The Distance-Constrained Matroid Median Problem

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Abstract Alamdari and Shmoys introduced the following variant of the k-median problem. In this variant, we are given an instance of the k-median problem and a threshold value. Then this variant is the same as the k-median problem except that if the distance between a client i and a facility j is more than the threshold value, then i is not allowed to be connected to j. In this paper, we consider a matroid generalization of this variant of the k-median problem. First, we introduce a generalization of this variant in which the constraint on the number of opened facilities is replaced by a matroid constraint. Then we propose a polynomial-time bicriteria approximation algorithm for this problem by combining the algorithm of Alamdari and Shmoys and the algorithm of Krishnaswamy, Kumar, Nagarajan, Sabharwal, and Saha for a matroid generalization of the k-median problem.

1 Introduction

The study of clustering problems is one of the central topics in the fields of combinatorial optimization and operations research. Among clustering problems, the k-center problem and the k-median problem are representative problems. The goal of the k-center problem is to locate k facilities so that the maximum distance between clients and facilities is minimized. On the other hand, the goal of the k-median problem is to locate k facilities so that the sum of the distances between clients and facilities is minimized. The k-center problem and the k-median problem have been extensively studied from the viewpoint of approximation algorithms. For example, 2-approximation algorithms for the k-center problem were given by Gonzalez [8] and Hochbaum and Shmoys [10]. For the k-median problem, approximation algorithms are still improved (see, e.g., [2–5, 11, 12, 14]). The current best approximation ratio is $1+\sqrt{3}+\varepsilon$ due to Li and Svensson [14].

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Several generalizations of the k-center problem and the k-median problem have been proposed. For example, in their matroid generalizations, the constraint on the number of opened facilities is replaced by a matroid constraint. Chen, Li, Liang, and Wang [7] considered a matroid generalization of the k-center problem, and proposed a 3-approximation algorithm for this problem. Krishnaswamy, Kumar, Nagarajan, Sabharwal, and Saha [13] considered a matroid generalization of the k-median problem, and proposed a 16-approximation algorithm for this problem. It should be noted that this problem is a generalization of the red-blue median problem considered by Hajiaghayi, Khandekar, and Kortsarz [9]. The approximation ratio for this matroid generalization of the k-median problem was improved by Charikar and Li [6] and Swamy [17], and the current best approximation ratio is 8 due to Swamy [17]. Alamdari and Shmoys [1] considered the following generalization of the k-median problem. In this problem, we are given an instance of the k-median problem and a threshold value. Then their problem is the same as the k-median problem except that if the distance between a client i and a facility j is more than the threshold value, then i is not allowed to be connected to j. For this problem, Alamdari and Shmoys [1] gave a $(4,8)$-bicriteria approximation algorithm (see Section 2 for the definition of a bicriteria approximation algorithm).

In this paper, we first introduce a matroid generalization of the problem considered by Alamdari and Shmoys [1]. Then we propose a polynomial-time $(11,16)$-bicriteria approximation algorithm for this problem. Our algorithm can be regarded as a generalization of the algorithm of [13].

2 Preliminaries

2.1 Notation and basics of matroids

Throughout this paper, we denote by $\mathbb{R}_+$ and $\mathbb{Q}_+$ the sets of non-negative real numbers and non-negative rational numbers, respectively. For each finite set U and each subset X of U, we define the vector χ_X in $\{0,1\}^U$ by

$$\chi_X(u) := \begin{cases} 1 & \text{for each element } u \text{ in } X \\ 0 & \text{for each element } u \text{ in } U \setminus X. \end{cases}$$

For each finite set U, each vector x in $\mathbb{R}_+^U$, and each subset X of U, we define

$$x(X) := \sum_{u \in X} x(u).$$

A pair $(U, \mathscr{I})$ of a finite set U and a family $\mathscr{I}$ of subsets of U is called a *matroid* if the following conditions are satisfied.

(I0) $\emptyset \in \mathscr{I}$.
(I1) For every pair of subsets I, J of U, if $I \in \mathscr{I}$ and $J \subseteq I$, then $J \in \mathscr{I}$.
(I2) For every pair of members I, J of $\mathscr{I}$ such that $|I| < |J|$, there exists a vertex u in $J \setminus I$ such that $I \cup \{u\} \in \mathscr{I}$.

Assume that we are given a matroid $\mathbf{M} = (U, \mathscr{I})$. A member of $\mathscr{I}$ is called an *independent set* of $\mathbf{M}$. Furthermore, an inclusion-wise maximal independent set of $\mathbf{M}$ is called a *base* of $\mathbf{M}$. Notice that (I2) implies that all the bases of $\mathbf{M}$ have the same size. For each subset X of U, we define

$$\mathscr{I}|X := \{I \in \mathscr{I} \mid I \subseteq X\}$$
$$\mathbf{M}|X := (X, \mathscr{I}|X).$$

It is well known (see, e.g., [15, p.20]) that for every subset X of U, $\mathbf{M}|X$ is a matroid. For each subset X of U, we denote by $\mathrm{rk}_{\mathbf{M}}(X)$ the size of a base of $\mathbf{M}|X$. Define $\mathbf{P}_{\mathbf{M}}$ as the set of vectors x in $\mathbb{R}_+^U$ such that $x(X) \le \mathrm{rk}_{\mathbf{M}}(X)$ for every subset X of U.

Lemma 1 (See, e.g., [16, Corollary 40.2b]) *Assume that we are given a matroid $\mathbf{M} = (U, \mathscr{I})$. For every subset X of U, $X \in \mathscr{I}$ if and only if $\chi_X \in \mathbf{P}_{\mathbf{M}}$.*

2.2 Problem formulation

In the *distance-constrained matroid median problem*, we are given a finite set V of vertices, a location cost function $f\colon V \to \mathbb{Q}_+$, a demand function $\omega\colon V \to \mathbb{Q}_+$, a matroid $\mathbf{M} = (V, \mathscr{I})$, a rational number L in $\mathbb{Q}_+$, and a function $d\colon V \times V \to \mathbb{Q}_+$ satisfying the following conditions.

(D1) For every pair of vertices u, v in V, $d(u, v) = 0$ if and only if $u = v$.
(D2) For every pair of vertices u, v in V, $d(u, v) = d(v, u)$.
(D3) For every triplet of vertices u, v, w in V, $d(u, v) + d(v, w) \ge d(u, w)$.

We assume that for every subset I of V, we can decide whether $I \in \mathscr{I}$ in time bounded by a polynomial in $|V|$ and the sizes of the input values.

Define Π as the set of pairs (X, o) of an independent set X of $\mathbf{M}$ and a mapping $o\colon V \to X$ such that $d(u, o(u)) \le L$ for every vertex u in V. For each pair (X, o) in Π, we define

$$\mathrm{cost}(X, o) := f(X) + \sum_{u \in V} \omega(u) d(u, o(u)).$$

Then the distance-constrained matroid median problem is defined as follows.

$$\begin{array}{ll} \text{Minimize} & \mathrm{cost}(X, o) \\ \text{subject to} & (X, o) \in \Pi. \end{array} \tag{1}$$

In what follows, we denote by Opt the optimal objective value of the problem (1). If $\Pi = \emptyset$, then we define $\mathsf{Opt} := \infty$.

Assume that we are given a pair (X, o) of an independent set X of $\mathbf{M}$ and a mapping $o\colon V \to X$. Then for each real number δ in $\mathbb{R}_+$, we call (X, o) a *δ-feasible solution of the problem* (1) if $d(u, o(u)) \le \delta L$ for every vertex u in V. For each pair (δ_1, δ_2) of real numbers in $\mathbb{R}_+$, we call (X, o) a *(δ_1, δ_2)-bicriteria approximation solution* of the problem (1) if (X, o) is a δ_1-feasible solution of the problem (1) and $\mathrm{cost}(X, o) \le \delta_2 \mathsf{Opt}$. In what follows, we propose a polynomial-time algorithm that outputs **null** (i.e., $\Pi = \emptyset$) or an $(11, 16)$-bicriteria approximation solution of the problem (1). (Notice that a bicriteria approximation solution does not mean that $\Pi \neq \emptyset$.)

