$ of isotopy classes of stable maps.*

In view of Proposition 6.3, we say that each Vassiliev 1-cocycle $c$ induces a *first order semi-local invariant* of stable maps. In general meaning of Remark 6.4 stated below, an isotopy invariant $L : \mathcal{M} \setminus \Gamma \to \mathbf{Z}$ is said to be of *first order* if $L$ can be extended to $\bar{L} : \mathcal{M} \to \mathbf{Z}$ satisfying the following conditions:

(1) $\bar{L}$ is constant on each connected component of $\Gamma_1$,
(2) $\bar{L}$ is constantly zero over $\Gamma \setminus \Gamma_1$,

(3) for each cooriented stratum $\Theta$ of $\Gamma_1$ and for any generic homotopy $F : M \times [-1,1] \to \mathbf{R}^2$ around $f_0$ such that $F(x,t) = f_t(x)$ and $F$ crosses $\Theta$ positively at $f_0$, it holds that $\bar{L}(f_0) = \bar{L}(f_1) - \bar{L}(f_{-1})$. Here, each stratum $\Theta$ of $\Gamma_1$ and its coorientation is the general meaning of Remark 6.4 stated below.

Therefore, if $c$ induces a semi-local invariant $L_c$ of isotopy classes of stable maps, then $L_c$ is a first order invariant.

*Proof of Proposition* 6.3. Let $c = \sum a_j \Theta_j$ be any Vassiliev 1-cocycle of $C^1(\mathcal{W}(\mathcal{M}); \mathbf{Z})$ and $F : M \times S^1 \to \mathbf{R}^2$ a generic loop in $\mathcal{M}$ in the sense of the definition just before (2) in Subsection 3.2. In order to prove that $c$ defines a semi-local invariant of isotopy classes of stable maps, we have only to show that $c(F) = 0$ holds. Since $\mathcal{M}$ is contractible, there exists a generic 2-parameter family $G : M \times D^2 \to \mathbf{R}^2$ such that $G|M \times \partial D^2 = F$ (see Proposition 3.10 and (2) in Subsection 3.2). We fix an orientation of $D^2$. By the definition of a generic 2-parameter family, $D^2$ is stratified into finitely many strata. This stratification is the same as that of $W \subset \mathbf{R}^2$ constructed in (2) of Subsection 3.2.

Let $p_1, \ldots, p_k$ be the 0-dimensional strata in $\mathrm{Int}D^2$. We take small disjoint $k$ disks $D(p_i)$ $(1 \leq i \leq k)$ centered at $p_i$ in $\mathrm{Int}D^2$. We may assume that each $\partial D(p_i)$ intersects the 1-dimensional strata of $D^2$ transversely. The orientation of $D(p_i)$ is induced from that of $D^2$. For each $p_i$, let $\Xi_i$ be the cooriented weak equivalence class of $g_{p_i}$, where $g_{p_i}$ is defined by $g_{p_i}(x) = G(x, p_i)$. The orientations of $\partial D^2$ and $\partial D(p_i)$ are induced from those of $D^2$ and $D(p_i)$ respectively. Note that $\partial D^2$ is homologous to $\sum_{i=1}^k \partial D(p_i)$ in $D^2$.

We define the generic loop $F_i : M \times \partial D(p_i) \to \mathbf{R}^2$ by $F_i = G|M \times \partial D(p_i)$ $(1 \leq i \leq k)$. It is easy to see that the intersection number $c(F) = \sum_j a_j \Theta_j(F)$ is equal to $\sum_{i=1}^k (\sum_j a_j \Theta_j(F_i))$. By the definition of the incidence coefficient, we have $\Theta_j(F_i) = [\Theta_j : \Xi_i]$ for each $i$ and $j$. Since $c = \sum_j a_j \Theta_j$ is a Vassiliev 1-cocycle, we have

$$\sum_j a_j [\Theta_j : \Xi_i] = 0$$

for each $i$ $(1 \leq i \leq k)$. Therefore, $c(F) = 0$ holds.

Conversely, suppose that for a non-zero element $c \in C^1(\mathcal{W}(\mathcal{M}); \mathbf{Z})$, we have $c(F) = 0$ for any generic loop $F : M \times S^1 \to \mathbf{R}^2$. For each element $\Xi \in C^2(\mathcal{W}(\mathcal{M}); \mathbf{Z})$, we take a generic 2-parameter family $G : M \times W \to \mathbf{R}^2$ around a representative of $\Xi$. Then $c(G|M \times \partial W) = 0$ holds by our assumption. Since $c(G|M \times \partial W)$ is equal to the coefficient of $\Xi$ for $\delta(c)$, we have $\delta(c) = 0$ and $c$ is a Vassiliev 1-cocycle.

This completes the proof. □

*Remark* 6.4. If we define another equivalence relation on $\Gamma_1$ and $\Gamma_2$, and if we have a well-defined coboundary operator, then we can construct another Vassiliev cochain complex. This means that a first order invariant depends on the equivalence relations on $\Gamma_1$ and

$\Gamma_2$, and on the definition of the coorientation on each equivalence class of $\Gamma_1$ (see [45], for example).

*Remark* 6.5. In [6], Birman and Lin reformulated order $i$ Vassiliev invariants of knots in $S^3$ in terms of a *skein relation* ($i \geq 1$). If we can define a suitable "skein relation" for generic maps $M \to N$ between manifolds, then we can construct *higher order finite-type invariants* for these generic maps. In [11, 12, 13], Ekholm determined the higher order finite-type invariants for immersions $S^k \looparrowright \mathbf{R}^{2k-1}$ ($k \geq 3$) and $S^k \looparrowright \mathbf{R}^{2k-2}$ ($k \geq 3$). In [22], Kamada determined the higher order finite-type invariants for immersions $M^2 \looparrowright \mathbf{R}^4$ of an oriented closed connected surface $M^2$. In [21], Januszkiewicz and Świątkowski determined the higher order finite-type invariants for immersions $M^n \looparrowright \mathbf{R}^{2n}$ of a closed connected $n$-dimensional manifold $M^n$. In [18], Habiro, Kanenobu and Shima determined the higher order finite-type invariants for ribbon 2-knots $S^2 \hookrightarrow \mathbf{R}^4$. Nowik determined the higher order finite-type invariants of immersions $M^2 \looparrowright \mathbf{R}^3$ of a closed surface $M^2$ in [36] and those of immersions $M^2 \looparrowright \mathbf{R}^3$ of a closed orientable surface $M^2$ in [37, 38].

But unfortunately, these higher order finite-type invariants are obtained as polynomials of first order finite-type invariants (see [11, 12, 13, 21, 22, 36, 37, 38]) or derivatives of the Alexander polynomial (see [18]). These results mean that we have not yet obtained essential second or higher order invariants when the dimension of the source manifold is strictly greater than one.

*Remark* 6.6. In this paper, we use the Vassiliev cochain complex to find first order invariants for stable maps. Kazarian [24, 25] used the Vassiliev cochain complex to study the characteristic classes of circle bundles. In this case, the Vassiliev cochain complex is constructed by considering fiberwise smooth functions on a total space. This is another application of the Vassiliev cochain complex.

## 7. First order semi-local invariants of stable maps

In this section, we determine the first order semi-local invariants of stable maps and clarify the geometric meanings of all the invariants.

### 7.1. Computation of the coboundary operator.

In this subsection, we determine the coboundary operator and all Vassiliev 1-cocycles for the Vassiliev cochain complex of the weak equivalence classes which has been defined in Section 5.

By looking at each bifurcation diagram $\mathrm{Int}W(\Xi)$ in Figure 14, we obtain the following.

**Proposition 7.1.** *The coboundary operator*

$$\delta : C^1(\mathcal{W}(\mathcal{M}); \mathbf{Z}) \to C^2(\mathcal{W}(\mathcal{M}); \mathbf{Z})$$

*of the Vassiliev cochain complex of the weak equivalence classes is represented by the following block matrix with respect to the natural bases of* $C^1(\mathcal{W}(\mathcal{M});\mathbf{Z})$ *and* $C^2(\mathcal{W}(\mathcal{M});\mathbf{Z})$:

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} & A_{1,5} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} & A_{2,5} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{24,1} & A_{24,2} & A_{24,3} & A_{24,4} & A_{24,5} \end{pmatrix}.$$

*Here, the natural (ordered) bases of* $C^1(\mathcal{W}(\mathcal{M});\mathbf{Z})$ *(resp.* $C^2(\mathcal{W}(\mathcal{M});\mathbf{Z})$*) corresponds to the weak equivalence classes as depicted in Figure* 12 *(resp. Figure* 13*). The non-zero blocks* $A_{i,j}$ *are given in Tables* 6–37. *The other blocks which are not in Tables* 6–37 *are the zero blocks. Note that* $A$ *is a* $389 \times 56$ *matrix.*

Tables $6, \cdots, 37$

By a direct calculation, we get the following.

**Theorem 7.2.** *The rank of* $\mathrm{Ker}(\delta)$ *is equal to seven and the following cochains* $c_1, c_2, \ldots, c_7$ *constitute a basis of* $\mathrm{Ker}(\delta)$:

(1) $c_1 = \mathrm{III}^a(l) + \mathrm{III}^a(b) + \mathrm{III}^b + \mathrm{III}^c + \mathrm{III}^d$,
(2) $c_2 = \mathrm{III}^b + \mathrm{III}^{0,a} + 2\mathrm{III}^{0,0}$,
(3) $c_3 = \mathrm{III}^{0,a} + \mathrm{III}^{1,a} + \mathrm{III}^e + 2\mathrm{III}^{0,1}$,
(4) $c_4 = \mathrm{III}^{1,a} + 2\mathrm{III}^{1,1} + \mathrm{III}^4_1 - \mathrm{III}^4_2 + \mathrm{III}^4_3$,
(5) $c_5 = \mathrm{III}^c + \mathrm{III}^e + 2\mathrm{III}^2 - \mathrm{III}^4_1 + \mathrm{III}^4_2 - \mathrm{III}^4_3 + 3\mathrm{III}^7_1 + \mathrm{III}^7_2$,
(6) $c_6 = \mathrm{III}^d + 2\mathrm{III}^3 - 3\mathrm{III}^7_1 - \mathrm{III}^7_2$,
(7) $c_7 = 2\mathrm{III}^3_2 - 2\mathrm{III}^7_1 + \mathrm{III}^8$.

*For* $\mathrm{III}^* = \mathrm{III}^{0,a}, \mathrm{III}^{1,a}, \mathrm{III}^e, \mathrm{III}^{0,0}, \mathrm{III}^{0,1}, \mathrm{III}^{1,1}, \mathrm{III}^2$ *and* $\mathrm{III}^3$, *we have set* $\mathrm{III}^* = \sum_\star \mathrm{III}^*_\star$.

The above proposition means that any Vassiliev 1-cocycle for the Vassiliev cochain complex of the weak equivalence classes is a linear combination of $c_1, c_2, \ldots, c_7$.

7.2. **Geometric interpretations of the 1-cocycles.** In this subsection, we give a geometric interpretation of each Vassiliev 1-cocycle $c_i \in \mathrm{Ker}(\delta)$ $(1 \le i \le 7)$ given in Theorem 7.2.

**Theorem 7.3.** *Let* $f$ *and* $g : M \to \mathbf{R}^2$ *be two stable maps of a closed orientable 3-dimensional manifold into the plane and* $F : M \times [-1,1] \to \mathbf{R}^2$ *a generic homotopy such that the initial stable map of* $F$ *is* $g$ *and the terminal one is* $f$. *Then we have the following for each* $c_i(F) \in \mathbf{Z}$ $(1 \le i \le 7)$.

(1) *The value* $c_1(F) \in \mathbf{Z}$ *is equal to* $(\sharp\mathrm{II}^a(f) - \sharp\mathrm{II}^a(g))/2$.
(2) *The value* $c_2(F) \in \mathbf{Z}$ *is equal to* $\sharp\mathrm{II}^{0,0}(f) - \sharp\mathrm{II}^{0,0}(g)$.
(3) *The value* $c_3(F) \in \mathbf{Z}$ *is equal to* $\sharp\mathrm{II}^{0,1}(f) - \sharp\mathrm{II}^{0,1}(g)$.

(4) *The value* $c_4(F) \in \mathbf{Z}$ *is equal to* $\sharp \mathrm{II}^{1,1}(f) - \sharp \mathrm{II}^{1,1}(g)$.

(5) *The value* $c_5(F) \in \mathbf{Z}$ *is equal to* $\sharp \mathrm{II}^2(f) - \sharp \mathrm{II}^2(g)$.

(6) *The value* $c_6(F) \in \mathbf{Z}$ *is equal to* $\sharp \mathrm{II}^3(f) - \sharp \mathrm{II}^3(g)$.

(7) *The value* $c_7(F) \in \mathbf{Z}$ *is equal to* $(\mathrm{rot}(f) - \mathrm{rot}(g))/2$.

*Here, we denote by* $\sharp \mathrm{II}^*(f)$ *the number of* $\mathrm{II}^*$*-type singular fibers of* $f$ *and by* $\mathrm{rot}(f)$ *the surgical rotation number of* $f(S(f))$.

By observing Figures 7 and 12 carefully, we can prove (1)–(6). Thus, the proof of (1)–(6) is easy and is left to the reader. To understand and prove (7), we first have to define the *surgical rotation number* of a singular value set. In the next subsection, we define the surgical rotation number, and after that we prove item (7) of Theorem 7.3.

7.3. **Surgical rotation number.** Let $M$ be a closed orientable 3-dimensional manifold and $f : M \to \mathbf{R}^2$ a stable map. The definition of the surgical rotation number of $f(S(f))$ is given as follows.

By the rule mentioned in Remark 4.6, the family of curves $f(S(f))$, except for its nodes and cusp points, is oriented. Let $N_f(\mathrm{II}^3)$ be the set of those nodes $y$ of $f(S(f))$ such that $f^{-1}(y)$ is weakly equivalent to the $\mathrm{II}^3$-type singular fiber. Then we can define the smoothing operation on $f(S(f))$ at $y \in N_f(\mathrm{II}^3)$ as follows.

The node $y$ is adjacent to four regions, say $\Omega_i$, $1 \le i \le 4$, of $\mathbf{R}^2 \setminus f(S(f))$, although there might be a repetition. We may assume that $n_f(\Omega_1) = n_f(\Omega_2) = a$ and $n_f(\Omega_3) = n_f(\Omega_4) = a + 1$ for some $a \ge 1$ (see Remark 4.6 and Figure 10, $\mathrm{II}^3$). Let us consider the smoothing operation which connects $\Omega_1$ and $\Omega_2$ as in Figure 15.

Figure 15

After the smoothing operation at each node in $N_f(\mathrm{II}^3)$, we obtain oriented plane curves with cusps $\hat{f} : \cup S^1 \to \mathbf{R}^2$. We define the *generalized rotation number* of $\hat{f}$ as the total degree of the tangent line map $(\hat{f})' : \cup S^1 \to \mathbf{R}P^1$ associated with $\hat{f}$ (for details, see below). Then we call the generalized rotation number of $\hat{f}$ the *surgical rotation number* of $f(S(f))$, and we denote it by $\mathrm{rot}(f)$.

The generalized rotation number is defined as follows. Let $h : \cup S^1 \to \mathbf{R}^2$ be oriented smooth curves such that each singular point $x \in \cup S^1$ is a cusp point. That is, around $x$ and $h(x)$, we can choose local coordinates $t$ and $(u, v)$ respectively such that $u \circ h = t^2$ and $v \circ h = t^3$ hold. We set $S(h) = \{x \in \cup S^1 | \operatorname{rank} dh_x = 0\}$ and call it the singular set of $h$. Fix the standard orientation on $\mathbf{R}^2$ and take the unit circle in $\mathbf{R}^2$ with the counterclockwise orientation. Orient the space of lines through the origin in $\mathbf{R}^2$, i.e. $\mathbf{R}P^1$, so that its double covering by the unit circle, which coincides with the space of directions, is orientation preserving. For the smooth curve $h$, we define the smooth map $h' : \cup S^1 \to \mathbf{R}P^1$ as follows. If $x \notin S(h)$, then $h'(x)$ is the derivative of $h$ at $x$, that is,

$h'(x)$ is the line through the origin in $\mathbf{R}^2$ parallel to the tangent line of $h$ at $h(x)$. It is easy to see that the map $h' : \cup S^1 \setminus S(h) \to \mathbf{R}P^1$ has a unique smooth extension to all of $\cup S^1$, which we denote by the same symbol $h'$. The *generalized rotation number* of $h$ is defined to be $\sum \deg(h'|S^1)$, where the summation runs over all the components $S^1$ of $\cup S^1$, the source of $h'$, and $\deg(h'|S^1)$ is the degree of $h'|S^1$. For more details, see [28].

7.4. **Proof of Theorem 7.3 (7).** In this subsection, we prove Theorem 7.3 (7).

Let $F : M \times [-1, 1] \to \mathbf{R}^2$ be a generic homotopy between two stable maps $f_{-1}$ and $f_1$ such that 0 is the unique codimension 1 bifurcation value, where $f_t$ is defined by $F(x,t) = f_t(x)$. We suppose that $F$ crosses $\Gamma_1$ positively at $f_0$. To prove Theorem 7.3 (7), we have only to check that $c_7(F) = (\mathrm{rot}(f_1) - \mathrm{rot}(f_{-1}))/2$ holds.

(1) When $f_0 \in \mathrm{III}^3_2$.
As is seen in Figure 16 (1), $f_1$ has exactly two additional nodefolds of type $\mathrm{II}^3$ when compared with $f_{-1}$. Then, looking at Figure 16 (1), we see easily that $\mathrm{rot}(f_1) - \mathrm{rot}(f_{-1}) = 4$. On the other hand, $c_7(F) = 2$ holds. Therefore, we have $c_7(F) = (\mathrm{rot}(f_1) - \mathrm{rot}(f_{-1}))/2$.

(2) When $f_0 \in \mathrm{III}^7_1$.
As is seen in Figure 16 (2), the number of nodefolds of type $\mathrm{II}^3$ for $f_1$ is smaller than that for $f_{-1}$ by three. Then, looking at Figure 16 (2), we see easily that $\mathrm{rot}(f_1) - \mathrm{rot}(f_{-1}) = -4$. On the other hand, $c_7(F) = -2$ holds. Therefore, we have $c_7(F) = (\mathrm{rot}(f_1) - \mathrm{rot}(f_{-1}))/2$.

(3) When $f_0 \in \mathrm{III}^8$.
As is seen in Figure 16 (3), the number of nodefolds of type $\mathrm{II}^3$ for $f_1$ is equal to that for $f_{-1}$. But, looking at Figure 16 (3), we see easily that $\mathrm{rot}(f_1) - \mathrm{rot}(f_{-1}) = 2$. On the other hand, $c_7(F) = 1$ holds. Therefore, we have $c_7(F) = (\mathrm{rot}(f_1) - \mathrm{rot}(f_{-1}))/2$.

For the case of the other weak equivalence classes, we see easily that $\mathrm{rot}(f_1) - \mathrm{rot}(f_{-1}) = 0$ holds. This completes the proof. □

Figure 16

*Remark* 7.4. For a stable map $f : M \to \mathbf{R}^2$ of a closed orientable 3-dimensional manifold, we have the quotient space $W_f$, where we identify points in the same connected component of each fiber of $f$. This space $W_f$ is called *Stein factorization* of $f$ (see [26, 29, 33, 43, 44, 45]). It is known that $W_f$ is a compact 2-dimensional polyhedron. It is easy to see that $\mathrm{rot}(f)/2 = \chi(W_f)$ holds for a stable map $f$, where $\chi(W_f)$ is the Euler characteristic of $W_f$. Therefore, we have $c_7(F) = \chi(W_{f_1}) - \chi(W_{f_{-1}})$ in Theorem 7.3 (7).

