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Djarot B. Darmadi
Joining and Welding Research Group (JWERG), Mechanical Engineering Department, Brawijaya University

Yuliati, Lilis
Fluid Machines Laboratory, Mechanical Engineering Department, Brawijaya University

Talice, Marco
Marco Talice, Pmsquared Engineering S.r.l.s

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Solutions of Hypergeometric ODE with *b* Equal to *c* for Any *x*

Djarot B. Darmadi[1,*], Lilis Yuliati[2], Marco Talice[3]
[1]Joining and Welding Research Group (JWERG), Mechanical Engineering Department, Brawijaya University, Indonesia
[2]Fluid Machines Laboratory, Mechanical Engineering Department, Brawijaya University, Indonesia
[3]Marco Talice, Pmsquared Engineering S.r.l.s, Cagliary – Italy
*E-mail: b_darmadi_djarot@ub.ac.id

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Abstract: The hypergeometric Ordinary Differential Equation (ODE) has wide application in the Mechanical Engineering field. The solution to the hypergeometric ODE is called a hypergeometric function, or also a hypergeometric series. Not all of the hypergeometric series converges to a simple defined algebraic function. It is a well-known fact that the hypergeometric series F(1, *c*, *c*; x) will converge to the Maclaurin series 1/(1-x). This article investigates the hypergeometric function F(*k*, *c*, *c*; x). It will be proven that the function F(*k*, *c*, *c*; x) will always converge to certain values. We can represent these values on a line of the form $A + m$x, whose coefficients *A* and m are functions of x.

Keywords: hypergeometric ODE; hypergeometric function; hypergeometric series.

1. Introduction

Engineering Mathematics is an expanding discipline that covers a wide range of areas. It's development and growing relevance are motivated by the engineer's need for a theoretical basis[1]. Mohd *et al.* used a numerical method based upon the Lattice Boltzmann approach to describe a three-dimensional free surface flow impacting on a cylindrical obstacle. The results shown a good agreement with the experiments[2]. Kumar *et al.* developed a mathematical model for a Rayleigh waves in thermoelastic medium. The model is based on stress and energy balance. It is found that, as wave number increases, the secular equation's determinant and the velocity value of Rayleigh waves decrease whilst the attenuation coefficient increase[3]. Abouelella *et al*. used mathematical modelling for computing mass balance, mass and heat transfer to describe adsorption of CO_2 in several materials[4]. Chiba used 2D Differential Transform Method to obtain the solution for natural convection between two finite vertical plates[5]. The solution describes the velocity profiles which depend on the temperature of the two plates. Bathavatchalam *et al*. developed a program using C language to analyze humidity and desalination[6]. Specific heat at constant pressure was modelled using polynomial equation which is retrieved from experiment's data. Mishra *et al*. evaluated the temperature distribution of workpiece in electric discharge machining using semi analytical approach[7]. The approach was based on the integral transformation method and numerical evaluation of the integral by Monte Carlo simulation. Chauhan and Khare analyze an ABB IRB 1520 robot movement[8]. The proposed mathematical model written in matrix form was evaluated using the RoboAnalyzer software. It is found that the cycloidal trajectory provides the smallest total time, with a smooth and vibration free movement. Seddiq and Maerefat found the solution for a cross-flow plate heat exchanger[9]. The solution was developed based on energy conservation in an incompressible medium without heat source and viscous dissipation. The obtained results have a good agreement with the experiment results. Muhammad *et al*. simulated the mixing action inside a photo bioreactor[10]. They defined a parameter which is called population balance model to quantitatively evaluate the environment quality for the growth of microalgae. Based on results obtained with the computational fluid dynamic software Ansys/Fluent, it is found that photo bioreactor with baffles improve the mixing action, which in turn improve the environment quality.

Second-order differential equations can describe many of the phenomena encountered in engineering. Depending on their complexity, second order ordinary differential equations may be solved either analytically or via some numerical technique[11]. Two main types of approaches do exist for obtaining the solution to an Ordinary Differential Equation (ODE) using numerical methods, namely the Power Series and the Frobenius methods, used when the ODE contains analytic and non-analytic coefficients, respectively.