If we are given a positive integer k and we define $\mathscr{I} := \{I \subseteq V \mid |I| \leq k\}$, then our problem is the same as the problem considered by Alamdari and Shmoys [1]. For this problem, Alamdari and Shmoys [1] gave a polynomial-time $(4,8)$-bicriteria approximation algorithm. If $L = \infty$, then our problem is the same as the matroid median problem introduced by Krishnaswamy, Kumar, Nagarajan, Sabharwal, and Saha [13]. For this problem, Krishnaswamy, Kumar, Nagarajan, Sabharwal, and Saha [13] proposed a polynomial-time $(1,16)$-bicriteria approximation algorithm. Swamy [17] and Charikar and Li [6] gave polynomial-time $(1,8)$-bicriteria and $(1,9)$-bicriteria approximation algorithms for this problem, respectively.

The contribution of this paper is a polynomial-time $(11,16)$-bicriteria approximation algorithm for the distance-constrained matroid median problem. Since our algorithm is a generalization of the algorithm of [13] for the matroid median problem, a large part of our proof is almost the same as the proof of the result of [13]. However, for the sake of completeness, we give proofs of all the lemmas except Lemma 22. The main differences between our algorithm and the algorithm of [13] are the integer programming formulation and the definition of the radiuses of the balls used to consolidate demands.

3 Integer Programming Formulation and Preprocessing

For each pair of vectors x in $\mathbb{R}_+^{V\times V}$ and y in $\mathbb{R}_+^V$, we define

$$\begin{aligned}
\mathsf{w}(x) &:= \sum_{u\in V} \omega(u) \sum_{v\in V} d(u,v)x(u,v)\\
\mathsf{f}(y) &:= \sum_{u\in V} f(u)y(u)\\
\mathsf{c}(x,y) &:= \mathsf{w}(x) + \mathsf{f}(y).
\end{aligned}$$

Then Lemma 1 implies that the distance-constrained matroid median problem can be formulated as the following problem (2).

$$\begin{aligned}
\text{Minimize}\quad & \mathsf{c}(x,y)\\
\text{subject to}\quad & \sum_{v\in V} x(u,v) = 1 \quad (\forall u \in V)\\
& x(u,v) \leq y(v) \quad (\forall u,v \in V)\\
& x(u,v) = 0 \quad (\forall u,v \in V \text{ s.t. } d(u,v) > L)\\
& (x,y) \in \{0,1\}^{V\times V} \times (\mathbf{P_M} \cap \{0,1\}^V).
\end{aligned} \tag{2}$$

Assume that we are given a pair (x,y) of vectors x in $\{0,1\}^{V\times V}$ and y in $\mathbf{P_M} \cap \{0,1\}^V$. Then for each real number δ in $\mathbb{R}_+$, we call (x,y) a *δ-feasible solution of the problem* (2) if (x,y) satisfies the first and second constraints of the problem (2) and $d(u,v) \leq \delta L$ for every pair of vertices u,v in V such that $x(u,v) > 0$. Furthermore, for each pair (δ_1,δ_2) of real numbers in $\mathbb{R}_+$, we call (x,y) a *(δ_1,δ_2)-bicriteria approximation solution* of the problem (2) if (x,y) is a δ_1-feasible solution of the problem (2) and $\mathsf{c}(x,y) \leq \delta_2 \mathsf{Opt}$.

We consider the following relaxation of the problem (2).

$$
\begin{aligned}
\text{Minimize} \quad & \mathsf{c}(x,y) \\
\text{subject to} \quad & \sum_{v\in V} x(u,v) = 1 \quad (\forall u \in V) \\
& x(u,v) \le y(v) \quad (\forall u,v \in V) \\
& x(u,v) = 0 \quad (\forall u,v \in V \text{ s.t. } d(u,v) > L) \\
& (x,y) \in \mathbb{R}_+^{V\times V} \times (\mathbf{P_M} \cap \mathbb{R}_+^V).
\end{aligned} \tag{3}
$$

It is known [13, Section 2.2] that the problem (3) can be solved in polynomial time. If there does not exist a feasible solution of the problem (3), then $\Pi = \emptyset$, and thus our algorithm outputs **null**. In what follows, we assume that there exists a feasible solution of the problem (3).

Define (x^*, y^*) as an optimal solution of the problem (3). For each vertex u in V, we define $\mathsf{L}(u)$ by

$$\mathsf{L}(u) := \sum_{v\in V} d(u,v)x^*(u,v).$$

Assume that $V = \{s_1, s_2, \dots, s_n\}$ and $\mathsf{L}(s_1) \le \mathsf{L}(s_2) \le \cdots \le \mathsf{L}(s_n)$. For each vertex u in V, we define $\mathbf{r}(u) := \min\{2\mathsf{L}(u), L\}$. Define the subsets $D, V_1, V_2, \dots, V_n$ of V and the function $\overline{\omega}\colon D \to \mathbb{R}_+$ by Algorithm 1.

Algorithm 1:

Set $V_i := \{s_i\}$ for each vertex s_i in V.
foreach $i \in \{1,2,\dots,n\}$ **do**
  if *there exists an integer j in $\{1,2,\dots,i-1\}$ such that $|V_j| \ge 1$ and $d(s_i,s_j) \le 2\mathbf{r}(s_i)$* **then**
    Set $V_j := V_j \cup \{s_i\}$ and $V_i := \emptyset$.
  end
end
Define $D := \{s_i \in V \mid |V_i| \ge 1\}$ and $\overline{\omega}(s_i) := \omega(V_i)$ for each vertex s_i in D.
Output $\overline{\omega}$, D, and $V_1, V_2, \dots, V_n$. Then halt.

Notice that for every pair of vertices s_i in D and u in V_i, $\mathsf{L}(s_i) \le \mathsf{L}(u)$. For each pair of vectors x in $\mathbb{R}_+^{D\times V}$ and y in $\mathbb{R}_+^V$, we define

$$
\begin{aligned}
\overline{\mathsf{w}}(x) &:= \sum_{s\in D} \overline{\omega}(s) \sum_{u\in V} d(s,u)x(s,u) \\
\overline{\mathsf{c}}(x,y) &:= \overline{\mathsf{w}}(x) + \mathsf{f}(y).
\end{aligned}
$$

We consider the following problem.

$$
\begin{aligned}
\text{Minimize} \quad & \overline{\mathsf{c}}(x,y) \\
\text{subject to} \quad & \sum_{u\in V} x(s,u) = 1 \quad (\forall s \in D) \\
& x(s,u) \le y(u) \quad (\forall s \in D,\ \forall u \in V) \\
& x(s,u) = 0 \quad (\forall s \in D,\ \forall u \in V \text{ s.t. } d(s,u) > L) \\
& (x,y) \in \mathbb{R}_+^{D\times V} \times (\mathbf{P_M} \cap \mathbb{R}_+^V).
\end{aligned} \tag{4}
$$

For each real number δ in $\mathbb{R}_+$, we call a pair (x,y) of vectors x in $\mathbb{R}_+^{D\times V}$ and y in $\mathbf{P_M} \cap \mathbb{R}_+^V$ a *δ-feasible solution* of the problem (4) if (x,y) satisfies the first and second constraints of the problem (4) and $d(s,u) \leq \delta L$ for every pair of vertices s in D and u in V such that $x(s,u) > 0$.

Lemma 2 *Assume that we are given a δ_1-feasible solution $(\bar{x},y)$ of the problem* (4) *such that $\bar{x} \in \{0,1\}^{D\times V}$, $y \in \{0,1\}^V$, and $\bar{\mathsf{c}}(\bar{x},y) \leq \delta_2 \mathsf{c}(x^*,y^*)$. Then a (δ_1+2,δ_2+4)-bicriteria approximation solution of the problem* (2) *can be constructed from $(\bar{x},y)$.*

Proof Define the vector x in $\{0,1\}^{V\times V}$ by $x(u,v) := \bar{x}(s_i,v)$ for each triplet of vertices s_i in D, u in V_i, and v in V. Then (x,y) satisfies the first and second constraints of the problem (2).