7.5. **Linear independence of the first order semi-local invariants.** Let $M$ be a closed orientable 3-dimensional manifold, $\mathcal{M} = C^\infty(M, \mathbf{R}^2)$ the mapping space and $\Gamma \subset$

$\mathcal{M}$ the set of all unstable maps. It is easy to see that the following eight invariants $L_i : \mathcal{M} \setminus \Gamma \to \mathbf{Z}$ $(0 \leq i \leq 7)$ are isotopy invariants of stable maps:

(0) $L_0(f) = 1$, (1) $L_1(f) = \sharp \mathrm{II}^a(f)/2$, (2) $L_2(f) = \sharp \mathrm{II}^{0,0}(f)$,

(3) $L_3(f) = \sharp \mathrm{II}^{0,1}(f)$, (4) $L_4(f) = \sharp \mathrm{II}^{1,1}(f)$, (5) $L_5(f) = \sharp \mathrm{II}^2(f)$,

(6) $L_6(f) = \sharp \mathrm{II}^3(f)$, (7) $L_7(f) = \mathrm{rot}(f)/2$.

Here, we denote by $\sharp \mathrm{II}^*(f)$ the number of $\mathrm{II}^*$-type singular fibers of $f$. By [48], $\sharp \mathrm{II}^a(f)$ is always even for any stable map $f : M \to \mathbf{R}^2$ (see also [27]). Furthermore, by using this property and the definition of the generalized rotation number of $f(S(f))$, we see that $\mathrm{rot}(f)$ is also always even. Therefore, both $L_1$ and $L_7$ are integer valued invariants. By Theorems 7.2 and 7.3, the seven invariants $L_1, L_2, \ldots, L_7$ are first order semi-local invariants and we have $L_i(f) = c_i(F) + L_i(g)$, where $g \in \mathcal{M} \setminus \Gamma$ is a distinguished stable map and $F : M \times [-1, 1] \to \mathbf{R}^2$ is a generic homotopy between $g$ and $f$.

In this subsection, we show that the eight isotopy invariants $L_0, L_1, \ldots, L_7$ are linearly independent for all $M$. To prove this, we construct eight examples of stable maps $f_1, f_2, \ldots, f_8 : S^3 \to \mathbf{R}^2$ such that the determinant of the matrix $(L_j(f_i))_{1 \leq i \leq 8, 0 \leq j \leq 7}$ is equal to 1.

If we can construct such examples for $M = S^3$, then we can prove the linear independence of $L_0, L_1, \ldots, L_7$ for an arbitrary $M$ as follows. Let $f : S^3 \to \mathbf{R}^2$ be a stable map of the 3-dimensional sphere and $g : M \to \mathbf{R}^2$ a stable map of a closed orientable 3-dimensional manifold. Then we can construct a stable map $f \sharp g : S^3 \sharp M \to \mathbf{R}^2$ such that $L_0(f \sharp g) = 1$, $L_i(f \sharp g) = L_i(f) + L_i(g)$ $(1 \leq i \leq 6)$ and $L_7(f \sharp g) = L_7(f) + L_7(g) - 1$, where $\sharp$ means connected sum of maps or manifolds. For a definition of the stable map $f \sharp g : S^3 \sharp M \to \mathbf{R}^2$, see [43, Lemma 5.4].

For any closed orientable 3-dimensional manifold $M$, let us define the isotopy invariant of stable maps $L : C^\infty(M, \mathbf{R}^2) \setminus \Gamma \to \mathbf{Z}^8$ by $L = (L_0, L_1, \ldots, L_7)$. For stable maps $f_i : S^3 \to \mathbf{R}^2$ $(1 \leq i \leq 8)$, suppose that the determinant of the $8 \times 8$ matrix

$$\widehat{L}(f_1, f_2, \ldots, f_8) = \begin{pmatrix} L(f_1) \\ \cdots\cdots \\ L(f_8) \end{pmatrix}$$

is non-zero. Then, we see that the determinant of $\widehat{L}(f_1 \sharp g, f_2 \sharp g, \ldots, f_8 \sharp g)$ is equal to the determinant of $\widehat{L}(f_1, f_2, \ldots, f_8)$ for the stable maps $f_i \sharp g : M \to \mathbf{R}^2$, where $g : M \to \mathbf{R}^2$ is any stable map. Therefore, for all $M$, $L_0, L_1, \ldots, L_7$ are linearly independent.

Set $S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbf{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$, and let $\pi : \mathbf{R}^4 \to \mathbf{R}^2$ be the standard projection defined by $\pi(x_1, x_2, x_3, x_4) = (x_1, x_2)$. Then we can construct the following examples:

(1) stable map $f_1 = \pi|S^3 : S^3 \to \mathbf{R}^2$ whose singular value set and associated fibers are as depicted in Figure 17 (1) with $L(f_1) = (1,0,0,0,0,0,0,1)$,
(2) stable map $f_2 : S^3 \to \mathbf{R}^2$ whose singular value set and associated fibers are as depicted in Figure 17 (2) with $L(f_2) = (1,1,0,0,0,0,0,1)$,
(3) stable map $f_3 : S^3 \to \mathbf{R}^2$ whose singular value set and associated fibers are as depicted in Figure 17 (3) with $L(f_3) = (1,1,1,0,0,0,0,1)$,
(4) stable map $f_4 : S^3 \to \mathbf{R}^2$ whose singular value set and associated fibers are as depicted in Figure 17 (4) with $L(f_4) = (1,1,1,1,0,0,0,1)$,
(5) stable map $f_5 : S^3 \to \mathbf{R}^2$ whose singular value set and associated fibers are as depicted in Figure 17 (5) with $L(f_5) = (1,1,0,1,1,0,0,1)$,
(6) stable map $f_6 : S^3 \to \mathbf{R}^2$ whose singular value set and associated fibers are as depicted in Figure 17 (6) with $L(f_6) = (1,1,0,1,0,1,0,1)$,
(7) stable map $f_7 : S^3 \to \mathbf{R}^2$ whose singular value set and associated fibers are as depicted in Figure 17 (7) with $L(f_7) = (1,2,0,0,0,0,1,1)$,
(8) stable map $f_8 : S^3 \to \mathbf{R}^2$ whose singular value set and associated fibers are as depicted in Figure 17 (8) with $L(f_8) = (1,0,0,0,0,0,0,2)$.

The stable map $f_1$ has neither $\mathrm{I}^1$- nor $\mathrm{II}^*$-type singular fibers. The stable map $f_2$ is obtained from $f_1$ by crossing $\mathrm{III}^a(l)$ positively. The stable map $f_3$ is obtained from $f_1$ by crossing $\mathrm{III}^b$ positively. The stable map $f_4$ is obtained from $f_2$ by crossing $\mathrm{III}_2^{0,a}$ positively. The stable map $f_5$ is obtained from $f_2$ by crossing $\mathrm{III}^a(l)$ positively twice, $\mathrm{III}_2^{1,a}$ positively once, and $\mathrm{III}^a(b)$ negatively twice. The stable map $f_6$ is obtained from $f_2$ by crossing $\mathrm{III}_2^e$ positively. The stable map $f_7$ is obtained from $f_2$ by crossing $\mathrm{III}^d$ positively. The stable map $f_8$ does not have $\mathrm{II}^*$-type singular fibers.

The determinant of $\widehat{L}(f_1, f_2, \ldots, f_8)$ is equal to 1. Therefore, the first order semi-local invariants $L_1, L_2, \ldots, L_7$ together with the constant invariant $L_0$ are linearly independent.

Figure 17

**Problem 7.5.** It is easy to construct a generic homotopy $F_i : S^3 \times [-1,1] \to \mathbf{R}^2$ between $f_1$ and $f_i$ ($2 \le i \le 7$). On the other hand, the stable map $f_8$ is directly constructed from a Heegaard splitting of $S^3$ (see [33] for example). The author does not know how to construct a generic homotopy $F_8 : S^3 \times [-1,1] \to \mathbf{R}^2$ between $f_1$ and $f_8$.

The following proposition is easy to see and its proof is left to the reader.

**Proposition 7.6.** *The seven first order semi-local invariants $L_1, L_2, \ldots, L_7$ are invariants of the $C^\infty$ right-left equivalence classes.*

This proposition means the following. Let $f$ and $g : M \to \mathbf{R}^2$ be two stable maps of a closed orientable 3-dimensional manifold such that $f$ and $g$ are $C^\infty$ right-left equivalent but are not necessarily isotopic. Then $L_i(f) = L_i(g)$ holds for each $i = 1, 2, \dots, 7$, and we conclude that these first order semi-local invariants cannot distinguish $f$ from $g$.

## 8. A NON-LOCAL FIRST ORDER INVARIANT OF STABLE MAPS

In this section, we subdivide the weak equivalence classes of unstable maps of codimensions 1 and 2 using their global properties. By using these new classifications of $\Gamma_1$ and $\Gamma_2$, we give a first order non-local invariant of stable maps and clarify the geometric meaning of this invariant. We use the adjective *"non-local"*, since we need to know the global behavior of the singular (value) set in order to classify $\Gamma_1$ and $\Gamma_2$ (see Remarks 6.2 and 8.3).

Let $f : M \to \mathbf{R}^2$ be a codimension 1 unstable map whose weak equivalence class is $\mathrm{III}^a(b)$ and $y \in \mathbf{R}^2$ the corresponding codimension 1 singular value of $f$. Let $F : M \times [-1, 1] \to \mathbf{R}^2$ be a generic homotopy around $f = f_0$, where $f_t : M \to \mathbf{R}^2$ is defined by $f_t(x) = F(x, t)$ and $F$ crosses $\Gamma_1$ positively at $f_0$.

Suppose $x_1$ and $x_2 \in S(f_1)$ are the new born cusps of $f_1$ in the generic homotopy $F$. The codimension 1 weak equivalence class $\mathrm{III}^a(b)$ of $\Gamma_1$ can be subdivided into the following three disjoint classes.

(a) The class $\mathrm{III}^a(b_1)$ consisting of the codimension 1 unstable maps in $\mathrm{III}^a(b)$ such that the new born cusps $x_1$ and $x_2$ belong to the same connected component of $S(f_1)$ and it corresponds to two connected components of $S(f_{-1})$. See Figure 18 (a).

(b) The class $\mathrm{III}^a(b_2)$ consisting of the codimension 1 unstable maps in $\mathrm{III}^a(b)$ such that the new born cusps $x_1$ and $x_2$ belong to the same connected component of $S(f_1)$ and it corresponds to exactly one connected component of $S(f_{-1})$. See Figure 18 (b).

(c) The class $\mathrm{III}^a(b_3)$ consisting of the codimension 1 unstable maps in $\mathrm{III}^a(b)$ such that the new born cusps belong to distinct connected components of $S(f_1)$. See Figure 18 (c).

Figure 18

We consider the coorientations of $\mathrm{III}^a(b_1)$, $\mathrm{III}^a(b_2)$ and $\mathrm{III}^a(b_3)$ are induced from that of $\mathrm{III}^a(b)$. According to this new classification of codimension 1 unstable maps, certain bifurcation diagrams in Figure 14 have to be modified. The class $\mathrm{III}^a(b)$ appears in the bifurcation diagrams of $\mathrm{IV}^a, \mathrm{IV}^b, \mathrm{IV}^c, \mathrm{IV}^h, \mathrm{IV}^i, \mathrm{IV}^j, \mathrm{IV}^k$, $\mathrm{IV}^{0,a}(b), \mathrm{IV}^{1,a}(b)$ and $\mathrm{IV}^e(b)$ in Figure 14.

(1) Stratum $\mathrm{IV}^a$.

The stratum $\mathrm{III}^a(b)$ can only be of type $\mathrm{III}^a(b_3)$. Therefore, $\mathrm{IV}^a$ is not subdivided.

(2) Stratum $\mathrm{IV}^b$ or $\mathrm{IV}^c$.

In the bifurcation diagrams for $\mathrm{IV}^b$ and $\mathrm{IV}^c$ (Figure 14 (a)), we have the following three cases:

(a) the stratum of type $\mathrm{III}^a(b)$ on the left hand side is of type $\mathrm{III}^a(b_1)$ and the stratum on the right hand side is of type $\mathrm{III}^a(b_3)$,

(b) the stratum of type $\mathrm{III}^a(b)$ on the left hand side is of type $\mathrm{III}^a(b_3)$ and the stratum on the right hand side is of type $\mathrm{III}^a(b_1)$,

(c) both of the two strata of type $\mathrm{III}^a(b)$ are of type $\mathrm{III}^a(b_2)$.

Therefore, the stratum $\mathrm{IV}^b$ (resp. $\mathrm{IV}^c$) is subdivided into $\mathrm{IV}^b_1$, $\mathrm{IV}^b_2$ and $\mathrm{IV}^b_3$ (resp. $\mathrm{IV}^c_1$, $\mathrm{IV}^c_2$ and $\mathrm{IV}^c_3$). The coorientations of $\mathrm{IV}^*_i$ $(i = 1, 2, 3)$ are induced from that of $\mathrm{IV}^*$. Here, $\mathrm{IV}^* = \mathrm{IV}^b$ or $\mathrm{IV}^c$.

(3) Stratum $\mathrm{IV}^h, \mathrm{IV}^i, \mathrm{IV}^j, \mathrm{IV}^k, \mathrm{IV}^{0,a}(b), \mathrm{IV}^{1,a}(b)$ or $\mathrm{IV}^e(b)$.

In the bifurcation diagrams for these classes (Figure 14 (a)), we have the following three cases:

(a) both of the two strata of type $\mathrm{III}^a(b)$ are of type $\mathrm{III}^a(b_1)$,

(b) both of the two strata of type $\mathrm{III}^a(b)$ are of type $\mathrm{III}^a(b_2)$.

(c) both of the two strata of type $\mathrm{III}^a(b)$ are of type $\mathrm{III}^a(b_3)$.

Therefore, the stratum $\mathrm{IV}^*$ (resp. $\mathrm{IV}^*(b)$) is subdivided into $\mathrm{IV}^*_1$, $\mathrm{IV}^*_2$ and $\mathrm{IV}^*_3$ (resp. $\mathrm{IV}^*_1(b)$, $\mathrm{IV}^*_2(b)$ and $\mathrm{IV}^*_3(b)$). The coorientations of $\mathrm{IV}^*_i$ or $\mathrm{IV}^*_i(b)$ $(i = 1, 2, 3)$ are induced from those of $\mathrm{IV}^*$ or $\mathrm{IV}^*_i(b)$ respectively. Here, $\mathrm{IV}^* = \mathrm{IV}^h, \mathrm{IV}^i, \mathrm{IV}^j$ or $\mathrm{IV}^k$ (resp. $\mathrm{IV}^*(b) = \mathrm{IV}^{0,a}(b), \mathrm{IV}^{1,a}(b)$ or $\mathrm{IV}^e(b)$).

Let us consider smooth maps with exactly two codimension 1 singular values such that the other singular values are all stable singular values. Let $f$ and $g : M \to \mathbf{R}^2$ be such smooth maps such that $y^f_1$ and $y^f_2 \in \mathbf{R}^2$ are codimension 1 singular values of $f$, and $y^g_1$ and $y^g_2 \in \mathbf{R}^2$ are codimension 1 singular values of $g$. These two maps $f$ and $g$ are weakly equivalent if $f^{-1}(y^f_i)$ and $g^{-1}(y^g_i)$ are *weakly equivalent* in the sense of Definition 4.1 (2) $(i = 1, 2$ and we exchange $y^g_1$ and $y^g_2$ if necessary). For the study of first order "non-local" invariants of stable maps, we have to study the bifurcation diagram of the weak equivalence class $\Xi = [f]_{\mathcal{W}}$ of such a map by fixing a coorientation (see Remarks 3.8 and 5.2). We have the following two cases to consider.

Let $f : M \to \mathbf{R}^2$ be a smooth map with exactly two codimension 1 singular values $y_1$ and $y_2 \in \mathbf{R}^2$ such that the other singular values are all stable singular values of $f$.

(4) Either $f^{-1}(y_1)$ or $f^{-1}(y_2)$ is not a $\mathrm{III}^a(b)$-type singular fiber.

By the definition of the subdivided codimension 1 weak equivalence classes, we see that for such a $\Xi$, the incidence coefficient $[\Theta : \Xi]$ is always zero for any subdivided codimension 1 weak equivalence class $\Theta$. Thus, we can ignore such a $\Xi$ in order to determine first order non-local invariants of stable maps.

(5) Both $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are $\mathrm{III}^a(b)$-type singular fibers.
Let $U_1$ and $U_2 \subset \mathbf{R}^2$ be sufficiently small open disk neighborhoods around $y_1$ and $y_2$ respectively. Since both $y_1$ and $y_2$ are codimension 1 singular values of $f$, there exists a homotopy $F_i : M \times [-1,1] \to \mathbf{R}^2$ such that $F_i|U_i \times [-1,1] : U_i \times [-1,1] \to \mathbf{R}^2$ is a generic homotopy around $f|U_i$ and $F_i(x,t) = f(x)$ holds for any $x \in M$ and $t \in [-1,1]$ $(i = 1,2)$. Suppose that $F_i|U_i \times [-1,1]$ crosses $\mathrm{III}^a(b)$ positively at $f|U_i$. Then, a smooth map $G : M \times [-1,1] \times [-1,1] \to \mathbf{R}^2$ is defined by

$$G(x,t,s) = \begin{cases} F_1(x,t) & \text{if } x \in U_1, \\ F_2(x,s) & \text{if } x \in U_2, \\ f(x) & \text{if } x \in M \setminus (U_1 \cup U_2). \end{cases}$$

Let $x_1^1$ and $x_1^2 \in S(f_{1,1})$ (resp. $x_2^1$ and $x_2^2 \in S(f_{1,1})$) be the new born cusps for $f^{-1}(y_1)$ (resp. $f^{-1}(y_2)$), where $f_{t,s} : M \to \mathbf{R}^2$ is defined by $f_{t,s}(x) = G(x,t,s)$. For these four points $x_1^1, x_1^2, x_2^1$ and $x_2^2 \in S(f_{1,1})$, we define the following equivalence relation. For $x_i^j$ and $x_k^l$, we define $x_i^j \sim x_k^l$ if and only if they are in the same connected component of $S(f_{1,1})$ $(i,j,k,l = 1,2)$. We denote by $\{*\}$ each equivalence class of $x_1^1, x_1^2, x_2^1$ and $x_2^2$ under this equivalence relation. The parameter space $\mathrm{Int}([-1,1] \times [-1,1])$ of $G$ is naturally stratified into four 2-dimensional strata, four 1-dimensional strata (i.e., the $t$- and $s$-axes) and one 0-dimensional stratum (i.e., the origin). We call this stratified parameter space $\mathrm{Int}([-1,1] \times [-1,1])$ of $G$ the bifurcation diagram of $\Xi$. For the bifurcation diagram of $\Xi$, we usually consider that each stratum contains some extra information on the stable (or codimension 1 or 2 unstable) maps corresponding to the stratum, such as their singular sets, their coorientations, etc. Then, essentially we have only to consider the following seven cases.

(i) For $\{x_1^1\}, \{x_1^2\}, \{x_2^1\}, \{x_2^2\}$, the bifurcation diagram of $\Xi$ is as depicted in Figure 19 (i).

(ii) For $\{x_1^1, x_1^2\}, \{x_2^1\}, \{x_2^2\}$, $\Xi$ is subdivided into two classes. The bifurcation diagrams of these subdivided classes are as depicted in Figures 19 (ii-1) and (ii-2).

(iii) For $\{x_1^1, x_2^1\}$ ,$\{x_1^2\}, \{x_2^2\}$, the bifurcation diagram of $\Xi$ is as depicted in Figure 19 (iii).

