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出版情報：SPACEFLIGHT MECHANICS 2009 : ADVANCES IN THE ASTRONAUTICAL SCIENCES. 134, pp.245-260, 2009. American Astronautical Society
バージョン：
権利関係：

AAS 09-117

A NEW OPTIMAL ORBIT CONTROL FOR TWO-POINT BOUNDARY-VALUE PROBLEM USING GENERATING FUNCTIONS

Mai Bando* and Hiroshi Yamakawa†

The optimal control problem of a spacecraft using impulsive and continuous thrust where the terminal state and time interval are explicitly given is considered. Using a recently developed technique based on Hamilton-Jacobi theory, we develop a method to approximate the solution of the Hamilton-Jacobi equation which can solve the two-point boundary-value problem. The proposed method is based on the successive approximation and Galerkin spectral method with Chebyshev polynomials. This approach is expected to derive the analytical solution of the optimal control problem in the large domain. Numerical simulation is given to illustrate the theory.

INTRODUCTION

The problem of optimizing trajectory using impulsive/continuous thrust in a central gravity field has been a topic of continual interest for many decades. It is well known that Pontryagin's maximum principle provides necessary conditions for the optimality of the control of a dynamical system. This principle renders the optimal control problem into two-point boundary-value problem of Hamiltonian system. However, the guidance law designed based on Pontryagin's maximum principle is obtained in an open-loop manner; the nominal control history is not able to respond to any perturbation that could alter the trajectory of the spacecraft. We employ the generating function approach[1,2] for canonical transformations in Hamiltonian dynamic systems in which the nonlinear motion of spacecraft is described as the solution to a two-point boundary value problem (TPBVP). This approach enables us to solve the optimal feedback control problem for impulsive/continuous control analytically. In Refs. 1 and 2, Hamilton-Jacobi (HJ) equation associaed with TPBVP is solved by perturbation method in which the motion of a spacecraft is modeled as a Hamiltonian dynamic system in the vicinity of a nominal trajectory. Then local solution to the generating function is constructed in the vicinity of a nominal trajectory using Taylor series. This technique is only applicable to systems that are small perturbations of a linear system. In addition, these methods are inherently tied to the convergence of a power series for which it is difficult to estimate the region of convergence.

In this paper, we investigate the method to construct generating functions associated with the original nonlinear original dynamics. The difficulty to tackle the nonlinear dynamics is in solving HJ equation which is 1st order nonlinear partial differential equation. The Hamilton-Jacobi-Bellman (HJB) equation in optimal control theory, whose solution is the value function of the optimal control problem, is a variant of the HJ equation of mechanics for the case of dynamics parameterized by a

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control variable. Motivated by this observation, our approach is based on the idea to solve HJ equation associated with canonical transformations as the infimum of optimal cost which is effectively solved by the Galerkin spectral method with Chebyshev polynomials.

PROBLEM STATEMENT

Consider a spacecraft subject to the central gravity field. The normalized equation of motion is described as:

$$\ddot{\boldsymbol{r}} = -\frac{\boldsymbol{r}}{r^3}. \tag{1}$$

Now we consider the following problem.

[Problem] Impulsive thrust orbital transfer (Lambert's Problem[3])
Consider a spacecraft located at $\boldsymbol{r}_0$ at $t = t_0$. Find the initial velocity which is required to transfer to $\boldsymbol{r}_f$ in time t_f.

CANONICAL TRANSFORMATION AND HAMILTON-JACOBI EQUATION

In this section, we briefly review how the solution of TPBVP are derived from generating functions. Let $H(\boldsymbol{q}, \boldsymbol{p}, t)$ be the Hamiltonian of the system. Consider the canonical transformations between a fixed terminal state $(q_f(t_f)$, $p_f(t_f))$ and moving initial state $(q(t)$, $p(t))$. Since the former set of coordinates $(q_f(t_f)$, $p_f(t_f))$ are constants, Hamiltonian of a fixed state can be defined as zero. Then the generating functions associated with the transformations can be written in following forms:[4]

$$F_1 = F_1(q_f, q, t), \;\; F_2 = F_2(q_f, p, t), \;\; F_3 = F_3(p_f, q, t), \;\; F_4 = F_4(p_f, p, t). \tag{2}$$

The generating functions provide connections for each coordinate system:

$$p_f = \frac{\partial F_1(q_f, q, t)}{\partial q_f}, \tag{3}$$

$$p(t) = -\frac{\partial F_1(q_f, q, t)}{\partial q}, \tag{4}$$

$$-\frac{\partial F_1(q_f, q, t)}{\partial t} + H(q, -\frac{\partial F_1(q_f, q, t)}{\partial q}, t) = 0, \tag{5}$$

$$p_f = \frac{\partial F_2(q_f, p, t)}{\partial q_f}, \tag{6}$$

$$q(t) = \frac{\partial F_2(q_f, p, t)}{\partial p}, \tag{7}$$

$$-\frac{\partial F_2(q_f, p, t)}{\partial t} + H(\frac{\partial F_2(q_f, p, t)}{\partial p}, p, t) = 0, \tag{8}$$

$$q_f = -\frac{\partial F_3(p_f, q, t)}{\partial p_f}, \tag{9}$$

$$p(t) = -\frac{\partial F_3(p_f, q, t)}{\partial q}, \tag{10}$$

$$-\frac{\partial F_3(p_f, q, t)}{\partial t} + H(q, -\frac{\partial F_3(p_f, q, t)}{\partial p}, t) = 0, \tag{11}$$

$$q_f = \frac{\partial F_4(p_f, p, t)}{\partial p_f}, \tag{12}$$

$$q(t) = -\frac{\partial F_1(p_f, p, t)}{\partial p}, \tag{13}$$

$$-\frac{\partial F_4(p_f, p, t)}{\partial t} + H(-\frac{\partial F_4(p_f, p, t)}{\partial p}, p, t) = 0, \tag{14}$$

