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<https://hdl.handle.net/2324/4481572>

出版情報 : Progress of Theoretical and Experimental Physics. 2021 (2), pp.023B05-, 2021-02.
Oxford University Press
バージョン :
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Received December 9, 2020; Accepted January 12, 2021; Published January 28, 2021

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The gradient flow bears a close resemblance to the coarse graining, the guiding principle of the renormalization group (RG). In the case of scalar field theory, a precise connection has been made between the gradient flow and the RG flow of the Wilson action in the exact renormalization group (ERG) formalism. By imitating the structure of this connection, we propose an ERG differential equation that preserves manifest gauge invariance in Yang–Mills theory. Our construction in continuum theory can be extended to lattice gauge theory.
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Subject Index B05, B32

1. Introduction

The gradient flow [1–6] is a continuous deformation of a gauge field configuration $A_\mu^a(x)$ along a fictitious time $t \geq 0$. It is given by a gauge-covariant diffusion equation

$$\partial_t B_\mu^a(t, x) = D_\nu G_{\nu\mu}^a(t, x), \quad B_\mu^a(t = 0, x) = A_\mu^a(x), \quad (1.1)$$

where

$$G_{\mu\nu}^a(t, x) \equiv \partial_\mu B_\nu^a(t, x) - \partial_\nu B_\mu^a(t, x) + f^{abc} B_\mu^b(t, x) B_\nu^c(t, x) \quad (1.2)$$

is the field strength of the flowed or diffused field $B_\mu^a(t, x)$, and

$$D_\mu X^a(t, x) \equiv \partial_\mu X^a(t, x) + f^{abc} B_\mu^b(t, x) X^c(t, x) \quad (1.3)$$

is the covariant derivative with respect to $B_\mu^a(t, x)$.¹ The gradient flow bears a close resemblance to the coarse graining along renormalization group (RG) flows [7]. This aspect of the gradient flow has been investigated from various perspectives [6, 8–18]. In this paper we further our understanding of how the gradient flows are related to the RG flows by using the exact renormalization group (ERG) formalism (for reviews of ERG, see for instance Refs. [19–21]).

In scalar field theory, the analogue of Eq. (1.1) would be [22]

$$\partial_t \varphi(t, x) = \partial_\mu \partial_\mu \varphi(t, x), \quad \varphi(t = 0, x) = \phi(x). \quad (1.4)$$

¹ f^{abc} is the structure constant defined from the anti-hermitian generator T^a of the gauge group by $[T^a, T^b] = f^{abc} T^c$.

It is actually possible to make a precise connection between the gradient flow and the flow of a Wilson action under ERG [16] (see also Ref. [18]). In D -dimensional Euclidean space, the ERG differential equation for the Wilson action $S_\tau[\phi]$ (the so-called Wilson–Polchinski equation [23]) reads, in terms of *dimensionless* variables,²

$$\begin{aligned} \frac{\partial}{\partial \tau} e^{S_\tau[\phi]} = \int_p \left(\left[\frac{\Delta(p)}{K(p)} + \frac{D+2}{2} - \frac{\eta_\tau}{2} \right] \phi(p) + p_\mu \frac{\partial}{\partial p_\mu} \phi(p) \right) \frac{\delta}{\delta \phi(p)} \\ + \frac{1}{p^2} \left[2 \frac{\Delta(p)}{K(p)} k(p) + 2p^2 \frac{dk(p)}{dp^2} - \eta_\tau k(p) \right] \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \right) e^{S_\tau[\phi]}, \quad (1.5) \end{aligned}$$

where K and k are cutoff functions satisfying

$$K(p) = \begin{cases} 1 & \text{for } |p| \rightarrow 0, \\ 0 & \text{for } |p| \rightarrow \infty, \end{cases}, \quad k(p) \xrightarrow{|p| \rightarrow 0} 0, \quad (1.6)$$

and

$$\Delta(p) \equiv -2p^2 \frac{dK(p)}{dp^2}. \quad (1.7)$$

The origin of the anomalous dimension η_τ in the above has been elucidated in Ref. [24]. Particularly for $K(p) = e^{-p^2}$, it has been shown [16] that the correlation functions of the diffused field $\varphi(t, x)$, defined by Eq. (1.4), calculated with the “bare” action $S_{\tau=0}[\phi]$ are essentially identical to the correlation functions of the elementary field $\phi(x)$ calculated with the Wilson action $S_\tau[\phi]$; the flow time t in Eq. (1.4) and the scale parameter τ in the ERG equation (1.5) are related by $t = e^{2\tau} - 1$.³ We will review this observation in the next section. The connection between the gradient flow and ERG can naturally explain [16] why the local products of the diffused field remain finite under the wave function renormalization of elementary fields [4,5]: we first obtain the Wilson action $S_\tau[\phi]$ by integrating over field modes whose momenta are higher than a cutoff (corresponding to the parameter τ), and then the correlation functions of the field $\phi(x)$ are obtained by integration of the field-modes with momenta less than the cutoff, and thus are finite.⁴

It is of great interest to find such a connection between the gradient and ERG flows in gauge theory; it would provide a natural understanding of the finiteness of the correlation functions of the diffused gauge field (1.1) in the continuum limit [4] (see also Ref. [25]). The manifest gauge covariance of the gradient flow (1.1) would suggest a manifestly gauge-invariant ERG formulation of gauge theory. It appears quite difficult, however, to make such a direct connection. The gradient flow equation in

² Throughout this paper, we use abbreviations,

$$\int_p \equiv \int \frac{d^D p}{(2\pi)^D}, \quad \delta(p) \equiv (2\pi)^D \delta^{(D)}(p).$$

³ In Ref. [16], a particular choice $k(p) = K(p)(1 - K(p))$ [23] has been made, but this restriction can be relaxed; see below.

⁴ The argument given for scalar field theory in Ref. [16] assumes the same flow time for the diffused fields (because the flow time is identified with the scale parameter in the Wilson action), but it somewhat extends the result of Refs. [4,5] for gauge theory, in that it applies not only to the continuum limit around the Gaussian fixed point but also to that around a non-trivial fixed point such as the Wilson–Fisher fixed point.

gauge theory (1.1) is highly non-linear compared with the flow equation (1.4) in scalar field theory, which is linear and solvable. The argument of Ref. [16] took advantage of this simplicity.

In this paper, we look at the problem from a different perspective. We first derive, on the basis of the result of Ref. [16], a representation of the Wilson action $S_\tau[\phi]$ directly in terms of the diffused field $\varphi(t, x)$ in Eq. (1.4). We can readily generalize this representation to the Yang–Mills theory, simply by replacing $\varphi(t, x)$ by the diffused gauge field $B_\mu^a(t, x)$ in Eq. (1.1). We regard this as a *definition* of the Wilson action. We will argue that our construction of the Wilson action effectively implements an ultraviolet cutoff in $S_\tau[A]$. From the representation of $S_\tau[A]$, we see that $S_\tau[A]$ and $S_{\tau=0}[A]$ give identical partition functions. The corresponding ERG transformation thus preserves the partition function, as is usually required for ERG. We can also see that $S_\tau[A]$ possesses manifest gauge invariance as long as the initial action $S_{\tau=0}[A]$ is gauge invariant; the ERG thus preserves gauge invariance. We then derive an ERG differential equation by taking the τ derivative of $S_\tau[A]$. The resulting ERG equation is written entirely in terms of $S_\tau[A]$, and once this ERG equation is obtained, we may forget about the original representation of $S_\tau[A]$ based on the gradient flow.

This paper is organized as follows. In Sect. 2, we review the argument of Ref. [16] and derive a representation, Eq. (2.13), of the Wilson action in terms of the diffused field; this representation becomes the basis of our construction of the Wilson action $S_\tau[A]$ in Yang–Mills theory in Sect. 3.1. We analyze the gauge invariance of $S_\tau[A]$ in Sect. 3.2; we show that $S_\tau[A]$ possesses manifest gauge invariance as long as the initial action $S_{\tau=0}[A]$ is gauge invariant. This implies that the ERG differential equation, Eq. (3.25), that we derive in Sect. 3.3 preserves gauge invariance. In Sect. 3.4, we solve the ERG equation in the lowest approximation, i.e., in the lowest order in a parameter λ (3.10). This parameter turns out to provide a convenient expansion parameter analogous to the conventional gauge coupling. In Sect. 4, we generalize the construction of the Wilson action in Sect. 3.1 to lattice gauge theory. We conclude the paper in Sect. 5. There is a short appendix to Sect. 3 about the normalization of the gauge field.

In this paper, we only present the basic idea and basic equations for our formulation of Yang–Mills theory; we defer possible applications for future studies.