(iv) For $\{x_1^1, x_1^2\}, \{x_2^1, x_2^2\}$, $\Xi$ is subdivided into three classes. The bifurcation diagrams of these subdivided classes are as depicted in Figures 19 (iv-1), (iv-2) and (iv-3).

(v) For $\{x_1^1, x_2^1\}, \{x_1^2, x_2^2\}$, $\Xi$ is subdivided into two classes. The bifurcation diagrams of these subdivided classes are as depicted in Figures 19 (v-1) and (v-2).

(vi) For $\{x_1^1, x_1^2, x_2^1, x_2^2\}$, if the cyclic order of these cusps on the corresponding component of the singular set is $x_1^1, x_1^2, x_2^1, x_2^2$, then $\Xi$ is subdivided into three classes. The bifurcation diagrams of these subdivided classes are as depicted in Figures 19 (vi-1), (vi-2) and (vi-3).

(vii) For $\{x_1^1, x_1^2, x_2^1, x_2^2\}$, if the cyclic order of these cusps on the corresponding component of the singular set is $x_1^1, x_2^1, x_1^2, x_2^2$, then $\Xi$ is subdivided into three classes. The bifurcation diagrams of these subdivided classes are as depicted in Figures 19 (vii-1), (vii-2) and (vii-3).

Figure 19

We consider the coorientations of strata in (5) are defined by the orientations of the corresponding parameter spaces.

Let $M$ be a closed orientable 3-dimensional manifold and we set $\mathcal{M} = C^\infty(M, \mathbf{R}^2)$. Let us denote by $C^1(\widetilde{\mathcal{W}}(\mathcal{M}); \mathbf{Z})$ the (finitely generated) free $\mathbf{Z}$-module generated by the subdivided weak equivalence classes of $\Gamma_1$. We denote by $C^2(\widetilde{\mathcal{W}}(\mathcal{M}); \mathbf{Z})$ the (finitely generated) free $\mathbf{Z}$-module generated by the subdivided weak equivalence classes of $\Gamma_2$ and by the cooriented weak equivalence classes of all unstable maps corresponding to (5). Here, $\Gamma_i$ is the set of all unstable maps of codimension $i$ and all the generators of $C^i(\widetilde{\mathcal{W}}(\mathcal{M}); \mathbf{Z})$ are cooriented ($i = 1, 2$). The rank of $C^1(\widetilde{\mathcal{W}}(\mathcal{M}); \mathbf{Z})$ is equal to 58, while the rank of $C^2(\widetilde{\mathcal{W}}(\mathcal{M}); \mathbf{Z})$ is equal to 422. Using these free $\mathbf{Z}$-modules, we naturally obtain the modified coboundary operator

$$\widetilde{\delta} : C^1(\widetilde{\mathcal{W}}(\mathcal{M}); \mathbf{Z}) \to C^2(\widetilde{\mathcal{W}}(\mathcal{M}); \mathbf{Z}).$$

The definition of $\widetilde{\delta}$ is the same as that of $\delta$ in Section 5. Then, we have the following propositions.

**Proposition 8.1.** *The rank of* $\mathrm{Ker}(\widetilde{\delta})$ *is equal to eight and the following cochains* $\widetilde{c}_1, \ldots, \widetilde{c}_8$ *constitute a basis of* $\mathrm{Ker}(\widetilde{\delta})$:

(1) $\widetilde{c}_1 = \mathrm{III}^a(l) + \mathrm{III}^a(b_1) + \mathrm{III}^a(b_2) + \mathrm{III}^a(b_3) + \mathrm{III}^b + \mathrm{III}^c + \mathrm{III}^d$,

(2) $\widetilde{c}_2 = \mathrm{III}^b + \mathrm{III}^{0,a} + 2\mathrm{III}^{0,0}$,

(3) $\widetilde{c}_3 = \mathrm{III}^{0,a} + \mathrm{III}^{1,a} + \mathrm{III}^e + 2\mathrm{III}^{0,1}$,

(4) $\widetilde{c}_4 = \mathrm{III}^{1,a} + 2\mathrm{III}^{1,1} + \mathrm{III}^4_1 - \mathrm{III}^4_2 + \mathrm{III}^4_3$,

(5) $\widetilde{c}_5 = \mathrm{III}^c + \mathrm{III}^e + 2\mathrm{III}^2 - \mathrm{III}^4_1 + \mathrm{III}^4_2 - \mathrm{III}^4_3 + 3\mathrm{III}^7_1 + \mathrm{III}^7_2$,

(6) $\widetilde{c}_6 = \mathrm{III}^d + 2\mathrm{III}^3 - 3\mathrm{III}^7_1 - \mathrm{III}^7_2$,

(7) $\widetilde{c}_7 = 2\mathrm{III}^3_2 - 2\mathrm{III}^7_1 + \mathrm{III}^8$,

(8) $\widetilde{c}_8 = \mathrm{III}^a(l) - \mathrm{III}^a(b_1) + \mathrm{III}^a(b_3)$.

Geometric interpretations of $\widetilde{c}_1, \widetilde{c}_2, \ldots, \widetilde{c}_7$ are the same as those of $c_1, c_2, \ldots, c_7$ in Theorem 7.3 respectively. A geometric interpretation of $\widetilde{c}_8$ is given as follows.

**Proposition 8.2.** *Let $f$ and $g : M \to \mathbf{R}^2$ be two stable maps of a closed orientable 3-dimensional manifold into the plane and $F : M \times [-1,1] \to \mathbf{R}^2$ a generic homotopy such that the initial stable map of $F$ is $g$ and the terminal one is $f$. Then the value $\widetilde{c}_8(F) \in \mathbf{Z}$ is equal to $\sharp S(f) - \sharp S(g)$, where $\sharp S(f)$ (or $\sharp S(g)$) is the number of connected components of the singular set of $f$ (resp. $g$).*

The proof of the above proposition is similar to that of Theorem 7.3, and is left to the reader.

*Remark* 8.3. The element $\widetilde{c}_8$ defines a first order *non-local invariant* in the sense as follows. We need to know the global behavior of the singular (value) set in order to decide which codimension 1 strata a generic homotopy crosses.

*Remark* 8.4. It is easy to see that the invariant $L_8 : \mathcal{M} \backslash \Gamma \to \mathbf{Z}$ defined by $L_8(f) = \sharp S(f)$ is an isotopy invariant of stable maps. By Proposition 8.1, this invariant is a first order non-local invariant and we have $L_8(f) = \widetilde{c}_8(F) + L_8(g)$, where $g \in \mathcal{M} \setminus \Gamma$ is a distinguished stable map and $F : M \times [-1,1] \to \mathbf{R}^2$ is a generic homotopy between $g$ and $f$. It is obvious that this isotopy invariant $L_8$ is invariant of the $C^\infty$ right-left equivalence classes (see Proposition 7.6).

## 9. First order semi-local invariants of stable fold maps

In this section, we determine those isotopy invariants for *stable fold maps* which are obtained as first order semi-local invariants.

Let $f : M \to \mathbf{R}^2$ be a smooth map of a closed orientable 3-dimensional manifold into the plane. If $f$ has only fold points as its singular points, then we call $f$ a *fold map*. If $f$ is a stable map and is a fold map (i.e., $f$ is a stable map without cusp points), then we call $f$ a *stable fold map*. Note that each singular fiber of a stable fold map is weakly equivalent to a fiber of type $\mathrm{I}^0$, $\mathrm{I}^1$, $\mathrm{II}^{0,0}$, $\mathrm{II}^{0,1}$, $\mathrm{II}^{1,1}$, $\mathrm{II}^2$ or $\mathrm{II}^3$.

*Remark* 9.1. By a theorem of Levine [27], any closed 3-dimensional manifold $M$ has a fold map $f : M \to \mathbf{R}^2$. Therefore, for any closed orientable 3-dimensional manifold $M$, first order semi-local invariants for stable fold maps make sense.

### 9.1. Computation of the Vassiliev quotient cochain complex for fold maps.

Let $M$ be a closed orientable 3-dimensional manifold and $\mathcal{F}$ the subspace of $\mathcal{M} = C^\infty(M, \mathbf{R}^2)$ which consists of all fold maps of $M$ into $\mathbf{R}^2$. It is easy to see that $\mathcal{F}$ is an open subspace of $\mathcal{M}$. By Remark 9.1, $\mathcal{F}$ is non-empty for any $M$. Let $\Gamma_i$ be the set of all codimension $i$

unstable maps in $\mathcal{M}$. Then, any element in $\mathcal{F} \cap \Gamma_i$ is called a *codimension $i$ unstable fold map* $(i = 1, 2)$.

Let $F : M \times I \to \mathbf{R}^2$ be a generic homotopy between two stable fold maps. If $f_t$ is a fold map for each $t \in I$, then we call $F$ a *generic fold homotopy*, and if $f_t$ is a stable fold map for each $t \in I$, then we call $F$ a *fold isotopy*. If $f_{t_0}$ is the unique codimension 1 unstable fold map in the 1-parameter family of fold maps $f_t$, we call such an $F$ a *generic fold homotopy around* $f_{t_0}$. Here, $f_t : M \to \mathbf{R}^2$ is defined by $F(x,t) = f_t(x)$ $(x \in M, t \in I)$ and $I$ is a closed interval in $\mathbf{R}$.

Let us construct the Vassiliev quotient cochain complex for the weak equivalence classes. We set $C^0(\mathcal{W}(\mathcal{F}); \mathbf{Z}) = C^3(\mathcal{W}(\mathcal{F}); \mathbf{Z}) = 0$. Let us denote by $C^i(\mathcal{W}(\mathcal{C}); \mathbf{Z})$ the (finitely generated) free $\mathbf{Z}$-module generated by the cooriented weak equivalence classes of $\Gamma_i \setminus (\Gamma_i \cap \mathcal{F})$, $i = 1, 2$. Let $\delta : C^1(\mathcal{W}(\mathcal{M}); \mathbf{Z}) \to C^2(\mathcal{W}(\mathcal{M}); \mathbf{Z})$ be the coboundary operator defined in Section 5. Since $\mathcal{F}$ is an open subset of $\mathcal{M}$, we see easily that $\delta(C^1(\mathcal{W}(\mathcal{C}); \mathbf{Z})) \subset C^2(\mathcal{W}(\mathcal{C}); \mathbf{Z})$ holds. Therefore, the cochain complex

$$0 \longrightarrow C^1(\mathcal{W}(\mathcal{C}); \mathbf{Z}) \xrightarrow{\delta} C^2(\mathcal{W}(\mathcal{C}); \mathbf{Z}) \longrightarrow 0$$

is a subcomplex of the Vassiliev cochain complex of the weak equivalence classes

$$0 \longrightarrow C^1(\mathcal{W}(\mathcal{M}); \mathbf{Z}) \xrightarrow{\delta} C^2(\mathcal{W}(\mathcal{M}); \mathbf{Z}) \longrightarrow 0$$

constructed in Section 5. We set $C^i(\mathcal{W}(\mathcal{F}); \mathbf{Z}) = C^i(\mathcal{W}(\mathcal{M}); \mathbf{Z})/C^i(\mathcal{W}(\mathcal{C}); \mathbf{Z})$. The rank of $C^1(\mathcal{W}(\mathcal{F}); \mathbf{Z})$ is equal to 45, while the rank of $C^2(\mathcal{W}(\mathcal{F}); \mathbf{Z})$ is equal to 301 (see Figures 12 and 13).

Let $p^1 : C^1(\mathcal{W}(\mathcal{M}); \mathbf{Z}) \to C^1(\mathcal{W}(\mathcal{F}); \mathbf{Z})$ and $p^2 : C^2(\mathcal{W}(\mathcal{M}); \mathbf{Z}) \to C^2(\mathcal{W}(\mathcal{F}); \mathbf{Z})$ be the canonical quotient maps. Then we can define the *Vassiliev quotient cochain complex* for the weak equivalence classes,

$$0 \longrightarrow C^1(\mathcal{W}(\mathcal{F}); \mathbf{Z}) \xrightarrow{\delta_{\mathcal{F}}} C^2(\mathcal{W}(\mathcal{F}); \mathbf{Z}) \longrightarrow 0, \tag{9.1}$$

so that the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & C^1(\mathcal{W}(\mathcal{M}); \mathbf{Z}) & \xrightarrow{\delta} & C^2(\mathcal{W}(\mathcal{M}); \mathbf{Z}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle p^1} & & \downarrow{\scriptstyle p^2} & & \\ 0 & \longrightarrow & C^1(\mathcal{W}(\mathcal{F}); \mathbf{Z}) & \xrightarrow{\delta_{\mathcal{F}}} & C^2(\mathcal{W}(\mathcal{F}); \mathbf{Z}) & \longrightarrow & 0 \end{array}$$

is commutative. Therefore, the Vassiliev quotient cochain complex (9.1) is well-defined.

By the definition of the Vassiliev quotient cochain complex for the weak equivalence classes, the elements $p^1(\mathrm{III}^*_\star)$ (or $p^2(\mathrm{IV}^*_\star)$) such that $*$ does not contain any alphabetical letter constitute the natural basis of $C^1(\mathcal{W}(\mathcal{F}); \mathbf{Z})$ (resp. $C^2(\mathcal{W}(\mathcal{F}); \mathbf{Z})$). We denote each element $p^1(\mathrm{III}^*_\star)$ (resp. $p^2(\mathrm{IV}^*_\star)$) by $[\mathrm{III}^*_\star]$ (resp. $[\mathrm{IV}^*_\star]$). Note that two codimension $i$ unstable fold maps $f$ and $g \in \mathcal{F} \cap \Gamma_i$ are weakly equivalent if and only if they represent the same element of $C^i(\mathcal{W}(\mathcal{F};\mathbf{Z}))$ $(i=1,2)$. By abuse of notation, we use the symbol $[\mathrm{III}^*_\star]$ (resp. $[\mathrm{IV}^*_\star]$) for the corresponding weak equivalence class of fold maps in $\mathcal{F}\cap\Gamma_1$ (resp. $\mathcal{F}\cap\Gamma_2$). Each weak equivalence class $[\mathrm{III}^*_\star]$ of $\mathcal{F}\cap\Gamma_1$ or $[\mathrm{IV}^*_\star]$ of $\mathcal{F}\cap\Gamma_2$ has the coorientation induced from that of $\mathrm{III}^*_\star$ or $\mathrm{IV}^*_\star$ respectively. By the definition of the coboundary operator $\delta_{\mathcal{F}}$, we have the following.

**Proposition 9.2.** *The coboundary operator*

$$\delta_{\mathcal{F}} : C^1(\mathcal{W}(\mathcal{F});\mathbf{Z}) \to C^2(\mathcal{W}(\mathcal{F});\mathbf{Z})$$

*of the Vassiliev quotient cochain complex for the weak equivalence classes is represented by the following block matrix $A_{\mathcal{F}}$ with respect to the natural bases of $C^1(\mathcal{W}(\mathcal{F});\mathbf{Z})$ and $C^2(\mathcal{W}(\mathcal{F});\mathbf{Z})$:*

$$A_{\mathcal{F}} = \begin{pmatrix} A_{4,2} & A_{4,3} & A_{4,4} & A_{4,5} \\ A_{8,2} & A_{8,3} & A_{8,4} & A_{8,5} \\ A_{9,2} & A_{9,3} & A_{9,4} & A_{9,5} \\ \cdots & \cdots & \cdots & \cdots \\ A_{24,2} & A_{24,3} & A_{24,4} & A_{24,5} \end{pmatrix}.$$

*Here, each block $A_{i,j}$ is the same as the block $A_{i,j}$ in Proposition 7.1. Note that $A_{\mathcal{F}}$ is a $301\times 45$ matrix.*

By a direct calculation, we we get the following.

**Theorem 9.3.** *The rank of* $\mathrm{Ker}(\delta_{\mathcal{F}})$ *is equal to seven and the following cochains $d_1, d_2, \ldots, d_7$ constitute a basis of* $\mathrm{Ker}(\delta_{\mathcal{F}})$:

(1) $d_1 = [\mathrm{III}^{0,0}]$,
(2) $d_2 = [\mathrm{III}^{0,1}]$,
(3) $d_3 = 2[\mathrm{III}^{1,1}] + [\mathrm{III}^4_1] - [\mathrm{III}^4_2] + [\mathrm{III}^4_3]$,
(4) $d_4 = 2[\mathrm{III}^2] - [\mathrm{III}^4_1] + [\mathrm{III}^4_2] - [\mathrm{III}^4_3] + 3[\mathrm{III}^7_1] + [\mathrm{III}^7_2]$,
(5) $d_5 = 2[\mathrm{III}^3] - 3[\mathrm{III}^7_1] - [\mathrm{III}^7_2]$,
(6) $d_6 = 2[\mathrm{III}^3_2] - 2[\mathrm{III}^7_1] + [\mathrm{III}^8]$,
(7) $d_7 = -[\mathrm{III}^{0,0}_1]$.

*For* $[\mathrm{III}^*] = [\mathrm{III}^{0,0}], [\mathrm{III}^{0,1}], [\mathrm{III}^{1,1}], [\mathrm{III}^2], [\mathrm{III}^3]$, *and* $[\mathrm{III}^7]$, *we have set* $[\mathrm{III}^*] = \sum_\star [\mathrm{III}^*_\star]$.

9.2. **Geometric interpretations of the 1-cocycles in** $\mathrm{Ker}(\delta_{\mathcal{F}})$**.** In this subsection, we give a geometric interpretation of each Vassiliev 1-cocycle $d_i \in \mathrm{Ker}(\delta_{\mathcal{F}})$ $(1\le i\le 7)$ given in Theorem 9.3.

Let $c = \widehat{c}_1 + \widehat{c}_2 \in C^1(\mathcal{W}(\mathcal{M});\mathbf{Z})$ be a 1-dimensional cochain such that $\widehat{c}_2 \in C^1(\mathcal{W}(\mathcal{C});\mathbf{Z})$ and $F : M\times[-1,1] \to \mathbf{R}^2$ a generic fold homotopy. By the definition, $c(F) = \widehat{c}_1(F) \in \mathbf{Z}$ and $p^1(c) = p^1(\widehat{c}_1) \in C^1(\mathcal{W}(\mathcal{F});\mathbf{Z})$ hold. Therefore, the integer value $[\mathrm{III}^*_\star](F) \in \mathbf{Z}$ defined by $[\mathrm{III}^*_\star](F) = \mathrm{III}^*_\star(F)$ for each $[\mathrm{III}^*_\star] \in C^1(\mathcal{W}(\mathcal{F}); \mathbf{Z})$ is well-defined. Let $\Theta_1, \Theta_2, \ldots, \Theta_{45}$ be the natural basis of $C^1(\mathcal{W}(\mathcal{F}); \mathbf{Z})$ and $d = \sum_{j=1}^{45} a_j \Theta_j$ a cocycle in $\mathrm{Ker}(\delta_{\mathcal{F}})$, where $a_j \in \mathbf{Z}$. Then $d$ defines an integer valued function on the set of all generic fold homotopies as follows:

$$d : \{\text{generic fold homotopies}\} \to \mathbf{Z},\ d(F) = \sum_{j=1}^{45} a_j \Theta_j(F).$$

It is easy to see that $\mathcal{F}$ is not connected. Let $\widetilde{\mathcal{F}}$ be any connected component of $\mathcal{F}$ and $F : M \times [-1, 1] \to \mathbf{R}^2$ any generic fold homotopy which connects two stable fold maps $f$ and $g : M \to \mathbf{R}^2$ in $\widetilde{\mathcal{F}}$. If we can show that the value $d(F)$ depends only on the fold isotopy classes of $f$ and $g$, then $d$ defines a semi-local invariant of fold isotopy classes for each connected component $\widetilde{\mathcal{F}}$ of $\mathcal{F}$. In fact, given such a $d$, taking a distinguished stable fold map $g \in \widetilde{\mathcal{F}} \setminus (\widetilde{\mathcal{F}} \cap \Gamma)$ and a constant $\alpha_0 \in \mathbf{Z}$, we can define the semi-local invariant $J_d^{\widetilde{\mathcal{F}}} : \widetilde{\mathcal{F}} \setminus (\widetilde{\mathcal{F}} \cap \Gamma) \to \mathbf{Z}$ of stable fold maps in $\widetilde{\mathcal{F}}$ by $J_d^{\widetilde{\mathcal{F}}}(f) = d(F) + \alpha_0$. Here, $F$ is any generic fold homotopy between $g$ and $f$. Note that $J_d^{\widetilde{\mathcal{F}}}$ is an isotopy invariant of stable fold maps in $\widetilde{\mathcal{F}}$ and $J_d^{\widetilde{\mathcal{F}}}(g) = \alpha_0$.