These functions are all related by the Legendre transformations:

$$F_2(q_f, p, t) = F_1(q_f, q, t) + p^T q, \tag{15}$$

$$F_3(p_f, q, t) = F_1(q_f, q, t) - p_f^T q_f, \tag{16}$$

$$F_4(p_f, p, t) = F_2(q_f, p, t) + p_f^T q_f. \tag{17}$$

Solution of TPBVP

Following Ref. 1, we introduce the canonical transformations between a moving initial state ($\boldsymbol{q}$, $\boldsymbol{p}, t$) and fixed final state ($\boldsymbol{q}_f, \boldsymbol{p}_f, t_f$). Then the solution to the TPBVP can be expressed by generating functions. For example, F_3 generating function satisfies $q_f = -\dfrac{\partial F_3}{\partial p_f}$, $p = -\dfrac{\partial F_3}{\partial q}$, and Hamitlon-Jacobi (HJ) equation $-\dfrac{\partial F_3(p_f, q, t_f; t)}{\partial t} + H(q, -\dfrac{\partial F_3}{\partial q}, t_f; t) = 0$. Then we can obtain solution of desired variables using Legendre transformation. In general, the TPBVP must be solved repetively for the varying boundary conditions. By the generating function techinique, it is possible to specify optimal solution by simple function evaluations with new boundary conditions.

PROPOSED METHOD

The focus of our method is motivated by the direct treatment of original nonlinear dynamics. In our approach, the HJ equation is approximated by 1st order linear partial differential equation using successive approximation, then it is solved by Galerkin spectral method with Chebyshev polynomials. Here we formulate the method to approximate solution to HJ equation. To apply the successive approximation to solve HJ equation, we relate the optimal control problem derived from the HJ equation. For this purpose, we need to specify boundary conditions of the generating function. Since at $t = t_f$ the current state are the same as the terminal state, the transformation induces the identity transformation, i.e.,

$$F_2(q_f, p, t_f; t_f) = q_f^T p, \tag{18}$$

$$F_3(p_f, q, t_f; t_f) = -p_f^T q. \tag{19}$$

However, such a transformation can only be described by F_2 and F_3 generating functions because singularity arises for F_1 and F_4 generating functions at $t = t_f$. Hence we wish to solve the HJ equation for F_2 or F_3 generating function. In the following, we show that only F_3 satisfy the sufficient conditions to obtain the solution to the HJ equation.

We show our approach with the following example. Consider the problem in one dimensional form of two-body central force problem: find initial and final velocity when initial and final position is in given time interval [t_0, t_f]. This problem is has an analytical solution.[3] Consider the

Hamiltonian descried by

$$H(q,p) = \frac{1}{2}p^2 + U(q), \tag{20}$$

where $U(q) \leq 0$ is a potential function, e.g. $U(q) = -\frac{1}{q}$. Note that

$$L(q,p,t) = \dot{q}p - H(q,p) = \frac{1}{2}p^2 + \frac{1}{q} \geq 0, \tag{21}$$

for all p. F_3 generating function must satisfy HJ equation (11):

$$\begin{aligned} &-\frac{\partial F_3}{\partial t} + H(q, -\frac{\partial F_3}{\partial p}, t) = 0 \\ \Longleftrightarrow &\frac{\partial F_3}{\partial t} - \frac{1}{2}\Big(\frac{\partial F_3}{\partial q}\Big)^2 - U(q) = 0. \end{aligned} \tag{22}$$

Note that Eq. (22) can be written as

$$\frac{\partial F_3}{\partial t} + \min_{\lambda}\Big[\frac{\partial F_3}{\partial q}\lambda + \frac{1}{2}\lambda^2 - U(q)\Big] = 0, \tag{23}$$

where substituting $\lambda = -\frac{\partial F_3}{\partial q}$ into Eq. (23) yields Eq. (22). We point out that the HJ equation (11) is expressed as the result of minimization of pre-Hamiltonian $\bar{H}(q,p,\lambda) = \frac{\partial F_3}{\partial q}\lambda + \frac{1}{2}\lambda^2 - U(q)$ and that's why we employ the F_3 generating function. We cannot use F_2 generating function for the lack of the minimizer, though HJ equation (8) is similar to (11).

Remark: When the canonical transformation between a fixed initial state $(q_0(t_0),\ p_0(t_0))$ and moving final state $(q(t),\ p(t))$ is employed, HJ equation for F_2 generating can be expressed as the result of minimization of pre-Hamiltonian. In this case we seek the F_2 generating function not F_3.

To apply the successive approximation to solve HJ equation, consider the affine nonlinear system

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + B\boldsymbol{u}, \tag{24}$$

$$\boldsymbol{x} = \begin{bmatrix} p_f \\ q \end{bmatrix}, \boldsymbol{u} = \begin{bmatrix} 0 \\ u \end{bmatrix}, f(\boldsymbol{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{25}$$

and optimal control problem to minimize the nonlinear performance index

$$J(\boldsymbol{u}; \boldsymbol{x}_0) = \int_{t_0}^{t_f} \frac{1}{2}\|\boldsymbol{u}\|^2 - U(q)dt - p_f q. \tag{26}$$

It is well known that the optimal value function obeys the following Hamilton-Jacobi-Bellman (HJB) partial differential equation:

$$\frac{\partial V}{\partial t} + \min_{\boldsymbol{u}}\Big[\frac{\partial V}{\partial \boldsymbol{x}}(f(\boldsymbol{x}) + B\boldsymbol{u}) + \frac{1}{2}\|\boldsymbol{u}\|^2 - U(q)\Big] = 0, \tag{27}$$

From Eq. (24) and (25), Eq. (27) becomes

$$\frac{\partial V}{\partial t} - \frac{1}{2}\Big(\frac{\partial V}{\partial q}\Big)^2 - U(q) = 0, \tag{28}$$

and boundary condition for Eq. (28) is given by

$$V(\boldsymbol{x}(t_f), t_f) = -p_f^T q. \tag{29}$$

Eqs. (28) and (29) are indeed the HJ equation (11) and boundary condition (19). Thus we obtain Eqs. (22) and (19) as a result of nonlinear optimal control problem.

