Some topics on variational problems for curves and surfaces

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Some topics on variational problems for curves and surfaces

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1 Introduction

There are a minimal surface and a constant mean curvature (CMC) surface as a solution of variational problem for surfaces. They have been studied for a long time. A minimal surface is a surface that mean curvature is equal to 0 everywhere. On the other hand, a CMC surface is a surface that mean curvature is constant (not always 0) everywhere. The former is a critical point of the area, the latter is a critical point of the area for all variations that preserve the enclosed volume and fix the boundary. Thus, these surfaces are sometimes called mathematical models of a soap film and a soap bubble respectively. In this paper, the following three results were obtained by joint work with Professor Miyuki Koiso (IMI, Kyushu University).

In Chapter 2, we introduce "Bifurcation and stability for surfaces with constant mean curvature bounded by two coaxial circles". The study of CMC surface with given curves as boundary is called "generalized Plateau problem". It is one of the most important problems of classical differential geometry. In the case where the boundary is a single Jordan curve, there are some known important results on the uniqueness of CMC surfaces (reference[1], [4], [6], [8], [21]). On the other hand, if the boundary has two or more connected components, it is only inferred from the observation of known examples that the existence and uniqueness of the solution are very complicated, and the general result is unknown. In this paper, we focused on the simplest case in this situation, that is, CMC surfaces bounded by two given coaxial circles with the same radius (we call the axis of the boundary circles the vertical axis). In particular, they are surfaces of revolution when we consider only embedded stable CMC surfaces spanned by two given coaxial circles and included in the domain bounded by two parallel planes. For all CMC surfaces of revolution, we used a highly original representation method where "the normal direction at the boundary" of surface is used as a parameter. As a result, we obtained various important geometrical representation formulas such as profile curves, enclosed volumes by surfaces and so on. In this study, moreover, we examined the following problem. A CMC surface is "a critical point of the area for all variations that preserve the enclosed volume and fix the boundary". It is important to judge whether the solution is stable or not. However, it is very difficult in general. We tried to determine the stability of all CMC surfaces of revolution. As a result, we obtained the conclusion of stability concerning "solutions those are close to known stable solutions with vertical symmetry" by complicated calculations using elliptic integrals and the bifurcation theory for CMC surfaces (Koiso-Palmer-Piccione, 2017, [16]). This study is very important from both theory and application because the CMC surface is also used as a mathematical model of a soap film and a micro-droplet which can ignore gravity.

In Chapter 3, we introduce "Construction of fundamental geometric theory that can be applied to all of smooth curves and surfaces, piecewise smooth curves and surfaces, discrete curves and surfaces". In differential geometry and discrete geometry, smooth curves and surfaces, discrete curves and surfaces have been treated separately. Therefore, for example, the curvature of a discrete surface was defined only for each face or each vertex. On the contrary, in this paper, we construct a method to handle them uniformly. For example, geometrical concepts such as curvatures and normal vectors are defined both at each smooth point and at each singular point. At any smooth point, they are defined in the usual way in the classical differential geometry. At each singular point, we define the "multi-valued unit normal" by extending the concept of "normal cone" known in convex geometry. As for the curvatures, we define the curvature for plane curves, the Gaussian curvature and the mean curvature for space surfaces so that generalizations of so-called the Steiner's formula and Minkowski's formula for smooth curves and surfaces hold also for piecewise-smooth curves and surfaces. The Steiner's formula is sometimes used to define curvatures of discrete curves and surfaces. However, it is a new idea to try to generalize the Minkowski's formula. This feature of our definition of the curvatures enables us to construct a variational method for piecewise-smooth curves and surfaces.

In Chapter 4, we introduce "Uniqueness of local minimizers for crystalline variational problems". Anisotropic substances, such as crystals and some kind of liquid crystals are considered to form a local minimizer of an anisotropic surface energy with volume constraint. The anisotropic surface energy is the integral of an energy density that depends on the normal direction of the surface. For a given energy density function, the energy minimizer is unique. It is a convex closed surface called the Wulff shape (J. E. Taylor, 1978, [23]). When the Wulff shape has singular points like a polyhedron, the energy density function has non-differentiable points and the classical variational method cannot be used. Thus, mathematical study about local minimizers of the energy has not progressed. Now, it is called crystalline variational problem to discuss local minimizers when the Wulff shape is a polyhedron. For any convex closed surface W including the origin inside in $\mathbb{R}^3$, there exists an energy density function γ such that W is the Wulff shape. In general, γ is not necessarily unique. However, there is a unique "convex" energy density function γ for each W. In particular, when W is smooth and strictly convex, γ is unique and the local minimizer of the anisotropic energy for all variations that preserve the enclosed volume is W up to homothety and translation (Palmer, 1998, [18]). Even when W is not smooth, in this paper, the uniqueness of local minimizer is obtained as follows. "Theorem 4.4.1 (Theorem 4.6.1): Let W be a regular polyhedron, let γ be the support function of W and let M be a piecewise-C^1 convex closed surface. Then, M is a local minimizer of the anisotropic energy for all variations that preserve the enclosed volume if and only if $M = W$ (up to homothety and translation)." We also mention an application of our Theorem 4.4.1 (Theorem 4.6.1) to material science (§4.8).

2 Bifurcation and stability for surfaces with constant mean curvature bounded by two coaxial circles

2.1 Introduction

It is interesting to find all axially-symmetric CMC surfaces bounded by given two circles with the same radius in two parallel planes. Below we will first study only CMC surfaces which are symmetric with respect to the plane which is parallel to the planes including the boundary circles. Moreover, in the case of unduloid, we will study only parts between two consecutive inflections. Later we should study more general cases so that we will be able to apply the results to study the bifurcation from a part of an unduloid between two consecutive necks (or bulges).

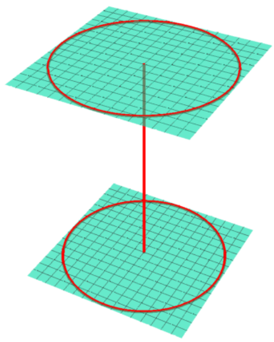


Figure 1: two coaxial circles with the same radius on two parallel planes

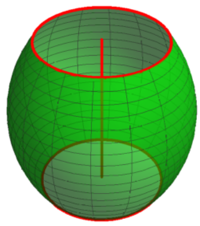


Figure 2: a surface of revolution

2.2 Representation formulas for Delaunay surfaces

Each Delaunay surface is parameterized as

$$X(s,\theta) = (x(s)e^{i\theta}, z(s)), \quad x \geq 0,$$

where s is arc length of the generating curve $(x(s), z(s))$. Note that the area element $d\Sigma$ of X is given by

$$d\Sigma = x ds d\theta$$

at any point where x does not vanish. For the meantime, we choose the "outward pointing" unit normal $\nu = (\nu_1, \nu_2, \nu_3)$ of X as

$$\nu(s,\theta) = (z'(s)\cos\theta, z'(s)\sin\theta, -x'(s)).$$

Then, the mean curvature H of X for this normal is

$$H = (x''z' - x'z'' - x^{-1}z')/2. \tag{1}$$

If H is constant, then the first integral of (1) is

$$Hx^2 = -xz' + \text{constant} =: -xz' + c. \tag{2}$$

In fact, by using (1) and $2(x'x'' + z'z'') = [(x')^2 + (z')^2]' = 0$, we have

$$(Hx^2)' = 2Hxx' = xx'x''z' - x(x')^2z'' - x'z' = -x(z')^2z'' - x(1-(z')^2)z'' - x'z' = -(xz')'. \tag{3}$$

Note that, by taking a new parameter $\tilde{s} := -s$, (2) becomes

$$H(x(\tilde{s}))^2 = x(\tilde{s})z'(\tilde{s}) + c.$$

So we may assume without loss of generality that $H \leq 0$ holds . With this normalization, we have the following cases:

- (I-1) $H = 0$ and $c = 0$: *horizontal plane.*
- (I-2) $H = 0$ and $c \neq 0$: *catenoid.*
- (II-1) $H < 0$ and $c = 0$: *round sphere* $S^2(|H|^{-1})$.
- (II-2) $H < 0$ and $c = (2|H|)^{-1}$: *cylinder* with radius $(2|H|)^{-1}$.
- (II-3) $H < 0$ and $(2|H|)^{-1} > c > 0$: *unduloid.*
- (II-4) $H < 0$ and $c < 0$: *nodoid.*

These surfaces are as follows.

- $H = 0$: minimal surface.

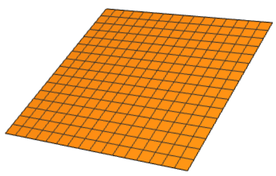


Figure 3: plane

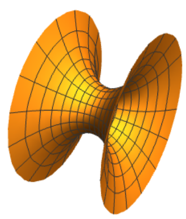


Figure 4: catenoid

- $H \equiv \text{constant}\,(\neq 0)$: CMC (constant mean curvature) surface.

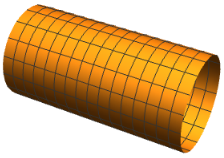


Figure 5: cylinder

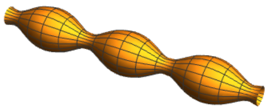


Figure 6: unduloid

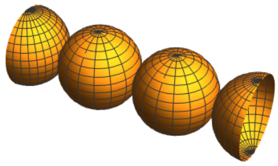


Figure 7: sphere

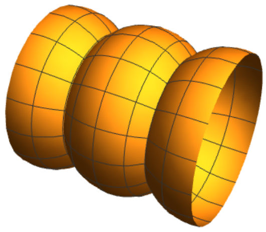


Figure 8: nodoid

In cases (I-2), (II-3) and (II-4), the respective generating curve is referred as a *catenary*, an *undulary*, and a *nodary*. In general, generating curves of Delaunay surfaces are called *Delaunay curves*.

Set $(u, v) := (z', -x')$. Then, (2) becomes

$$ux + Hx^2 = c, \tag{4}$$

where c is the integral constant. By considering the direction of the Gauss map for the case of nodoids, we have the following:

Proposition 2.2.1 (cf. [14]). *Let*

$$\sigma \mapsto (u(\sigma), v(\sigma)) := (\cos\sigma, \sin\sigma), \quad \sigma \in (-\infty, \infty),$$

be the profile curve of the unit sphere S^2. Let $X(s,\theta) = (x(s)e^{i\theta}, z(s))$ be a Delaunay surface with mean curvature $H \leq 0$. The unit normal $(z'(s), -x'(s))$ of the profile curve $(x(s), z(s))$ can be written as $(z'(s), -x'(s)) = (u(\sigma(s)), v(\sigma(s)))$. Then the immersion X is given as follows.

(i) When X is an catenoid,

$$x = c/u$$

for some nonzero constant c.

(ii) When X is an unduloid,

$$x = \frac{u \pm \sqrt{u^2 + 4Hc}}{-2H}$$

for some constants $c > 0$ and $H < 0$, where $x = x(u(\sigma))$ is defined in $\{\sigma | u \geq \sqrt{-4Hc}\}$.

(iii) When X is an nodoid,

$$x = \frac{u + \sqrt{u^2 + 4Hc}}{-2H}$$

for some constants $c < 0$ and $H < 0$, where $x = x(u(\sigma))$ is defined in $\{-\infty < \sigma < \infty\}$.

In all cases above, z is given by

$$z = \int^u v_u x_u \, du. \tag{5}$$

Conversely, define x and z as in (i) $\sim$ (iii) and (5). Then $X(s,\theta) = (x(s)e^{i\theta}, z(s))$ is a Delaunay surface which satisfies

$$z'x + Hx^2 = c,$$

where s is the arc length of (x, z), and H is supposed to be zero for Case (i).

If we study unduloid and nodoid, sometimes it is convenient to use the parameter

$$\alpha := 4cH \tag{6}$$

instead of c, because α is invariant under homotheties.

Remark 2.2.1. *For each undulary and nodary $(x(s), z(s))$, there are a unique local maximum B and a unique local minimum N of x, which we will call a bulge and a neck respectively. For the unduloid given in Proposition 2.2.1, B and N are given by*

$$B = \frac{1+\sqrt{1+\alpha}}{-2H}, \quad N = \frac{1-\sqrt{1+\alpha}}{-2H}, \quad (-1 < \alpha < 0,\ H < 0).$$

For the nodoid given in Proposition 2.2.1, they are given by

$$B = \frac{1+\sqrt{1+\alpha}}{-2H}, \quad N = \frac{-1+\sqrt{1+\alpha}}{-2H}, \quad (\alpha > 0,\ H < 0).$$

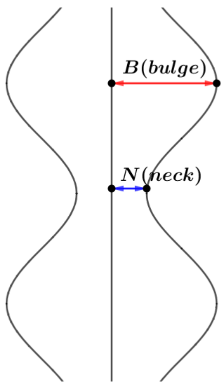


Figure 9: bulge and neck (unduloid)

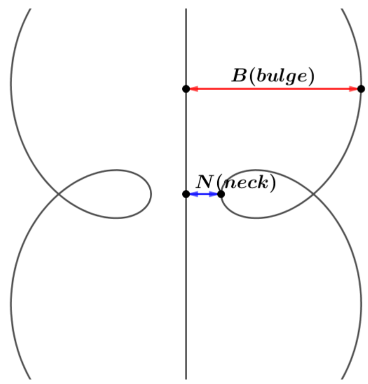


Figure 10: bulge and neck (nodoid)

2.3 Delaunay surfaces bounded by two circles

Let $r > 0$, $h > 0$. Let C_-, C_+ be the circles

$$C_- := \{(re^{i\theta}, -h) \mid \theta \in [0, 2\pi)\}, \quad C_+ := \{(re^{i\theta}, h) \mid \theta \in [0, 2\pi)\}$$

with radius r which lie in parallel horizontal planes $\{z = h\}$, $\{z = -h\}$.

We consider the part of a Delaunay surface which is bounded by C_- and C_+. Then the surface is parameterized as

$$X(s, \theta) = (x(s)e^{i\theta}, z(s)), \quad x \geq 0, \qquad s \in [-L, L],$$
$$z(-L) < z(L),$$

where s is arc length of the generating curve $(x(s), z(s))$. We assume that the surface is symmetric with respect to the plane $\{z = 0\}$. This implies that $X(0, \theta)$ is either a neck or a bulge. Let $\nu(s, \theta)$ be the unit normal along X which satisfies $\nu(0, 0) = (1, 0, 0)$.

2.3.1 Representation formulas for Delaunay surfaces bounded by two circles

Since the surface X is symmetric with respect to the plane $\{z=0\}$, we consider only its "half"

$$X(s,\theta) = (x(s)e^{i\theta}, z(s)), \qquad s \in [0, L], \tag{7}$$

By using Proposition 2.2.1, we have the following representation formulas for the profile curve $\Gamma := (x, z)$ of the surface (7).

Case(I-1) X is a non-convex nodoid: $H > 0$, $0 \le u \le 1$, $\alpha > 0$,

$$x = \frac{u - \sqrt{u^2+\alpha}}{-2H}, \tag{8}$$

$$z = \frac{1}{-2H}\int_1^u \frac{-\tilde{u}}{\sqrt{1-\tilde{u}^2}} \cdot \frac{\tilde{u} - \sqrt{\tilde{u}^2+\alpha}}{\sqrt{\tilde{u}^2+\alpha}}\, d\tilde{u}. \tag{9}$$

Case(I-2) X is a non-convex unduloid: $H < 0$, $\sqrt{-\alpha} \le u \le 1$, $-1 < \alpha < 0$,

$$x = \frac{u - \sqrt{u^2+\alpha}}{-2H}, \tag{10}$$

$$z = \frac{1}{-2H}\int_1^u \frac{-\tilde{u}}{\sqrt{1-\tilde{u}^2}} \cdot \frac{\tilde{u} - \sqrt{\tilde{u}^2+\alpha}}{\sqrt{\tilde{u}^2+\alpha}}\, d\tilde{u}. \tag{11}$$

Case(II-1) X is a convex nodoid: $H < 0$, $0 \le u \le 1$, $\alpha > 0$,

$$x = \frac{u + \sqrt{u^2+\alpha}}{-2H}, \tag{12}$$

$$z = \frac{1}{-2H}\int_1^u \frac{-\tilde{u}}{\sqrt{1-\tilde{u}^2}} \cdot \frac{\tilde{u} + \sqrt{\tilde{u}^2+\alpha}}{\sqrt{\tilde{u}^2+\alpha}}\, d\tilde{u}. \tag{13}$$

Case(II-2) X is a convex unduloid: $H < 0$, $\sqrt{-\alpha} \le u \le 1$, $-1 < \alpha < 0$,

$$x = \frac{u + \sqrt{u^2+\alpha}}{-2H}, \tag{14}$$

$$z = \frac{1}{-2H}\int_1^u \frac{-\tilde{u}}{\sqrt{1-\tilde{u}^2}} \cdot \frac{\tilde{u} + \sqrt{\tilde{u}^2+\alpha}}{\sqrt{\tilde{u}^2+\alpha}}\, d\tilde{u}. \tag{15}$$

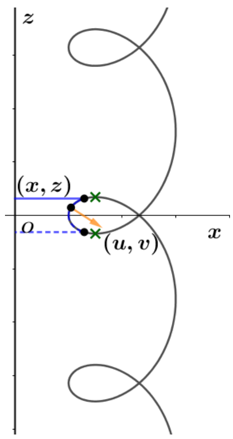


Figure 11: Case(I-1)

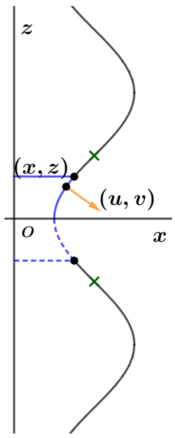


Figure 12: Case(I-2)

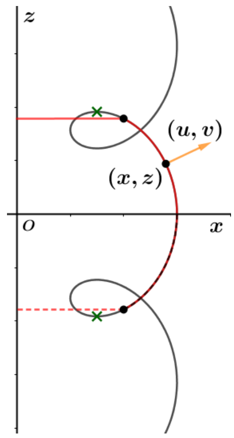


Figure 13: Case(II-1)

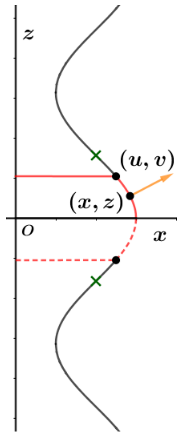


Figure 14: Case(II-2)

Remark 2.3.1 (Sphere)**.** *Both* $\lim_{\alpha\to+0}(\mathrm{II}-1)$ $\lim_{\alpha\to-0}(\mathrm{II}-2)$ *give the sphere with mean curvature* H*.*

Remark 2.3.2 (Catenoid)**.** *Recall* $\alpha = 4cH$*. For* $c \neq 0$*, Both* $\lim_{H\to+0}(\mathrm{I}-1)$ $\lim_{H\to-0}(\mathrm{I}-2)$ *give the catenary*

$$x = \frac{c}{u}, \quad z = c\log\frac{1+\sqrt{1-u^2}}{u},$$

that is

$$x = \frac{c}{2}(e^{z/c} + e^{-z/c}).$$

Remark 2.3.3 (Cylinder)**.** *The cylinder with radius* $1/|2H|$ *appears when* $\alpha \to -1+0$ *for unduloids (I-2), (II-2). In this case,* $u \equiv 1$*.*

2.3.2 The ratio z/x for the profile curves

The ratio z/x of x, z studied in the last subsection ((x, z) is a profile curve of a part of an undulary or a nodary) is a function of u and α. So we set

$$\varphi(u, \alpha) := z/x,$$

and study the equation "$\varphi(u, \alpha) = $ constant".

We will try to determine parts of Delaunay surfaces bounded by two circles

$$C_- := \{(re^{i\theta}, -h) \mid \theta \in [0, 2\pi)\}, \quad C_+ := \{(re^{i\theta}, h) \mid \theta \in [0, 2\pi)\}.$$

with radius r which lie in parallel horizontal planes $\{z = h\}$, $\{z = -h\}$. Assume that (u_0, α) is a solution of the equation

$$\varphi(u, \alpha) = h/r.$$

Define H by

$$r = \begin{cases} \dfrac{u_0 - \sqrt{u_0^2 + \alpha}}{-2H}, & (\mathrm{Case}(\mathrm{I}-1), (\mathrm{I}-2)), \\ \dfrac{u_0 + \sqrt{u_0^2 + \alpha}}{-2H}, & (\mathrm{Case}(\mathrm{II}-1), (\mathrm{II}-2)). \end{cases} \tag{16}$$

Then, x, z defined in the last subsection 2.3.1 determine axially-symmetric CMC surfaces bounded by $C_{\pm}$. Note that the neck size N and the bulge size B are

$$\begin{cases} N = \dfrac{1 - \sqrt{1 + \alpha}}{-2H}, & (\mathrm{Case}(\mathrm{I}-1), (\mathrm{I}-2)), \\ B = \dfrac{1 + \sqrt{1 + \alpha}}{-2H}, & (\mathrm{Case}(\mathrm{II}-1), (\mathrm{II}-2)). \end{cases} \tag{17}$$

Below we will sometimes write (x_0, z_0) instead of (r, h).