In the following theorem, we give a geometric interpretation of each Vassiliev 1-cocycle $d_i \in \mathrm{Ker}(\delta_{\mathcal{F}})$ $(1 \le i \le 7)$.

**Theorem 9.4.** *Let $f$ and $g : M \to \mathbf{R}^2$ be two stable fold maps of a closed orientable 3-dimensional manifold into the plane such that $f$ and $g$ are in the same connected component of $\mathcal{F}$. Suppose that $F : M \times [-1, 1] \to \mathbf{R}^2$ is a generic fold homotopy such that the initial stable fold map of $F$ is $g$ and the terminal one is $f$. Then we have the following for each $d_i(F) \in \mathbf{Z}$ $(1 \le i \le 7)$.*

1. *The value $d_1(F) \in \mathbf{Z}$ is equal to $(\sharp\mathrm{II}^{0,0}(f) - \sharp\mathrm{II}^{0,0}(g))/2$.*
2. *The value $d_2(F) \in \mathbf{Z}$ is equal to $(\sharp\mathrm{II}^{0,1}(f) - \sharp\mathrm{II}^{0,1}(g))/2$.*
3. *The value $d_3(F) \in \mathbf{Z}$ is equal to $\sharp\mathrm{II}^{1,1}(f) - \sharp\mathrm{II}^{1,1}(g)$.*
4. *The value $d_4(F) \in \mathbf{Z}$ is equal to $\sharp\mathrm{II}^{2}(f) - \sharp\mathrm{II}^{2}(g)$.*
5. *The value $d_5(F) \in \mathbf{Z}$ is equal to $\sharp\mathrm{II}^{3}(f) - \sharp\mathrm{II}^{3}(g)$.*
6. *The value $d_6(F) \in \mathbf{Z}$ is equal to $(\mathrm{rot}(f) - \mathrm{rot}(g))/2$.*
7. *The value $d_7(F) \in \mathbf{Z}$ is equal to $(Bq(f) - Bq(g))/2$.*

*Here, we denote by $\sharp\mathrm{II}^*(f)$ the number of $\mathrm{II}^*$-type singular fibers of $f$, by $\mathrm{rot}(f)$ the surgical rotation number of $f(S(f))$ and by $Bq(f)$ the Bennequin invariant of the definite fold of $f$.*

Theorem 9.4 (1)–(6) can be proved as in the proof of Theorem 7.3. To prove Theorem 9.4 (7), we first have to define the *Bennequin invariant* of the definite fold. In the next subsection, we define the Bennequin invariant and after that we prove Theorem 9.4 (7).

9.3. **Bennequin invariant.** Let $f : M \to \mathbf{R}^2$ be a stable fold map and $S_0(f)$ the set of all definite fold points of $f$. Each connected component of $S_0(f)$ is a circle and the restriction $f|S_0(f) : S_0(f) \to \mathbf{R}^2$ is an immersion with normal crossings. Let us consider the orientation on each connected component $S_i^1$ of $S_0(f) = \cup_i S_i^1$ which is induced from the orientation of $f(S_i^1) \subset \mathbf{R}^2$ (see Remark 4.6).

Let $\mathbf{R}^2$ be the oriented plane. For $S_0(f) = \cup_i S_i^1$ and the parameterization $f|S_0(f) = \cup_i(f|S_i^1) : \cup_i S_i^1 \to \mathbf{R}^2$ of $f(S_0(f))$, we define the embedding lift $F : \cup_i S_i^1 \to \mathbf{R}^2 \times S^1$ of $f|S_0(f)$ as follows. For $x \in S_i^1$, let $n(x)$ be the unit vector normal to the positive unit tangent vector $t(x)$ of $f(S_i^1)$ at $(f|S_i^1)(x)$ such that the frame $\{n(x), t(x)\}$ is positive on $\mathbf{R}^2$. Then $F|S_i^1$ is defined by $(F|S_i^1)(x) = ((f|S_i^1)(x), n(x))$. We call $F = \cup_i(F|S_i^1) : S_0(f) \to \mathbf{R}^2 \times S^1$ the *Legendrian link* of $f|S_0(f)$. Since $f|S_0(f)$ is an immersion with normal crossings, the Legendrian link $F$ is an embedding.

We slightly shift each embedding lift $F|S_i^1 \subset \mathbf{R}^2 \times S^1$ to the direction of the positive unit normal vector field $n(x)$, $x \in S_i^1$, of $f|S_i^1$. As a result, we obtain the embedding $\widetilde{F}|S_i^1$ defined by $(\widetilde{F}|S_i^1)(x) = ((f|S_i^1)(x) + \varepsilon n(x), n(x))$, $x \in S_i^1$, where $\varepsilon > 0$ is a sufficiently small positive real number. Since $F$ is an embedding and $\varepsilon$ is a sufficiently small positive real number, each embedding $\widetilde{F}|S_i^1$ does not intersect $F(S_0(f))$ in $\mathbf{R}^2 \times S^1$.

To define the Bennequin invariant, we first define the linking number of $F|S_i^1$ and $\widetilde{F}|S_j^1$ for each pair $i, j$. We define the embedding $\iota : \mathbf{R}^2 \times S^1 \to \mathbf{R}^3 = \mathbf{C} \times \mathbf{R}^1$ defined by $(x, y, \theta) \mapsto (\rho, z) = (e^x\theta, y)$, where we use coordinates $(x, y, \theta)$ on $\mathbf{R}^2 \times S^1$ and coordinates $(\rho, z)$ on $\mathbf{C} \times \mathbf{R}^1$, and we identify $S^1$ with the unit circle in $\mathbf{C}$. We define the *linking number* $lk(F|S_i^1, \widetilde{F}|S_j^1)$ as the standard linking number of the oriented knots $\iota \circ (F|S_i^1)$ and $\iota \circ (\widetilde{F}|S_j^1)$ in $\mathbf{R}^3$. For more details, see [4, 7, 8]. Then the Bennequin invariant $Bq(f)$ of the Legendrian link $F$ of $f|S_0(f)$ is defined by $Bq(f) = \sum_{i,j} lk(F|S_i^1, \widetilde{F}|S_j^1)$. We call it the *Bennequin invariant* of the definite fold of $f$. The Bennequin invariant is defined only when $f$ is a fold map.

9.4. **Proof of Theorem 9.4 (7).** In this subsection, we prove Theorem 9.4 (7). Let $F : M \times [-1, 1] \to \mathbf{R}^2$ be a generic fold homotopy around $f_0$ in a connected component $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ which connects two stable fold maps $f_{-1}$ and $f_1$, where $f_t$ is defined by $F(x, t) = f_t(x)$. We suppose that the positive coorientation of the weak equivalence class of $f_0$ coincides with the canonical positive direction of $[-1, 1]$. To prove Theorem 9.4 (7), we have only to check that when $F$ crosses each $\mathrm{III}_\star^* \cap \widetilde{\mathcal{F}} \subset \Gamma_1 \cap \widetilde{\mathcal{F}}$ positively, $d_7(F) = (Bq(f_1) - Bq(f_{-1}))/2$ holds. Here, $*$ does not contain any alphabetical letter.

Suppose that $f_0 \in \mathrm{III}_1^{0,0}$. As is shown in Figure 20, we see that $Bq(f_1) - Bq(f_{-1}) = -2$ whether or not the two tangent definite fold arcs are in the same connected component of $S_0(f_0)$. On the other hand, $d_7(F) = -1$ holds. Therefore, we have $d_7(F) = (Bq(f_1) - Bq(f_{-1}))/2$.

Figure 20

In Figure 20, parts of the curves $f_1(S^1_i)$ and $f_1(S^1_j)$ (resp. $f_{-1}(S^1_i)$ and $f_{-1}(S^1_j)$) of $f_1(S_0(f_1))$ (resp. $f_{-1}(S_0(f_{-1}))$) are drawn in each $\mathbf{R}^2$. The corresponding parts of the Legendrian link $F_1(S^1_i)$, $F_1(S^1_j)$ (resp. $F_{-1}(S^1_i)$, $F_{-1}(S^1_j)$) and its shift $\widetilde{F}_1(S^1_i)$, $\widetilde{F}_1(S^1_j)$ (resp. $\widetilde{F}_{-1}(S^1_i)$ $\widetilde{F}_{-1}(S^1_j)$) are drawn in each $\mathbf{R}^2 \times S^1$. Note that in Figure 20, the positive direction of the $y$-axis enters from the face of the sheet and comes out of the reverse side. Thus, while $F_{-1}(S^1_i)$ is under $F_{-1}(S^1_j)$, $F_1(S^1_i)$ is on $F_1(S^1_j)$

If $f_0$ belongs to the other weak equivalence classes of $\Gamma_1 \cap \widetilde{\mathcal{F}}$, we see that $Bq(f_1) - Bq(f_{-1}) = 0$ holds. This completes the proof. □

By Theorem 9.4 and the definition of the Bennequin invariant, we see that each value $d_i(F)$ depends only on $f$ and $g$ ($1 \leq i \leq 7$). Thus, we have the following.

**Corollary 9.5.** *Let $d \in \mathrm{Ker}(\delta_{\mathcal{F}})$ be any element of the kernel of the coboundary operator*

$$\delta_{\mathcal{F}} : C^1(\mathcal{W}(\mathcal{F}); \mathbf{Z}) \to C^2(\mathcal{W}(\mathcal{F}); \mathbf{Z}).$$

*Then for each connected component $\widetilde{\mathcal{F}}$ of $\mathcal{F}$, $d$ induces a first order semi-local invariant $J^{\widetilde{\mathcal{F}}}_d$ of stable fold maps in $\widetilde{\mathcal{F}}$.*

Since we do not know if the space $\widetilde{\mathcal{F}}$ is contractible or not, we cannot directly use Proposition 6.3 in general. Thus, we cannot prove that each Vassiliev 1-cocycle $d \in \mathrm{Ker}(\delta_{\mathcal{F}})$ defines a first order semi-local invariant of stable fold maps only by the construction of the coboundary operator $\delta_{\mathcal{F}}$.

*Remark* 9.6. Let $M$ be a closed orientable 3-dimensional manifold and $\mathcal{F}$ the subspace of $C^\infty(M, \mathbf{R}^2)$ which consists of all fold maps of $M$ into $\mathbf{R}^2$. Suppose that $\widetilde{\mathcal{F}}$ is any connected component of $\mathcal{F}$. As in the case of stable maps, the following (modified) seven first order semi-local invariants of stable fold maps are also invariants of the $C^\infty$ right-left equivalence classes in $\widetilde{\mathcal{F}}$ (see Proposition 7.6):

$$\begin{aligned}
J^{\widetilde{\mathcal{F}}}_1(f) &= \sharp \mathrm{II}^{0,0}(f)/2 \text{ or } = (\sharp \mathrm{II}^{0,0}(f) + 1)/2,\\
J^{\widetilde{\mathcal{F}}}_2(f) &= \sharp \mathrm{II}^{0,1}(f)/2 \text{ or } = (\sharp \mathrm{II}^{0,1}(f) + 1)/2,\\
J^{\widetilde{\mathcal{F}}}_3(f) &= \sharp \mathrm{II}^{1,1}(f),\\
J^{\widetilde{\mathcal{F}}}_4(f) &= \sharp \mathrm{II}^{2}(f),\\
J^{\widetilde{\mathcal{F}}}_5(f) &= \sharp \mathrm{II}^{3}(f),\\
J^{\widetilde{\mathcal{F}}}_6(f) &= \mathrm{rot}(f)/2,\\
J^{\widetilde{\mathcal{F}}}_7(f) &= Bq(f)/2 \text{ or } = (Bq(f) + 1)/2.
\end{aligned}$$

The values of $J^{\widetilde{\mathcal{F}}}_1(f)$, $J^{\widetilde{\mathcal{F}}}_2(f)$ and $J^{\widetilde{\mathcal{F}}}_7(f)$ depend only on the connected component $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ which contains $f \in \mathcal{F}$. All the invariants $J^{\widetilde{\mathcal{F}}}_i$ can be extended to integer-valued invariants

$J_i : \mathcal{F} \setminus (\mathcal{F} \cap \Gamma) \to \mathbf{Z}$ of the $C^\infty$ right-left equivalence classes of stable fold maps in $\mathcal{F}$ $(1 \leq i \leq 7)$.

**Example 9.7.** Let $\mathcal{F}(S^3)$ be the subspace of $C^\infty(S^3, \mathbf{R}^2)$ which consists of all fold maps of $S^3$ into $\mathbf{R}^2$. Suppose that $f_1 : S^3 \to \mathbf{R}^2$ is the stable fold map defined in Subsection 7.5. Let $\widetilde{\mathcal{F}}(S^3)$ be the connected component of $\mathcal{F}(S^3)$ which contains $f_1$. It is easy to see that the following seven invariants $J_i^{\widetilde{\mathcal{F}}(S^3)} : \widetilde{\mathcal{F}}(S^3) \setminus (\widetilde{\mathcal{F}}(S^3) \cap \Gamma) \to \mathbf{Z}$ $(1 \leq i \leq 7)$ are isotopy invariants of stable fold maps in $\widetilde{\mathcal{F}}(S^3)$:

(1) $J_1^{\widetilde{\mathcal{F}}(S^3)}(f) = \sharp\mathrm{II}^{0,0}(f)/2$, (2) $J_2^{\widetilde{\mathcal{F}}(S^3)}(f) = \sharp\mathrm{II}^{0,1}(f)/2$, (3) $J_3^{\widetilde{\mathcal{F}}(S^3)}(f) = \sharp\mathrm{II}^{1,1}(f)$,
(4) $J_4^{\widetilde{\mathcal{F}}(S^3)}(f) = \sharp\mathrm{II}^{2}(f)$, (5) $J_5^{\widetilde{\mathcal{F}}(S^3)}(f) = \sharp\mathrm{II}^{3}(f)$, (6) $J_6^{\widetilde{\mathcal{F}}(S^3)}(f) = \mathrm{rot}(f)/2$,
(7) $J_7^{\widetilde{\mathcal{F}}(S^3)}(f) = (Bq(f)+1)/2$.

Here, we denote by $\sharp\mathrm{II}^*(f)$ the number of $\mathrm{II}^*$-type singular fibers of $f$. By using Theorem 9.4 and the fact that $J_i^{\widetilde{\mathcal{F}}(S^3)}(f_1) \in \mathbf{Z}$ $(1 \leq i \leq 7)$, we see that all these are integer valued invariants. By Theorems 9.3, 9.4 and Corollary 9.5, these seven invariants are first order semi-local invariants of stable fold maps in $\widetilde{\mathcal{F}}$. Furthermore, we have $J_i^{\widetilde{\mathcal{F}}(S^3)}(f) = d_i(F) + J_i^{\widetilde{\mathcal{F}}(S^3)}(g)$, where $g \in \widetilde{\mathcal{F}}(S^3) \setminus (\widetilde{\mathcal{F}}(S^3) \cap \Gamma)$ is a distinguished stable fold map in $\widetilde{\mathcal{F}}(S^3)$ and $F : M \times [-1,1] \to \mathbf{R}^2$ is a generic fold homotopy in $\widetilde{\mathcal{F}}(S^3)$ between $g$ and $f$.

Let us define the isotopy invariant $J^{\widetilde{\mathcal{F}}(S^3)} : \widetilde{\mathcal{F}}(S^3) \setminus (\widetilde{\mathcal{F}}(S^3) \cap \Gamma) \to \mathbf{Z}^7$ by $J^{\widetilde{\mathcal{F}}(S^3)} = (J_1^{\widetilde{\mathcal{F}}(S^3)}, J_2^{\widetilde{\mathcal{F}}(S^3)}, \ldots, J_7^{\widetilde{\mathcal{F}}(S^3)})$. For any stable fold map $f$ in $\widetilde{\mathcal{F}}(S^3) \setminus (\widetilde{\mathcal{F}}(S^3) \cap \Gamma)$, we have $J^{\widetilde{\mathcal{F}}(S^3)}(f) = (a+b, 0, 0, 0, 0, 1, 1-b) \in \mathbf{Z}^7$ for some $a$ and $b$ with $a + b \geq 0$.

This statement is proved as follows. Since $f_1$ has only definite fold points, there is no indefinite fold in any stable fold map $f$ in $\widetilde{\mathcal{F}}(S^3)$. Thus, $J_i^{\widetilde{\mathcal{F}}(S^3)}(f) = 0$ for $i = 2, 3, 4, 5$. By Corollary 10.1 in the next section, $J_6^{\widetilde{\mathcal{F}}(S^3)}(f) = 1$ holds. It is easy to see that the weak equivalence classes of maps belonging to $\widetilde{\mathcal{F}}(S^3) \cap \Gamma_1$ are either $\mathrm{III}_k^{0,0}$ $(k = 1, 2, 3)$ or $\mathrm{III}_l^{0,0,0}$ $(l = 1, 2)$. Let $F : S^3 \times [-1,1] \to \mathbf{R}^2$ be a generic fold homotopy in $\widetilde{\mathcal{F}}(S^3)$ between $f_1$ and $f$. We remark that even if $F$ crosses $\mathrm{III}_l^{0,0,0}$, both of the values $J_1^{\widetilde{\mathcal{F}}(S^3)}$ and $J_7^{\widetilde{\mathcal{F}}(S^3)}$ remain unchanged. Suppose that $F$ crosses $\mathrm{III}_k^{0,0}$ positively $\alpha_k^+$ times and that $F$ crosses $\mathrm{III}_k^{0,0}$ negatively $\alpha_k^-$ times. Then we have $d_1(F) = \sum_{k=1}^3 (\alpha_k^+ - \alpha_k^-)$ and $d_7(F) = -(\alpha_1^+ - \alpha_1^-)$. Since each $J_i^{\widetilde{\mathcal{F}}(S^3)}$ is a first order semi-local invariant, the values $d_1(F)$ and $d_7(F) \in \mathbf{Z}$ depend only on $f_1$ and $f$ only. Thus, if we put $a = \sum_{k=2}^3 (\alpha_k^+ - \alpha_k^-)$ and $b = \alpha_1^+ - \alpha_1^-$, we have $J_1^{\widetilde{\mathcal{F}}(S^3)}(f) = a + b$ and $J_7^{\widetilde{\mathcal{F}}(S^3)}(f) = 1 - b$. By definition, $J_1^{\widetilde{\mathcal{F}}(S^3)}(f)$ is non-negative for any $f \in \widetilde{\mathcal{F}}(S^3)$. Thus, $a + b \geq 0$ holds.

Furthermore, for any non-negative integers $a$ and $b$, we can construct a stable fold map $f : S^3 \to \mathbf{R}^2$ in $\widetilde{\mathcal{F}}(S^3)$ such that $J^{\widetilde{\mathcal{F}}(S^3)}(f) = (a+b, 0, 0, 0, 0, 1, 1-b)$ except for

$(a, b) = (0, b)$ with $b \geq 1$. The singular value set and the associated fibers of such a map $f$ is described in Figure 21.