2. Scalar field theory

As pointed out in Ref. [26], the change of a Wilson action S_τ under a change of the cutoff scale in Eq. (1.5) can be formulated as an equality of *modified* correlation functions. In terms of *dimensionless* variables, Eq. (38) of Ref. [26] with $t \rightarrow 0$, $\Delta t \rightarrow \tau$, and $e^{\Delta t \gamma} \rightarrow Z_\tau^{1/2}$ reads

$$\langle\langle \phi(p_1 e^\tau) \cdots \phi(p_n e^\tau) \rangle\rangle_{S_\tau}^{K,k} = e^{-\tau n(D+2)/2} Z_\tau^{n/2} \langle\langle \phi(p_1) \cdots \phi(p_n) \rangle\rangle_{S_{\tau=0}}^{K,k}. \quad (2.1)$$

The anomalous dimension in Eq. (1.5) and the wave function renormalization factor Z_τ are related by

$$\eta_\tau = \frac{\partial}{\partial \tau} \ln Z_\tau. \quad (2.2)$$

Here, the modified correlation functions are defined by [26]

$$\langle\langle \phi(p_1) \cdots \phi(p_n) \rangle\rangle_S^{K,k} \equiv \prod_{i=1}^n \frac{1}{K(p_i)} \left\langle \exp \left[- \int_p \frac{k(p)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \right] \phi(p_1) \cdots \phi(p_n) \right\rangle_S, \quad (2.3)$$

where the ordinary correlation functions are denoted with single brackets:

$$\langle \phi(p_1) \cdots \phi(p_n) \rangle_S \equiv \int [d\phi] \phi(p_1) \cdots \phi(p_n) e^{S[\phi]}. \quad (2.4)$$

In terms of ordinary correlation functions, Eq. (2.1) reads

$$\begin{aligned} & \left\langle \exp \left[- \int_p \frac{k(p)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} \right] \phi(p_1 e^\tau) \cdots \phi(p_n e^\tau) \right\rangle_{S_\tau} \\ &= e^{-\tau n(D+2)/2} Z_\tau^{n/2} \prod_{i=1}^n \frac{K(p_i e^\tau)}{K(p_i)} \left\langle \exp \left[- \int_p \frac{k(p)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} \right] \phi(p_1) \cdots \phi(p_n) \right\rangle_{S_{\tau=0}}. \end{aligned} \quad (2.5)$$

Now, let us choose the Gaussian

$$K(p) = e^{-p^2} \quad (2.6)$$

as the cutoff function K . We then have

$$\begin{aligned} & \left\langle \exp \left[- \int_p \frac{k(p)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} \right] \phi(p_1 e^\tau) \cdots \phi(p_n e^\tau) \right\rangle_{S_\tau} \\ &= e^{-\tau n(D+2)/2} Z_\tau^{n/2} \left\langle \exp \left[- \int_p \frac{k(p)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} \right] \varphi(t, p_1) \cdots \varphi(t, p_n) \right\rangle_{S_{\tau=0}}, \end{aligned} \quad (2.7)$$

where

$$\varphi(t, p) \equiv e^{-tp^2} \phi(p), \quad t \equiv e^{2\tau} - 1, \quad (2.8)$$

is the diffused scalar field in Eq. (1.4) given in momentum space. In terms of functional integrals, this reads

$$\begin{aligned} & \int [d\phi] \phi(p_1) \cdots \phi(p_n) \exp \left[- \int_p \frac{k(p)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} \right] e^{S_\tau[\phi]} \\ &= e^{-\tau n(D+2)/2} Z_\tau^{n/2} \\ & \times \int [d\phi] \varphi(t, p_1 e^{-\tau}) \cdots \varphi(t, p_n e^{-\tau}) \exp \left[- \int_p \frac{k(p)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} \right] e^{S_{\tau=0}[\phi]}. \end{aligned} \quad (2.9)$$

Using field variables in coordinate space

$$\phi(x) = \int_p e^{ipx} \phi(p), \quad \varphi(t, x) = \int_p e^{ipx} \varphi(t, p), \quad (2.10)$$

we get $\delta/[\delta\phi(p)] = \int d^D x e^{ipx} (\delta/[\delta\phi(x)])$ and $\delta/[\delta\varphi(t, p)] = \int d^D x e^{ipx} (\delta/[\delta\varphi(t, x)])$. Hence, we can rewrite Eq. (2.9) as

$$\begin{aligned} & \int [d\phi] \phi(x_1) \cdots \phi(x_n) \exp \left[- \int d^D x \int d^D y \mathcal{D}(x-y) \frac{1}{2} \frac{\delta^2}{\delta\phi(x)\delta\phi(y)} \right] e^{S_\tau[\phi]} \\ &= e^{\tau n(D-2)/2} Z_\tau^{n/2} \end{aligned}$$

$$\times \int [d\phi] \varphi(t, x_1 e^\tau) \cdots \varphi(t, x_n e^\tau) \exp \left[- \int d^D x \int d^D y \mathcal{D}(x-y) \frac{1}{2} \frac{\delta^2}{\delta \phi(x) \delta \phi(y)} \right] e^{S_{\tau=0}[\phi]}, \quad (2.11)$$

where

$$\mathcal{D}(x) \equiv \int_p e^{ipx} \frac{k(p)}{p^2}. \quad (2.12)$$

This leads to a representation of the Wilson action $S_\tau[\phi]$,

$$\begin{aligned} e^{S_\tau[\phi]} &= \exp \left[\int d^D x \int d^D y \mathcal{D}(x-y) \frac{1}{2} \frac{\delta^2}{\delta \phi(x) \delta \phi(y)} \right] \\ &\times \int [d\phi'] \prod_{x'} \delta \left(\phi(x) - e^{\tau(D-2)/2} Z_\tau^{1/2} \varphi'(t, x' e^\tau) \right) \\ &\times \exp \left[- \int d^D x'' \int d^D y'' \mathcal{D}(x''-y'') \frac{1}{2} \frac{\delta^2}{\delta \phi'(x'') \delta \phi'(y'')} \right] e^{S_{\tau=0}[\phi']}. \end{aligned} \quad (2.13)$$

Note that the field $\varphi'(t, x' e^\tau)$ in the delta function results from diffusion of the integration variable ϕ' by the flow equation (1.4). It is easy to check Eq. (2.13) simply by substituting it into Eq. (2.11). Written with the diffused field in coordinate space, this representation admits straightforward generalization to the other systems whose gradient flow equation may be non-linear in fields. Yang–Mills theory is such an example.⁵ Equation (2.13) is the basis of our construction in the next section.

Before discussing generalization to Yang–Mills theory, let us verify that Eq. (2.13) satisfies the ERG equation (1.5). Recalling $t = e^{2\tau} - 1$ (Eq. (2.8)) and the flow equation (1.4), we find

$$\begin{aligned} &\frac{\partial}{\partial \tau} e^{S_\tau[\phi]} \\ &= \exp \left[\int d^D x \int d^D y \mathcal{D}(x-y) \frac{1}{2} \frac{\delta^2}{\delta \phi(x) \delta \phi(y)} \right] \\ &\times \int [d\phi'] \int d^D x' \left[-\frac{D-2}{2} - \frac{\eta_\tau}{2} - 2\Delta_{x'} - x'_\mu \frac{\partial}{\partial x'_\mu} \right] e^{\tau(D-2)/2} Z_\tau^{1/2} \varphi'(t, x' e^\tau) \\ &\times \frac{\delta}{\delta \phi(x')} \prod_x \delta \left(\phi(x) - e^{\tau(D-2)/2} Z_\tau^{1/2} \varphi'(t, x e^\tau) \right) \\ &\times \exp \left[- \int d^D x \int d^D y \mathcal{D}(x-y) \frac{1}{2} \frac{\delta^2}{\delta \phi'(x) \delta \phi'(y)} \right] e^{S_{\tau=0}[\phi']} \\ &= \exp \left[\int d^D x \int d^D y \mathcal{D}(x-y) \frac{1}{2} \frac{\delta^2}{\delta \phi(x) \delta \phi(y)} \right] \\ &\times \int d^D x' \frac{\delta}{\delta \phi(x')} \left[-2\Delta_{x'} - \frac{D-2}{2} - \frac{\eta_\tau}{2} - x'_\mu \frac{\partial}{\partial x'_\mu} \right] \phi(x') \\ &\times \int [d\phi'] \prod_x \delta \left(\phi(x) - e^{\tau(D-2)/2} Z_\tau^{1/2} \varphi'(t, x e^\tau) \right) \end{aligned}$$

⁵ We can also generalize this to the $O(N)$ non-linear sigma model [27–30].

$$\begin{aligned}
& \times \exp \left[- \int d^D x \int d^D y \mathcal{D}(x-y) \frac{1}{2} \frac{\delta^2}{\delta \phi'(x) \delta \phi'(y)} \right] e^{S_{\tau=0}[\phi']} \\
& = \exp \left[\int d^D x \int d^D y \mathcal{D}(x-y) \frac{1}{2} \frac{\delta^2}{\delta \phi(x) \delta \phi(y)} \right] \\
& \quad \times \int d^D x' \left[-2\Delta_{x'} - \frac{D-2}{2} - \frac{\eta_\tau}{2} - x'_\mu \frac{\partial}{\partial x'_\mu} \right] \phi(x') \cdot \frac{\delta}{\delta \phi(x')} \\
& \quad \times \int [d\phi'] \prod_x \delta \left(\phi(x) - e^{\tau(D-2)/2} Z_\tau^{1/2} \varphi'(t, x e^\tau) \right) \\
& \quad \times \exp \left[- \int d^D x \int d^D y \mathcal{D}(x-y) \frac{1}{2} \frac{\delta^2}{\delta \phi'(x) \delta \phi'(y)} \right] e^{S_{\tau=0}[\phi']}. \tag{2.14}
\end{aligned}$$

The first equality is obvious. In the second equality, we have made the replacement, $e^{\tau(D-2)/2} Z_\tau^{1/2} \varphi'(t, x' e^\tau) \rightarrow \phi(x')$, which is justified in front of the delta function. Then, we have interchanged $\delta/[\delta \phi(x')]$ and $\phi(x')$ neglecting an infinite constant $\frac{\delta}{\delta \phi(x')} \phi(x') = \delta^{(D)}(x=0)$ because this contributes only to the constant term in $S_\tau[\phi]$. Finally, using the relation

$$\begin{aligned}
& \exp \left[\int d^D x \int d^D y \mathcal{D}(x-y) \frac{1}{2} \frac{\delta^2}{\delta \phi(x) \delta \phi(y)} \right] \phi(x') \\
& = \left[\phi(x') + \int d^D x \mathcal{D}(x-x') \frac{\delta}{\delta \phi(x)} \right] \exp \left[\int d^D x \int d^D y \mathcal{D}(x-y) \frac{1}{2} \frac{\delta^2}{\delta \phi(x) \delta \phi(y)} \right], \tag{2.15}
\end{aligned}$$

we obtain an ERG equation

$$\begin{aligned}
& \frac{\partial}{\partial \tau} e^{S_\tau[\phi]} \\
& = \int d^D x' \left(-2\Delta_{x'} - \frac{D-2}{2} - \frac{\eta_\tau}{2} - x'_\mu \frac{\partial}{\partial x'_\mu} \right) \left[\phi(x') + \int d^D x \mathcal{D}(x-x') \frac{\delta}{\delta \phi(x)} \right] \\
& \quad \times \frac{\delta}{\delta \phi(x')} e^{S_\tau[\phi]}. \tag{2.16}
\end{aligned}$$