Case(I-1) and (I-2): Non-convex unduloid and non-convex nodoid:

By using the formulas obtained in the last subsection, we have

$$\varphi(u, \alpha) = \frac{1}{u - \sqrt{u^2 + \alpha}} \int_1^u \frac{-\tilde{u}}{\sqrt{1 - \tilde{u}^2}} \cdot \frac{\tilde{u} - \sqrt{\tilde{u}^2 + \alpha}}{\sqrt{\tilde{u}^2 + \alpha}} \, d\tilde{u}, \tag{18}$$

where

$$\begin{cases} H > 0, \ 0 \leq u \leq 1, \ \alpha > 0, & (\mathrm{Case}(\mathrm{I}-1) : \mathrm{non-convex\ nodoid}), \\ H < 0, \ \sqrt{-\alpha} \leq u \leq 1, \ -1 < \alpha < 0, & (\mathrm{Case}(\mathrm{I}-2) : \mathrm{non-convex\ unduloid}). \end{cases} \tag{19}$$

By standard computations, we obtain

$$\lim_{\alpha \to 0} \varphi(u, \alpha) = u \log \frac{1 + \sqrt{1 - u^2}}{u}. \tag{20}$$

Remark 2.3.4. *Assume that* (u_0, α) *is a solution of*

$$\varphi(u, \alpha) := z_0/x_0, \quad x_0 > 0, \ z_0 > 0.$$

Then,

$$x_0 = \frac{u_0 - \sqrt{u_0^2 + \alpha}}{-2H}, \quad N = \frac{1 - \sqrt{1 + \alpha}}{-2H}. \tag{21}$$

Since $\alpha = 4cH$ *((6)),*

$$x_0 = \frac{u_0 - \sqrt{u_0^2 + 4cH}}{-2H}, \quad N = \frac{1 - \sqrt{1 + 4cH}}{-2H}. \tag{22}$$

If we have a smooth 1-parameter family $\{(u_t, \alpha_t) \in \mathbb{R}^2 \mid 0 < t < \varepsilon\}$, $(\varepsilon > 0)$*, such that*

$$\varphi(u_t, \alpha_t) = z_0/x_0, \ \alpha_t = 4c_t H_t, \ H_t \in C^1 \text{in } t, \quad \forall t \in (0, \varepsilon), \tag{23}$$

$$\lim_{t \to +0} H_t = 0, \ \lim_{t \to +0} u_t = u_0, \ \lim_{t \to +0} c_t = c_0. \tag{24}$$

Then,

$$x_0 = \lim_{t \to 0} \frac{u_t - \sqrt{u_t^2 + 4c_t H_t}}{-2H_t} = \frac{c_0}{u_0}, \tag{25}$$

$$N_0 := \lim_{t \to 0} \frac{1 - \sqrt{1 + 4c_t H_t}}{-2H_t} = c_0. \tag{26}$$

Case(II-1) and (II-2): Convex unduloid and convex nodoid:

By using the formulas obtained in the subsection 2.3.1, we have

$$\varphi(u, \alpha) = \frac{1}{u + \sqrt{u^2 + \alpha}} \int_1^u \frac{-\tilde{u}}{\sqrt{1 - \tilde{u}^2}} \cdot \frac{\tilde{u} + \sqrt{\tilde{u}^2 + \alpha}}{\sqrt{\tilde{u}^2 + \alpha}} \, d\tilde{u}, \tag{27}$$

where

$$\begin{cases} H < 0, \ 0 \le u \le 1, \ \alpha > 0, & (\mathrm{Case(II-1) : convex\ nodoid}), \\ H < 0, \ \sqrt{-\alpha} \le u \le 1, \ -1 < \alpha < 0, & (\mathrm{Case(II-2) : convex\ unduloid}). \end{cases} \tag{28}$$

2.3.3 How to find the solutions of $\varphi(u, \alpha) = \text{constant}$?

As we saw in the previous subsection 2.3.2, the solutions of "$\varphi(u, \alpha) = \text{constant} = h/r$" determine (all?) axially-symmetric CMC surfaces bounded by $C_\pm$. These solutions are given by the level curves $\{z = \text{constant}\}$ of the surface $z = \varphi(u, \alpha)$. So we will try to draw pictures of the surface $z = \varphi(u, \alpha)$.

Case(I-1) and (I-2): Non-convex unduloid and non-convex nodoid:

Recall that we obtained the following formula in subsection 2.3.1.

$$\varphi(u, \alpha) = \frac{1}{u - \sqrt{u^2 + \alpha}} \int_1^u \frac{-\tilde{u}}{\sqrt{1 - \tilde{u}^2}} \cdot \frac{\tilde{u} - \sqrt{\tilde{u}^2 + \alpha}}{\sqrt{\tilde{u}^2 + \alpha}} \, d\tilde{u}, \tag{29}$$

where

$$\begin{cases} H > 0, \ 0 \le u \le 1, \ \alpha > 0, & (\mathrm{Case(I-1) : non-convex\ nodoid}), \\ H < 0, \ \sqrt{-\alpha} \le u \le 1, \ -1 < \alpha < 0, & (\mathrm{Case(I-2) : non-convex\ unduloid}). \end{cases} \tag{30}$$

We compute

$$\begin{aligned} \varphi(u, \alpha) &= \frac{1}{u - \sqrt{u^2 + \alpha}} \int_1^u \frac{-\tilde{u}}{\sqrt{1 - \tilde{u}^2}} \cdot \frac{\tilde{u} - \sqrt{\tilde{u}^2 + \alpha}}{\sqrt{\tilde{u}^2 + \alpha}} \, d\tilde{u} & (31) \\ &= \frac{1}{u - \sqrt{u^2 + \alpha}} \int_1^u \left(\frac{-\tilde{u}^2}{\sqrt{1 - \tilde{u}^2}\sqrt{\tilde{u}^2 + \alpha}} + \frac{\tilde{u}}{\sqrt{1 - \tilde{u}^2}} \right) d\tilde{u} & (32) \\ &= \frac{1}{u - \sqrt{u^2 + \alpha}} \int_1^u \left(\frac{-\tilde{u}^2 - \alpha + \alpha}{\sqrt{1 - \tilde{u}^2}\sqrt{\tilde{u}^2 + \alpha}} \right) d\tilde{u} + \frac{-\sqrt{1 - u^2}}{u - \sqrt{u^2 + \alpha}} & (33) \end{aligned}$$

$$= \frac{1}{u-\sqrt{u^2+\alpha}}\left(\int_1^u \frac{-\sqrt{\tilde{u}^2+\alpha}}{\sqrt{1-\tilde{u}^2}}\,d\tilde{u} + \int_1^u \frac{\alpha}{\sqrt{1-\tilde{u}^2}\sqrt{\tilde{u}^2+\alpha}}\,d\tilde{u}\right) + \frac{-\sqrt{1-u^2}}{u-\sqrt{u^2+\alpha}} \tag{34}$$

$$=: \frac{1}{u-\sqrt{u^2+\alpha}}\left(-\mathcal{I}_1(u,\alpha)+\alpha\mathcal{I}_2(u,\alpha)\right) + \frac{-\sqrt{1-u^2}}{u-\sqrt{u^2+\alpha}}, \tag{35}$$

where

$$\mathcal{I}_1(u,\alpha) := \int_1^u \frac{\sqrt{\tilde{u}^2+\alpha}}{\sqrt{1-\tilde{u}^2}}\,d\tilde{u}, \tag{36}$$

$$\mathcal{I}_2(u,\alpha) := \int_1^u \frac{1}{\sqrt{1-\tilde{u}^2}\sqrt{\tilde{u}^2+\alpha}}\,d\tilde{u}. \tag{37}$$

It is known that the integrals $\mathcal{I}_1(u,\alpha)$, $\mathcal{I}_2(u,\alpha)$ can be written in terms of standard elliptic integrals. The elliptic integral of the first kind $F(k,\phi)$ and that of the second kind $E(k,\phi)$ are defined as follows.

$$F(k,\phi) := \int_0^\phi \frac{1}{\sqrt{1-k^2\sin^2\psi}}\,d\psi = \int_0^{\sin\phi} \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}\,dt, \tag{38}$$

$$E(k,\phi) := \int_0^\phi \sqrt{1-k^2\sin^2\psi}\,d\psi = \int_0^{\sin\phi} \sqrt{\frac{1-k^2t^2}{1-t^2}}\,dt. \tag{39}$$

The followings are notations in the software Mathematica (cf. [24]).

$$\mathrm{EllipticF}[\phi,m] := \int_0^\phi \frac{1}{\sqrt{1-m\sin^2\psi}}\,d\psi, \quad \mathrm{EllipticK}[m] := \int_0^{\pi/2} \frac{1}{\sqrt{1-m\sin^2\psi}}\,d\psi, \tag{40}$$

$$\mathrm{EllipticE}[\phi,m] := \int_0^\phi \sqrt{1-m\sin^2\psi}\,d\psi, \quad \mathrm{EllipticE}[m] := \int_0^{\pi/2} \sqrt{1-m\sin^2\psi}\,d\psi. \tag{41}$$

The Mathematica gives the followings:

$$\int^u \frac{\sqrt{\tilde{u}^2+\alpha}}{\sqrt{1-\tilde{u}^2}}\,d\tilde{u} = -\sqrt{1+\alpha}\mathrm{EllipticE}[\mathrm{ArcCos}[u], \frac{1}{1+\alpha}] = -\sqrt{1+\alpha}\; E(\frac{1}{\sqrt{1+\alpha}}, \arccos u), \tag{42}$$

$$\int^u \frac{1}{\sqrt{1-\tilde{u}^2}\sqrt{\tilde{u}^2+\alpha}}\,d\tilde{u} = -\frac{1}{\sqrt{1+\alpha}}\mathrm{EllipticF}[\mathrm{ArcCos}[u], \frac{1}{1+\alpha}] = -\frac{1}{\sqrt{1+\alpha}}\; F(\frac{1}{\sqrt{1+\alpha}}, \arccos u). \tag{43}$$

By using (34), (42), and (43), we obtain

$$\begin{aligned}
\varphi(u,\alpha) &= \frac{1}{u-\sqrt{u^2+\alpha}}\left(\int_1^u \frac{-\sqrt{\tilde{u}^2+\alpha}}{\sqrt{1-\tilde{u}^2}}\,d\tilde{u} + \int_1^u \frac{\alpha}{\sqrt{1-\tilde{u}^2}\sqrt{\tilde{u}^2+\alpha}}\,d\tilde{u}\right) + \frac{-\sqrt{1-u^2}}{u-\sqrt{u^2+\alpha}} \\
&= \frac{1}{u-\sqrt{u^2+\alpha}}\Bigg(\sqrt{1+\alpha}\; E(\frac{1}{\sqrt{1+\alpha}}, \arccos u) \\
&\quad -\frac{\alpha}{\sqrt{1+\alpha}}\; F(\frac{1}{\sqrt{1+\alpha}}, \arccos u) - \sqrt{1-u^2}\Bigg)
\end{aligned} \tag{44}$$

Case(II-1) and (II-2): Convex unduloid and convex nodoid:

$$\varphi(u,\alpha) = \frac{1}{u+\sqrt{u^2+\alpha}} \int_1^u \frac{-\tilde{u}}{\sqrt{1-\tilde{u}^2}} \cdot \frac{\tilde{u}+\sqrt{\tilde{u}^2+\alpha}}{\sqrt{\tilde{u}^2+\alpha}}\, d\tilde{u} \tag{45}$$

where

$$\begin{cases} H<0,\ 0 \le u \le 1,\ \alpha > 0, & (\mathrm{Case(II-1)} : \text{convex nodoid}), \\ H<0,\ \sqrt{-\alpha} \le u \le 1,\ -1<\alpha<0, & (\mathrm{Case(II-2)} : \text{convex unduloid}). \end{cases} \tag{46}$$

we obtain

$$\begin{aligned} \varphi(u,\alpha) &= \frac{1}{u+\sqrt{u^2+\alpha}} \int_1^u \frac{-\tilde{u}}{\sqrt{1-\tilde{u}^2}} \cdot \frac{\tilde{u}+\sqrt{\tilde{u}^2+\alpha}}{\sqrt{\tilde{u}^2+\alpha}}\, d\tilde{u} \qquad (47) \\ &= \frac{1}{u+\sqrt{u^2+\alpha}} \Bigg(\sqrt{1+\alpha}\; E(\frac{1}{\sqrt{1+\alpha}}, \arccos u) \\ &\qquad - \frac{\alpha}{\sqrt{1+\alpha}}\; F(\frac{1}{\sqrt{1+\alpha}}, \arccos u) + \sqrt{1-u^2} \Bigg) \qquad (48) \end{aligned}$$

Recall that we want to know the level curves $\{z = \text{constant}\}$ of the surface $z = \varphi(u,\alpha)$. So we will try to draw pictures of the surface $z = \varphi(u,\alpha)$ and its level curves by using the formulas (44) and (48). As the examples, we drew the following pictures by using Mathematica.

- Case(I-1) : Non-convex nodoid ($h/r = 0.4$).
- Case(I-2) : Non-convex unduloid ($h/r = 0.4$).
- Case(II-1) : Convex nodoid ($h/r = 5$).
- Case(II-2) : Convex unduloid ($h/r = 5$).

Please pay attention to the followings. Firstly, for Case(I-2) and (II-2) (in cases of unduloid), the intersection of the surface and the plane denote a family of inflection points. That is,

$$u = \sqrt{-\alpha} \Leftrightarrow u^2 = -\alpha \Leftrightarrow \alpha = -u^2$$

denote a family of inflection points.

Secondly, from Remark 2.3.1, for Case(II-1) and (II-2) (in cases of convex nodoid and convex unduloid), each picture shows that it is a sphere when $\alpha \to \pm 0$.

Thirdly, from Remark 2.3.2, for Case(I-1) and (I-2) (in cases of non-convex nodoid and non-convex unduloid), each picture shows that it is a catenoid when $\alpha = 0$. When we gradually increase the height of $\{z = \text{constant}\}$, the number of intersections of the surface $z = \varphi(u,\alpha)$ and the plane at $\alpha = 0$ changes. It means that the number of intersections is equivalent to the number of solutions.

Lastly, from Remark 2.3.3, for Case(I-2) and (II-2) (in cases of unduloid), each picture shows that it is a cylinder when $\alpha \to -1+0$ because $u \equiv 1$.

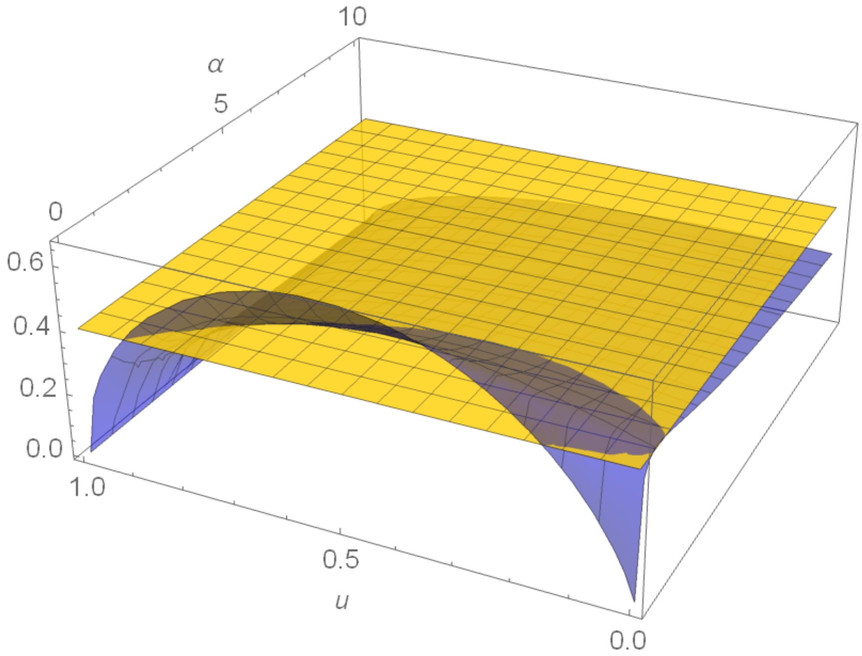


Figure 15: Case(I-1): Non-convex nodoid ($h/r = 0.4$)

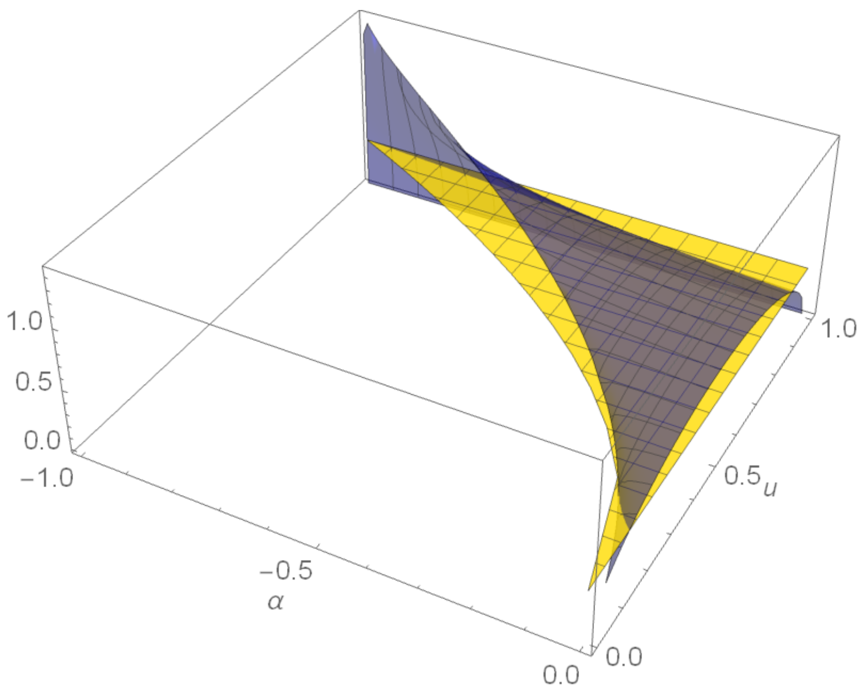


Figure 16: Case(I-2): Non-convex unduloid ($h/r = 0.4$)

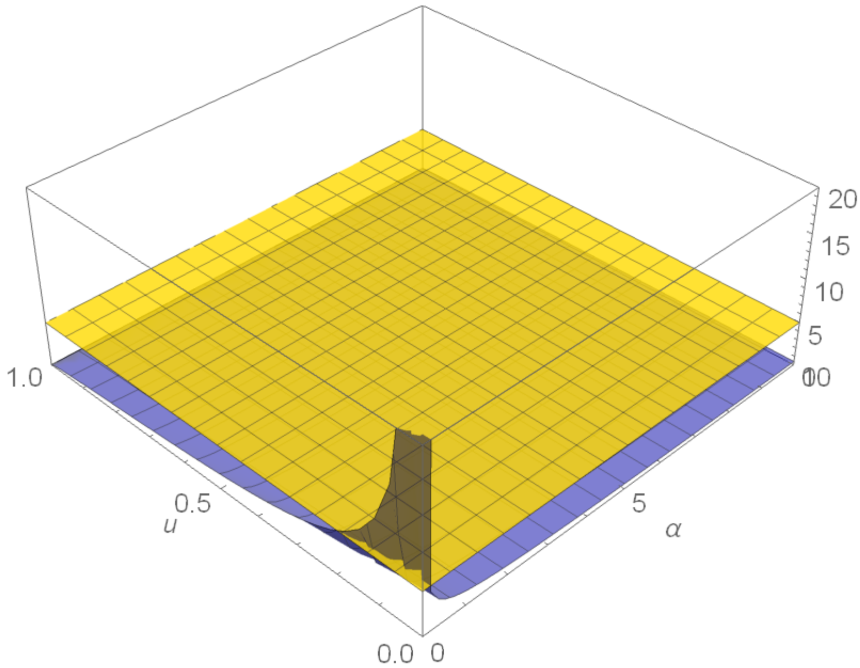


Figure 17: Case(II-1): Convex nodoid $(h/r = 5)$

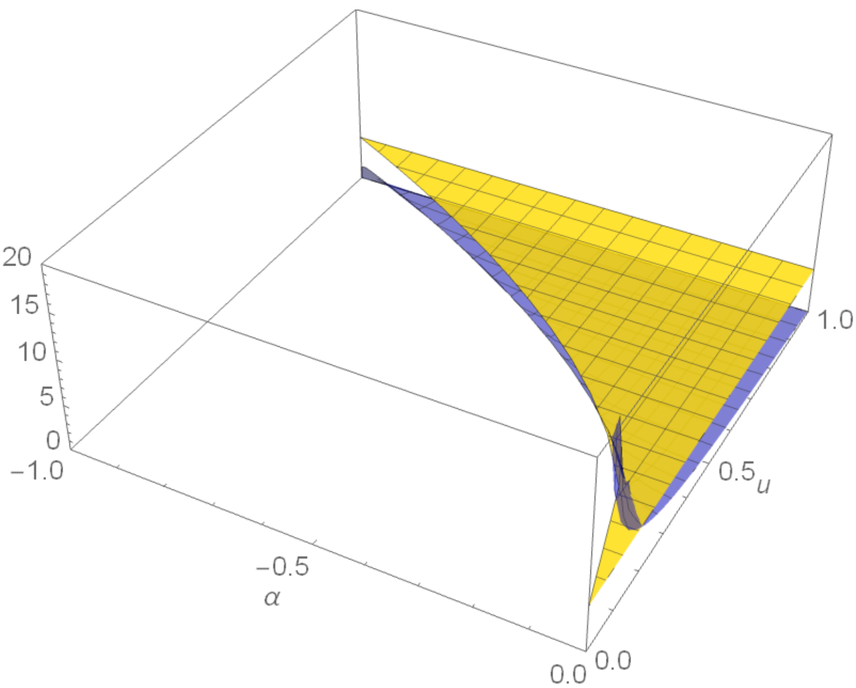


Figure 18: Case(II-2): Convex unduloid $(h/r = 5)$

2.4 Representation formulas for unduloids

2.4.1 Unduloids which are symmetric with respect to the plane $\{z = 0\}$

In this subsection, we deal with only a part of $\{z \geq 0\}$ because we consider unduloids which are symmetric with respect to the plane $\{z = 0\}$.

