UNLINKING SINGULAR LOCI FROM REGULAR FIBERS AND ITS APPLICATION TO SUBMERSIONS

Saeki, Osamu
Institute of Mathematics for Industry, Kyushu University

<http://hdl.handle.net/2324/4485669>

出版情報 : Journal of Singularities. 22, pp.92-103, 2020. Worldwide Center of Mathematics, LLC
バージョン :
権利関係 :

UNLINKING SINGULAR LOCI FROM REGULAR FIBERS AND ITS APPLICATION TO SUBMERSIONS

OSAMU SAEKI

Dedicated to Professor Maria Aparecida Soares Ruas on the occasion of her 70th birthday

ABSTRACT. Given a null-cobordant oriented framed link L in a closed oriented 3-manifold M , we study the condition for the existence of a generic smooth map of M to the plane that has L as an oriented framed regular fiber such that the singular point set is unlinked with L . As an application, we give a singularity theoretical proof to the theorem, originally proved by Hector, Peralta-Salas and Miyoshi, about the realization of a link in an open oriented 3-manifold as a regular fiber of a submersion to the plane.

1. INTRODUCTION

Let M be a smooth closed oriented 3-dimensional manifold and $f : M \rightarrow \mathbf{R}^2$ a smooth map. If $y \in f(M) \subset \mathbf{R}^2$ is a regular value, then $f^{-1}(y)$ is an oriented link in M and is naturally framed. Furthermore, if f is generic enough, then the singular point set $S(f)$ of f is an unoriented link in $M \setminus f^{-1}(y)$. In our previous paper [18], for an oriented framed link L in M , we characterized those unoriented links in $M \setminus L$ which arise as the singular point set of a generic map that has L as an oriented framed regular fiber. Such a characterization was given in terms of a relative Stiefel–Whitney class, or an obstruction to extending the trivialization of $TM|_L$ induced by the framing over the whole manifold M .

In this paper, we first study the obstruction class more in detail, and give a more practical characterization in terms of $\mathbf{Z}_2$ linking numbers. We also clarify the components of L which have non-trivial $\mathbf{Z}_2$ linking numbers with the singular point set. Then, as an application of such studies, we consider submersions of open oriented 3-manifolds to $\mathbf{R}^2$ that realize given oriented framed links as regular fibers. The idea is to consider a generic map f whose singular point set $S(f)$ is unlinked with a given oriented framed regular fiber and to delete a neighborhood of the singular point set $S(f)$ for obtaining a submersion. In this way, we get a singularity theoretical proof to the characterization theorem, originally due to Hector and Peralta-Salas [8] and Miyoshi [13], of those oriented (framed) links in $\mathbf{R}^3$ that arise as regular fibers of submersions. Recall that their proofs used the h-principle for submersions due to Phillips [15]. Instead, in this paper, we arrange the singular point set by using Levine’s cusp elimination techniques [11] (see also [17, 18]) in a controlled way and push it to infinity, so that we get a submersion.

The paper is organized as follows. In §2, we recall several definitions and terminologies together with our main theorem in [18], which describes the characterization of singular point sets as unoriented links in terms of a certain obstruction class. In §3, we study the obstruction class more in detail, especially for closed oriented

Date: January 14, 2019.

2000 Mathematics Subject Classification. Primary 57R45; Secondary 57R30, 58K30, 57M25, 57R20.

Key words and phrases. Submersion, link, 3-manifold, excellent map, singular point set, regular fiber, relative Stiefel–Whitney class, framing.

3-manifolds M with $H_*(M; \mathbf{Z}) \cong H_*(S^3; \mathbf{Z})$. In such a case, we can identify the obstruction class in terms of $\mathbf{Z}_2$ linking numbers. Then, we can describe the condition for the obstruction class to vanish in terms of $\mathbf{Z}_2$ linking numbers. Finally in §4, we apply these results to submersions of open oriented 3-manifolds to $\mathbf{R}^2$. We will see that our singularity theoretical proof works well for punctured 3-manifolds, i.e. open 3-manifolds of the form $M^\circ = M \setminus D^3$ obtained from a closed 3-manifold M by removing a small 3-disk D^3 in M . For a general open oriented 3-manifold, we need to use an “absolute version” of the h-principle due to Phillips. Recall that the original proof due to Hector and Peralta-Salas [8] or Miyoshi [13] used the “relative version”, stronger than the “absolute version”, of the h-principle [6].

Throughout the paper, manifolds and maps are differentiable of class C^∞ unless otherwise indicated. All (co)homology groups are with $\mathbf{Z}_2$ -coefficients unless otherwise indicated. The symbol “ $\cong$ ” means an appropriate isomorphism between algebraic objects or a diffeomorphism between smooth manifolds.

2. PRELIMINARIES

Let M (resp. N) be a closed 3-dimensional manifold (resp. a surface) and consider a map $f : M \rightarrow N$. We denote by $S(f)$ the set of singular points of f . A point in $S(f)$ is a *fold singularity* (or a *cuspl singularity*) of f if the map germ of f at that point is modeled on the map germ $(x, y, z) \mapsto (x, y^2 \pm z^2)$ (resp. $(x, y, z) \mapsto (x, y^3 + xy - z^2)$) at the origin. We say that a fold singularity is *definite* (resp. *indefinite*) if it is modeled on the map germ $(x, y, z) \mapsto (x, y^2 + z^2)$ (resp. $(x, y, z) \mapsto (x, y^2 - z^2)$). We say that f is *excellent* if $S(f)$ consists only of fold and cuspl singularities. It is known that the set of excellent maps is always open and dense in the mapping space $C^\infty(M, N)$ endowed with the Whitney C^∞ topology (for example, see [5, 20]). If f is an excellent map, then $S(f)$ is an (unoriented) link in M , i.e. a finite disjoint union of smoothly embedded circles.

