SCHUBERT CALCULUS, SEEN FROM TORUS EQUIVARIANT TOPOLOGY

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バージョン：

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ABSTRACT. We survey the classical (ordinary) Schubert calculus in the first half of this note. Then we lift everything up to an equivariant setting; we see three descriptions of the equivariant cohomology of flag varieties and investigate their relation. The example given in §10 will be helpful to read this note. (This version fixed some errors in the published one.)

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1. Introduction: Schubert's quiz

"How many lines are there in the three dimensional space which intersects all the four given lines ?" Hermann Schubert (1848-1911) considered this kind of problems in an insightful but not rigorous way. His method goes as follows. First, we define a series of logical symbols concerning a line in the space: A line

$\Omega_{(1234)}$: without any restriction
$\Omega_{(1324)}$: intersecting a given line
$\Omega_{(2314)}$: goes through a given point
$\Omega_{(1423)}$: lying on a given plane
$\Omega_{(2413)}$: lying on a given plane and goes through a given point on the plane
$\Omega_{(3412)}$: lying on a given line, i.e. the line itself

Then we can do logical calculation such as:

$$\Omega_{(1324)} \cap \Omega_{(1324)} = \Omega_{(2314)} \cup \Omega_{(1423)},$$

which means in the usual language that "A line intersecting two given lines is either 1) going through the intersection point of the two, or 2) lying on the plane spanned by the two." Note that for this "calculation," we have to assume that the problem doesn't lose generality if we move the two given lines so that they have an intersection. This assumption is what Schubert called the "principle of continuity," which we accept for the present.

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To solve Schubert's quiz, what we want to know is $\Omega_{(1324)} \cap \Omega_{(1324)} \cap \Omega_{(1324)} \cap \Omega_{(1324)}$ and the calculation proceeds as:

$$\begin{aligned}
&=(\Omega_{(2314)} \cup \Omega_{(1423)}) \cap (\Omega_{(2314)} \cup \Omega_{(1423)}) \\
&=(\Omega_{(2314)} \cap \Omega_{(2314)}) \cup 2(\Omega_{(2314)} \cap \Omega_{(1423)}) \cup (\Omega_{(1423)} \cap \Omega_{(1423)}) \qquad (\clubsuit) \\
&=\Omega_{(3412)} \cup \emptyset \cup \Omega_{(3412)} \\
&=2\Omega_{(3412)}.
\end{aligned}$$

This kind of counting problems belong to *enumerative geometry*. Hilbert asked for a rigorous foundation for it as the 15th problem in his 1900 lecture and now Schubert's quiz can be rephrased in terms of intersection theory of a Grassmaniann manifold (see [27]).

First we need to consider the problem in $\mathbb{C}P^3$ instead of $\mathbb{R}^3$ (this is justified by [38]), to allow an "intersection at infinity" and to work in algebro-geometric setting. The space of projective lines in $\mathbb{C}P^3$ is identified with the Grassmannian manifold $Gr(2,4)$ of two dimensional linear sub-spaces of $\mathbb{C}^4$, and the conditions are replaced by its sub-varieties, called the (classical) *Schubert varieties*, which is indexed by a certain subset of permutations.

Definition 1.1. *We denote an element* $w = \begin{pmatrix} 1 & 2 & \cdots & n+m \\ w(1) & w(2) & \cdots & w(n+m) \end{pmatrix}$ *of the permutation group* S_{n+m} *of* $n+m$*-letters by* one-line notation $(w(1), w(2), \ldots, w(n+m))$.

A set $W^{P_n} \subset S_{n+m}$ *of* Grassmaniann permutations with a descent at n *is defined to be*

$$W^{P_n} := \{w = (i_1, i_2, \ldots, i_{n+m}) \in S_{n+m} \mid i_1 \le i_2 \le \cdots \le i_n, i_{n+1} \le i_{n+2} \le \cdots \le i_{n+m}\}.$$

Then, the Schubert variety in $Gr(n, n+m)$ corresponding to a Grassmaniann permutation is defined[1] by the *incidence condition*:

Definition 1.2.

$$\Omega_w := \left\{V_n \in Gr(n, n+m) \mid \dim(V_n \cap \mathbb{C}^i) \ge \#\{j \mid n < j \le n+m, w(j) \le i\}\right\}, \quad w \in W^{P_n}.$$

Example 1.3. $\Omega_{(1324)} \subset Gr(2,4)$ *is the set of those* V_2 *such that*

$$\dim(V_2 \cap \mathbb{C}^1) \ge 0, \dim(V_2 \cap \mathbb{C}^2) \ge 1, \dim(V_2 \cap \mathbb{C}^3) \ge 1, \dim(V_2 \cap \mathbb{C}^4) \ge 2,$$

which means that the projective line $\overline{V_2}$ *has intersection at a projective point with the fixed projective line* $\overline{\mathbb{C}^2}$*, and intersection at a projective point with the fixed projective plane* $\overline{\mathbb{C}^3}$*, and intersection at a projective line with the whole space* $\mathbb{C}P^3 = \overline{\mathbb{C}^4}$*. The latter two are redundant since they are automatically satisfied.*

Remark 1.4. *Usually Schubert varieties in Grasmaniann manifolds are indexed by* (n,m)-partitions $\{(\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n) \mid m \ge \lambda_1, \lambda_n \ge 0\}$*, or in other words,* Young diagram. *Correspondence between Grassmaniann permutations and partitions is given by*

$$(\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n) \mapsto (1 + \lambda_n, 2 + \lambda_{n-1}, \ldots, i + \lambda_{n+1-i}, \ldots, n + \lambda_1, j_1, j_2, \ldots, j_m),$$

where $j_1 \le j_2 \le \cdots \le j_m$ *are the those numbers not appearing as* $i + \lambda_{n+1-i}$.

In this setting, we can think of $\cap$ and $\cup$ in the calculation ($\clubsuit$) as the intersection product and the sum in the intersection cohomology (the Chow ring), which makes Schubert's argument rigorous.

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[1] W^{P_n} can be thought of as the *minimal length left coset representatives* of $W/W_{P_n} = S_{n+m}/S_n \times S_m$. Ω_w is independent of the choice of a representative w in the coset W/W_{P_n}.

2. What is Schubert calculus

In the calculation (♣) we had to resort to geometric intuition to calculate the product of two Schubert varieties. A long-standing open problem in Schubert calculus is to give an "good" algorithm for the *structure constants*. We first see the precise statement of the problem in a general setting.

Let G be a connected complex Lie group of rank r, B be its Borel sub-group. Then the (right quotient) homogeneous space G/B (or more generally, G/P where $B \subset P$) is known to be a smooth projective variety and called the (generalized) *flag variety*. Alternatively, if we take $K \subset G$ to be a maximal compact connected sub-group, then by Iwasawa decomposition we have a diffeomorphism $K/T \to G/B$ induced by the inclusion $K \hookrightarrow G$. We use both forms interchangeably, in particular, flag varieties are easily seen to be compact from the latter.

Example 2.1. *Let $G = GL_{n+m}(\mathbb{C})$ and B be the sub-group of upper-triangular matrices. The* space of flags[2]

$$Fl_{n+m} = \{0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{n+m} = \mathbb{C}^{n+m} \mid \dim_{\mathbb{C}}(V_i) = i\}$$

admits a transitive G action and B fixes the base point $(0 \subset \mathbb{C}^1 \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^{n+m})$. So $Fl_{n+m} \cong G/B$. We can take a maximal compact sub-group $U(n+m) \subset GL_{n+m}(\mathbb{C})$ and then $Fl_{n+m} \cong U(n+m)/T$.

Similarly, if we take a sub-group $B \subset P_n \subset GL_{n+m}(\mathbb{C})$ as

$$P_n := \left\{ A \in GL_{n+m}(\mathbb{C}) \mid A = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}, X \in GL_n(\mathbb{C}), Z \in GL_m(\mathbb{C}) \right\},$$

then the quotient GL_{n+m}/P_n is the Grassmannian manifold $Gr(n, n+m)$.