Engineering problems such as those encountered in the fields of electrical circuits[12], resonance[13], matrix[14], forensic sciences[15], chemistry[16], risk analysis in generator[17], sampling application[18], Design optimization of Francis turbine[19], delamination of composite[20], thermal analysis of heat sink[21], and stochastic computing[22] can be solved using a hypergeometric differential equation, whose solution is a hypergeometric function. A hypergeometric ODE is solved using the Frobenius Methods since its coefficients in the ODE standard form are not analytic.

Mubeen *et al*. and Li & Dong found the solution for a class of modified hypergeometric ODE called *k*-Hypergeometric ODE[23, 24]. They found solutions for the *k*-Hypergeometric ODE around all its regular singular points. Yilmazer *et al.* found discrete fractional solutions for both homogeneous and nonhomogeneous confluent hypergeometric function[25]. They used the Nabla fractional calculus operator to solve the integration as it is used in the classical methods. The discrete fractional calculus requires a massive computational effort, and the use of a hypergeometric ODE offers an alternative way to obtain the solution. Iskhanyan and Iskhanyan solved the Heun confluent equation using generalized confluent hypergeometric solutions[26]. The Heun equation is widely encountered in contemporary physics research such as atomic and particle physics, theory of black holes, general relativity and cosmology.

Not all solutions of the hypergeometric ODE converge to a simple defined algebraic function. This article will discuss a specific solution to the hypergeometric ODE, which does not converge to a simple algebraic function.

2. The Hypergeometric ODE

In its original form, a hypergeometric ODE can be expressed as:

$$x(1-x)\frac{d^2x}{dy^2} + \{c-(a+b+1)x\}\frac{dy}{dx} - aby \quad (1)$$

Using the Frobenius method to express y and its derivatives, one can write $y = \sum_{m=0}^{\infty} a_m x^{m+r}$; $\frac{dy}{dx} = \sum_{m=0}^{\infty}(m+r)a_m x^{m+r-1}$ and $\frac{d^2y}{dx^2} = \sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-2}$. Substitution of these equations into (1) and grouping the x within terms of same order, results in:

$$\sum_{m=0}^{\infty}\{r^2+(2m+c-1)r+(m^2+cm-m)\}a_m x^{m+r-1} - \sum_{m=0}^{\infty}\{r^2+(2m+a+b)r+(m^2+am+bm+ab)\}a_m x^{m+r} = 0 \quad (2)$$

Equating the coefficient of minimum order of x to zero (i.e., x^{r-1}) brings to what is called an indicial equation:

$$r(r+c-1)r = 0$$

whose solutions are $r = r_1 = 0$ and $r = r_2 = 1 - c$. We are interested in the case of $r = r_1$. Substitution into the (2), brings to:

$$\sum_{m=0}^{\infty}(m^2+cm-m)a_m x^{m-1} - \sum_{m=0}^{\infty}(m^2+am+bm+ab)a_m x^m = 0$$

and substituting s = m-1 for the first series, and m = s for the second series one obtains:

$$\sum_{s=-1}^{\infty}\{(s^2+2s+1)+(cs+c)-(s+1)\}a_{s+1}x^s - \sum_{s=0}^{\infty}(s^2+as+bs+ab)a_s x^s = 0$$

And, since for s = -1 the coefficient of the first series is equal to zero, then:

$$\sum_{s=0}^{\infty}\{(s^2+2s+1)+(cs+c)-(s+1)\}a_{s+1}x^s - \sum_{s=0}^{\infty}(s^2+as+bs+ab)a_s x^s = 0$$

Finally, equating the coefficients of x^s to zero, results in the recursive formula:

$$a_{s+1} = \frac{(a+s)(b+s)}{(c+s)(s+1)}a_s \quad (3)$$

3. Radius of Convergence

The solution in the form of a series: $\sum_{m=0}^{\infty} a_m x^{m+r}$ is only meaningful for practical purposes if it converges. Values of the variable x for which convergence can be obtained are determined by the radius of convergence. The radius of convergence can be calculated using the following expression:

$$R = \frac{1}{\lim\limits_{s\to\infty}\left|\frac{a_{s+1}}{a_s}\right|} = 1\Big/\frac{(a+\infty)(b+\infty)}{(c+\infty)(1+\infty)} = 1\Big/\frac{\infty\,.\infty}{\infty\,.\infty} = 1$$

Thus, the solution of the hypergeometric function (1) only converges for -1 < x < 1.