Let $f : M \rightarrow N$ be a map. For a regular value $y \in f(M) \subset N$, we call $L = f^{-1}(y)$ a *regular fiber*, which is a link in $M \setminus S(f)$. Note that L is naturally *framed*: its framing is given as the pull-back of the trivial normal framing of the point y in N . Furthermore, when M and N are oriented, L is naturally oriented.

In the following, we fix an orientation for $\mathbf{R}^2$ once and for all. For excellent maps of closed oriented 3-manifolds into $\mathbf{R}^2$, we have the following (for details, see [16, Proposition 5.1] and [18]).

LEMMA 2.1. *Let L be an oriented framed link in a closed oriented 3-manifold M . Then, it is realized as an oriented framed regular fiber of an excellent map $f : M \rightarrow \mathbf{R}^2$ if and only if it is framed null-cobordant: i.e. there exists a compact oriented normally framed surface V embedded in M whose framed boundary coincides with L .*

REMARK 2.2. Let L be an oriented link in a closed oriented 3-manifold M . Then, we can easily show that it bounds a compact oriented surface in M if and only if L represents zero in $H_1(M; \mathbf{Z})$. This can be proved by considering a certain map $M \setminus L \rightarrow S^1$. In particular, if $H_1(M; \mathbf{Z}) = 0$, then every oriented link bounds a compact oriented surface embedded in M .

REMARK 2.3. In [3], it is shown that every link in S^3 is a regular fiber of the restriction to S^3 of a certain polynomial map $\mathbf{C}^2 \rightarrow \mathbf{C}$.

Now, let L be an oriented framed link in a closed oriented 3-manifold. If L is realized as a framed regular fiber of an excellent map $f : M \rightarrow \mathbf{R}^2$, then $S(f)$ is a link in $M \setminus L$. Thus, it is natural to ask the following.

QUESTION 2.4. Which links in $M \setminus L$ appear as the singular point set $S(f)$ of an excellent map $f : M \rightarrow \mathbf{R}^2$ such that $f^{-1}(y)$ coincides with L as oriented framed links for some regular value $y \in \mathbf{R}^2$?

In order to answer to the above question, let us prepare some notations and terminologies. For an (unoriented) link J in $M \setminus L$, we denote by $[J]_2 \in H_1(M \setminus L)$ the $\mathbf{Z}_2$ -homology class represented by J . Let $N(L)$ be a small tubular neighborhood of L in M disjoint from J . Since L is a framed link, we have a natural trivialization of $TM|_{N(L)}$. The obstruction to extending it over M is the relative Stiefel–Whitney class (see [9]), denoted by $w_2(M, L)$, which is an element of the $\mathbf{Z}_2$ -cohomology group $H^2(M, N(L)) \cong H^2(M, L)$. Note that by excision and Poincaré–Lefschetz duality, we have

$$H^2(M, N(L)) \cong H^2(M \setminus \text{Int } N(L), \partial N(L)) \cong H_1(M \setminus \text{Int } N(L)) \cong H_1(M \setminus L).$$

The following characterization, which answers to Question 2.4, has been proved in [18]. Recall that the proof was singularity theoretical in the sense that we used a result of Thom [19] about the homology class represented by the singular locus, and a cusp elimination result by Levine [11] for arranging the singular locus of an excellent map.

THEOREM 2.5. *Let L be an oriented null-cobordant framed link in a closed oriented 3-manifold M , and J an unoriented link in $M \setminus L$. Then, there exist an excellent map $f : M \rightarrow \mathbf{R}^2$ and a regular value $y \in \mathbf{R}^2$ such that $f^{-1}(y)$ coincides with L as oriented framed links and that $S(f) = J$ if and only if $[J]_2 \in H_1(M \setminus L)$ is Poincaré dual to $w_2(M, L) \in H^2(M, L)$.*

3. CASE OF INTEGRAL HOMOLOGY 3-SPHERES

In this section, we mainly consider closed oriented 3-manifolds M with

$$H_*(M; \mathbf{Z}) \cong H_*(S^3; \mathbf{Z})$$

and replace the condition described by the obstruction class $w_2(M, L)$ in Theorem 2.5 with that of $\mathbf{Z}_2$ linking numbers.

First, let M be an arbitrary closed oriented 3-manifold and L an oriented framed link in M . For the inclusion $j : (M, \emptyset) \rightarrow (M, L)$, the induced homomorphism $j^* : H^2(M, L) \rightarrow H^2(M)$ sends $w_2(M, L)$ to the second Stiefel–Whitney class $w_2(M)$ of M , which vanishes. By the cohomology exact sequence

$$H^1(L) \xrightarrow{\delta} H^2(M, L) \xrightarrow{j^*} H^2(M),$$

we have that $w_2(M, L) = \delta(\alpha)$ for some $\alpha \in H^1(L)$, although such an α may not be unique. In fact, such a class can be explicitly given as follows.

Set $L = L_1 \cup L_2 \cup \cdots \cup L_\mu$, where L_s are the components of L , $s = 1, 2, \dots, \mu$. It is known that the tangent bundle TM of a closed oriented 3-manifold M is always trivial. Once a trivialization τ of TM is fixed, we can compare it with the specific trivialization of $TM|_{L_s}$ associated with the framing given for each component L_s of the framed link L . (We consider the trivialization given by the ordered vector fields v_1, v_2 and v_3 , where v_1 is tangent to L_s consistent with the orientation, and v_2, v_3 are consistent with the framing.) This defines a well-defined element a_s in $\pi_1(SO(3)) \cong \mathbf{Z}_2$ for each s . Then, we have proved the following in [18].