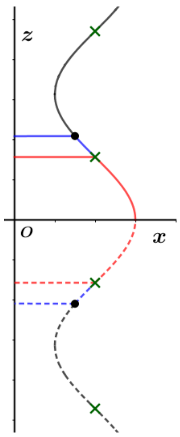


Figure 19: (G_+)

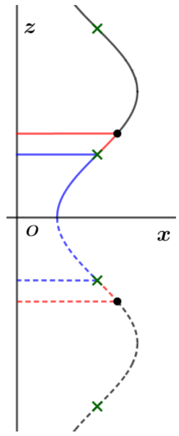


Figure 20: (G_-)

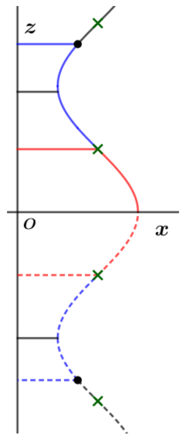


Figure 21: (R_+)

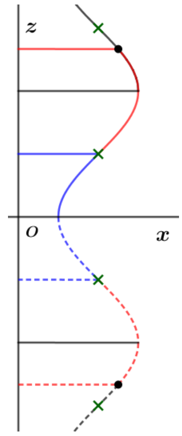


Figure 22: (R_-)

(G_+): A profile curve exceeds a inflection point. However, it doesn't exceed a neck: $H < 0$, $\sqrt{-\alpha} \leq u \leq 1$, $-1 < \alpha < 0$,

$$\begin{cases} x = \dfrac{u + \sqrt{u^2 + \alpha}}{-2H}, & \text{(convex)} \\ x = \dfrac{u - \sqrt{u^2 + \alpha}}{-2H}, & \text{(non-convex)} \end{cases}$$

$$\begin{aligned} z &= \frac{1}{-2H}\left(\int_1^{\sqrt{-\alpha}} \frac{-\tilde{u}}{\sqrt{1-\tilde{u}^2}} \cdot \frac{\tilde{u} + \sqrt{\tilde{u}^2 + \alpha}}{\sqrt{\tilde{u}^2 + \alpha}}\, d\tilde{u} - \int_{\sqrt{-\alpha}}^{u} \frac{-\tilde{u}}{\sqrt{1-\tilde{u}^2}} \cdot \frac{\tilde{u} - \sqrt{\tilde{u}^2 + \alpha}}{\sqrt{\tilde{u}^2 + \alpha}}\, d\tilde{u}\right) \\ &= \frac{1}{-2H}\left(-\frac{2\alpha}{\sqrt{1+\alpha}}\, F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) + 2\sqrt{1+\alpha}\, E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha})\right. \\ &\quad \left. + \frac{\alpha}{\sqrt{1+\alpha}}\, F(\frac{1}{\sqrt{1+\alpha}}, \arccos u) - \sqrt{1+\alpha}\, E(\frac{1}{\sqrt{1+\alpha}}, \arccos u) + \sqrt{1-u^2}\right) \end{aligned}$$

(G_-): A profile curve exceeds a inflection point. However, it doesn't exceed a bulge: $H < 0$,

$\sqrt{-\alpha} \leq u \leq 1$, $-1 < \alpha < 0$,

$$\begin{cases} x = \dfrac{u + \sqrt{u^2+\alpha}}{-2H}, & \text{(convex)} \\ x = \dfrac{u - \sqrt{u^2+\alpha}}{-2H}, & \text{(non-convex)} \end{cases}$$

$$\begin{aligned} z &= \frac{1}{-2H}\left(\int_1^{\sqrt{-\alpha}} \frac{-\tilde{u}}{\sqrt{1-\tilde{u}^2}} \cdot \frac{\tilde{u} - \sqrt{\tilde{u}^2+\alpha}}{\sqrt{\tilde{u}^2+\alpha}}\, d\tilde{u} - \int_{\sqrt{-\alpha}}^{u} \frac{-\tilde{u}}{\sqrt{1-\tilde{u}^2}} \cdot \frac{\tilde{u} + \sqrt{\tilde{u}^2+\alpha}}{\sqrt{\tilde{u}^2+\alpha}}\, d\tilde{u}\right) \\ &= \frac{1}{-2H}\left(-\frac{2\alpha}{\sqrt{1+\alpha}}\, F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) + 2\sqrt{1+\alpha}\, E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha})\right. \\ &\quad \left. + \frac{\alpha}{\sqrt{1+\alpha}}\, F(\frac{1}{\sqrt{1+\alpha}}, \arccos u) - \sqrt{1+\alpha}\, E(\frac{1}{\sqrt{1+\alpha}}, \arccos u) - \sqrt{1-u^2}\right) \end{aligned}$$

(R_+): A profile curve exceeds both a inflection point and a neck. However, it doesn't exceed a next inflection point: $H < 0$, $\sqrt{-\alpha} \leq u \leq 1$, $-1 < \alpha < 0$,

$$\begin{cases} x = \dfrac{u + \sqrt{u^2+\alpha}}{-2H}, & \text{(convex)} \\ x = \dfrac{u - \sqrt{u^2+\alpha}}{-2H}, & \text{(non-convex)} \end{cases}$$

$$\begin{aligned} z &= \frac{1}{-2H}\left\{\left(\int_1^{\sqrt{-\alpha}} \frac{-\tilde{u}}{\sqrt{1-\tilde{u}^2}} \cdot \frac{\tilde{u} + \sqrt{\tilde{u}^2+\alpha}}{\sqrt{\tilde{u}^2+\alpha}}\, d\tilde{u}\right)\right. \\ &\quad \left. + \left(-\int_{\sqrt{-\alpha}}^{1} \frac{-\tilde{u}}{\sqrt{1-\tilde{u}^2}} \cdot \frac{\tilde{u} - \sqrt{\tilde{u}^2+\alpha}}{\sqrt{\tilde{u}^2+\alpha}}\, d\tilde{u} + \int_1^{u} \frac{-\tilde{u}}{\sqrt{1-\tilde{u}^2}} \cdot \frac{\tilde{u} - \sqrt{\tilde{u}^2+\alpha}}{\sqrt{\tilde{u}^2+\alpha}}\, d\tilde{u}\right)\right\} \\ &= \frac{1}{-2H}\left(-\frac{2\alpha}{\sqrt{1+\alpha}}\, F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) + 2\sqrt{1+\alpha}\, E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha})\right. \\ &\quad \left. - \frac{\alpha}{\sqrt{1+\alpha}}\, F(\frac{1}{\sqrt{1+\alpha}}, \arccos u) + \sqrt{1+\alpha}\, E(\frac{1}{\sqrt{1+\alpha}}, \arccos u) - \sqrt{1-u^2}\right) \end{aligned}$$

(R_-): A profile curve exceeds both a inflection point and a bulge. However, it doesn't exceed a next inflection point: $H < 0$, $\sqrt{-\alpha} \leq u \leq 1$, $-1 < \alpha < 0$,

$$\begin{cases} x = \dfrac{u + \sqrt{u^2+\alpha}}{-2H}, & \text{(convex)} \\ x = \dfrac{u - \sqrt{u^2+\alpha}}{-2H}, & \text{(non-convex)} \end{cases}$$

$$
\begin{aligned}
z &= \frac{1}{-2H}\left\{\left(\int_1^{\sqrt{-\alpha}} \frac{-\tilde{u}}{\sqrt{1-\tilde{u}^2}}\cdot\frac{\tilde{u}-\sqrt{\tilde{u}^2+\alpha}}{\sqrt{\tilde{u}^2+\alpha}}\,d\tilde{u}\right)\right.\\
&\quad \left.+\left(-\int_{\sqrt{-\alpha}}^1 \frac{-\tilde{u}}{\sqrt{1-\tilde{u}^2}}\cdot\frac{\tilde{u}+\sqrt{\tilde{u}^2+\alpha}}{\sqrt{\tilde{u}^2+\alpha}}\,d\tilde{u}+\int_1^u \frac{-\tilde{u}}{\sqrt{1-\tilde{u}^2}}\cdot\frac{\tilde{u}+\sqrt{\tilde{u}^2+\alpha}}{\sqrt{\tilde{u}^2+\alpha}}\,d\tilde{u}\right)\right\}\\
&= \frac{1}{-2H}\left(-\frac{2\alpha}{\sqrt{1+\alpha}}\,F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})+2\sqrt{1+\alpha}\;E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right.\\
&\quad \left.-\frac{\alpha}{\sqrt{1+\alpha}}\,F(\frac{1}{\sqrt{1+\alpha}},\arccos u)+\sqrt{1+\alpha}\;E(\frac{1}{\sqrt{1+\alpha}},\arccos u)+\sqrt{1-u^2}\right)
\end{aligned}
$$

2.4.2 Unduloids which are asymmetric with respect to the plane $\{z = 0\}$

In this subsection, we deal with a part of $z \in [-h, h] (h > 0)$ because we consider unduloids which are asymmetric with respect to the plane $\{z = 0\}$. Now, we introduce (B_+) and (B_-). We can consider upside-down unduloid in both cases, however, these are essentially the same.

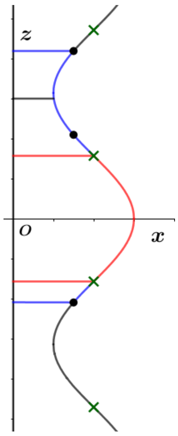


Figure 23: (B_+)

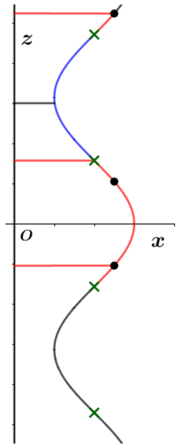


Figure 24: (B_-)

(B_+): A unduloid which is asymmetric with respect to the plane $\{z = 0\}$ where the radius of boundary circles are equal to $\dfrac{u-\sqrt{u^2+\alpha}}{-2H}$: $H < 0$, $\sqrt{-\alpha} \leq u \leq 1$, $-1 < \alpha < 0$,

$$
\begin{cases}
x = \dfrac{u+\sqrt{u^2+\alpha}}{-2H}, & \text{(convex)}\\
x = \dfrac{u-\sqrt{u^2+\alpha}}{-2H}, & \text{(non-convex)}
\end{cases}
$$

$$
\begin{aligned}
z &= 2\left((G_+) + \mathrm{Case}(\mathrm{I}-2)\right)\\
&= 2\cdot\left(-\frac{1}{2H}\right)\cdot\left\{\left(-\frac{2\alpha}{\sqrt{1+\alpha}}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha}) + 2\sqrt{1+\alpha}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right.\right.\\
&\quad \left. + \frac{\alpha}{\sqrt{1+\alpha}}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos u) - \sqrt{1+\alpha}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos u) + \sqrt{1-u^2}\right)\\
&\quad \left. + \left(-\frac{\alpha}{\sqrt{1+\alpha}}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos u) + \sqrt{1+\alpha}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos u) - \sqrt{1-u^2}\right)\right\}\\
&= -\frac{2}{H}\left(-\frac{\alpha}{\sqrt{1+\alpha}}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha}) + \sqrt{1+\alpha}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right)
\end{aligned}
$$

(B_-): A unduloid which is asymmetric with respect to the plane $\{z=0\}$ where the radius of boundary circles are equal to $\dfrac{u+\sqrt{u^2+\alpha}}{-2H}$: $H<0$, $\sqrt{-\alpha}\le u\le 1$, $-1<\alpha<0$,

$$
\begin{cases}
x = \dfrac{u+\sqrt{u^2+\alpha}}{-2H}, & \text{(convex)}\\
x = \dfrac{u-\sqrt{u^2+\alpha}}{-2H}, & \text{(non-convex)}
\end{cases}
$$

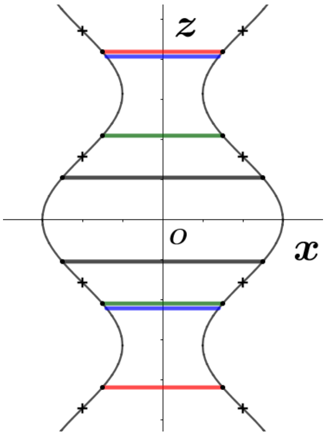
$$
\begin{aligned}
z &= 2\left(\mathrm{Case}(\mathrm{II}-2) + (G_-)\right)\\
&= 2\cdot\left(-\frac{1}{2H}\right)\cdot\left\{\left(-\frac{\alpha}{\sqrt{1+\alpha}}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos u) + \sqrt{1+\alpha}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos u) + \sqrt{1-u^2}\right)\right.\\
&\quad + \left(-\frac{2\alpha}{\sqrt{1+\alpha}}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha}) + 2\sqrt{1+\alpha}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right.\\
&\quad \left.\left. + \frac{\alpha}{\sqrt{1+\alpha}}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos u) - \sqrt{1+\alpha}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos u) - \sqrt{1-u^2}\right)\right\}\\
&= -\frac{2}{H}\left(-\frac{\alpha}{\sqrt{1+\alpha}}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha}) + \sqrt{1+\alpha}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right)
\end{aligned}
$$

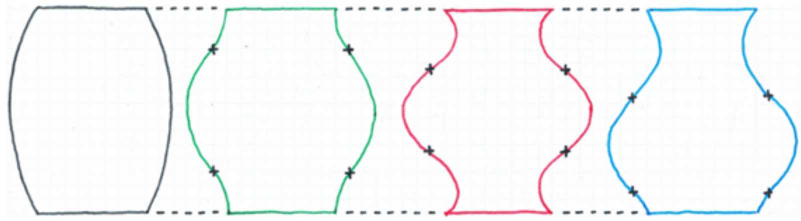
Remark 2.4.1. *The equation z in (B_+) and (B_-) are as follows.*

$$
z = -\frac{2}{H}\left(-\frac{\alpha}{\sqrt{1+\alpha}}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha}) + \sqrt{1+\alpha}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right)
$$

That is, both cases are the same.

- Delaunay surfaces bounded by two coaxial circles: Around one period of an unduloid between two necks

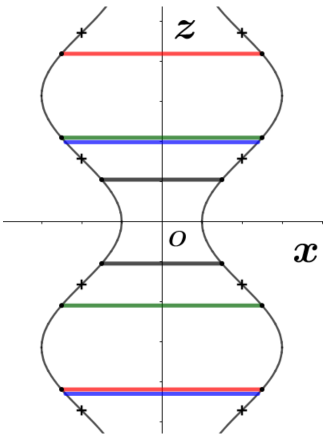


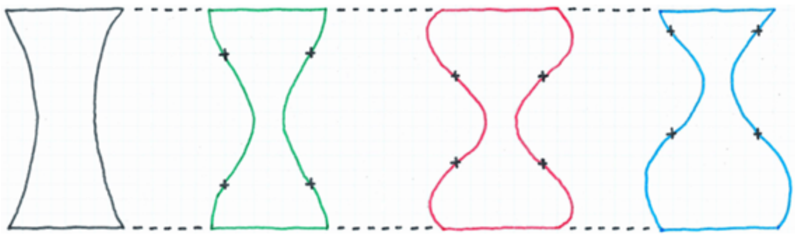


(II-2) (G_+) (R_+) (B_+)

	(II-2)	(G_+)	(R_+)	(B_+)
inflection point	0	2	2	2
neck	0	0	2	1
bulge	1	1	1	1

- Delaunay surfaces bounded by two coaxial circles: Around one period of an unduloid between two bulges





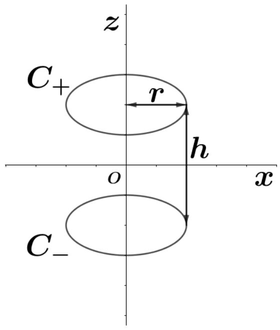
(I-2) (G_-) (R_-) (B_-)

	(I-2)	(G_-)	(R_-)	(B_-)
inflection point	0	2	2	2
neck	1	1	1	1
bulge	0	0	2	1

Remark 2.4.2. *In the case of (B_+) and (B_-), the number of inflection point, neck and bulge are the same. They can be classified into whether between inflection points are neck or bulge.*

2.5 Graph of function $h/r = \varphi(u, \alpha)$

The surface below is the graph of function $h/r = \varphi(u, \alpha)$. The level curves of the surface (two curves of intersection of the surface and the plane) show that Delaunay surfaces bounded by two circles.



$\alpha = -1$	$-1 < \alpha < 0$	$\alpha = 0$
cylinder	non-convex unduloids	catenoid

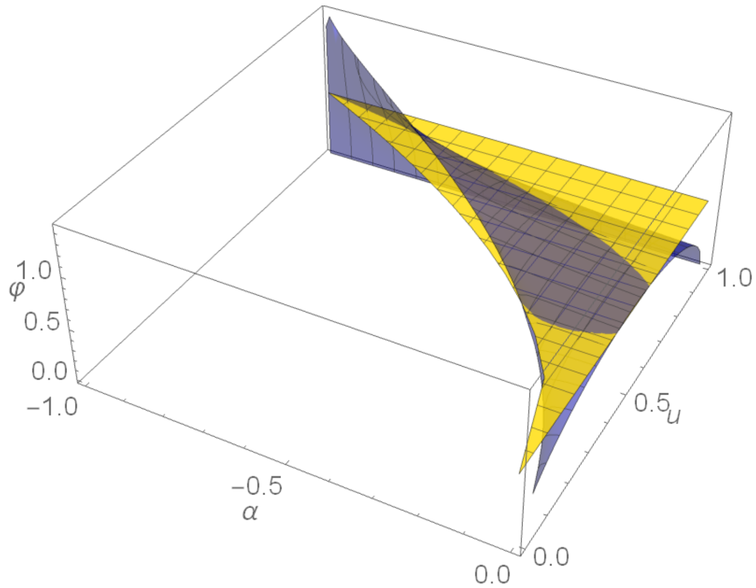
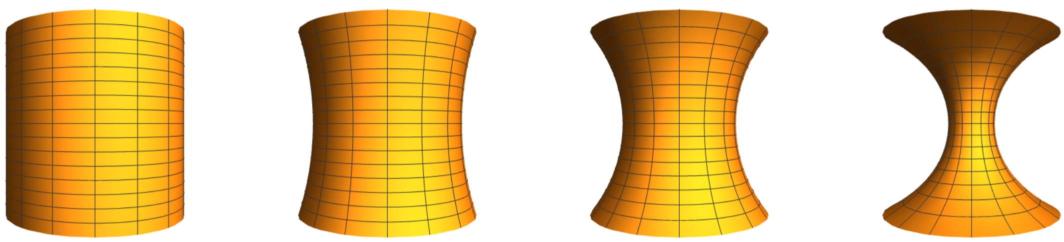
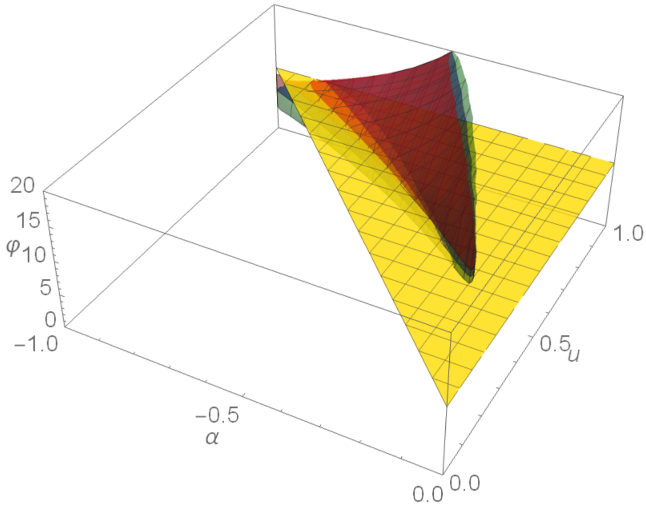
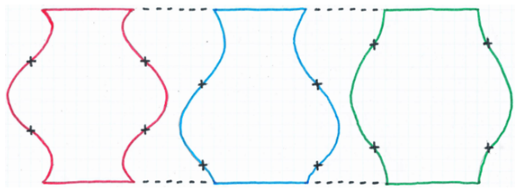


Figure 25: Case(I-2): Non-convex unduloid



- Similarly, three surfaces are the graphs of function $\varphi(u, \alpha)$ of (R_+), (B_+) and (G_+). Each of the level curves of the surfaces show that Delaunay surfaces bounded by two circles.

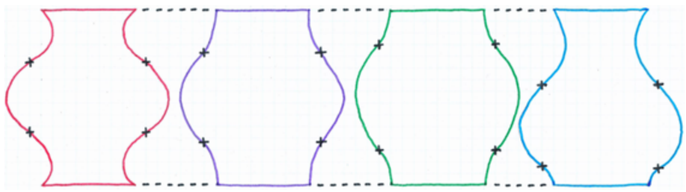




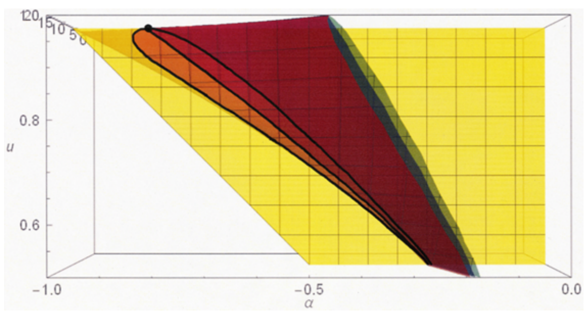
(R_+) (B_+) (G_+)

Figure 26: The surfaces that the radius of boundary circles are equal to $\dfrac{u-\sqrt{u^2+\alpha}}{-2H}$ and its level curves

- The bottom picture is an enlarged picture of the level curves when we watch Figure 26 from above. The surface of branch point of three level curves is the unduloid which is just from a neck to a neck.