4. The Hypergeometric Function

Simulating the recursive formula for s = 0, 1, 2, 3, … one obtains the coefficients:

$$a_1 = \frac{ab}{1!c}a_0$$

$$a_2 = \frac{(a+1)ab(b+1)}{2!c(c+1)}a_0$$

$$a_3 = \frac{(a+2)(a+1)ab(b+1)(b+2)}{2!c(c+1)(c+2)}a_0$$

…

Let's take $r = r_1 = 0$. Following Frobenius, the solution is given by: $y = \sum_{m=1}^{\infty} a_m x^{m+r}$. Thus, the sought solution for the hypergeometric ODE is:

$$y = a_0\left(1 + \frac{ab}{1!c}x + \frac{(a+1)ab(b+1)}{2!c(c+1)}x^2 + \frac{(a+2)(a+1)ab(b+1)(b+2)}{2!c(c+1)(c+2)}x^3 + \cdots\right) \quad (4)$$

The terms in parenthesis is called the hypergeometric series or hypergeometric function and denoted as F(*a*, *b*, *c* ; x). Thus,

$$F(a,b,c;x) = 1 + \frac{ab}{1!c}x + \frac{(a+1)ab(b+1)}{2!c(c+1)}x^2 + \frac{(a+2)(a+1)ab(b+1)(b+2)}{2!c(c+1)(c+2)}x^3 + \cdots \quad (5)$$

It should be noted here that the values of the coefficient *c* are generally nor equal to 0 nor are negative integer numbers, which results into non-analytic solutions to equation (5).

5. F(1, 1, 1; x)

The function F(1, 1, 1; x) is a special case, since it converges to a simple algebraic function. The hypergeometric function which is obtained assuming a, b, and c equal to 1 in equation (4) is: $y = 1 + x + x^2 + x^3 + \cdots$. This expression is a Maclaurin series that converges to the simple algebraic function: $y = \frac{1}{1-x}$. It is worth noting that for convergence to be achievable, the value of x should strictly be in the interval -1 < x < 1. Figure 1 plots the function F(1, 1, 1; x), where the abscissa represents the number of terms used in the hypergeometric function of equation (5).

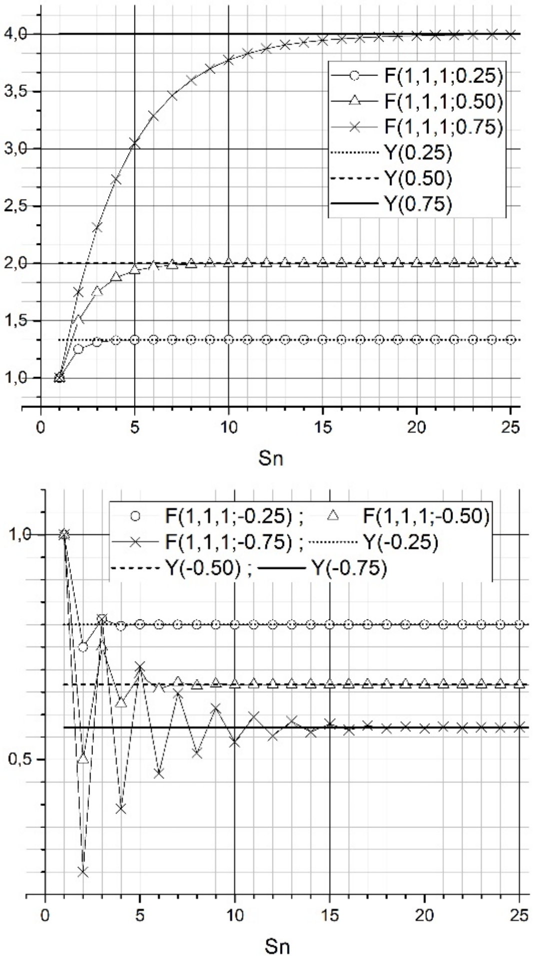

Figure 1. The hypergeometric function for a=b=c=1 and varied x.