Figure 21

Let $M$ be a closed orientable 3-dimensional manifold and $\mathcal{F}$ the subspace of $C^\infty(M, \mathbf{R}^2)$ which consists of all fold maps of $M$ into $\mathbf{R}^2$. Suppose that $\widetilde{\mathcal{F}}$ is any connected component of $\mathcal{F}$. Let $J_1^{\widetilde{\mathcal{F}}}, J_2^{\widetilde{\mathcal{F}}}, \ldots, J_7^{\widetilde{\mathcal{F}}}$ be first order semi-local invariants of stable fold maps defined in Remark 9.6. In general, these seven first order semi-local invariants of stable fold maps are not linearly independent (see Example 9.7 and compare Subsection 7.5). Therefore, we have the following problem.

**Problem 9.8.** Do there exist a closed orientable 3-dimensional manifold $M$ and a connected component $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ such that the seven first order semi-local invariants of stable fold maps $J_1^{\widetilde{\mathcal{F}}}, J_2^{\widetilde{\mathcal{F}}}, \ldots, J_7^{\widetilde{\mathcal{F}}}$ are linearly independent?

The stable fold maps $f_1$ and $f_8 : S^3 \to \mathbf{R}^2$ constructed in Subsection 7.5 show that the invariants $J_6^{\widetilde{\mathcal{F}}(S^3)}$ and $J_7^{\widetilde{\mathcal{F}}(S^3)}$ are linearly independent. In fact, $(J_6^{\widetilde{\mathcal{F}}(S^3)}(f_1), J_7^{\widetilde{\mathcal{F}}(S^3)}(f_1)) = (1, 1)$ and $(J_6^{\widetilde{\mathcal{F}}(S^3)}(f_8), J_7^{\widetilde{\mathcal{F}}(S^3)}(f_8)) = (2, 1)$.

## 10. Invariants of the connected components of the space of fold maps

In this section, we study the connected components of the space of all fold maps $\mathcal{F}$ using results of the previous sections.

Let $f : M \to \mathbf{R}^2$ be a stable fold map of a closed orientable 3-dimensional manifold $M$ into the plane. We denote by $S_0(f)$ the set of all definite fold points of $f$ and by $S_1(f)$ the set of all indefinite fold points of $f$. It is easy to see that each component of $S_0(f)$ or $S_1(f)$ is a circle and $S_0(f) \cap S_1(f) = \emptyset$. We denote by $\sharp S_j(f)$ the number of connected components of $S_j(f)$ and by $\mathrm{rot}(f|S_j(f))$ the surgical rotation number of $f(S_j(f))$ $(j = 0, 1)$. By the definition of the surgical rotation number, $\mathrm{rot}(f) = \mathrm{rot}(f|S_0(f)) + \mathrm{rot}(f|S_1(f))$ holds.

Note that for a stable fold map $f : M \to \mathbf{R}^2$, the rotation number of $f|S_0(f)$ in the usual sense is equal to one half the surgical rotation number $\mathrm{rot}(f|S_0(f))$.

Let $F : M \times [-1, 1] \to \mathbf{R}^2$ be a generic fold homotopy between two stable fold maps $f_{-1}$ and $f_1$ such that 0 is the unique codimension 1 bifurcation value of $F$, where $f_t$ is defined by $F(x, t) = f_t(x)$. Then, it is easy to check that $\sharp S_0(f_t), \sharp S_1(f_t)$ and $\mathrm{rot}(f_t|S_0(f_t))$ remain unchanged under the generic fold homotopy (see Figures 7, 12 and the definition of a generic homotopy in Subsection 3.2). Therefore we have the following.

**Corollary 10.1.** *Let $f$ and $g : M \to \mathbf{R}^2$ be two stable fold maps of a closed orientable 3-dimensional manifold $M$ into the plane. If $f$ and $g$ are in the same connected component of $\mathcal{F}$, then $\sharp S_0(f) = \sharp S_0(g)$, $\sharp S_1(f) = \sharp S_1(g)$ and $\mathrm{rot}(f|S_0(f)) = \mathrm{rot}(g|S_0(g))$ hold.*

For a generic fold homotopy $F : M \times [-1,1] \to \mathbf{R}^2$ around $[\mathrm{III}^3_2]$, $[\mathrm{III}^7_1]$ or $[\mathrm{III}^8]$, we have $\mathrm{rot}(f_1|S_1(f_1)) \neq \mathrm{rot}(f_{-1}|S_1(f_{-1}))$. Therefore, $\mathrm{rot}(f|S_1(f))$ is not an invariant of connected component of $\mathcal{F}$. Here, $f_t \in \mathcal{F}$ is defined by $f_t(x) = F(x,t)$.

Let $V_k = S^1_1 \cup \cdots \cup S^1_k$ be a disjoint union of $k$ copies of the circle embedded in $M$ ($k \geq 1$). We denote by $\mathcal{F}(V_k)$ the set of all fold maps $f : M \to \mathbf{R}^2$ such that $S_0(f) = V_k$. Note that $\mathcal{F}(V_k)$ is a subspace of $\mathcal{F}$. By [44], $\mathcal{F}(V_k)$ is always non-empty.

We can modify the above invariants to obtain invariants of the connected components of $\mathcal{F}(V_k)$. Let $f : M \to \mathbf{R}^2$ be a stable fold map in $\mathcal{F}(V_k)$ ($k \geq 1$). We consider the $k$-tuple vector $(\mathrm{rot}(f|S^1_1), \ldots, \mathrm{rot}(f|S^1_k)) \in \mathbf{Z}^k$, where $\mathrm{rot}(f|S^1_i)$ denotes the usual rotation number of the immersion $f|S^1_i$ ($1 \leq i \leq k$). Then we have the following.

**Corollary 10.2.** *Let $f$ and $g : M \to \mathbf{R}^2$ be two stable fold maps of a closed orientable 3-dimensional manifold into the plane and $V_k = S^1_1 \cup \cdots \cup S^1_k$ a disjoint union of $k$ copies of the circle embedded in $M$ ($k \geq 1$). If $f$ and $g$ are in the same connected component of $\mathcal{F}(V_k)$, then $\sharp S_1(f) = \sharp S_1(g)$ and $(\mathrm{rot}(f|S^1_1), \ldots, \mathrm{rot}(f|S^1_k)) = (\mathrm{rot}(g|S^1_1), \ldots, \mathrm{rot}(g|S^1_k)) \in \mathbf{Z}^k$ hold.*

To prove this corollary, we need a relative version of the parameterized multi-transversality theorem as follows.

Let $M$ be an $n$-dimensional manifold and $P$ a $p$-dimensional manifold. We fix an $m$-dimensional (resp. $q$-dimensional) properly embedded submanifold $N$ of $M$ (resp. $Q$ of $P$). Then a relative map $f : (M,N) \to (P,Q)$ is a smooth map of $M$ into $P$ with $f(N) \subset Q$. We denote by $C^\infty(M,N;P,Q)$ the space of all relative maps $f : (M,N) \to (P,Q)$ such that $f : M \to P$ is proper. We endow this set with the topology induced from the Whitney $C^\infty$ topology on $C^\infty(M,P)$. In the jet bundle $J^r(M,P)$, we consider the submanifold $J^r(M,N;P,Q)$ of $r$-jets along $N$ of relative maps in $C^\infty(M,N;P,Q)$. For a relative map $f : (M,N) \to (P,Q)$, the jet section $j^rf : M \to J^r(M,P)$ maps $N$ to $J^r(M,N;P,Q)$. We fix a positive integer $s$. For each pair $(k,l)$ of non-negative integers with $k+l \leq s$, we set ${}_{k,l}J^r(M,N;P,Q) = J^r(M,N;P,Q)^k \times J^r(M,P)^l$ and denote by $N^{(k)} \times (M \setminus N)^{(l)}$ the set of $(a_1,\ldots,a_k;b_1,\ldots,b_l) \in M^{k+l}$ such that $a_1,\ldots,a_k$ (resp. $b_1,\ldots,b_l$) are distinct points in $N$ (resp. $M \setminus N$). For a relative map $f : (M,N) \to (P,Q)$, we define the relative multi-jet section ${}_{k,l}j^rf : N^{(k)} \times (M \setminus N)^{(l)} \to {}_{k,l}J^r(M,N;P,Q)$ by

$$ {}_{k,l}j^rf(a_1,\ldots,a_k;b_1,\ldots,b_l) = (j^rf(a_1),\ldots,j^rf(a_k);j^rf(b_1),\ldots,j^rf(b_l)), $$

and ${}_kj^r(f|N) : N^{(k)} \to J^r(N,Q)^k$ by

$$ {}_kj^r(f|N)(a_1,\ldots,a_k) = (j^r(f|N)(a_1),\ldots,j^r(f|N)(a_k)). $$

Then we have the following.

**Proposition 10.3** (Relative multi-transversality theorem [19]). *Let $r$ be a non-negative integer and $s$ a positive integer. For given countable families of submanifolds $S_{k,l}$ of ${}_{k,l}J^r(M,N;P,Q)$, $k+l \leq s$, and $U_k$ of $J^r(N,Q)^k$, $k \leq s$, the set*

$$R = \left\{ F \in C^\infty(M,N;P,Q) \;\middle|\; \begin{array}{l} {}_{k,l}j^r f \text{ is transverse to } S_{k,l}, k+l \leq s, \text{ and} \\ {}_k j^r(f|N) \text{ is transverse to } U_k, k \leq s \end{array} \right\}$$

*is a residual subset and is dense in $C^\infty(M,N;P,Q)$.*

Let $Z$ be a manifold and $F : (M \times Z, N \times Z) \to (P,Q)$ a relative smooth map. For each $z \in Z$, the relative smooth map $F_z : (M,N) \to (P,Q)$ is defined by $F_z(x) = F(x,z)$. We set ${}_{k,l}J^r(M,N;P,Q;Z) = (J^r(M,N;P,Q)^k \times Z^k) \times (J^r(M,P)^l \times Z^l)$. We define the parameterized jet extension ${}_{k,l}j^r F : (N^{(k)} \times Z^k) \times ((M \setminus N)^{(l)} \times Z^l) \to_{k,l} J^r(M,N;P,Q;Z)$ by

$$\begin{aligned} &{}_{k,l}j^r F(a_1,\dots,a_k,z_1,\dots,z_k;b_1,\dots,b_l,z_{k+1},\dots,z_{k+l}) \\ =&(j^r F_{z_1}(a_1),\dots,j^r F_{z_k}(a_k),z_1,\dots,z_k;j^r F_{z_{k+1}}(b_1),\dots,j^r F_{z_{k+l}}(b_l),z_{k+1},\dots,z_{k+l}), \end{aligned}$$

and ${}_k j^r(F|N \times Z) : N^{(k)} \times Z^k \to J^r(N,Q)^k \times Z^k$ by

$${}_k j^r(F|N \times Z)(a_1,\dots,a_k,z_1,\dots,z_k) = (j^r(F_{z_1}|N)(a_1),\dots,j^r(F_{z_k}|N)(a_k),z_1,\dots,z_k).$$

Then, the parameterized relative multi-transversality theorem is stated as follows.

**Proposition 10.4** (Parameterized relative multi-transversality theorem). *Let $r$ be a non-negative integer and $s$ a positive integer. For given countable families of submanifolds $S_{k,l}$ of ${}_{k,l}J^r(M,N;P,Q;Z)$, $k+l \leq s$, and $U_k$ of $J^r(N,Q)^k \times Z^k$, $k \leq s$, the set*

$$R = \left\{ F \in C^\infty(M \times Z, N \times Z;P,Q) \;\middle|\; \begin{array}{l} {}_{k,l}j^r F \text{ is transverse to } S_{k,l}, k+l \leq s, \text{ and} \\ {}_k j^r(F|N \times Z) \text{ is transverse to } U_k, k \leq s \end{array} \right\}$$

*is a residual subset and is dense in $C^\infty(M \times Z, N \times Z;P,Q)$.*

*Proof of Corollary* 10.2. We put $N = V_k$, $P = Q = \mathbf{R}^2$ and apply the relative version of the parameterized multi-transversality theorem. Then, we see that there exists a generic fold homotopy $F : M \times [-1,1] \to \mathbf{R}^2$ in the space $\mathcal{F}(V_k)$ which connects the two stable fold maps $f$ and $g$. By an argument similar to that for Corollary 10.1, we see that $\operatorname{rot}(f_t|S_i^1)$ and $\sharp S_1(f_t)$ remain unchanged during the generic fold homotopy $F$ $(1 \leq i \leq k)$, where $f_t$ is defined by $F(x,t) = f_t(x)$. This completes the proof. □

Note that if the hypothesis of Corollary 10.2 holds, then the two links $S_1(f)$ and $S_1(g)$ are isotopic in $M$. We can check this by studying the behavior of the singular set during a generic fold homotopy between $g$ and $f$.

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| type | normal form $f(x,y,z)$ |
|---|---|
| definite fold (D fold) | $(x, y^2+z^2)$ |
| indefinite fold (I fold) | $(x, y^2-z^2)$ |
| cusp | $(x, y^3+xy+z^2)$ |
| D&D nodefold | $(x_1, y_1^2+z_1^2), (y_2^2+z_2^2, x_2)$ |
| D&I nodefold | $(x_1, y_1^2+z_1^2), (y_2^2-z_2^2, x_2)$ |
| I&I nodefold | $(x_1, y_1^2-z_1^2), (y_2^2-z_2^2, x_2)$ |

TABLE 1. stable ($\mathcal{A}_e$-codimension 0) germs

| type | | normal form $G(x,y,z,t)$ |
|---|---|---|
| lips | | $(x, y^3+y(x^2-t)+z^2)$ |
| beaks | | $(x, y^3-y(x^2-t)+z^2)$ |
| D swallowtail | | $(x, y^4+xy-ty^2+z^2)$ |
| I swallowtail | | $(x, y^4+xy-ty^2-z^2)$ |
| cusp-plus-D fold | (type 1) | $(x_1, y_1^3+x_1y_1+z^2), (y_2^2+z_2^2-t, x_2)$ |
| | (type 2) | $(x_1, y_1^3+x_1y_1+z^2), (-y_2^2-z_2^2-t, x_2)$ |
| cusp-plus-I fold | | $(x_1, y_1^3+x_1y_1+z^2), (y_2^2-z_2^2-t, x_2)$ |
| D&D tacnodefold | (type 1) | $(x_1, y_1^2+z_1^2+t), (x_2, x_2^2+y_2^2+z_2^2)$ |
| | (type 2) | $(x_1, -y_1^2-z_1^2+t), (x_2, x_2^2+y_2^2+z_2^2)$ |
| | (type 3) | $(x_1, y_1^2+z_1^2+t), (x_2, x_2^2-y_2^2-z_2^2)$ |
| D&I tacnodefold | (type 1) | $(x_1, y_1^2+z_1^2+t), (x_2, x_2^2+y_2^2-z_2^2)$ |
| | (type 2) | $(x_1, -y_1^2-z_1^2+t), (x_2, x_2^2+y_2^2-z_2^2)$ |
| I&I tacnodefold | | $(x_1, y_1^2-z_1^2+t), (x_2, x_2^2+y_2^2-z_2^2)$ |
| D&D&D triplefold | (type 1) | $(x_1+y_1^2+z_1^2, x_1-y_1^2-z_1^2+t), (x_2, y_2^2+z_2^2), (-y_3^2-z_3^2, x_3)$ |
| | (type 2) | $(x_1+y_1^2+z_1^2, x_1-y_1^2-z_1^2+t), (x_2, y_2^2+z_2^2), (y_3^2+z_3^2, x_3)$ |
| D&D&I triplefold | (type 1) | $(x_1+y_1^2-z_1^2, x_1-y_1^2+z_1^2+t), (x_2, y_2^2+z_2^2), (-y_3^2-z_3^2, x_3)$ |
| | (type 2) | $(x_1+y_1^2-z_1^2, x_1-y_1^2+z_1^2+t), (x_2, y_2^2+z_2^2), (y_3^2+z_3^2, x_3)$ |
| D&I&I triplefold | | $(x_1+y_1^2-z_1^2, x_1-y_1^2+z_1^2+t), (x_2, y_2^2+z_2^2), (y_3^2-z_3^2, x_3)$ |
| I&I&I triplefold | | $(x_1+y_1^2-z_1^2, x_1-y_1^2+z_1^2+t), (x_2, y_2^2-z_2^2), (y_3^2-z_3^2, x_3)$ |

TABLE 2. 1-parameter unfoldings

| type | | normal form $H(x,y,z,a,b)$ or $H(x,y,z,a,b,t)$ |
|---|---|---|
| goose | | $(x, y^3 + x^3y + z^2 + ax + bxy)$ |
| butterfly | | $(x, xy + y^5 + y^7 + z^2 + ay^3 + by^2)$ |
| D gulls | | $(x, xy^2 + y^4 + y^5 + z^2 + axy + by)$ |
| I gulls | | $(x, xy^2 + y^4 + y^5 - z^2 + axy + by)$ |
| $D_4^+$ | | $(x, xy + y^3 + ty^2z + z^3 + z^5 + az + by^2)$ |
| | | $(t > -(27/4)^{1/3}, t \neq 0, (27/4)^{1/3})$ |
| $D_4^-$ | | $(x, xy + y^3 + ty^2z + z^3 + z^5 + az + by^2)$ |
| | | $(t < -(27/4)^{1/3})$ |
| lips-plus-D fold | | $(x_1, y_1^3 + y_1(x_1^2 - a) + z_1^2),\ (y_2^2 + z_2^2 + b, x_2)$ |
| lips-plus-I fold | | $(x_1, y_1^3 + y_1(x_1^2 - a) + z_1^2),\ (y_2^2 - z_2^2 + b, x_2)$ |
| beaks-plus-D fold | | $(x_1, y_1^3 - y_1(x_1^2 - a) + z_1^2),\ (y_2^2 + z_2^2 + b, x_2)$ |
| beaks-plus-I fold | | $(x_1, y_1^3 - y_1(x_1^2 - a) + z_1^2),\ (y_2^2 - z_2^2 + b, x_2)$ |
| D swallowtail-plus-D fold | | $(x_1, y_1^4 + x_1y_1 - ay_1^2 + z_1^2),\ (y_2^2 + z_2^2 + b, x_2)$ |
| D swallowtail-plus-I fold | | $(x_1, y_1^4 + x_1y_1 - ay_1^2 + z_1^2),\ (y_2^2 - z_2^2 + b, x_2)$ |
| I swallowtail-plus-D fold | | $(x_1, y_1^4 + x_1y_1 - ay_1^2 - z_1^2),\ (y_2^2 + z_2^2 + b, x_2)$ |
| I swallowtail-plus-I fold | | $(x_1, y_1^4 + x_1y_1 - ay_1^2 - z_1^2),\ (y_2^2 - z_2^2 + b, x_2)$ |
| cusp-plus-cusp | (type 1) | $(x_1 + a, y_1^3 + x_1y_1 + z_1^2),\ (y_2^3 + x_2y_2 + z_2^2, x_2 + b)$ |
| | (type 2) | $(x_1 + a, y_1^3 + x_1y_1 + z_1^2),\ (y_2^3 + x_2y_2 - z_2^2, x_2 + b)$ |
| | (type 3) | $(x_1 + a, y_1^3 + x_1y_1 - z_1^2),\ (y_2^3 + x_2y_2 - z_2^2, x_2 + b)$ |
| cusp-plus-D fold tangency | (type 1) | $(x_1 + a, y_1^3 + x_1y_1 + z_1^2),\ (x_2, y_2^2 + z_2^2 + ax_2 + b)$ |
| | (type 2) | $(x_1 + a, y_1^3 + x_1y_1 + z_1^2),\ (x_2, -y_2^2 - z_2^2 + ax_2 + b)$ |
| cusp-plus-I fold tangency | | $(x_1 + a, y_1^3 + x_1y_1 + z_1^2),\ (x_2, y_2^2 - z_2^2 + ax_2 + b)$ |
| D&D flecnodefold | (type 1) | $(x_1, x_1^3 - ax_1 + y_1^2 + z_1^2),\ (x_2, y_2^2 + z_2^2 + b)$ |
| | (type 2) | $(x_1, x_1^3 - ax_1 + y_1^2 + z_1^2),\ (x_2, -y_2^2 - z_2^2 + b)$ |
| D&I flecnodefold | | $(x_1, x_1^3 - ax_1 + y_1^2 + z_1^2),\ (x_2, y_2^2 - z_2^2 + b)$ |
| I&I flecnodefold | | $(x_1, x_1^3 - ax_1 + y_1^2 - z_1^2),\ (x_2, y_2^2 - z_2^2 + b)$ |