Here, the derivative with respect to x' does not act on x' in $\delta/[\delta \phi(x')]$. Switching back to momentum space, we get

$$\begin{aligned}
\frac{\partial}{\partial \tau} e^{S_\tau[\phi]} & = \int_p \left\{ \left[\left(2p^2 + \frac{D+2}{2} - \frac{\eta_\tau}{2} \right) \phi(p) + p_\mu \frac{\partial}{\partial p_\mu} \phi(p) \right] \frac{\delta}{\delta \phi(p)} \right. \\
& \quad \left. + \frac{1}{p^2} \left[4p^2 k(p) + 2p^2 \frac{dk(p)}{dp^2} - \eta_\tau k(p) \right] \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \right\} e^{S_\tau[\phi]}. \tag{2.17}
\end{aligned}$$

Since $\Delta(p)$ in Eq. (1.7) is given by $2p^2 e^{-p^2}$ for our choice (2.6), this equation coincides precisely with the ERG equation in momentum space, Eq. (1.5).

3. Yang–Mills theory

3.1. Wilson action

A natural generalization of Eq. (2.13) to Yang–Mills theory is given by

$$e^{S_\tau[A]} = \exp \left[\int d^D x \frac{1}{2} \frac{\delta^2}{\delta A_\mu^a(x) \delta A_\mu^a(x)} \right]$$

$$\begin{aligned}
& \times \int [dA'] \prod_{x', v, b} \delta \left(A_v^b(x') - e^{\tau(D-2)/2} B_v^{b'}(t, x' e^\tau) \right) \\
& \times \exp \left[- \int d^D x'' \frac{1}{2} \frac{\delta^2}{\delta A_{\rho'}^c(x'') \delta A_{\rho'}^c(x'')} \right] e^{S_{\tau=0}[A']}, \quad (3.1)
\end{aligned}$$

where, as in Eq. (2.8), we identify the flow time t and the scale parameter τ by

$$t \equiv e^{2\tau} - 1. \quad (3.2)$$

The field $B_v^{b'}(t, x' e^\tau)$ in the delta function is diffused from the integration variable A' by the flow equation

$$\partial_t B_\mu^a(t, x) = D_\nu G_{\nu\mu}^a(t, x) + \alpha_0 D_\mu \partial_\nu B_\nu^a(t, x), \quad B_\mu^a(t=0, x) = A_\mu^a(x). \quad (3.3)$$

Note that we have added a “gauge-fixing term” with the parameter α_0 [3,4] to the original flow equation (1.1); this term suppresses the gauge degrees of freedom along the diffusion and guarantees the finiteness of gauge non-invariant correlation functions of the diffused gauge field in perturbation theory [4]. This somewhat peculiar addition is due to our tacit assumption of perturbation theory in this section. In fact, we exclude this term in lattice gauge theory discussed in the next section. In transcribing Eq. (2.13) to gauge theory, we have set $Z_\tau = 1$ because the diffused field does not receive wave function renormalization [4]; we will see that this choice is consistent with an effective presence of a cutoff in the Wilson action. We have also adopted $k(p) = p^2$, which yields $\mathcal{D}(x) = \delta^{(D)}(x)$ in Eq. (2.12).

Under a change of the scale parameter τ , Eq. (3.1) preserves the partition function:

$$\begin{aligned}
\int [dA] e^{S_\tau[A]} &= \int [dA] \exp \left[- \int d^D x \frac{1}{2} \frac{\delta^2}{\delta A_\mu^a(x) \delta A_\mu^a(x)} \right] e^{S_\tau[A]} \\
&= \int [dA] \exp \left[- \int d^D x \frac{1}{2} \frac{\delta^2}{\delta A_\mu^a(x) \delta A_\mu^a(x)} \right] e^{S_{\tau=0}[A]} \\
&= \int [dA] e^{S_{\tau=0}[A]}. \quad (3.4)
\end{aligned}$$

The first equality follows from the vanishing of a total derivative $\int [dA] (\delta / [\delta A_\mu^a(x)]) \mathcal{F}[A] = 0$ for any well-behaved functional $\mathcal{F}[A]$; for the second equality, we have used Eq. (3.1). The invariance of the partition function, expected of a Wilson action, remains formal unless the functional integral in the most right-hand side of Eq. (3.4) is regularized. In perturbation theory, at least, we can give a gauge-invariant meaning to the last integral by dimensional regularization. With the lattice transcription of Eq. (3.1) in the next section, the invariance of the partition function can be given a rigorous meaning.

Another important relation that follows immediately from Eq. (3.1) is

$$\begin{aligned}
& \left\langle \exp \left[- \int d^D x \frac{1}{2} \frac{\delta^2}{\delta A_\mu^a(x) \delta A_\mu^a(x)} \right] A_{\mu_1}^{a_1}(x_1) \cdots A_{\mu_n}^{a_n}(x_n) \right\rangle_{S_\tau} \\
&= e^{\tau n(D-2)/2} \left\langle \exp \left[- \int d^D x \frac{1}{2} \frac{\delta^2}{\delta A_\mu^a(x) \delta A_\mu^a(x)} \right] B_{\mu_1}^{a_1}(t, x_1 e^\tau) \cdots B_{\mu_n}^{a_n}(t, x_n e^\tau) \right\rangle_{S_{\tau=0}}. \quad (3.5)
\end{aligned}$$

This is analogous to Eq. (2.7) in scalar field theory. As for the right-hand side, note that the flow equation (3.3) can be written as an integral equation [3,4]:

$$B_\mu^a(t, x) = \int d^D y \left[K_t(x - y)_{\mu\nu} A_\nu^a(y) + \int_0^t ds K_{t-s}(x - y)_{\mu\nu} R_\nu^a(s, y) \right], \quad (3.6)$$

where

$$K_t(x)_{\mu\nu} \equiv \int_p \frac{e^{ipx}}{p^2} \left[(\delta_{\mu\nu} p^2 - p_\mu p_\nu) e^{-tp^2} + p_\mu p_\nu e^{-\alpha_0 t p^2} \right] \quad (3.7)$$

is the integration kernel of a linear diffusion, and

$$R_\mu^a \equiv f^{abc} \left[2B_\nu^b \partial_\nu B_\mu^c - B_\nu^b \partial_\mu B_\nu^c + (\alpha_0 - 1) B_\mu^b \partial_\nu B_\nu^c + f^{cde} B_\nu^b B_\nu^d B_\mu^e \right]. \quad (3.8)$$

Using Eq. (3.6), we can express $\delta B / \delta A$, necessary on the right-hand side of Eq. (3.5), as a power series in B . The right-hand side of Eq. (3.5) is then given by correlation functions of the diffused field B .

We now suppose that the “bare” action $S_{\tau=0}[A]$ contains a gauge coupling g_0 . Setting $g_0 = \mu^\epsilon Z_g(\epsilon)g$, where μ is an arbitrary mass scale and $D = 4 - 2\epsilon$, we take $\epsilon \rightarrow 0$ for a continuum limit.⁶ By a general theorem [4], the right-hand of Eq. (3.5) has a finite limit. Hence, the correlation functions with respect to $S_\tau[A]$ on the left-hand side of Eq. (3.5) are finite in the continuum limit. This suggests that our definition of the Wilson action (3.1) implements effectively an ultraviolet cutoff for the Wilson action.⁷

3.2. Gauge invariance

We next show that $S_\tau[A]$ defined by Eq. (3.1) is invariant under any infinitesimal gauge transformation of the scaled gauge potential

$$\tilde{A}_\mu^a(x) \equiv \lambda A_\mu^a(x), \quad (3.9)$$

where

$$\lambda \equiv e^{-\tau(D-4)/2}. \quad (3.10)$$

The τ -dependent factor λ acts like a coupling constant: An infinitesimal gauge transformation on \tilde{A} is

$$\tilde{A}_\mu^a(x) \longrightarrow \tilde{A}_\mu^a(x) + \partial_\mu^x \omega^a(xe^\tau) + f^{abc} \tilde{A}_\mu^b(x) \omega^c(xe^\tau), \quad (3.11)$$

⁶ Here, $Z_g(\epsilon) = 1 - [g^2/(4\pi)^2](\beta_0/2\epsilon) + O(g^4)$ and $\beta_0 = (11/3)C_A$, where C_A is the Casimir of the adjoint representation, $f^{abc}f^{bcd} = C_A\delta^{ab}$.

⁷ In a lattice transcription of Eq. (3.1) in the next section, the presence of an ultraviolet cutoff in the Wilson action is obvious.

but the corresponding gauge transformation on A is modified by λ as

$$A_\mu^a(x) \longrightarrow A_\mu^a(x) + \lambda^{-1} \partial_\mu^x \omega^a(xe^\tau) + f^{abc} A_\mu^b(x) \omega^c(xe^\tau). \quad (3.12)$$

(See the Appendix for an alternative normalization of A .)

