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An Improved Algorithm for Testing Substitutability of Weak Preferences

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Abstract

In this paper, we consider the problem of testing substitutability of weak preferences. For this problem, Aziz, Brill, and Harrenstein proposed an $O(\ell^3 u^2 + \ell^2 u^2 s^2)$-time algorithm, where u is the size of the ground set, ℓ is the number of acceptable sets, and s is the maximum size of an equivalent class. In this paper, we propose an $O(\ell^3 u + \ell^2 u^2 s)$-time algorithm for this problem. Our algorithm is based on a generalization of the characterization of substitutability of strict preferences given by Croitoru and Mehlhorn.

Keywords: choice function, substitutability, weak preference

1. Introduction

The two-sided matching market model introduced by Gale and Shapley [1] is one of the most fundamental mathematical models for assignment problems. When we consider many-to-many matching markets, we frequently assume that the preference lists of agents are given in the form of choice functions. (If there exist ties in the preference lists, then they are given in the form of choice correspondences.) In matching models with choice functions, the property called substitutability plays an important role (see, e.g., [2, 3]). For example, in several matching models, if choice functions are substitutable, then a stable matching always exists (see, e.g., [4]).

In this paper, we consider substitutability of choice functions from the algorithmic viewpoint. More specifically, we consider the problem of checking whether a given preference list (i.e., the choice function induced by it) is substitutable. This line of research was initiated by Hatfield, Immorlica, and Kominers [4], and they proposed an $O(\ell^3 u^3)$-time algorithm in the strict preference case, where u is the size of the ground set and ℓ is the number of acceptable sets. Then Aziz, Brill, and Harrenstein [5] considered the weak preference case (i.e., the preference lists contain ties), and they proposed an $O(\ell^3 u^2 + \ell^2 u^2 s^2)$-time algorithm, where s is the maximum size of an equivalent class.

In this paper, we propose an $O(\ell^3 u + \ell^2 u^2 s)$-time algorithm in the weak preference case. Our algorithm is based on a generalization of the characterization of substitutability of strict preferences given by Croitoru and Mehlhorn [6].

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2. Preliminaries

Throughout this paper, we fix a finite set U and a transitive and complete relation $\succsim$ on 2^U. The relation $\succsim$ is also called a *weak preference*. We denote by $\mathcal{A}$ the family of subsets X of U such that $X \succsim \emptyset$. That is, $\mathcal{A}$ is the family of acceptable subsets of U. For each pair of subsets X, Y of U, we write $X \succ Y$ (resp., $X \sim Y$) if $X \succsim Y$ and $Y \not\succsim X$ (resp., $X \succsim Y$ and $Y \succsim X$). Since the relation $\succsim$ is transitive and complete, it is not difficult to see that $\mathcal{A}$ can be uniquely partitioned into $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$ satisfying the following conditions.

- For every integer i in $\{1, 2, \ldots, k\}$ and every pair of members X, Y of $\mathcal{A}_i$, we have $X \sim Y$.
- For every pair of integers i, j in $\{1, 2, \ldots, k\}$ such that $i < j$, every member X of $\mathcal{A}_i$, and every member Y of $\mathcal{A}_j$, we have $X \succ Y$.

We assume that $\succsim$ is given in the form of $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$. Define $u := |U|$, $\ell := |\mathcal{A}|$, and $s := \max\{|\mathcal{A}_1|, |\mathcal{A}_2|, \ldots, |\mathcal{A}_k|\}$.

The *choice correspondence* $\mathsf{C}\colon 2^U \to 2^{2^U}$ induced by $\succsim$ is defined by

$$\mathsf{C}(X) := \{Y \subseteq X \mid Y \succsim Z \text{ for every subset } Z \text{ of } X\}.$$

Notice that since $\succsim$ is transitive, $\mathsf{C}(X)$ is not empty for any subset X of U. Furthermore, notice that for every subset X of U, since $\emptyset \subseteq X$, $\mathsf{C}(X) \subseteq \mathcal{A}$.