TABLE 3. 2-parameter unfoldings (1) (for $D_4^\pm$, $t$ is the parameter of modality)

| type | | normal form $H(x,y,z,a,b)$ |
|---|---|---|
| D&D nodefold-plus-cusp | (type 1) | $(x_1, y_1^3 + x_1y_1 + z_1^2)$, $(x_2 + y_2^2 + z_2^2, x_2 - y_2^2 - z_2^2 + a)$, $(x_3 + y_3^2 + z_3^2, -x_3 + y_3^2 + z_3^2 + b)$ |
| | (type 2) | $(x_1, y_1^3 + x_1y_1 + z_1^2)$, $(x_2 + y_2^2 + z_2^2, x_2 - y_2^2 - z_2^2 + a)$, $(x_3 - y_3^2 - z_3^2, -x_3 - y_3^2 - z_3^2 + b)$ |
| | (type 3) | $(x_1, y_1^3 + x_1y_1 - z_1^2)$, $(x_2 + y_2^2 + z_2^2, x_2 - y_2^2 - z_2^2 + a)$, $(x_3 - y_3^2 - z_3^2, -x_3 - y_3^2 - z_3^2 + b)$ |
| | (type 4) | $(x_1, y_1^3 + x_1y_1 + z_1^2)$, $(x_2 - y_2^2 - z_2^2, x_2 + y_2^2 + z_2^2 + a)$, $(x_3 - y_3^2 - z_3^2, -x_3 - y_3^2 - z_3^2 + b)$ |
| D&I nodefold-plus-cusp | (type 1) | $(x_1, y_1^3 + x_1y_1 + z_1^2)$, $(x_2 + y_2^2 + z_2^2, x_2 - y_2^2 - z_2^2 + a)$, $(x_3 + y_3^2 - z_3^2, -x_3 + y_3^2 - z_3^2 + b)$ |
| | (tyep 2) | $(x_1, y_1^3 + x_1y_1 - z_1^2)$, $(x_2 + y_2^2 + z_2^2, x_2 - y_2^2 - z_2^2 + a)$, $(x_3 + y_3^2 - z_3^2, -x_3 + y_3^2 - z_3^2 + b)$ |
| | (tyep 3) | $(x_1, y_1^3 + x_1y_1 + z_1^2)$, $(x_2 - y_2^2 - z_2^2, x_2 + y_2^2 + z_2^2 + a)$, $(x_3 - y_3^2 + z_3^2, -x_3 - y_3^2 + z_3^2 + b)$ |
| | (type 4) | $(x_1, y_1^3 + x_1y_1 - z_1^2)$, $(x_2 - y_2^2 - z_2^2, x_2 + y_2^2 + z_2^2 + a)$, $(x_3 - y_3^2 + z_3^2, -x_3 - y_3^2 + z_3^2 + b)$ |
| I&I nodefold-plus-cusp | | $(x_1, y_1^3 + x_1y_1 + z_1^2)$, $(x_2 + y_2^2 - z_2^2, x_2 - y_2^2 + z_2^2 + a)$, $(x_3 + y_3^2 - z_3^2, -x_3 + y_3^2 - z_3^2 + b)$ |
| D&D tacnodefold-plus-D fold | (type 1) | $(x_1, x_1^2 + y_1^2 + z_1^2)$, $(x_2, y_2^2 + z_2^2 + a)$, $(y_3^2 + z_3^2 + b, x_3)$ |
| | (type 2) | $(x_1, x_1^2 + y_1^2 + z_1^2)$, $(x_2, -y_2^2 - z_2^2 + a)$, $(y_3^2 + z_3^2 + b, x_3)$ |
| | (type 3) | $(x_1, x_1^2 - y_1^2 - z_1^2)$, $(x_2, y_2^2 + z_2^2 + a)$, $(y_3^2 + z_3^2 + b, x_3)$ |
| D&D tacnodefold-plus-I fold | (tyep 1) | $(x_1, x_1^2 + y_1^2 + z_1^2)$, $(x_2, y_2^2 + z_2^2 + a)$, $(y_3^2 - z_3^2 + b, x_3)$ |
| | (tyep 2) | $(x_1, x_1^2 + y_1^2 + z_1^2)$, $(x_2, -y_2^2 - z_2^2 + a)$, $(y_3^2 - z_3^2 + b, x_3)$ |
| | (type 3) | $(x_1, x_1^2 - y_1^2 - z_1^2)$, $(x_2, y_2^2 + z_2^2 + a)$, $(y_3^2 - z_3^2 + b, x_3)$ |
| D&I tacnodefold-plus-D fold | (type 1) | $(x_1, x_1^2 + y_1^2 + z_1^2)$, $(x_2, y_2^2 - z_2^2 + a)$, $(y_3^2 + z_3^2 + b, x_3)$ |
| | (type 2) | $(x_1, x_1^2 - y_1^2 - z_1^2)$, $(x_2, y_2^2 - z_2^2 + a)$, $(y_3^2 + z_3^2 + b, x_3)$ |
| D&I tacnodefold-plus-I fold | (type 1) | $(x_1, x_1^2 + y_1^2 + z_1^2)$, $(x_2, y_2^2 - z_2^2 + a)$, $(y_3^2 - z_3^2 + b, x_3)$ |
| | (type 2) | $(x_1, x_1^2 - y_1^2 - z_1^2)$, $(x_2, y_2^2 - z_2^2 + a)$, $(y_3^2 - z_3^2 + b, x_3)$ |
| I&I tacnodefold-plus-D fold | | $(x_1, x_1^2 + y_1^2 - z_1^2)$, $(x_2, y_2^2 - z_2^2 + a)$, $(y_3^2 + z_3^2 + b, x_3)$ |
| I&I tacnodefold-plus-I fold | | $(x_1, x_1^2 + y_1^2 - z_1^2)$, $(x_2, y_2^2 - z_2^2 + a)$, $(y_3^2 - z_3^2 + b, x_3)$ |

TABLE 4. 2-parameter unfoldings (2)

| type | | normal form $H(x,y,z,a,b,t)$ |
|---|---|---|
| D&D&D&D quadruplefold | (type 1) | $(x_1+y_1^2+z_1^2, x_1+y_1^2+z_1^2+a)$, $(x_2, y_2^2+z_2^2)$, $(y_3^2+z_3^2, x_3)$, $(x_4+y_4^2+z_4^2, -tx_4+y_4^2+z_4^2+b)$ |
| | (type 2) | $(x_1+y_1^2+z_1^2, x_1+y_1^2+z_1^2+a)$, $(x_2, y_2^2+z_2^2)$, $(-y_3^2-z_3^2, x_3)$, $(x_4+y_4^2+z_4^2, -tx_4+y_4^2+z_4^2+b)$ |
| D&D&D&I quadruplefold | (type 1) | $(x_1+y_1^2+z_1^2, x_1+y_1^2+z_1^2+a)$, $(x_2, y_2^2+z_2^2)$, $(y_3^2+z_3^2, x_3)$, $(x_4+y_4^2-z_4^2, -tx_4+y_4^2-z_4^2+b)$ |
| | (type 2) | $(x_1+y_1^2+z_1^2, x_1+y_1^2+z_1^2+a)$, $(x_2, y_2^2+z_2^2)$, $(-y_3^2-z_3^2, x_3)$, $(x_4+y_4^2-z_4^2, -tx_4+y_4^2-z_4^2+b)$ |
| | (type 3) | $(x_1+y_1^2+z_1^2, x_1+y_1^2+z_1^2+a)$, $(x_2, y_2^2-z_2^2)$, $(y_3^2+z_3^2, x_3)$, $(x_4+y_4^2+z_4^2, -tx_4+y_4^2+z_4^2+b)$ |
| D&D&I&I quadruplefold | (type 1) | $(x_1+y_1^2+z_1^2, x_1+y_1^2+z_1^2+a)$, $(x_2, y_2^2+z_2^2)$, $(y_3^2-z_3^2, x_3)$, $(x_4+y_4^2-z_4^2, -tx_4+y_4^2-z_4^2+b)$ |
| | (type 2) | $(x_1+y_1^2+z_1^2, x_1+y_1^2+z_1^2+a)$, $(x_2, y_2^2-z_2^2)$, $(y_3^2+z_3^2, x_3)$, $(x_4+y_4^2-z_4^2, -tx_4+y_4^2-z_4^2+b)$ |
| | (type 3) | $(x_1+y_1^2+z_1^2, x_1+y_1^2+z_1^2+a)$, $(x_2, y_2^2-z_2^2)$, $(y_3^2-z_3^2, x_3)$, $(x_4+y_4^2+z_4^2, -tx_4+y_4^2+z_4^2+b)$ |
| D&I&I&I quadruplefold | | $(x_1+y_1^2+z_1^2, x_1+y_1^2+z_1^2+a)$, $(x_2, y_2^2-z_2^2)$, $(y_3^2-z_3^2, x_3)$, $(x_4+y_4^2-z_4^2, -tx_4+y_4^2-z_4^2+b)$ |
| I&I&I&I quadruplefold | | $(x_1+y_1^2-z_1^2, x_1+y_1^2-z_1^2+a)$, $(x_2, y_2^2-z_2^2)$, $(y_3^2-z_3^2, x_3)$, $(x_4+y_4^2-z_4^2, -tx_4+y_4^2-z_4^2+b)$ |

TABLE 5. 2-parameter unfoldings (3) (for these cases, $t$ is the parameter of modality)

| | $III^a(l)$ | $III^a(b)$ | $III^b$ | $III^c$ | $III^d$ | $III_1^{0,a}$ | $III_2^{0,a}$ | $III_1^{1,a}$ | $III_2^{1,a}$ | $III_1^e$ | $III_2^e$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $IV^a$ | 1 | −1 | | | | | | | | | |
| $IV^f$ | | | 1 | −1 | | | −1 | | | | 1 |
| $IV^b$ | | −2 | 2 | | | | | | | | |
| $IV^c$ | | −2 | | 2 | | | | | | | |
| $IV^g$ | | | | | | | | | | | |
| $IV^h$ | | | | 1 | −1 | | | | | | |
| $IV^i$ | | | | | | | | | | | |
| $IV^j$ | | | | | | | | | | | |
| $IV^k$ | | | | | | | | | | | |
| $IV^{0,a}(l)$ | | | | | | −1 | 1 | | | | |
| $IV^{1,a}(l)$ | | | | | | | | −1 | 1 | | |
| $IV^e(l)$ | | | | | | | | | | −1 | 1 |
| $IV^{0,a}(b)$ | | | | | | 1 | −1 | | | | |
| $IV^{1,a}(b)$ | | | | | | | | 1 | −1 | | |
| $IV^e(b)$ | | | | | | | | | | 1 | −1 |

TABLE 6. The block $A_{1,1}$ of the block matrix $A$

| | $III_1^{0,0}$ | $III_2^{0,0}$ | $III_3^{0,0}$ | $III_1^{0,1}$ | $III_2^{0,1}$ | $III_3^{0,1}$ | $III_4^{0,1}$ | $III_1^{1,1}$ | $III_2^{1,1}$ | $III_3^{1,1}$ | $III_1^2$ | $III_2^2$ | $III_3^2$ | $III_1^3$ | $III_2^3$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $IV^a$ | | | | | | | | | | | | | | | |
| $IV^f$ | | | | | | | | | | | | | | | |
| $IV^b$ | −1 | | | | | | | | | | | | | | |
| $IV^c$ | | | | | | | | | | | −1 | | | | |
| $IV^g$ | | | | | | | | | | | | | | | |
| $IV^h$ | | | | | | | | | | | | | | | |
| $IV^i$ | | | | | | | | | | | | | | | |
| $IV^j$ | | | | | | | | | | | | | | | |
| $IV^k$ | | | | | | | | | | | | | | | |
| $IV^{0,a}(l)$ | | | | | | | | | | | | | | | |
| $IV^{1,a}(l)$ | | | | | | | | | | | | | | | |
| $IV^e(l)$ | | | | | | | | | | | | | | | |
| $IV^{0,a}(b)$ | | | | | | | | | | | | | | | |
| $IV^{1,a}(b)$ | | | | | | | | | | | | | | | |
| $IV^e(b)$ | | | | | | | | | | | | | | | |

TABLE 7. The block $A_{1,2}$ of the block matrix $A$

| | $III_1^4$ | $III_2^4$ | $III_3^4$ | $III_1^5$ | $III_2^5$ | $III_1^6$ | $III_2^6$ | $III_1^7$ | $III_2^7$ | $III^8$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $IV^a$ | | | | | | | | | | |
| $IV^f$ | | | | | | | | | | |
| $IV^b$ | | | | | | | | | | |
| $IV^c$ | | | | | | | | | | |
| $IV^g$ | | | | | | | | | | |
| $IV^h$ | | | | | −1 | | | | −1 | |
| $IV^i$ | | | | | | | | | | |
| $IV^j$ | | | | | | | | | | |
| $IV^k$ | | | | | | | | | | |
| $IV^{0,a}(l)$ | | | | | | | | | | |
| $IV^{1,a}(l)$ | | | | | | | | | | |
| $IV^e(l)$ | | | | | | | | | | |
| $IV^{0,a}(b)$ | | | | | | | | | | |
| $IV^{1,a}(b)$ | | | | | | | | | | |
| $IV^e(b)$ | | | | | | | | | | |

TABLE 8. The block $A_{1,5}$ of the block matrix $A$

| | $III^a(l)$ | $III^a(b)$ | $III^b$ | $III^c$ | $III^d$ | $III_1^{0,a}$ | $III_2^{0,a}$ | $III_1^{1,a}$ | $III_2^{1,a}$ | $III_1^e$ | $III_2^e$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $IV^{0,b}$ | | | | | | −1 | 1 | | | | |
| $IV^{1,b}$ | | | | | | | | −1 | 1 | | |
| $IV^{0,c}$ | | | | | | −1 | 1 | | | | |
| $IV^{1,c}$ | | | | | | | | −1 | 1 | | |
| $IV^{0,d}$ | | | | | | −1 | 1 | | | | |
| $IV^{1,d}$ | | | | | | | | −1 | 1 | | |
| $IV^l$ | | | | | | | | −1 | | | 1 |
| $IV^m$ | | | | | | | | | | −1 | 1 |
| $IV^n$ | | | | 1 | −1 | | | | | −1 | 1 |

TABLE 9. The block $A_{2,1}$ of the block matrix $A$

| | $III_1^{0,0,0}$ | $III_2^{0,0,0}$ | $III_1^{0,0,1}$ | $III_2^{0,0,1}$ | $III_3^{0,0,1}$ | $III_1^{0,1,1}$ | $III_2^{0,1,1}$ | $III_3^{0,1,1}$ | $III_1^{1,1,1}$ | $III_2^{1,1,1}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $IV^{0,b}$ | | $-1$ | | | | | | | | |
| $IV^{1,b}$ | | | | $-1$ | | | | | | |
| $IV^{0,c}$ | | | | | | | | | | |
| $IV^{1,c}$ | | | | | | | | | | |
| $IV^{0,d}$ | | | | | | | | | | |
| $IV^{1,d}$ | | | | | | | | | | |
| $IV^{l}$ | | | | | | | | | | |
| $IV^{m}$ | | | | | | | | | | |
| $IV^{n}$ | | | | | | | | | | |

TABLE 10. The block $A_{2,3}$ of the block matrix $A$

| | $III_1^{0,2}$ | $III_2^{0,2}$ | $III_3^{0,2}$ | $III_1^{1,2}$ | $III_2^{1,2}$ | $III_3^{1,2}$ | $III_1^{0,3}$ | $III_2^{0,3}$ | $III_1^{1,3}$ | $III_2^{1,3}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $IV^{0,b}$ | | | | | | | | | | |
| $IV^{1,b}$ | | | | | | | | | | |
| $IV^{0,c}$ | | $-1$ | | | | | | | | |
| $IV^{1,c}$ | | | | | $-1$ | | | | | |
| $IV^{0,d}$ | | | | | | | | $-1$ | | |
| $IV^{1,d}$ | | | | | | | | | | $-1$ |
| $IV^{l}$ | | | | | | | | | | |
| $IV^{m}$ | | | | | | | | | | |
| $IV^{n}$ | | | | | | | | | | |

TABLE 11. The block $A_{2,4}$ of the block matrix $A$

| | $III_1^4$ | $III_2^4$ | $III_3^4$ | $III_1^5$ | $III_2^5$ | $III_1^6$ | $III_2^6$ | $III_1^7$ | $III_2^7$ | $III^8$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $IV^{0,b}$ | | | | | | | | | | |
| $IV^{1,b}$ | | | | | | | | | | |
| $IV^{0,c}$ | | | | | | | | | | |
| $IV^{1,c}$ | | | | | | | | | | |
| $IV^{0,d}$ | | | | | | | | | | |
| $IV^{1,d}$ | | | | | | | | | | |
| $IV^{l}$ | | $-1$ | | | | | | | | |
| $IV^{m}$ | | | | | | | $-1$ | | | |
| $IV^{n}$ | | | | | | | | | $-1$ | |

TABLE 12. The block $A_{2,5}$ of the block matrix $A$

| | $III^a(l)$ | $III^a(b)$ | $III^b$ | $III^c$ | $III^d$ | $III_1^{0,a}$ | $III_2^{0,a}$ | $III_1^{1,a}$ | $III_2^{1,a}$ | $III_1^e$ | $III_2^e$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $IV_1^{a,a}$ | | | | | | $-1$ | $1$ | $1$ | $-1$ | | |
| $IV_2^{a,a}$ | | | | | | | | | | | |
| $IV_3^{a,a}$ | | | | | | | | | | | |
| $IV_1^o$ | | | | | | $-1$ | $1$ | | | $1$ | $-1$ |
| $IV_2^o$ | | | | | | | | | | | |
| $IV_3^o$ | | | | | | | | | | | |
| $IV_1^{0,a}$ | | | | | | $-1$ | $-1$ | | | | |
| $IV_2^{0,a}$ | | | | | | $-1$ | $-1$ | | | | |
| $IV_1^{1,a}$ | | | | | | | | $-1$ | $-1$ | | |
| $IV_2^{1,a}$ | | | | | | | | $-1$ | $-1$ | | |
| $IV_1^e$ | | | | | | | | | | $-1$ | $-1$ |
| $IV_2^e$ | | | | | | | | | | $-1$ | $-1$ |

TABLE 13. The block $A_{3,1}$ of the block matrix $A$

| | $III_1^{0,0}$ | $III_2^{0,0}$ | $III_3^{0,0}$ | $III_1^{0,1}$ | $III_2^{0,1}$ | $III_3^{0,1}$ | $III_4^{0,1}$ | $III_1^{1,1}$ | $III_2^{1,1}$ | $III_3^{1,1}$ | $III_1^2$ | $III_2^2$ | $III_3^2$ | $III_1^3$ | $III_2^3$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $IV_1^{a,a}$ | | | | | | | | | | | | | | | |
| $IV_2^{a,a}$ | | | | | | | | | | | | | | | |
| $IV_3^{a,a}$ | | | | | | | | | | | | | | | |
| $IV_1^o$ | | | | | | | | | | | | | | | |
| $IV_2^o$ | | | | | | | | | | | | | | | |
| $IV_3^o$ | | | | | | | | | | | | | | | |
| $IV_1^{0,a}$ | | | $1$ | $1$ | | | | | | | | | | | |
| $IV_2^{0,a}$ | $1$ | | | | | $1$ | | | | | | | | | |
| $IV_1^{1,a}$ | | | | | | $1$ | | $1$ | | | | | | | |
| $IV_2^{1,a}$ | | | | | | | $1$ | | | $1$ | | | | | |
| $IV_1^e$ | | | | | | $1$ | | | | | $1$ | | | | |
| $IV_2^e$ | | | | | | | $1$ | | | | | | $1$ | | |