LEMMA 3.1. *Let $\alpha \in H^1(L)$ be the unique cohomology class such that the Kronecker product $\langle \alpha, [L_s]_2 \rangle \in \mathbf{Z}_2$ coincides with a_s for each component L_s of L . Then, we have $\delta(\alpha) = w_2(M, L)$.*

Note that the trivialization τ of TM may not be unique. The set of homotopy classes of such trivializations is in one-to-one correspondence with the homotopy

set $[M, SO(3)]$. If we consider the set of homotopy classes of trivializations on the 2-skeleton of M , then each such trivialization up to homotopy defines a *spin structure* on M , and the set of spin structures is in one-to-one correspondence with $H^1(M)$ (see [12]).

By the cohomology exact sequence,

$$(3.1) \quad H^1(M) \xrightarrow{i^*} H^1(L) \xrightarrow{\delta} H^2(M, L) \xrightarrow{j^*} H^2(M),$$

we see that for an arbitrary element $\beta \in \text{Im } i^*$, we could choose $\alpha + \beta$ instead of α , where $i : L \rightarrow M$ is the inclusion map. The observation in the previous paragraph shows that this corresponds to choosing another trivialization which is “twisted along β ”.

The following proposition has also been proved in [18].

LEMMA 3.2. *Let L be an oriented framed link which bounds a compact oriented surface V consistent with the framing. Let $\alpha \in H^1(L)$ be an element such that $\delta(\alpha) = w_2(M, L)$. Then, we have*

$$\begin{aligned} \langle w_2(M, L), [V, \partial V]_2 \rangle &= \langle \delta(\alpha), [V, \partial V]_2 \rangle \\ &= \langle \alpha, [L]_2 \rangle \\ &\equiv \sharp L \pmod{2}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the Kronecker product, $[V, \partial V]_2 \in H_2(M, L)$ is the fundamental class of V in $\mathbf{Z}_2$ -coefficients, and $\sharp L$ denotes the number of components of L .

Note that the above lemma is applicable for an arbitrary null-cobordant framed link L and that the value $\langle \alpha, [L]_2 \rangle \in \mathbf{Z}_2$ does not depend on a particular choice of α . Furthermore, if L has an odd number of components, then the obstruction $w_2(M, L)$ never vanishes.

Let us now consider the case of a local knot component. Suppose that the oriented framed link L contains a component L_s that lies in the interior of a closed 3-disk D embedded in M . Set $U = \text{Int } D$, which is an open set of M diffeomorphic to $\mathbf{R}^3$. In the following, let us identify U with $\mathbf{R}^3$. In this case, up to homotopy, we may assume that the trivialization τ of TM over U is given by the standard one of $T\mathbf{R}^3$.

Let $\pi : \mathbf{R}^3 \rightarrow H$ be the orthogonal projection onto a generic hyperplane $H \cong \mathbf{R}^2$ in the sense that $\pi|_{L_s}$ is an immersion with normal crossings. Recall that the first vector field defining the trivialization $TM|_{L_s}$ associated with the framing on L_s is tangent to L_s consistent with the orientation. Since $\pi|_K$ is an immersion, we may assume that at each point x of L_s the remaining two vector fields give a 2-framing that is a basis for a 2-plane $N_x \subset T_x \mathbf{R}^3$ transverse to $T_x L_s$ containing the direction $H^\perp$ perpendicular to H . Then, we count the number of times modulo 2 the 2-framing rotates in N_x with respect to a fixed positive direction of $H^\perp$ while x goes once around L_s . This number is denoted by $t_v(L_s)$, which is an element in $\mathbf{Z}_2$. Then, we have proved the following in [18].

LEMMA 3.3. *Let $\alpha \in H^1(L)$ be an arbitrary element such that $\delta(\alpha) = w_2(M, L)$. Then, we have*

$$\langle \alpha, [L_s]_2 \rangle \equiv t_v(L_s) + c(L_s) + 1 \pmod{2},$$

where $c(L_s)$ denotes the number of crossings of the immersion $\pi|_{L_s} : L_s \rightarrow H$ with normal crossings.

From now on, we will consider integral homology 3-spheres for M in this section. Let us start with the following.

DEFINITION 3.4. For an oriented link L in a closed oriented 3-manifold M with $H_1(M; \mathbf{Z}) = 0$, we always have a *Seifert surface*, i.e. a compact oriented surface V

FIGURE 1. Seifert algorithm for positive and negative crossings

embedded in M such that $\partial V = L$. Such a Seifert surface is not unique; however, it is known that the induced framing on L is uniquely determined (for example, see [8, §3.6.1]). In the following, such a framing is said to be *preferred*.

Then, for oriented links with preferred framings in the 3-sphere S^3 , we have the following. In the following, we fix an orientation for S^3 once and for all.

PROPOSITION 3.5. *Let $L = L_1 \cup L_2 \cup \cdots \cup L_\mu$ be an oriented link in S^3 , on which a preferred framing is given. Then $w_2(S^3, L) = 0$ if and only if for each s with $1 \leq s \leq \mu$, we have*

$$\sum_{t \neq s} \text{lk}(L_s, L_t) \equiv 1 \pmod{2},$$

where lk denotes the linking number.

Proof. First, note that by the exact sequence (3.1) with $M = S^3$, we see that δ is injective and that $\alpha \in H^1(L)$ with $\delta(\alpha) = w_2(S^3, L)$ is uniquely determined. Therefore, $w_2(S^3, L) = 0$ if and only if $\langle \alpha, [L_s]_2 \rangle = 0$ for all s .