2.1. **Cell cohomology.** The cohomology group $H^*(G/B;\mathbb{Z})$ (or equivalently, the Chow group $A^*(G/B)$) has a distinguished basis as we will see here. Let B_- be the Borel sub-group opposite to B, i.e. $B \cap B_-$ is the maximal algebraic torus, and W be the Weyl group of G. W is a finite group generated by the simple reflections $s_1, \ldots, s_r$ corresponding to the simple roots $\alpha_1, \ldots, \alpha_r$. The *length* $l(w) \in \mathbb{Z}_{\geq 0}$ for $w \in W$ is the minimal length of the presentation of w by a product of $s_1, \ldots, s_r$.

The Bruhat decomposition $G \cong \coprod_{w\in W} B_-wB$ induces a left T-stable cell decomposition $G/B \cong \coprod_{w\in W} B_-wB/B$ which has even cells only. It is known that $B_-wB/B \cong \mathbb{C}^{l(w_0)-l(w)}$ ([6]), where w_0 is the *longest element* of W. In particular, the real dimension of G/B is $\dim_{\mathbb{R}}(\mathbb{C}^{l(w_0)}) = 2l(w_0)$.

Example 2.2. *When $G = GL_r(\mathbb{C})$, T can be taken as the sub-group of the diagonal matrices, B be the sub-group of the upper triangular matrices, B_- be the sub-group of the lower triangular matrices, and $W = S_r$ is the sub-group of the permutation matrices. The simple root α_i $(1 \leq i \leq r-1)$ is identified as $t_{i+1} - t_i$, where t_i's are the coordinates of $\mathbb{C}^r$.*

Any matrix A in $GL_r(\mathbb{C})$ can be decomposed as $A = LPU$, where L is a lower triangular matrix, P is a permutation, and U is an upper triangular matrix. This is often referred to as the LPU-decomposition of the invertible matrices.

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[2]Imagine drawing a picture of (base point on the ground ⊂ flagpole ⊂ entire flag), then you'll know why it is named "flags."

Example 2.3. *Let G be $GL_4(\mathbb{C})$ and $w = (3412) = s_2s_1s_3s_2 \in W = S_4$. Then $l(w) = 4$ and $l(w_0) = l(s_3s_2s_1s_3s_2s_3) = 6$.*

$$
\begin{aligned}
w &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \in GL_4(\mathbb{C}) \\
B_-w &= \begin{pmatrix} 0 & 0 & * & 0 \\ 0 & 0 & * & * \\ * & 0 & * & * \\ * & * & * & * \end{pmatrix} \subset GL_4(\mathbb{C}) \\
B_-wB/B &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & * & 1 \\ 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \end{pmatrix} \cong \mathbb{C}^2 \subset GL_4(\mathbb{C})/B
\end{aligned}
$$

Definition 2.4. *The closure of the cell $\overline{B_-wB/B}$ becomes a sub-variety (possibly with rational singularity) of real codimension $2l(w)$ called the* Schubert variety *and denoted by Ω_w.*

The Poincare dual of its fundamental class determines a cohomology class $[\Omega_w] \in H^{2l(w)}(G/B; \mathbb{Z})$, called the Schubert class and denoted by Z_w.

Remark 2.5. *Here we adopt convention of taking left B_- orbit. The original definition of a Schubert variety is the left B orbit $\overline{BwB/B}$, which is dimension $2l(w)$ instead of codimension $2l(w)$. If we take the Kronecker (instead of Poincare) dual of $\overline{BwB/B}$, it gives the same class Z_w, since $\overline{BwB/B}$ and $\overline{B_-wB/B}$ are dual to each other. (That is, $\overline{BwB/B} \cap \overline{B_-wB/B} = pt$.)*

Example 2.6. *For $G = GL_r(\mathbb{C})$, the Schubert variety $\overline{B_-wB/B}$ is defined by the incidence condition on the flags:*

$$\left\{0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_r = \mathbb{C}^r \mid \dim(V_i \cap \mathbb{C}^j) \geq r_w(i,j) := \#\{p \mid r-i < p \leq r, w(p) \leq j\}\right\}$$

and has codimension $2l(w)$, twice the number of inversions in w.

Schubert varieties for general G/P cases are defined as:

$$\Omega_w := \overline{B_-wP/P}, \quad w \in W^P,$$

where W^P is the left coset W/W_P. Here W_P is the Weyl group of P. Note that this coincide with the definition for Grassmaniann manifolds $Gr(n, n+m) \cong GL_{n+m}/P_n$.

Since the Bruhat decomposition involves only even dimensional cells,

Theorem 2.7 (Basis Theorem)**.** *$H^*(G/P; \mathbb{Z})$ is a free $\mathbb{Z}$-module generated by Schubert classes, i.e.*

$$H^*(G/P; \mathbb{Z}) \cong \bigoplus_{w \in W^P} \mathbb{Z}\langle Z_w \rangle, \text{ in particular, } H^*(G/B; \mathbb{Z}) \cong \bigoplus_{w \in W} \mathbb{Z}\langle Z_w \rangle,$$

Remark 2.8 (See [9])**.** *From Morse theoretic point of view, Schubert classes can be considered as follows. Let $\mathfrak{t}$ and $\mathfrak{g}$ be the Lie algebra for T and K respectively. If we take $X_0 \in \mathfrak{t}$, where X_0 an* *internal point of the Weyl chamber, K/T is identified with the adjoint orbit $\{gX_0g^{-1} \mid g \in K\} \subset \mathfrak{g}$. Then we have a perfect Morse function*

$$h : K/T \to \mathbb{R}, \quad X \mapsto |X - X_0|^2$$

with the critical points $\{wX_0 \mid w \in W\}$ of index $\dim(K/T) - 2l(w)$.

By the Basis Theorem, the cup product[3] of two Schubert classes can be represented by a linear combination of Schubert classes

$$Z_u \cup Z_v = \sum_{w \in W^P} c_{uv}^w Z_w, \quad c_{uv}^w \in \mathbb{Z},$$

where c_{uv}^w is called the *structure constant*. If we replace the ordinary cohomology H^* by the K-theory K^*, the quantum cohomology QH^*, or their equivariant versions H_T^*, K_T^*, QH_T^*, we have corresponding problem of the structure constants $c_{uv}^w \in h^*(pt)$, where $h^* = H^*, K^*, H_T^*, K_T^*$ or QH_T^*.

Remark 2.9. *Since the (ordinary) structure constant can be regarded as counting a certain number of solutions, it is known to be a positive integer. It also has other interpretations in representation theory and combinatorics of symmetric functions (see* [35]*).*

Question 2.10. *Give a combinatorial algorithm for the structure constant.*

A lot of partial answers are known so far. For example:

- Classical Pieri, Monk and Chevalley rules (see [20])
- Littlewood-Richardson rule for $H^*(Gr(n, n+m))$ [34]
- Knutson and Tao's Puzzle rule for $H_T^*(Gr(n, n+m))$ [28]
- Coskun's formula for $H^*(GL_r(\mathbb{C})/B)$ [14] and (combined with [12]) for $QH^*(Gr(n, n+m))$.

Note that since the projection $p : G/B \to G/P$ induces an injection on the cohomology rings (and the Chow rings)

$$\begin{array}{ccc} H^*(G/P; \mathbb{Z}) & \xrightarrow{p^*} & H^*(G/B; \mathbb{Z}) \\ Z_w & \mapsto & Z_w, \end{array}$$

the problem for G/P is a sub-problem for G/B. This is also true for other cohomology theories. So from now on, we only take up the case of full flag varieties G/B.

3. Schubert calculus in Bott tower

There are a lot of ways to attack the problem, such as by investigating intersections geometrically, employing combinatorial technique, and reducing the problem to that of polynomials, which we will pursue here.

Before proceeding further, let us consider a corresponding problem of structure constants in a familiar setting of Bott manifolds.

[3] In cohomology, we use the symbol $\cup$ for product, while $\cap$ in intersection theory.