Furthermore, evaluation of equation (5) shows that F(*c*, 1, *c*; x) and F(1, *c*, *c*; x) also converge to $y = \frac{1}{1-x}$ because terms *c* at the denominator cancel terms a or b at the numerator. Finally, Figure 1 above, demonstrates that the value to which the hypergeometric series converges depend on the values of x.

6. F(*k*, *c*, *c*; x) or F(*c*, *k*, *c*; x)

The discussion about the hypergeometric series presented above can be found in many books of Engineering Mathematics. We want to make now a step further and investigate the case in which values for coefficients a or b are set to a generic constant *k* instead than to one, resulting in the two identical functions F(*k*, *c*, *c*; x) and F(*c*, *k*, *c*; x), respectively. Since it has already been observed that the values to which the hypergeometric function will converge depend on x, the fact that now coefficients a and b are no longer set to one, but to an arbitrary value *k*, results in a whole lot of possible functions (or curves in a x-y line diagram).

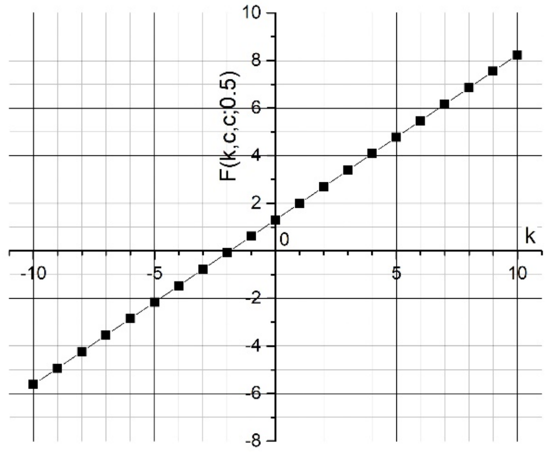

Figure 2. The values of F(*k*, *c*, *c*; 0.5).

For sake of simplicity, we evaluate first the hypergeometric series obtained by taking x = 0.5, which are F(*k*, *c*, *c*; 0.5) and F(*c*, *k*, *c*; 0.5), and which both result in the same values. If equation (5) is applied to simulate the series behavior, it is found that the hypergeometric series converge to the values shown in Figure 2. The convergence was evaluated by using up to 500 terms in equation (5), which is considered enough to reach a satisfactory convergence level. As shown by Figure 2, it is interesting to observe that the hypergeometric final values will fall on a line on the *k*-F plane for changing values of the free parameter *k*. The implication of this findings is that once the values of the function F is known for two values of the parameter k, any other value of F can be obtained by linear interpolation or extrapolation of the known values. All of the computed F's values will lie on the same line.

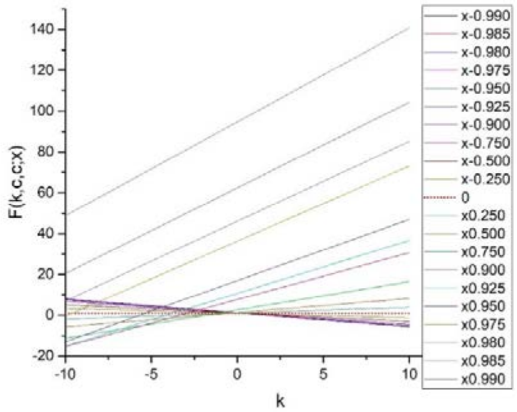

Figure 3. The values of F(*k*, *c*, *c*; x) as a function of *k*.

As it has been previously mentioned, the values to which the hypergeometric function converges depends on the value of x. Figure 2 presents the evaluation of F made for x equal to 0.5. If the same method is applied to other values of x (under the constrain of -1<x<1) the results are those presented in Figure 3.