Consider the following example. (This example was introduced in [5].)

$$\begin{aligned} &U = \{a, b, c, d\}, \\ &\{a, b, d\} \sim \{b, c, d\} \succ \{a, b\} \sim \{b, c\} \sim \{a, c\} \succ \emptyset. \end{aligned} \tag{1}$$

In this example,

$$\mathcal{A}_1 = \{\{a, b, d\}, \{b, c, d\}\}$$
$$\mathcal{A}_2 = \{\{a, b\}, \{b, c\}, \{a, c\}\}$$
$$\mathcal{A}_3 = \{\emptyset\}.$$

Furthermore, $\mathsf{C}(U) = \{\{a, b, c\}, \{b, c, d\}\}$ and $\mathsf{C}(\{a\}) = \{\emptyset\}$.

The relation $\succsim$ is said to be *substitutable* if the following conditions are satisfied.

(S1) Assume that we are given non-empty subsets A, B of U such that $B \subseteq A$. Then for every member X of $\mathsf{C}(A)$, there exists a member Y of $\mathsf{C}(B)$ such that $X \cap B \subseteq Y$.

(S2) Assume that we are given non-empty subsets A, B of U such that $B \subseteq A$. Then for every member Y of $\mathsf{C}(B)$, there exists a member X of $\mathsf{C}(A)$ such that $X \cap B \subseteq Y$.

The above definition of substitutability was introduced by Sotomayor [7]. See also [5, Section 3] for this definition. Notice that the relation in (1) is not substitutable. If we set $A := U$ and $B := \{a, b, c\}$, then (S2) is not satisfied. In this case,

$$\mathsf{C}(A) = \{\{a, b, d\}, \{b, c, d\}\}$$
$$\mathsf{C}(B) = \{\{a, b\}, \{b, c\}, \{a, c\}\}.$$

However, $X \cap B \not\subseteq \{a, c\}$ for any member X of $\mathsf{C}(A)$.

In this paper, we consider the problem of checking whether the relation $\succsim$ is substitutable. Aziz, Brill, and Harrenstein [5] proposed an $O(\ell^3 u^2 + \ell^2 u^2 s^2)$-time algorithm for this problem. As a special case of this problem, the case of $s = 1$ was considered. We call this special case the *strict preference case*. Hatfield, Immorlica, and Kominers [4] proposed an $O(\ell^3 u^3)$-time algorithm in the strict preference case. Notice that the time complexity of Aziz, Brill, and Harrenstein [5] in the strict preference case (i.e., the case of $s = 1$) is $O(\ell^3 u^2)$, which improved the time complexity of the algorithm of [4]. Furthermore, in the strict preference case, Croitoru and Mehlhorn [6] gave a new characterization of substitutability of strict preferences, and proved that their characterization naturally leads to an $O(\ell^3 u + \ell^2 u^2)$-time algorithm. By improving this algorithm, Croitoru and Mehlhorn [6] also proposed an $O(\ell^2 u^2)$-time algorithm in the strict preference case.

In this paper, we propose an $O(\ell^3 u + \ell^2 u^2 s)$-time algorithm for the above problem. Our algorithm improves the time complexity of the algorithm of [5]. Our algorithm is based on a new characterization of substitutability of weak preferences that is a generalization of the characterization of substitutability in the strict preference case given by Croitoru and Mehlhorn [6].

3. Characterization

The following lemma was proved in [5]. For completeness, we give its proof.

Lemma 1 (Aziz, Brill, and Harrenstein [5, Lemma 1]). *Assume that we are given subsets A, B of U such that $B \subseteq A$. If $\mathsf{C}(A) \cap 2^B \neq \emptyset$, then $\mathsf{C}(B) = \mathsf{C}(A) \cap 2^B$.*

Proof. Let X be a member of $\mathsf{C}(A) \cap 2^B$. Since $X \in \mathsf{C}(A)$ and $B \subseteq A$, we have $X \succsim Y$ for every subset Y of B. Since $X \subseteq B$, this implies that $X \in \mathsf{C}(B)$.

Let X be a member of $\mathsf{C}(B)$. Since $X \subseteq B$, if we can prove that $X \in \mathsf{C}(A)$, then $X \in \mathsf{C}(A) \cap 2^B$. Let Y be a member of $\mathsf{C}(A) \cap 2^B$. Then $Y \in \mathsf{C}(B)$ (see above). Thus, since $X \in \mathsf{C}(B)$, $X \sim Y$. This implies that $X \in \mathsf{C}(A)$. This completes the proof. □

A pair (X, Y) of members of $\mathcal{A}$ is called a *witness* (to non-substitutability) if the following conditions are satisfied.