Definition 3.1. *A* Bott tower *is an iterated $\mathbb{C}P^1$-bundle*

$$\begin{array}{ccccccccc}
\mathbb{C}P^1 & & \mathbb{C}P^1 & & & & \mathbb{C}P^1 & & \\
\downarrow & & \downarrow & & & & \downarrow\cong & & \\
B_n & \xrightarrow{\pi_n} & B_{n-1} & \xrightarrow{\pi_{n-1}} & \cdots & \xrightarrow{\pi_2} & B_1 & \xrightarrow{\pi_1} & B_0 = pt,
\end{array}$$

where each $\mathbb{C}P^1$-bundle structure comes from a projectivization of a line bundle L_i, i.e. $B_n = P(L_i \oplus \mathbb{C})$ with $\mathbb{C}$ the trivial bundle over B_{n-1}.

The highest total space B_n is called a Bott manifold*, which is a $2n$-dimensional toric variety with the canonical action of T^n.*

Let σ_i be the 0-section of π_i. Then a cell decomposition of B_n indexed by $I = (I_1, I_2, \dots, I_n) \in \{0,1\}^n$ is given by

$$B_n = \bigcup_{I \in \{0,1\}^n} b(I_n) \circ b(I_{n-1}) \circ \cdots \circ b(I_1)(pt),$$

where $b(0) = \sigma_i$ and $b(1) = \pi_i^{-1}$. Hence if we put Γ_I as the class (Kronecker) dual to $\overline{b(I_n) \circ b(I_{n-1}) \circ \cdots b(I_1)(pt)}$, we have $H^*(B_n;\mathbb{Z}) = \bigoplus_{I\in\{0,1\}^n} \mathbb{Z}\langle\Gamma_I\rangle$. Note that $\Gamma_I \in H^{2l(I)}(B_n;\mathbb{Z})$, where $l(I)$ is the number of 1's in I. Here we encounter a problem similar to Schubert calculus (and in fact, closely related to it as we'll see in Theorem 3.3), that is, to determine the structure constant c_{IJ}^K for

$$\Gamma_I \cup \Gamma_J = \sum_{K\in\{0,1\}^n} c_{IJ}^K \Gamma_K.$$

This is solved as follows. Let x_i be the class corresponding to $(0, 0, \dots, 1, \dots, 0)$ with the only 1 at i-th entry, i.e. the class coming from the i-th fiber $\mathbb{C}P^1$, and $e_i \in H^*(B_{i-1};\mathbb{Z})$ be the Euler class of L_i. Then we have the following description for the cohomology ring:

$$\textbf{(A)} \qquad H^*(B_n;\mathbb{Z}) \cong \frac{\mathbb{Z}[x_1, \dots, x_n]}{(x_i^2 + e_i x_i)},$$

and an identification **(B)** $\Gamma_I = x^I = x_1^{I_1} x_2^{I_2} \cdots x_n^{I_n}$ in this presentation. We can write $e_i = -\sum_{j<i} a_{ji} x_j, a_{ji} \in \mathbb{Z}$ so that the problem is now reduced to calculation of polynomials.

Example 3.2. *Let B_n be the Bott tower defined*[4] *by* $(a_{ji}) = \begin{pmatrix} 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. *Then,*

$$x_1^2 = 0,\ x_2^2 = x_1x_2,\ x_3^2 = x_1x_3,\ x_4^2 = -2x_1x_4 + x_2x_4 + x_3x_4 \in H^*(B_n;\mathbb{Z})$$

$$\begin{aligned}
(\Gamma_{(1000)} + \Gamma_{(0001)})^4 &= (x_1 + x_4)^4 \\
&= x_4^4 + 4x_1x_4^3 = (-2x_1 + x_2 + x_3)^3 x_4 + 4(x_1x_2x_4^2 + x_1x_3x_4^2) \\
&= (8 - 12 + 6)x_1x_2x_3x_4 = 2x_1x_2x_3x_4 \\
&= 2\Gamma_{(1111)}
\end{aligned}$$

[4] Euler classes determine each step of $\mathbb{C}P^1$-bundles, and hence, the Bott tower.

The number 2 in this example is corresponding to the answer for the quiz in §1 by the following Theorem.

Theorem 3.3 (Bott-Samelson [9], Duan [16], Willems [40])**.** *Let $w = s_{k_1}s_{k_2}\cdots s_{k_{l(w)}} \in W$ be a reduced (minimal length) expression, and $B_{l(w)}$ be the Bott manifold determined by the upper triangular matrix (a_{ji}), where*

$$a_{ji} = \begin{cases} -2\dfrac{\langle \alpha_{k_j}, \alpha_{k_i}\rangle}{|\alpha_{k_j}|^2} & (j < i) \\ 0 & (j \geq i). \end{cases}$$

For $I \in \{0,1\}^{l(w)}$, $w^I \in W$ is defined to be $s_{k_1}^{I_1}s_{k_2}^{I_2}\cdots s_{k_{l(w)}}^{I_{l(w)}}$. Then

$$\sum_{w^I=u}\Gamma_I \cup \sum_{w^J=v}\Gamma_J = c_{uv}^w\Gamma_{(11\cdots 1)},$$

where c_{uv}^w is the structure constant[5] *for $H^*(G/B;\mathbb{Z})$.*

Skech of proof. See either [9], [16], or [40] for detail.

$B_{l(w)}$ is geometrically constructed as $P_{k_1}\times_B P_{k_2}\times_B\cdots\times_B P_{k_{l(w)}}/B$, where P_i is the minimal parabolic sub-group corresponding to α_i so that $P_i/B \cong \mathbb{C}P^1$. Then the multiplication map $\psi_w : B_{l(w)} \to G/B$ induces in cohomology $\psi_w^*(Z_u) = \sum_{w^I=u}\Gamma_I$ and hence

$$\sum_{w^I=u}\Gamma_I \cup \sum_{w^J=v}\Gamma_J = \psi_w^*(Z_uZ_v) = \psi_w^*\left(\sum_{l(w')=l(u)+l(v)} c_{uv}^{w'}Z_{w'}\right) = \sum_{l(w')=l(u)+l(v)} c_{uv}^{w'}\psi_w^*(Z_{w'}) = c_{uv}^w\Gamma_{(11\cdots 1)}.$$

□

Since we know how to compute the LHS, we can obtain the structure constant only from the information of the root system of G.

Example 3.4. *Let $G = GL_4(\mathbb{C})$ and $w = s_2s_3s_1s_2 \in S_4$. Since $\langle\alpha_i,\alpha_j\rangle = \begin{cases} 0 & (|i-j|>2) \\ -1 & (|i-j|=1), \\ 2 & (i=j) \end{cases}$ we have $(a_{ji}) = \begin{pmatrix} 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, as in Example 3.2. If we put $u = s_2$, then $u = w^I$ iff $I =$* $(1000), (0001)$. *So the coefficient of Z_w in the expansion Z_u^4 is calculated by*

$$\left(\sum_{w^I=u}\Gamma_I\right)^4 = (x_1+x_4)^4 = 2x_1x_2x_3x_4 = 2\Gamma_{(11\cdots 1)}.$$

Theorem 3.3 gives an algorithm for the structure constant in a uniform way for all Lie types. However, as we saw in Example 3.2, it is not *positive*, i.e. it contains a lot of cancellation on the way of computation. A positive formula is yet to be found.

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[5] Replacing the relations in (A) $x_i^2 = -e_ix_i$ by $x_i^2 = \alpha_{k_i}x_i - e_ix_i$, we have a formula for the equivariant cohomology.

4. Schubert polynomials

As we saw in the previous section, the problem of structure constant can be solved by reducing it to that for polynomials. The key facts which enables us to do the calculation in Bott towers are (A) a polynomial description for the cohomology ring and (B) polynomial representatives for the basis classes Γ_I. So for the flag varieties, the first step should be to give a polynomial description for the cohomology ring and second is to find a polynomial representative for a Schubert class.

Fortunately enough, a handy presentation for $H^*(G/B)$ is known for more than fifty years, but we have to be a little careful for its coefficients. Let R be a ring in which all the *torsion primes* of G invertible, i.e. the primes such that $H_*(G;\mathbb{Z})$ has p-torsion are invertible.