Let u,v be vertices in V such that $x(u,v) = 1$ and $u \in V_i$. Then $\bar{x}(s_i,v) = 1$. Since $(\bar{x},y)$ is a δ_1-feasible solution of the problem (4), $d(s_i,v) \leq \delta_1 L$. Thus,

$$d(u,v) \leq d(u,s_i) + d(s_i,v) \leq 2\mathbf{r}(u) + \delta_1 L \leq 2L + \delta_1 L.$$

Thus, x is a (δ_1+2)-feasible solution of the problem (2). Furthermore,

$$\begin{aligned}
\mathsf{w}(x) &= \sum_{u\in V} \omega(u) \sum_{v\in V} d(u,v)x(u,v) = \sum_{s_i\in D}\sum_{u\in V_i} \omega(u) \sum_{v\in V} d(u,v)\bar{x}(s_i,v) \\
&\leq \sum_{s_i\in D}\sum_{u\in V_i} \omega(u)d(u,s_i) \sum_{v\in V} \bar{x}(s_i,v) + \sum_{s_i\in D} \overline{\omega}(s_i) \sum_{v\in V} d(s_i,v)\bar{x}(s_i,v) \\
&\leq 4 \sum_{s_i\in D}\sum_{u\in V_i} \omega(u)\mathsf{L}(u) + \overline{\mathsf{w}}(\bar{x}) = 4 \sum_{u\in V} \omega(u) \sum_{v\in V} d(u,v)x^*(u,v) + \overline{\mathsf{w}}(\bar{x}),
\end{aligned}$$

where the first inequality follows from (D3) and the second inequality follows from $d(u,s_i) \leq 2\mathbf{r}(u) \leq 4\mathsf{L}(u)$ and the first constraint of (4). Thus,

$$\mathsf{c}(x,y) \leq 4\mathsf{w}(x^*) + \bar{\mathsf{c}}(\bar{x},y) \leq 4\mathsf{c}(x^*,y^*) + \delta_2 \mathsf{c}(x^*,y^*) \leq (\delta_2+4)\mathsf{Opt}.$$

This completes the proof. □

Notice that the proof of Lemma 2 implies a polynomial-time algorithm for finding the desired solution of the problem (2).

4 Algorithm

In this section, we construct a 9-feasible solution $(\bar{x},y)$ of the problem (4) such that $\bar{x} \in \{0,1\}^{D\times V}$, $y \in \{0,1\}^V$, and $\bar{\mathsf{c}}(\bar{x},y) \leq 12\mathsf{c}(x^*,y^*)$ in polynomial time. This and Lemma 2 imply that we can find an $(11,16)$-bicriteria approximation solution of the distance-constrained matroid median problem.

First, we construct new vectors $x_1,x_2,\ldots,x_5$ in $\mathbb{R}_+^{D\times V}$. Then we construct a pair $(x_{\mathrm{f}},y_{\mathrm{f}})$ of vectors x_{f} in $\{0,1\}^{D\times V}$ and y_{f} in $\{0,1\}^V$ such that $(x_{\mathrm{f}},y_{\mathrm{f}})$ is a 9-feasible solution of the problem (4) and $\bar{\mathsf{c}}(x_{\mathrm{f}},y_{\mathrm{f}}) \leq 12\mathsf{c}(x^*,y^*)$.

Intuitively speaking, our algorithm works as follows. First, we construct the vector x_1 by restricting the domain of x^* to $D\times V$. Second, we construct the vectors x_2,x_3,x_4,x_5 by reconnecting vertices in D to vertices in V that are sufficiently close to vertices in D. Lastly, we solve some related linear program, and construct a solution. Then we evaluate the cost of the solution by using x_5.

4.1 Construction of x_1

Define the vector x_1 in $\mathbb{R}_+^{D \times V}$ by $x_1(s,u) := x^*(s,u)$ for each pair of vertices s in D and u in V. Then since (x^*, y^*) is a feasible solution of the problem (3), (x_1, y^*) is clearly a feasible solution of the problem (4).

Lemma 3 $\overline{\mathsf{c}}(x_1, y^*) \leq \mathsf{c}(x^*, y^*)$.

Proof It is sufficient to prove that $\overline{\mathsf{w}}(x_1) \leq \mathsf{w}(x^*)$. We have

$$\begin{aligned}\overline{\mathsf{w}}(x_1) &= \sum_{s_i \in D} \overline{\omega}(s_i) \sum_{u \in V} d(s_i,u) x^*(s_i,u) = \sum_{s_i \in D} \overline{\omega}(s_i) \mathsf{L}(s_i) \\ &= \sum_{s_i \in D} \sum_{u \in V_i} \omega(u) \mathsf{L}(s_i) \leq \sum_{s_i \in D} \sum_{u \in V_i} \omega(u) \mathsf{L}(u) = \sum_{u \in V} \omega(u) \mathsf{L}(u) = \mathsf{w}(x^*),\end{aligned}$$

where the inequality follows from $\mathsf{L}(s_i) \leq \mathsf{L}(u)$. This completes the proof. □

For each vertex s in D, we define $\mathsf{B}(s) := \{u \in V \mid d(s,u) \leq \mathbf{r}(s)\}$.

Lemma 4 *For every vertex s in D,*

$$\sum_{u \in \mathsf{B}(s)} x_1(s,u) \geq 0.5 \quad \Big(\textit{i.e.,} \sum_{u \in V \setminus \mathsf{B}(s)} x_1(s,u) \leq 0.5\Big).$$

Proof If $\mathsf{L}(s) = 0$, then (D1) implies that $x_1(s,s) = 1$. Thus, since $s \in \mathsf{B}(s)$, this completes the proof. In the rest of this proof, we assume that $\mathsf{L}(s) > 0$.

First, we consider the case in which $L \leq 2\mathsf{L}(s)$, that is, $\mathbf{r}(s) = L$. Since the third constraint of the problem (4) implies that $x_1(s,u) = 0$ for every vertex u in $V \setminus \mathsf{B}(s)$, the first constraint of the problem (4) implies that

$$\sum_{u \in \mathsf{B}(s)} x_1(s,u) = \sum_{u \in V} x_1(s,u) = 1.$$

This completes the proof.

Next, we consider the case in which $2\mathsf{L}(s) < L$, that is, $\mathbf{r}(s) = 2\mathsf{L}(s)$. Since

$$\begin{aligned}\mathsf{L}(s) &= \sum_{u \in V} d(s,u) x_1(s,u) = \sum_{u \in \mathsf{B}(s)} d(s,u) x_1(s,u) + \sum_{u \in V \setminus \mathsf{B}(s)} d(s,u) x_1(s,u) \\ &\geq \sum_{u \in V \setminus \mathsf{B}(s)} 2\mathsf{L}(s) x_1(s,u) = 2\mathsf{L}(s) \sum_{u \in V \setminus \mathsf{B}(s)} x_1(s,u),\end{aligned}$$

we have

$$\sum_{u \in V \setminus \mathsf{B}(s)} x_1(s,u) \leq 0.5.$$

Thus, the first constraint of the problem (4) completes the proof. □

Lemma 5 *For every vertex s in D, $y^*(\mathsf{B}(s)) \geq 0.5$.*

Proof Since (x_1, y^*) is a feasible solution of the problem (4), Lemma 4 implies this lemma. □

Lemma 6 *For every pair of distinct vertices s_i, s_j in D,*

$$d(s_i, s_j) > 2\max\{\mathbf{r}(s_i), \mathbf{r}(s_j)\}.$$

Proof Assume that $i > j$. Then since $s_i, s_j \in D$, $d(s_i, s_j) > 2\mathbf{r}(s_i)$. Thus, since $\mathbf{r}(s_j) \leq \mathbf{r}(s_i)$, this completes the proof. □

Lemma 7 *For every pair of distinct vertices s_i, s_j in D, $\mathsf{B}(s_i) \cap \mathsf{B}(s_j) = \emptyset$.*

Proof Assume that $i > j$. If $\mathsf{B}(s_i) \cap \mathsf{B}(s_j) \neq \emptyset$, then

$$d(s_i, s_j) \leq d(u, s_i) + d(u, s_j) \leq \mathbf{r}(s_i) + \mathbf{r}(s_j) \leq \mathbf{r}(s_i) + \mathbf{r}(s_i) = 2\mathbf{r}(s_i)$$

for every vertex u in $\mathsf{B}(s_i) \cap \mathsf{B}(s_j)$. This contradicts Lemma 6. □