The HJB equation is approximated via two steps. The first step is to reduce the HJB equation, which is a nonlinear partial differential equation, to a sequence of linear partial differential equations, called the generalized Hamilton-Jacobi-Bellman (GHJB) equation. The second step is to use the Galerkin spectral method with Chebyshev polynomials to approximate the GHJB equation.

Successive Approximation of the HJ Equation

The successive approximation method is originally proposed to approximate the solution of HJB equation which arises in dynamic programming of optimal control theory.[5] The basic idea is to start with an initial stabilizing control, find the cost of this control with respect to the cost index, and then update the control to reduce this cost.

To obtain the approximate solution of (28), given an initial control $\boldsymbol{u}^{(0)}$, consider the performance index

$$J^{(k)}(\boldsymbol{u}^{(k)}; \boldsymbol{x}_0) = \int_{t_0}^{t_f} \frac{1}{2}\|\boldsymbol{u}^{(k)}(t)\|^2 - U(\boldsymbol{x})dt + g(\boldsymbol{x}(t_f)) \tag{30}$$

where $U(\boldsymbol{x}) \leq 0$ and the linear partial differential equation

$$\frac{\partial V^{(k)}}{\partial t} + \frac{\partial V^{(k)}}{\partial \boldsymbol{x}}(f + B\boldsymbol{u}^{(k)}) - U(\boldsymbol{x}) + \frac{1}{2}\|\boldsymbol{u}^{(k)}\|^2 = 0, \tag{31}$$

$$V^{(k)}(\boldsymbol{x}(t_f), t_f) = g(\boldsymbol{x}(t_f)), \tag{32}$$

which is called the generalized HJB equation (GHJB) in Ref. 5. The recursive algorithm is summarized below.

[successive approximation]

1. For $k = 0$, define $u^{(0)}$.
2. Solve GHJB (31) to compute $V^{(k)}$.
3. Update $u^{(k)}$ by $u^{(k+1)} = -B^T \dfrac{\partial V^{(k)}}{\partial x}$.
4. Set $k = k + 1$ and repeat Step 2 and 3.

It is shown that the successive approximation improves the performance index of the previous control in Ref. 5. The method is also shown to monotonically converge pointwise to the optimal solution, i.e., to the solution of the HJB equation.

For the HJ equation (22), GHJB equation becomes

$$\frac{\partial F_3^{(k)}}{\partial t} + \frac{\partial F_3^{(k)}}{\partial q} p^{(k)} + \frac{\|p^{(k)}\|^2}{2} + \frac{1}{q} = 0, \tag{33}$$

$$F_3^{(k)}(\boldsymbol{x}(t_f), t_f) = -p_f^T q. \tag{34}$$

The $(k+1)$th momentum $p^{(k+1)}$ is obtained by

$$p^{(k+1)} = \frac{\partial F_3^{(k)}}{\partial q}. \tag{35}$$

Spectral Method using Chabyshev Polynomials

The spectral method is based on the idea to assume the solution can be approximated by a sum of $N+1$ basis functions $\phi_n(x)$.[6] Since the difference between the exact solution and approximated solution is identically equal to zero, the problem is to choose the the series coefficient so that the difference is minimized. The Galerkin spectral method requires that the residual function $R(x; c_0, \cdots, c_N) = V_N^{(k)} - V^{(k)}$ is small in the sense that the first $(N+1)$ terms of its spectral series are zero and HJ equation reduce to ordinary differential equations for spectral coefficients. Among many types of of basis functions, a natural basis for the approximation of functions on a finite interval employs the Chebyshev polynomials $T_n(x)$. In this section, we seek an approximating solution to Eq. (33) of the form:

$$F_3^{(k)}(\boldsymbol{p}_f, \boldsymbol{q}, t) \triangleq \sum_{n_1=0}^{N} \cdots \sum_{n_{2n}=0}^{N} c_{n_1,\cdots,n_{2n}}^{(k)}(t) T_{n_1}(x_1) \cdots T_{n_{2n}}(x_{2n}), \tag{36}$$

where $[\boldsymbol{p}_f^T, \boldsymbol{q}^T] = [x_1, \cdots, x_{2n}]$, n denotes dimension of state and $c_{n_1,\cdots,2n}^{(k)}(t)$ represents the time-varying coefficient of Chebyshev polynomial $T_{n_1}(x_1)T_{n_2}(x_2)\cdots T_{n_{2n}}(x_{2n})$.

