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NASH EQUILIBRIUM STRATEGIES IN KALAI AND STANFORD MODEL WITH A COST FUNCTION AND CONSISTENT CONJECTURAL VARIATIONS

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1. Introduction

The game theoretic approach is useful for analizing interdependences among oligopolistic firms over more than one period. However, folk theorem shows that even if there is only one Nash equilibrium in the one-shot game, there are infinite perfect Nash equilibrium strategies when the game is infinitely repeated. A focus of recent studies of infinitely repeated game is fixed on examining what conditions are needed to narrow these perfect Nash equilibrium strategies. Kalai and Stanford 1985 present a family of Nash equilibrium strategies with constant conjectural variations in the duopoly supergame supported by a pair of quantities in stationary equilibrium.

The conjectural variation begining with Bowley 1924 is the conjecture of one firm about the change of other firm's behavior that the change of his behavior will induce. Therefore, it is to analizing competition between firms over more than one period that we can give the significant application of conjectual variation. In one-shot game such as Cournot game, it is not necessary for one firm to consider other firm's reaction. This implies that his conjectural variation is zero. In the game over more than one period it is not zero in general. Kalai and Stanford suppose that conjectural variation d_i constituting Nash equilibrium strategies in their duopoly game is between the perfectly competitive and the perfectly collusive, that is, in the interval $[-1, 1]$.

We will see that in the duopoly game under complete information, duopolistic firms' conjectural variations should be consistent, that is, one firm's conjecture should coincide with other firm's behavior. This implies that the conjectural variation and the slope of reaction function are equated. Bresnahan 1981 has proved that there exists an unique pair of consistent conjectures in the duopoly game on condition that both firms' cost functions are quadratic increasing.

The purpose of this paper is two-fold. First we will examine the existence of a family of Nash equilibrium strategies in the case where we introduce a quadratic increasing cost function into Kalai and Stanford model which has no cost function. Our second purpose is to consider the economic implication of conjectual variation under complete information.

This paper is organized as follows. In the following section, we define details of the Kalai and Stanford model with a cost function, the supergame consisting of countably many repetition of the linear demand duopoly game. The main results are stated in Section 3. In Section 4, it is shown that under complete information both firms' conjectural variations should be consistent, and they exist in the case where both firms' cost functions are quadratic increasing. Section 5 presents concluding remarks.

2. The Model

We consider the supergame consisting of countably many repetitions of the linear demand duopoly game under complete information.

We now introduce a cost function into Kalai and Stanford model.[1] We assume increasing cost for the two firms producing homogeneous goods. Let C_i $(i=1, 2)$ be firm i's cost function,

$$C_1 = C_1(x) = \frac{1}{2}c_1x^2 \quad (c_1>0), \qquad C_2 = C_2(y) = \frac{1}{2}c_2y^2 \quad (c_2>0),$$

where x is an output level of firm 1, y is an output level of firm 2, and c_i $(i=1, 2)$ is constant. Supposing that p is a price of goods, our linear inverse demand function is given by, for $z=x+y$,

$$p = p(z) = a-b(x+y).$$

where a and b are both positive constants.

The object of each firm is to maximize the sum of discounted profits by choosing its own output level of the product at each stage. Thus, for example, the profit function for firm 1 at each stage can be defined as:

$$\pi_1 = x(a-b(x+y))-\frac{1}{2}c_1x^2$$

$$= -\left(b+\frac{1}{2}c_1\right)x^2+ax-bxy.$$

So, aggregate profits, π_{1T}, π_{2T}, of firm 1, 2 to period T are given by

$$\pi_{1T} = \sum_{t=1}^{T}\alpha^{t-1}\left(x_t(a-b(x_t+y_t))-\frac{1}{2}c_1x_t^2\right)$$

$$\pi_{2T} = \sum_{t=1}^{T}\alpha^{t-1}\left(y_t(a-b(x_t+y_t))-\frac{1}{2}c_2y_t^2\right),$$

respectively, where $\alpha\in[0, 1)$ is a discount parameter. To be convenient, we take the strategy set for the game to be the same for both firms. Taking restraint of productive equipment into account, we restrict outputs to non-negative quantities bounded above. Thus, at each stage, each firm will choose output quantities in $I=[0, la/b]$ for $l\geqq 1$.

1) Introduction of the cost function has a closed relation with the range and the consistency of conjecture, as seen in Section 4.