Lemma 7 implies that for every vertex u in V, the number of vertices s in D such that $u \in \mathsf{B}(s)$ is at most one. For each vertex u in V, we define $\pi(u)$ as follows. If there exists a vertex s in D such that $u \in \mathsf{B}(s)$, then we define $\pi(u) := s$. If $u \notin \mathsf{B}(s)$ holds for every vertex s in D, then $\pi(u) := \emptyset$. Define $M := \{u \in V \mid \pi(u) \neq \emptyset\}$ and $N := V \setminus M$. For each vertex s in D, we define $\mathsf{N}(s)$ as the set of vertices u in N such that $x_1(s,u) > 0$. For each vertex s in D, we define $\overline{\mathsf{F}}(s)$ as the set of vertices u in $\mathsf{N}(s)$ satisfying the condition that $d(s,u) \leq d(t,u)$ for every vertex t in $D \setminus \{s\}$ such that $u \in \mathsf{N}(t)$. For each vertex s_i in D, we define $\mathsf{F}(s_i)$ as the set of vertices u in $\overline{\mathsf{F}}(s_i)$ satisfying the condition that $i < j$ for every vertex s_j in $D \setminus \{s_i\}$ such that $u \in \overline{\mathsf{F}}(s_j)$. Define $\mathsf{S}(s) := \mathsf{N}(s) \setminus \mathsf{F}(s)$ for each vertex s in D. Define S as the set of vertices u in V such that $u \in \mathsf{S}(s)$ for some vertex s in D. For every vertex u in S, the number of vertices s in D such that $u \in \mathsf{F}(s)$ is equal to one. For each vertex u in S, we define $\phi(u)$ as the vertex s in D such that $u \in \mathsf{F}(s)$. For each vertex s in D, we define $\mathsf{P}(s) := \mathsf{B}(s) \cup \mathsf{F}(s)$ (see Figure 1).

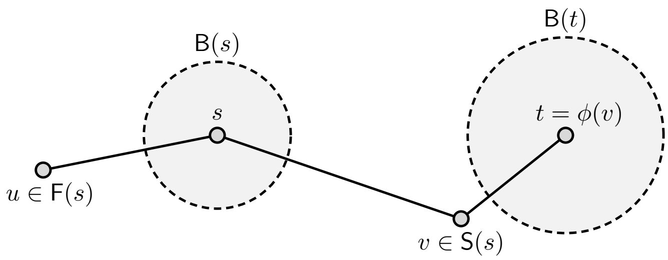

Fig. 1 An illustration of $\mathsf{P}(s)$ and $\mathsf{S}(s)$. We assume that $x_1(s,u), x_1(s,v), x_1(t,v) > 0$.

Lemma 8 *For every pair of vertices s in D and u in $\mathsf{S}(s)$, $d(s, \phi(u)) \leq 2d(s,u) \leq 2L$.*

Proof Since $d(\phi(u), u) \leq d(s,u)$ follows from the definition of $\phi(u)$,

$$d(s, \phi(u)) \leq d(s,u) + d(u, \phi(u)) \leq 2d(s,u).$$

Furthermore, the second inequality follows from the third constraint of the problem (4) and $x_1(s,u) > 0$. This completes the proof. □

Lemma 9 *For every pair of vertices s in D and u in $M \setminus \mathsf{B}(s)$,*
(a) $d(s,\pi(u)) \leq 2d(s,u)$, *and*
(b) $d(s,u) \leq 1.5d(s,\pi(u))$ *and* $d(s,u) \leq d(s,\pi(u)) + L$.

Proof First, we prove the statement (a). Lemma 6 implies that

$$2d(s,u) \geq 2d(s,\pi(u)) - 2\mathbf{r}(\pi(u)) \geq 2d(s,\pi(u)) - d(s,\pi(u)) = d(s,\pi(u)).$$

Next, we prove the statement (b). Lemma 6 implies that

$$d(s,u) \leq d(s,\pi(u)) + \mathbf{r}(\pi(u)) \leq \begin{cases} d(s,\pi(u)) + 0.5d(s,\pi(u)) \\ d(s,\pi(u)) + L. \end{cases}$$

This completes the proof. □

4.2 Construction of x_2

The vector x_2 in $\mathbb{R}_+^{D \times V}$ is defined as follows.

- Define $x_2(s,u) := x_1(s,u)$ for each pair of vertices s in D and u in $\mathsf{P}(s)$.
- Define $x_2(s,u) := 0$ for each pair of vertices s in D and u in $N \setminus \mathsf{F}(s)$.
- Define $x_2(s,u)$ for each pair of vertices s in D and u in $M \setminus \mathsf{B}(s)$ so that the following conditions are satisfied.
  - $x_2(s,u) \leq y^*(u)$ for every pair of vertices s in D and u in $M \setminus \mathsf{B}(s)$.
  - For every pair of vertices s in D and t in $D \setminus \{s\}$,

$$\sum_{u \in \mathsf{B}(t)} x_2(s,u) = \sum_{u \in \mathsf{S}(s)\colon \phi(u)=t} x_1(s,u) + \sum_{u \in \mathsf{B}(t)} x_1(s,u).$$

Notice that Lemmas 4 and 5 imply that there exists a vector in $\mathbb{R}_+^{D \times V}$ satisfying the above conditions. Furthermore, x_2 clearly satisfies the first and second constraints of the problem (4). For each vertex s in D, we define

$$\mathsf{D}(s) := \{t \in D \setminus \{s\} \mid x_2(s,u) > 0 \text{ for some vertex } u \text{ in } \mathsf{B}(t)\}.$$

Lemma 10 *For every pair of vertices s in D and t in $\mathsf{D}(s)$, $d(s,t) \leq 2L$.*

Proof If there exists a vertex u in $\mathsf{B}(t)$ such that $x_1(s,u) > 0$, then the third constraint of the problem (4) implies that $d(s,u) \leq L$. This and Lemma 9(a) imply that $d(s,t) \leq 2L$. If $x_1(s,u) = 0$ for every vertex u in $\mathsf{B}(t)$, then there exists a vertex v in $\mathsf{S}(s)$ such that $\phi(v) = t$. Thus, Lemma 8 implies that $d(s,t) \leq 2L$. □

Define the function $d^\circ \colon D \times V \to \mathbb{R}_+$ by

$$d^\circ(s,u) := \begin{cases} d(s,t) & \text{if } u \in \mathsf{B}(t) \text{ for some vertex } t \text{ in } D \setminus \{s\} \\ d(s,u) & \text{otherwise.} \end{cases}$$

For each pair of vectors x in $\mathbb{R}_+^{D \times V}$ and y in $\mathbb{R}_+^{V}$, we define

$$\overline{\mathsf{w}}^\circ(x) := \sum_{s \in D} \overline{\varpi}(s) \sum_{u \in V} d^\circ(s,u) x(s,u)$$
$$\overline{\mathsf{c}}^\circ(x,y) := \overline{\mathsf{w}}^\circ(x) + \mathsf{f}(y).$$

Lemma 11 $\overline{\mathsf{c}}^{\circ}(x_2, y^*) \le 2\mathsf{c}(x^*, y^*)$.