(W1) $X \in \mathsf{C}(X \cup Y)$.

(W2) There exists an element x in $X \setminus Y$ satisfying the following conditions.

(a) $Y \in \mathsf{C}(Y \cup \{x\})$.

(b) At least one of the following conditions holds.

(i) For any member Z of $\mathsf{C}(Y \cup \{x\})$, $X \cap (Y \cup \{x\})$ is not a subset of Z.

(ii) For any member Z of $\mathsf{C}(X \cup Y)$, $Z \cap (Y \cup \{x\})$ is not a subset of Y.

Intuitively speaking, $X \cup Y$ plays the role of A, and $Y \cup \{x\}$ that of B, in the definition of substitutability. This definition of a witness is a generalization of the definition of a witness in the strict preference case proposed by Croitoru and Mehlhorn [6]. The definition of a witness by Croitoru and Mehlhorn [6] consists of only (W1) and (W2)-(a). It should be noted that (W1) implies that for every witness (X, Y), $X \succsim Y$. In the strict preference case, (W1) implies that $X \succ Y$. Croitoru and Mehlhorn [6] included this in the definition of a witness. However, this is redundant. Thus, we remove this condition.

Theorem 1. *The relation $\succsim$ is not substitutable if and only if there exists a witness.*

Proof. Assume that the relation $\succsim$ is not substitutable. Then at least one of (S1) and (S2) does not hold.

First, we consider the case in which (S1) does not hold. In this case, there exist subsets A, B of U satisfying the following conditions.

(P1) $B \subseteq A$.

(P2) There exists a member X of $\mathsf{C}(A)$ such that $X \cap B$ is not a subset of Z for any member Z of $\mathsf{C}(B)$.

Let Y be a member of $\mathsf{C}(B)$ such that $X \cap Y$ is inclusion-wise maximal among all members of $\mathsf{C}(B)$. Then (P2) implies that $X \cap B \not\subseteq Y$. Since $X \in \mathsf{C}(A)$ and $Y \in \mathsf{C}(B)$, we have $X, Y \in \mathcal{A}$. Thus, what remains is to prove that (W1) and (W2) are satisfied. Since $X \in \mathsf{C}(A) \cap 2^{X \cup Y}$ and $Y \subseteq B \subseteq A$ (i.e., $X \cup Y \subseteq A$), Lemma 1 implies that $\mathsf{C}(X \cup Y) = \mathsf{C}(A) \cap 2^{X \cup Y}$, which implies that $X \in \mathsf{C}(X \cup Y)$. That is, (X, Y) satisfies (W1). Since $X \cap B \not\subseteq Y$, there exists an element x in $(X \cap B) \setminus Y$. Since $Y \subseteq B$ and $x \in B$, we have $Y \cup \{x\} \subseteq B$. Thus, since $Y \in \mathsf{C}(B) \cap 2^{Y \cup \{x\}}$, Lemma 1 implies that $\mathsf{C}(Y \cup \{x\}) = \mathsf{C}(B) \cap 2^{Y \cup \{x\}}$, which implies that $Y \in \mathsf{C}(Y \cup \{x\})$. That is, (X, Y) satisfies (W2)-(a). Let Z be a member of $\mathsf{C}(Y \cup \{x\})$. Since $\mathsf{C}(Y \cup \{x\}) = \mathsf{C}(B) \cap 2^{Y \cup \{x\}}$,

$Z \in \mathsf{C}(B)$. Thus, the choice of Y implies that $X \cap Y \not\subseteq X \cap Z$ or $X \cap Y = X \cap Z$. If $X \cap Y \not\subseteq X \cap Z$, then there exists an element y in $X \cap Y$ such that $y \notin Z$. If $X \cap Y = X \cap Z$, then since $x \in X$ and $x \notin Y$, we have $x \notin Z$. In both cases,

$$X \cap (Y \cup \{x\}) = (X \cap Y) \cup \{x\} \not\subseteq Z.$$

This implies that (X, Y) satisfies (W2)-(b)-(i).