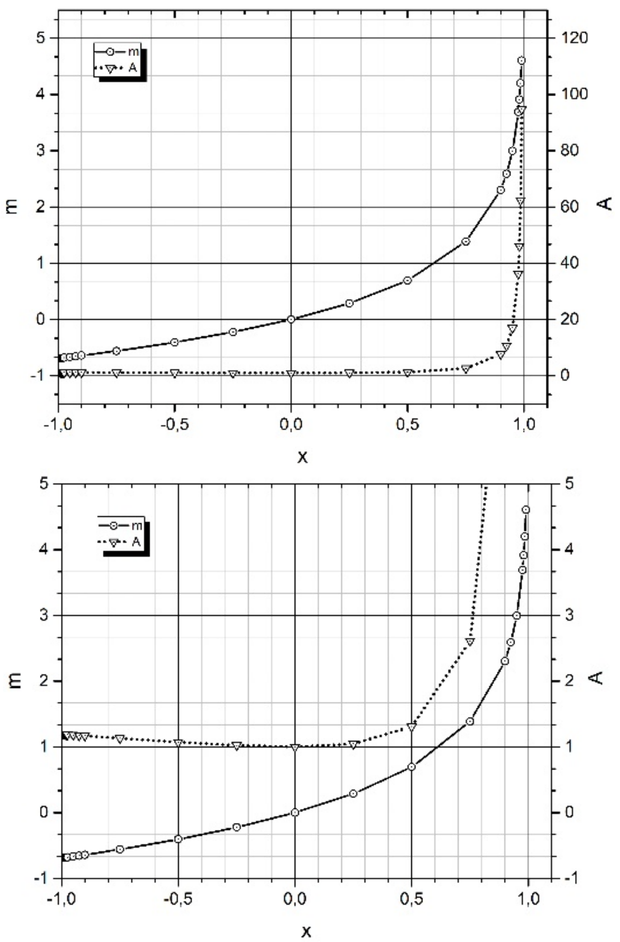

Figure 4. Slope (*m*) and intercept values (*A*) for the lines of Figure 3; plotted values of *A* and *m* are valid for all values of *k*; the plot on the upper of the figure is rescaled for convenience.

Figure 3 shows that when the free parameter k changes and for each value of x, all the F(*k*, *c*, *c*; x) fall on a straight line. Since the equation for a line is y = *A* + *m*x, where m is the line's slope and *A* is the value at which the line intercepts the ordinate axis, then all lines in Figure 3 can be expressed as F(*k*, *c*, *c*; x) = *A*(x) + *m*(x)·*k*. Figure 4 plots the values of *A*(x) and *m*(x) for different values of the independent variable x. It is worth noting here that the plot reported in Figure 4 is valid for any value of the free parameter *k*.

A smoother curve representation is given in Figure 5, where more positive values of the independent variable x have been used for the evaluation of *A* and m. Table 1 reports the data upon which Figure 5 is based.

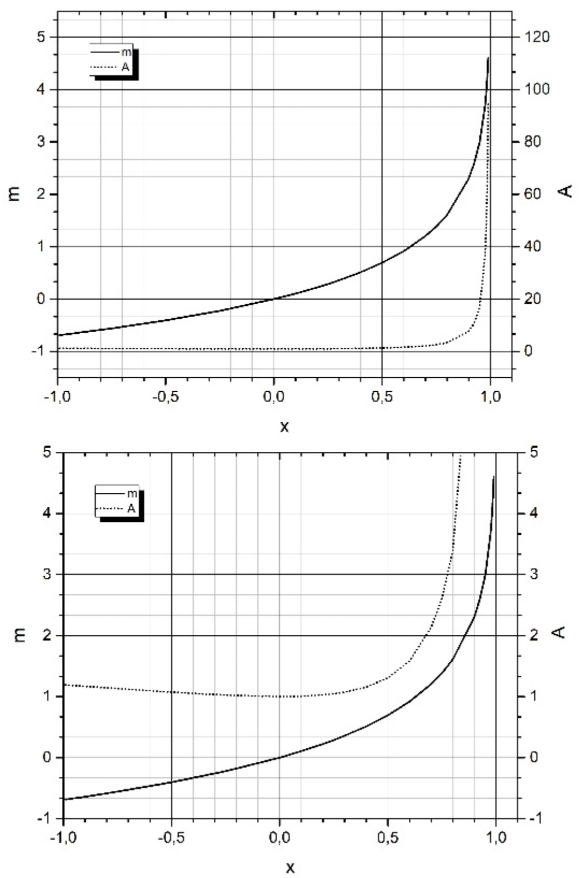

Figure 5. Slope (*m*) and intercept values (*A*) for the function F(*k*, *c*, *c*; x); a higher number of x values have been used for plotting; the plot on the right of the figure is rescaled for convenience.