G	$GL_n(\mathbb{C})$	$SO_{2n+1}(\mathbb{C})$	$Sp_{2n}(\mathbb{C})$	$SO_{2n}(\mathbb{C})$
K/T	$U(n)/T$	$SO(2n+1)/T$	$Sp(n)/T$	$SO(2n)/T$
torsion primes	$\emptyset$	2	ϕ	2

G	$G_2^{\mathbb{C}}$	$F_4^{\mathbb{C}}$	$E_6^{\mathbb{C}}$	$E_7^{\mathbb{C}}$	$E_8^{\mathbb{C}}$
K/T	G_2/T	F_4/T	E_6/T	E_7/T	E_8/T
torsion primes	2	2, 3	2, 3	2, 3	2, 3, 5

Table 1. Torsion primes (see [7])

Theorem 4.1 ([8]).

$$H^*(G/P;R) \cong \frac{H^*(BT;R)^{W_P}}{(H^+(BT;R)^W)}.$$

In particular, the cohomology of a flag variety is isomorphic to the coinvariant algebra

$$H^*(G/B;R) \cong \frac{R[x_1,\dots,x_r]}{(R^+[x_1,\dots,x_r]^W)},$$

which we will denote by $R_W[x]$.

Example 4.2. *For* $G = GL_{n+m}(\mathbb{C})$ *and* $W = S_{n+m}$*, the above Theorem holds for the integral coefficients. We have*

$$H^*(GL_{n+m}(\mathbb{C})/B;\mathbb{Z}) \cong \frac{\mathbb{Z}[x_1,\dots,x_{n+m}]}{(c_1,\dots,c_{n+m})},$$

where c_i *is the i-th elementary symmetric function in the* $n+m$ *variables* $x_1,\dots,x_{n+m}$. *Furthermore, for* $Gr(n,n+m) = GL_{n+m}(\mathbb{C})/P_n$,

$$H^*(Gr(n,n+m);\mathbb{Z}) \cong \frac{\mathbb{Z}[c'_1,\dots,c'_n,c''_1,\dots,c''_m]}{(c_1,\dots,c_{n+m})} = H^*(GL_{n+m}(\mathbb{C})/B;\mathbb{Z})^{S_n\times S_m},$$

where c'_i *and* c''_i *are the elementary symmetric functions respectively in* $x_1,\dots,x_n$ *and in* $x_{n+1},\dots,x_{n+m}$.

Geometrically, x_i *can be considered as the first Chern class for the canonical line bundle over* $Fl_{n+m} = GL_{n+m}(\mathbb{C})/B$ *with the fiber* V_i/V_{i-1} *and hence* c_i *is the i-th Chern class for the trivial bundle with fiber* $\bigoplus_i V_i/V_{i-1} = \mathbb{C}^n$. c'_i *and* c''_i *are the i-th Chern classes for the bundle over* $Gr(n,n+m)$ *with fiber* V_n *and* $V_n^\perp$ *respectively.*

Next thing to do is to find a representative for a Schubert class. Such a polynomial representative is called the *Schubert polynomial*. There are choices for Schubert polynomials and several definitions are given so far. For example,

- Schur functions for $H^*(Gr(n, n+m))$
- Lascoux-Schützenberger's original Schubert polynomials for $H^*(GL_n(\mathbb{C})/B)$ [32].
- Q- and $\tilde{Q}$-Schur functions for $H^*(Sp(n)/U(n))$ and $H^*(SO(2n+1)/U(n))$ [37, 31].
- Billey-Haiman [4], Fomin-Kirillov [19], etc. for $H^*(G/B)$, where G is a classical group.
- Lascoux-Schützenberger's original double Schubert polynomials for $H_T^*(GL_n(\mathbb{C})/B)$ [32].
- Ikeda-Mihalcea-Naruse [25], Kresch-Tamvakis [29] for $H_T^*(G/B)$, where G is a classical group.
- Grothendieck polynomials for $K(GL_n/B)$ [33].
- Quantum Schubert polynomials for $QH^*(GL_n(\mathbb{C})/B)$ [18, 17].

Lascoux-Schützenberger's Schubert polynomial for type A Lie group is definitive; it has all the desirable combinatorial properties. For other classical types, each definition has both advantages and disadvantages. For exceptional types, no definition is known nor even "What is a desirable property ?"

5. Equivariant cohomology of flag varieties

As is often the case with manifolds carrying a group symmetry, the geometry of flag varieties become easier to access when we take an additional equivariant structure into account. From now on, we consider the problem in an equivariant setting and provide some machinery to attack it.

A flag variety G/B is equipped with a left T-action induced by the left multiplication. The *Borel construction* according to this action is defined as the following fiber bundle:

$$G/B \to ET \times_T G/B \to BT,$$

where ET is the universal G-bundle (which also serves as the universal T-bundle) and $ET \times_T G/B = \{[e, gB] \mid e \in ET, gB \in G/B, [e, gB] = [te, tgB], \forall t \in T\}$. The ordinary cohomology of $ET \times_T G/B$ is called the equivariant cohomology of G/B and denoted by $H_T^*(G/B; \mathbb{Z})$. $H_T^*(G/B; \mathbb{Z})$ is a $H_T^*(pt; \mathbb{Z}) = H^*(BT; \mathbb{Z}) = \mathbb{Z}[t_1, \dots, t_r]$ algebra induced by the equivariant map $G/B \to pt$.

Since Schubert varieties are T-stable sub-varieties, they also form a basis for the equivariant cohomology:

Theorem 5.1 (Basis Theorem). *$H_T^*(G/B; \mathbb{Z})$ is a free $\mathbb{Z}[t_1, \dots, t_r]$-module generated by Schubert classes, i.e.*

$$H_T^*(G/B; \mathbb{Z}) \cong \bigoplus_{w \in W} \mathbb{Z}[t_1, \dots, t_r]\langle Z_w \rangle.$$

Here again, the ring structure according to this distinguished basis is the problem. In other words,

Question 5.2. *Give an algorithm for the structure constants $c_{uv}^w(t) \in \mathbb{Z}[t_1, \dots, t_r]$ for $H_T^*(G/B; \mathbb{Z})$, where*

$$Z_u \cup Z_v = \sum_{w \in W} c_{uv}^w(t) Z_w.$$

By [23]*, they are again known to be "positive," i.e. c^w_{uv} is a polynomial in the simple roots with positive coefficients. Therefore, a positive algorithm is desirable.*

Remark 5.3. *We can recover the ordinary cohomology from the equivariant one by the forgetting homomorphism*

$$H^*_T(G/B;\mathbb{Z}) = H^*(ET \times_T G/B;\mathbb{Z}) \to H^*(G/B;\mathbb{Z})$$

induced by the fiber inclusion of the Borel construction. So the equivariant structure constant reduces to the ordinary one by evaluating at $t_i = 0$ $(1 \le i \le r)$.

5.1. **Polynomial description.** Just as in the case of the ordinary cohomology, a polynomial description of the equivariant cohomology is useful. For $H^*_T(G/P;R)$, we have an analogous result to Theorem 4.1.

Proposition 5.4. *As $H^*(BT;R)$-algebras,*

$$H^*_T(G/P;R) \cong H^*(BT;R) \otimes_{H^*(BG;R)} H^*(BT;R)^{W_P}.$$

In particular, $H^*_T(G/B;R) \cong H^*(BT;R) \otimes_{H^*(BG;R)} H^*(BT;R)$.

Proof. Consider the Eilenberg-Moore spectral sequence (see [36]) for the following pullback

$$\begin{array}{ccccc}
G/P & = & G/P & & \\
\downarrow & & \downarrow & & \\
ET \times_T G/P & \longrightarrow & EG \times_G G/P & = & BP \\
\downarrow & & \downarrow & & \\
BT & \longrightarrow & BG & &
\end{array}$$

with the E_2-term $\mathrm{Tor}_{H^*(BG;R)}(H^*(BP;R), H^*(BT;R))$, converging to $H^*_T(G/P;R) \cong H^*(ET \times_T G/P;R)$. Recall from [8] that $H^*(BG;R) \cong H^*(BT;R)^W$ and $H^*(BP;R) \cong H^*(BT;R)^{W_P}$. Since $H^*(BT;R)$ is free over $H^*(BG;R)$, there are only non-trivial entries in the 0-th column and so $E_2 \cong H^*_T(G/P;R)$ as algebras. Here $E_2 \cong \mathrm{Tor}_{H^*(BG;R)}(H^*(BP;R), H^*(BT;R))$ is just the tensor product $H^*(BP;R) \otimes_{H^*(BG;R)} H^*(BT;R)$. □

We denote the polynomial algebra in t_i's (x_i's) by $R[t]$ (respectively $R[x]$)[6]. Then as $R[t]$-algebras, $H^*(BT;R) \otimes_{H^*(BG;R)} H^*(BT;R) \cong \dfrac{R[t_1,\ldots,t_r] \otimes R[x_1,\ldots,x_r]}{J}$, where J is the ideal generated by $f(t_1,\ldots,t_r) - f(x_1,\ldots,x_r)$ for all positive degree W-invariant polynomials f. We denote it by $R_W[t;x]$. Since $H^*_T(G/B;\mathbb{Z})$ is torsion-free, $H^*_T(G/B;\mathbb{Z})$ can be regarded as a $\mathbb{Z}[t]$-sub-algebra of $H^*_T(G/B;R) \cong R_W[t;x]$. This is the key point in the later discussion.