$(R_+) \longleftarrow \boxed{\text{branch point}} \longrightarrow (G_+)$ (B_+)



- Secondly, three surfaces are the graphs of function $\varphi(u, \alpha)$ of (R_-), (B_-) and (G_-). Each of the level curves of the surfaces show that Delaunay surfaces bounded by two circles.

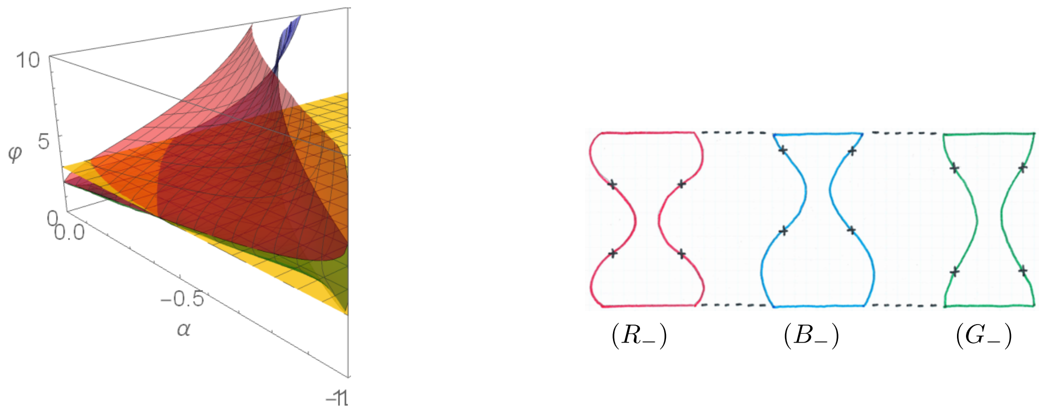
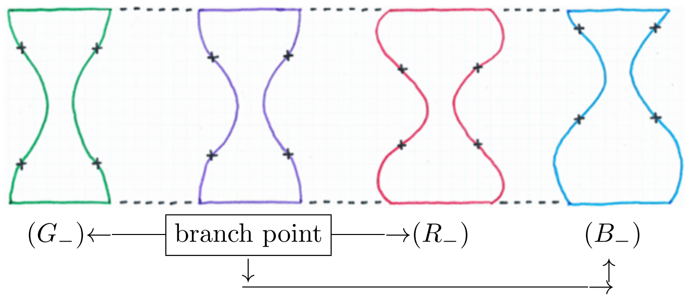
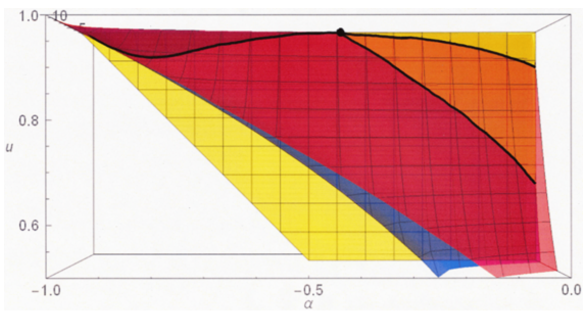


Figure 27: The surfaces that the radius of boundary circles are equal to $\dfrac{u+\sqrt{u^2+\alpha}}{-2H}$ and its level curves

- The bottom picture is an enlarged picture of the level curves when we watch Figure 27 from above. The surface of branch point of three level curves is the unduloid which is just from a bulge to a bulge.





2.6 Volume of unduloid

We will examine stability of (G_+), (R_+), (B_+), (G_-), (R_-) and (B_-) later. Hence, in this section, we compute volume of unduloid in each case.

2.6.1 Volume V_- of Case(I-2) and volume V_+ of Case(II-2)

In this subsection, we will calculate volume of (G_+), (R_+), (B_+), (G_-), (R_-) and (B_-) later. Thus, firstly, we compute volume V_- of Case(I-2) and volume V_+ of Case(II-2).

Case(I-2) X is a non-convex unduloid: $H < 0$, $\sqrt{-\alpha} \leq u \leq 1$, $-1 < \alpha < 0$,

$$\begin{aligned} x &= \frac{u - \sqrt{u^2+\alpha}}{-2H} \\ z &= \frac{1}{-2H}\int_1^u \frac{-\tilde{u}}{\sqrt{1-\tilde{u}^2}} \cdot \frac{\tilde{u} - \sqrt{\tilde{u}^2+\alpha}}{\sqrt{\tilde{u}^2+\alpha}}\, d\tilde{u} \end{aligned}$$

Hence, volume V_- of Case(I-2) is as follows.

$$\begin{aligned} V_- &= \pi \int_1^u x^2 \frac{dz}{du} du & (49) \\ &= \pi \int_1^u \left(\frac{2u^2 - 2u\sqrt{u^2+\alpha} + \alpha}{4H^2}\right)\left\{-\frac{1}{2H}\left(\frac{-u}{\sqrt{1-u^2}} \cdot \frac{u - \sqrt{u^2+\alpha}}{\sqrt{u^2+\alpha}}\right)\right\} du & (50) \\ &= \frac{\pi}{8H^3}\int_1^u \left(\frac{4u^4}{\sqrt{1-u^2}\sqrt{u^2+\alpha}} + \frac{-4u^3}{\sqrt{1-u^2}} + \frac{3\alpha u^2}{\sqrt{1-u^2}\sqrt{u^2+\alpha}} + \frac{-\alpha u}{\sqrt{1-u^2}}\right) du & (51) \end{aligned}$$

Proposition 2.6.1.

$$\begin{aligned} \int_1^u \frac{4u^4}{\sqrt{1-u^2}\sqrt{u^2+\alpha}} du &= \frac{4}{3}\left\{-u\sqrt{1-u^2}\sqrt{u^2+\alpha} - 2(1-\alpha)\sqrt{1+\alpha}\ E(\frac{1}{\sqrt{1+\alpha}}, \arccos u)\right. \\ &\quad \left. + \frac{\alpha(1-2\alpha)}{\sqrt{1+\alpha}}\ F(\frac{1}{\sqrt{1+\alpha}}, \arccos u)\right\} \end{aligned}$$

Therefore, from (51) and Proposition 2.6.1,

$$\begin{aligned} & \frac{\pi}{8H^3}\int_1^u \left(\frac{4u^4}{\sqrt{1-u^2}\sqrt{u^2+\alpha}} + \frac{-4u^3}{\sqrt{1-u^2}} + \frac{3\alpha u^2}{\sqrt{1-u^2}\sqrt{u^2+\alpha}} + \frac{-\alpha u}{\sqrt{1-u^2}}\right) du \\ = & -\frac{\pi}{24H^3}\left\{4\sqrt{1-u^2}\left(u\sqrt{u^2+\alpha} - 2 - u^2 - \frac{3}{4}\alpha\right)\right. \\ & \left. + (8+\alpha)\sqrt{1+\alpha}\ E(\frac{1}{\sqrt{1+\alpha}}, \arccos u) - \frac{\alpha(4+\alpha)}{\sqrt{1+\alpha}}\ F(\frac{1}{\sqrt{1+\alpha}}, \arccos u)\right\} \end{aligned}$$

Consequently, we obtain the following.

Theorem 2.6.1. *Case(I-2) X is a non-convex unduloid:* $H<0$, $\sqrt{-\alpha}\le u\le 1$, $-1<\alpha<0$,

$$\begin{aligned} x &= \frac{u-\sqrt{u^2+\alpha}}{-2H}, \\ z &= \frac{1}{-2H}\int_1^u \frac{-\tilde{u}}{\sqrt{1-\tilde{u}^2}}\cdot\frac{\tilde{u}-\sqrt{\tilde{u}^2+\alpha}}{\sqrt{\tilde{u}^2+\alpha}}\,d\tilde{u}. \end{aligned}$$

Thus, volume V_- *of Case(I-2) is as follows.*

$$\begin{aligned} V_- &= -\frac{\pi}{24H^3}\left\{4\sqrt{1-u^2}\left(u\sqrt{u^2+\alpha}-2-u^2-\frac{3}{4}\alpha\right)\right. \\ &\quad \left. +(8+\alpha)\sqrt{1+\alpha}\;E(\frac{1}{\sqrt{1+\alpha}},\arccos u)-\frac{\alpha\,(4+\alpha)}{\sqrt{1+\alpha}}\;F(\frac{1}{\sqrt{1+\alpha}},\arccos u)\right\} \end{aligned}$$

Corollary 2.6.1. *In Case(I-2), volume* Vi_- *which is from a neck to a inflection point is the following.*

$$\begin{aligned} Vi_- &= -\frac{\pi}{24H^3}\left\{\sqrt{1+\alpha}\,(\alpha-8)\right. \\ &\quad \left. +(8+\alpha)\sqrt{1+\alpha}\;E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})-\frac{\alpha\,(4+\alpha)}{\sqrt{1+\alpha}}\;F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\} \end{aligned}$$

Similarly, we obtain the following.

Theorem 2.6.2. *Case(II-2) X is a convex unduloid:* $H<0$, $\sqrt{-\alpha}\le u\le 1$, $-1<\alpha<0$,

$$\begin{aligned} x &= \frac{u+\sqrt{u^2+\alpha}}{-2H}, \\ z &= \frac{1}{-2H}\int_1^u \frac{-\tilde{u}}{\sqrt{1-\tilde{u}^2}}\cdot\frac{\tilde{u}+\sqrt{\tilde{u}^2+\alpha}}{\sqrt{\tilde{u}^2+\alpha}}\,d\tilde{u}. \end{aligned}$$

Thus, volume V_+ *of Case(II-2) is as follows.*

$$\begin{aligned} V_+ &= -\frac{\pi}{24H^3}\left\{4\sqrt{1-u^2}\left(u\sqrt{u^2+\alpha}+2+u^2+\frac{3}{4}\alpha\right)\right. \\ &\quad \left. +(8+\alpha)\sqrt{1+\alpha}\;E(\frac{1}{\sqrt{1+\alpha}},\arccos u)-\frac{\alpha\,(4+\alpha)}{\sqrt{1+\alpha}}\;F(\frac{1}{\sqrt{1+\alpha}},\arccos u)\right\} \end{aligned}$$

Corollary 2.6.2. *In Case(II-2), volume* Vi_+ *which is from a bulge to a inflection point is the following.*

$$\begin{aligned} Vi_+ &= -\frac{\pi}{24H^3}\left\{\sqrt{1+\alpha}\,(8-\alpha)\right. \\ &\quad \left. +(8+\alpha)\sqrt{1+\alpha}\;E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})-\frac{\alpha\,(4+\alpha)}{\sqrt{1+\alpha}}\;F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\} \end{aligned}$$

2.6.2 Volume of (G_+)

(G_+): A profile curve exceeds a inflection point. However, it doesn't exceed a neck.
We calculate volume $V_{(G_+)}$ in (G_+).

$$\begin{aligned}
\frac{V_{(G_+)}}{2} &= (Vi_+) + (Vi_-) - (V_-) \\
&= -\frac{\pi}{24H^3}\left[-4\sqrt{1-u^2}\left(u\sqrt{u^2+\alpha} - 2 - u^2 - \frac{3}{4}\alpha\right)\right. \\
&\quad + (8+\alpha)\sqrt{1+\alpha}\left\{2\ E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) - E(\frac{1}{\sqrt{1+\alpha}}, \arccos u)\right\} \\
&\quad \left. - \frac{\alpha(4+\alpha)}{\sqrt{1+\alpha}}\left\{2\ F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) - F(\frac{1}{\sqrt{1+\alpha}}, \arccos u)\right\}\right]
\end{aligned}$$

Consequently, we obtain the following.

Theorem 2.6.3.

$$\begin{aligned}
V_{(G_+)} &= -\frac{\pi}{12H^3}\left[-4\sqrt{1-u^2}\left(u\sqrt{u^2+\alpha} - 2 - u^2 - \frac{3}{4}\alpha\right)\right. \\
&\quad + (8+\alpha)\sqrt{1+\alpha}\left\{2\ E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) - E(\frac{1}{\sqrt{1+\alpha}}, \arccos u)\right\} \\
&\quad \left. - \frac{\alpha(4+\alpha)}{\sqrt{1+\alpha}}\left\{2\ F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) - F(\frac{1}{\sqrt{1+\alpha}}, \arccos u)\right\}\right]
\end{aligned}$$

2.6.3 Volume of (R_+)

(R_+): A profile curve exceeds both a inflection point and a neck. However, it doesn't exceed a next inflection point.
We calculate volume $V_{(R_+)}$ in (R_+).

$$\begin{aligned}
\frac{V_{(R_+)}}{2} &= (Vi_+) + (Vi_-) + (V_-) \\
&= -\frac{\pi}{24H^3}\left[4\sqrt{1-u^2}\left(u\sqrt{u^2+\alpha} - 2 - u^2 - \frac{3}{4}\alpha\right)\right. \\
&\quad + (8+\alpha)\sqrt{1+\alpha}\left\{2\ E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) + E(\frac{1}{\sqrt{1+\alpha}}, \arccos u)\right\} \\
&\quad \left. - \frac{\alpha(4+\alpha)}{\sqrt{1+\alpha}}\left\{2\ F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) + F(\frac{1}{\sqrt{1+\alpha}}, \arccos u)\right\}\right]
\end{aligned}$$

Consequently, we obtain the following.

Theorem 2.6.4.

$$
\begin{aligned}
V_{(R_+)} &= -\frac{\pi}{12H^3}\left[4\sqrt{1-u^2}\left(u\sqrt{u^2+\alpha}-2-u^2-\frac{3}{4}\alpha\right)\right.\\
&\quad+(8+\alpha)\sqrt{1+\alpha}\left\{2\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})+E(\frac{1}{\sqrt{1+\alpha}},\arccos u)\right\}\\
&\quad\left.-\frac{\alpha\,(4+\alpha)}{\sqrt{1+\alpha}}\left\{2\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})+F(\frac{1}{\sqrt{1+\alpha}},\arccos u)\right\}\right]
\end{aligned}
$$

2.6.4 Volume of (B_+)

(B_+): A unduloid which is asymmetric with respect to the plane $\{z=0\}$ where the radius of boundary circles are equal to $\dfrac{u-\sqrt{u^2+\alpha}}{-2H}$.
We calculate volume $V_{(B_+)}$ in (B_+).

$$
\begin{aligned}
\frac{V_{(B_+)}}{2} &= \frac{V_{(G_+)}}{2}+(V_-)\\
&= -\frac{\pi}{12H^3}\left\{(8+\alpha)\sqrt{1+\alpha}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})-\frac{\alpha\,(4+\alpha)}{\sqrt{1+\alpha}}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}
\end{aligned}
$$

Consequently, we obtain the following.

Theorem 2.6.5.

$$
V_{(B_+)} = -\frac{\pi}{6H^3}\left\{(8+\alpha)\sqrt{1+\alpha}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})-\frac{\alpha\,(4+\alpha)}{\sqrt{1+\alpha}}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}
$$

2.6.5 Volume of (G_-)

(G_-): A profile curve exceeds a inflection point. However, it doesn't exceed a bulge.
We calculate volume $V_{(G_-)}$ in (G_-).

$$
\begin{aligned}
\frac{V_{(G_-)}}{2} &= (Vi_-)+(Vi_+)-(V_+)\\
&= -\frac{\pi}{24H^3}\left[-4\sqrt{1-u^2}\left(u\sqrt{u^2+\alpha}+2+u^2+\frac{3}{4}\alpha\right)\right.\\
&\quad+(8+\alpha)\sqrt{1+\alpha}\left\{2\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})-E(\frac{1}{\sqrt{1+\alpha}},\arccos u)\right\}\\
&\quad\left.-\frac{\alpha\,(4+\alpha)}{\sqrt{1+\alpha}}\left\{2\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})-F(\frac{1}{\sqrt{1+\alpha}},\arccos u)\right\}\right]
\end{aligned}
$$

Consequently, we obtain the following.

Theorem 2.6.6.

$$\begin{aligned} V_{(G_-)} &= -\frac{\pi}{12H^3}\left[-4\sqrt{1-u^2}\left(u\sqrt{u^2+\alpha}+2+u^2+\frac{3}{4}\alpha\right)\right. \\ &\quad +(8+\alpha)\sqrt{1+\alpha}\left\{2\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})-E(\frac{1}{\sqrt{1+\alpha}},\arccos u)\right\} \\ &\quad \left.-\frac{\alpha(4+\alpha)}{\sqrt{1+\alpha}}\left\{2\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})-F(\frac{1}{\sqrt{1+\alpha}},\arccos u)\right\}\right] \end{aligned}$$

2.6.6 Volume of (R_-)

(R_-): A profile curve exceeds both a inflection point and a bulge. However, it doesn't exceed a next inflection point.
We calculate volume $V_{(R_-)}$ in (R_-).

$$\begin{aligned} \frac{V_{(R_-)}}{2} &= (Vi_-)+(Vi_+)+(V_+) \\ &= -\frac{\pi}{24H^3}\left[4\sqrt{1-u^2}\left(u\sqrt{u^2+\alpha}+2+u^2+\frac{3}{4}\alpha\right)\right. \\ &\quad +(8+\alpha)\sqrt{1+\alpha}\left\{2\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})+E(\frac{1}{\sqrt{1+\alpha}},\arccos u)\right\} \\ &\quad \left.-\frac{\alpha(4+\alpha)}{\sqrt{1+\alpha}}\left\{2\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})+F(\frac{1}{\sqrt{1+\alpha}},\arccos u)\right\}\right] \end{aligned}$$

Consequently, we obtain the following.

Theorem 2.6.7.

$$\begin{aligned} V_{(R_-)} &= -\frac{\pi}{12H^3}\left[4\sqrt{1-u^2}\left(u\sqrt{u^2+\alpha}+2+u^2+\frac{3}{4}\alpha\right)\right. \\ &\quad +(8+\alpha)\sqrt{1+\alpha}\left\{2\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})+E(\frac{1}{\sqrt{1+\alpha}},\arccos u)\right\} \\ &\quad \left.-\frac{\alpha(4+\alpha)}{\sqrt{1+\alpha}}\left\{2\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})+F(\frac{1}{\sqrt{1+\alpha}},\arccos u)\right\}\right] \end{aligned}$$

2.6.7 Volume of (B_-)

(B_-): A unduloid which is asymmetric with respect to the plane $\{z=0\}$ where the radius of boundary circles are equal to $\dfrac{u+\sqrt{u^2+\alpha}}{-2H}$.
We calculate volume $V_{(B_-)}$ in (B_-).

$$\begin{aligned} \frac{V_{(B_-)}}{2} &= \frac{V_{(G_-)}}{2}+(V_+) \\ &= -\frac{\pi}{12H^3}\left\{(8+\alpha)\sqrt{1+\alpha}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})-\frac{\alpha(4+\alpha)}{\sqrt{1+\alpha}}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\} \end{aligned}$$

Consequently, we obtain the following.

Theorem 2.6.8.

$$V_{(B_-)} \;=\; -\frac{\pi}{6H^3}\left\{(8+\alpha)\sqrt{1+\alpha}\;E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha}) - \frac{\alpha\,(4+\alpha)}{\sqrt{1+\alpha}}\;F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}$$

Remark 2.6.1. *$V_{(B_+)} = V_{(B_-)}$ holds.*

2.7 Partially differentiation of elliptic integral of the first (second) kind with respect to u or α

In this section, in order to calculate partial differentiation of function $h/r = \varphi\,(u,\alpha)$ later, we must partially differentiate elliptic integral of the first (second) kind with respect to u or α. We need to calculate 8 types.