Letting

$$X_1 \in I, \qquad X_t : (I \times I)^{t-1} \to I \qquad \text{for } t \geqq 2$$

$$Y_1 \in I, \qquad Y_t : (I \times I)^{t-1} \to I \qquad \text{for } t \geqq 2,$$

in general, we can write strategies of firms as follows

$$\{X_t\}_{t=1}^{\infty}, \qquad \{Y_t\}_{t=1}^{\infty},$$

respectively.

We are interested to consider the family of strategies that each firm's strategy is a choice of output at each stage, where each choice is dependent on past history only through rival's output in the prior period. And so we define the strategies of firms. Let $h_t = (x_1, y_1, x_2, y_2, \ldots, x_t, y_t) \in (I \times I)^t$ be a vector describing the history of production quantities for both firms from period 1 to period t. Then, for $\tilde{X}_{t+1} : I \to I$, $X_{t+1} : (I \times I)^t \to I$, $\tilde{Y}_{t+1} : I \to I$, $Y_{t+1} : (I \times I)^t \to I$, $t \geqq 1$, there exist functions $f : R \to I$ and $g_i : I \to R$ $(i=1, 2)$, such that

$$\begin{aligned} X_{t+1}(h_t) &= X_{t+1}(x_1, y_1, x_2, y_2, \ldots, x_t, y_t) \\ &= \tilde{X}_{t+1}(y_t) = f(g_1(y_t)). \\ Y_{t+1}(h_t) &= Y_{t+1}(x_1, y_1, x_2, y_2, \ldots, x_t, y_t) \\ &= \tilde{Y}_{t+1}(x_t) = f(g_2(x_t)). \end{aligned}$$

Next, we give three definitions.

Definition 1 *For* $\alpha \in [0, 1)$,

$$\bar{x} = \frac{a(b + c_1 + \alpha b d_2)}{(2b + c_1 + \alpha b d_2)\,(2b + c_2 + \alpha b d_1) - b^2}, \quad \bar{y} = \frac{a(b + c_2 + \alpha b d_1)}{(2b + c_1 + \alpha b d_2)\,(2b + c_2 + \alpha b d_1) - b^2}.$$

where $d_i \in (-1, 0)$, $i = 1, 2$[2].

Definition 2 *For* $g_i : I \to R$ $(i=1, 2)$,

$$g_1(y_t) = \bar{x} + d_1(y_t - \bar{y}), \qquad g_2(x_t) = \bar{y} + d_2(x_t - \bar{x}).$$

Definition 3 (truncated function) *For* $f : R \to I$,

$$\begin{aligned} f(x) &= 0 \quad \textit{if} \quad x < 0 \\ &= 0 \quad \textit{if} \quad x \in I \\ &= la/b \quad \textit{otherwise}. \end{aligned}$$

We provide the following Lemma which will characterize truncated function and will serve to prove the theorem in the next section.

Lemma *For* $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_t, \ldots) \in I^\infty$, *there exists* $\bar{\sigma} = (\bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_t, \ldots) \in I^\infty$ *such that*

$$\bar{\sigma}_1(a - b(\bar{\sigma}_1 + \bar{y})) - \frac{1}{2} c_1 \bar{\sigma}_1^2 + \sum_{t=2}^{\infty} \alpha^{t-1} \Big(\bar{\sigma}_t (a - b(\bar{\sigma}_t + \bar{y} + d_2(\bar{\sigma}_{t-1} - \bar{x})) - \frac{1}{2} c_1 \bar{\sigma}_t^2 \Big)$$

2) We will present a reason why $d_i \in (-1, 0)$, $i = 1, 2$ in Section 4.

$$\geqq \sigma_1(a-b(\sigma_1+\bar{y}))-\frac{1}{2}c_1\sigma_1^2+\sum_{t=2}^{\infty}\alpha^{t-1}\left(\sigma_t(a-b(\sigma_t+f(\bar{y}+d_2(\sigma_{t-1}-\bar{x})))-\frac{1}{2}c_1\sigma_t^2\right). \quad (1)$$