Now, we may assume that L is contained in $U \subset S^3$ as above, and let us consider the generic projection $\pi|_L : L \rightarrow H$. By the so-called Seifert algorithm, we can construct a compact oriented surface $V \subset S^3$ with $\partial V = L$ (see Fig. 1). Then, by construction, we see that when $\pi(x)$ goes once around $\pi(L_s)$, each time it goes through a positive (resp. negative) crossing point, it contributes $+1/2$ (resp. $-1/2$) to $t_v(L_s)$. Since the number of crossing points of $\pi(L_s)$ and $\pi(L_t)$ is even for each $t \neq s$, and $\pi(x)$ goes through each self-crossing point of $\pi(L_s)$ twice, we have

$$t_v(L_s) \equiv \frac{1}{2} \sum_{t \neq s} \tilde{c}(L_s, L_t) + \tilde{c}(L_s) \pmod{2}$$

for each s , where $\tilde{c}(L_s, L_t)$ is the sum of the signs of crossing points of $\pi(L_s)$ and $\pi(L_t)$, and $\tilde{c}(L_s)$ is the sum of the signs of self-crossing points of $\pi(L_s)$. Then, since $\tilde{c}(L_s) \equiv c(L_s) \pmod{2}$, by Lemma 3.3, we have

$$\begin{aligned} \langle \alpha, [L_s]_2 \rangle &\equiv \frac{1}{2} \sum_{t \neq s} \tilde{c}(L_s, L_t) + 1 \pmod{2} \\ &\equiv \sum_{t \neq s} \text{lk}(L_s, L_t) + 1 \pmod{2}, \end{aligned}$$

by the definition of linking numbers. Hence, the result follows. $\square$

In fact, we have the following more general result.

PROPOSITION 3.6. *Let M be a closed connected oriented 3-manifold with*

$$H_1(M; \mathbf{Z}) = 0$$

and $L = L_1 \cup L_2 \cup \cdots \cup L_\mu$ be an oriented link in M , on which a preferred framing is given. Then, $w_2(M, L) = 0$ if and only if for each s with $1 \leq s \leq \mu$, we have

$$(3.2) \quad \sum_{t \neq s} \text{lk}(L_s, L_t) \equiv 1 \pmod{2}.$$

Proof. Since $H_1(M; \mathbf{Z}) = 0$, there exists a Seifert surface V for L , which is a compact oriented surface embedded in M with $\partial V = L$. By definition, this is consistent with the framing of L . Set $V' = V \setminus \text{Int } N(L)$ and $\tilde{L}_s = V' \cap N(L_s)$ for each s , where $N(L)$ is a small tubular neighborhood of L in M , $N(L_s)$ is the component of $N(L)$ containing L_s , $\partial N(L)$ intersects V transversely, and $V \cap N(L)$ is a collar neighborhood of ∂V in V . Note that $\tilde{L}_s$ is a knot parallel to L_s , and we orient $\tilde{L}_s$ consistently with L_s . Then, the oriented link $\hat{L}_s = L \setminus L_s$ is $\mathbf{Z}$ -homologous to $-\tilde{L}_s$ in $M \setminus L_s$, where $-\tilde{L}_s$ denotes $\tilde{L}_s$ with the opposite orientation.

Now, suppose $w_2(M, L) = 0$. In this case, the given framing of L extends over M . Let us suppose that a Seifert surface V_s for L_s is consistent with the given framing of L_s for some s . Then, by Lemma 3.2 applied to L_s , $w_2(M, L_s) \in H^2(M, L_s)$ does not vanish, as we obviously have $\sharp L_s = 1$. This implies that $a_s \in \mathbf{Z}_2$ as appears in Lemma 3.1 does not vanish. This contradicts our assumption that the framing of L extends over M . Therefore, an arbitrary Seifert surface V_s for L_s is not consistent with the given framing of L_s for each s . Since V is consistent with the framing of L_s , the linking number of L_s and $\tilde{L}_s$ must be an odd integer. Since $-\tilde{L}_s$ is $\mathbf{Z}$ -homologous to $\hat{L}_s$ in $M \setminus L_s$, we have the congruence (3.2).

Conversely, suppose (3.2) holds for each s . Then, by the above argument we see that $a_s = 0$ for each s . Hence, by Lemma 3.1, we have $w_2(M, L) = 0$. This completes the proof. $\square$

In fact, the above argument implies the following.

PROPOSITION 3.7. *Let M be a closed connected oriented 3-manifold with*

$$H_1(M; \mathbf{Z}) = 0$$

and $L = L_1 \cup L_2 \cup \cdots \cup L_\mu$ be an oriented link in M , on which a preferred framing is given. For each s with $1 \leq s \leq \mu$, define $a_s \in \mathbf{Z}_2$ by

$$a_s = \sum_{t \neq s} \text{lk}(L_s, L_t) + 1 \pmod{2}.$$

Let $\alpha \in H^1(L)$ be the unique cohomology class such that $\langle \alpha, [L_s]_2 \rangle = a_s$ for all s . Then, we have $\delta(\alpha) = w_2(M, L)$.

When $H_1(M; \mathbf{Z}) = 0$, we have $H^1(M) = 0 = H^2(M)$, and hence the exact sequence (3.1) implies that we have the isomorphism $\delta : H^1(L) \rightarrow H^2(M, L)$. We easily see that its composition with the isomorphism $H^2(M, L) \rightarrow H_1(M \setminus L)$ corresponds to the Alexander duality whose inverse isomorphism is given by taking $\mathbf{Z}_2$ linking numbers. This observation together with Theorem 2.5 leads to the following, which answers to Question 2.4 for oriented framed links in integral homology 3-spheres.

THEOREM 3.8. *Let M be a closed connected oriented 3-manifold with*

$$H_1(M; \mathbf{Z}) = 0,$$

$L = L_1 \cup L_2 \cup \cdots \cup L_\mu$ be an oriented link in M , and J be an unoriented link in $M \setminus L$. Then, there exists an excellent map $f : M \rightarrow \mathbf{R}^2$ such that $L = f^{-1}(y)$ for

a regular value $y \in \mathbf{R}^2$ and $J = S(f)$ if and only if for each s with $1 \leq s \leq \mu$, the $\mathbf{Z}_2$ linking number of J with L_s coincides with

$$\sum_{t \neq s} \text{lk}(L_s, L_t) + 1 \pmod{2}.$$

Proof. By the above observations, we see that $[J]_2 \in H_1(M \setminus L)$ is Poincaré dual to $w_2(M, L) \in H^2(M, L)$ if and only if it satisfies the condition on $\mathbf{Z}_2$ linking numbers in the theorem. Thus, the result follows from Theorem 2.5. $\square$

Let us observe the following.