For simplicity, we illustrate for one dimensional problem, i.e., $n=1$. Then Eq. (36) reduces to $F_3^{(k)}(p_f, q, t) = \sum_{n_1=0}^{N} \sum_{n_2=0}^{N} c_{n_1,n_2}^{(k)}(t) T_{n_1}(p_f) T_{n_2}(q)$. The n-th order Chebyshev polynomial is given by

$$\cos n\theta \triangleq T_n(\cos\theta) = T_n(x), x = \cos\theta,\ x \in [-1,\ 1]. \tag{37}$$

Although $T_n(x)$ are polynomial in x, a Chebyshev series is a Fourier cosine expansion with change of variables. Since Chebyshev polynomials are defined in interval $[-1,\ 1]$, we have to change variables to satisfy the condition. The Chebyshev polynomials form a complete orthogonal system on the interval $[-1,\ 1]$ with respect to the the weighting function $\frac{1}{\sqrt{1-x^2}}$, i.e.,

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_k(x) T_l(x) dx = \begin{cases} \pi & (k=l=0) \\ \frac{\pi}{2} & (k=l\neq 0) \\ 0 & (k\neq l). \end{cases} \tag{38}$$

If the Chebyshev expansion of the continuous function $f(x)$ is given by

$$f(x) = \sum_{n=0}^{N} c_n T_n(x),\ x \in [-1,\ 1]. \tag{39}$$

then the Chebyshev coefficients c_n are found by exploiting the orthogonality (38)

$$c_n = \begin{cases} \dfrac{1}{\pi}\displaystyle\int_{-1}^{1} f(x)T_n(x)\frac{dx}{\sqrt{1-x^2}}, & n = 0, \\ \dfrac{2}{\pi}\displaystyle\int_{-1}^{1} f(x)T_n(x)\frac{dx}{\sqrt{1-x^2}}, & n = 1, \cdots N. \end{cases} \tag{40}$$

The important relation of Chebyshev polynomials are

$$T_i(x)T_j(x) = \frac{1}{2}(T_{i+j}(x) + T_{|i-j|}(x)) \tag{41}$$

$$2T_k(x) = \frac{1}{k+1}\frac{dT_{k+1}(x)}{dx} - \frac{1}{k-1}\frac{dT_{k-1}(x)}{dx}. \tag{42}$$

Each term of (31) can be expanded to

$$\frac{\partial F_3^{(k)}}{\partial t}(p_f, q) = \sum_{n_1,n_2=0}^{N} \dot{c}_{n_1,n_2}^{(k)}(t)T_{n_1}(p_f)T_{n_2}(q), \tag{43}$$

$$u^{(k)} = \frac{\partial F_3^{(k-1)}}{\partial q} \tag{44}$$

$$= \sum_{n_1,n_2=0}^{N} {c'_{n_1,n_2}}^{(k-1)}(t)T_{n_1}(p_f)T_{n_2}(q), \tag{45}$$

$$\frac{\partial F_3^{(k)}}{\partial q}u^{(k)} \triangleq \sum_{n_1,n_2=0}^{N} (P{c_{n_1,n_2}}^{(k)})(t)T_{n_1}(p_f)T_{n_2}(q), \tag{46}$$

$$U(q) \triangleq \sum_{n_1,n_2=0}^{N} l_{n_1,n_2}T_{n_1}(p_f)T_{n_2}(q), \tag{47}$$

$$\|u^{(k)}\|^2 = \Big(\sum_{n_1,n_2=0}^{N} {c'_{n_1,n_2}}^{(k-1)}(t)T_{n_1}(p_f)T_{n_2}(q)\Big)^2 \tag{48}$$

$$\triangleq \sum_{n_1,n_2=0}^{N} a_{n_1,n_2}(t)T_{n_1}(p_f)T_{n_2}(q). \tag{49}$$

where c'_{n_1,n_2} denotes the coefficients of the Chebyshev expansion:

$$\sum_{n_1,n_2=0}^{N} \frac{\partial}{\partial q}\big(c_{n_1,n_2}T_{n_1}T_{n_2}\big) \triangleq \sum_{n_1,n_2=0}^{N} c'_{n_1,n_2}T_{n_1}T_{n_2}. \tag{50}$$

and P is a contant matrix which is obtained by matching terms in Eq. (46). The computation of $\frac{\partial}{\partial x}(\cdot)(x)$ involves global spectral information which makes the straightforward formulation computationally inefficient. However, using the relations (41) and (42), Chebyshev coefficients can be obtained without integration in Eq. (40) except $U(q)$. This is the main advantage of using Chebyshev polynomial to solve our problem. More details are shown in Appendix B.

Since

$$\alpha_{n_i,n_i}\langle T_{n_i}, T_{n_i}\rangle = 0 \implies \alpha_{n_i,n_i} = 0, \ \forall i$$
$$\langle T_{n_i}, T_{n_j}\rangle = 0 \ \ if \ \ i \neq j,$$

taking an inner product with $T_{n_1,n_2}(0 \leq n_1, n_2 \leq N)$, Eq. (31) gives the following linear ordinary differential equation:

$$\frac{d{c_{n_1,n_2}}^{(k)}}{dt}(t) + P{c_{n_1,n_2}}^{(k)}(t) + l_n + a_{n_1,n_2}(t) = 0. \quad 0 \leq n \leq N. \tag{51}$$

From (19) the boundary conditions are given by

$$\begin{cases} c_{1,1} = -1, \\ c_{n_1,n_2} = 0, \quad n_1 \neq 1, n_2 \neq 1. \end{cases} \tag{52}$$

SIMULATION

One-dimensional motion Consider one-dimensional motion described by the Hamiltonian $H(q,p) = \frac{p^2}{2} - \frac{1}{q}$. The equations of motion are given by

$$\dot{q} = p, \quad \dot{p} = -\frac{1}{q^2}. \tag{53}$$

We solve two-point boundary-value problem by finding F_3 generating function. GHJB equation (33) are solved using Chebyshev polynomials with $N = 5$. Figures 1 and 2 show the trajectory for specific boundary condition $q_0 = 1.02$ and $p_f = -0.07$. We can see that the approximate solution converges after 5 iterations. Now consider the perturbed motion described by the Hamiltonian $H_2(q,p) = \frac{p^2}{2} - \frac{1}{q} - \frac{1}{q^3} - \frac{1}{q^5}$. Our method can also give the solution in analytical form for the perturbed problem. Figures 3 and 4 show the trajectory for the boundary condition ($q_0 = 1.02$, $p_f = 0$) with the perturbed Hamiltonian H_2. It is shown that the solution for the TBVBP are obtained for the perturbed problem.