2.7.1 Definition of elliptic integral of the first (second) kind

Definition 2.7.1. *Elliptic integral of the first kind $F(k,\phi)$ and elliptic integral of the second kind $E(k,\phi)$ are defined as follows.*

$$F(k,\phi) := \int_0^{\phi} \frac{1}{\sqrt{1-k^2\sin^2\theta}}\;d\theta = \int_0^{\sin\phi} \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}\;dt,$$

$$E(k,\phi) := \int_0^{\phi} \sqrt{1-k^2\sin^2\theta}\;d\theta = \int_0^{\sin\phi} \sqrt{\frac{1-k^2t^2}{1-t^2}}\;dt.$$

2.7.2 Partially differentiation of elliptic integral of the first (second) kind with respect to u

Proposition 2.7.1. *The following four formulas hold.*

$$\frac{\partial}{\partial u}\;F(\frac{1}{\sqrt{1+\alpha}},\arccos u) = -\frac{1}{\sqrt{1-u^2}\sqrt{\dfrac{\alpha+u^2}{1+\alpha}}}, \tag{52}$$

$$\frac{\partial}{\partial u}\;E(\frac{1}{\sqrt{1+\alpha}},\arccos u) = -\frac{\sqrt{\dfrac{\alpha+u^2}{1+\alpha}}}{\sqrt{1-u^2}}, \tag{53}$$

$$\frac{\partial}{\partial u}\;F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha}) = 0, \tag{54}$$

$$\frac{\partial}{\partial u}\;E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha}) = 0. \tag{55}$$

2.7.3 Partially differentiation of elliptic integral of the first (second) kind with respect to α

Definition 2.7.2. *Elliptic integral of the third kind is defined as follows.*

$$\Pi(k,n,\varphi) := \int_0^{\varphi} \frac{1}{(1+n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}} d\theta$$

Claim 2.7.1. *The following formula holds.*

$$\left(\Pi(k,-k^2,\varphi) =\right) \int_0^{\varphi} \frac{1}{\left(1-k^2\sin^2\theta\right)^{3/2}} d\theta = \frac{1}{1-k^2}\left\{ E(k,\varphi) - \frac{k^2\sin 2\varphi}{2\sqrt{1-k^2\sin^2\varphi}} \right\}$$

Proposition 2.7.2. *The following four formulas hold. In particular, the equation (56) can be proved by using Claim 2.7.1.*

$$\begin{aligned} &\frac{\partial}{\partial\alpha} F(\frac{1}{\sqrt{1+\alpha}}, \arccos u) \\ = \quad & \frac{1}{2(1+\alpha)} F(\frac{1}{\sqrt{1+\alpha}}, \arccos u) - \frac{1}{2\alpha}\left\{ E(\frac{1}{\sqrt{1+\alpha}}, \arccos u) - \frac{\sin 2(\arccos u)}{2(1+\alpha)\sqrt{\dfrac{\alpha+u^2}{1+\alpha}}} \right\}, \end{aligned} \tag{56}$$

$$\frac{\partial}{\partial\alpha} E(\frac{1}{\sqrt{1+\alpha}}, \arccos u) = \frac{1}{2(1+\alpha)}\left\{F(\frac{1}{\sqrt{1+\alpha}}, \arccos u) - E(\frac{1}{\sqrt{1+\alpha}}, \arccos u)\right\}, \tag{57}$$

$$\frac{\partial}{\partial\alpha} F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) = \frac{1}{2(1+\alpha)} F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) - \frac{1}{2\alpha} E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}), \tag{58}$$

$$\frac{\partial}{\partial\alpha} E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) = \frac{1}{2(1+\alpha)}\left\{F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) - E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha})\right\}. \tag{59}$$

2.8 Partially differentiation of function $h/r = \varphi(u,\alpha)$ (Gradient of tangential line of level curve on α-u plane)

2.8.1 Partially differentiation of function $h/r = \varphi(u,\alpha)$ in (G_+) with respect to u

Firstly, $\varphi(u,\alpha)$ in (G_+) is as follows.

$$\begin{aligned} \varphi(u,\alpha) \quad = \quad & \frac{2}{u-\sqrt{u^2+\alpha}}\left\{-\frac{2\alpha}{\sqrt{1+\alpha}} F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) + 2\sqrt{1+\alpha}\, E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) \right. \\ & \left. + \frac{\alpha}{\sqrt{1+\alpha}} F(\frac{1}{\sqrt{1+\alpha}}, \arccos u) - \sqrt{1+\alpha}\, E(\frac{1}{\sqrt{1+\alpha}}, \arccos u) + \sqrt{1-u^2}\right\} \end{aligned}$$

Thus, we partially differentiate both-hand sides of "$\varphi(u,\alpha) = \text{constant}$" with respect to α,

$$\varphi_u u_\alpha + \varphi_\alpha = 0 \Leftrightarrow u_\alpha = -\frac{\varphi_\alpha}{\varphi_u}$$

holds. Hence, we must calculate φ_u and φ_α. Therefore, we compute φ_u. From (52), (53), (54), (55),

$$\frac{\partial}{\partial u}\left\{-\frac{2\alpha}{\sqrt{1+\alpha}}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\} = 0\cdot\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha}) - \frac{2\alpha}{\sqrt{1+\alpha}}\cdot 0 = 0$$

$$\frac{\partial}{\partial u}\left\{2\sqrt{1+\alpha}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\} = 0\cdot\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha}) + 2\sqrt{1+\alpha}\cdot 0 = 0$$

$$\frac{\partial}{\partial u}\left\{\frac{\alpha}{\sqrt{1+\alpha}}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos u)\right\} = \frac{\alpha}{\sqrt{1+\alpha}}\left(-\frac{1}{\sqrt{1-u^2}\sqrt{\dfrac{\alpha+u^2}{1+\alpha}}}\right) = -\frac{\alpha}{\sqrt{1+\alpha}\sqrt{1-u^2}\sqrt{\dfrac{\alpha+u^2}{1+\alpha}}}$$

$$\frac{\partial}{\partial u}\left\{-\sqrt{1+\alpha}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos u)\right\} = -\sqrt{1+\alpha}\left(-\frac{\sqrt{\dfrac{\alpha+u^2}{1+\alpha}}}{\sqrt{1-u^2}}\right) = \frac{\sqrt{1+\alpha}\sqrt{\dfrac{\alpha+u^2}{1+\alpha}}}{\sqrt{1-u^2}}$$

$$\frac{\partial}{\partial u}\left(\sqrt{1-u^2}\right) = \frac{-2u}{2\sqrt{1-u^2}} = -\frac{u}{\sqrt{1-u^2}}$$

$$\frac{\partial}{\partial u}\left(u-\sqrt{u^2+\alpha}\right) = 1-\frac{2u}{2\sqrt{u^2+\alpha}} = 1-\frac{u}{\sqrt{u^2+\alpha}} = \frac{\sqrt{u^2+\alpha}-u}{\sqrt{u^2+\alpha}} = -\frac{u-\sqrt{u^2+\alpha}}{\sqrt{u^2+\alpha}}$$

Thus,

$$\begin{aligned}\frac{1}{2}\frac{\partial\varphi}{\partial u} \quad = \quad & \frac{1}{u-\sqrt{u^2+\alpha}}\left[\left\{\frac{-\alpha+(1+\alpha)\left(\dfrac{\alpha+u^2}{1+\alpha}\right)-u\sqrt{1+\alpha}\sqrt{\dfrac{\alpha+u^2}{1+\alpha}}}{\sqrt{1+\alpha}\sqrt{1-u^2}\sqrt{\dfrac{\alpha+u^2}{1+\alpha}}}\right\}\right.\\ & -\left\{-\frac{2\alpha}{\sqrt{1+\alpha}}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha}) + 2\sqrt{1+\alpha}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right.\\ & \left.\left.+\ \frac{\alpha}{\sqrt{1+\alpha}}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos u) - \sqrt{1+\alpha}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos u) + \sqrt{1-u^2}\right\}\cdot\left(-\frac{1}{\sqrt{u^2+\alpha}}\right)\right]\end{aligned}$$

Now,

$$\lim_{u\to 1-0}\frac{1}{u-\sqrt{u^2+\alpha}}\cdot\frac{-\alpha+(1+\alpha)\left(\dfrac{\alpha+u^2}{1+\alpha}\right)-u\sqrt{1+\alpha}\sqrt{\dfrac{\alpha+u^2}{1+\alpha}}}{\sqrt{1+\alpha}\sqrt{1-u^2}\sqrt{\dfrac{\alpha+u^2}{1+\alpha}}} \quad = \quad \frac{1}{\sqrt{1+\alpha}}\lim_{u\to 1-0}\frac{1}{\sqrt{1-u^2}} \to +\infty$$

Consequently,

$$\lim_{u\to 1-0}\frac{1}{2}\frac{\partial\varphi}{\partial u} = \frac{1}{\sqrt{1+\alpha}}\lim_{u\to 1-0}\frac{1}{\sqrt{1-u^2}} \to +\infty \qquad (60)$$

holds.

2.8.2 The other cases ((G_+), (R_+), (B_+))

In the case of (G_+),

$$\lim_{u\to 1-0}\frac{1}{2}\frac{\partial\varphi}{\partial\alpha}=\frac{1}{\left(1-\sqrt{1+\alpha}\right)^2}\left\{\frac{1-\sqrt{1+\alpha}}{1+\alpha}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})+\frac{1}{\sqrt{1+\alpha}}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}>0 \tag{61}$$

holds.

In the case of (R_+),

$$\lim_{u\to 1-0}\frac{1}{2}\frac{\partial\varphi}{\partial u}=-\frac{1}{\sqrt{1+\alpha}}\lim_{u\to 1-0}\frac{1}{\sqrt{1-u^2}}\to-\infty \tag{62}$$

and

$$\lim_{u\to 1-0}\frac{1}{2}\frac{\partial\varphi}{\partial\alpha}=\frac{1}{\left(1-\sqrt{1+\alpha}\right)^2}\left\{\frac{1-\sqrt{1+\alpha}}{1+\alpha}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})+\frac{1}{\sqrt{1+\alpha}}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}>0 \tag{63}$$

hold.

In the case of (B_+),

$$\lim_{u\to 1-0}\frac{1}{4}\frac{\partial\varphi}{\partial u}=\frac{1}{\sqrt{1+\alpha}\left(1-\sqrt{1+\alpha}\right)}\left\{-\frac{\alpha}{\sqrt{1+\alpha}}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})+\sqrt{1+\alpha}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}>0 \tag{64}$$

and

$$\lim_{u\to 1-0}\frac{1}{4}\frac{\partial\varphi}{\partial\alpha}=\frac{1}{2\left(1-\sqrt{1+\alpha}\right)^2}\left\{\frac{1-\sqrt{1+\alpha}}{1+\alpha}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})+\frac{1}{\sqrt{1+\alpha}}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}>0 \tag{65}$$

hold.

2.9 Partially differentiation of volume $V(u,\alpha)$

Firstly, we have already obtained that radius of boundary circles $r=\dfrac{u-\sqrt{u^2+\alpha}}{-2H}$. Thus,

$$r=\frac{u-\sqrt{u^2+\alpha}}{-2H}\Leftrightarrow H(u,\alpha)=\frac{u-\sqrt{u^2+\alpha}}{-2r} \tag{66}$$

From (66), we can compute as follows.

$$-\frac{\pi}{12H^3}=-\frac{\pi}{12\left(\dfrac{u-\sqrt{u^2+\alpha}}{-2r}\right)^3}=-\frac{\pi}{12\cdot\dfrac{\left(u-\sqrt{u^2+\alpha}\right)^3}{-8r^3}}=(-\pi)\cdot\frac{-8r^3}{12\left(u-\sqrt{u^2+\alpha}\right)^3}=\frac{2\pi r^3}{3\left(u-\sqrt{u^2+\alpha}\right)^3} \tag{67}$$

Similarly,

$$-\frac{\pi}{6H^3}=-\frac{\pi}{12H^3}\cdot 2=\frac{4\pi r^3}{3\left(u-\sqrt{u^2+\alpha}\right)^3} \tag{68}$$

2.9.1 Partially differentiation of volume $V(u,\alpha)$ in (G_+) with respect to u

Firstly, $V(u,\alpha) := V_{(G_+)}(u,\alpha)$ in (G_+) is as follows.

$$\begin{aligned}
V(u,\alpha) &= -\frac{\pi}{12H^3}\left[-4\sqrt{1-u^2}\left(u\sqrt{u^2+\alpha}-2-u^2-\frac{3}{4}\alpha\right)\right.\\
&+(8+\alpha)\sqrt{1+\alpha}\left\{2\,E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})-E(\frac{1}{\sqrt{1+\alpha}},\arccos u)\right\}\\
&\left.-\frac{\alpha(4+\alpha)}{\sqrt{1+\alpha}}\left\{2\,F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})-F(\frac{1}{\sqrt{1+\alpha}},\arccos u)\right\}\right]
\end{aligned}$$

From (67),

$$\begin{aligned}
V(u,\alpha) &= \frac{2\pi r^3}{3\left(u-\sqrt{u^2+\alpha}\right)^3}\left[-4\sqrt{1-u^2}\left(u\sqrt{u^2+\alpha}-2-u^2-\frac{3}{4}\alpha\right)\right.\\
&+(8+\alpha)\sqrt{1+\alpha}\left\{2\,E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})-E(\frac{1}{\sqrt{1+\alpha}},\arccos u)\right\}\\
&\left.-\frac{\alpha(4+\alpha)}{\sqrt{1+\alpha}}\left\{2\,F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})-F(\frac{1}{\sqrt{1+\alpha}},\arccos u)\right\}\right]
\end{aligned}$$

That is,

$$\begin{aligned}
\frac{3}{2\pi r^3}V(u,\alpha) &= \frac{1}{\left(u-\sqrt{u^2+\alpha}\right)^3}\left[-4\sqrt{1-u^2}\left(u\sqrt{u^2+\alpha}-2-u^2-\frac{3}{4}\alpha\right)\right.\\
&+(8+\alpha)\sqrt{1+\alpha}\left\{2\,E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})-E(\frac{1}{\sqrt{1+\alpha}},\arccos u)\right\}\\
&\left.-\frac{\alpha(4+\alpha)}{\sqrt{1+\alpha}}\left\{2\,F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})-F(\frac{1}{\sqrt{1+\alpha}},\arccos u)\right\}\right]
\end{aligned}$$

Now, we partially differentiate $\frac{3}{2\pi r^3}V(u,\alpha)$ with respect to u. Therefore, we compute $\frac{3}{2\pi r^3}V_u$.
From (52), (53), (54), (55),

$$\begin{aligned}
&\frac{\partial}{\partial u}\left\{-4\sqrt{1-u^2}\left(u\sqrt{u^2+\alpha}-2-u^2-\frac{3}{4}\alpha\right)\right\}\\
=\ &\frac{4u}{\sqrt{1-u^2}}\left(u\sqrt{u^2+\alpha}-2-u^2-\frac{3}{4}\alpha\right)-4\sqrt{1-u^2}\left(\sqrt{u^2+\alpha}+\frac{u^2}{\sqrt{u^2+\alpha}}-2u\right)
\end{aligned}$$

$$\frac{\partial}{\partial u}\left[(8+\alpha)\sqrt{1+\alpha}\left\{2\,E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})-E(\frac{1}{\sqrt{1+\alpha}},\arccos u)\right\}\right] = (8+\alpha)\sqrt{1+\alpha}\left(\frac{\sqrt{\dfrac{\alpha+u^2}{1+\alpha}}}{\sqrt{1-u^2}}\right)$$

$$\frac{\partial}{\partial u}\left[-\frac{\alpha(4+\alpha)}{\sqrt{1+\alpha}}\left\{2\,F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})-F(\frac{1}{\sqrt{1+\alpha}},\arccos u)\right\}\right] = -\frac{\alpha(4+\alpha)}{\sqrt{1+\alpha}}\left(\frac{1}{\sqrt{1-u^2}\sqrt{\dfrac{\alpha+u^2}{1+\alpha}}}\right)$$

$$\frac{\partial}{\partial u}\left(u-\sqrt{u^2+\alpha}\right)^3 \quad = \quad \left(u-\sqrt{u^2+\alpha}\right)^3\cdot\left(-\frac{3}{\sqrt{u^2+\alpha}}\right)$$

Hence,

$$\begin{aligned}
\frac{3}{2\pi r^3}\frac{\partial V}{\partial u} &= \frac{1}{\left(u-\sqrt{u^2+\alpha}\right)^6}\left[\left\{\frac{4u}{\sqrt{1-u^2}}\left(u\sqrt{u^2+\alpha}-2-u^2-\frac{3}{4}\alpha\right)-4\sqrt{1-u^2}\left(\sqrt{u^2+\alpha}+\frac{u^2}{\sqrt{u^2+\alpha}}-2u\right)\right.\right.\\
&+(8+\alpha)\sqrt{1+\alpha}\left(\frac{\sqrt{\frac{\alpha+u^2}{1+\alpha}}}{\sqrt{1-u^2}}\right)-\frac{\alpha(4+\alpha)}{\sqrt{1+\alpha}}\left.\left(\frac{1}{\sqrt{1-u^2}\sqrt{\frac{\alpha+u^2}{1+\alpha}}}\right)\right\}\cdot\left(u-\sqrt{u^2+\alpha}\right)^3\\
&-\left\{-4\sqrt{1-u^2}\left(u\sqrt{u^2+\alpha}-2-u^2-\frac{3}{4}\alpha\right)\right.\\
&+(8+\alpha)\sqrt{1+\alpha}\left(2\,E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})-E(\frac{1}{\sqrt{1+\alpha}},\arccos u)\right)\\
&-\left.\left.\frac{\alpha(4+\alpha)}{\sqrt{1+\alpha}}\left(2\,F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})-F(\frac{1}{\sqrt{1+\alpha}},\arccos u)\right)\right\}\cdot\left(u-\sqrt{u^2+\alpha}\right)^3\cdot\left(-\frac{3}{\sqrt{u^2+\alpha}}\right)\right]
\end{aligned}$$

Now,

$$\begin{aligned}
&\lim_{u\to 1-0}\frac{1}{\left(u-\sqrt{u^2+\alpha}\right)^6}\left\{\frac{4u}{\sqrt{1-u^2}}\left(u\sqrt{u^2+\alpha}-2-u^2-\frac{3}{4}\alpha\right)\right.\\
&\quad+(8+\alpha)\sqrt{1+\alpha}\left(\frac{\sqrt{\frac{\alpha+u^2}{1+\alpha}}}{\sqrt{1-u^2}}\right)-\frac{\alpha(4+\alpha)}{\sqrt{1+\alpha}}\left.\left(\frac{1}{\sqrt{1-u^2}\sqrt{\frac{\alpha+u^2}{1+\alpha}}}\right)\right\}\cdot\left(u-\sqrt{u^2+\alpha}\right)^3\\
&= \frac{3}{\sqrt{1+\alpha}}\lim_{u\to 1-0}\frac{1}{\sqrt{1-u^2}}\to+\infty
\end{aligned}$$

Consequently,

$$\lim_{u\to 1-0}\frac{3}{2\pi r^3}\frac{\partial V}{\partial u} = \frac{3}{\sqrt{1+\alpha}}\lim_{u\to 1-0}\frac{1}{\sqrt{1-u^2}}\to+\infty \tag{69}$$

holds.

2.9.2 The other cases ((G_+), (R_+), (B_+))

In the case of (G_+),

$$\begin{aligned}
\lim_{u\to 1-0}\frac{3}{2\pi r^3}\frac{\partial V}{\partial \alpha} \quad &= \quad \frac{3}{\left(1-\sqrt{1+\alpha}\right)^4}\left[\left\{-\frac{\alpha\left(1+\alpha+3\sqrt{1+\alpha}\right)}{(1+\alpha)^{3/2}}\right\}F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right.\\
&\left.+\left(\frac{2+\alpha+6\sqrt{1+\alpha}}{\sqrt{1+\alpha}}\right)E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right]>0
\end{aligned} \tag{70}$$

holds.

In the case of (R_+),

$$\lim_{u\to 1-0} \frac{3}{2\pi r^3} \frac{\partial V}{\partial u} = -\frac{3}{\sqrt{1+\alpha}} \lim_{u\to 1-0} \frac{1}{\sqrt{1-u^2}} \to -\infty \tag{71}$$

and

$$\begin{aligned} \lim_{u\to 1-0} \frac{3}{2\pi r^3} \frac{\partial V}{\partial \alpha} &= \frac{3}{\left(1-\sqrt{1+\alpha}\right)^4} \left[\left\{ -\frac{\alpha\left(1+\alpha+3\sqrt{1+\alpha}\right)}{(1+\alpha)^{3/2}} \right\} F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) \right. \\ &\quad \left. + \left(\frac{2+\alpha+6\sqrt{1+\alpha}}{\sqrt{1+\alpha}} \right) E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) \right] > 0 \end{aligned} \tag{72}$$

hold.

In the case of (B_+),

$$\begin{aligned} \lim_{u\to 1-0} \frac{3}{4\pi r^3} \frac{\partial V}{\partial u} &= \frac{3}{\sqrt{1+\alpha}\left(1-\sqrt{1+\alpha}\right)^3} \left\{ (8+\alpha)\sqrt{1+\alpha}\; E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) \right. \\ &\quad \left. - \frac{\alpha\,(4+\alpha)}{\sqrt{1+\alpha}} F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) \right\} > 0 \end{aligned} \tag{73}$$

and

$$\begin{aligned} \lim_{u\to 1-0} \frac{3}{4\pi r^3} \frac{\partial V}{\partial \alpha} &= \frac{3}{2\left(1-\sqrt{1+\alpha}\right)^4} \left[\left\{ -\frac{\alpha\left(1+\alpha+3\sqrt{1+\alpha}\right)}{(1+\alpha)^{3/2}} \right\} F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) \right. \\ &\quad \left. + \left(\frac{2+\alpha+6\sqrt{1+\alpha}}{\sqrt{1+\alpha}} \right) E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) \right] > 0 \end{aligned} \tag{74}$$

hold.