LEMMA 3.9. *If the congruence (3.2) holds, then the number of components of L must be even.*

Proof. Consider the sum of all linking numbers

$$\sum_{s=1}^{\mu} \sum_{t \neq s} \text{lk}(L_s, L_t) \in \mathbf{Z}$$

over all s and t with $s \neq t$. Since $\text{lk}(L_s, L_t) = \text{lk}(L_t, L_s)$, the above sum must be even. On the other hand, the congruence (3.2) implies that the above sum has the same parity as the number of components of L . Thus the result follows. $\square$

The above lemma together with Theorem 3.8 implies that for an integral homology 3-sphere M and an excellent map $f : M \rightarrow \mathbf{R}^2$, if $L = f^{-1}(y)$ has an odd number of components for a regular value $y \in \mathbf{R}^2$, then $S(f)$ has a non-trivial linking number with a component of L .

In order to get a more general result, let us introduce the following definition.

DEFINITION 3.10. Let M be a closed connected oriented 3-manifold and L, L' be non-empty disjoint links in M . We say that L and L' are *not linked* if there exists an embedded 2-sphere in $M \setminus (L \cup L')$ which separates M into two components in such a way that one of them contains L and the other contains L' . If such a 2-sphere does not exist, then we say that L and L' are *linked*.

LEMMA 3.11. *Let M be a closed connected oriented 3-manifold containing an embedded 2-sphere S which separates M into two components M_1 and M_2 , where M_1 and M_2 are the closures of the connected components of $M \setminus S$. If a framed link L is contained in $\text{Int } M_1$ and is framed null-cobordant in M , then it is also framed null-cobordant in $\text{Int } M_1$.*

Proof. Let V be a compact oriented normally framed surface in M which bounds L and is consistent with the framing of L . We may assume that V and S intersect each other transversely. Then, $V \cap S$ consists of a finite number of simple closed curves in the 2-sphere S . By considering $V \cap M_1$, adding 2-disks bounded by the simple closed curves in S , and by slightly translating the 2-disks in a parallel manner using the inner-most argument, we get a compact oriented surface embedded in $\text{Int } M_1$. This gives a desired framed null-cobordism for L in $\text{Int } M_1$. $\square$

We have the following as a result of Lemma 3.11.

PROPOSITION 3.12. *Let M be a closed connected oriented 3-manifold and $f : M \rightarrow \mathbf{R}^2$ an excellent map. For a regular value $y \in \mathbf{R}^2$, if $L = f^{-1}(y)$ is non-empty and has an odd number of connected components, then L is necessarily linked with $S(f)$.*

Proof. Suppose that there exists a 2-sphere S that separates L and $S(f)$. Let M_1 and M_2 be the closures of the two components of $M \setminus S$ such that $L \subset \text{Int } M_1$ and $S(f) \subset \text{Int } M_2$. Since L is framed null-cobordant in M , it is also framed null-cobordant in $\text{Int } M_1$ by Lemma 3.11. Therefore, there exists a compact oriented

normally framed surface in $\text{Int } M_1$ that bounds L . Let $\widehat{M}_1$ be the closed oriented 3-manifold obtained by attaching a 3-disk to M_1 along the boundary S . Then, since $f|_{M_1}$ is a submersion and $\pi_2(SO(3))$ vanishes, we see that the trivialization of $T\widehat{M}_1|_L$ extends to $\widehat{M}_1$, and hence $w_2(\widehat{M}_1, L)$ vanishes. Then, by Lemma 3.2 applied to $L \subset \widehat{M}_1$, this leads to a contradiction, since $\sharp L$ is odd by our assumption. Therefore, L and $S(f)$ are necessarily linked. This completes the proof. $\square$

In the case of integral homology 3-spheres, by Theorem 3.8 we have the following.

PROPOSITION 3.13. *Let M be a closed connected oriented 3-manifold with*

$$H_1(M; \mathbf{Z}) = 0$$

and $L = L_1 \cup L_2 \cup \dots \cup L_\mu$ be an oriented link in M . For an arbitrary excellent map $f : M \rightarrow \mathbf{R}^2$ such that $L = f^{-1}(y)$ for a regular value $y \in \mathbf{R}^2$, $S(f)$ necessarily links with each component L_s of L with

$$(3.3) \quad \sum_{t \neq s} \text{lk}(L_s, L_t) \equiv 0 \pmod{2}.$$

Compare the above proposition with [18, Problem 5.1]. For example, if the congruence (3.3) holds for all s , then for an excellent map $f : M \rightarrow \mathbf{R}^2$ such that $f^{-1}(y) = L$ for a regular value $y \in \mathbf{R}^2$, each component of L links with at least one component of $S(f)$.

We do not know if the results in this section for M with $H_1(M; \mathbf{Z}) = 0$ also hold for M with $H_1(M) = 0$ in $\mathbf{Z}_2$ -coefficients.

4. SUBMERSIONS OF OPEN 3-MANIFOLDS TO $\mathbf{R}^2$

In this section, as an application of our results in [18] and in the previous sections of the present paper, we consider submersions of open orientable 3-manifolds to $\mathbf{R}^2$.

First, let us recall the following fundamental theorem for submersions of $\mathbf{R}^3$ to $\mathbf{R}^2$ obtained in [8].