Two-dimensional motion Consider the target spacecraft in a circular orbit of radius R_0. The orbit rate is given by $n = \sqrt{\mu_e/R_0^3}$, where $\mu_e \equiv GM_e$ is the gravitational parameter of the Earth, G the universal gravitational constant and M_e the mass of the Earth. To introduce the dynamics relative to the circular orbit, the right-handed coordinate system $(\tilde{x}, \tilde{y}, \tilde{z})$ fixed at the center of mass of the target is used, where $\tilde{x}$ axis is along the radial direction, $\tilde{y}$ axis along the flight direction of

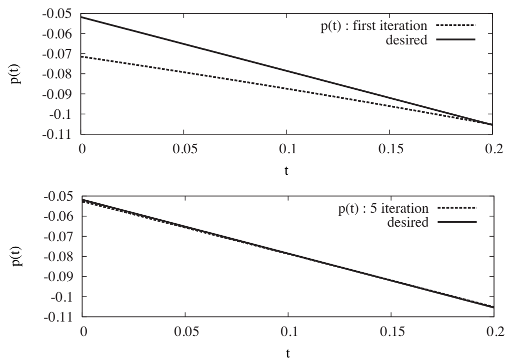

Figure 1 Trajectory of $p^{(k)}$ with $k = 0$ (upper), $k = 5$ (lower).

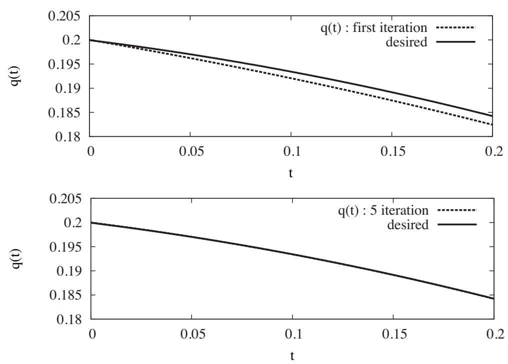

Figure 2 Trajectory of $q^{(k)}$ evaluated by $\dot{q} = p^{(k)}$, $q(0) = 1.02$ with $k = 0$ (upper), $k = 5$ (lower).

the target, and $\tilde{z}$ axis is out of the orbit plane (see Figure 8). The Newton's equation of motion gives

$$\ddot{\tilde{x}} = 2n\dot{\tilde{y}} + n^2(R_0 + \tilde{x}) - \frac{\mu_e(R_0 + \tilde{x})}{[(R_0 + \dot{\tilde{x}})^2 + \tilde{y}^2 + \tilde{z}^2]^{\frac{3}{2}}}, \tag{54}$$

$$\ddot{\tilde{y}} = -2n\dot{\tilde{x}} + n^2\tilde{y} - \frac{\mu_e\tilde{y}}{[(R_0 + \tilde{x})^2 + \tilde{y}^2 + \tilde{z}^2]^{\frac{3}{2}}}, \tag{55}$$

$$\ddot{\tilde{z}} = -\frac{\mu_e\tilde{z}}{[(R_0 + \tilde{x})^2 + \tilde{y}^2 + \tilde{z}^2]^{\frac{3}{2}}}. \tag{56}$$

Here we consider the two-dimensional problem, where z is set to 0. Nondimensionalizing with reference legth R_0 and reference time n yields

$$\ddot{\tilde{x}} = 2\dot{\tilde{y}} + (1 + \tilde{x}) - \frac{(1 + \tilde{x})}{[(1 + \tilde{x})^2 + \tilde{y}^2]^{\frac{3}{2}}}, \tag{57}$$

$$\ddot{\tilde{y}} = -2\dot{\tilde{x}} + \tilde{y} - \frac{\tilde{y}}{[(1 + \tilde{x})^2 + \tilde{y}^2]^{\frac{3}{2}}}. \tag{58}$$

As an example, we consider the reference length $R_0 = 4.23 \times 10^4$ km and the reference time $\bar{t} = 1.38 \times 10^4$sec, which is the geosynchronous orbit with the frequency $\omega = 2\pi/1day = 7.27 \times$

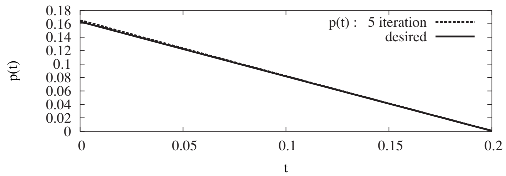

Figure 3 Trajectory of $p^{(5)}$ for H_2.

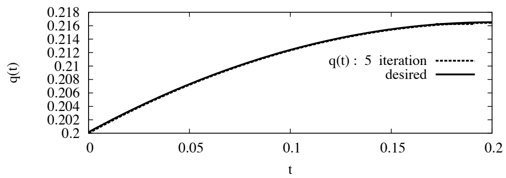

Figure 4 Trajectory of $q^{(5)}$ evaluated by $\dot{q} = p^{(k)}$, $q(0) = 1.02$ for H_2.

10^5 rad/sec. To use Chebyshev polynomials, the state variables $(\tilde{x}, \tilde{y})$ are transformed to $(x, y) \in [-1, 1]$. Using these variables, the potential function is given by $U(x, y) = \dfrac{1}{\sqrt{(1 + d_1 x)^2 + d_2^2 y^2}}$. $U(x, y)$ with $d_1 = d_2 = 0.5$ is shown in Figure 5. We solve the GHJB equation with Chebyshev polynomials with $N = 2$. See the Appendix A for a detailed derivation of the HJ equation and GHJB equation for this problem.