Proof. The proof proceeds by an induction argument. Since $\alpha\in[0, 1)$, it is obvious that $\lim_{t\to\infty}\alpha^t=0$. Thus, both sides of (1) converge to some values. Therefore, the proof will be established, if we can show that for any $\sigma=(\sigma_1, \sigma_2, ..., \sigma_t, ...)\in I^\infty$, any $T\geqq 2$, there exists $\bar{\sigma}=(\bar{\sigma}_1, \bar{\sigma}_2, ...)\in I^\infty$

such that

$$\bar{\sigma}_1(a-b(\bar{\sigma}_1+\bar{y}))-\frac{1}{2}c_1\bar{\sigma}_1^2+\sum_{t=2}^{T-1}\alpha^{t-1}\left(\bar{\sigma}_t(a-b(\bar{\sigma}_t+\bar{y}+d_2(\bar{\sigma}_{t-1}-\bar{x})))-\frac{1}{2}c_1\bar{\sigma}_t^2\right)$$

$$+\alpha^{T-1}\left(\bar{\sigma}_T(a-b(\bar{\sigma}_T+\bar{y}+d_2(\bar{\sigma}_{T-1}-\bar{x})))-\frac{1}{2}c_1\bar{\sigma}_T^2\right)$$

$$\geqq \sigma_1(a-b(\sigma_1+\bar{y}))-\frac{1}{2}c_1\sigma_1^2+\sum_{t=2}^{T}\alpha^{t-1}\left(\sigma_t(a-b(\sigma_t+f(\bar{y}+d_2(\sigma_{t-1}-\bar{x})))-\frac{1}{2}c_1\sigma_t^2\right).$$

Let

$$\bar{\sigma}_{t-1}=\sigma_{t-1} \qquad \textit{if} \quad \bar{y}+d_2(\sigma_{t-1}-\bar{x})\geqq 0$$

$$\bar{\sigma}_{t-1}=\bar{x}-\bar{y}/d_2 \qquad \textit{if} \quad \bar{y}+d_2(\sigma_{t-1}-\bar{x})<0.$$

Let t^* be the first t for which $\bar{y}+d_2(\sigma_{t-1}-\bar{x})<0$. We first consider the case $t^*=2$. Since $d_2\in(-1, 0)$, we have

$$\sigma_1<\bar{\sigma}=\bar{x}=\bar{y}/d_2.$$

Also define

$$W(x)=x(a-b(x+\bar{y}))-\frac{1}{2}c_1x^2.$$

Then, there exists an unique point, x^*, such that maximizes W(x), because W(x) is a quadratic function which describes a unimodal curve. We get

$$x^*=\frac{a-b\bar{y}}{2b+c_1}.$$

Thus by elementary calculation, we have

$$\bar{\sigma}_1=x-\frac{\bar{y}}{d_2}=\frac{a((d_2-1)b+d_2c_2+\alpha bd_1d_2-c_1-\alpha bd_2)}{d_2A},$$

Where $A=(2b+c_1+\alpha bd_2)(2b+c_2+\alpha bd_1)-b^2$. Moreover, we obtain

$$\bar{\sigma}_1-x^*=\frac{-a}{(2b+c_1)d_2A}(\alpha bd_2c_1+3bc_1+\alpha b^2d_2^2+\alpha^2b^2d_1d_2+2b^2+2\alpha b^2d_2+c_1^2+\alpha bc_2d_2^2).$$

Since $A>0$, $a>0$, $b>0$, $c_1>0$, $c_2>0$, d_1, $d_2\in(-1, 0)$, we have that

$$\bar{\sigma}_1-x^*>0.$$

Thereby $x^*<\bar{\sigma}_1<\sigma_1$. This implies that $W(x^*)>W(\bar{\sigma}_1)>W(\sigma_1)$. Hence we have

$$\bar{\sigma}_1(a-b(\bar{\sigma}_1+\bar{y}))-\frac{1}{2}c_1\bar{\sigma}_1^2+\alpha c_2\sigma_2\left(a-b(\sigma_2+\bar{y}+d_2(\bar{\sigma}_1-\bar{x}))-\frac{1}{2}c_1\sigma_2^2\right)$$

$$\geqq \sigma_1(a-b(\sigma_1+\bar{y}))-\frac{1}{2}c_1\sigma_1^2+\alpha c_2\sigma_2\left(a-b(\sigma_2+f(\bar{y}+d_2(\sigma_1-\bar{x})))-\frac{1}{2}c_1\sigma_2^2\right),$$

or

$$\bar{\sigma}_1(a-b(\bar{\sigma}_1+\bar{y}))-\frac{1}{2}c_1\bar{\sigma}_1^2\geqq \sigma_1(a-b(\sigma_1+\bar{y}))-\frac{1}{2}c_1\sigma_1^2.$$

The case $t^*>2$ is proved similarly. ∎

3. Nash Equilibrium strategies

In this section we define a family of strategies and show that it is a Nash equilibrium in the duopoly supergame.