THEOREM 4.1 (Hector and Peralta-Salas, 2012). *Let $L = L_1 \cup L_2 \cup \dots \cup L_\mu \subset \mathbf{R}^3$ be an oriented link in $\mathbf{R}^3$. Then, there exists a submersion $f : \mathbf{R}^3 \rightarrow \mathbf{R}^2$ such that $f^{-1}(y) = L$ for some $y \in \mathbf{R}^2$ if and only if for each s with $1 \leq s \leq \mu$, we have*

$$\sum_{t \neq s} \text{lk}(L_s, L_t) \equiv 1 \pmod{2}.$$

Recall that in [8], the authors used the h-principle for submersions [6, 15] for the proof. Here, we give a new proof to the above theorem using our singularity theoretical techniques.

Proof of Theorem 4.1. Let L be an oriented link in $\mathbf{R}^3$ which satisfies the condition about the linking numbers as in the theorem. By identifying the interior of an embedded 3-disk D in S^3 with $\mathbf{R}^3$, we may assume that $L \subset \text{Int } D \subset S^3$. Then, by Proposition 3.5, we have $w_2(S^3, L) = 0$ with respect to the preferred framing on L . Therefore, for an arbitrary non-empty link J in $S^3 \setminus D$, there exists an excellent map $g : S^3 \rightarrow \mathbf{R}^2$ and a regular value $y \in \mathbf{R}^2$ such that $L = g^{-1}(y)$ and $J = S(g)$. By restricting g to $\mathbf{R}^3 = \text{Int } D$, we get a submersion $f : \mathbf{R}^3 \rightarrow \mathbf{R}^2$ which has L as a regular fiber.

Conversely, suppose that we have a submersion $f : \mathbf{R}^3 \rightarrow \mathbf{R}^2$ and a regular value $y \in \mathbf{R}^2$ such that $f^{-1}(y) = L$. Then, we can find an embedded 3-disk $D \subset \mathbf{R}^3$ whose interior contains L . Note that $f|_D : D \rightarrow \mathbf{R}^2$ is a submersion which has L as a regular fiber. By embedding D into S^3 , we can extend $f|_D$ to a smooth map $g_1 : S^3 \rightarrow \mathbf{R}^2$. Here, $f(\partial D)$ misses $y \in \mathbf{R}^2$, and since the second homotopy group of $\mathbf{R}^2 \setminus \{y\}$ is trivial, $f|_{\partial D}$ is null-homotopic inside $\mathbf{R}^2 \setminus \{y\}$. Therefore, we can

arrange the smooth map g_1 in such a way that g_1 has $y \in \mathbf{R}^2$ as a regular value and that $g_1^{-1}(y) = L \subset \text{Int } D$. Then, by slightly perturbing g_1 on a neighborhood of $S^3 \setminus \text{Int } D$, we get an excellent map $g_2 : S^3 \rightarrow \mathbf{R}^2$ such that $y \in \mathbf{R}^2$ is a regular value, that $g_2^{-1}(y) = L$, and that $S(g_2)$ is contained in $S^3 \setminus \text{Int } D$. In particular, $S(g_2)$ is $\mathbf{Z}_2$ null-homologous in $S^3 \setminus L$, and hence we have $w_2(S^3, L) = 0$. Then, by Proposition 3.5, we get the result. $\square$

REMARK 4.2. More generally, instead of $\mathbf{R}^3$, the above theorem holds also for an arbitrary open 3-manifold of the form $M \setminus D^3$ for a closed connected orientable 3-dimensional manifold M with $H_1(M; \mathbf{Z}) = 0$, where D^3 is a small closed 3-disk embedded in M .

In the case of a link with an odd number of components, we have the following.

REMARK 4.3. Let $f : \mathbf{R}^3 \rightarrow \mathbf{R}^2$ be an excellent map, and suppose that $y \in \mathbf{R}^2$ is a regular value such that $L = f^{-1}(y)$ is compact and has an odd number of components. Then, by Proposition 3.12 together with an argument similar to the above, we see that the singular point set $S(f)$ necessarily links with L (see also the paragraph just after [14, Theorem 10]): in other words, we can find no 2-sphere embedded in $\mathbf{R}^3$ that separates L and $S(f)$. This implies, in particular, that such an f can never be a submersion.

In fact, we have the following.

PROPOSITION 4.4. *Let M be a closed connected orientable 3-manifold with*

$$H_1(M; \mathbf{Z}) = 0$$

and set $M^\circ = M \setminus D^3$. Let $L = L_1 \cup L_2 \cup \dots \cup L_\mu \subset M^\circ$ be an oriented link such that $f^{-1}(y) = L$ for some excellent map $f : M^\circ \rightarrow \mathbf{R}^2$ and a regular value $y \in \mathbf{R}^2$. Then, each component L_s of L with

$$(4.1) \quad \sum_{t \neq s} \text{lk}(L_s, L_t) \equiv 0 \pmod{2}$$

links with at least one component of $S(f)$. In particular, such an f can never be a submersion.

Compare the above proposition with [18, Problem 5.1]. See also [1, 2, 4, 10] for related physical results.

Proof of Proposition 4.4. First note that each component of $S(f)$ is diffeomorphic to a circle or a real line. Furthermore, $S(f)$ is a closed submanifold of M° which may have infinitely many connected components.

Let V_s be a Seifert surface for L_s in M , where L_s satisfies (4.1). We may assume that $L_s \subset M^\circ$ and that $S(f)$ intersects L_s transversely at finitely many points. We have only to show that there are an odd number of intersection points.

Let $\tilde{D}$ be a 3-disk in M such that $\text{Int } \tilde{D} \supset D^3$, $L \cap \tilde{D} = \emptyset$, $V_s \cap \tilde{D} = \emptyset$, and that $\partial \tilde{D}$ intersects $S(f)$ transversely at finitely many points. Then, by an argument similar to that in the proof of Theorem 4.1, we can construct an excellent map $g : M \rightarrow \mathbf{R}^2$ such that $g|_{M \setminus \text{Int } \tilde{D}} = f|_{M \setminus \text{Int } \tilde{D}}$ and that $g^{-1}(y) = L$. By our assumption (4.1), we have that L_s has a non-trivial $\mathbf{Z}_2$ linking number with $S(g)$ by Theorem 3.8. Therefore, $S(g)$ intersects V_s transversely at an odd number of points. By construction of g , this implies that $S(f)$ also intersects V_s transversely at an odd number of points. This completes the proof. $\square$

The following is a special case of a theorem proved by Miyoshi [13], who used a relative version of the h-principle for submersions [6]. Here, we use our singularity theoretical arguments in order to prove the theorem for punctured 3-manifolds.