2.10 Partially differentiation $\tilde{V}_u$ of volume of unduloid fixed boundary circles with respect to u

Now, we can express $V = V(u, \alpha(u))$. We set $\tilde{V}(u) := V(u, \alpha(u))$, and partially differentiate it with respect to u. We examine whether $\lim_{u\to 1-0} \tilde{V}_u$ is positive or not. We can compute $\tilde{V}_u$ as follows.

$$\tilde{V}_u = V_u + V_\alpha \cdot \alpha_u = V_u + V_\alpha \cdot \frac{1}{u_\alpha} = V_u - V_\alpha \cdot \frac{\varphi_u}{\varphi_\alpha} = \frac{1}{\varphi_\alpha}\left(\varphi_\alpha V_u - \varphi_u V_\alpha\right)$$

2.10.1 Partially differentiation $\tilde{V}_u$ in (G_+)

Firstly, we examine sign of

$$\lim_{u\to 1-0} \tilde{V}_u = \lim_{u\to 1-0} \frac{1}{\varphi_\alpha}\left(\varphi_\alpha V_u - \varphi_u V_\alpha\right)$$

in (G_+). From (61), (69), (60), (70),

$$\lim_{u\to 1-0}\frac{1}{2}\frac{\partial\varphi}{\partial\alpha}=\frac{1}{\left(1-\sqrt{1+\alpha}\right)^2}\left\{\frac{1-\sqrt{1+\alpha}}{1+\alpha}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})+\frac{1}{\sqrt{1+\alpha}}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}>0$$

$$\lim_{u\to 1-0}\frac{3}{2\pi r^3}\frac{\partial V}{\partial u}=\frac{3}{\sqrt{1+\alpha}}\lim_{u\to 1-0}\frac{1}{\sqrt{1-u^2}}\to+\infty$$

$$\lim_{u\to 1-0}\frac{1}{2}\frac{\partial\varphi}{\partial u}=\frac{1}{\sqrt{1+\alpha}}\lim_{u\to 1-0}\frac{1}{\sqrt{1-u^2}}\to+\infty$$

$$\begin{aligned}\lim_{u\to 1-0}\frac{3}{2\pi r^3}\frac{\partial V}{\partial\alpha} &= \frac{3}{\left(1-\sqrt{1+\alpha}\right)^4}\left[\left\{-\frac{\alpha\left(1+\alpha+3\sqrt{1+\alpha}\right)}{(1+\alpha)^{3/2}}\right\}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right.\\ &\quad\left.+\left(\frac{2+\alpha+6\sqrt{1+\alpha}}{\sqrt{1+\alpha}}\right)\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right]>0\end{aligned}$$

Now, we examine sign of

$$A(\alpha):=\lim_{u\to 1-0}\frac{3}{4\pi r^3}\left(\varphi_\alpha V_u-\varphi_u V_\alpha\right).\quad (\text{i.e. }\lim_{u\to 1-0}\tilde{V}_u=\lim_{u\to 1-0}\frac{1}{\varphi_\alpha}\cdot\frac{4\pi r^3}{3}\cdot A(\alpha).)$$

Firstly, both "$\lim_{u\to 1-0}\varphi_\alpha V_u$" and "$\lim_{u\to 1-0}\varphi_u V_\alpha$" are positive. Thus, we can't find out sign of $A(\alpha)$ soon. Hence, we will try to compute $A(\alpha)$.

Remark 2.10.1. *Now, both $\lim_{u\to 1-0}\frac{3}{2\pi r^3}V_u$ and $\lim_{u\to 1-0}\frac{1}{2}\varphi_u$ diverges in the same order. Therefore, we set*

$$\lim_{u\to 1-0}\frac{3}{2\pi r^3}V_u:=\frac{3}{\sqrt{1+\alpha}},\quad \lim_{u\to 1-0}\frac{1}{2}\varphi_u:=\frac{1}{\sqrt{1+\alpha}}.$$

$$\begin{aligned}A(\alpha) &= \frac{1}{\left(1-\sqrt{1+\alpha}\right)^2}\left\{\frac{1-\sqrt{1+\alpha}}{1+\alpha}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})+\frac{1}{\sqrt{1+\alpha}}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}\cdot\left(\frac{3}{\sqrt{1+\alpha}}\right)\\ &\quad-\left(\frac{1}{\sqrt{1+\alpha}}\right)\cdot\frac{3}{\left(1-\sqrt{1+\alpha}\right)^4}\left[\left\{-\frac{\alpha\left(1+\alpha+3\sqrt{1+\alpha}\right)}{(1+\alpha)^{3/2}}\right\}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right.\\ &\quad\left.+\left(\frac{2+\alpha+6\sqrt{1+\alpha}}{\sqrt{1+\alpha}}\right)\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right]\\ &= \frac{3}{\sqrt{1+\alpha}\left(1-\sqrt{1+\alpha}\right)^2}\left\{-\frac{2\sqrt{1+\alpha}\left(3\sqrt{1+\alpha}+1\right)\left(1-\sqrt{1+\alpha}\right)}{(1+\alpha)^{3/2}\left(1-\sqrt{1+\alpha}\right)^2}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right.\\ &\quad\left.+\frac{-8}{\left(1-\sqrt{1+\alpha}\right)^2}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}\end{aligned}$$

Now,

$$\frac{3}{\sqrt{1+\alpha}\left(1-\sqrt{1+\alpha}\right)^{2}}>0,\quad -\frac{2\sqrt{1+\alpha}\left(3\sqrt{1+\alpha}+1\right)\left(1-\sqrt{1+\alpha}\right)}{(1+\alpha)^{3/2}\left(1-\sqrt{1+\alpha}\right)^{2}}<0,\quad \frac{-8}{\left(1-\sqrt{1+\alpha}\right)^{2}}<0,$$

$$F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})>E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})>0.$$

Hence,

$$-\frac{2\sqrt{1+\alpha}\left(3\sqrt{1+\alpha}+1\right)\left(1-\sqrt{1+\alpha}\right)}{(1+\alpha)^{3/2}\left(1-\sqrt{1+\alpha}\right)^{2}}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})+\frac{-8}{\left(1-\sqrt{1+\alpha}\right)^{2}}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})<0$$

Thus, $A(\alpha)<0\,(\to-\infty)$. Hence, we concluded $\lim_{u\to 1-0}\tilde{V}_u<0\,(\to-\infty)$ because $\lim_{u\to 1-0}\varphi_\alpha>0$. Therefore, volume $\tilde{V}$ of unduloid fixed two boundary circles in (G_+) is monotone decreasing because $\lim_{u\to 1-0}\tilde{V}_u<0\,(\to-\infty)$.

2.10.2 Partially differentiation $\tilde{V}_u$ in (R_+)

Firstly, we examine sign of

$$\lim_{u\to 1-0}\tilde{V}_u=\lim_{u\to 1-0}\frac{1}{\varphi_\alpha}\left(\varphi_\alpha V_u-\varphi_u V_\alpha\right)$$

in (R_+). From (63), (71), (62), (72),

$$\lim_{u\to 1-0}\frac{1}{2}\frac{\partial\varphi}{\partial\alpha}=\frac{1}{\left(1-\sqrt{1+\alpha}\right)^{2}}\left\{\frac{1-\sqrt{1+\alpha}}{1+\alpha}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})+\frac{1}{\sqrt{1+\alpha}}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}>0$$

$$\lim_{u\to 1-0}\frac{3}{2\pi r^3}\frac{\partial V}{\partial u}=-\frac{3}{\sqrt{1+\alpha}}\lim_{u\to 1-0}\frac{1}{\sqrt{1-u^2}}\to-\infty$$

$$\lim_{u\to 1-0}\frac{1}{2}\frac{\partial\varphi}{\partial u}=-\frac{1}{\sqrt{1+\alpha}}\lim_{u\to 1-0}\frac{1}{\sqrt{1-u^2}}\to-\infty$$

$$\begin{aligned}\lim_{u\to 1-0}\frac{3}{2\pi r^3}\frac{\partial V}{\partial\alpha} &= \frac{3}{\left(1-\sqrt{1+\alpha}\right)^{4}}\left[\left\{-\frac{\alpha\left(1+\alpha+3\sqrt{1+\alpha}\right)}{(1+\alpha)^{3/2}}\right\}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right.\\ &\quad\left.+\left(\frac{2+\alpha+6\sqrt{1+\alpha}}{\sqrt{1+\alpha}}\right)\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right]>0\end{aligned}$$

Now, we examine sign of

$$A(\alpha):=\lim_{u\to 1-0}\frac{3}{4\pi r^3}\left(\varphi_\alpha V_u-\varphi_u V_\alpha\right).\quad (\text{i.e. }\lim_{u\to 1-0}\tilde{V}_u=\lim_{u\to 1-0}\frac{1}{\varphi_\alpha}\cdot\frac{4\pi r^3}{3}\cdot A(\alpha).)$$

Firstly, both "$\lim_{u\to 1-0}\varphi_\alpha V_u$" and "$\lim_{u\to 1-0}\varphi_u V_\alpha$" are negative. Thus, we can't find out sign of $A(\alpha)$ soon. Hence, we will try to compute $A(\alpha)$.

Remark 2.10.2. *Now, both* $\lim_{u\to 1-0}\dfrac{3}{2\pi r^3}V_u$ *and* $\lim_{u\to 1-0}\dfrac{1}{2}\varphi_u$ *diverges in the same order. Therefore, we set*

$$\lim_{u\to 1-0}\frac{3}{2\pi r^3}V_u := -\frac{3}{\sqrt{1+\alpha}},\qquad \lim_{u\to 1-0}\frac{1}{2}\varphi_u := -\frac{1}{\sqrt{1+\alpha}}.$$

$$\begin{aligned}
A(\alpha) &= \frac{1}{\left(1-\sqrt{1+\alpha}\right)^2}\left\{\frac{1-\sqrt{1+\alpha}}{1+\alpha}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})+\frac{1}{\sqrt{1+\alpha}}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}\cdot\left(-\frac{3}{\sqrt{1+\alpha}}\right)\\
&\quad -\left(-\frac{1}{\sqrt{1+\alpha}}\right)\cdot\frac{3}{\left(1-\sqrt{1+\alpha}\right)^4}\left[\left\{-\frac{\alpha\left(1+\alpha+3\sqrt{1+\alpha}\right)}{(1+\alpha)^{3/2}}\right\}F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right.\\
&\quad \left.+\left(\frac{2+\alpha+6\sqrt{1+\alpha}}{\sqrt{1+\alpha}}\right)E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right]\\
&= -\frac{3}{\sqrt{1+\alpha}\left(1-\sqrt{1+\alpha}\right)^2}\left\{-\frac{2\sqrt{1+\alpha}\left(3\sqrt{1+\alpha}+1\right)\left(1-\sqrt{1+\alpha}\right)}{(1+\alpha)^{3/2}\left(1-\sqrt{1+\alpha}\right)^2}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right.\\
&\quad \left.+\frac{-8}{\left(1-\sqrt{1+\alpha}\right)^2}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}
\end{aligned}$$

Now,

$$-\frac{3}{\sqrt{1+\alpha}\left(1-\sqrt{1+\alpha}\right)^2}<0,\quad -\frac{2\sqrt{1+\alpha}\left(3\sqrt{1+\alpha}+1\right)\left(1-\sqrt{1+\alpha}\right)}{(1+\alpha)^{3/2}\left(1-\sqrt{1+\alpha}\right)^2}<0,\quad \frac{-8}{\left(1-\sqrt{1+\alpha}\right)^2}<0,$$

$$F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})>E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})>0.$$

Hence,

$$-\frac{2\sqrt{1+\alpha}\left(3\sqrt{1+\alpha}+1\right)\left(1-\sqrt{1+\alpha}\right)}{(1+\alpha)^{3/2}\left(1-\sqrt{1+\alpha}\right)^2}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})+\frac{-8}{\left(1-\sqrt{1+\alpha}\right)^2}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})<0$$

Thus, $A(\alpha)>0\,(\to+\infty)$. Hence, we concluded $\lim_{u\to 1-0}\tilde{V}_u>0\,(\to+\infty)$ because $\lim_{u\to 1-0}\varphi_\alpha>0$. Therefore, volume $\tilde{V}$ of unduloid fixed two boundary circles in (R_+) is monotone increasing because $\lim_{u\to 1-0}\tilde{V}_u>0\,(\to+\infty)$.

2.10.3 Partially differentiation $\tilde{V}_u$ in (B_+)

Firstly, we examine sign of

$$\lim_{u\to 1-0}\tilde{V}_u=\lim_{u\to 1-0}\frac{1}{\varphi_\alpha}\left(\varphi_\alpha V_u-\varphi_u V_\alpha\right)$$

in (B_+). From (65), (73), (64), (74),

$$\lim_{u\to 1-0}\frac{1}{4}\frac{\partial\varphi}{\partial\alpha}=\frac{1}{2\left(1-\sqrt{1+\alpha}\right)^2}\left\{\frac{1-\sqrt{1+\alpha}}{1+\alpha}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})+\frac{1}{\sqrt{1+\alpha}}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}>0$$

$$\begin{aligned}\lim_{u\to 1-0}\frac{3}{4\pi r^3}\frac{\partial V}{\partial u} &= \frac{3}{\sqrt{1+\alpha}\left(1-\sqrt{1+\alpha}\right)^3}\left\{(8+\alpha)\sqrt{1+\alpha}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right.\\ &\quad\left.-\frac{\alpha\,(4+\alpha)}{\sqrt{1+\alpha}}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}>0\end{aligned}$$

$$\lim_{u\to 1-0}\frac{1}{4}\frac{\partial\varphi}{\partial u}=\frac{1}{\sqrt{1+\alpha}\left(1-\sqrt{1+\alpha}\right)}\left\{-\frac{\alpha}{\sqrt{1+\alpha}}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})+\sqrt{1+\alpha}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}>0$$

$$\begin{aligned}\lim_{u\to 1-0}\frac{3}{4\pi r^3}\frac{\partial V}{\partial\alpha} &= \frac{3}{2\left(1-\sqrt{1+\alpha}\right)^4}\left[\left\{-\frac{\alpha\left(1+\alpha+3\sqrt{1+\alpha}\right)}{(1+\alpha)^{3/2}}\right\}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right.\\ &\quad\left.+\left(\frac{2+\alpha+6\sqrt{1+\alpha}}{\sqrt{1+\alpha}}\right)\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right]>0\end{aligned}$$

Now, we examine sign of

$$A(\alpha):=\lim_{u\to 1-0}\frac{3}{16\pi r^3}\left(\varphi_\alpha V_u-\varphi_u V_\alpha\right).\quad(\text{i.e. }\lim_{u\to 1-0}\tilde{V}_u=\lim_{u\to 1-0}\frac{1}{\varphi_\alpha}\cdot\frac{16\pi r^3}{3}\cdot A(\alpha).)$$

Firstly, both "$\lim_{u\to 1-0}\varphi_\alpha V_u$" and "$\lim_{u\to 1-0}\varphi_u V_\alpha$" are positive. Thus, we can't find out sign of $A(\alpha)$ soon. Hence, we will try to compute $A(\alpha)$.

$$\begin{aligned}A(\alpha) &= \frac{3}{2\sqrt{1+\alpha}\left(1-\sqrt{1+\alpha}\right)^5}\left[-\frac{\alpha\,(4+\alpha)\left(1-\sqrt{1+\alpha}\right)}{(1+\alpha)^{3/2}}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right.\\ &\quad+\left\{\frac{\left(1-\sqrt{1+\alpha}\right)(8+\alpha)\sqrt{1+\alpha}}{1+\alpha}-\frac{\alpha\,(4+\alpha)}{1+\alpha}\right\}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\\ &\quad\left.+\frac{(8+\alpha)\sqrt{1+\alpha}}{\sqrt{1+\alpha}}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right]\\ &\quad-\frac{3}{2\sqrt{1+\alpha}\left(1-\sqrt{1+\alpha}\right)^5}\left[\frac{\alpha^2\left(1+\alpha+3\sqrt{1+\alpha}\right)}{(1+\alpha)^2}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right.\\ &\quad+\left\{\frac{-\alpha\left(2+\alpha+6\sqrt{1+\alpha}\right)}{1+\alpha}-\frac{\alpha\sqrt{1+\alpha}\left(1+\alpha+3\sqrt{1+\alpha}\right)}{(1+\alpha)^{3/2}}\right\}\\ &\quad\times F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\end{aligned}$$

$$
\begin{aligned}
& + \frac{\left(2+\alpha+6\sqrt{1+\alpha}\right)\sqrt{1+\alpha}}{\sqrt{1+\alpha}} \; E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) \; E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha})\Bigg] \\
= \quad & \frac{3}{2\sqrt{1+\alpha}\left(1-\sqrt{1+\alpha}\right)^5} \left[\frac{4\alpha\left(1-\sqrt{1+\alpha}\right)}{1+\alpha} \; \left\{ F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) \right\}^2 \right. \\
& + \frac{2\left(1-\sqrt{1+\alpha}\right)(-4-5\alpha)}{1+\alpha} \; F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) \; E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) \\
& \left. + 6\left(1-\sqrt{1+\alpha}\right) \; \left\{ E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) \right\}^2 \right] \\
=: \quad & \frac{3}{2\sqrt{1+\alpha}\left(1-\sqrt{1+\alpha}\right)^5} \left[a \; \left\{ F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) \right\}^2 \right. \\
& \left. + b \; F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) \; E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) + c \; \left\{ E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) \right\}^2 \right] \\
=: \quad & \frac{3}{2\sqrt{1+\alpha}\left(1-\sqrt{1+\alpha}\right)^5} \cdot M
\end{aligned}
$$

Now,

$$
a := \frac{4\alpha\left(1-\sqrt{1+\alpha}\right)}{1+\alpha} < 0, \quad b := \frac{2\left(1-\sqrt{1+\alpha}\right)(-4-5\alpha)}{1+\alpha}, \quad c := 6\left(1-\sqrt{1+\alpha}\right) > 0.
$$

Thus,

$$
a+b+c = \frac{-2}{1+\alpha}\left(1-\sqrt{1+\alpha}\right) < 0. \tag{75}
$$

Now,

$$
\begin{aligned}
M \quad := \quad & a\left\{ F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) \right\}^2 \\
& + b \; F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) \; E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) + c\left\{ E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) \right\}^2 \\
= \quad & a\left\{ E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) \right\}^2 \left[\left\{ \frac{F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha})}{E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha})} \right\}^2 + \frac{b}{a}\left\{ \frac{F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha})}{E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha})} \right\} + \frac{c}{a} \right]
\end{aligned}
$$

We set the following.

$$
x := \frac{F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha})}{E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha})} \; (> 1)
$$

Then, we can express the following.

$$
M = a\left\{ E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) \right\}^2 \left(x^2 + \frac{b}{a}x + \frac{c}{a} \right) =: a\left\{ E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) \right\}^2 f(x)
$$

Now, we set $f(x) := x^2 + \dfrac{b}{a}x + \dfrac{c}{a}$. We remark that $a\left\{E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha})\right\}^2 < 0$ because $a < 0$. We define discriminant of $f(x)$ as D.

$$D = \left(\frac{b}{a}\right)^2 - 4\cdot 1\cdot\left(\frac{c}{a}\right) = \frac{b^2 - 4ac}{a^2}$$

Denominator of discriminant D is positive, thus we compute $b^2 - 4ac$,

$$\begin{aligned} b^2 - 4ac &= \left\{\frac{2\left(1-\sqrt{1+\alpha}\right)(-4-5\alpha)}{1+\alpha}\right\}^2 - 4\cdot\frac{4\alpha\left(1-\sqrt{1+\alpha}\right)}{1+\alpha}\cdot 6\left(1-\sqrt{1+\alpha}\right) \\ &= \frac{4\left(1-\sqrt{1+\alpha}\right)^2}{1+\alpha}\left\{\frac{\alpha^2 + 16(1+\alpha)}{1+\alpha}\right\} > 0 \end{aligned}$$

We turned out that quadratic equation $f(x)$ has two different real solution because $D > 0$. Now, we want to show $f(x) > 0$ when $x > 1$.

$$f(x) = 0 \Leftrightarrow x = \frac{-\frac{b}{a} \pm \sqrt{\left(\frac{b}{a}\right)^2 - 4\cdot 1\cdot\frac{c}{a}}}{2\cdot 1} = \frac{b \mp \sqrt{b^2 - 4ac}}{-2a} =: x_{\mp}$$

Now, $a < 0 \Leftrightarrow -2a > 0$ and $\sqrt{b^2 - 4ac} > |b|$. Thus, $b + \sqrt{b^2 - 4ac} > 0$ and $b - \sqrt{b^2 - 4ac} < 0$. Moreover, from $a < 0$ and (75), $f(1) = 1 + \dfrac{b}{a} + \dfrac{c}{a} = \dfrac{a+b+c}{a} > 0$. Hence, we showed that $f(x) > 0$ when $x > 1$. Therefore,

$$\begin{aligned} f(x) > 0 \quad &\Leftrightarrow \quad a\left\{E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha})\right\}^2 f(x) < 0 \,(\Leftrightarrow M < 0) \\ &\Leftrightarrow \quad A(\alpha) = \frac{3}{2\sqrt{1+\alpha}\left(1-\sqrt{1+\alpha}\right)^5}\cdot M < 0 \end{aligned}$$

Now,

$$\frac{3}{2\sqrt{1+\alpha}\left(1-\sqrt{1+\alpha}\right)^5} > 0$$

Thus, $A(\alpha)\,(=\text{finite}) < 0$. Hence, we concluded $\lim\limits_{u\to 1-0}\tilde{V}_u\,(=\text{finite}) < 0$ because $\lim\limits_{u\to 1-0}\varphi_\alpha > 0$. Therefore, volume $\tilde{V}$ of unduloid fixed two boundary circles in (B_+) is monotone decreasing because $\lim\limits_{u\to 1-0}\tilde{V}_u\,(=\text{finite}) < 0$.