5.2. **GKM description.** A major advantage of considering Schubert calculus in the torus equivariant setting is the availability of the localization technique.

By the definition of the Weyl group $W = N(T)/T$, we can easily see that the fixed point set of the left torus action on G/B is $\{wB/B \mid w \in W\}$. Since the inclusion $i_w : wB/B \hookrightarrow G/B$ is an equivariant morphism, we have the *localization map*

$$H^*_T(G/B;\mathbb{Z}) \xrightarrow{\bigoplus_{w\in W} i^*_w} \bigoplus_{w\in W} H^*_T(wB/B;\mathbb{Z}) \cong \bigoplus_{w\in W} H^*(BT;\mathbb{Z}).$$

[6] We consider the degree of generators t_i and x_i to be 2 to match the degree of the cohomology ring.

Recall that in the Bruhat decomposition $G/B \cong \coprod_{w\in W} B_{-}wB/B$, each cell $B_{-}wB/B$ is equivariantly contractible to the fixed point wB/B. Hence by the Mayer-Vietoris sequence

$$0 \to H_T^*(G/B;\mathbb{Z}) \to \bigoplus_{w\in W} H_T^*(B_{-}wB/B;Z) \cong \bigoplus_{w\in W} H_T^*(wB/B;\mathbb{Z}) \cong \bigoplus_{w\in W} H^*(BT;\mathbb{Z}),$$

we see that the localization map is injective.

To identify the image of the localization map, we introduce the following oriented graph:

Definition 5.5 ([1, 22])**.** *The GKM graph for a flag variety G/B has a vertex set W the Weyl group of G. There is a labeled oriented edge $v \xrightarrow{\beta} w$ for a positive root β iff $w = s_\beta v$ and $l(w) > l(v)$. We denote $v \leq w$ if there is an oriented path from v to w. This partial order on the Weyl group is called the (left)* weak Bruhat order.

Theorem 5.6 ([1, 22])**.** *The image of the localization map in $\bigoplus_{w\in W} H^*(BT;\mathbb{Z})$ is precisely the list of polynomials $\{h_w \in H^*(BT;\mathbb{Z}) \cong \mathbb{Z}[t] \mid w \in W\}$ which satisfies the following condition (referred to as the* GKM condition*):*

$$h_w - h_v \in \langle\beta\rangle \text{ if } v \xrightarrow{\beta} w.$$

Now we have three descriptions for the equivariant cohomology of G/B:

- Additive description $\bigoplus_{w\in W} \mathbb{Z}[t_1,\dots,t_r]\langle Z_w\rangle$
- Polynomial description $R_W[t;x] = \dfrac{R[t_1,\dots,t_r]\otimes R[x_1,\dots,x_r]}{J}$
- GKM description $\{\{h_w \in \mathbb{Z}[t] \mid w \in W\} \mid \text{satisfying the GKM condition}\}$

In the following sections, we'll investigate their relationship.

6. Left $W \times W$-action on $H_T^*(G/B;\mathbb{Z})$

To investigate the relationship between the three description for $H_T^*(G/B;\mathbb{Z})$, we make use of a right $W \times W$-action on $ET \times_T G/B \simeq ET \times_T K/T$ defined by:

$$\begin{array}{rcl}(ET \times_T K/T)\times(W\times W) & \to & (ET \times_T K/T) \\ [e, gT]\times(w', v) & \mapsto & \left[w'^{-1}e, w'^{-1}gvT\right].\end{array}$$

Note that this action is well-defined because $w \in W = N(T)/T$. For notational convenience, we always use primed letters $w', v', \dots$ for the element of the first factor of $W \times W$ while $w, v, \dots$ for the second factor.

Consider the following pull-back diagram:

$$\begin{array}{ccccccc}
 & & K/T & = & K/T & & \\
 & & \downarrow & & \downarrow & & \\
 & & ET \times_T K/T & \xrightarrow{p_2} & EK \times_K K/T & = & BT \\
 & & \downarrow{\scriptstyle p_1} & & \downarrow & & \\
K/T & \hookrightarrow & BT & \longrightarrow & BK & &
\end{array}$$

Since $p_1([e, gT]) = [e] \in BT, p_2([e, gT]) = [e, gT] = [g^{-1}e, T] = [g^{-1}e] \in ET \times_K K/T \cong BT$, the $W \times W$-action is compatible with the standard **right** action on $BT \times BT$ via (p_1, p_2) :

$ET \times_T K/T \to BT \times BT$. Moreover, since $i_w([pt]) = [e, wT]$ and $w([e, T]) = [e, wT]$, we have $i_w = w \circ i_1$. Hence we have,

Proposition 6.1. *The induced* **left** *$W \times W$-action on $H^*_T(G/B; R)$ is represented on $f(t; x) \in R_W[t; x]$ by $f(w'^{-1}(t); w^{-1}(x))$ and the restriction map i^*_w is represented by*
$i^*_w(f(t; x)) = i^*_1(f(t; w^{-1}(x))) = f(t; w^{-1}(t)) \in R[t]$.

Since $i^*_w(f(t; x)) - i^*_{s_\beta w}(f(t; x)) = f(t; w^{-1}(t)) - f(t; w^{-1} s_\beta(t))$ is divisible by a multiple of β, this partially explains the GKM condition in Theorem 5.6. Theorem 5.6 says much more since it holds with the integral coefficients; a class $f(t; x) \in R_W[t; x] \cong H^*_T(G/B; R)$ is integral iff $f(t; w^{-1}(t)) - f(t; w^{-1} s_\beta(t))$ is divisible by β for all positive roots β.

Corollary 6.2 (cf. [25])**.** *On the GKM description, the induced left $W \times W$-action is represented by*

$$((u', v)h)_w (t) = h_{u'^{-1}wv}(u'^{-1}(t)).$$

Proof. The commutativity of the action and the localization gives the proof; for a class $\{h_w(t)\} \in \bigoplus_{w \in W} H^*(BT; \mathbb{Z})$, choose a representative $f(t; x) \in R_W[t; x]$ such that $i^*_w(f(t; x)) = f(t; w^{-1}(t)) = h_w(t)$. Then from the previous Proposition,

$$\begin{aligned} ((u', v)h)_w (t) &= i^*_w((u', v) f(t; x)) \\ &= i^*_w f(u'^{-1}(t); v^{-1}(x)) \\ &= f(u'^{-1}(t); v^{-1} w^{-1}(t)) \\ &= u' \left(f(t; v^{-1} w^{-1} u'(t)) \right) \\ &= u' \left(h_{u'^{-1}wv}(t) \right) \\ &= h_{u'^{-1}wv}(u'^{-1}(t)) \end{aligned}$$

□

7. Divided difference operators

To handle Schubert calculus combinatorially, a powerful tool called the *divided difference operator* will be defined using the $W \times W$-action of the previous section. Then we can investigate a characterization of the representative for a Schubert class both in the GKM and the polynomial descriptions.

For a simple root α_i, there is the associated minimal parabolic sub-group P_i whose Weyl group is generated by s_i, and $P_i/B \cong \mathbb{C}P^1$.