THEOREM 4.5. *Let M be a closed orientable 3-manifold and L a compact oriented framed link in $M^\circ = M \setminus D^3$. Then, there exists a submersion $f : M^\circ \rightarrow \mathbf{R}^2$ such that $f^{-1}(y)$ coincides with L as oriented framed links for some $y \in \mathbf{R}^2$ if and only if L bounds a proper normally framed surface in M° and the trivialization of $TM^\circ|_L$ induced by the framing of L extends over M° .*

Proof. If there exists a submersion f as in the theorem, then the inverse image by f of the half line $[y_1, \infty) \times \{y_2\} \subset \mathbf{R}^2$ is a proper normally framed surface in M° that bounds L , where $y = (y_1, y_2)$. Furthermore, since f is a submersion, we can pull-back the natural trivialization of $T\mathbf{R}^2$ to M° by f in such a way that the pull-back naturally extends the trivialization of $TM^\circ|_L$ induced by the framing of L .

Conversely, suppose that L bounds a proper normally framed surface V in M° and the trivialization of $TM^\circ|_L$ induced by the framing of L extends over M° . Let $\tilde{D}$ be a small 3-disk neighborhood of D^3 whose interior contains D^3 such that $\tilde{D} \subset M \setminus N(L)$ for a small tubular neighborhood $N(L)$ of L in M . Then, we may assume that V intersects $\partial\tilde{D}$ transversely along finitely many embedded circles. Note that then $V \cap \partial\tilde{D}$ bounds a compact oriented surface V' in $\tilde{D}$. Then, by replacing $V \cap \tilde{D}$ by V' , we see that L is framed null-cobordant in M . Furthermore, by our assumption, the trivialization of $TM^\circ|_L$ induced by the framing of L extends over M° . Since $\pi_2(SO(3))$ vanishes, this implies that it also extends over M . Therefore, we have that the obstruction $w_2(M, L)$ vanishes. Hence, by Theorem 2.5, there exists an excellent map $f : M \rightarrow \mathbf{R}^2$ and a regular value $y \in \mathbf{R}^2$ such that $f^{-1}(y)$ coincides with L as oriented framed links and that $S(f)$ is contained in $\text{Int } D^3$. Then, f restricted to $M^\circ = M \setminus D^3$ is a desired submersion. $\square$

In fact, if we use the “absolute version” of the h-principle [15] in order to treat the end of an open 3-manifold, we can prove the following. Note again that the following theorem was originally proved by Miyoshi [13] by using a “relative version” of the h-principle [6].

THEOREM 4.6. *Let M be an open orientable 3-manifold and L a compact oriented framed link in M . Then, there exists a submersion $f : M \rightarrow \mathbf{R}^2$ such that $f^{-1}(y)$ coincides with L as oriented framed links for some $y \in \mathbf{R}^2$ if and only if L bounds a proper normally framed surface in M and the trivialization of $TM|_L$ induced by the framing of L extends over M .*

Proof. Necessity can be proved by the same argument as in the proof of Theorem 4.5.

Conversely, suppose that there exists a proper normally framed surface V in M that bounds L as described in the theorem. Let Q be a compact 3-dimensional submanifold of M with boundary such that $\text{Int } Q \supset L$ and that ∂Q intersects V transversely along finitely many embedded circles.

Let us first construct a smooth map $g_1 : M \rightarrow \mathbf{R}^2$ as follows. Let $h : V \rightarrow [0, \infty)$ be a smooth function such that $h^{-1}(0) = \partial V = L$ and that h is non-singular near ∂V . Let $N(V) \cong V \times I$ be a tubular neighborhood of V in M , where $I = [-1, 1]$ and the I -factor is consistent with the normal orientation of V . Then, we define g_1 on $N(V)$ by

$$N(V) \cong V \times I \xrightarrow{h \times \text{id}_I} [0, \infty) \times I \subset \mathbf{R}^2,$$

where id_I is the identity map of I . We can extend $g_1|_{N(V)}$ to $N(V) \cup N(L)$ in such a way that $g_1|_{N(L)}$ is a submersion, that the origin 0 is a regular value, and that the framed regular fiber $g_1^{-1}(0)$ coincides with L . Then, since $\mathbf{R}^2 \setminus g_1(N(V) \cup N(L))$ is contractible, we can extend g_1 to the whole manifold M in such a way that 0 is still a regular value and that the framed regular fiber $g_1^{-1}(0)$ coincides with L .

Set $Q' = Q \setminus \text{Int } N(L)$, which is a compact 3-manifold with boundary $\partial Q \cup \partial N(L)$. Note that $g_1(Q') \subset \mathbf{R}^2 \setminus \text{Int } D$, where D is a small 2-disk neighborhood of the origin.

By our assumption, the framing on L extends over M . Using such a framing, we can construct a bundle epimorphism $T(M \setminus \text{Int } Q) \rightarrow T(\mathbf{R}^2 \setminus \text{Int } D)$ covering $g_1|_{M \setminus \text{Int } Q}$. Then, by the h-principle for submersions, g_1 is homotopic to a smooth map $g_2 : M \rightarrow \mathbf{R}^2$ such that

- (1) g_2 is a submersion over $M \setminus \text{Int } Q$,
- (2) $g_2 = g_1$ over $N(L)$,
- (3) $g_2(M \setminus \text{Int } N(L)) \subset \mathbf{R}^2 \setminus \text{Int } D$.