Let the boundary condition for $F_3(\boldsymbol{p_f}, \boldsymbol{q})$ be

$$(\boldsymbol{p_f}, \boldsymbol{q}) = (0, 0, \tilde{x}, \tilde{y}), \quad \tilde{x}, \tilde{y} \in [-0.5,\ 0.5], \tag{59}$$

and t_f is set to be 0.5. Figure 6 shows the errors using our method.Our solution was compared with those founded by the method in Ref. 2.The F_3 generating function is expanded as power series about a circular nominal trajectory. Figure 7 shows the final velocity errors using the Taylor series solution of order 2. From Figure 7, the final velocity error increases as the distance from the origin becomes large, especially in smaller x domain. It is seen in Figure 6 that the approximation error is small in the same domain when spectral method with Chebyshev polynomials is used. It turns out that there exist approximation errors in the Chebyshev expansion of $U(\tilde{x}, \tilde{y})$ when $N = 2$ is used. For the sufficiently large N implementation is limited by computer memory constraints, however, the effect of the spectral method can be seen in the Figures.

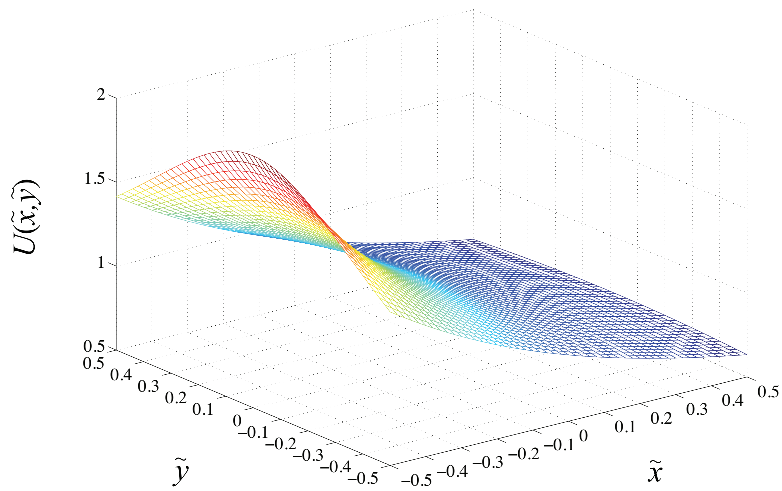

Figure 5 Potential function $U(\tilde{x}, \tilde{y})$

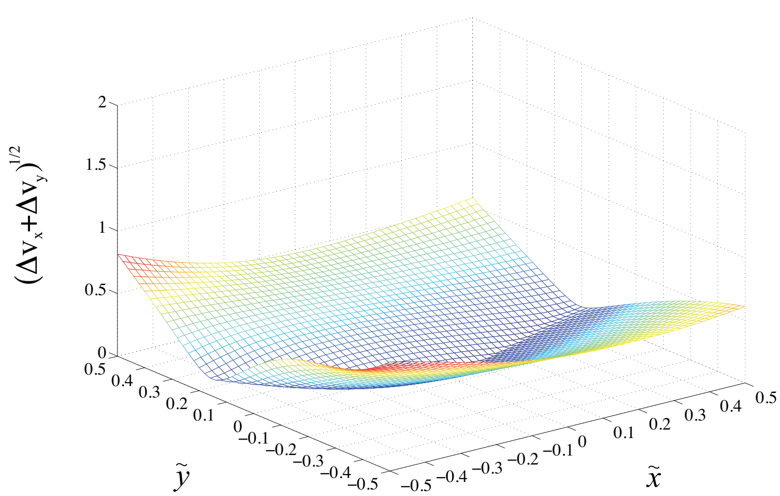

Figure 6 Error of final velocity using Chebyshev polynomials (N=2)

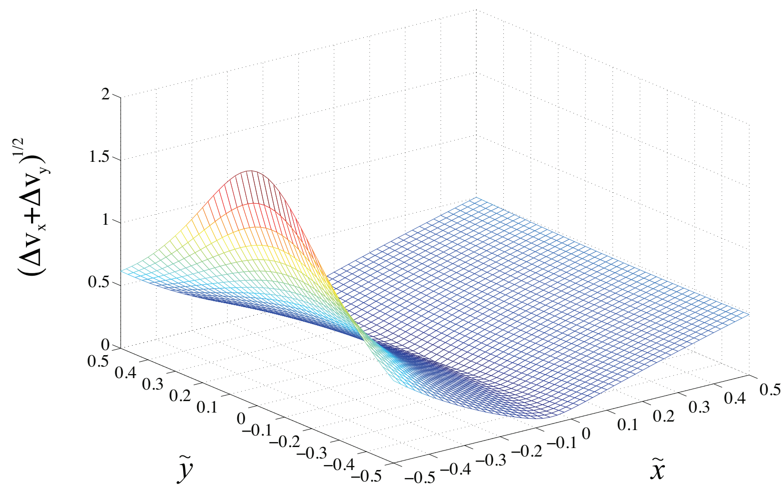

Figure 7   Error of final velocity using a power series (N=2)

CONCLUSION

New procedure to derive an analytical solution to the optimal orbital transfer is developed. It is shown that the among four possible forms of generating functions, F_3 generating function is obtained by successive approximation and converges to the solution of the HJ equation. The approach enables us to solve the TPBVP for nonlinear dynamics without linearization about the nominal trajectory.