2.10.4 Bifurcation around one period of an unduloid between two necks

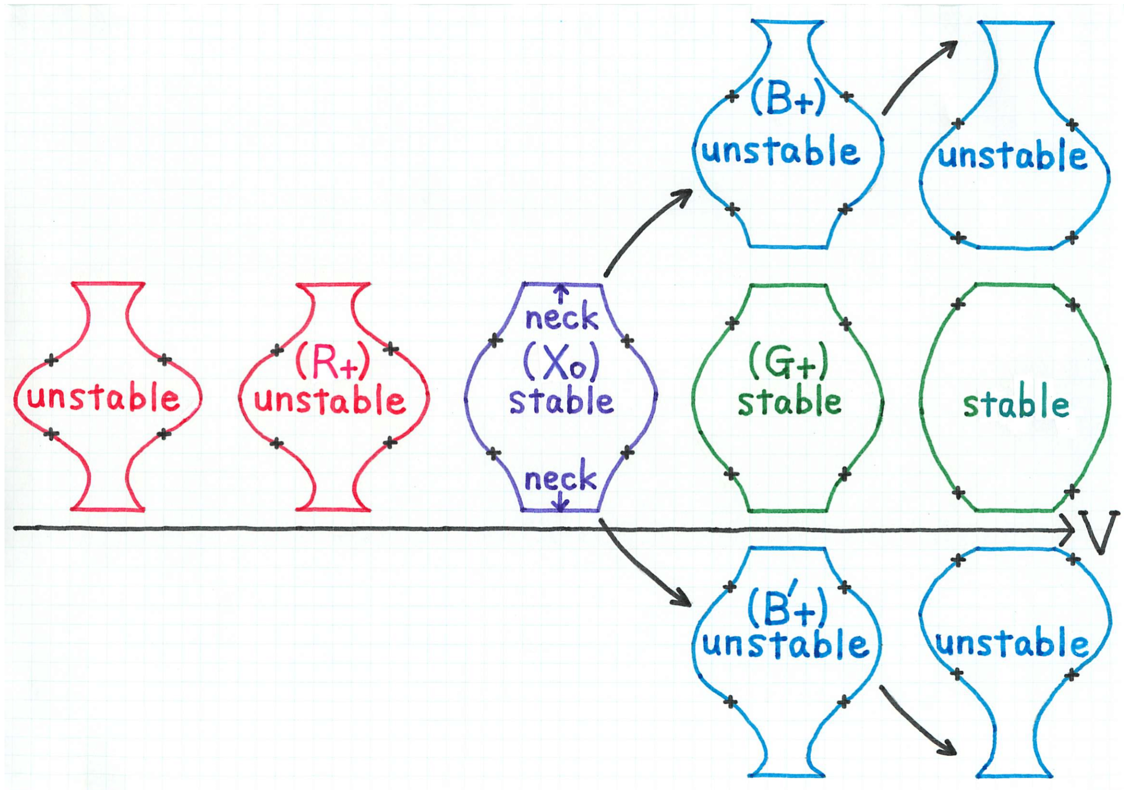
Definition 2.10.1. *A CMC surface is stable.*
$\underset{\text{def.}}{\Longleftrightarrow}$ *The second variation of area is non-negative for all variations that preserve the enclosed volume and fix the boundary.*

Remark 2.10.3. *We remark the following.*

(1) A CMC surface is an equilibrium point of the area for all variations that preserve the enclosed volume and fix the boundary.

(2) If a surface is a local minimizer of area among all surfaces enclosing the same volume and bounded by the same boundary curves, then this surface is a stable CMC surface. (The converse is NOT necessarily true.)

Theorem 2.10.1. *Thanks to a representation formula (Koiso-Palmer, 2005, [14]), the bifurcation theory for CMC surfaces (Koiso-Palmer-Piccione, 2017, [16]) and volume based on elliptic integrals (new idea), we prove the following.*



This picture shows that bifurcation around one period of an unduloid between two necks. It has already known that the volume increases $(R_+)\to(X_0)\to(G_+)$. It also has already known that "(R_+) is unstable", "(X_0) is just stable" and "(G_+) is stable" (Koiso-Palmer, 2017 [15]). However, when we compared the volume (B_+) and (X_0), it is unknown which volume is bigger. In our research, we used next the bifurcation theory for CMC surfaces (Koiso-Palmer-Piccione, 2017 [16]).

"Volume of (B_+) < Volume of $(X_0) \Rightarrow (B_+)$ is stable."
"Volume of (B_+) > Volume of $(X_0) \Rightarrow (B_+)$ is unstable."

By using the this theory, we calculated in §2.10 (§2.10.1, 2.10.2, 2.10.3). Consequently, we proved that "Volume of (B_+) > Volume of (X_0)", therefore, we obtained (B_+) is unstable. This theorem holds $\forall\alpha \in (-1, 0)$. This bifurcation is called "subcritical pitchfork bifurcation".

2.11 Partially differentiation of function $h/r = \varphi(u, \alpha)$ (Gradient of tangential line of level curve on α-u plane)

2.11.1 Partially differentiation of function $h/r = \varphi(u, \alpha)$ in (G_-) with respect to u

Firstly, $\varphi(u, \alpha)$ in (G_-) is as follows.

$$\begin{aligned}\varphi(u,\alpha) &= \frac{2}{u+\sqrt{u^2+\alpha}}\left\{-\frac{2\alpha}{\sqrt{1+\alpha}}\ F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) + 2\sqrt{1+\alpha}\ E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha})\right.\\ &\quad \left. + \frac{\alpha}{\sqrt{1+\alpha}}\ F(\frac{1}{\sqrt{1+\alpha}}, \arccos u) - \sqrt{1+\alpha}\ E(\frac{1}{\sqrt{1+\alpha}}, \arccos u) - \sqrt{1-u^2}\right\}\end{aligned}$$

Thus, we partially differentiate both-hand sides of "$\varphi(u, \alpha) = \text{constant}$" with respect to α,

$$\varphi_u u_\alpha + \varphi_\alpha = 0 \Leftrightarrow u_\alpha = -\frac{\varphi_\alpha}{\varphi_u}$$

holds. Hence, we must calculate φ_u and φ_α. Therefore, we compute φ_u. From (52), (53), (54), (55),

$$\frac{\partial}{\partial u}\left\{-\frac{2\alpha}{\sqrt{1+\alpha}}\ F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha})\right\} = 0\cdot\ F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) - \frac{2\alpha}{\sqrt{1+\alpha}}\cdot 0 = 0$$

$$\frac{\partial}{\partial u}\left\{2\sqrt{1+\alpha}\ E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha})\right\} = 0\cdot\ E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) + 2\sqrt{1+\alpha}\cdot 0 = 0$$

$$\frac{\partial}{\partial u}\left\{\frac{\alpha}{\sqrt{1+\alpha}}\ F(\frac{1}{\sqrt{1+\alpha}}, \arccos u)\right\} = \frac{\alpha}{\sqrt{1+\alpha}}\left(-\frac{1}{\sqrt{1-u^2}\sqrt{\dfrac{\alpha+u^2}{1+\alpha}}}\right) = -\frac{\alpha}{\sqrt{1+\alpha}\sqrt{1-u^2}\sqrt{\dfrac{\alpha+u^2}{1+\alpha}}}$$

$$\frac{\partial}{\partial u}\left\{-\sqrt{1+\alpha}\ E(\frac{1}{\sqrt{1+\alpha}}, \arccos u)\right\} = -\sqrt{1+\alpha}\left(-\frac{\sqrt{\dfrac{\alpha+u^2}{1+\alpha}}}{\sqrt{1-u^2}}\right) = \frac{\sqrt{1+\alpha}\sqrt{\dfrac{\alpha+u^2}{1+\alpha}}}{\sqrt{1-u^2}}$$

$$\frac{\partial}{\partial u}\left(-\sqrt{1-u^2}\right) = -\frac{-2u}{2\sqrt{1-u^2}} = \frac{u}{\sqrt{1-u^2}}$$

$$\frac{\partial}{\partial u}\left(u+\sqrt{u^2+\alpha}\right) = 1 + \frac{2u}{2\sqrt{u^2+\alpha}} = 1 + \frac{u}{\sqrt{u^2+\alpha}} = \frac{\sqrt{u^2+\alpha}+u}{\sqrt{u^2+\alpha}} = \frac{u+\sqrt{u^2+\alpha}}{\sqrt{u^2+\alpha}}$$

Thus,

$$\frac{1}{2}\frac{\partial\varphi}{\partial u} = \frac{1}{u+\sqrt{u^2+\alpha}}\left[\left\{\frac{-\alpha+(1+\alpha)\left(\dfrac{\alpha+u^2}{1+\alpha}\right)+u\sqrt{1+\alpha}\sqrt{\dfrac{\alpha+u^2}{1+\alpha}}}{\sqrt{1+\alpha}\sqrt{1-u^2}\sqrt{\dfrac{\alpha+u^2}{1+\alpha}}}\right\}\right.$$
$$-\left\{-\frac{2\alpha}{\sqrt{1+\alpha}}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})+2\sqrt{1+\alpha}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right.$$
$$\left.\left.+\ \frac{\alpha}{\sqrt{1+\alpha}}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos u)-\sqrt{1+\alpha}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos u)-\sqrt{1-u^2}\right\}\cdot\left(\frac{1}{\sqrt{u^2+\alpha}}\right)\right]$$

Now,

$$\lim_{u\to1-0}\frac{1}{u+\sqrt{u^2+\alpha}}\cdot\frac{-\alpha+(1+\alpha)\left(\dfrac{\alpha+u^2}{1+\alpha}\right)+u\sqrt{1+\alpha}\sqrt{\dfrac{\alpha+u^2}{1+\alpha}}}{\sqrt{1+\alpha}\sqrt{1-u^2}\sqrt{\dfrac{\alpha+u^2}{1+\alpha}}}=\frac{1}{\sqrt{1+\alpha}}\lim_{u\to1-0}\frac{1}{\sqrt{1-u^2}}\to+\infty$$

Consequently,

$$\lim_{u\to1-0}\frac{1}{2}\frac{\partial\varphi}{\partial u}=\frac{1}{\sqrt{1+\alpha}}\lim_{u\to1-0}\frac{1}{\sqrt{1-u^2}}\to+\infty \tag{76}$$

holds.

2.11.2 The other cases ((G_-), (R_-), (B_-))

In the case of (G_-),

$$\lim_{u\to1-0}\frac{1}{2}\frac{\partial\varphi}{\partial\alpha}=\frac{1}{\left(1+\sqrt{1+\alpha}\right)^2}\left\{-\frac{1+\sqrt{1+\alpha}}{1+\alpha}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})+\frac{1}{\sqrt{1+\alpha}}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}<0 \tag{77}$$

holds.

In the case of (R_-),

$$\lim_{u\to1-0}\frac{1}{2}\frac{\partial\varphi}{\partial u}=-\frac{1}{\sqrt{1+\alpha}}\lim_{u\to1-0}\frac{1}{\sqrt{1-u^2}}\to-\infty \tag{78}$$

and

$$\lim_{u\to1-0}\frac{1}{2}\frac{\partial\varphi}{\partial\alpha}=\frac{1}{\left(1+\sqrt{1+\alpha}\right)^2}\left\{-\frac{1+\sqrt{1+\alpha}}{1+\alpha}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})+\frac{1}{\sqrt{1+\alpha}}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}<0 \tag{79}$$

hold.

In the case of (B_-),

$$\lim_{u\to1-0}\frac{1}{4}\frac{\partial\varphi}{\partial u}=\frac{1}{\sqrt{1+\alpha}\left(1+\sqrt{1+\alpha}\right)}\left\{\frac{\alpha}{\sqrt{1+\alpha}}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})-\sqrt{1+\alpha}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}<0 \tag{80}$$

and

$$\lim_{u\to 1-0}\frac{1}{4}\frac{\partial\varphi}{\partial\alpha}=\frac{1}{2\left(1+\sqrt{1+\alpha}\right)^2}\left\{-\frac{1+\sqrt{1+\alpha}}{1+\alpha}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})+\frac{1}{\sqrt{1+\alpha}}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}<0 \tag{81}$$

hold.

2.12 Partially differentiation of volume $V\left(u,\alpha\right)$

Firstly, we have already obtained that radius of boundary circles $r=\dfrac{u+\sqrt{u^2+\alpha}}{-2H}$. Thus,

$$r=\frac{u+\sqrt{u^2+\alpha}}{-2H}\Leftrightarrow H\left(u,\alpha\right)=\frac{u+\sqrt{u^2+\alpha}}{-2r} \tag{82}$$

From (82), we can compute as follows.

$$-\frac{\pi}{12H^3}=-\frac{\pi}{12\left(\dfrac{u+\sqrt{u^2+\alpha}}{-2r}\right)^3}=-\frac{\pi}{12\cdot\dfrac{\left(u+\sqrt{u^2+\alpha}\right)^3}{-8r^3}}=(-\pi)\cdot\frac{-8r^3}{12\left(u+\sqrt{u^2+\alpha}\right)^3}=\frac{2\pi r^3}{3\left(u+\sqrt{u^2+\alpha}\right)^3} \tag{83}$$

Similarly,

$$-\frac{\pi}{6H^3}=-\frac{\pi}{12H^3}\cdot 2=\frac{4\pi r^3}{3\left(u+\sqrt{u^2+\alpha}\right)^3} \tag{84}$$

2.12.1 Partially differentiation of volume $V\left(u,\alpha\right)$ in (G_-) with respect to u

Firstly, $V\left(u,\alpha\right):=V_{(G_-)}\left(u,\alpha\right)$ in (G_-) is as follows.

$$\begin{aligned}V\left(u,\alpha\right) &= -\frac{\pi}{12H^3}\left[-4\sqrt{1-u^2}\left(u\sqrt{u^2+\alpha}+2+u^2+\frac{3}{4}\alpha\right)\right.\\ &\quad+(8+\alpha)\sqrt{1+\alpha}\left\{2\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})-E(\frac{1}{\sqrt{1+\alpha}},\arccos u)\right\}\\ &\quad\left.-\frac{\alpha\left(4+\alpha\right)}{\sqrt{1+\alpha}}\left\{2\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})-F(\frac{1}{\sqrt{1+\alpha}},\arccos u)\right\}\right]\end{aligned}$$

From (83),

$$\begin{aligned}V\left(u,\alpha\right) &= \frac{2\pi r^3}{3\left(u+\sqrt{u^2+\alpha}\right)^3}\left[-4\sqrt{1-u^2}\left(u\sqrt{u^2+\alpha}+2+u^2+\frac{3}{4}\alpha\right)\right.\\ &\quad+(8+\alpha)\sqrt{1+\alpha}\left\{2\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})-E(\frac{1}{\sqrt{1+\alpha}},\arccos u)\right\}\\ &\quad\left.-\frac{\alpha\left(4+\alpha\right)}{\sqrt{1+\alpha}}\left\{2\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})-F(\frac{1}{\sqrt{1+\alpha}},\arccos u)\right\}\right]\end{aligned}$$

That is,

$$\frac{3}{2\pi r^3}V(u,\alpha) = \frac{1}{\left(u+\sqrt{u^2+\alpha}\right)^3}\Bigg[-4\sqrt{1-u^2}\left(u\sqrt{u^2+\alpha}+2+u^2+\frac{3}{4}\alpha\right)$$
$$+(8+\alpha)\sqrt{1+\alpha}\left\{2\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})-E(\frac{1}{\sqrt{1+\alpha}},\arccos u)\right\}$$
$$-\frac{\alpha(4+\alpha)}{\sqrt{1+\alpha}}\left\{2\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})-F(\frac{1}{\sqrt{1+\alpha}},\arccos u)\right\}\Bigg]$$

Now, we partially differentiate $\frac{3}{2\pi r^3}V(u,\alpha)$ with respect to u. Therefore, we compute $\frac{3}{2\pi r^3}V_u$. From (52), (53), (54), (55),

$$\frac{\partial}{\partial u}\left\{-4\sqrt{1-u^2}\left(u\sqrt{u^2+\alpha}+2+u^2+\frac{3}{4}\alpha\right)\right\}$$
$$= \frac{4u}{\sqrt{1-u^2}}\left(u\sqrt{u^2+\alpha}+2+u^2+\frac{3}{4}\alpha\right)-4\sqrt{1-u^2}\left(\sqrt{u^2+\alpha}+\frac{u^2}{\sqrt{u^2+\alpha}}+2u\right)$$

$$\frac{\partial}{\partial u}\left[(8+\alpha)\sqrt{1+\alpha}\left\{2\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})-E(\frac{1}{\sqrt{1+\alpha}},\arccos u)\right\}\right] = (8+\alpha)\sqrt{1+\alpha}\left(\frac{\sqrt{\frac{\alpha+u^2}{1+\alpha}}}{\sqrt{1-u^2}}\right)$$

$$\frac{\partial}{\partial u}\left[-\frac{\alpha(4+\alpha)}{\sqrt{1+\alpha}}\left\{2\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})-F(\frac{1}{\sqrt{1+\alpha}},\arccos u)\right\}\right] = -\frac{\alpha(4+\alpha)}{\sqrt{1+\alpha}}\left(\frac{1}{\sqrt{1-u^2}\sqrt{\frac{\alpha+u^2}{1+\alpha}}}\right)$$

$$\frac{\partial}{\partial u}\left(u+\sqrt{u^2+\alpha}\right)^3 = \left(u+\sqrt{u^2+\alpha}\right)^3\cdot\left(\frac{3}{\sqrt{u^2+\alpha}}\right)$$

Hence,

$$\frac{3}{2\pi r^3}\frac{\partial V}{\partial u} = \frac{1}{\left(u+\sqrt{u^2+\alpha}\right)^6}\Bigg[\Bigg\{\frac{4u}{\sqrt{1-u^2}}\left(u\sqrt{u^2+\alpha}+2+u^2+\frac{3}{4}\alpha\right)-4\sqrt{1-u^2}\left(\sqrt{u^2+\alpha}+\frac{u^2}{\sqrt{u^2+\alpha}}+2u\right)$$
$$+(8+\alpha)\sqrt{1+\alpha}\left(\frac{\sqrt{\frac{\alpha+u^2}{1+\alpha}}}{\sqrt{1-u^2}}\right)-\frac{\alpha(4+\alpha)}{\sqrt{1+\alpha}}\left(\frac{1}{\sqrt{1-u^2}\sqrt{\frac{\alpha+u^2}{1+\alpha}}}\right)\Bigg\}\cdot\left(u+\sqrt{u^2+\alpha}\right)^3$$
$$-\Bigg\{-4\sqrt{1-u^2}\left(u\sqrt{u^2+\alpha}+2+u^2+\frac{3}{4}\alpha\right)$$
$$+(8+\alpha)\sqrt{1+\alpha}\left(2\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})-E(\frac{1}{\sqrt{1+\alpha}},\arccos u)\right)$$
$$-\frac{\alpha(4+\alpha)}{\sqrt{1+\alpha}}\left(2\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})-F(\frac{1}{\sqrt{1+\alpha}},\arccos u)\right)\Bigg\}\cdot\left(u+\sqrt{u^2+\alpha}\right)^3\cdot\left(\frac{3}{\sqrt{u^2+\alpha}}\right)\Bigg]$$

Now,

$$\lim_{u\to 1-0}\frac{1}{\left(u+\sqrt{u^2+\alpha}\right)^6}\left\{\frac{4u}{\sqrt{1-u^2}}\left(u\sqrt{u^2+\alpha}+2+u^2+\frac{3}{4}\alpha\right)\right.$$

$$\left.+\,(8+\alpha)\sqrt{1+\alpha}\left(\frac{\sqrt{\dfrac{\alpha+u^2}{1+\alpha}}}{\sqrt{1-u^2}}\right)-\frac{\alpha\,(4+\alpha)}{\sqrt{1+\alpha}}\left(\frac{1}{\sqrt{1-u^2}\sqrt{\dfrac{\alpha+u^2}{1+\alpha}}}\right)\right\}\cdot\left(u+\sqrt{u^2+\alpha}\right)^3$$

$$=\frac{3}{\sqrt{1+\alpha}}\lim_{u\to 1-0}\frac{1}{\sqrt{1-u^2}}\to+\infty$$

Consequently,

$$\lim_{u\to 1-0}\frac{3}{2\pi r^3}\frac{\partial V}{\partial u}=\frac{3}{\sqrt{1+\alpha}}\lim_{u\to 1-0}\frac{1}{\sqrt{1-u^2}}\to+\infty \tag{85}$$

holds.

2.12.2 The other cases ((G_-), (R_-), (B_-))

In the case of (G_-),

$$\begin{aligned}\lim_{u\to 1-0}\frac{3}{2\pi r^3}\frac{\partial V}{\partial \alpha} &= \frac{3}{\left(1+\sqrt{1+\alpha}\right)^4}\left[\left\{-\frac{\alpha\left(1+\alpha-3\sqrt{1+\alpha}\right)}{(1+\alpha)^{3/2}}\right\}F\Big(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha}\Big)\right.\\ &\left.+\left(\frac{2+\alpha-6\sqrt{1+\alpha}}{\sqrt{1+\alpha}}\right)E\Big(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha}\Big)\right]<0\end{aligned} \tag{86}$$

holds.

In the case of (R_-),

$$\lim_{u\to 1-0}\frac{3}{2\pi r^3}\frac{\partial V}{\partial u}=-\frac{3}{\sqrt{1+\alpha}}\lim_{u\to 1-0}\frac{1}{\sqrt{1-u^2}}\to-\infty \tag{87}$$

and

$$\begin{aligned}\lim_{u\to 1-0}\frac{3}{2\pi r^3}\frac{\partial V}{\partial \alpha} &= \frac{3}{\left(1+\sqrt{1+\alpha}\right)^4}\left[\left\{-\frac{\alpha\left(1+\alpha-3\sqrt{1+\alpha}\right)}{(1+\alpha)^{3/2}}\right\}F\Big(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha}\Big)\right.\\ &\left.+\left(\frac{2+\alpha-6\sqrt{1+\alpha}}{\sqrt{1+\alpha}}\right)E\Big(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha}\Big)\right]<0\end{aligned} \tag{88}$$

hold.

In the case of (B_-),

$$\begin{aligned}\lim_{u\to 1-0}\frac{3}{4\pi r^3}\frac{\partial V}{\partial u} &= \frac{3}{\sqrt{1+\alpha}\left(1+\sqrt{1+\alpha}\right)^3}\left\{-\,(8+\alpha)\sqrt{1+\alpha}\;E\Big(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha}\Big)\right.\\ &\left.+\frac{\alpha\,(4+\alpha)}{\sqrt{1+\alpha}}\;F\Big(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha}\Big)\right\}<0\end{aligned} \tag{89}$$

and

$$\lim_{u\to 1-0}\frac{3}{4\pi r^3}\frac{\partial V}{\partial \alpha} = \frac{3}{2\left(1+\sqrt{1+\alpha}\right)^4}\left[\left\{-\frac{\alpha\left(1+\alpha-3\sqrt{1+\alpha}\right)}{(1+\alpha)^{3/2}}\right\} F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) + \left(\frac{2+\alpha-6\sqrt{1+\alpha}}{\sqrt{1+\alpha}}\right) E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha})\right] < 0 \tag{90}$$

hold.