Proof Lemma 3 implies that it is sufficient to prove that $\overline{\mathsf{w}}^{\circ}(x_2) \le 2\overline{\mathsf{w}}(x_1)$. It is not difficult to see that

$$\begin{aligned}
&\overline{\mathsf{w}}^{\circ}(x_2) = \sum_{s \in D} \overline{\varpi}(s) \sum_{u \in V} d^{\circ}(s,u) x_2(s,u) = \\
&\sum_{s \in D} \overline{\varpi}(s) \sum_{u \in \mathsf{P}(s)} d^{\circ}(s,u) x_2(s,u) + \sum_{s \in D} \overline{\varpi}(s) \sum_{t \in D \setminus \{s\}} \sum_{u \in \mathsf{B}(t)} d^{\circ}(s,u) x_2(s,u) \\
&= \sum_{s \in D} \overline{\varpi}(s) \sum_{u \in \mathsf{P}(s)} d(s,u) x_1(s,u) + \sum_{s \in D} \overline{\varpi}(s) \sum_{t \in D \setminus \{s\}} d(s,t) \sum_{u \in \mathsf{B}(t)} x_2(s,u).
\end{aligned}$$

Lemmas 8 and 9(a) imply that for every vertex s in D,

$$\begin{aligned}
&\sum_{t \in D \setminus \{s\}} d(s,t) \sum_{u \in \mathsf{B}(t)} x_2(s,u) \\
&= \sum_{t \in D \setminus \{s\}} d(s,t) \sum_{u \in \mathsf{S}(s) \colon \phi(u) = t} x_1(s,u) + \sum_{t \in D \setminus \{s\}} d(s,t) \sum_{u \in \mathsf{B}(t)} x_1(s,u) \\
&= \sum_{t \in D \setminus \{s\}} \sum_{u \in \mathsf{S}(s) \colon \phi(u) = t} d(s,t) x_1(s,u) + \sum_{t \in D \setminus \{s\}} \sum_{u \in \mathsf{B}(t)} d(s, \pi(u)) x_1(s,u) \\
&\le \sum_{t \in D \setminus \{s\}} \sum_{u \in \mathsf{S}(s) \colon \phi(u) = t} 2d(s,u) x_1(s,u) + \sum_{t \in D \setminus \{s\}} \sum_{u \in \mathsf{B}(t)} 2d(s,u) x_1(s,u) \\
&= 2 \sum_{u \in \mathsf{S}(s)} d(s,u) x_1(s,u) + 2 \sum_{t \in D \setminus \{s\}} \sum_{u \in \mathsf{B}(t)} d(s,u) x_1(s,u) \\
&= 2 \sum_{u \in V \setminus \mathsf{P}(s)} d(s,u) x_1(s,u).
\end{aligned}$$

Thus,

$$\overline{\mathsf{w}}^{\circ}(x_2) = \sum_{s \in D} \overline{\varpi}(s) \sum_{u \in \mathsf{P}(s)} d(s,u) x_1(s,u) + 2 \sum_{s \in D} \overline{\varpi}(s) \sum_{u \in V \setminus \mathsf{P}(s)} d(s,u) x_1(s,u). \tag{5}$$

The right-hand side of (5) is at most $2\overline{\mathsf{w}}(x_1)$. This completes the proof. $\square$

4.3 Construction of x_3

For each vertex s in D, we define $\xi'(s)$ as follows. If $\mathsf{D}(s) \neq \emptyset$, then we define $\xi'(s)$ as a vertex t in $D \setminus \{s\}$ such that for every vertex t' in $D \setminus \{s\}$, $d(s,t) \le d(s,t')$. If $\mathsf{D}(s) = \emptyset$, then we define $\xi'(s) := s$. Define $E := \{s \in D \mid \xi'(s) = s\}$.

Assume that we are given distinct vertices $s^1, s^2, \ldots, s^k$ in D such that $k \ge 2$ and a mapping $q \colon D \to D$. Then we call the vertices $s^1, s^2, \ldots, s^k$ a *cycle with respect to* q if for every integer i in $\{1, 2, \ldots, k\}$, $q(s^i) = s^{i+1}$, where we define $s^{k+1} := s^1$. If for every integer i in $\{1, 2, \ldots, k\}$,

$$d(s^i, s^{i+1}) < d(s^{i-1}, s^i),$$

then $d(s^k, s^1) < d(s^k, s^1)$ holds, where we define $s^0 := s^k$. This is a contradiction. Thus, there exists an integer j in $\{1, 2, \ldots, k\}$ such that

$$d(s^j, s^{j+1}) \geq d(s^j, s^{j-1}). \tag{6}$$

For each vertex s in D, $\xi(s)$ is defined by the following Procedure 1.

Procedure 1

Step 1: Set $\ell := 1$. Define $q_1(s) := \xi'(s)$ for each vertex s in D.

Step 2: If there exists a cycle $s^1, s^2, \ldots, s^k$ with respect to q_ℓ such that $k \geq 3$, then do the following steps (2-1) to (2-3).

(2-1) Set j to be an integer satisfying (6).

(2-2) Define $q_{\ell+1}(s^j) := s^{j-1}$ and $q_{\ell+1}(s) := q_\ell(s)$ for each vertex s in $D \setminus \{s^j\}$.

(2-3) Set $\ell := \ell + 1$, and go back to the beginning of Step 2.

Step 3: Define $\xi(s) := q_\ell(s)$ for each vertex s in D.

Notice that for every cycle $s^1, s^2, \ldots, s^k$ with respect to ξ, we have $k = 2$. Furthermore, for every vertex s in D, $\xi(s) = s$ if and only if $s \in E$.

The vector x_3 in $\mathbb{R}_+^{D \times V}$ is defined as follows.

- Define $x_3(s,u) := x_2(s,u)$ for each pair of vertices s in D and u in $N \cup \mathsf{B}(s)$.
- Define $x_3(s,u) := 0$ for each pair of vertices s in D and u in $M \setminus (\mathsf{B}(s) \cup \mathsf{B}(\xi(s)))$.
- Define $x_3(s,u)$ for each pair of vertices s in $D \setminus E$ and u in $\mathsf{B}(\xi(s))$ so that the following conditions are satisfied.
  - $x_3(s,u) \leq y^*(u)$ for every pair of vertices s in $D \setminus E$ and u in $\mathsf{B}(\xi(s))$.
  - For every vertex s in $D \setminus E$,

$$\sum_{u \in \mathsf{B}(\xi(s))} x_3(s,u) = \sum_{t \in D \setminus \{s\}} \sum_{u \in \mathsf{B}(t)} x_2(s,u).$$

Notice that Lemmas 4 and 5 imply that there exists a vector in $\mathbb{R}_+^{D \times V}$ satisfying the above conditions. Furthermore, x_3 clearly satisfies the first and second constraints of the problem (4).

Lemma 12 *For every pair of vertices s in D and t in $D \setminus \{s\}$, $d(s, \xi(s)) \leq d(s,t)$.*

Proof This lemma immediately follows from the definition of $\xi'(s)$ and (6). □

Lemma 13 *For every vertex s in D, $d(s, \xi(s)) \leq 2L$.*

Proof If $\mathsf{D}(s) \neq \emptyset$, then Lemmas 10 and 12 imply that $d(s, \xi(s)) \leq 2L$. □

Lemma 14 $\overline{c}^\circ(x_3, y^*) \leq \overline{c}^\circ(x_2, y^*)$.

Proof Clearly, we have

$$\sum_{s \in E} \overline{\omega}(s) \sum_{u \in V} d^\circ(s,u) x_3(s,u) = \sum_{s \in E} \overline{\omega}(s) \sum_{u \in V} d^\circ(s,u) x_2(s,u). \tag{7}$$

Furthermore, since for every triplet of vertices s in $D \setminus E$, t in $D \setminus \{s\}$, and u in $\mathsf{B}(t)$, $d(s,\xi(s)) \le d(s,t) = d^\circ(s,u)$, we have

$$\begin{aligned}
&\sum_{s \in D\setminus E} \overline{\omega}(s) \sum_{u \in V} d^\circ(s,u) x_3(s,u) \\
&= \sum_{s \in D\setminus E} \overline{\omega}(s) \sum_{u \in \mathsf{P}(s)} d^\circ(s,u) x_3(s,u) + \sum_{s \in D\setminus E} \overline{\omega}(s) d(s,\xi(s)) \sum_{u \in \mathsf{B}(\xi(s))} x_3(s,u) \\
&= \sum_{s \in D\setminus E} \overline{\omega}(s) \sum_{u \in \mathsf{P}(s)} d^\circ(s,u) x_2(s,u) + \sum_{s \in D\setminus E} \overline{\omega}(u) d(s,\xi(s)) \sum_{t \in D\setminus\{s\}} \sum_{u \in \mathsf{B}(t)} x_2(s,u) \\
&\le \sum_{s \in D\setminus E} \overline{\omega}(s) \sum_{u \in \mathsf{P}(s)} d^\circ(s,u) x_2(s,u) + \sum_{s \in D\setminus E} \overline{\omega}(s) \sum_{t \in D\setminus\{s\}} \sum_{u \in \mathsf{B}(t)} d^\circ(s,u) x_2(s,u).
\end{aligned} \tag{8}$$

Then (7) and (8) imply that $\overline{\mathsf{w}}^\circ(x_3) \le \overline{\mathsf{w}}^\circ(x_2)$. This completes the proof. □

For each vertex s in D, we define $\varphi(s)$ by the following Procedure 2.