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APPENDIX A: DYNAMICS OF TWO-BODY PROBLEM IN ROTATIONAL FRAME

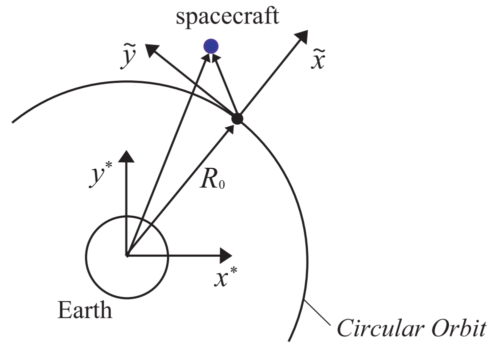

Figure 8 Inertial frame and rotational frame.

Consider the dynamics of two-body problem in two reference frame shown in Figure 8. The first reference frame is the inertial frame and is expressed in terms of the Cartesian coordinates (x^*, y^*, z^*) having its origin at the center of Earth. The second reference frame is the rotational frame and is expressed in terms of the Cartesian coordinates ($\tilde{x}$, $\tilde{y}$, $\tilde{z}$) having its origin at the reference circular orbit. Consider the Lagrangian and Hamiltonian of the system in two reference frame under the influence of the potential function $U(r) = -\dfrac{1}{\sqrt{x^{*2} + y^{*2}}}$.

Lagrangian in the inertial frame:

$$L(x^*, y^*, \dot{x}^*, \dot{y}^*) = \frac{1}{2}(\dot{x}^{*2} + \dot{y}^{*2}) + \frac{1}{\sqrt{x^{*2} + y^{*2}}} \tag{60}$$

$$\begin{bmatrix} x^* \\ y^* \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} 1 + \tilde{x} \\ \tilde{y} \end{bmatrix} \tag{61}$$

Lagrangian in the rotational frame:

$$L(\tilde{x}, \tilde{y}, \dot{\tilde{x}}, \dot{\tilde{y}}) = \frac{1}{2}[(\dot{\tilde{x}} - \tilde{y})^2 + (\dot{\tilde{y}} - 1 - \tilde{x})^2] + \frac{1}{\sqrt{(1 + \tilde{x})^2 + \tilde{y}^2}} \tag{62}$$

$$= \frac{1}{2}[\dot{\tilde{x}}^2 + \dot{\tilde{y}}^2 + (1 + \tilde{x})^2 + \tilde{y}^2 - 2(\dot{\tilde{x}}\tilde{y} - \dot{\tilde{y}} - \tilde{x}\dot{\tilde{y}})] + \frac{1}{\sqrt{(1 + \tilde{x})^2 + \tilde{y}^2}} \tag{63}$$

State transformation to normalized state variables:

$$\tilde{x}, \tilde{y} \rightarrow x, y \quad x, y \in [-1, 1]$$

$$\begin{cases} \tilde{x} = d_1 x \\ \tilde{y} = d_2 y \\ \dot{\tilde{x}} = d_1 \dot{x} \\ \dot{\tilde{y}} = d_2 \dot{y} \end{cases}$$

Canonical equations for normalized state variables:

$$L(x, y, \dot{x}, \dot{y}) = \frac{1}{2}[(d_1\dot{x} - d_2 y)^2 + (d_2\dot{y} - 1 - d_1 x)^2] + \frac{1}{\sqrt{(1 + d_1 x)^2 + d_2^2 y^2}} \tag{64}$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = d_1(d_1\dot{x} - d_2 y) \tag{65}$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = d_2(d_2\dot{y} + 1 + d_1 x) \tag{66}$$

$$H(x, y, p_x, p_y) = \begin{bmatrix} \dot{x} & \dot{y} \end{bmatrix} \begin{bmatrix} p_{\tilde{x}} \\ p_{\tilde{y}} \end{bmatrix} + \frac{1}{2}(p_{\tilde{x}}^2 + p_{\tilde{y}}^2) - \frac{1}{\sqrt{(1 + d_1 x)^2 + d_2^2 y^2}} \tag{67}$$

$$= \frac{1}{2}[\frac{p_x^2}{d_1^2} + \frac{p_y^2}{d_2^2} + 2\frac{d_2 y}{d_1}p_x - 2\frac{(1 + d_1 x)}{d_2}p_y] - \frac{1}{\sqrt{(1 + d_1 x)^2 + d_2^2 y^2}} \tag{68}$$

$$\begin{cases} \dot{x} &= \frac{\partial H}{\partial p_x} = \frac{p_x}{d_1^2} + \frac{d_2}{d_1}y \\ \dot{y} &= \frac{\partial H}{\partial p_y} = \frac{p_y}{d_2^2} + \frac{1}{d_2}(1 + d_1 x) \\ \dot{p}_x &= -\frac{\partial H}{\partial x} = -\frac{d_1}{d_2}p_y - \frac{d_1(1 + d_1 x)}{[(1 + d_1 x)^2 + d_2^2 y]^{\frac{3}{2}}} \\ \dot{p}_y &= -\frac{\partial H}{\partial y} = -\frac{d_2}{d_1}p_x - \frac{d_2^2 y}{[(1 + d_1 x)^2 + d_2^2 y]^{\frac{3}{2}}} \end{cases} \tag{69}$$