Table 1. Slope (*m*) and intercept values (*A*) for the function F(*k*, *c*, *c*; x); tabulated values are valid for every *k*.

	x-0.99	x-0.985	x-0.98	x-0.975	x-0.95
A	1.1939	1.1897	1.1882	1.1869	1.1806
m	-0.6881	-0.6856	-0.6831	-0.6806	-0.6678
	x-0.925	x-0.9	x-0.75	x-0.5	x-0.25
A	1.1744	1.1682	1.1310	1.0721	1.0231
m	-0.6549	-0.6419	-0.5596	-0.4055	-0.2231
	x0	x0.1	x0.2	x0.25	x0.3
A	1.0000	1.0058	1.0269	1.0457	1.0719
m	0.0000	0.1054	0.2231	0.2877	0.3567
	x0.4	x0.5	x0.6	x0.7	x0.75
A	1.1558	1.3069	1.5837	2.1294	2.6137
m	0.5108	0.6931	0.9163	1.2040	1.3863
	x0.8	x0.9	x0.925	x0.95	x0.975
A	3.3906	7.6974	10.7431	17.0043	36.3110
m	1.6094	2.3026	2.5903	2.9957	3.6889
	x0.98	x0.985	x0.99		
A	46.0860	62.4327	94.7455		
m	3.9120	4.1996	4.6041		

In order to test the correctness of Figure 5 and/or Table 1, we will now evaluate the values of the hypergeometric function F(*k*, *c*, *c*; x) when *k* = 1 and for three values of the independent variable x, namely the functions F(1, *c*, *c*; 0.25) , F(1, *c*, *c*; 0. 5), and F(1, *c*, *c*; 0.75). Their values are equal to 1⅓, 2 and, 4 respectively. These values can be calculated directly from the Maclaurin series which, as it has been already discussed earlier, converges to $\frac{1}{1-x}$. Because Figure 5 (or its equivalent Table 1) is valid for every values of *k*, one can always use Figure 5 or Table 1 to retrieve the values of A and m for every value of x. If now we assume x to be equal to 0.25, 0.5, and 0.75 the corresponding value of *A* are 1.0457, 1.3069 and 2.6137, respectively, and those of *m* are 0.2877, 0.6931 and 1.3863, respectively. If now we put those values of A and m into the line equation F(1, *c*, *c*; x) = *A* + *mk* for x equal to 0.25, 0.5 and 0.75, and *k* = 1, we can evaluate the values of F(1, *c*, *c*; 0.25), F(1, *c*, *c*; 0. 5), and F(1, *c*, *c*; 0.75) to be equal to 1.3333, 2.0000, and 4.0000, respectively, which is consistent with the values obtained using the Maclaurin series and thus confirms the correctness of Figure 5 and/or Table 1. The test can be repeated for the evaluation of the hypergeometric function in the negative x realm, such as F(1, *c*, *c*; -0.25) , F(1, *c*, *c*; -0. 5), and F(1, *c*, *c*; -0.75). The corresponding Maclaurin series yields: $\frac{1}{1.25} = 0.8$, $\frac{1}{1.5} = 0.6667$, and $\frac{1}{1.75} = 0.5714$. If one repeat the previous steps by retrieving the values of *A* and *m* from Figure 5 or Table 1, and using those values in the line equation F(*k*, *c*, *c*; x) = *A* + *mk*, the corresponding values of the three functions F(1, *c*, *c*; -0.25) , F(1, *c*, *c*; -0. 5), and F(1, *c*, *c*; -0.75) are found to be equal to 0.8000, 0.6667, and 0.5714, respectively, which, once again, confirms the correctness of the proposed approach.

Because the function F(*k*, *c*, *c*; x) covers the all range of *c*, a solution always exists even when *c* is equal to zero or is a negative integer, since the form of the terms of equation (5) prevents the denominators from going to zero When x is equal to zero, the slope m will be zero (horizontal line) and the function F(*k*, *c*, *c*; 0) will always be equal to 1, which is also confirmed by the Maclaurin series.

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