We consider the following two $\mathbb{C}P^1$-bundles:

$$\begin{array}{ccccc} & & \mathbb{C}P^1 & = & P_i/B \\ & & \downarrow & & \downarrow \\ \mathbb{C}P^1 & \hookrightarrow & ET \times_T G/B & \xrightarrow{p_2} & ET \times_T G/P_i \\ \| & & \downarrow{\scriptstyle p_1} & & \downarrow \\ P_i/B & \hookrightarrow & EP_i \times_{P_i} G/B & \longrightarrow & EP_i \times_{P_i} G/P_i \end{array} ,$$

Then we have two maps:

$$\Delta_i : H_T^*(G/B;\mathbb{Z}) \xrightarrow{(p_2)_*} H_T^{*-2}(G/P_i;\mathbb{Z}) \xrightarrow{p_2^*} H_T^{*-2}(G/B;\mathbb{Z}),$$

$$\delta_i : H_T^*(G/B;\mathbb{Z}) \xrightarrow{(p_1)_*} H_{P_i}^{*-2}(G/B;\mathbb{Z}) \xrightarrow{p_1^*} H_T^{*-2}(G/B;\mathbb{Z}),$$

where $(p_1)_*$ and $(p_2)_*$ are the push-forward maps.

Definition 7.1 (cf. [2, 15]). *The right divided difference operator for $w \in W$ is defined as*

$$\Delta_w = \Delta_{i_1} \circ \cdots \circ \Delta_{i_k} : H_T^*(G/B;\mathbb{Z}) \to H_T^{*-2l(w)}(G/B;\mathbb{Z}).$$

where $w = s_{i_1} \cdots s_{i_k} \in W$ is a reduced expression. Δ_w is independent of the choice of a reduced expression.

The left divided difference operator δ_w is defined similarly and also independent of the choice of a reduced expression.

Proposition 7.2. *Δ_i operates on $f(t;x) \in R_W[t;x]$ as:*

$$\Delta_i f(t;x) = \frac{f(t;x) - f(t;s_i(x))}{-\alpha_i(x)},$$

where $\alpha_i(x) \in R[x] \cong H^*(BT;R)$ is the i-th simple root expressed in the x variable.

Sketch of proof. By Leray-Hirsch Theorem, $H_T^*(G/B;R)$ is a free $H_T^*(G/P_i;R)$-module generated by $\{1, \omega_i\}$, where ω_i is the i-th fundamental weight. Hence any element of $H_T^*(G/B;R)$ can be written as $a + b\omega_i$, where $a, b \in H_T^*(G/P_i;R)$. Since $(p_2)_*(a + b\omega_i) = -b$, we have $\Delta_i(a + b\omega_i) = -b$. On the other hand, $s_i(a + b\omega_i) = a + b(\omega_i - \alpha_i(x))$ and $\alpha_i(x) \in H^0(BT;R) \otimes H^2(BT;R) \subset H_T^2(G/B;R)$, so we have the conclusion. □

Similarly,

$$\delta_i f(t;x) = \frac{f(t;x) - f(s_i(t);x)}{\alpha_i(t)}.$$

Corollary 7.3. *On the GKM description, we have*

$$\Delta_i(h)_w(t) = \frac{h_w(t) - h_{ws_i}(t)}{-\alpha_i(w^{-1}(t))}$$

and

$$\delta_i(h)_w(t) = \frac{h_w(t) - h_{s_i w}(s_i(t))}{\alpha_i(t)}.$$

Proof. Choose $f(t;x) \in R_W[t;x]$ such that $i_w^*(f(t;x)) = f(t;w^{-1}(t)) = h_w(t)$. Then

$$\begin{aligned}\Delta_i(h)_w(t) &= i_w^*(\Delta_i f(t;x)) \\ &= i_w^*\left(\frac{f(t;x) - f(t;s_i(x))}{-\alpha_i(x)}\right) \\ &= \frac{f(t;w^{-1}(t)) - f(t;s_i w^{-1}(t))}{-\alpha_i(w^{-1}(t))} \\ &= \frac{h_w(t) - h_{ws_i}(t)}{-\alpha_i(w^{-1}(t))}.\end{aligned}$$

□

Note that Δ_i is a (degree -2) $R[t]$-module morphism while δ_i is not. This implies that δ_i is peculiar to the equivariant cohomology, while Δ_i can act on the ordinary cohomology.

The following remarkable Proposition reveals a hierarchy of Schubert classes in such a way that it enables us the induction on the Bruhat order of the Weyl group.

Proposition 7.4 (cf. [2, 25])**.** $\Delta_w Z_v = \begin{cases} Z_{vw^{-1}} & (l(vw^{-1}) = l(v) - l(w)) \\ 0 & (\textit{otherwise}). \end{cases}$, *and* $\delta_w Z_v = \begin{cases} Z_{wv} & (l(wv) = l(v) - l(w)) \\ 0 & (\textit{otherwise}). \end{cases}$.

Sketch of proof. For the statement for the right divided difference operator, we only have to show $\Delta_i Z_w = Z_{ws_i}$ when $l(ws_i) = l(w) - 1$. Note that in this case, there is a reduced word for w of the form $s_{j_1} s_{j_2} \cdots s_{j_{l(w)-1}} s_i$. As in the proof of Theorem 3.3, we take $\psi_w : B_{l(w)} \to G/B$. Then

$$\begin{array}{ccc} ET \times_T B_{l(w)} & \xrightarrow{\psi_w} & ET \times_T G/B \\ \downarrow \pi & & \downarrow p_2 \\ ET \times_T B_{l(w)}/P_i & \xrightarrow{\overline{\psi_w}} & ET \times_T G/P_i. \end{array}$$

Since $\pi^*\pi_*([\Gamma_{(11\cdots1)}]) = [\Gamma_{(11\cdots0)}] = \psi_w^*(Z_{ws_i})$, by the commutativity of push-forward map, we have the result.

The statement for the left divided difference operator follows similarly from the equivalence $ET \times_{P_i} B_{l(w)} \cong ET \times_T B_{l(s_{j'_1} s_{j'_2} \cdots s_{j'_{l(w)-1}})}$ for $w = s_i s_{j'_1} s_{j'_2} \cdots s_{j'_{l(w)-1}}$.

□

Proposition 7.5 ([1, 30])**.** *The image under the localization map* $\{h_v = i_v^*(Z_w) \mid v \in W\}$ *of a Schubert class* Z_w *is characterized by the following three conditions:*

(1) h_v *is homogeneous of degree* $2l(w)$*, satisfying the GKM condition.*
(2) $h_v = 0$ *if* $l(v) < l(w)$ *or* ($l(v) = l(w)$ *and* $v \neq w$)*.*
(3) $h_w = \prod_{\exists v, v \xrightarrow{\beta} w} \beta$*.*

Proof. (1) Z_w represents a class in degree $2l(w)$.

(2) Since $\overline{B_- wB/B} = \coprod_{v \geq w} B_- vB/B$, those fixed points which lies in Ω_w are $\{vB/B \mid v \geq w\}$. Hence $i_v^*(Z_w) \neq 0$ only when $v \geq w$.

(3) When $l(ws_i) = l(w) + 1$, by Corollary 7.3 and Proposition 7.4 we have

$$w(\alpha_i) h_w = w(\alpha_i) i_w^*(\Delta_i(Z_{ws_i})) = -i_w^*(Z_{ws_i}) + i_{ws_i}^*(Z_{ws_i}).$$

From (2) above, $i_w^*(Z_{ws_i}) = 0$ and hence $w(\alpha_i) i_w^*(Z_w) = i_{ws_i}^*(Z_{ws_i})$. For $w = s_{i_1} \cdots s_{i_{l(w)}}$, we obtain inductively

$$i_w^*(Z_w) = \prod_{k=1}^{l(w)} s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k} = \prod_{\exists v, v \xrightarrow{\beta} w} \beta.$$

On the other hand, suppose that there are two lists of polynomials $\{h_v\}, \{h'_v\}$ satisfying the two conditions. $h_v - h'_v$ vanishes on all $v, l(v) \leq l(w)$. If $h_u - h'_u \neq 0$ for some u, then by the GKM condition, it should be divisible by $\prod_{\exists u', u' \xrightarrow{\beta} u} \beta$ and so it has a degree at least $2l(u) > 2l(w)$, which contradicts $|h_u - h'_u| = 2l(w)$. □

By Proposition 7.4, Schubert classes are obtained by applying the divided difference operator to a higher Schubert class in the Bruhat order. The *top class* $Z_{w_0} \in H_T^{2l(w_0)}(G/B; \mathbb{Z})$ corresponding to the longest element $w_0 \in W$ produces the other Schubert classes by $Z_w = \Delta_{w^{-1}w_0} Z_{w_0}$.