Definiton 4 For $h_t=(x_1,\ y_1,\ x_2,\ y_2,\ \dots,\ x_t,\ y_t)\in(I\times I)^t$.

$$X_1=\bar{x}\in I, \qquad Y_1=\bar{y}\in I$$

$$X_{t+1}(h_t)=f(g_1(y_t))=f(\bar{x}+d_1(y_t-\bar{y})) \qquad Y_{t+1}(h_t)=f(g_2(x_t))=f(\bar{y}+d_2(x_t-\bar{x})). \tag{2}$$

Theorem *Under definitions 1, 2, 3, 4, any pair of strategies defined as in* (2) *is a Nash equilibrium in the duopoly supergame. Moreover a pair $(\bar{x},\ \bar{y})$ is unique in the sense that no pair $(\tilde{x},\ \tilde{y})$ differents from $(\bar{x},\ \bar{y})$ can constitute Nash equilibrium in* (2).

Proof. In view of Lemma, we can transform the strategy of firm 2 into a optimization problem of firm 1. It will be shown as follows,

$$\max_{\sigma\in I^\infty} x_1(a-b(x_1+\bar{y}))-\frac{1}{2}c_1x_1^2+\sum_{t=2}^{\infty}\alpha^{t-1}\left(x_t(a-b(x_t+\bar{y}+d_2(x_{t-1}-\bar{x})))-\frac{1}{2}c_1x_t^2\right),$$

where $\sigma=(x_1,\ x_2,\ \dots,\ x_t,\ \dots,)\in I^\infty$. Therefore, if we have that for any $\sigma=(x_1,\ x_2,\ \dots,\ x_t,\ \dots,)\in I^\infty$,

$$\bar{x}(a-b(\bar{x}+\bar{y}))-\frac{1}{2}c_1\bar{x}^2+\sum_{t=2}^{\infty}\alpha^{t-1}\left(\bar{x}(a-b(\bar{x}+\bar{y}))-\frac{1}{2}c_1\bar{x}^2\right)$$

$$\geqq x_1(a-b(x_1+\bar{y}))-\frac{1}{2}c_1x_1^2+\sum_{t=2}^{\infty}\alpha^{t-1}\left(x_t(a-b(x_t+\bar{y}+d_2(x_{t-1}-\bar{x})))-\frac{1}{2}c_1x_t^2\right),$$

then we will establish that the vector $\bar{\sigma}=(\bar{x},\ \bar{x},\ \dots,\ \bar{x},\ \dots,)$ is a best response of firm 1 to the strategies of firm 2 and it is a solution of optimization problem above. The firm 2's optimization problem is solved similarly.

For any vector of output $\sigma==\in I^\infty$, let

$$H_1^N(\sigma)=x_1(a-b(x_1+\bar{y}))-\frac{1}{2}c_1x_1^2+\sum_{t=2}^{N}\alpha^{t-1}\left(x_t(a-b(x_t+\bar{y}+d_2(x_{t-1}-\bar{x})))-\frac{1}{2}c_1x_t^2\right).$$

If we can show that for all $\sigma==\in I^\infty$, $H_1^\infty(\bar{\sigma})\geqq H_1^\infty(\sigma)$, we complete the proof. Here let

$$\bar{H}_1^N(\sigma)=H_1^N(\sigma)-\alpha^N bd_2\bar{x}x_N$$

$H_1^\infty(\sigma) = \lim_{N\to\infty} H_1^N(\sigma)$.

We assume that there exists a $\sigma = \in I^\infty$ such that $H_1^\infty(\sigma) > H_1^\infty(\bar\sigma)$. Now we demonstrate that this assumptin results a contradiction. To prove this, suppose that

$H_1^\infty(\sigma) > H_1^\infty(\bar\sigma)$, for $\exists\, \sigma \in I^\infty$.

This implies that ther exists $\varepsilon > 0$ such that

$H_1^\infty(\bar\sigma) + \varepsilon = H_1^\infty(\sigma)$.