HJ equation for F_3 generating function:

$$-\frac{\partial F_3}{\partial t} + H(q, -\frac{\partial F_3}{\partial p}, t) = 0$$

$$\Longleftrightarrow \frac{\partial F_3}{\partial t} - \frac{1}{2d_1^2}\Big(\frac{\partial F_3}{\partial x}\Big)^2 - \frac{1}{2d_2^2}\Big(\frac{\partial F_3}{\partial y}\Big)^2 + \frac{d_2 y}{d_1}\frac{\partial F_3}{\partial x} - \frac{(1 + d_1 x)}{d_2}\frac{\partial F_3}{\partial y} - U(\boldsymbol{q}) = 0 \tag{70}$$

$$\Longleftrightarrow \frac{\partial F_3}{\partial t} + \min_{\boldsymbol{u}}\Big[\frac{1}{2d_1^2}p_x^2 + \frac{1}{2d_2^2}p_y^2 + \frac{\partial F_3}{\partial x}\Big(\frac{p_x}{d_1^2} + \frac{d_2 y}{d_1}\Big) + \frac{\partial F_3}{\partial y}\Big(\frac{p_y}{d_2^2} - \frac{1 + d_1 x}{d_2}\Big) - U(\boldsymbol{q})\Big] = 0 \tag{71}$$

Nonlinear affine system:

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial p_x} = \dfrac{p_x}{d_1^2} + \dfrac{d_2}{d_1}y \\ \qquad\quad \triangleq \dfrac{1}{d_1}u_x + \dfrac{d_2}{d_1}y \\ \dot{y} = \dfrac{\partial H}{\partial p_y} = \dfrac{p_y}{d_2^2} - \dfrac{1}{d_2}(1 + d_1 x) \\ \qquad\quad \triangleq \dfrac{1}{d_2}u_y - \dfrac{1}{d_2}(1 + d_1 x) \end{cases}$$

$$\iff \qquad \dot{\boldsymbol{q}} = f(\boldsymbol{q}) + B\boldsymbol{u}, \tag{72}$$

$$\boldsymbol{q} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \boldsymbol{u} = \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} \frac{p_x}{d_1} \\ \frac{p_y}{d_2} \end{bmatrix} \tag{73}$$

$$f(\boldsymbol{q}) = \begin{bmatrix} \dfrac{d_2}{d_1}y \\ -\dfrac{1}{d_2}(1 + d_1 x) \end{bmatrix}, B = \begin{bmatrix} \dfrac{1}{d_1} & 0 \\ 0 & \dfrac{1}{d_2} \end{bmatrix} \tag{74}$$

Performance index:

$$J = \int_{t_0}^{t_f} \frac{1}{2}\|u\|^2 - U(q)dt \tag{75}$$

$$= \int_{t_0}^{t_f} \frac{1}{2}(|u_x|^2 + |u_y|^2) - U(q)dt \tag{76}$$

$$= \int_{t_0}^{t_f} \frac{1}{2}(\frac{|p_x|^2}{d_1^2} + \frac{|p_y|^2}{d_2^2}) - U(q)dt \tag{77}$$

GHJB equation for (72) **and** (77)**:**

$$\frac{\partial F_3^{(k)}}{\partial t} + \frac{\partial F_3^{(k)}}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}} + \frac{\|\boldsymbol{u}\|^2}{2} - U(\boldsymbol{q}) = 0 \tag{78}$$

$$\begin{aligned} \iff \frac{\partial F_3^{(k)}}{\partial t} + \frac{\partial F_3^{(k)}}{\partial x}\frac{d_2 y}{d_1} - \frac{\partial F_3^{(k)}}{\partial y}\frac{1 + d_1 x}{d_2} + \frac{\partial F_3^{(k)}}{\partial x}p_x^{(k-1)} + \frac{\partial F_3^{(k)}}{\partial y}p_y^{(k-1)} \\ + \frac{|p_x^{(k-1)}|^2}{2d_1^2} + \frac{|p_y^{(k-1)}|^2}{2d_2^2} + \frac{1}{\sqrt{(1+\tilde{x})^2 + \tilde{y}^2}} = 0 \end{aligned} \tag{79}$$

$$F_3^{(k)}(\boldsymbol{p}_f, \boldsymbol{q}, t_f) = -\boldsymbol{p}_f^T \boldsymbol{q} = -p_{x_f} x - p_{y_f} y, \tag{80}$$

$$p_x^{(k+1)} = -\frac{\partial F_3^{(k)}}{\partial x} \tag{81}$$

$$p_y^{(k+1)} = -\frac{\partial F_3^{(k)}}{\partial y} \tag{82}$$

APPENDIX B: RELATION OF CHEBYSHEV COEFFICIENTS

Using relations (41) and (42), Chebyshev coefficients can be obtained without integration in (40) except potential term $U(x)$. Consider an approximating expansion of the form:

$$f(x) = \sum_{n=0}^{N} a_n T_n(x),$$
$$g(x) = \sum_{n=0}^{N} b_n T_n(x),$$

Let the derivative of $f(x)$ be

$$f'(x) \triangleq \sum_{n=0}^{N} a'_n T_n(x). \tag{83}$$

Then the corresponding coefficients are given by

$$a'_n = \begin{cases} \displaystyle\sum_{j=n+1,\ j-n:odd}^{N} 2ja_j & (n \geq 1) \\ \displaystyle\sum_{j=1,\ j-n:odd}^{N} ja_j & (n = 0) \end{cases} \tag{84}$$

Also, let the product of $f(x)$ and $g(x)$ be

$$f(x)g(x) = \Big(\sum_{n=0}^{N} a_n T_n(x)\Big)\Big(\sum_{n=0}^{N} b_n T_n(x)\Big) \tag{85}$$

$$\triangleq \sum_{n=0}^{N} c_n T_n(x), \tag{86}$$

and the corresponding coefficient are

$$c_n = \frac{1}{2} \sum_{i,j=0,i}^{N} \sum_{+j=n,|i-j|=n}^{N} a_i\, b_j. \tag{87}$$

The generalisation to more than two variables is straight forward. Using Eq. (84) and (87), Chebyshev coefficients can be obtained without integration. This is the main advantage of using Chebyshev polynomial to solve our problem.