Corollary 7.6. (1) *The localization image of the top class is given as:*

$$i_v^*(Z_{w_0}) = \begin{cases} \prod_{\beta:\text{positive roots}} \beta & (v = w_0) \\ 0 & (v \neq w_0) \end{cases}.$$

(2) *The localization image of a Schubert class is given by:*

$$i_v^*(Z_w) = c_{w,v}^v.$$

In particular, $i_v^*(Z_w)$ is a polynomial of the simple roots with positive coefficients. (An explicit formula is given in [3].*)*

(3) ***(Newton Interpolation formula)*** $f(t; x) = \sum_{w \in W} \Delta_w(f)(t; t) \cdot Z_w$

Proof. (1) Since w_0 is the longest element, for any positive root β, $l(s_\beta w_0) < l(w_0)$. So the assertion follows from Proposition 7.5 (3).

(2) Since $i_u^*(Z_v Z_w) = i_u^*(Z_v) i_u^*(Z_w) = 0$ unless $u \geq v, w$ by the proof of Proposition 7.5 (2), the product $Z_v Z_w$ should expand as $\sum_{u \geq v,w} c_{v,w}^u Z_u$. Applying localization, we obtain $i_v^*(Z_v Z_w) = \sum_{u \geq v,w} c_{v,w}^u i_v^*(Z_u)$. Again by Proposition 7.5 (2), $i_v^*(Z_u) = 0$ unless $v \geq u$. So $i_v^*(Z_v Z_w) = c_{v,w}^v i_v^*(Z_v)$ and $i_v^*(Z_w) = i_v^*(Z_v Z_w)/i_v^*(Z_v) = c_{v,w}^v$.

(3) Suppose $f(t; x) = \sum_{v \in W} a_v(t) \cdot Z_v$. Then $\Delta_w(f)(t; x) = \sum_{v \in W} a_v(t) \cdot \Delta_w(Z_v)$. Since $i_e^*(Z_v) = \begin{cases} 1 & (v = e) \\ 0 & (v \neq e) \end{cases}$ and $i_e^*(\Delta_w(Z_v)) = 0$ unless $w = v$, we have $\Delta_w(f)(t; t) = i_e^*(\Delta_w(f)) = a_w(t)$.

□

If we find polynomial representatives $\mathfrak{S}_w(t; x) \in R_W[t; x]$ for Z_w, we can calculate the structure constants $c_{u,v}^w = \Delta_w(\mathfrak{S}_u \cdot \mathfrak{S}_v)(t; t)$ by (3) above.

Definition 7.7. *A representative $\mathfrak{S}_w(t; x) \in R_W[t; x]$ (or more precisely its lift to $R[t; x]$) of a Schubert class Z_w is called the double Schubert polynomial for $w \in W$.*

The problem of finding such a polynomial is often referred to as Giambelli problem.

By Proposition 7.5, a polynomial $f(t; x) \in R[t; x]$ of degree $2l(w)$ represents the Schubert class Z_w iff

$$f(t; v^{-1}(t)) = 0 \ (\forall v \neq w, l(v) \leq l(w)), \quad f(t; w^{-1}(t)) = \prod_{\exists v, v \xrightarrow{\beta} w} \beta.$$

On the other hand, by Proposition 7.4 (1), a representative $\mathfrak{S}_w(t; x)$ can be obtained by $\mathfrak{S}_w(t; x) = \Delta_{w^{-1}w_0}(\mathfrak{S}_{w_0}(t; x))$. Thus, a representative $\mathfrak{S}_{w_0}(t; x)$ for the top class Z_{w_0} produces all the others. The top class $\mathfrak{S}_{w_0}(t; x) \in R_W[t; x]$ is characterized by $\mathfrak{S}_{w_0}(t; w^{-1}(t)) = \begin{cases} \prod_{\beta:\text{positive roots}} \beta & (w = w_0) \\ 0 & (w \neq w_0) \end{cases}$, however, there are no known method to produce such a polynomial representative in general.

Example 7.8. *Lascoux and Schützenberger defined in* [32] *the double Schubert polynomials for $GL_r(\mathbb{C})/B$ recursively as follows:*

$$\mathfrak{S}_{w_0}(t;x) = \prod_{i+j<l}(x_i - t_j)$$
$$\mathfrak{S}_w(t;x) = \Delta_{w^{-1}w_0}(\mathfrak{S}_{w_0}(t;x))$$

The localization of the top class satisfies $\mathfrak{S}_{w_0}(t;w^{-1}(t)) = \prod_{i+j<l}(t_{w^{-1}(i)} - t_j) = \begin{cases} \prod_{i>j}(t_i - t_j) & (w = w_0) \\ 0 & (w \neq w_0) \end{cases}$.
It can be verified that for a Grassmann permutation, $\mathfrak{S}_w$ is identified with a double Schur function.

More concretely, when $n = 3$,

$$\mathfrak{S}_{(123)} = 1, \mathfrak{S}_{(312)} = (x_1 - t_2)(x_1 - t_1),$$
$$\mathfrak{S}_{(213)} = x_1 - t_1, \mathfrak{S}_{(132)} = x_2 - t_2 + x_1 - t_1,$$
$$\mathfrak{S}_{(231)} = (x_2 - t_1)(x_1 - t_1), \mathfrak{S}_{(321)} = (x_2 - t_1)(x_1 - t_2)(x_1 - t_1),$$

and by Corollary 7.6 (3), we can calculate for example,

$$Z_{(213)}^2 = (t_1 - x_1)^2 = \sum_{w \in S_3} \Delta_w((x_1 - t_1)^2)(t;t) \cdot Z_w = (t_2 - t_1)Z_{(213)} + Z_{(312)}.$$

Example 7.9. *For G of type B_n, C_n and D_n, Fulton and Pragacz* [21] *give a representative for the top class*[7]. *Recall that the Weyl group for $G = SO_{2n+1}(\mathbb{C}), G = Sp_{2n}(\mathbb{C})$ is the signed permutations of n-letters, and that for $G = SO_{2n}(\mathbb{C})$, it is the signed permutations of n-letters with even number of negative signs.*

Let w_0 be the longest element in W, that is,

$$w_0 = \begin{cases} (-1, -2, \ldots, -n) & (G = SO_{2n+1}(\mathbb{C}), Sp_{2n}(\mathbb{C}), SO_{4n'}(\mathbb{C}))) \\ (1, -2, -3, \ldots, -n) & (G = SO_{4n'+2}(\mathbb{C})) \end{cases}.$$

Then

$$\mathfrak{S}_{w_0}(t;x) = w_0\left(\det(E) \prod_{i>j}(x_i - t_j)\right),$$

where E is an $n \times n$-matrix (e_{ij}) with

$$e_{ij} = \begin{cases} \frac{1}{2}(c_{n+1+j-2i}(x) + c_{n+1+j-2i}(t)) & (G = SO_{2n+1}(\mathbb{C})) \\ c_{n+1+j-2i}(x) + c_{n+1+j-2i}(t) & (G = Sp_{2n}(\mathbb{C})) \\ \frac{1}{2}(c_{n+j-2i}(x) + c_{n+j-2i}(t)) & (G = SO_{2n}(\mathbb{C})) \end{cases}$$

Example 7.10. *For $G_2^{\mathbb{C}}/B$, the Weyl group $W = \langle s_1, s_2 \rangle$ is the dihedral group of order* 12 *and $R = \mathbb{Z}[\frac{1}{2}]$. We can take generators of $R[x] = R[x_1, x_2]$ such that*

$$s_1(x_1) = -x_1, s_1(x_2) = 3x_1 + x_2, s_2(x_1) = x_1 + x_2, s_2(x_2) = -x_2.$$

[7] They consider in the context of degeneracy locus of flag bundles and the formula is a bit different.