Procedure 2

Step 1: For each vertex s in D, we set $\varphi(s) := \xi(s)$.

Step 2: If there does not exist a vertex s in D satisfying the following conditions, then output φ and halt.

- $\varphi(s) \ne s$ holds and $\varphi(\varphi(s)) \ne s$ holds.
- There exists a vertex t in $D \setminus \{s\}$ such that $\varphi(t) = s$.
- For any vertex t in $D \setminus \{s\}$ such that $\varphi(t) = s$, there does not exist a vertex t' in $D \setminus \{s,t\}$ such that $\varphi(t') = t$.

If there exists a vertex s in D satisfying the above conditions, then we set $\bar{s}$ to be a vertex in D satisfying these conditions.

Step 3: Set $\bar{t}$ to be a vertex in $D \setminus \{\bar{s}\}$ such that $\varphi(\bar{t}) = \bar{s}$ and

$$d(\bar{s},\bar{t}) = \min\{d(\bar{s},s) \mid s \in D \setminus \{\bar{s}\}, \varphi(s) = \bar{s}\}.$$

Step 4: If $d(\bar{s},\bar{t}) \le 2d(\bar{s},\varphi(\bar{s}))$, then we set $\varphi(\bar{s}) := \bar{t}$, and go back to Step 2.

Step 5: Set $\varphi(s) := \varphi(\bar{s})$ for each vertex s in $D \setminus \{\bar{s}\}$ such that $\varphi(s) = \bar{s}$, and go back to Step 2.

Notice that for every vertex s in D, $\varphi(s) = s$ if and only if $s \in E$. Each connected component in the directed graph obtained from φ is one of the following types A, B, and C in Figure 2. (In this graph, for each vertex s in $D \setminus E$, there exists an arc from s to $\varphi(s)$.) In the type A, there exist distinct vertices s,t in D such that $\varphi(s) = t$, $\varphi(t) = s$, $\varphi(s') \in \{s,t\}$ for every vertex s' in this component except s,t, and there exist vertices s'',t'' in $D \setminus \{s,t\}$ such that $\varphi(s'') = s$ and $\varphi(t'') = t$. In this type, the definition of Procedure 2 implies that $\xi(s) = t$ and $\xi(t) = s$. In the type B, there exist distinct vertices s,t in D such that $\varphi(s) = t$, $\varphi(t) = s$, and there exists a vertex s' in $\{s,t\}$ such that $\varphi(s'') = s'$ for every vertex s'' in this component except s,t. In the type C, there exists a vertex s in D such that $\varphi(s) = s$ and $\varphi(t) = s$ for every vertex t in this component except s.

Lemma 15 *For every pair of vertices s in D and t in $D \setminus \{s\}$ such that $\varphi(t) = s$, we have $d(s,\varphi(s)) \le d(s,t)$.*

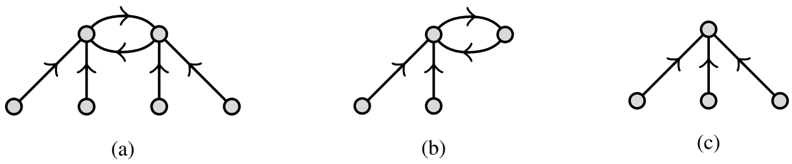

Fig. 2 (a) Type A. (b) Type B. (c) Type C.

Proof Let C be the connected component containing s in the directed graph obtained from φ. If $\varphi(s) = s$ (i.e., the type of C is C), the this lemma clearly holds. If the type of C is A, then this lemma follows from Lemma 12. If the type of C is B, then this lemma follows from the definition of $\bar{t}$ in Step 3 of Procedure 2. □

Lemma 16 *For every vertex s in D, $d(s,\varphi(s)) \leq 2d(s,\xi(s))$.*

Proof Let s be a vertex in D. If $\varphi(s) = \xi(s)$, then this lemma clearly holds. Thus, we consider the case in which $\varphi(s) \neq \xi(s)$. If $\varphi(s)$ is updated in Step 4, then this lemma clearly holds. Thus, we consider the case in which $\varphi(s)$ is not updated in Step 4. In this case, it is not difficult to see that there exist distinct vertices $s^1, s^2, \ldots, s^k$ in D such that $s^1 = s$, $s^k = \varphi(s)$, $\xi(s^i) = s^{i+1}$ for every integer i in $\{1,2,\ldots,k-1\}$, and

$$d(s,\varphi(s)) \leq \sum_{i=1}^{k-1} d(s^i, s^{i+1}) \leq \sum_{i=1}^{k-1} \frac{d(s^1,s^2)}{2^{i-1}} \leq 2d(s^1,s^2) = 2d(s,\xi(s)).$$

This completes the proof. □

Lemma 17 *For every vertex s in D, $d(s,\varphi(s)) \leq 4L$.*

Proof This lemma immediately follows from Lemmas 13 and 16. □

4.4 Construction of x_4

Define Z the set of vertices s in D such that $\varphi(s) \neq \xi(s)$. Then the vector x_4 in $\mathbb{R}_+^{D\times V}$ is defined as follows.

- Define $x_4(s,u) := x_3(s,u)$ for each pair of vertices s in $D \setminus Z$ and u in V.
- Define $x_4(s,u) := x_3(s,u)$ for each pair of vertices s in Z and u in V such that $u \notin \mathsf{B}(\xi(s)) \cup \mathsf{B}(\varphi(s))$.
- Define $x_4(s,u) := 0$ for each pair of vertices s in Z and u in $\mathsf{B}(\xi(s))$.
- Define $x_4(s,u)$ for each pair of vertices s in Z and u in $\mathsf{B}(\varphi(s))$ so that the following conditions are satisfied.
  - $x_4(s,u) \leq y^*(u)$ for each pair of vertices s in Z and u in $\mathsf{B}(\varphi(s))$.
  - For every vertex s in Z,

$$\sum_{v\in \mathsf{B}(\varphi(s))} x_4(s,u) = \sum_{v\in \mathsf{B}(\xi(s))} x_3(s,u).$$

Notice that Lemmas 4 and 5 imply that there exists a vector in $\mathbb{R}_+^{D\times V}$ satisfying the above conditions. Furthermore, x_4 clearly satisfies the first and second constraints of the problem (4).

Lemma 18 $\overline{\mathsf{c}}^\circ(x_4,y^*) \le 2\overline{\mathsf{c}}^\circ(x_3,y^*)$.

Proof It is sufficient to prove that $\overline{\mathsf{w}}^\circ(x_4) \le 2\overline{\mathsf{w}}^\circ(x_3)$. Lemma 16 implies that

$$\begin{aligned}
&\sum_{s\in D\setminus E} \overline{\varpi}(s) \sum_{u\in V} d^\circ(s,u)x_4(s,u)\\
&= \sum_{s\in D\setminus E} \overline{\varpi}(s) \sum_{u\in \mathsf{P}(s)} d^\circ(s,u)x_4(s,u) + \sum_{s\in D\setminus E} \overline{\varpi}(s) \sum_{u\in \mathsf{B}(\varphi(s))} d^\circ(s,\varphi(s))x_4(u,u)\\
&\le \sum_{s\in D\setminus E} \overline{\varpi}(s) \sum_{u\in \mathsf{P}(s)} d^\circ(s,u)x_3(s,u) + 2\sum_{s\in D\setminus E} \overline{\varpi}(s) \sum_{u\in \mathsf{B}(\xi(s))} d^\circ(s,\xi(s))x_3(s,u).
\end{aligned}$$

This completes the proof. □

4.5 Construction of x_5

For each vertex s in D, we define the subset $\mathsf{G}(s)$ of $\mathsf{F}(s)$ as follows. First, we assume that $\varphi(s) \neq s$. Then we define $\mathsf{G}(s)$ as the set of vertices u in $\mathsf{F}(s)$ such that $d(s,u) > d(s,\varphi(s))$. Second, we consider the case in which $\varphi(s) = s$. In this case, we define $\mathsf{G}(s)$ as the set of vertices u in $\mathsf{F}(s)$ satisfying the condition that $d(s,u) > d(s,t)$ for some vertex t in $D\setminus\{s\}$ such that $\varphi(t) = s$.