To see the invariance of $S_\tau[A]$, we first note that the first factor in Eq. (3.1),

$$\exp \left[\int d^D x \frac{1}{2} \frac{\delta^2}{\delta A_\mu^a(x) \delta A_\mu^a(x)} \right], \quad (3.13)$$

is invariant under the transformation (3.12) because the functional derivative transforms in the adjoint representation under Eq. (3.12):

$$\frac{\delta}{\delta A_\mu^a(x)} \longrightarrow f^{abc} \frac{\delta}{\delta A_\mu^b(x)} \omega^c(xe^\tau). \quad (3.14)$$

We next examine the argument of the delta function in Eq. (3.1). Under the transformation (3.12), we find (we write x' as x for simplicity)

$$\begin{aligned} & A_\nu^b(x) - e^{\tau(D-2)/2} B_\nu'^b(t, xe^\tau) \\ & \longrightarrow A_\nu^b(x) + \lambda^{-1} \partial_\nu^x \omega^b(xe^\tau) + f^{bcd} A_\nu^c(x) \omega^d(xe^\tau) - e^{\tau(D-2)/2} B_\nu'^b(t, xe^\tau) \\ & = A_\nu^b(x) - e^{\tau(D-2)/2} \left[B_\nu'^b(t, xe^\tau) - e^{-\tau} \partial_\nu^x \omega^b(xe^\tau) - f^{bcd} e^{-\tau(D-2)/2} A_\nu^c(x) \omega^d(xe^\tau) \right] \\ & = A_\nu^b(x) - e^{\tau(D-2)/2} \left[B_\nu'^b(t, xe^\tau) - \partial_\nu \omega^b(xe^\tau) - f^{bcd} B_\nu'^c(t, xe^\tau) \omega^d(xe^\tau) \right] \\ & = A_\nu^b(x) - e^{\tau(D-2)/2} \left[B_\nu'^b(t, xe^\tau) - D_\nu' \omega^b(xe^\tau) \right]. \end{aligned} \quad (3.15)$$

In the third line above, we can replace $e^{-\tau(D-2)/2} A_\nu^c(x)$ by $B_\nu'^c(t, xe^\tau)$ since ω is infinitesimal, and the two are equal when $\omega = 0$. The last line implies that the gauge transformation (3.12) on the external variable A induces a gauge transformation on $B_\nu'^b(t, xe^\tau)$ with the gauge function $-\omega^b(xe^\tau)$:

$$B_\mu'^a(t, x) \longrightarrow B_\mu'^a(t, x) - D_\mu' \omega^a(x). \quad (3.16)$$

In the functional integral (3.1), the integration variable A' and the diffused gauge field B' are related by the flow equation (3.3). We wish to show that there is a gauge transformation on A' that gives the gauge transformed B' , given by Eq. (3.16), as the solution of the diffusion equation (3.3). To show this, let us consider an infinitesimal gauge transformation on the diffused field B that *depends* on the flow time s (we save t for $t = e^{2\tau} - 1$):

$$B_\mu^a(s, x) \longrightarrow B_\mu^a(s, x) - D_\mu \xi^a(s, x). \quad (3.17)$$

This changes the flow equation (3.3) to

$$\partial_s B_\mu^a(s, x) = D_\nu G_{\nu\mu}^a(s, x) + \alpha_0 D_\mu \partial_\nu B_\nu^a(s, x) + D_\mu (\partial_s - \alpha_0 D_\nu \partial_\nu) \xi^a(s, x). \quad (3.18)$$

If we choose ξ as the solution to the linear diffusion equation,

$$(\partial_s - \alpha_0 D_\nu \partial_\nu) \xi^a(s, x) = 0, \quad \xi^a(s = t, x) = \omega^a(x), \quad (3.19)$$

Eq. (3.18) reduces to the original diffusion equation (3.3) (with s replacing t). Note that we must solve Eq. (3.19) *backward* against the flow time; ξ is specified at $s = t$ rather than the usual $s = 0$. Thus, if we gauge-transform the integration variable A' by

$$A'^a_\mu(x) \longrightarrow A'^a_\mu(x) - D'_\mu \xi^a(s=0, x), \quad (3.20)$$

the diffusion equation (3.3) gives the gauge-transformed B' given by Eq. (3.16).

We have shown that the gauge transformation (3.12) on the external variable A induces the ordinary gauge transformation (3.20) on the integration variable A' . Now, the functional measure $[dA']$ in Eq. (3.1) can be and is defined to be gauge invariant (by dimensional regularization, for example). The factor

$$\exp \left[- \int d^D x'' \frac{1}{2} \frac{\delta^2}{\delta A'^c_\rho(x'') \delta A'^c_\rho(x'')} \right] \quad (3.21)$$

is invariant just as the factor (3.13) is. We thus conclude that, if the original “bare” action $S_{\tau=0}[A]$ in Eq. (3.1) is invariant under the gauge transformation, then the Wilson action $S_\tau[A]$ is invariant under the λ -dependent (hence τ -dependent) gauge transformation (3.12). This is how our definition of the Wilson action preserves manifest gauge invariance.⁸

3.3. ERG equation

We now derive an ERG differential equation satisfied by the above Wilson action (3.1). By using Eqs. (3.2) and (3.3), calculations analogous to Eq. (2.14) yield

$$\begin{aligned} & \frac{\partial}{\partial \tau} e^{S_\tau[A]} \\ &= \exp \left[\int d^D x \frac{1}{2} \frac{\delta^2}{\delta A^a_\mu(x) \delta A^a_\mu(x)} \right] \\ & \quad \times \int d^D x' \frac{\delta}{\delta \widetilde{A}^b_v(x')} \left[-2D_\rho \widetilde{F}^b_{\rho v}(x') - 2\alpha_0 \widetilde{D}_v \partial_\rho \widetilde{A}^b_\rho(x') - \left(\frac{D-2}{2} + x'_\rho \partial'_\rho \right) \widetilde{A}^b_v(x') \right] \\ & \quad \times \int [dA'] \prod_{x'', \rho, c} \delta \left(A^c_\rho(x'') - e^{\tau(D-2)/2} B^c_\rho(t, x'') e^\tau \right) \\ & \quad \times \exp \left[- \int d^D x''' \frac{1}{2} \frac{\delta^2}{\delta A'^d_\lambda(x''') \delta A'^d_\lambda(x''')} \right] e^{S_{\tau=0}[A']} \\ &= \exp \left[\lambda^2 \int d^D x \frac{1}{2} \frac{\delta^2}{\delta \widetilde{A}^a_\mu(x) \delta \widetilde{A}^a_\mu(x)} \right] \\ & \quad \times \int d^D x' \frac{\delta}{\delta \widetilde{A}^b_v(x')} \left[-2D_\rho \widetilde{F}^b_{\rho v}(x') - 2\alpha_0 \widetilde{D}_v \partial_\rho \widetilde{A}^b_\rho(x') - \left(\frac{D-2}{2} + x'_\rho \partial'_\rho \right) \widetilde{A}^b_v(x') \right] \end{aligned}$$

⁸ To compute the correlation functions of elementary fields such as Eq. (3.5) in perturbation theory, we need to add a gauge-fixing term to $S_{\tau=0}[A]$, which breaks the gauge invariance. This breaking propagates to $S_\tau[A]$. In lattice gauge theory in the next section, however, such breaking of gauge invariance by gauge fixing is unnecessary.

$$\times \exp \left[-\lambda^2 \int d^D x'' \frac{1}{2} \frac{\delta^2}{\delta \widetilde{A}_\sigma^c(x'') \delta \widetilde{A}_\sigma^c(x'')} \right] e^{S_\tau[A]}, \quad (3.22)$$

where the gauge potential $A_\mu^a(x)$ under the tilde ($\widetilde{}$) is replaced by the rescaled potential, Eq. (3.9).

Using a relation analogous to Eq. (2.15) (with $\delta^{(D)}(x)$ replacing $\mathcal{D}(x)$):

$$\exp \left[\lambda^2 \int d^D x \frac{1}{2} \frac{\delta^2}{\delta \widetilde{A}_\mu^a(x) \delta \widetilde{A}_\mu^a(x)} \right] \widetilde{A}_\nu^b(x') = \widehat{\widetilde{A}}_\nu^b(x') \exp \left[\lambda^2 \int d^D x \frac{1}{2} \frac{\delta^2}{\delta \widetilde{A}_\mu^a(x) \delta \widetilde{A}_\mu^a(x)} \right], \quad (3.23)$$

where we define the hat ($\widehat{}$) by

$$\widehat{\widetilde{A}}_\mu^a(x) \equiv \widetilde{A}_\mu^a(x) + \lambda^2 \frac{\delta}{\delta \widetilde{A}_\mu^a(x)}, \quad (3.24)$$

we can rewrite Eq. (3.22) compactly as

$$\frac{\partial}{\partial \tau} e^{S_\tau[A]} = \int d^D x \frac{\delta}{\delta \widetilde{A}_\mu^a(x)} \left[-2 \widehat{\widetilde{D}}_\nu F_{\nu\mu}^a(x) - 2\alpha_0 \widehat{\widetilde{D}}_\mu \partial_\nu \widetilde{A}_\nu^b(x) - \left(\frac{D-2}{2} + x_\nu \partial_\nu \right) \widehat{\widetilde{A}}_\mu^a(x) \right] e^{S_\tau[A]}. \quad (3.25)$$

Here, the gauge potential $\widetilde{A}_\mu^a(x)$ is replaced by the combination (3.24) if it appears under the hat. This is our ERG equation for Yang–Mills theory.