It is immediate that there is $m \in \mathrm{R}$ such that for all $(x_{t-1}, x_t) \in I^2$,

$$\left| x_t(a - b(x_t + \bar y + d_2(x_{t-1} - \bar x))) - \frac{1}{2} c_1 x_t^2 \right| \leqq m.$$

Take N_1 such that if $k \geqq N_1$, $\sum_{t=k}^{\infty} \alpha^{t-1} \leqq \varepsilon/8$, and N_2 such that if $k \geqq N_2$, $\alpha^k b c_2 \bar x x_k \leqq \varepsilon/8$ for all $x_k \in I$.

Letting $N = \max\{N_1, N_2\}$, we have

$$|H_1^k(\sigma) - H_1^\infty(\sigma)| = \left| \sum_{t=k}^{\infty} \alpha^t (x_{t+1}(a - b(x_{t+1} + \bar y + d_2(x_t - \bar x))) \right|$$

$$\leqq m \sum_{t=k}^{\infty} \alpha^t \leqq \varepsilon/8,$$

$$|H_1^k(\sigma) - \bar H_1^k(\sigma)| = \alpha^k b d_2 \bar x x_k \leqq \varepsilon/8,$$

$$|H_1^\infty(\bar\sigma) - H_1^k(\bar\sigma)| = \left| \sum_{t=k}^{\infty} \alpha^t \left(\bar x(a - b(\bar x + \bar y)) - \frac{1}{2} c_1 \bar x^2 \right) \right|$$

$$\leqq m \sum_{t=k}^{\infty} \alpha^t \leqq \varepsilon/8,$$

$$|H_1^k(\bar\sigma) - \bar H_1^k(\bar\sigma)| = \alpha^k b d_2 \bar x x_k \leqq \varepsilon/8.$$

Hence we obtain

$$\varepsilon/4 \geqq |H_1^\infty(\bar\sigma) - H_1^k(\bar\sigma)| + |H_1^k(\bar\sigma) - \bar H_1^k(\bar\sigma)|$$
$$\geqq |H_1^\infty(\bar\sigma) - \bar H_1^k(\bar\sigma)|$$
$$\varepsilon/4 \geqq |H_1^\infty(\sigma) - H_1^k(\sigma)| + |H_1^k(\sigma) - \bar H_1^k(\sigma)|$$
$$\geqq |H_1^\infty(\sigma) - \bar H_1^k(\sigma)|.$$

These prove following inequality.

$$\varepsilon/2 \geqq |H_1^\infty(\bar\sigma) - \bar H_1^k(\bar\sigma)| + |H_1^k(\sigma) - \bar H_1^k(\sigma)|$$
$$= |H_1^\infty(\sigma) - \varepsilon - \bar H_1^k(\bar\sigma)| + |H_1^\infty(\bar\sigma) - H_1^k(\sigma)|$$
$$\geqq |\bar H_1^k(\bar\sigma) - \varepsilon - \bar H_1^k(\sigma)|.$$

If $\bar H_1^k(\bar\sigma) \geqq \bar H_1^k(\sigma)$ holds for all such k, it follows that above assumption yields a contradiction.

We consider first order conditions with $\bar H_1^k$ are given by

$$\frac{\partial \bar H_1^k(\sigma)}{\partial x_1} = -(2b + c_1)x_1 + a - b\bar y - \alpha b d_2 x_2 = 0,$$

$$\frac{\partial \bar H_1^k(\sigma)}{\partial x_t} = \alpha^{t-1}\{a - (2b + c_1)x_t - b\bar y - bd_2(x_{t-1} - \bar x) + \alpha d b_2 x_{t+1}\} = 0,$$

$$t=2, \ldots, k-1 \tag{3}$$

$$\frac{\partial \bar{H}_1^k(\sigma)}{\partial x_k}=\alpha^{k-1}\{a-(2b+c_1)x_k-by-bd_2(x_{k-1}-\bar{x})+\alpha bd_2\bar{x}\} = 0.$$

Let $\bar{y}=a(b+c_2+\alpha bd_1)/[(2b+c_1+\alpha bd_2)(2b+c_2+\alpha bd_1)-b^2]$. We can find that $x_1=x_2=\ldots=x_k=\bar{x}=a(b+c_1+\alpha bd_2)/[(2b+c_1+\alpha bd_2)(2b+c_2+\alpha bd_1)-b^2]$ is a solution to the simultanuous equation (3).