Next we assume that (S2) does not hold. In this case, there exist subsets A, B of U satisfying the following conditions.

(Q1) $B \subseteq A$.

(Q2) There exists a member Y of $\mathsf{C}(B)$ such that $Z \cap B$ is not a subset of Y for any member Z of $\mathsf{C}(A)$.

Let X be a member of $\mathsf{C}(A)$ such that $X \setminus Y$ is inclusion-wise minimal among all members of $\mathsf{C}(A)$. Then (Q2) implies that $X \cap B \not\subseteq Y$. Since $X \in \mathsf{C}(A)$ and $Y \in \mathsf{C}(B)$, we have $X, Y \in \mathcal{A}$. Thus, what remains is to prove that (W1) and (W2) are satisfied. Since $X \in \mathsf{C}(A) \cap 2^{X \cup Y}$ and $Y \subseteq B \subseteq A$ (i.e., $X \cup Y \subseteq A$), Lemma 1 implies that $\mathsf{C}(X \cup Y) = \mathsf{C}(A) \cap 2^{X \cup Y}$, which implies that $X \in \mathsf{C}(X \cup Y)$. That is, (X, Y) satisfies (W1). Since $X \cap B \not\subseteq Y$, there exists an element x in $(X \cap B) \setminus Y$. Since $Y \subseteq B$ and $x \in B$, we have $Y \cup \{x\} \subseteq B$. Thus, since $Y \in \mathsf{C}(B) \cap 2^{Y \cup \{x\}}$, Lemma 1 implies that $\mathsf{C}(Y \cup \{x\}) = \mathsf{C}(B) \cap 2^{Y \cup \{x\}}$, which implies that $Y \in \mathsf{C}(Y \cup \{x\})$. That is, (X, Y) satisfies (W2)-(a). Let Z be a member of $\mathsf{C}(X \cup Y)$. Since $\mathsf{C}(X \cup Y) = \mathsf{C}(A) \cap 2^{X \cup Y}$, $Z \in \mathsf{C}(A)$. Since $Z \in \mathsf{C}(X \cup Y)$, $Z \subseteq X \cup Y$. This implies that $Z \setminus Y \subseteq X \setminus Y$. Thus, the choice of X implies that $Z \setminus Y = X \setminus Y$. Since $x \in X$ and $x \notin Y$, we have $x \in X \setminus Y = Z \setminus Y$. Thus,

$$Z \cap (Y \cup \{x\}) = (Z \cap Y) \cup \{x\} \not\subseteq Y.$$

This implies that (X, Y) satisfies (W2)-(b)-(ii).

To prove the other direction, we assume that there exists a witness, that is, there exists a pair (X, Y) of members of $\mathcal{A}$ satisfying (W1) and (W2). Let x be an element in $X \setminus Y$ satisfying (W2). Define $A := X \cup Y$ and $B := Y \cup \{x\}$. Then $B \subseteq A$. Furthermore, (W1) implies that $X \in \mathsf{C}(A)$, and (W2)-(a) implies that $Y \in \mathsf{C}(B)$. First, we assume that (W2)-(b)-(i) holds. That is, we assume that $X \cap B \not\subseteq Z$ for any member Z of $\mathsf{C}(B)$. This implies that (S1) does not hold. Next we assume that (W2)-(b)-(ii) holds. That is, we assume that $Z \cap B \not\subseteq Y$ for any member Z of $\mathsf{C}(A)$. This implies that (S2) does not hold. This completes the proof. □

4. Algorithm

In this section, we propose an algorithm for the problem of checking whether the relation $\succsim$ is substitutable. See Algorithm 1. This algorithm is based on Theorem 1. If the relation $\succsim$ is substitutable, then this algorithm outputs Yes. Otherwise, it outputs No.

First, we prove the correctness of Algorithm 1. For this, we prove that at Line 15 (resp., 18), it is sufficient to consider only members of $\mathcal{A}_{i(Y)}$ (resp., $\mathcal{A}_{i(X)}$). At Line 15, since $\mathsf{sens}(x, Y) =$ False, $Y \in \mathsf{C}(Y \cup \{x\})$. This implies that for every member Z of $\mathsf{C}(Y \cup \{x\})$, we have $Y \sim Z$. Thus, at Line 15, it is sufficient to consider only members of $\mathcal{A}_{i(Y)}$. Furthermore, at Line 18, $X \in \mathsf{C}(X \cup Y)$ (see Line 12). This implies that for every member Z of $\mathsf{C}(X \cup Y)$, we have $X \sim Z$. Thus, at Line 18, it is sufficient to consider only members of $\mathcal{A}_{i(X)}$.