Note that without the hat, Eq. (3.25) would involve only the first order differentials of S_τ , and our ERG equation would be merely a change of variables. It is the differential operator in the hat (3.24), whose origin is the exponentiated second-order differentials in Eq. (3.22), that introduces higher-order differentials in Eq. (3.25).

Once the ERG equation (3.25) has been obtained, we may forget the original construction (3.1) and the gradient flow behind it. Under the ERG flow, the gauge invariance is preserved in the sense explained in Sect. 3.2.

For completeness, we give a little more explicit form of the ERG equation (3.25):

$$\begin{aligned} & \frac{\partial}{\partial \tau} e^{S_\tau[A]} \\ &= \int d^D x \frac{\delta}{\delta \widetilde{A}_\mu^a(x)} \\ & \times \left\{ -2 \widetilde{D}_\nu \left[\widetilde{F}_{\nu\mu}^a(x) + \lambda^2 \widetilde{D}_\nu \frac{\delta}{\delta \widetilde{A}_\mu^a(x)} - \lambda^2 \widetilde{D}_\mu \frac{\delta}{\delta \widetilde{A}_\nu^a(x)} + \lambda^4 f^{abc} \frac{\delta}{\delta \widetilde{A}_\nu^b(x)} \frac{\delta}{\delta \widetilde{A}_\mu^c(x)} \right] \right. \\ & \quad - 2\lambda^2 f^{abc} \frac{\delta}{\delta \widetilde{A}_\nu^b(x)} \left[\widetilde{F}_{\nu\mu}^c(x) + \lambda^2 \widetilde{D}_\nu \frac{\delta}{\delta \widetilde{A}_\mu^c(x)} - \lambda^2 \widetilde{D}_\mu \frac{\delta}{\delta \widetilde{A}_\nu^c(x)} + \lambda^4 f^{cde} \frac{\delta}{\delta \widetilde{A}_\nu^d(x)} \frac{\delta}{\delta \widetilde{A}_\mu^e(x)} \right] \\ & \quad - 2\alpha_0 \left[\widetilde{D}_\mu \partial_\nu \widetilde{A}_\nu^a(x) + \lambda^2 \partial_\mu \partial_\nu \frac{\delta}{\delta \widetilde{A}_\nu^a(x)} + \lambda^2 f^{abc} \widetilde{A}_\mu^b(x) \partial_\nu \frac{\delta}{\delta \widetilde{A}_\nu^c(x)} \right. \\ & \quad \quad \left. + \lambda^2 f^{abc} \frac{\delta}{\delta \widetilde{A}_\mu^b(x)} \partial_\nu \widetilde{A}_\nu^c(x) + \lambda^4 f^{abc} \frac{\delta}{\delta \widetilde{A}_\mu^b(x)} \partial_\nu \frac{\delta}{\delta \widetilde{A}_\nu^c(x)} \right] \\ & \quad \left. - \left(\frac{D-2}{2} + x_\nu \partial_\nu \right) \left[\widetilde{A}_\mu^a(x) + \lambda^2 \frac{\delta}{\delta \widetilde{A}_\mu^a(x)} \right] \right\} e^{S_\tau[A]}. \quad (3.26) \end{aligned}$$

In deriving this, we have interchanged the order of $\delta/[\delta\tilde{A}_\nu^b(x)]$ and $\tilde{A}_\mu^c(x)$ in the combination $f^{abc}(\delta/[\delta\tilde{A}_\nu^b(x)])\tilde{A}_\mu^c(x)$; this is justified because f^{abc} is anti-symmetric in $b \leftrightarrow c$.

To write a differential equation for S_τ , we multiply e^{-S_τ} from the left of Eq. (3.26) and write covariant derivatives explicitly to obtain

$$\begin{aligned}
& \frac{\partial}{\partial \tau} S_\tau[A] \\
&= e^{-S_\tau[A]} \int d^D x \frac{\delta}{\delta A_\mu^a(x)} \\
& \quad \times \left\{ -2\partial_\nu \left[\partial_\nu A_\mu^a(x) - \partial_\mu A_\nu^a(x) + \partial_\nu \frac{\delta}{\delta A_\mu^a(x)} - \partial_\mu \frac{\delta}{\delta A_\nu^a(x)} \right. \right. \\
& \quad \quad + \lambda f^{abc} A_\nu^b(x) A_\mu^c(x) + \lambda f^{abc} \left[A_\nu^b(x) \frac{\delta}{\delta A_\mu^c(x)} - A_\mu^b(x) \frac{\delta}{\delta A_\nu^c(x)} \right] \\
& \quad \quad \left. + \lambda f^{abc} \frac{\delta}{\delta A_\nu^b(x)} \frac{\delta}{\delta A_\mu^c(x)} \right] \\
& \quad - 2\lambda f^{abc} \left[A_\nu^b(x) + \frac{\delta}{\delta A_\nu^b(x)} \right] \\
& \quad \times \left[\partial_\nu A_\mu^c(x) - \partial_\mu A_\nu^c(x) + \partial_\nu \frac{\delta}{\delta A_\mu^c(x)} - \partial_\mu \frac{\delta}{\delta A_\nu^c(x)} \right. \\
& \quad \quad + \lambda f^{cde} A_\nu^d(x) A_\mu^e(x) + \lambda f^{cde} \left[A_\nu^d(x) \frac{\delta}{\delta A_\mu^e(x)} - A_\mu^d(x) \frac{\delta}{\delta A_\nu^e(x)} \right] \\
& \quad \quad \left. + \lambda f^{cde} \frac{\delta}{\delta A_\nu^d(x)} \frac{\delta}{\delta A_\mu^e(x)} \right] \\
& \quad - 2\alpha_0 \left[\partial_\mu \partial_\nu A_\nu^a(x) + \partial_\mu \partial_\nu \frac{\delta}{\delta A_\nu^a(x)} + \lambda f^{abc} A_\mu^b(x) \partial_\nu A_\nu^c(x) + \lambda f^{abc} A_\mu^b(x) \partial_\nu \frac{\delta}{\delta A_\nu^c(x)} \right. \\
& \quad \quad \left. + \lambda f^{abc} \frac{\delta}{\delta A_\mu^b(x)} \partial_\nu A_\nu^c(x) + \lambda f^{abc} \frac{\delta}{\delta A_\mu^b(x)} \partial_\nu \frac{\delta}{\delta A_\nu^c(x)} \right] \\
& \quad \left. - \left(\frac{D-2}{2} + x_\nu \partial_\nu \right) \left[A_\mu^a(x) + \frac{\delta}{\delta A_\mu^a(x)} \right] \right\} e^{S_\tau[A]}. \tag{3.27}
\end{aligned}$$

Differentiating e^{S_τ} further, we obtain a non-linear ERG equation that involves up to quartic differentials of S_τ :

$$\begin{aligned}
& \frac{\partial}{\partial \tau} S_\tau[A] \\
&= \int d^D x \left[\frac{\delta S_\tau}{\delta A_\mu^a(x)} + \frac{\delta}{\delta A_\mu^a(x)} \right] \\
& \quad \times \left(-2\partial_\nu \left\{ \partial_\nu A_\mu^a(x) - \partial_\mu A_\nu^a(x) + \partial_\nu \frac{\delta S_\tau}{\delta A_\mu^a(x)} - \partial_\mu \frac{\delta S_\tau}{\delta A_\nu^a(x)} \right. \right. \\
& \quad \quad \left. \left. + \lambda f^{abc} A_\nu^b(x) A_\mu^c(x) + \lambda f^{abc} \left[A_\nu^b(x) \frac{\delta S_\tau}{\delta A_\mu^c(x)} - A_\mu^b(x) \frac{\delta S_\tau}{\delta A_\nu^c(x)} \right] \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \lambda f^{abc} \left[\frac{\delta^2 S_\tau}{\delta A_\nu^b(x) \delta A_\mu^c(x)} + \frac{\delta S_\tau}{\delta A_\nu^b(x)} \frac{\delta S_\tau}{\delta A_\mu^c(x)} \right] \Big\} \\
& - 2\lambda f^{abc} \left[A_\nu^b(x) + \frac{\delta S_\tau}{\delta A_\nu^b(x)} + \frac{\delta}{\delta A_\nu^b(x)} \right] \\
& \times \left\{ \partial_\nu A_\mu^c(x) - \partial_\mu A_\nu^c(x) + \partial_\nu \frac{\delta S_\tau}{\delta A_\mu^c(x)} - \partial_\mu \frac{\delta S_\tau}{\delta A_\nu^c(x)} \right. \\
& + \lambda f^{cde} A_\nu^d(x) A_\mu^e(x) + \lambda f^{cde} \left[A_\nu^d(x) \frac{\delta S_\tau}{\delta A_\mu^e(x)} - A_\mu^d(x) \frac{\delta S_\tau}{\delta A_\nu^e(x)} \right] \\
& + \lambda f^{cde} \left[\frac{\delta^2 S_\tau}{\delta A_\nu^d(x) \delta A_\mu^e(x)} + \frac{\delta S_\tau}{\delta A_\nu^d(x)} \frac{\delta S_\tau}{\delta A_\mu^e(x)} \right] \Big\} \\
& - 2\alpha_0 \left\{ \partial_\mu \partial_\nu A_\nu^a(x) + \partial_\mu \partial_\nu \frac{\delta S_\tau}{\delta A_\nu^a(x)} + \lambda f^{abc} A_\mu^b(x) \partial_\nu A_\nu^c(x) + \lambda f^{abc} A_\mu^b(x) \partial_\nu \frac{\delta S_\tau}{\delta A_\nu^c(x)} \right. \\
& + \lambda f^{abc} \left[\frac{\delta S_\tau}{\delta A_\mu^b(x)} + \frac{\delta}{\delta A_\mu^b(x)} \right] \partial_\nu A_\nu^c(x) \\
& + \lambda f^{abc} \left[\frac{\delta S_\tau}{\delta A_\mu^b(x)} \partial_\nu \frac{\delta S_\tau}{\delta A_\nu^c(x)} + \frac{\delta}{\delta A_\mu^b(x)} \partial_\nu \frac{\delta S_\tau}{\delta A_\nu^c(x)} \right] \Big\} \\
& - \left(\frac{D-2}{2} + x_\nu \partial_\nu \right) \left[A_\mu^a(x) + \frac{\delta S_\tau}{\delta A_\mu^a(x)} \right]. \tag{3.28}
\end{aligned}$$