TABLE 14. The block $A_{3,2}$ of the block matrix $A$

| | $III_1^{0,0}$ | $III_2^{0,0}$ | $III_3^{0,0}$ | $III_1^{0,1}$ | $III_2^{0,1}$ | $III_3^{0,1}$ | $III_4^{0,1}$ | $III_1^{1,1}$ | $III_2^{1,1}$ | $III_3^{1,1}$ | $III_1^2$ | $III_2^2$ | $III_3^2$ | $III_1^3$ | $III_2^3$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $IV_1^{0,0}$ | | | | | | | | | | | | | | | |
| $IV_2^{0,0}$ | | −1 | 1 | | | | | | | | | | | | |
| $IV_1^{0,1}$ | | | | 1 | | | −1 | | | | | | | | |
| $IV_2^{0,1}$ | | | | | −1 | 1 | | | | | | | | | |
| $IV_1^{1,1}$ | | | | | | | | | | | | | | | |
| $IV_2^{1,1}$ | | | | | | | | | −1 | 1 | | | | | |
| $IV_1^2$ | | | | | | | | | | | | | | | |
| $IV_2^2$ | | | | | | | | | | | | −1 | 1 | | |
| $IV_1^3$ | | | | | | | | | | | | | | | |
| $IV_2^3$ | | | | | | | | | | | | | | | |

TABLE 15. The block $A_{4,2}$ of the block matrix $A$

| | $III_1^{0,0,0}$ | $III_2^{0,0,0}$ | $III_1^{0,0,1}$ | $III_2^{0,0,1}$ | $III_3^{0,0,1}$ | $III_1^{0,1,1}$ | $III_2^{0,1,1}$ | $III_3^{0,1,1}$ | $III_1^{1,1,1}$ | $III_2^{1,1,1}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $IV_1^{0,0,a}$ | | −1 | | 1 | | | | | | |
| $IV_2^{0,0,a}$ | | 1 | 1 | | | | | | | |
| $IV_3^{0,0,a}$ | −1 | | | | −1 | | | | | |
| $IV_4^{0,0,a}$ | | −1 | | 1 | | | | | | |
| $IV_1^{0,1,a}$ | | | | −1 | | | 1 | | | |
| $IV_2^{0,1,a}$ | | | | 1 | | −1 | | | | |
| $IV_3^{0,1,a}$ | | | −1 | | | | −1 | | | |
| $IV_4^{0,1,a}$ | | | | | −1 | | | 1 | | |
| $IV_5^{0,1,a}$ | | | | | −1 | | | 1 | | |
| $IV_6^{0,1,a}$ | | | −1 | | | | −1 | | | |
| $IV_7^{0,1,a}$ | | | | 1 | | 1 | | | | |
| $IV_8^{0,1,a}$ | | | | −1 | | | 1 | | | |
| $IV_1^{1,1,a}$ | | | | | | | −1 | | | 1 |
| $IV_2^{1,1,a}$ | | | | | | | | 1 | 1 | |
| $IV_3^{1,1,a}$ | | | | | | −1 | | | | −1 |
| $IV_4^{1,1,a}$ | | | | | | | −1 | | | 1 |

TABLE 16. The block $A_{5,3}$ of the block matrix $A$

| | $III_1^{0,0,0}$ | $III_2^{0,0,0}$ | $III_1^{0,0,1}$ | $III_2^{0,0,1}$ | $III_3^{0,0,1}$ | $III_1^{0,1,1}$ | $III_2^{0,1,1}$ | $III_3^{0,1,1}$ | $III_1^{1,1,1}$ | $III_2^{1,1,1}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $IV_1^{2,a}$ | | | | | | | | | | |
| $IV_2^{2,a}$ | | | | | | | | | | |
| $IV_3^{2,a}$ | | | | | | | | | | |
| $IV_4^{2,a}$ | | | | | | | | | | |
| $IV_1^{3,a}$ | | | | | | | | | | |
| $IV_2^{3,a}$ | | | | | | | | | | |
| $IV_1^{0,e}$ | | | | | −1 | | | | | |
| $IV_2^{0,e}$ | | | | 1 | | | | | | |
| $IV_3^{0,e}$ | | | −1 | | | | | | | |
| $IV_4^{0,e}$ | | | | −1 | | | | | | |
| $IV_5^{0,e}$ | | | | −1 | | | | | | |
| $IV_6^{0,e}$ | | | −1 | | | | | | | |
| $IV_7^{0,e}$ | | | | 1 | | | | | | |
| $IV_8^{0,e}$ | | | | | −1 | | | | | |
| $IV_1^{1,e}$ | | | | | | | −1 | | | |
| $IV_2^{1,e}$ | | | | | | | | 1 | | |
| $IV_3^{1,e}$ | | | | | | −1 | | | | |
| $IV_4^{1,e}$ | | | | | | | −1 | | | |
| $IV_5^{1,e}$ | | | | | | | −1 | | | |
| $IV_6^{1,e}$ | | | | | | −1 | | | | |
| $IV_7^{1,e}$ | | | | | | | | 1 | | |
| $IV_8^{1,e}$ | | | | | | | −1 | | | |

TABLE 17. The block $A_{6,3}$ of the block matrix $A$

| | $III_1^{0,2}$ | $III_2^{0,2}$ | $III_3^{0,2}$ | $III_1^{1,2}$ | $III_2^{1,2}$ | $III_3^{1,2}$ | $III_1^{0,3}$ | $III_2^{0,3}$ | $III_1^{1,3}$ | $III_2^{1,3}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $IV_1^{2,a}$ | | $-1$ | | | $1$ | | | | | |
| $IV_2^{2,a}$ | | | $1$ | $1$ | | | | | | |
| $IV_3^{2,a}$ | $-1$ | | | | | $-1$ | | | | |
| $IV_4^{2,a}$ | | $-1$ | | | $1$ | | | | | |
| $IV_1^{3,a}$ | | | | | | | $-1$ | | $1$ | |
| $IV_2^{3,a}$ | | | | | | | | $-1$ | | $1$ |
| $IV_1^{0,e}$ | | | $1$ | | | | | | | |
| $IV_2^{0,e}$ | $1$ | | | | | | | | | |
| $IV_3^{0,e}$ | | $-1$ | | | | | | | | |
| $IV_4^{0,e}$ | | $1$ | | | | | | | | |
| $IV_5^{0,e}$ | | $1$ | | | | | | | | |
| $IV_6^{0,e}$ | | $-1$ | | | | | | | | |
| $IV_7^{0,e}$ | $1$ | | | | | | | | | |
| $IV_8^{0,e}$ | | | $1$ | | | | | | | |
| $IV_1^{1,e}$ | | | | | | $1$ | | | | |
| $IV_2^{1,e}$ | | | | $1$ | | | | | | |
| $IV_3^{1,e}$ | | | | | $-1$ | | | | | |
| $IV_4^{1,e}$ | | | | | $1$ | | | | | |
| $IV_5^{1,e}$ | | | | | $1$ | | | | | |
| $IV_6^{1,e}$ | | | | | $-1$ | | | | | |
| $IV_7^{1,e}$ | | | | $1$ | | | | | | |
| $IV_8^{1,e}$ | | | | | | $1$ | | | | |

TABLE 18. The block $A_{6,4}$ of the block matrix $A$

| | $III^a(l)$ | $III^a(b)$ | $III^b$ | $III^c$ | $III^d$ | $III_1^{0,a}$ | $III_2^{0,a}$ | $III_1^{1,a}$ | $III_2^{1,a}$ | $III_1^e$ | $III_2^e$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $IV_1^p$ | | | | | | | | $-1$ | | $1$ | |
| $IV_2^p$ | | | | | | | | | $-1$ | | $1$ |
| $IV_3^p$ | | | | | | | | $1$ | | $-1$ | |
| $IV_4^p$ | | | | | | | | | $1$ | | $-1$ |
| $IV_5^p$ | | | | | | | | $1$ | | $-1$ | |
| $IV_6^p$ | | | | | | | | | $1$ | | $-1$ |
| $IV_7^p$ | | | | | | | | $-1$ | | $1$ | |
| $IV_8^p$ | | | | | | | | | $-1$ | | $1$ |
| $IV_1^q$ | | | | | | | | | | | |
| $IV_2^q$ | | | | | | | | | | | |
| $IV_3^q$ | | | | | | | | | | | |
| $IV_4^q$ | | | | | | | | | | | |
| $IV_1^r$ | | | | | | | | | | | |
| $IV_2^r$ | | | | | | | | | | | |

TABLE 19. The block $A_{7,1}$ of the block matrix $A$

| | $III_1^{0,2}$ | $III_2^{0,2}$ | $III_3^{0,2}$ | $III_1^{1,2}$ | $III_2^{1,2}$ | $III_3^{1,2}$ | $III_1^{0,3}$ | $III_2^{0,3}$ | $III_1^{1,3}$ | $III_2^{1,3}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $IV_1^p$ | | $-1$ | | | | | | | | |
| $IV_2^p$ | | | $1$ | | | | | | | |
| $IV_3^p$ | $-1$ | | | | | | | | | |
| $IV_4^p$ | | $-1$ | | | | | | | | |
| $IV_5^p$ | | $-1$ | | | | | | | | |
| $IV_6^p$ | $-1$ | | | | | | | | | |
| $IV_7^p$ | | | $1$ | | | | | | | |
| $IV_8^p$ | | $-1$ | | | | | | | | |
| $IV_1^q$ | | $-1$ | | | | | | | | |
| $IV_2^q$ | | | $1$ | | | | | | | |
| $IV_3^q$ | $-1$ | | | | | | | | | |
| $IV_4^q$ | | $-1$ | | | | | | | | |
| $IV_1^r$ | | | | | | | | $-1$ | | |
| $IV_2^r$ | | | | | | | $-1$ | | | |

TABLE 20. The block $A_{7,4}$ of the block matrix $A$

| | $III^4_1$ | $III^4_2$ | $III^4_3$ | $III^5_1$ | $III^5_2$ | $III^6_1$ | $III^6_2$ | $III^7_1$ | $III^7_2$ | $III^8$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $IV^p_1$ | | | 1 | | | | | | | |
| $IV^p_2$ | 1 | | | | | | | | | |
| $IV^p_3$ | | | $-1$ | | | | | | | |
| $IV^p_4$ | | 1 | | | | | | | | |
| $IV^p_5$ | | 1 | | | | | | | | |
| $IV^p_6$ | | | $-1$ | | | | | | | |
| $IV^p_7$ | 1 | | | | | | | | | |
| $IV^p_8$ | | | 1 | | | | | | | |
| $IV^q_1$ | | | | | 1 | | | | | |
| $IV^q_2$ | | | | 1 | | | | | | |
| $IV^q_3$ | | | | | $-1$ | | | | | |
| $IV^q_4$ | | | | | 1 | | | | | |
| $IV^r_1$ | | | | | | | 1 | | | |
| $IV^r_2$ | | | | | | 1 | | | | |

TABLE 21. The block $A_{7,5}$ of the block matrix $A$

| | $III_1^{0,0,0}$ | $III_2^{0,0,0}$ | $III_1^{0,0,1}$ | $III_2^{0,0,1}$ | $III_3^{0,0,1}$ | $III_1^{0,1,1}$ | $III_2^{0,1,1}$ | $III_3^{0,1,1}$ | $III_1^{1,1,1}$ | $III_2^{1,1,1}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $IV_1^{0,0,0}$ | | $-2$ | | | | | | | | |
| $IV_2^{0,0,0}$ | $-1$ | 1 | | | | | | | | |
| $IV_3^{0,0,0}$ | $-1$ | 1 | | | | | | | | |
| $IV_1^{0,0,1}$ | | | | $-2$ | | | | | | |
| $IV_2^{0,0,1}$ | | | $-1$ | | 1 | | | | | |
| $IV_3^{0,0,1}$ | | | $-1$ | | 1 | | | | | |
| $IV_4^{0,0,1}$ | | | | $-1$ | $-1$ | | | | | |
| $IV_5^{0,0,1}$ | | | $-1$ | 1 | | | | | | |
| $IV_6^{0,0,1}$ | | | $-1$ | 1 | | | | | | |
| $IV_7^{0,0,1}$ | | | | $-1$ | $-1$ | | | | | |
| $IV_1^{0,1,1}$ | | | | | | | $-1$ | $-1$ | | |
| $IV_2^{0,1,1}$ | | | | | | $-1$ | 1 | | | |
| $IV_3^{0,1,1}$ | | | | | | $-1$ | 1 | | | |
| $IV_4^{0,1,1}$ | | | | | | | $-1$ | $-1$ | | |
| $IV_5^{0,1,1}$ | | | | | | | $-2$ | | | |
| $IV_6^{0,1,1}$ | | | | | | $-1$ | | 1 | | |
| $IV_7^{0,1,1}$ | | | | | | $-1$ | | 1 | | |
| $IV_1^{1,1,1}$ | | | | | | | | | | $-2$ |
| $IV_2^{1,1,1}$ | | | | | | | | | $-1$ | 1 |
| $IV_3^{1,1,1}$ | | | | | | | | | $-1$ | 1 |

TABLE 22. The block $A_{8,3}$ of the block matrix $A$

| | $III_1^{0,0}$ | $III_2^{0,0}$ | $III_3^{0,0}$ | $III_1^{0,1}$ | $III_2^{0,1}$ | $III_3^{0,1}$ | $III_4^{0,1}$ | $III_1^{1,1}$ | $III_2^{1,1}$ | $III_3^{1,1}$ | $III_1^2$ | $III_2^2$ | $III_3^2$ | $III_1^3$ | $III_2^3$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $IV_1^{0,2}$ | | | | | | | | | | | | | | | |
| $IV_2^{0,2}$ | | | | | | | | | | | | | | | |
| $IV_3^{0,2}$ | | | | | | | | | | | | | | | |
| $IV_4^{0,2}$ | | | | | | | | | | | | | | | |
| $IV_5^{0,2}$ | | | | | | | | | | | | | | | |
| $IV_6^{0,2}$ | | | | | | | | | | | | | | | |
| $IV_7^{0,2}$ | | | | | | | | | | | | | | | |
| $IV_1^{1,2}$ | | | | | | | | | | | | | | | |
| $IV_2^{1,2}$ | | | | | | | | | | | | | | | |
| $IV_3^{1,2}$ | | | | | | | | | | | | | | | |
| $IV_4^{1,2}$ | | | | | | | | | | | | | | | |
| $IV_5^{1,2}$ | | | | | | | | | | | | | | | |
| $IV_6^{1,2}$ | | | | | | | | | | | | | | | |
| $IV_7^{1,2}$ | | | | | | | | | | | | | | | |
| $IV_1^{0,3}$ | | | | | | | | | | | | | | | |
| $IV_2^{0,3}$ | | | | | | | | | | | | | | | |
| $IV_3^{0,3}$ | | | | $-1$ | 1 | | | | | | | | | | |
| $IV_4^{0,3}$ | | | | | | $-1$ | 1 | | | | | | | | |
| $IV_1^{1,3}$ | | | | | | | | | | | | | | | |
| $IV_2^{1,3}$ | | | | | | | | | | | | | | | |
| $IV_3^{1,3}$ | | | | | | | | $-1$ | 1 | | | | | | |
| $IV_4^{1,3}$ | | | | | | | | 1 | | $-1$ | | | | | |

TABLE 23. The block $A_{9,2}$ of the block matrix $A$

| | $III_1^{0,2}$ | $III_2^{0,2}$ | $III_3^{0,2}$ | $III_1^{1,2}$ | $III_2^{1,2}$ | $III_3^{1,2}$ | $III_1^{0,3}$ | $III_2^{0,3}$ | $III_1^{1,3}$ | $III_2^{1,3}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $IV_1^{0,2}$ | | −2 | | | | | | | | |
| $IV_2^{0,2}$ | −1 | | 1 | | | | | | | |
| $IV_3^{0,2}$ | −1 | | 1 | | | | | | | |
| $IV_4^{0,2}$ | | −1 | −1 | | | | | | | |
| $IV_5^{0,2}$ | −1 | 1 | | | | | | | | |
| $IV_6^{0,2}$ | −1 | 1 | | | | | | | | |
| $IV_7^{0,2}$ | | −1 | −1 | | | | | | | |
| $IV_1^{1,2}$ | | | | | −2 | | | | | |
| $IV_2^{1,2}$ | | | | −1 | | 1 | | | | |
| $IV_3^{1,2}$ | | | | −1 | | 1 | | | | |
| $IV_4^{1,2}$ | | | | | −1 | −1 | | | | |
| $IV_5^{1,2}$ | | | | −1 | 1 | | | | | |
| $IV_6^{1,2}$ | | | | −1 | 1 | | | | | |
| $IV_7^{1,2}$ | | | | | −1 | −1 | | | | |
| $IV_1^{0,3}$ | | | | | | | | −2 | | |
| $IV_2^{0,3}$ | | | | | | | −2 | | | |
| $IV_3^{0,3}$ | | | | | | | −1 | 1 | | |
| $IV_4^{0,3}$ | | | | | | | 1 | −1 | | |
| $IV_1^{1,3}$ | | | | | | | | | | −2 |
| $IV_2^{1,3}$ | | | | | | | | | −2 | |
| $IV_3^{1,3}$ | | | | | | | | | −1 | 1 |
| $IV_4^{1,3}$ | | | | | | | | | 1 | −1 |

TABLE 24. The block $A_{9,4}$ of the block matrix $A$

| | $III_1^{0,0}$ | $III_2^{0,0}$ | $III_3^{0,0}$ | $III_1^{0,1}$ | $III_2^{0,1}$ | $III_3^{0,1}$ | $III_4^{0,1}$ | $III_1^{1,1}$ | $III_2^{1,1}$ | $III_3^{1,1}$ | $III_1^2$ | $III_2^2$ | $III_3^2$ | $III_1^3$ | $III_2^3$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $IV_1^4$ | | | | | | | | | | | | | | | |
| $IV_2^4$ | | | | | | | | | | | | | | | |
| $IV_3^4$ | | | | | | | | | | | | | | | |
| $IV_4^4$ | | | | | | | | | | | | | | | |
| $IV_5^4$ | | | | | | | | 1 | | | −1 | | | | |
| $IV_6^4$ | | | | | | | | | 1 | | | −1 | | | |
| $IV_7^4$ | | | | | | | | | | 1 | | | −1 | | |
| $IV_1^5$ | | | | | | | | | | | | | | | |
| $IV_2^5$ | | | | | | | | | | | | | | | |
| $IV_3^5$ | | | | | | | | | | | | | | | |
| $IV_1^6$ | | | | | | | | | | | 1 | −1 | | | |
| $IV_2^6$ | | | | | | | | | | | −1 | | 1 | | |
| $IV_3^6$ | | | | | | | | | | | | | | | |
| $IV_4^6$ | | | | | | | | | | | | | | | |
| $IV_1^7$ | | | | | | | | | | | 1 | | | −1 | |
| $IV_2^7$ | | | | | | | | | | | | 1 | | | −1 |
| $IV_3^7$ | | | | | | | | | | | | | 1 | | −1 |
| $IV^8$ | | | | | | | | | | | | | | −1 | 1 |