We will present the check of second order conditions in appendix.

It remains to prove the uniquness of $(\bar{x}, \bar{y})$. Let

$$G(x, y, \bar{x}, \bar{y}) = x(a-b(x+\bar{y}))-\frac{1}{2}c_1x^2+\sum_{t=2}^{N}\alpha^{t-1}\left(x(a-b(x+\bar{y}+d_2(x-\bar{x})))-\frac{1}{2}c_1x^2\right),$$

$$J(x, y, \bar{x}, \bar{y}) = y(a-b(\bar{x}+y))-\frac{1}{2}c_2y^2+\sum_{t=2}^{N}\alpha^{t-1}\left(y(a-b(\bar{x}+y+d_1(y-\bar{y})))-\frac{1}{2}c_2y^2\right).$$

Take a vector $(\tilde{x}, \tilde{y})$ different from $(\bar{x}, \bar{y})$. If (2) with $(\tilde{x}, \tilde{y})$ substituted for $(\bar{x}, \bar{y})$ is a Nash equilibrium, as seen in the proof of theorem, it holds that

$$G_x(\tilde{x}) = \left.\frac{\partial G(x, y, \tilde{x}, \tilde{y})}{\partial x}\right|_{x=\tilde{x}} = 0, \quad J_y(\tilde{y}) = \left.\frac{\partial J(x, y, \tilde{x}, \tilde{y})}{\partial y}\right|_{y=\tilde{y}} = 0.$$

Since G is linear with $\bar{x}$, $\bar{y}$, and quadratic with x, G_x is linear with x, $\bar{x}$, $\bar{y}$. This argument is similar to J and J_y. These imply that $(\tilde{x}, \tilde{y})$ sutisfying $G_x(\tilde{x})=0, J_y(\tilde{y})=0$ will coinside with $(\bar{x}, \bar{y})$ sutisfying $G_x(\bar{x})=0$, $J_y(\bar{y})=0$. This proves the uniqueness of $(\bar{x}, \bar{y})$. ∎

4. The economic implication of d_i under complete information

In this section we will present a reason why $d_i \in(-1, 0)$, which so far we have postulated. Kalai and Stanford have cosidered that conjectual variation d_i is between the perfectly competitive and the perfectly collusive, that is, $d_i\in[-1, 1]$.

In contrast with the assumption of Kalai and Stanford, we find that the range of d_i can be narrowed to $(-1, 0)$ on the condition of complete information. We see that under completet information the consistent conjecture shown by Bresnahan 1981 will be established in equilibrium.[3)]

Let dy/dx be a conjectural variation, namely, an anticipation of firm 1 to firm 2' output change yielded by firm 1' output alteration. Reaction functions of both firms are given with x, y by

$$\frac{d\pi_1}{dx} = h_1(x, y) = \frac{d}{dx}(xp(z)-C_1(x))$$

3) There are consistent conjectures in the case of three firms on a specific condition, but not in general. See M. Koreeda 1991.

$$= p+xp'\left(1+\frac{dy}{dx}\right)-C_1' = 0, \tag{4}$$

$$\frac{d\pi_2}{dx} = h_2(x,\ y) = \frac{d}{dy}(yp(z)-C_2(y))$$

$$= p+yp'\left(1+\frac{dx}{dy}\right)-C_2' = 0. \tag{5}$$

Solving (4), we have a function with y

$$x = \rho_1(y), \tag{6}$$

and solving (5), we have a function with x

$$y = \rho_2(x). \tag{7}$$

Under complete in formatiön, both firms know (4) and (5), and know that each other knows (4) and (5). Therefore, for example, if firm 1 is rational, he considers that if firm 2 is rational, he should make dx/dy correspond to the the slope of (6), conjectureing that firm 1 should make dy/dx correspond to the slope of (7). Hence firm 1 thinks that firm 2 will let $dx/dy=\partial\rho_1/\partial y$, $dy/dx=\partial\rho_2/\partial x$. This argument is completely symmetric when we consider firm 2' conjecture. Thus if both firms are rational, they know that

$$\frac{dy}{dx} = \frac{\partial\rho_2(x)}{\partial x} = -\frac{\partial h_1/\partial y}{\partial h_1/\partial x}, \qquad \frac{dx}{dy} = \frac{\partial\rho_1(y)}{\partial y} = -\frac{\partial h_2/\partial x}{\partial h_2/\partial y}. \tag{8}$$