3.4. Approximate solution to $O(\lambda^0)$

From Eq. (3.28), we see that the parameter λ , whose original definition is Eq. (3.10), provides a convenient expansion parameter which organizes terms in the ERG equation. We expand the Wilson action in powers of λ as

$$S_\tau[A] \equiv \sum_{n=2}^{\infty} \lambda^{n-2} \frac{1}{n!} \int d^D x_1 \cdots \int d^D x_n w_{n,\mu_1 \cdots \mu_n}^{a_1 \cdots a_n}(x_1, \dots, x_n) A_{\mu_1}^{a_1}(x_1) \cdots A_{\mu_n}^{a_n}(x_n), \tag{3.29}$$

where $w_n = O(\lambda^0)$. By substituting this into the right-hand side of Eq. (3.28), we obtain terms of the form

$$\sum_{n=2}^{\infty} \lambda^{n-2} \frac{1}{n!} \int d^D x_1 \cdots \int d^D x_n W_{n,\mu_1 \cdots \mu_n}^{a_1 \cdots a_n}(x_1, \dots, x_n) A_{\mu_1}^{a_1}(x_1) \cdots A_{\mu_n}^{a_n}(x_n). \tag{3.30}$$

Therefore, the expansion of the Wilson action in the form (3.29) is consistent with the ERG equation (3.28).

In this paper, we study only the lowest-order $O(\lambda^0)$ terms in some detail, postponing the higher-order calculations for future studies.⁹ We thus set

$$S_\tau[A] = \frac{1}{2} \int d^D x \int d^D y w_{2,\mu\nu}^{ab}(x, y) A_\mu^a(x) A_\nu^b(y). \tag{3.31}$$

⁹ This is the only term for the abelian gauge theory.

Equation (3.28) then gives

$$\begin{aligned} & \frac{\partial}{\partial \tau} \frac{1}{2} w_{2,\mu\nu}^{ab}(x, y) \\ &= -2\partial_\rho \partial_\rho w_{2,\mu\nu}^{ab}(x, y) + (1 - \alpha_0) \left[\partial_\mu \partial_\rho w_{2,\rho\nu}^{ab}(x, y) + \partial_\nu \partial_\rho w_{2,\mu\rho}^{ab}(x, y) \right] \\ &+ \left[\frac{D+2}{2} + \frac{1}{2}(x-y)_\rho \partial_\rho \right] w_{2,\mu\nu}^{ab}(x, y) \\ &+ \int d^D z w_{2,\mu\rho}^{ac}(x, z) \left[\delta_{\rho\sigma} (-2\partial_\lambda^z \partial_\lambda^z + 1) + 2(1 - \alpha_0) \partial_\rho^z \partial_\sigma^z \right] w_{2,\sigma\nu}^{cb}(z, y). \end{aligned} \quad (3.32)$$

In deriving this, we have neglected $\delta^{(D)}(x=0)$ assuming dimensional regularization. Imposing the translational and rotational invariance and global gauge invariance, we can write

$$w_{2,\mu\nu}^{ab}(x, y) = \delta^{ab} \int_p e^{ip(x-y)} [T(p)(p^2 \delta_{\mu\nu} - p_\mu p_\nu) + L(p)p_\mu p_\nu], \quad (3.33)$$

where $T(p)$ and $L(p)$ are functions of p^2 . Equation (3.32) then gives

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial \tau} T &= -p^2 \frac{\partial}{\partial p^2} T + p^2(2p^2 + 1)T^2 + 2p^2 T, \\ \frac{1}{2} \frac{\partial}{\partial \tau} L &= -p^2 \frac{\partial}{\partial p^2} L + p^2(2\alpha_0 p^2 + 1)L^2 + 2\alpha_0 p^2 L. \end{aligned} \quad (3.34)$$

The general solution is given by

$$T(\tau, p) = -\frac{1}{C(pe^{-\tau})e^{-2p^2} + p^2}, \quad L(\tau, p) = -\frac{1}{D(pe^{-\tau})e^{-2\alpha_0 p^2} + p^2}, \quad (3.35)$$

where $C(p)$ and $D(p)$ are arbitrary functions of p^2 . Locality demands that $C(p)$ and $D(p)$ can be expanded in powers of p^2 at $p=0$:

$$C(p) = C_0 + C_1 p^2 + \frac{1}{2} C_2 (p^2)^2 + \cdots, \quad D(p) = D_0 + D_1 p^2 + \frac{1}{2} D_2 (p^2)^2 + \cdots. \quad (3.36)$$

Unitarity demands $C_0 > 0$ and $D_0 > 0$.

As $\tau \rightarrow +\infty$, the action $S_\tau[A]$ approaches an infrared fixed point $S^*[A]$, corresponding to constants C_0 and D_0 :

$$T^*(p) = -\frac{1}{C_0 e^{-2p^2} + p^2}, \quad L^*(p) = -\frac{1}{D_0 e^{-2\alpha_0 p^2} + p^2}. \quad (3.37)$$

Since $C_0 > 0$ and $D_0 > 0$ are arbitrary, their variations give marginal operators:

$$\delta T(p) = \frac{\delta C_0 e^{-2p^2}}{(C_0 e^{-2p^2} + p^2)^2}, \quad \delta L(p) = \frac{\delta D_0 e^{-2\alpha_0 p^2}}{(D_0 e^{-2\alpha_0 p^2} + p^2)^2}. \quad (3.38)$$

It can be seen that these correspond to the change of normalization of the gauge field A (see the Appendix).¹⁰ Infinitesimal C_n and D_n , on the other hand, give

$$\delta T(\tau, p) \equiv T(\tau, p) - T^*(p) \simeq \frac{C_n (p^2 e^{-2\tau})^n e^{-2p^2}}{(C_0 e^{-2p^2} + p^2)^2},$$

¹⁰ δD_0 corresponds to an infinitesimal change of the gauge-fixing parameter.

$$\delta L(\tau, p) \equiv L(\tau, p) - L^*(p) \simeq \frac{D_n(p^2 e^{-2\tau})^n e^{-2\alpha_0 p^2}}{(D_0 e^{-2\alpha_0 p^2} + p^2)^2}, \quad (3.39)$$

where $n = 1, 2, \dots$, which correspond to irrelevant operators at the fixed point.

If we make the particular choice of $C_0 = 1$ and $D_0 = \infty$ in Eq. (3.36), the fixed-point action becomes transverse:

$$S_\tau^*[A] = -\frac{1}{2} \int d^D x \int d^D y \int_p e^{ip(x-y)} \frac{1}{e^{-2p^2} + p^2} (p^2 \delta_{\mu\nu} - p_\mu p_\nu) A_\mu^a(x) A_\nu^a(y), \quad (3.40)$$

and the marginal operator at the fixed point is given by

$$\mathcal{O}_0 = \int d^D x \int d^D y \int_p e^{ip(x-y)} \frac{e^{-2p^2}}{(e^{-2p^2} + p^2)^2} (p^2 \delta_{\mu\nu} - p_\mu p_\nu) A_\mu^a(x) A_\nu^a(y). \quad (3.41)$$

It is important to pursue the above analysis to higher orders in λ to see how the ordinary beta function arises in our formalism.

4. Lattice gauge theory

In the previous section, we have constructed a gauge-invariant Wilson action and its associated ERG equation for a generic Yang–Mills theory in continuum \mathbb{R}^4 . We now tailor the construction for lattice gauge theory.¹¹ For simplicity, we consider an infinite volume lattice \mathbb{Z}^4 . The discrete coordinates on \mathbb{Z}^4 render our ERG transformation discrete. This discreteness is introduced through “block-spins.” Let us pick a fixed “block-spin” factor b from one of the integers 2, 3, \dots . We then define a “block-spin” link variable by

$$\mathcal{U}(x, \mu) \equiv U(x, \mu) U(x + \hat{\mu}, \mu) \cdots U(x + (b-1)\hat{\mu}, \mu), \quad x \in b\mathbb{Z}^4, \quad (4.1)$$

where $U(x, \mu)$ is a conventional link variable on the \mathbb{Z}^4 lattice; here, $\hat{\mu}$ denotes the unit vector in the μ direction. This $\mathcal{U}(x, \mu)$ is regarded as a link variable on the coarse lattice $b\mathbb{Z}^4$ scaled by the factor b .

We then divide the range of the scale factor τ , originally continuous in $0 \leq \tau < \infty$, into the contiguous intervals

$$n\Delta\tau < \tau \leq (n+1)\Delta\tau, \quad n = 0, 1, 2, \dots, \quad (4.2)$$

where

$$\Delta\tau \equiv \ln b. \quad (4.3)$$

The n th interval corresponds to the scaling of x by a factor between b^n and b^{n+1} . Multiplying a lattice coordinate $x \in \mathbb{Z}^4$ by $e^{\Delta\tau} = b$ gives the coordinate bx on the coarse lattice $b\mathbb{Z}^4$.