Then we find a polynomial representative

$$\mathfrak{S}_{w_0}(t;x) = \frac{1}{2}(x_1+t_1)(x_1-t_1-t_2)(x_1-2t_1-t_2)(x_1+t_1+t_2)(x_1+2t_1+t_2)(x_2+3t_1+t_2).$$

8. Reduction to the ordinary cohomology

Since $H_T^*(G/B;\mathbb{Z})$ is a free $H^*(BT;\mathbb{Z})$-module, the equivariant cohomology recovers all the information of the ordinary cohomology. In fact, the augmentation

$$r_1 : H_T^*(G/B;\mathbb{Z}) \to \frac{H_T^*(G/B;\mathbb{Z})}{(H^+(BT;\mathbb{Z}))} \cong H^*(G/B;\mathbb{Z})$$

gives a map from $H_T^*(G/B;\mathbb{Z})$ to $H^*(G/B;\mathbb{Z})$, which is represented on the polynomial description as

$$r_1 : R_W[t;x] \ni f(t;x) \mapsto f(0;x) \in R_W[x].$$

Since r_1 is compatible with the right divided difference operators, it maps the equivariant Schubert classes to the ordinary ones. In other words, if we know polynomial representatives $\mathfrak{S}_w(t;x)$ for the equivariant Schubert classes, then we obtain representatives for the ordinary Schubert classes by $\mathfrak{S}_w(x) = \mathfrak{S}_w(0;x)$.

On the other hand, we can consider the following map

$$\begin{array}{ccc} H_T^*(G/B;\mathbb{Q}) & \to & H_T^*(G/B;\mathbb{Q})^W \cong H_T^*(G/G;\mathbb{Q}) = H_T^*(pt;\mathbb{Q}) = H^*(BT;\mathbb{Q}) \\ p & \mapsto & \frac{1}{|W|}\sum_{w\in W} w(p) \end{array}$$

On the polynomial description, it is represented as

$$r_2 : \mathbb{Q}_W[t;x] \ni f(t;x) \mapsto \frac{1}{|W|}\sum_{w\in W} f(t;w^{-1}(x)) = \frac{1}{|W|}\sum_{w\in W} f(t;w^{-1}(t)) \in \mathbb{Q}[t].$$

Here $\sum_{w\in W} f(t;w^{-1}(x)) = \sum_{w\in W} f(t;w^{-1}(t))$ in $\mathbb{Q}_W[t;x]$ because $\sum_{w\in W} f(t;w^{-1}(x))$ is W-invariant. It is easily seen that $\Delta_i \circ r_2 = r_2 \circ \delta_i$, and hence $r_2(Z_w)$ represents $Z_{w^{-1}}$ in the ordinary cohomology. By Proposition 6.1, $f(t;w^{-1}(t)) = i_w^*(f)$ so r_2 is equal to the composition

$$H_T^*(G/B;\mathbb{Q}) \xrightarrow{i^*} \bigoplus_{w\in W} H^*(BT;\mathbb{Q}) \xrightarrow{\text{sum}} H^*(BT;\mathbb{Q}).$$

Note that by Corollary 7.6 (2),

$$r_2(\mathfrak{S}_{w^{-1}}) = \frac{1}{|W|}\sum_{v\in W} \mathfrak{S}_{w^{-1}}(t;v^{-1}(t)) = \frac{1}{|W|}\sum_{v\in W} i_v^*(\mathfrak{S}_{w^{-1}}) = \frac{1}{|W|}\sum_{v\in W} c^v_{vw^{-1}}(t) \in \mathbb{Q}[t]$$

is always a **positive** polynomial representative of Z_w in the ordinary cohomology for any representative $\mathfrak{S}_{w^{-1}} \in \mathbb{Q}_W[t;x]$ of $Z_{w^{-1}}$ in the equivariant cohomology.

9. Some open problems

Question 9.1.
- *How to find a polynomial representative of the top Schubert class.*
- *An appropriate characterization of the double Schubert polynomials of exceptional types.*
- *Similar problems for other cohomology theories (e.g. double Grothendieck polynomials in the equivariant K-theory).*

10. Example

We list properties in a fundamental example of $GL_r(\mathbb{C})/B$:

G	$GL_r(\mathbb{C})$
B	upper triangular matrices
K	$U(r)$
T	diagonal matrices
$G/B \cong K/T$	$Fl_r = \{0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_r = \mathbb{C}^r \mid \dim_{\mathbb{C}}(V_i) = i\}$
W	symmetric group S_r
$H^*(BT;\mathbb{Z})$	$\mathbb{Z}[t_1,\ldots,t_r]$
simple roots	$\alpha_i = t_{i+1} - t_i \quad (1 \le i \le r-1)$
simple reflections	$s_i = (i, i+1) \quad (1 \le i \le r-1)$
positive roots	$t_i - t_j \quad (i > j)$
longest element of W	$w_0 = (r, r-1, \ldots, 1)$
$H^*(G/B;\mathbb{Z})$	$\mathbb{Z}_W(t;x) \cong \dfrac{\mathbb{Z}[t_1,\ldots,t_r,x_1,\ldots,x_r]}{(c_1(t)-c_1(x),\ldots,c_r(t)-c_r(x))}$
right divided difference	$\Delta_i(f(t;x)) = \dfrac{f(t_1,\ldots,t_r,x_1,\ldots,x_{i-1},x_{i+1},x_i,x_{i+2},\ldots,x_r)}{x_{i+1}-x_i}$
left divided difference	$\delta_i(f(t;x)) = \dfrac{f(t_1,\ldots,t_{i-1},t_{i+1},t_i,t_{i+2},\ldots,t_r,x_1,\ldots,x_r)}{t_{i+1}-t_i}$
top class in $H_T^*(G/B;\mathbb{Z})$	$\mathfrak{S}_{w_0}(t;x) = \prod_{i+j<r}(x_i - t_j)$
localization $i_w^*(Z_{w_0})$	$\mathfrak{S}_{w_0}(t;w^{-1}(t)) = \prod_{i+j<r}(t_{w^{-1}(i)} - t_j) = \begin{cases} \prod_{i>j}(t_i - t_j) & (w = w_0) \\ 0 & (w \neq w_0) \end{cases}$

type	A_{n-1}	B_n	C_n	D_n
G	$GL_n(\mathbb{C})$	$SO_{2n+1}(\mathbb{C})$	$Sp_{2n}(\mathbb{C})$	$SO_{2n}(\mathbb{C})$
K/T	$U(n)/T$	$SO(2n+1)/T$	$Sp(n)/T$	$SO(2n)/T$
$\dim(G/B)$	$n(n-1)$	$2n^2$	$2n^2$	$2n(n-1)$
$\#W$	$n!$	$2^n n!$	$2^n n!$	$2^{n-1}n!$

type	G_2	F_4	E_6	E_7	E_8
G	$G_2^{\mathbb{C}}$	$F_4^{\mathbb{C}}$	$E_6^{\mathbb{C}}$	$E_7^{\mathbb{C}}$	$E_8^{\mathbb{C}}$
K/T	G_2/T	F_4/T	E_6/T	E_7/T	E_8/T
$\dim(G/B)$	12	48	72	126	240
$\#W$	12	1152	51840	2903040	696729600

11. Further readings

Here is a list for the references:

- A note by Jonah Blasiak (`http://math.berkeley.edu/~hutching/teach/215b-2005/blasiak.pdf`) is a charming invitation for Schubert calculus of the complex Grassmannian.

- Kleiman and Laksov's survey [27] is a classical and definitive introduction, where geometrical aspects are stressed.
- Fulton's book [20] is a comprehensive text for the subject.
- Kumar's book [30] is another comprehensive text. In particular, Chapter XI gives a detailed account for §7 of this note.
- Manivel's book [35] deals with newer materials, mainly from a combinatorial perspective.
- Fulton's lecture note (`http://www.math.washington.edu/~dandersn/eilenberg/`) is the only resource I know which provides a systematic treatment for the equivariant Schubert calculus.
- Brion's lecture note [11] is written from a view point of algebraic geometry, and surveys singularity of Schubert varieties.

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