TABLE 25. The block $A_{10,2}$ of the block matrix $A$

| | $III_1^4$ | $III_2^4$ | $III_3^4$ | $III_1^5$ | $III_2^5$ | $III_1^6$ | $III_2^6$ | $III_1^7$ | $III_2^7$ | $III^8$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $IV_1^4$ | | −1 | −1 | | | | | | | |
| $IV_2^4$ | −1 | | 1 | | | | | | | |
| $IV_3^4$ | −1 | | 1 | | | | | | | |
| $IV_4^4$ | | −1 | −1 | | | | | | | |
| $IV_5^4$ | | | −2 | | | | | | | |
| $IV_6^4$ | −1 | 1 | | | | | | | | |
| $IV_7^4$ | −1 | 1 | | | | | | | | |
| $IV_1^5$ | | | | | −2 | | | | | |
| $IV_2^5$ | | | | −1 | 1 | | | | | |
| $IV_3^5$ | | | | −1 | 1 | | | | | |
| $IV_1^6$ | | | | | | 1 | −1 | | | |
| $IV_2^6$ | | | | | | −1 | 1 | | | |
| $IV_3^6$ | | | | | | | −2 | | | |
| $IV_4^6$ | | | | | | −2 | | | | |
| $IV_1^7$ | | | | | | | | | −2 | |
| $IV_2^7$ | | | | | | | | −1 | 1 | |
| $IV_3^7$ | | | | | | | | −1 | 1 | |
| $IV^8$ | | | | | | | | | | −2 |

TABLE 26. The block $A_{10,5}$ of the block matrix $A$

| | $III_1^{0,0,0}$ | $III_2^{0,0,0}$ | $III_1^{0,0,1}$ | $III_2^{0,0,1}$ | $III_3^{0,0,1}$ | $III_1^{0,1,1}$ | $III_2^{0,1,1}$ | $III_3^{0,1,1}$ | $III_1^{1,1,1}$ | $III_2^{1,1,1}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $IV_1^{0,0,3}$ | | | | | | | | | | |
| $IV_2^{0,0,3}$ | | | | | | | | | | |
| $IV_3^{0,0,3}$ | | | | | | | | | | |
| $IV_4^{0,0,3}$ | | | | | | | | | | |
| $IV_5^{0,0,3}$ | | | $-1$ | 2 | 1 | | | | | |
| $IV_1^{0,1,3}$ | | | | | | | | | | |
| $IV_2^{0,1,3}$ | | | | | | | | | | |
| $IV_3^{0,1,3}$ | | | | | | | | | | |
| $IV_4^{0,1,3}$ | | | | | | | | | | |
| $IV_5^{0,1,3}$ | | | | | | $-1$ | 2 | 1 | | |
| $IV_6^{0,1,3}$ | | | | | | 1 | $-2$ | $-1$ | | |
| $IV_1^{1,1,3}$ | | | | | | | | | | |
| $IV_2^{1,1,3}$ | | | | | | | | | | |
| $IV_3^{1,1,3}$ | | | | | | | | | | |
| $IV_4^{1,1,3}$ | | | | | | | | | | |
| $IV_5^{1,1,3}$ | | | | | | | | | $-1$ | 3 |

TABLE 27. The block $A_{12,3}$ of the block matrix $A$

| | $III_1^{0,0,0}$ | $III_2^{0,0,0}$ | $III_1^{0,0,1}$ | $III_2^{0,0,1}$ | $III_3^{0,0,1}$ | $III_1^{0,1,1}$ | $III_2^{0,1,1}$ | $III_3^{0,1,1}$ | $III_1^{1,1,1}$ | $III_2^{1,1,1}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $IV_1^{0,4}$ | | | | | | | 1 | | | |
| $IV_2^{0,4}$ | | | | | | | | $-1$ | | |
| $IV_3^{0,4}$ | | | | | | | $-1$ | | | |
| $IV_4^{0,4}$ | | | | | | 1 | | | | |
| $IV_5^{0,4}$ | | | | | | | $-1$ | | | |
| $IV_6^{0,4}$ | | | | | | | 1 | | | |
| $IV_7^{0,4}$ | | | | | | | | $-1$ | | |
| $IV_8^{0,4}$ | | | | | | | 1 | | | |
| $IV_9^{0,4}$ | | | | | | | | 1 | | |
| $IV_{10}^{0,4}$ | | | | | | $-1$ | | | | |
| $IV_{11}^{0,4}$ | | | | | | 1 | | | | |
| $IV_{12}^{0,4}$ | | | | | | | $-1$ | | | |
| $IV_1^{1,4}$ | | | | | | | | | | 1 |
| $IV_2^{1,4}$ | | | | | | | | | | $-1$ |
| $IV_3^{1,4}$ | | | | | | | | | | $-1$ |
| $IV_4^{1,4}$ | | | | | | | | | 1 | |
| $IV_5^{1,4}$ | | | | | | | | | | $-1$ |
| $IV_6^{1,4}$ | | | | | | | | | | 1 |
| $IV_7^{1,4}$ | | | | | | | | | | $-1$ |
| $IV_8^{1,4}$ | | | | | | | | | | 1 |
| $IV_9^{1,4}$ | | | | | | | | | | 1 |
| $IV_{10}^{1,4}$ | | | | | | | | | $-1$ | |
| $IV_{11}^{1,4}$ | | | | | | | | | 1 | |
| $IV_{12}^{1,4}$ | | | | | | | | | | $-1$ |

TABLE 28. The block $A_{13,3}$ of the block matrix $A$

| | $III_1^{0,2}$ | $III_2^{0,2}$ | $III_3^{0,2}$ | $III_1^{1,2}$ | $III_2^{1,2}$ | $III_3^{1,2}$ | $III_1^{0,3}$ | $III_2^{0,3}$ | $III_1^{1,3}$ | $III_2^{1,3}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $IV_1^{0,4}$ | | | | | −1 | | | | | |
| $IV_2^{0,4}$ | | | | | | 1 | | | | |
| $IV_3^{0,4}$ | | | | | 1 | | | | | |
| $IV_4^{0,4}$ | | | | −1 | | | | | | |
| $IV_5^{0,4}$ | | | | | 1 | | | | | |
| $IV_6^{0,4}$ | | | | | −1 | | | | | |
| $IV_7^{0,4}$ | | | | | | 1 | | | | |
| $IV_8^{0,4}$ | | | | | −1 | | | | | |
| $IV_9^{0,4}$ | | | | | | −1 | | | | |
| $IV_{10}^{0,4}$ | | | | 1 | | | | | | |
| $IV_{11}^{0,4}$ | | | | −1 | | | | | | |
| $IV_{12}^{0,4}$ | | | | | 1 | | | | | |
| $IV_1^{1,4}$ | | | | | −1 | | | | | |
| $IV_2^{1,4}$ | | | | | | 1 | | | | |
| $IV_3^{1,4}$ | | | | | 1 | | | | | |
| $IV_4^{1,4}$ | | | | −1 | | | | | | |
| $IV_5^{1,4}$ | | | | | 1 | | | | | |
| $IV_6^{1,4}$ | | | | | −1 | | | | | |
| $IV_7^{1,4}$ | | | | | | 1 | | | | |
| $IV_8^{1,4}$ | | | | | −1 | | | | | |
| $IV_9^{1,4}$ | | | | | | −1 | | | | |
| $IV_{10}^{1,4}$ | | | | 1 | | | | | | |
| $IV_{11}^{1,4}$ | | | | −1 | | | | | | |
| $IV_{12}^{1,4}$ | | | | | 1 | | | | | |

TABLE 29. The block $A_{13,4}$ of the block matrix $A$

| | $III_1^{0,2}$ | $III_2^{0,2}$ | $III_3^{0,2}$ | $III_1^{1,2}$ | $III_2^{1,2}$ | $III_3^{1,2}$ | $III_1^{0,3}$ | $III_2^{0,3}$ | $III_1^{1,3}$ | $III_2^{1,3}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $IV_1^{0,6}$ | | | | | | | | | | |
| $IV_2^{0,6}$ | | | | | | | | | | |
| $IV_3^{0,6}$ | | | | | | | | | | |
| $IV_4^{0,6}$ | | | | | | | | | | |
| $IV_5^{0,6}$ | −1 | 2 | 1 | | | | | | | |
| $IV_6^{0,6}$ | 1 | −2 | −1 | | | | | | | |
| $IV_1^{1,6}$ | | | | | | | | | | |
| $IV_2^{1,6}$ | | | | | | | | | | |
| $IV_3^{1,6}$ | | | | | | | | | | |
| $IV_4^{1,6}$ | | | | | | | | | | |
| $IV_5^{1,6}$ | | | | −1 | 2 | 1 | | | | |
| $IV_6^{1,6}$ | | | | 1 | −2 | −1 | | | | |
| $IV_1^{0,7}$ | −1 | −1 | 1 | | | | 2 | 1 | | |
| $IV_2^{0,7}$ | | −1 | | | | | | 1 | | |
| $IV_3^{0,7}$ | | 1 | | | | | | −1 | | |
| $IV_4^{0,7}$ | | −1 | | | | | | 1 | | |
| $IV_1^{1,7}$ | | | | −1 | −1 | 1 | | | 2 | 1 |
| $IV_2^{1,7}$ | | | | | −1 | | | | | 1 |
| $IV_3^{1,7}$ | | | | | 1 | | | | | −1 |
| $IV_4^{1,7}$ | | | | | −1 | | | | | 1 |
| $IV^{0,8}$ | | | | | | | 1 | −1 | | |
| $IV^{1,8}$ | | | | | | | | | 1 | −1 |

TABLE 30. The block $A_{15,4}$ of the block matrix $A$

| | $III_1^{0,2}$ | $III_2^{0,2}$ | $III_3^{0,2}$ | $III_1^{1,2}$ | $III_2^{1,2}$ | $III_3^{1,2}$ | $III_1^{0,3}$ | $III_2^{0,3}$ | $III_1^{1,3}$ | $III_2^{1,3}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $IV_1^{2,3}$ | | | | | | | | | | |
| $IV_2^{2,3}$ | | | | | | | | | | |
| $IV_3^{2,3}$ | | | | | | | | | | |
| $IV_4^{2,3}$ | | | | | | | | | | |
| $IV_5^{2,3}$ | | | | −1 | 2 | 1 | | | | |

TABLE 31. The block $A_{17,4}$ of the block matrix $A$

| | $III_1^{0,2}$ | $III_2^{0,2}$ | $III_3^{0,2}$ | $III_1^{1,2}$ | $III_2^{1,2}$ | $III_3^{1,2}$ | $III_1^{0,3}$ | $III_2^{0,3}$ | $III_1^{1,3}$ | $III_2^{1,3}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $IV_1^9$ | | | | | | | | | | |
| $IV_2^9$ | | | | | $-1$ | 1 | | | | |
| $IV_3^9$ | | | | | | | | | | |
| $IV_4^9$ | | | | $-1$ | $-1$ | | | | | |
| $IV_5^9$ | | | | | | | | | | |
| $IV_6^9$ | | | | | | | | | | |
| $IV_7^9$ | | | | | $-2$ | | | | | |
| $IV_8^9$ | | | | | 1 | 1 | | | | |
| $IV_9^9$ | | | | $-1$ | 1 | | | | | |
| $IV_{10}^9$ | | | | 1 | | $-1$ | | | | |
| $IV_{11}^9$ | | | | | | | | | | |
| $IV_{12}^9$ | | | | | 1 | $-1$ | | | | |
| $IV_{13}^9$ | | | | | | | | | | |
| $IV_{14}^9$ | | | | $-1$ | $-1$ | | | | | |
| $IV_{15}^9$ | | | | | | | | | | |
| $IV_{16}^9$ | | | | | | | | | | |
| $IV_1^{10}$ | | | | | $-1$ | | | | | |
| $IV_2^{10}$ | | | | | | 1 | | | | |
| $IV_3^{10}$ | | | | | $-1$ | | | | | |
| $IV_4^{10}$ | | | | | 1 | | | | | |
| $IV_5^{10}$ | | | | $-1$ | | | | | | |
| $IV_6^{10}$ | | | | | 1 | | | | | |
| $IV_7^{10}$ | | | | | | $-1$ | | | | |
| $IV_8^{10}$ | | | | 1 | | | | | | |
| $IV_9^{10}$ | | | | | 1 | | | | | |
| $IV_{10}^{10}$ | | | | | | $-1$ | | | | |
| $IV_{11}^{10}$ | | | | | $-1$ | | | | | |
| $IV_{12}^{10}$ | | | | 1 | | | | | | |

TABLE 32. The block $A_{19,4}$ of the block matrix $A$

| | $III_1^4$ | $III_2^4$ | $III_3^4$ | $III_1^5$ | $III_2^5$ | $III_1^6$ | $III_2^6$ | $III_1^7$ | $III_2^7$ | $III^8$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $IV_1^9$ | | | | | | | | | | |
| $IV_2^9$ | | | | | | | | | | |
| $IV_3^9$ | | | | | | | | | | |
| $IV_4^9$ | 1 | 1 | | | | | | | | |
| $IV_5^9$ | | | | | | | | | | |
| $IV_6^9$ | | | | | | | | | | |
| $IV_7^9$ | | 1 | 1 | | | | | | | |
| $IV_8^9$ | | $-1$ | $-1$ | | | | | | | |
| $IV_9^9$ | 1 | | $-1$ | | | | | | | |
| $IV_{10}^9$ | $-1$ | | 1 | | | | | | | |
| $IV_{11}^9$ | | | | | | | | | | |
| $IV_{12}^9$ | | | | | | | | | | |
| $IV_{13}^9$ | | | | | | | | | | |
| $IV_{14}^9$ | 1 | 1 | | | | | | | | |
| $IV_{15}^9$ | | | | | | | | | | |
| $IV_{16}^9$ | | | | | | | | | | |
| $IV_1^{10}$ | | | | | 1 | | | | | |
| $IV_2^{10}$ | | | | | $-1$ | | | | | |
| $IV_3^{10}$ | | | | | 1 | | | | | |
| $IV_4^{10}$ | | | | | $-1$ | | | | | |
| $IV_5^{10}$ | | | | 1 | | | | | | |
| $IV_6^{10}$ | | | | | $-1$ | | | | | |
| $IV_7^{10}$ | | | | | 1 | | | | | |
| $IV_8^{10}$ | | | | $-1$ | | | | | | |
| $IV_9^{10}$ | | | | | $-1$ | | | | | |
| $IV_{10}^{10}$ | | | | | 1 | | | | | |
| $IV_{11}^{10}$ | | | | | 1 | | | | | |
| $IV_{12}^{10}$ | | | | $-1$ | | | | | | |

TABLE 33. The block $A_{19,5}$ of the block matrix $A$

| | $III_1^{0,2}$ | $III_2^{0,2}$ | $III_3^{0,2}$ | $III_1^{1,2}$ | $III_2^{1,2}$ | $III_3^{1,2}$ | $III_1^{0,3}$ | $III_2^{0,3}$ | $III_1^{1,3}$ | $III_2^{1,3}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $IV_1^{11}$ | | | | | | | | | $-1$ | |
| $IV_2^{11}$ | | | | | | | | | | $-1$ |
| $IV_3^{11}$ | | | | | | | | | 1 | |
| $IV_4^{11}$ | | | | | | | | | | 1 |
| $IV_5^{11}$ | | | | | | | | | 1 | |
| $IV_6^{11}$ | | | | | | | | | | 1 |

TABLE 34. The block $A_{20,4}$ of the block matrix $A$

| | $III_1^4$ | $III_2^4$ | $III_3^4$ | $III_1^5$ | $III_2^5$ | $III_1^6$ | $III_2^6$ | $III_1^7$ | $III_2^7$ | $III^8$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $IV_1^{11}$ | | | | | | 1 | | | | |
| $IV_2^{11}$ | | | | | | | 1 | | | |
| $IV_3^{11}$ | | | | | | $-1$ | | | | |
| $IV_4^{11}$ | | | | | | | $-1$ | | | |
| $IV_5^{11}$ | $-1$ | 1 | 2 | | | $-1$ | | | | |
| $IV_6^{11}$ | 1 | $-1$ | $-2$ | | | | $-1$ | | | |

TABLE 35. The block $A_{20,5}$ of the block matrix $A$

| | $III_1^4$ | $III_2^4$ | $III_3^4$ | $III_1^5$ | $III_2^5$ | $III_1^6$ | $III_2^6$ | $III_1^7$ | $III_2^7$ | $III^8$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $IV_1^{13}$ | | | | | | | | | | |
| $IV_2^{13}$ | | | | | | | | | | |
| $IV_3^{13}$ | | | | | | | | | | |
| $IV_4^{13}$ | | | | | | | | | | |
| $IV_5^{13}$ | | | | | | | | | | |
| $IV_6^{13}$ | | | | | | | | | | |
| $IV_7^{13}$ | | | | | | | | | | |
| $IV_8^{13}$ | | | | | | | | | | |
| $IV_9^{13}$ | | | | $-1$ | 3 | | | | | |
| $IV_{10}^{13}$ | | | | 1 | $-3$ | | | | | |
| $IV_1^{14}$ | | | | | | | | | | |
| $IV_2^{14}$ | | | | | | | | | | |
| $IV_3^{14}$ | | | | | | | | | | |
| $IV_4^{14}$ | | | | | | | | | | |
| $IV_5^{14}$ | $-1$ | 1 | 2 | | | | | | | |
| $IV_1^{15}$ | | | | | | | | | | |
| $IV_2^{15}$ | | | | | | | | | | |
| $IV_3^{15}$ | $-1$ | $-1$ | | 1 | 1 | | | | | |
| $IV_4^{15}$ | | | | | | | | | | |
| $IV_5^{15}$ | $-1$ | | 1 | | 2 | | | | | |
| $IV_1^{16}$ | | | | | $-1$ | | 1 | | | |
| $IV_2^{16}$ | | | | | 1 | | $-1$ | | | |
| $IV_3^{16}$ | | | | | $-1$ | | 1 | | | |
| $IV_4^{16}$ | | | | | 1 | | $-1$ | | | |
| $IV_5^{16}$ | $-1$ | $-1$ | | | 1 | 2 | 1 | | | |
| $IV_6^{16}$ | 1 | 1 | | $-1$ | | | $-1$ | | | |
| $IV_7^{16}$ | | | | 1 | | $-2$ | $-1$ | | | |
| $IV_8^{16}$ | | | | | $-1$ | | 1 | | | |
| $IV_9^{16}$ | | $-1$ | $-1$ | | 1 | | 1 | | | |
| $IV_{10}^{16}$ | | 1 | 1 | | $-1$ | | | | | |
| $IV_{11}^{16}$ | $-1$ | | 1 | | $-1$ | 2 | 1 | | | |
| $IV_{12}^{16}$ | $-1$ | | 1 | 1 | | | $-1$ | | | |
| $IV_1^{17}$ | | | | | | $-1$ | 1 | | | |
| $IV_2^{17}$ | | | | | | 1 | $-1$ | | | |
| $IV_3^{17}$ | | | | | | $-1$ | 1 | | | |
| $IV_4^{17}$ | | | | | | 1 | $-1$ | | | |

TABLE 36. The block $A_{22,5}$ of the block matrix $A$

| | $III_1^4$ | $III_2^4$ | $III_3^4$ | $III_1^5$ | $III_2^5$ | $III_1^6$ | $III_2^6$ | $III_1^7$ | $III_2^7$ | $III^8$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $IV_1^{21}$ | | | | | | | | | | |
| $IV_2^{21}$ | | | | | | | | | | |
| $IV_3^{21}$ | | | | | | | | | | |
| $IV_4^{21}$ | | | | | | | | | | |
| $IV_5^{21}$ | | | | | | $-1$ | 1 | 1 | $-3$ | 2 |
| $IV^{22}$ | | | | | | | | | | |

TABLE 37. The block $A_{24,5}$ of the block matrix $A$