Now, we consider a continuous change of the Wilson action within one of the intervals in Eq. (4.2). A natural extension of Eq. (3.1) for the interval $\tau = (n\Delta\tau, (n+1)\Delta\tau]$ would be the discrete

¹¹ Many versions of the renormalization group transformation have been proposed for lattice gauge theory. We cite Refs. [31,32] as the pioneering works. Some of the more recent works are Refs. [33,34].

transformation from S_n to S_{n+1} , given by¹²

$$e^{S_{n+1}[U]} = \exp \left(\sum_{x,\mu,a} \frac{1}{2} \partial_{x,\mu}^a \partial_{x,\mu}^a \right) \int [dU'] \prod_{x',\nu} \delta(U(x',\nu) - W'_{\Delta\tau}(bx',\nu)) \\ \times \exp \left(- \sum_{x'',\rho,b} \frac{1}{2} \partial_{x'',\rho}^b \partial_{x'',\rho}^b \right) e^{S_n[U']}. \quad (4.4)$$

This needs a fair amount of explanation, which we give below.

First, $\partial_{x,\mu}^a$ is a link differential operator defined by (see also Appendix A of Ref. [3])

$$\partial_{x,\mu}^a \mathcal{F}[U] \equiv \frac{d}{ds} \mathcal{F}[e^{sX} U] \Big|_{s=0}, \quad X(y,\nu) = \begin{cases} T^a & \text{if } (y,\nu) = (x,\mu), \\ 0 & \text{otherwise,} \end{cases} \quad (4.5)$$

where T^a denotes a (anti-hermitian) generator of the gauge group. The exponentiated link differential operator in Eq. (4.4) is an analogue of the exponentiated functional differential operator in Eq. (3.1).

Secondly, $W'_\tau(bx',\nu)$ in Eq. (4.4) is the solution of the lattice flow equation [2,3] on the coarse lattice $x \in b\mathbb{Z}^4$:

$$\frac{\partial}{\partial\tau} W'_\tau(x,\mu) = -2\partial_{x,\mu} S_w[W'_\tau] \cdot W'_\tau(x,\mu), \quad (4.6)$$

where $\partial_{x,\mu} \equiv T^a \partial_{x,\mu}^a$. The initial value at $\tau = 0$ is given by the “block-spin” link variable (4.1) constructed from the integration variable U' defined on \mathbb{Z}^4 :

$$W'_{\tau=0}(x,\mu) = \mathcal{U}'(x,\mu) \equiv U'(x,\mu) U'(x + \hat{\mu}, \mu) \cdots U'(x + (b-1)\hat{\mu}, \mu), \quad x \in b\mathbb{Z}^4. \quad (4.7)$$

It is the value of W'_τ at $\tau = \Delta\tau$ that appears in the delta function. A possible choice of $S_w[W]$ is the plaquette action,

$$S_w[W] \equiv \sum_p \text{Re tr}[1 - W(p)], \quad (4.8)$$

where the sum runs over the plaquettes p belonging to the coarse lattice $b\mathbb{Z}^4$, and $W(p)$ is the product of the “block-spin” link variables around p . Note that the lattice flow equation (4.6) is written in terms of the scale factor τ rather than the flow time $t = b^{2n}e^{2\tau} - 1$. We have used $\partial/\partial t = b^{-2n}e^{-2\tau}(\partial/2\partial\tau)$ and absorbed the factor $b^{2n}e^{2\tau}$ into the right-hand side; this prescription is natural because we have rescaled the lattice coordinates by the factor $b^{2n}e^{2\tau}$ compared with $n = 0$. Thanks to this prescription, the ERG transformation (4.4) from S_n to S_{n+1} does not depend on n explicitly.

We obtain the lattice Wilson action $S_{n+1}[U]$ by successive applications of Eq. (4.4) on the “bare” action $S_0[U]$. The preservation of the partition function and the gauge invariance, both demonstrated in Sect. 3 on the basis of perturbation theory, now hold true non-perturbatively, as we explain below.

First, we consider the partition function. If $[dU]$ is the group-invariant Haar measure such that $[d(e^\eta U)] = [dU]$ for infinitesimal Lie algebra elements $\eta_\mu(x)$, we find, for any functional $\mathcal{F}[U]$,

$$\int [dU] \mathcal{F}[U] = \int [d(e^\eta U)] \mathcal{F}[e^\eta U]$$

¹² Note that the formula (3.1) can be used to relate the Wilson actions between two non-zero τ values.

$$\begin{aligned}
&= \int [dU] \mathcal{F}[e^\eta U] \\
&= \int [dU] \left[\mathcal{F}[U] + \sum_x \eta_\mu^a(x) \partial_{x,\mu}^a \mathcal{F}[U] \right].
\end{aligned} \tag{4.9}$$

This implies $\int [dU] \partial_{x,\mu}^a \mathcal{F}[U] = 0$. Using this identity for Eq. (4.4), we obtain

$$\int [dU] e^{S_{n+1}[U]} = \int [dU] e^{S_n[U]}. \tag{4.10}$$

Hence, the partition function is preserved just as in Eq. (3.4).

As for the gauge invariance, we first note that a gauge transformation is given by

$$U(x, \mu) \longrightarrow U^g(x, \mu) \equiv g(x) U(x, \mu) g(x + \hat{\mu})^{-1}, \quad g(x) \equiv e^{\omega(x)}. \tag{4.11}$$

If ω is infinitesimal, the link differential operator transforms in the adjoint representation,

$$(\partial_{x,\mu}^a \mathcal{F}[U])_{U \rightarrow U^g} = \partial_{x,\mu}^a \mathcal{F}[U^g] + f^{abc} \omega^b(x) \partial_{x,\mu}^c \mathcal{F}[U], \tag{4.12}$$

where the link differential operator acts on U^g on the left-hand side, but it acts on U of U^g on the right. This shows that $(\partial_{x,\mu}^a \partial_{x,\mu}^a \mathcal{F}[U])_{U \rightarrow U^g} = \partial_{x,\mu}^a \partial_{x,\mu}^a \mathcal{F}[U^g]$, and in Eq. (4.4) the gauge transformation on U and the first exponentiated link differential operator commute.

The gauge transformation (4.11) acts on the delta function in Eq. (4.4) as (we set $x' \rightarrow x$ for simplicity)

$$\begin{aligned}
&\delta(U(x, \nu) - W'_{\Delta\tau}(bx, \nu)) \\
&\longrightarrow \delta(g(x) U(x, \nu) g(x + \hat{\nu})^{-1} - W'_{\Delta\tau}(bx, \nu)) \\
&= \delta(U(x, \nu) - g(x)^{-1} W'_{\Delta\tau}(bx, \nu) g(x + \hat{\nu})).
\end{aligned} \tag{4.13}$$

This shows that the gauge transformation (4.11) on U induces an inverse gauge transformation $W_{\Delta\tau}^{g^{-1}}$ on $W'_{\Delta\tau}$ defined on the coarse lattice $b\mathbb{Z}^4$. Now, if W'_τ is the solution of the lattice flow equation (4.6) with the initial condition \mathcal{U}' , given by Eq. (4.7), then $W'^{g^{-1}}_\tau$ is the solution with the initial condition $\mathcal{U}'^{g^{-1}}$ as long as g does not depend on τ ; this follows from the property (4.12). Hence, the gauge transformation g on U induces the inverse gauge transformation g^{-1} on the initial condition \mathcal{U}' . To obtain this transformation on $b\mathbb{Z}^4$, we can introduce the following gauge transformation on \mathbb{Z}^4 :

$$U'(x, \mu) \longrightarrow h(x)^{-1} U'(x, \mu) h(x + \hat{\mu}), \quad h(x) = \begin{cases} g(y) & \text{if } x = by \text{ for } y \in \mathbb{Z}^4, \\ 1 & \text{otherwise.} \end{cases} \tag{4.14}$$

This gauge transformation commutes with the second exponentiated link differential operator in Eq. (4.4) and, as long as $S_n[U']$ is gauge invariant, the resulting Wilson action $S_{n+1}[U]$ is also gauge invariant. This completes our argument for the gauge invariance of the lattice ERG transformation.

The structure of our Wilson action defined recursively by Eq. (4.4) resembles the “lattice effective action” that has been advocated and studied in Refs. [8,9]. Our definition is different in two crucial aspects, however: Eq. (4.4) has exponentiated link differential operators, and the lattice points are rescaled in each step of the ERG transformation. As we have emphasized in the previous section, these two are essential ingredients for obtaining an ERG differential equation that is non-linear in the Wilson action and entails scale transformation of space.