Define

$$\frac{dy}{dx} = r_{12}, \qquad \frac{dx}{dy} = r_{21}. \tag{9}$$

Then, we obtain

$$\rho_{21} = \frac{\partial\rho_2(x)}{\partial x} = -\frac{b}{2b+br_{12}+C_1''}$$

$$\rho_{12} = \frac{\partial\rho_1(y)}{\partial y} = -\frac{b}{2b+br_{21}+C_2''}.$$

From the consistency of conjectures it follows that

$$\rho_{12} = r_{21}, \qquad \rho_{21} = r_{12}.$$

By elementary calculation, we have

$$r_{21} = \frac{-\beta\pm[\beta(\beta-4b^2)]^{1/2}}{2b(2b+c_2)} \tag{10}$$

$$r_{12} = \frac{-\beta\pm[\beta(\beta-4b^2)]^{1/2}}{2b(2b+c_1)} \tag{11}$$

where $\beta=(2b+c_1)(2b+c_2)$.[4)]

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4) If $p-C'>0$, from (4) we get
$-c_1(2b+c_2)\pm[\beta(\beta-4b^2)]^{1/2}>0$.
Moreover we have that $[\beta(\beta-4b^2)]^{1/2}-c_1(2b+c_2)>0$. Therefore if $p-C'>0$, we can abandon negative radicals of (10) and (11).

Now by (7) and (8), we obtain that

$$C_1'' = -b\left(2+r_{12}+\frac{1}{r_{21}}\right)$$

$$C_2'' = -b\left(2+r_{21}+\frac{1}{r_{12}}\right).$$

Since $C_i''=c_i>0(i=1,\ 2)$, $b>0$, it is obvious that

$$2+r_{12}+\frac{1}{r_{21}}<0, \qquad 2+r_{21}+\frac{1}{r_{12}}<0.$$

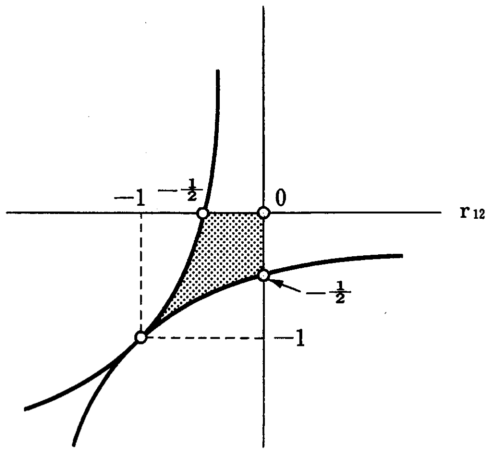

Figure. 1

If $p-C_i''>0(i=1,\ 2)$, then $r_{ij}>-1(i\neq j)$. Therefore there exists r_{ij} in the shadowed area shown in the Figure 1 above.

Now we summarize above results as two propositions.

Proposition 1. Under complete information both firms' conjectures are consistent in the duopoly game over more than one period.

Proposition 2. Suppose that the inverse demand function is linear and cost functions of both firms are quadratic increasing. If the price always exceeds marginal cost, then there exists an unique pair $(r_{12},\ r_{21})$ *of consistent conjectures, and* $0>r_{ij}>-1\quad(i\neq j)$.

5. Concluding Remarks

We have analyzed Nash Equilibrium strategies with a conjectural variation in the duopoly supergame introduced a cost function into Kalai and Stanford model. In one-shot game, where it is not necessary for firms to consider each other's reaction, Cournot equilibrium is rational in the sense that it is an equilibrium with self-fulfilling expectation, in other words, it is an equilibrium such that it gives a optimal response for firms. It is obvious that both conjectual variations of firms are zero in this case. In repeated duopoly game, one competitor determines his outputs considering that it gives rise to other firm's output change. In this case, in general, the conjectural variations of both firms are not zero, unlike Cournot.

In section 4, we have seen that under complete information duopolistic firms' conjectural variations should be consistent, that is, one firm's conjecture should be coincided with other firm' s behavior. Introduction of a quadratic increasing cost function gives a possibility of the existense of consistent conjectures $r_{ij}(i \neq j)$. In contrast with assumptions of Kalai and Stanford, $d_i \in [-1, 1]$ $i=1, 2$, we have shown that there exists an unique consistent conjecture pair (r_{12}, r_{21}), and that $r_{ij} \in (-1, 0)$, $i \neq j$.