Define $\overline{\varphi}(s)$ for each vertex s in D as follows. For each vertex s in D such that at least one of $\mathsf{G}(s) = \emptyset$ and $\varphi(s) \neq s$ holds, we define $\overline{\varphi}(s) := \varphi(s)$. For each vertex s in D such that $|\mathsf{G}(s)| \ge 1$ and $\varphi(s) = s$, we define $\overline{\varphi}(s)$ as a vertex t in $D\setminus\{s\}$ satisfying the condition that $\varphi(t) = s$ and $d(s,t) \le d(s,t')$ for every vertex t' in $D\setminus\{s\}$ such that $\varphi(t') = s$.

The vector x_5 in $\mathbb{R}_+^{D\times V}$ is defined as follows.

- Define $x_5(s,u) := x_4(s,u)$ for each pair of vertices s in D such that $\mathsf{G}(s) = \emptyset$ and u in V.
- Define $x_5(s,u) := x_4(s,u)$ for each pair of vertices s in D such that $|\mathsf{G}(s)| \ge 1$ and u in V such that $u \notin \mathsf{B}(\overline{\varphi}(s)) \cup \mathsf{G}(s)$.
- Define $x_5(s,u) := 0$ for each pair of vertices s in D such that $|\mathsf{G}(s)| \ge 1$ and u in $\mathsf{G}(s)$.
- Define $x_5(s,u)$ for each pair of vertices s in D such that $|\mathsf{G}(s)| \ge 1$ and u in $\mathsf{B}(\overline{\varphi}(s))$ so that the following conditions are satisfied.
  - $x_5(s,u) \le y^*(u)$ for each pair of vertices s in D such that $|\mathsf{G}(s)| \ge 1$ and u in $\mathsf{B}(\overline{\varphi}(s))$.
  - For every vertex s in D such that $|\mathsf{G}(s)| \ge 1$,
$$\sum_{u\in \mathsf{B}(\overline{\varphi}(s))} x_5(s,u) = \sum_{u\in \mathsf{B}(\overline{\varphi}(s))} x_4(s,u) + \sum_{u\in \mathsf{G}(s)} x_4(s,u).$$

Notice that Lemmas 4 and 5 imply that there exists a vector in $\mathbb{R}_+^{D\times V}$ satisfying the above conditions. Furthermore, x_5 clearly satisfies the first and second constraints of the problem (4). Define $\overline{E}$ as the set of vertices s in D such that $\overline{\varphi}(s) = s$.

Lemma 19 *For every pair of vertices s in D and t in $D \setminus \{s\}$ such that $\overline{\varphi}(t) = s$, we have $d(s, \overline{\varphi}(s)) \leq d(s,t)$.*

Proof This lemma immediately follows from Lemma 15 and the definition of $\overline{\varphi}(s)$ for vertices s in D such that $\varphi(s) = s$. □

For each vertex s in D, we define $\mathsf{F}^*(s) := \mathsf{F}(s) \setminus \mathsf{G}(s)$.

Lemma 20 *For every triplet of vertices s in D, u in $\mathsf{F}^*(s)$, and t in $D \setminus \{s\}$ such that $\overline{\varphi}(t) = s$, we have $d(s,u) \leq d(s,t)$.*

Proof If $s \in \overline{E}$, then $\mathsf{G}(s) = \emptyset$. This implies that $d(s,u) \leq d(s,t)$. Assume that $s \in D \setminus \overline{E}$. Then the definition of $\overline{\varphi}(s)$ implies that $d(s,u) \leq d(s, \overline{\varphi}(s))$, and Lemma 19 implies that $d(s, \overline{\varphi}(s)) \leq d(s,t)$. Thus, $d(s,u) \leq d(s,t)$. □

Lemma 21 $\overline{c}^{\circ}(x_5, y^*) \leq \overline{c}^{\circ}(x_4, y^*)$.

Proof This lemma immediately follows from the definition of $\overline{\varphi}$. □

4.6 Construction of $(x_{\mathrm{f}}, y_{\mathrm{f}})$

For each vertex s in D, we define $\mathsf{Q}(s) := \mathsf{B}(s) \cup \mathsf{F}^*(s)$. Define

$$\mathbf{R}_1 := \{\{s\} \mid s \in \overline{E}\}, \quad \mathbf{R}_2 := \{\{s,t\} \subseteq D \setminus \overline{E} \mid \overline{\varphi}(s) = t,\ \overline{\varphi}(t) = s\}.$$

Define $\mathbf{R} := \mathbf{R}_1 \cup \mathbf{R}_2$. For each vector z in $\mathbb{R}_+^V$ and each vertex s in D, we define

$$\mathsf{p}(z,s) := \sum_{u \in \mathsf{Q}(s)} d(s,u)z(u) + d(s, \overline{\varphi}(s)) \Big(1 - \sum_{u \in \mathsf{Q}(s)} z(u)\Big).$$

For each vector z in $\mathbb{R}_+^V$, we define $\mathsf{h}_1(z)$ and $\mathsf{h}_2(z)$ by

$$\mathsf{h}_1(z) := \sum_{s \in D} \overline{\varpi}(s) \mathsf{p}(z,s), \quad \mathsf{h}_2(z) := \sum_{u \in V} f(u) z(u).$$

We consider the following problem.

$$\begin{array}{ll} \text{Minimize} & \mathsf{h}_1(z) + \mathsf{h}_2(z) \\ \text{subject to} & z(\mathsf{Q}(s)) \leq 1 \quad (\forall s \in D) \\ & \displaystyle\sum_{s \in R} z(\mathsf{Q}(s)) \geq 1 \quad (\forall R \in \mathbf{R}) \\ & z \in \mathbf{P}_{\mathbf{M}} \cap \mathbb{R}_+^V. \end{array} \tag{9}$$

Lemma 22 ([13, Lemma 2]) *There exists an optimal solution z^* of the problem* (9) *such that $z^* \in \{0,1\}^V$.*

It is known [13, p.455] that we can compute z^* in polynomial time.

Lemma 23 $\mathsf{h}_1(z^*) + \mathsf{h}_2(z^*) \leq 4\mathsf{c}(x^*, y^*)$.

Proof Define the vector z in $\mathbb{R}_+^V$ as follows.

- Define $z(u) := 0$ for each vertex u in V such that $u \notin \mathsf{Q}(s)$ holds for every vertex s in D.
- Define $z(u) := x_5(s,u)$ for each pair of vertices s in D and u in $\mathsf{Q}(s)$.

The first constraint of the problem (4) implies that z satisfies the first constraint of the problem (9). The first constraint of the problem (4) and Lemma 4 imply that z satisfies the second constraint of the problem (9). Lastly, the second constraint of the problem (4) implies that $z \in \mathbf{P_M} \cap \mathbb{R}_+^V$. Since $\mathsf{h}_1(z) = \overline{\mathsf{w}}^\circ(x_5)$ and $\mathsf{h}_2(z) \le \mathsf{f}(y^*)$, Lemmas 11, 14, 18, and 21 imply that

$$\mathsf{h}_1(z) + \mathsf{h}_2(z) \le \overline{\mathsf{c}}^\circ(x_5, y^*) \le 2\overline{\mathsf{c}}^\circ(x_2, y^*) \le 4\mathsf{c}(x^*, y^*).$$

This completes the proof. □

For each vertex s in D, we define $\sigma(s)$ as follows. If there exists a vertex u in $\mathsf{Q}(s)$ such that $z^*(u) = 1$, then we define $\sigma(s) := s$. Assume that $z^*(u) = 0$ for every vertex u in $\mathsf{Q}(s)$. Let R be a member of $\mathbf{R}$ such that $\overline{\varphi}(s) = t$ for some vertex t in R. Notice that there exists a member R of $\mathbf{R}$ satisfying this condition (see Figure 2). Then we define $\sigma(s)$ as a vertex u in $\bigcup_{t \in R} \mathsf{Q}(t)$ such that $z^*(u) = 1$.