Finally, let us derive an ERG differential equation in lattice gauge theory that follows from the definition (4.4) of the Wilson action. For this, we define $S_{n+1}(\tau)[U]$ by

$$e^{S_{n+1}(\tau)[U]} \equiv \exp \left(\sum_{x,\mu,a} \frac{1}{2} \partial_{x,\mu}^a \partial_{x,\mu}^a \right) \int [dU'] \prod_{x',v} \delta(U(x',v) - W'_\tau(bx',v)) \\ \times \exp \left(- \sum_{x'',\rho,b} \frac{1}{2} \partial_{x'',\rho}^b \partial_{x'',\rho}^b \right) e^{S_n[U']}. \quad (4.15)$$

We have introduced a diffusion factor τ so that

$$S_{n+1}(\Delta\tau)[U] = S_{n+1}[U]. \quad (4.16)$$

As $\tau \rightarrow 0+$, $S_{n+1}(\tau)$ reduces essentially to S_n , written for the block-spin link variables \mathcal{U} defined by Eq. (4.7):

$$e^{S_{n+1}(\tau \rightarrow 0+)[U]} \equiv \exp \left(\sum_{x,\mu,a} \frac{1}{2} \partial_{x,\mu}^a \partial_{x,\mu}^a \right) \int [dU'] \prod_{x',v} \delta(U(x',v) - \mathcal{U}'(bx',v)) \\ \times \exp \left(- \sum_{x'',\rho,b} \frac{1}{2} \partial_{x'',\rho}^b \partial_{x'',\rho}^b \right) e^{S_n[U']}. \quad (4.17)$$

The dependence of $S_{n+1}(\tau)$ on the diffusion factor τ is given by the differential equation,

$$\frac{\partial}{\partial \tau} e^{S_{n+1}(\tau)[U]} \\ = \exp \left(\sum_{x,\mu,a} \frac{1}{2} \partial_{x,\mu}^a \partial_{x,\mu}^a \right) \int [dU'] \sum_{y,\sigma,c} (-2) \partial_{y,\sigma}^c S_w[W'_\tau] \cdot \partial_{y,\sigma}^{lc} \prod_{x',v} \delta(U(x',v) - W'_\tau(bx',v)) \\ \times \exp \left(- \sum_{x'',\rho,b} \frac{1}{2} \partial_{x'',\rho}^b \partial_{x'',\rho}^b \right) e^{S_n[U']} \\ = 2 \exp \left(\sum_{x,\mu,a} \frac{1}{2} \partial_{x,\mu}^a \partial_{x,\mu}^a \right) \sum_{y,\sigma,c} \partial_{y,\sigma}^c \left(\partial_{y,\sigma}^c S_w[U] \right) \int [dU'] \prod_{x',v} \delta(U(x',v) - W'_\tau(bx',v)) \\ \times \exp \left(- \sum_{x'',\rho,b} \frac{1}{2} \partial_{x'',\rho}^b \partial_{x'',\rho}^b \right) e^{S_n[U']}. \quad (4.18)$$

For the first equality above, we have used the lattice flow equation (4.6) in evaluating $\frac{\partial}{\partial \tau} \mathcal{F}[W'_\tau] = \sum_{y,\sigma,c} [(\partial/\partial \tau) W'_\tau(y,\sigma) \cdot W'_\tau(y,\sigma)]^c \partial_{y,\sigma}^{lc} \mathcal{F}[W'_\tau]$, which follows from the definition of the link differential operator (4.5). It is understood that the operator $\partial_{y,\sigma}^{lc}$ acts on W'_τ . For the second equality, we have rewritten $\partial_{y,\sigma}^{lc}$ as the derivative on U , $\partial_{y,\sigma}^{lc} \rightarrow -\partial_{y,\sigma}^c$; this identity holds because the link differential operator acts on the delta function as $d/ds \delta(U(x',v) - e^{sT^a} W'_\tau(bx',v)) = d/ds \delta(e^{-sT^a} U(x',v) - W'_\tau(bx',v))$. This link differential operator on U can be put outside to act on the integral over U' . Then, we can replace $\partial_{y,\sigma}^c S_w[W'_\tau]$ by $\partial_{y,\sigma}^c S_w[U]$ thanks to the delta function.

Therefore, from Eq. (4.15), we get an ERG differential equation:

$$\begin{aligned} \frac{\partial}{\partial \tau} e^{S_{n+1}(\tau)[U]} &= \exp \left(\sum_{x,\mu,a} \frac{1}{2} \partial_{x,\mu}^a \partial_{x,\mu}^a \right) \sum_{x',v,b} \partial_{x',v}^b \left[\partial_{x',v}^b S_w[U] \right] \\ &\times \exp \left(- \sum_{x'',\rho,c} \frac{1}{2} \partial_{x'',\rho}^c \partial_{x'',\rho}^c \right) e^{S_{n+1}(\tau)[U]}. \end{aligned} \quad (4.19)$$

By integrating this from $\tau = 0+$ to $\tau = \Delta\tau$, we restore the finite change of the Wilson action in Eq. (4.4).

Thus, our ERG transformation in lattice gauge theory consists of the rescaling of lattice points by Eq. (4.17) and the diffusion from $\tau = 0+$ to $\tau = \Delta\tau$ by Eq. (4.19) (see Eq. (4.16)). As we have shown, this transformation preserves the partition function and manifest gauge-invariance of the Wilson action. It is important to note that neither Eq. (4.17) nor Eq. (4.19) depends explicitly on n . This implies a possibility of finding a fixed point solution, $S_{n+1} = S_n$. The technique in Ref. [2] appears helpful to the study of such questions.

5. Conclusion

Imitating the structure of the Wilson action in scalar field theory, expressed by the field diffused by the flow equation, we have constructed a manifestly gauge-invariant Wilson action and its associated ERG differential equation in Yang–Mills theory. The construction, extended to lattice gauge theory, provides a non-perturbative gauge-invariant Wilson action of Yang–Mills theory. We have presented only the basic idea and basic relations in this paper; we expect many future applications including analytical or numerical searches for non-trivial RG fixed points in gauge theory. We can also expect extensions in various directions, such as inclusion of matter fields and search for a reparametrization-invariant ERG formulation of quantum gravity. It should be also interesting to clarify a possible relation to the other gauge-invariant ERG formulations of gauge theory [35–39].

Acknowledgements

This work was initiated during the 10th International Conference on Exact Renormalization Group 2020 (ERG2020). We would like to thank the organizers and the Yukawa Institute for Theoretical Physics at Kyoto University for support (YITP-W-20-09). This work was partially supported by Japan Society for the Promotion of Science (JSPS) Grant-in-Aid for Scientific Research Grant Numbers JP16H03982 and JP20H01903.

Funding

Open Access funding: SCOAP³.

Appendix A. Normalization of the gauge field

In Sect. 3, we have normalized the gauge field $A_\mu^a(x)$ so that the rescaled field $\widetilde{A}_\mu^a(x) \equiv \lambda A_\mu^a(x)$, defined by Eq. (3.10), has the ordinary gauge transformation (3.11). In fact this is not the only choice of normalization. We can change the normalization of $A_\mu^a(x)$ arbitrarily so that the rescaled field is given by

$$\widetilde{A}_\mu^a(x) = \lambda z(\tau) A_\mu^a(x). \quad (A.1)$$

Let $S_{z,\tau}[A]$ be the Wilson action of this field. We should then obtain

$$\begin{aligned} & z(\tau)^n \left\langle \exp \left[-\frac{1}{2} \int d^D x \frac{\delta^2}{\delta A_\mu^a(x) \delta A_\mu^a(x)} \right] A_{\mu_1}^{a_1}(x_1) \cdots A_{\mu_n}^{a_n}(x_n) \right\rangle_{S_{z,\tau}} \\ &= \left\langle \exp \left[-\frac{1}{2} \int d^D x \frac{\delta^2}{\delta A_\mu^a(x) \delta A_\mu^a(x)} \right] A_{\mu_1}^{a_1}(x_1) \cdots A_{\mu_n}^{a_n}(x_n) \right\rangle_{S_\tau}. \end{aligned} \quad (\text{A.2})$$

This implies [26]

$$e^{S_{z,\tau}[A]} = \exp \left[\frac{1 - 1/z(\tau)^2}{2} \int d^D x \frac{\delta^2}{\delta A_\mu^a(x) \delta A_\mu^a(x)} \right] \exp(S_\tau[z(\tau)A]). \quad (\text{A.3})$$

For

$$z(\tau) = 1 + \epsilon \quad (\text{A.4})$$

where ϵ is infinitesimal, we obtain

$$\begin{aligned} S_{z,\tau}[A] - S_\tau[A] &= \epsilon \int d^D x \left\{ \left[\frac{\delta S_\tau}{\delta A_\mu^a(x)} \frac{\delta S_\tau}{\delta A_\mu^a(x)} + \frac{\delta^2 S_\tau}{\delta A_\mu^a(x) \delta A_\mu^a(x)} \right] + A_\mu^a(x) \frac{\delta S_\tau}{\delta A_\mu^a(x)} \right\} \\ &\equiv -\epsilon \mathcal{N}_\tau[A]. \end{aligned} \quad (\text{A.5})$$

Hence, $S_{z,\tau}$ satisfies the same ERG equation (3.25) as S_τ except with the addition of

$$-\frac{dz(\tau)}{d\tau} \mathcal{N}_\tau[A] \quad (\text{A.6})$$

on the right-hand side. We can interpret $-[dz(\tau)]/d\tau$ as the anomalous dimension of the gauge field.

The marginal operator $\mathcal{O}_0(p)$ (3.41) that we have found at the end of Sect. 3 is in fact the operator \mathcal{N} ; we find

$$\begin{aligned} \mathcal{N}^*[A] &= - \int d^D x \left\{ \left[\frac{\delta S_\tau}{\delta A_\mu^a(x)} \frac{\delta S_\tau}{\delta A_\mu^a(x)} + \frac{\delta^2 S_\tau}{\delta A_\mu^a(x) \delta A_\mu^a(x)} \right] + A_\mu^a(x) \frac{\delta S_\tau}{\delta A_\mu^a(x)} \right\} \\ &= \int d^D x \int d^D y A_\mu^a(x) A_\nu^a(y) \int_p e^{ip(x-y)} (p^2 \delta_{\mu\nu} - p_\mu p_\nu) \left[-\frac{p^2}{(e^{-2p^2} + p^2)^2} + \frac{1}{e^{-2p^2} + p^2} \right] \\ &= \int d^D x \int d^D y A_\mu^a(x) A_\nu^a(y) \int_p e^{ip(x-y)} (p^2 \delta_{\mu\nu} - p_\mu p_\nu) \frac{e^{-2p^2}}{(e^{-2p^2} + p^2)^2} \\ &= \mathcal{O}_0. \end{aligned} \quad (\text{A.7})$$

We believe that the right choice of the anomalous dimension is necessary to obtain a fixed point of the ERG transformation.

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