Appendix

Here we prove that $(\bar{x}, \bar{y})$ satisfies second order conditions for the maximum problem.

Let $[F_t]$ be a Hessian matrix of $\bar{H}_t$, that is,

$$[F_t] = \begin{bmatrix} -(b+c) & -abd & 0 & 0 & & \\ -abd & -2a(b+c) & -a^2bd & 0 & & \\ 0 & -a^2bd & -2a^2(b+c) & -a^3bd & & \\ & & \cdots\cdots\cdots\cdots & & & \\ & & -a^{t-2}bd & -2a^{t-2}(b+c) & -a^{t-1}bd \\ & & & -a^{t-1}bd & -2a^{t-1}(b+c) \end{bmatrix},$$

where $c=c_1/2$, $d=d_2$. Note that $[F_t]$ is a symmetric matrix. For $u=(u_1, u_2, \ldots, u_n) \in R^n$, define

$$G_t = {}^t u[F_t]u,$$

where ${}^t u$ is a transposed vector of u. If $G_t<0$ for any $u \in R^n (u \neq 0)$, then $[F_t]$ is a negative definite matrix. This is equivalent to

$|F_t|<0$ if t is odd,

$|F_t|>0$ if t is even.

This implies that with $\bar{H}_1^N$ as objective, second order conditions are satisfied. We can show that $[F_t]$ is a negative definite matrix, as follows.

We have

$$\begin{aligned} G_t &= {}^t u[F_t]u \\ &= -2(b+c)u_1^2 - \alpha\ bdu_1u_2 \\ &\quad -\alpha\ bdu_1u_2 - 2\alpha\ (b+c)u_2^2 - \alpha^2 bdu_2u_3 \\ &\quad -\alpha^2 bdu_2u_3 - 2\alpha^2(b+c)u_3^2 - \alpha^3 bdu_3u_4 \\ &\qquad \cdots\cdots\cdots\cdots\cdots \\ &\quad -\alpha^{t-2}bdu_{t-1}u_{t-2} - 2\alpha^{t-2}(b+c)u_{t-1}^2 - \alpha^{t-1}bdu_{t-1}u_t \\ &\quad -\alpha^{t-1}bdu_{t-1}u_t - 2\alpha^{t-1}(b+c)u_t^2 \\ &= -\left(\sum_{n=1}^{t} 2\alpha^{n-1}(b+c)u_n^2 + \sum_{m=1}^{t-1} 2\alpha^m bdu_m u_{m+1}\right). \end{aligned}$$

Then we rewrite so that

$$G_t = -(b+c)u_1^2 - \left(\sum_{l=1}^{t-1} \alpha^{l-1}(b+c)u_l^2 + 2\alpha bdu_l u_{l+1} + \alpha(b+c)u_{l+1}^2\right) - \alpha^{t-1}(b+c)u_t^2.$$

The proof proceeds by cases.

Case 1. $u_l u_{l+1} > 0$ $(1 \leqq l \leqq t-1)$. If $\alpha = 0$, It is obvious that $G_t < 0$. Let $\alpha \in (1, 0)$. Since $c > 0$, $d \in (-1, 0)$, we have

$$\begin{aligned} &(b+c)u_l^2 + 2\alpha bdu_l u_{l+1} + \alpha(b+c)u_{l+1}^2 \\ &\geqq bu_l^2 + 2\alpha bdu_l u_{l+1} + \alpha^2 bu_{l+1}^2 \\ &> bu_l^2 - 2\alpha bu_l u_{l+1} + \alpha^2 bu_{l+1}^2 = b(u_l - \alpha u_{l+1})^2 \geqq 0. \end{aligned}$$

Therefore we have that

$$G_t \leqq -(b+c)u_1^2 - \sum_{l=1}^{t-1} \alpha^{l-1} b(u_l - \alpha u_{l+1})^2 - \alpha^{t-1}(b+c)u_t^2 < 0.$$

Case 2. $u_l u_{l+1} < 0$ $(1 \leqq l \leqq t-1)$. Since $d \in (-1, 0)$, it is obvious that $G_t < 0$. ∎

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