Lemma 24 *For every vertex s in D, the following statements hold.*

(a) $d(\overline{\varphi}(s), \sigma(s)) \le d(s, \overline{\varphi}(s))$.
(b) $d(s, \sigma(s)) \le 2d(s, \overline{\varphi}(s)) \le 8L$.

Proof First, we prove the statement (a). If $\sigma(s) \in \{s, \overline{\varphi}(s)\}$, then this statement clearly holds. Thus, we consider the case in which $\sigma(s) \notin \{s, \overline{\varphi}(s)\}$. In this case, $\overline{\varphi}(\overline{\varphi}(s)) = \sigma(s)$ (see Figure 2). Thus, Lemma 19 implies this statement.

Next, we prove the statement (b). The statement (a) implies that

$$d(s, \sigma(s)) \le d(s, \overline{\varphi}(s)) + d(\overline{\varphi}(s), \sigma(s)) \le 2d(s, \overline{\varphi}(s)).$$

If $s \in E \setminus \overline{E}$, then since $\varphi(\overline{\varphi}(s)) = s$, Lemma 17 implies that

$$d(s, \overline{\varphi}(s)) = d(\varphi(\overline{\varphi}(s)), \overline{\varphi}(s)) \le 4L.$$

Thus, the second inequality of the statement (b) follows from Lemma 17. □

The pair $(x_\mathrm{f}, y_\mathrm{f})$ of vectors x_f in $\{0,1\}^{D \times V}$ and y_f in $\{0,1\}^V$ is defined as follows.

- Define $x_\mathrm{f}(s,u) := z^*(u)$ for each pair of vertices s in D and u in $\mathsf{Q}(\sigma(s))$.
- Define $x_\mathrm{f}(s,u) := 0$ for each pair of vertices s in D and u in $V \setminus \mathsf{Q}(\sigma(s))$.
- Define $y_\mathrm{f}(u) := z^*(u)$ for each vertex u in V.

Clearly, $(x_\mathrm{f}, y_\mathrm{f})$ satisfies the first and second constraints of the problem (2).

Lemma 25 *For every pair of vertices s in D and u in V such that $x_\mathrm{f}(s,u) > 0$, we have $d(s,u) \le 9L$.*

Proof If $\sigma(s) = s$, then $u \in \mathsf{Q}(s)$. Thus, $d(s,u) \leq L$. Assume that $\sigma(s) \neq s$. In this case, Lemma 24(b) implies that $d(s,\sigma(s)) \leq 8L$. If $u \in \mathsf{B}(\sigma(s))$, then this lemma follows from Lemma 9(b). If $u \in \mathsf{F}^*(\sigma(s))$, then $x_1(\sigma(s),u) > 0$ implies that $d(\sigma(s),u) \leq L$. Thus,

$$d(s,u) \leq d(s,\sigma(s)) + d(\sigma(s),u) \leq 8L + L \leq 9L.$$

This completes the proof. □

Lemma 26 $\overline{\mathsf{c}}(x_{\mathrm{f}}, y_{\mathrm{f}}) \leq 12\mathsf{c}(x^*, y^*)$.

Proof Define

$$A_1 := \{s \in D \mid \text{there exists a vertex } u \text{ in } \mathsf{Q}(s) \text{ such that } z^*(u) = 1\}$$
$$A_2 := \{s \in D \setminus A_1 \mid \sigma(s) = \overline{\varphi}(s)\}$$
$$A_3 := \{s \in D \setminus A_1 \mid \sigma(s) \neq \overline{\varphi}(s)\}.$$

For each vertex s in D, we define

$$\mathsf{p}^*(s) := \mathsf{p}(z^*, s), \quad \mathsf{q}(s) := \sum_{u \in V} d(s,u) x_{\mathrm{f}}(s,u).$$

For every vertex s in A_1, since there exists a vertex u in $\mathsf{Q}(s)$ such that $x_{\mathrm{f}}(s,u) = z^*(u) = 1$, we have $\mathsf{q}(s) = d(s,u) = \mathsf{p}^*(s)$.

Let s be a vertex in A_2. Then $\mathsf{p}^*(s) = d(s,\overline{\varphi}(s))$. First, we assume that there exists a vertex u in $\mathsf{B}(\overline{\varphi}(s))$ such that $x_{\mathrm{f}}(s,u) = 1$. In this case, Lemma 9(b) implies that

$$\mathsf{q}(s) = d(s,u) \leq 1.5 d(s,\overline{\varphi}(s)) = 1.5\mathsf{p}^*(s).$$

Next, we assume that there exists a vertex u in $\mathsf{F}^*(\overline{\varphi}(s))$ such that $x_{\mathrm{f}}(s,u) = 1$. In this case, Lemma 20 implies that $d(\overline{\varphi}(s),u) \leq d(\overline{\varphi}(s),s)$. Thus,

$$\mathsf{q}(s) = d(s,u) \leq d(s,\overline{\varphi}(s)) + d(\overline{\varphi}(s),u) \leq 2d(s,\overline{\varphi}(s)) = 2\mathsf{p}^*(s).$$

Let s be a vertex in A_3. Then $\mathsf{p}^*(s) = d(s,\overline{\varphi}(s))$. First, we assume that there exists a vertex u in $\mathsf{B}(\sigma(s))$ such that $x_{\mathrm{f}}(s,u) = 1$. Then Lemmas 9(b) and 24(b) imply that

$$\mathsf{q}(s) = d(s,u) \leq 1.5 d(s,\sigma(s)) \leq 3d(s,\overline{\varphi}(s)) = 3\mathsf{p}^*(s).$$

Next, we assume that there exists a vertex u in $\mathsf{F}^*(\sigma(s))$ such that $x_{\mathrm{f}}(s,u) = 1$. In this case, since $\overline{\varphi}(\overline{\varphi}(s)) = \sigma(s)$, Lemma 20 implies that $d(\sigma(s),u) \leq d(\overline{\varphi}(s),\sigma(s))$. Thus, Lemma 24(a) implies that

$$d(\sigma(s),u) \leq d(\overline{\varphi}(s),\sigma(s)) \leq d(\overline{\varphi}(s),s).$$

This and Lemma 24(b) imply that

$$\mathsf{q}(s) = d(s,u) = d(s,\sigma(s)) + d(\sigma(s),u) \leq 3d(s,\overline{\varphi}(s)) = 3\mathsf{p}^*(s).$$

Since

$$\overline{\mathsf{w}}(x_{\mathrm{f}}) = \sum_{s \in A_1 \cup A_2 \cup A_3} \overline{\varpi}(s)\mathsf{q}(s) \leq 3 \sum_{s \in D} \overline{\varpi}(s)\mathsf{p}^*(s) = 3\mathsf{h}_1(z^*)$$

and $\mathsf{f}(y_{\mathrm{f}}) = \mathsf{h}_2(z^*)$, Lemma 23 implies that

$$\overline{\mathsf{c}}(x_{\mathrm{f}}, y_{\mathrm{f}}) \leq 3\mathsf{h}_1(z^*) + \mathsf{h}_2(z^*) \leq 12\mathsf{c}(x^*, y^*).$$

This completes the proof. □

Define the vector x_{o} in $\{0,1\}^{V \times V}$ by $x_{\mathrm{o}}(u,v) := x_{\mathrm{f}}(s_i,v)$ for each triplet of vertices s_i in D, u in V_i, and v in V. Define $y_{\mathrm{o}}(u) := y_{\mathrm{f}}(u)$ for each vertex u in V.

Theorem 1 $(x_{\mathrm{o}}, y_{\mathrm{o}})$ *is an* $(11,16)$*-bicriteria approximation solution of the problem* (2).

Proof This theorem follows from Lemmas 2, 25, and 26. □

5 Conclusion

In this paper, we first introduce a matroid generalization of the variant of the k-median problem considered by Alamdari and Shmoys [1]. Then we propose a polynomial-time $(11,16)$-bicriteria approximation algorithm for this problem. An apparent future work is to consider whether the approximation ratio can be improved by using the ideas of the faster algorithms for the matroid median problem proposed by Swamy [17] and Charikar and Li [6].

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