Then, we can approximate g_2 by an excellent map g_3 that enjoys the same properties as g_2 described above. Then, $S(g_3)$ is a closed subset of Q , which is compact. Therefore, $S(g_3)$ is an unoriented link in $Q \setminus \text{Int } N(L)$. Furthermore, as we started with a framing that extends over M , the obstruction to extending the framing on $\partial(Q \setminus \text{Int } N(L))$ induced by g_3 to the whole Q vanishes. This implies that the $\mathbf{Z}_2$ -homology class represented by $S(g_3)$ vanishes in Q . Then, by our techniques developed in [18] using Levine's cusp eliminations (see [11, 17]), we can homotope g_3 to an excellent map g_4 that satisfies the properties described above such that $S(g_4)$ is unlinked from L : more precisely, there exists an embedded 3-disk $B \subset \text{Int } Q \setminus N(L)$ such that $\text{Int } B \supset S(g_4)$. Then, for an appropriate embedded arc $A \subset M \setminus N(L)$ that "connects" B to infinity, we see that M is diffeomorphic to $M \setminus (A \cup B)$ by a diffeomorphism that is the identity on $N(L)$ (for example, see [13]). Then, the restriction of g_4 to $M \setminus (A \cup B)$ gives the desired submersion. This completes the proof. $\square$

REMARK 4.7. It is known that there exist open 3-manifolds that cannot be embedded in compact 3-manifolds [7].

We finish this paper by posing an open problem.

PROBLEM 4.8. Is there a polynomial map $\mathbf{R}^3 \rightarrow \mathbf{R}^2$ that has a compact regular fiber as in Theorem 4.1 ?

Compare the above problem with Remark 2.3.

One can find some relevant open problems in [8, §4] as well.

ACKNOWLEDGMENT

The author would like to express his sincere gratitude to Professor David Chillingworth for stimulating discussions and for indicating the works of Hector and Peralta-Salas [8, 14]. The author has been supported in part by JSPS KAKENHI Grant Numbers JP16K13754, JP16H03936, JP17H06128.

REFERENCES

- [1] M.V. Berry and M.R. Dennis, *Knotted and linked phase singularities in monochromatic waves*, Proc. R. Soc. Lond. A **457** (2001), 2251–2263.
- [2] M.V. Berry and M.R. Dennis, *Topological events on wave dislocation lines: birth and death of loops, and reconnection*, J. Physics A: Mathematical and Theoretical **40** (2007), 65–74.
- [3] B. Bode and M.R. Dennis, *Constructing a polynomial whose nodal set is any prescribed knot or link*, preprint, arXiv:1612.06328 [math.GT].
- [4] M.R. Dennis, *Local phase structure of wave dislocation lines: twist and twirl*, J. Opt. A: Pure Appl. Opt. **6** (2004), S202–S208.
- [5] M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Grad. Texts in Math., Vol. 14, Springer-Verlag, New York, Heidelberg, Berlin, 1973.
- [6] A. Haefliger, *Lectures on the theorem of Gromov*, Proceedings of Liverpool Singularities Symposium, II (1969/1970) (Berlin), Lecture Notes in Math., vol. 209, Springer-Verlag, 1971, pp. 128–141.

- [7] W. Haken, *Some results on surfaces in 3-manifolds*, in “1968 Studies in Modern Topology”, pp. 39–98.
- [8] G. Hector and D. Peralta-Salas, *Integrable embeddings and foliations*, Amer. J. Math. **134** (2012), 773–825.
- [9] M.A. Kervaire, *Relative characteristic classes*, Amer. J. Math. **79** (1957), 517–558.
- [10] H. Larocque, D. Sugic, D. Mortimer, A.J. Taylor, R. Fickler, R.W. Boyd, M.R. Dennis, and E. Karimi, *Reconstructing the topology of optical polarization knots*, Nature Physics (2018), <https://doi.org/10.1038/s41567-018-0229-2>.
- [11] H.I. Levine, *Elimination of cusps*, Topology **3**, Suppl. 2 (1965), 263–296.
- [12] J. Milnor, *Spin structures on manifolds*, Enseignement Math. (2) **9** (1963), 198–203.
- [13] S. Miyoshi, *Links and submersions to the plane on an open 3-manifold*, J. Topol. Anal. **8** (2016), 677–690.
- [14] D. Peralta-Salas, *Realization problems in the theory of foliations*, In: S. Ibáñez, J.S. Pérez del Río, A. Pumariño, J.A. Rodríguez (eds.), Progress and Challenges in Dynamical Systems, Springer Proceedings in Mathematics & Statistics, vol. 54, pp. 309–323, Springer, Berlin, Heidelberg, 2013.
- [15] A. Phillips, *Submersions of open manifolds*, Topology **6** (1967), 171–206.
- [16] O. Saeki, *Stable maps and links in 3-manifolds*, Kodai Math. J. **17** (1994), 518–529.
- [17] O. Saeki, *Constructing generic smooth maps of a manifold into a surface with prescribed singular loci*, Ann. Inst. Fourier (Grenoble) **45** (1995), 1135–1162.
- [18] O. Saeki, *Linking between singular locus and regular fibers*, preprint, arXiv:1804.00359 [math.GT].
- [19] R. Thom, *Les singularités des applications différentiables*, Ann. Inst. Fourier, Grenoble **6** (1955–1956), 43–87.
- [20] H. Whitney, *On singularities of mappings of Euclidean spaces. I. Mappings of the plane into the plane*, Ann. of Math. (2) **62** (1955), 374–410.

INSTITUTE OF MATHEMATICS FOR INDUSTRY, KYUSHU UNIVERSITY, MOTOOKA 744, NISHI-KU, FUKUOKA 819-0395, JAPAN

E-mail address: `saeki@imi.kyushu-u.ac.jp`