Algorithm 1:

```
1  for each member X of 𝒜 do
2      Define i(X) as the integer i in {1, 2, ..., k} such that X ∈ 𝒜_i.
3      for each element x of U \ X do
4          if X ∉ C(X ∪ {x}) then
5              Define sens(x, X) := True.
6          else
7              Define sens(x, X) := False.
8          end
9      end
10 end
11 for each pair of members X, Y of 𝒜 do
12     if X ∈ C(X ∪ Y) then
13         for each element x in X \ Y do
14             if sens(x, Y) = False then
15                 if X ∩ (Y ∪ {x}) ⊈ Z for any member Z of 𝒜_{i(Y)} then
16                     Output No, and halt.
17                 end
18                 if Z ∩ (Y ∪ {x}) ⊈ Y for any member Z of 𝒜_{i(X)} then
19                     Output No, and halt.
20                 end
21             end
22         end
23     end
24 end
25 Output Yes, and halt.
```

What remains is to evaluate the time complexity of Algorithm 1. First, we need to specify how the input data is given. We assume that for each integer i in $\{1, 2, \dots, k\}$, $\mathcal{A}_i$ is given in the form of a list L_i. That is, each element of L_i is a member of $\mathcal{A}_i$ (i.e., some subset of U). Each member X of $\mathcal{A}$ is given in the form of a one-dimensional array a_X indexed by elements in U. For each element x in U, if $x \in X$, then $a_X[x] = 1$. Otherwise, $a_X[x] = 0$. Thus, for every pair of members X, Y of $\mathcal{A}$, we can find $X \cup Y$ and $X \cap Y$ in $O(u)$ time, and we can check whether $X \subseteq Y$ in $O(u)$ time.

We are now ready to evaluate the time complexity of Algorithm 1. First, we evaluate the time complexity of Lines 1 to 10. Let X be a member of $\mathcal{A}$. Then we consider the time complexity of Lines 2 to 9. By searching the list L_i for all integers i in $\{1, 2, \dots, k\}$, we can compute $i(X)$ in $O(\ell u)$ time. For each element x in $U \setminus X$, we can compute $\mathsf{C}(X \cup \{x\})$ in $O(\ell u)$ time by searching the list L_i for all integers i in $\{1, 2, \dots, k\}$. Thus, for each element x in $U \setminus X$, we can compute $\mathsf{sens}(x, X)$ in $O(\ell u)$ time. Since $|U \setminus X| \le u$, the time complexity of Lines 3 to 9 is

$O(\ell u^2)$. This implies that the time complexity of Lines 2 to 9 is $O(\ell u^2)$. Thus, the time complexity of Lines 1 to 10 is $O(\ell^2 u^2)$.

Next, we evaluate the time complexity of Lines 11 to 24. Let X, Y be members of $\mathcal{A}$. Then we consider the time complexity of Lines 12 to 23. In the same way as above, we can complete Line 12 in $O(\ell u)$ time. Let x be an element in $X \setminus Y$. Then we evaluate the time complexity of Lines 14 to 21. Clearly, we can complete Line 14 in $O(1)$ time. (Notice that if we do not prepare $\mathsf{sens}(\cdot, \cdot)$, then this part needs $O(\ell u)$ time. Thus, we can not achieve the desired time complexity without $\mathsf{sens}(\cdot, \cdot)$.) Furthermore, the time complexity of Line 15 is $O(us)$. Similarly, the time complexity of Line 18 is $O(us)$. Thus, the time complexity of Lines 14 to 21 is $O(us)$. This implies that the time complexity of Lines 12 to 23 is $O(\ell u + u^2 s)$. Thus, the time complexity of Lines 11 to 24 is $O(\ell^3 u + \ell^2 u^2 s)$. This implies that the time complexity of Algorithm 1 is $O(\ell^3 u + \ell^2 u^2 s)$.

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