2.13 Stability of one period of unduloid from a bulge to the next bulge

In this section, we examine the stability of one period of unduloid from a bulge to the next bulge. From now on, we call this unduloid as $\mathcal{U}$.

If $\mathcal{U}$ is close to a cylinder, it is known to be stable. Conversely, if $\mathcal{U}$ is close to a sphere (two hemispheres), it is known to be unstable.

Now, we examine sign of $I(\alpha) := \int_{\mathcal{U}} \eta\, d\Sigma$ for $\mathcal{U}$. The following has already known: if $I(\alpha) \geq 0$, $\mathcal{U}$ is stable. Conversely, if $I(\alpha) < 0$, $\mathcal{U}$ is unstable.

2.13.1 Calculation of $I(\alpha) := \int_{\mathcal{U}} \eta\, d\Sigma$

Firstly, we define as follows.

$$\eta := aq - \tilde{q}, \quad a := \frac{1}{\sqrt{1+\alpha}} > 1 \quad (-1 < \alpha < 0)$$

Now, q, $\tilde{q}$ are support function of $\mathcal{U}$(unduloid), $\mathcal{N}$(nodoid) respectively. Then,

$$q(s) = rz' - r'z, \quad \tilde{q}(s) = \tilde{r}\tilde{z}' - \tilde{r}'\tilde{z}$$

where

$$\tilde{r} := ar, \quad \tilde{z} := \int_0^s \pm\sqrt{1-(ar')^2}\, ds.$$

Moreover,

$$d\Sigma = r\, dsd\theta, \quad r = \frac{u \pm \sqrt{u^2+\alpha}}{-2H}, \quad \frac{dr}{du} = \frac{1}{-2H}\left(1 \pm \frac{u}{\sqrt{u^2+\alpha}}\right).$$

Now,

$$z = \int_0^z dz = \int_1^u \frac{dz}{ds}\frac{ds}{dr}\frac{dr}{du}\, du = \int_1^u u \cdot \frac{1}{-v}\cdot\frac{1}{-2H}\left(1 \pm \frac{u}{\sqrt{u^2+\alpha}}\right) du = \int_1^u \frac{u}{\sqrt{1-u^2}}\cdot\frac{1}{-2H}\left(1 \pm \frac{u}{\sqrt{u^2+\alpha}}\right) du$$

Moreover,

$$ds = \frac{ds}{dr}\frac{dr}{du}\, du = \frac{1}{\sqrt{1-u^2}}\cdot\frac{1}{-2H}\left(1 \pm \frac{u}{\sqrt{u^2+\alpha}}\right) du$$

Let s_B denote the first positive value of s at which a bulge of $\mathcal{U}$ occurs. Then the first bulge on $\mathcal{N}$ also occurs at $s = s_B$. Thus,

$$\begin{aligned}
I(\alpha) &:= \int_{\mathcal{U}} \eta \, d\Sigma = \int_{\mathcal{U}} (aq - \tilde{q})\, r \, dsd\theta = \int_{\mathcal{U}} (arq - r\tilde{q}) \, dsd\theta \\
&= \left(\int_0^{2\pi} d\theta\right) \left\{ 2\int_0^{s_B} (arq - r\tilde{q}) \, ds \right\} = 2\pi \cdot 2 \left(\int_0^{s_B} arq \, ds - \int_0^{s_B} r\tilde{q} \, ds \right) \\
&= 4\pi \left\{ a \int_0^{s_B} r\,(rz' - r'z) \, ds - \int_0^{s_B} r\,(\tilde{r}\tilde{z}' - \tilde{r}'\tilde{z}) \, ds \right\} \\
&= 4\pi \left\{ a \left(\int_0^{s_B} r^2 z' \, ds - \int_0^{s_B} rr'z \, ds \right) - \left(\int_0^{s_B} r\tilde{r}\tilde{z}' \, ds - \int_0^{s_B} r\tilde{r}'\tilde{z} \, ds \right) \right\} \\
&=: 4\pi \left\{ a\,(A - B) - (C - D) \right\}
\end{aligned}$$

Hence, A, B, C, D are as follows.

$$A := \int_0^{s_B} r^2 z' \, ds, \quad B := \int_0^{s_B} rr'z \, ds, \quad C := \int_0^{s_B} r\tilde{r}\tilde{z}' \, ds, \quad D := \int_0^{s_B} r\tilde{r}'\tilde{z} \, ds$$

Proposition 2.13.1. *The following result holds.*

$$\begin{aligned}
A &:= \int_0^{s_B} r^2 z' \, ds \\
&= \int_1^{\sqrt{-\alpha}} \left(\frac{u - \sqrt{u^2+\alpha}}{-2H} \right)^2 \cdot u \cdot \left\{ \frac{1}{\sqrt{1-u^2}} \cdot \frac{1}{-2H} \left(1 - \frac{u}{\sqrt{u^2+\alpha}} \right) \right\} \, du \\
&\quad + \int_{\sqrt{-\alpha}}^1 \left(\frac{u + \sqrt{u^2+\alpha}}{-2H} \right)^2 \cdot u \cdot \left\{ \frac{1}{\sqrt{1-u^2}} \cdot \frac{1}{-2H} \left(1 + \frac{u}{\sqrt{u^2+\alpha}} \right) \right\} \, du \\
&= \frac{1}{-12H^3} \left\{ (8+\alpha)\sqrt{1+\alpha} \; E\Big(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}\Big) - \frac{\alpha\,(4+\alpha)}{\sqrt{1+\alpha}} \; F\Big(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}\Big) \right\}
\end{aligned}$$

Proposition 2.13.2. *The following result holds.*

$$\begin{aligned}
B &:= \int_0^{s_B} rr'z \, ds \\
&= \int_1^{\sqrt{-\alpha}} \left(\frac{u - \sqrt{u^2+\alpha}}{-2H} \right) \cdot \sqrt{1-u^2} \cdot \left\{ \int_1^u u \cdot \frac{1}{\sqrt{1-u^2}} \cdot \frac{1}{-2H} \left(1 - \frac{u}{\sqrt{u^2+\alpha}} \right) \, du \right\} \\
&\quad \times \left\{ \frac{1}{\sqrt{1-u^2}} \cdot \frac{1}{-2H} \left(1 - \frac{u}{\sqrt{u^2+\alpha}} \right) \right\} \, du \\
&\quad + \int_{\sqrt{-\alpha}}^1 \left(\frac{u + \sqrt{u^2+\alpha}}{-2H} \right) \cdot \sqrt{1-u^2} \\
&\quad \times \left\{ \int_1^{\sqrt{-\alpha}} u \cdot \frac{1}{\sqrt{1-u^2}} \cdot \frac{1}{-2H} \left(1 - \frac{u}{\sqrt{u^2+\alpha}} \right) \, du + \int_{\sqrt{-\alpha}}^u u \cdot \frac{1}{\sqrt{1-u^2}} \cdot \frac{1}{-2H} \left(1 + \frac{u}{\sqrt{u^2+\alpha}} \right) \, du \right\}
\end{aligned}$$

$$\times \left\{ \frac{1}{\sqrt{1-u^2}} \cdot \frac{1}{-2H} \left(1+\frac{u}{\sqrt{u^2+\alpha}}\right)\right\} \; du$$

$$= \frac{1}{-12H^3} \left\{ \sqrt{1+\alpha}\left(-1+3\sqrt{1+\alpha}+\alpha\right) \; E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) - \alpha\left(\sqrt{1+\alpha}+3\right) \; F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) \right\}$$

Proposition 2.13.3. *The following result holds.*

$$\begin{aligned}
C &:= \int_0^{s_B} r\tilde{r}\tilde{z}' \; ds \\
&= \int_1^{\sqrt{-\alpha}} a \left(\frac{u-\sqrt{u^2+\alpha}}{-2H}\right)^2 \left(-\sqrt{1-a^2\left(1-u^2\right)}\right) \left\{ \frac{1}{\sqrt{1-u^2}} \cdot \frac{1}{-2H} \left(1-\frac{u}{\sqrt{u^2+\alpha}}\right)\right\} \; du \\
&\quad + \int_{\sqrt{-\alpha}}^{1} a \left(\frac{u+\sqrt{u^2+\alpha}}{-2H}\right)^2 \left(\sqrt{1-a^2\left(1-u^2\right)}\right) \left\{ \frac{1}{\sqrt{1-u^2}} \cdot \frac{1}{-2H} \left(1+\frac{u}{\sqrt{u^2+\alpha}}\right)\right\} \; du \\
&= \frac{a}{-12H^3} \left\{ (8+7\alpha) \; E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) - 4\alpha \; F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) \right\}
\end{aligned}$$

Proposition 2.13.4. *The following result holds.*

$$\begin{aligned}
D &:= \int_0^{s_B} r\tilde{r}'\tilde{z} \; ds \\
&= \int_1^{\sqrt{-\alpha}} \left(\frac{u-\sqrt{u^2+\alpha}}{-2H}\right) \cdot \left(a\sqrt{1-u^2}\right) \cdot \left[\int_1^{u} -\sqrt{1-a^2\left(1-u^2\right)} \left\{ \frac{1}{\sqrt{1-u^2}} \cdot \frac{1}{-2H} \left(1-\frac{u}{\sqrt{u^2+\alpha}}\right)\right\} \; du\right] \\
&\quad \times \left\{ \frac{1}{\sqrt{1-u^2}} \cdot \frac{1}{-2H} \left(1-\frac{u}{\sqrt{u^2+\alpha}}\right)\right\} \; du \\
&\quad + \int_{\sqrt{-\alpha}}^{1} \left(\frac{u+\sqrt{u^2+\alpha}}{-2H}\right) \cdot \left(a\sqrt{1-u^2}\right) \\
&\quad \times \left[\int_1^{\sqrt{-\alpha}} -\sqrt{1-a^2\left(1-u^2\right)} \left\{ \frac{1}{\sqrt{1-u^2}} \cdot \frac{1}{-2H} \left(1-\frac{u}{\sqrt{u^2+\alpha}}\right)\right\} \; du \right. \\
&\quad \left. + \int_{\sqrt{-\alpha}}^{u} \sqrt{1-a^2\left(1-u^2\right)} \left\{ \frac{1}{\sqrt{1-u^2}} \cdot \frac{1}{-2H} \left(1+\frac{u}{\sqrt{u^2+\alpha}}\right)\right\} \; du\right] \\
&\quad \times \left\{ \frac{1}{\sqrt{1-u^2}} \cdot \frac{1}{-2H} \left(1+\frac{u}{\sqrt{u^2+\alpha}}\right)\right\} \; du \\
&= \frac{a}{-12H^3} \left\{ \left(-1-2\alpha+3\sqrt{1+\alpha}\right) \; E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) + 2\alpha \; F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) \right\}
\end{aligned}$$

From Proposition 2.13.1, 2.13.2, 2.13.3 and 2.13.4,

$$\begin{aligned}
I(\alpha) &:= \int_{\mathcal{U}} \eta \; d\Sigma = 4\pi \left\{ a \left(\int_0^{s_B} r^2 z' \; ds - \int_0^{s_B} rr'z \; ds \right) - \left(\int_0^{s_B} r\tilde{r}\tilde{z}' \; ds - \int_0^{s_B} r\tilde{r}'\tilde{z} \; ds \right) \right\} \\
&=: 4\pi \left\{ a(A-B) - (C-D) \right\} = 4\pi \left(aA - aB - C + D \right)
\end{aligned}$$

$$
\begin{aligned}
= \quad & 4\pi \left[\frac{a}{-12H^3} \left\{ (8+\alpha)\sqrt{1+\alpha}\ E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) - \frac{\alpha\,(4+\alpha)}{\sqrt{1+\alpha}}\ F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) \right\} \right. \\
& - \frac{a}{-12H^3} \left\{ \sqrt{1+\alpha}\left(-1+3\sqrt{1+\alpha}+\alpha\right)\ E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) \right. \\
& \quad \left. - \alpha\left(\sqrt{1+\alpha}+3\right)\ F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) \right\} \\
& - \frac{a}{-12H^3} \left\{ (8+7\alpha)\ E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) - 4\alpha\ F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) \right\} \\
& \left. + \frac{a}{-12H^3} \left\{ \left(-1-2\alpha+3\sqrt{1+\alpha}\right)\ E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) + 2\alpha\ F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) \right\} \right] \\
= \quad & \frac{\pi}{-H^3} \left\{ 4\left(1-\sqrt{1+\alpha}\right)\ E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) + \frac{-\alpha\left(1-3\sqrt{1+\alpha}\right)}{1+\alpha}\ F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) \right\} \\
= \quad & \frac{\pi}{-H^3} \left[4\left(1-\sqrt{1+\alpha}\right) \cdot \frac{1}{\sqrt{1+\alpha}} \left\{ \ E(\sqrt{1+\alpha}) + \alpha\ K(\sqrt{1+\alpha}) \right\} + \frac{-\alpha\left(1-3\sqrt{1+\alpha}\right)}{1+\alpha} \cdot \sqrt{1+\alpha}\ K(\sqrt{1+\alpha}) \right] \\
= \quad & \frac{\pi}{-H^3\sqrt{1+\alpha}} \left\{ 4\left(1-\sqrt{1+\alpha}\right)\ E(\sqrt{1+\alpha}) + \alpha\left(3-\sqrt{1+\alpha}\right)\ K(\sqrt{1+\alpha}) \right\} \\
= \quad & \frac{-\alpha\pi}{-H^3} \left\{ \frac{4}{1+\sqrt{1+\alpha}}\ E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) + \frac{1-3\sqrt{1+\alpha}}{1+\alpha}\ F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) \right\} =: \frac{-\alpha\pi}{-H^3} \cdot f\,(\alpha)
\end{aligned}
$$

Proposition 2.13.5. *The following properties hold.*

(i) $\lim_{\alpha \to -1+0} f\,(\alpha) = +\infty$

(ii) $\lim_{\alpha \to -0} f\,(\alpha) = -\infty$

(iii) $f'\,(\alpha) < 0, \quad \forall \alpha \in (-1, 0)$

(iv) $f\,(-0.9) > 0, \quad f\,(-0.8) < 0$

where

$$
f\,(\alpha) := \frac{4}{1+\sqrt{1+\alpha}}\ E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) + \frac{1-3\sqrt{1+\alpha}}{1+\alpha}\ F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) \tag{91}
$$

Now, from $\dfrac{-\alpha\pi}{-H^3} > 0$ and Proposition 2.13.5, we found out that stability of one period of unduloid from a bulge to the next bulge.

$\exists 1 \alpha_0 \in (-0.9, -0.8)\,, (\alpha_0 \fallingdotseq -0.83)$ such that when $-1 < \alpha \leq \alpha_0$, $I\,(\alpha) \geq 0$ holds. Therefore, the unduloid $\mathcal{U}$ is stable. Conversely, when $\alpha_0 < \alpha < 0$, $I\,(\alpha) < 0$ holds. Hence, the unduloid $\mathcal{U}$ is unstable.

2.14 Partially differentiation $\tilde{V}_u$ of volume of unduloid fixed boundary circles with respect to u

Now, we can express $V = V\,(u, \alpha\,(u))$. We set $\tilde{V}\,(u) := V\,(u, \alpha\,(u))$, and partially differentiate it with respect to u. We examine whether $\lim_{u \to 1-0} \tilde{V}_u$ is positive or not. We can compute $\tilde{V}_u$ as follows.

$$\tilde{V}_u = V_u + V_\alpha \cdot \alpha_u = V_u + V_\alpha \cdot \frac{1}{u_\alpha} = V_u - V_\alpha \cdot \frac{\varphi_u}{\varphi_\alpha} = \frac{1}{\varphi_\alpha}\left(\varphi_\alpha V_u - \varphi_u V_\alpha\right)$$

2.14.1 Partially differentiation $\tilde{V}_u$ in (G_-)

Firstly, we examine sign of

$$\lim_{u\to 1-0} \tilde{V}_u = \lim_{u\to 1-0} \frac{1}{\varphi_\alpha}\left(\varphi_\alpha V_u - \varphi_u V_\alpha\right)$$

in (G_-). From (77), (85), (76), (86),

$$\lim_{u\to 1-0} \frac{1}{2}\frac{\partial\varphi}{\partial\alpha} = \frac{1}{\left(1+\sqrt{1+\alpha}\right)^2}\left\{-\frac{1+\sqrt{1+\alpha}}{1+\alpha}\ F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) + \frac{1}{\sqrt{1+\alpha}}\ E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha})\right\} < 0$$

$$\lim_{u\to 1-0} \frac{3}{2\pi r^3}\frac{\partial V}{\partial u} = \frac{3}{\sqrt{1+\alpha}} \lim_{u\to 1-0} \frac{1}{\sqrt{1-u^2}} \to +\infty$$

$$\lim_{u\to 1-0} \frac{1}{2}\frac{\partial\varphi}{\partial u} = \frac{1}{\sqrt{1+\alpha}} \lim_{u\to 1-0} \frac{1}{\sqrt{1-u^2}} \to +\infty$$

$$\begin{aligned}\lim_{u\to 1-0} \frac{3}{2\pi r^3}\frac{\partial V}{\partial\alpha} \quad = \quad & \frac{3}{\left(1+\sqrt{1+\alpha}\right)^4}\left[\left\{-\frac{\alpha\left(1+\alpha-3\sqrt{1+\alpha}\right)}{\left(1+\alpha\right)^{3/2}}\right\}\ F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha})\right. \\ & \left. + \left(\frac{2+\alpha-6\sqrt{1+\alpha}}{\sqrt{1+\alpha}}\right)\ E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha})\right] < 0\end{aligned}$$

Now, we examine sign of

$$A\left(\alpha\right) := \lim_{u\to 1-0} \frac{3}{4\pi r^3}\left(\varphi_\alpha V_u - \varphi_u V_\alpha\right). \quad (\text{i.e.} \lim_{u\to 1-0} \tilde{V}_u = \lim_{u\to 1-0} \frac{1}{\varphi_\alpha} \cdot \frac{4\pi r^3}{3} \cdot A\left(\alpha\right).)$$

Firstly, both "$\lim_{u\to 1-0} \varphi_\alpha V_u$" and "$\lim_{u\to 1-0} \varphi_u V_\alpha$" are negative. Thus, we can't find out sign of $A\left(\alpha\right)$ soon. Hence, we will try to compute $A\left(\alpha\right)$.

Remark 2.14.1. *Now, both $\lim_{u\to 1-0} \frac{3}{2\pi r^3} V_u$ and $\lim_{u\to 1-0} \frac{1}{2}\varphi_u$ diverges in the same order. Therefore, we set*

$$\lim_{u\to 1-0} \frac{3}{2\pi r^3} V_u := \frac{3}{\sqrt{1+\alpha}}, \quad \lim_{u\to 1-0} \frac{1}{2}\varphi_u := \frac{1}{\sqrt{1+\alpha}}.$$

$$
\begin{aligned}
A(\alpha) &= \frac{1}{\left(1+\sqrt{1+\alpha}\right)^2}\left\{-\frac{1+\sqrt{1+\alpha}}{1+\alpha}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})+\frac{1}{\sqrt{1+\alpha}}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}\cdot\left(\frac{3}{\sqrt{1+\alpha}}\right) \\
&\quad -\left(\frac{1}{\sqrt{1+\alpha}}\right)\cdot\frac{3}{\left(1+\sqrt{1+\alpha}\right)^4}\left[\left\{-\frac{\alpha\left(1+\alpha-3\sqrt{1+\alpha}\right)}{(1+\alpha)^{3/2}}\right\}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right. \\
&\quad \left.+\left(\frac{2+\alpha-6\sqrt{1+\alpha}}{\sqrt{1+\alpha}}\right)\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right] \\
&= \frac{6}{\sqrt{1+\alpha}\left(1+\sqrt{1+\alpha}\right)^3}\left\{\frac{1-3\sqrt{1+\alpha}}{1+\alpha}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})+\frac{4}{1+\sqrt{1+\alpha}}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}
\end{aligned}
$$

Now,

$$\frac{6}{\sqrt{1+\alpha}\left(1+\sqrt{1+\alpha}\right)^3}>0$$

and from (91),

$$\frac{1-3\sqrt{1+\alpha}}{1+\alpha}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})+\frac{4}{1+\sqrt{1+\alpha}}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})=:f(\alpha)$$

Thus, $\exists 1\alpha_0\in(-0.9,-0.8)\,,(\alpha_0\fallingdotseq -0.83)$ such that when $-1<\alpha\le\alpha_0$, $A(\alpha)>0\,(\to+\infty)$. Hence, we concluded $\lim_{u\to 1-0}\tilde{V}_u<0\,(\to-\infty)$ because $\lim_{u\to 1-0}\varphi_\alpha<0$. Therefore, volume $\tilde{V}$ of unduloid fixed two boundary circles in (G_-) is monotone decreasing because $\lim_{u\to 1-0}\tilde{V}_u<0\,(\to-\infty)$. Conversely, when $\alpha_0<\alpha<0$, $A(\alpha)<0\,(\to-\infty)$. Hence, we concluded $\lim_{u\to 1-0}\tilde{V}_u>0\,(\to+\infty)$ because $\lim_{u\to 1-0}\varphi_\alpha<0$. Therefore, volume $\tilde{V}$ of unduloid fixed two boundary circles in (G_-) is monotone increasing because $\lim_{u\to 1-0}\tilde{V}_u>0\,(\to+\infty)$.

2.14.2 Partially differentiation $\tilde{V}_u$ in (R_-)

Firstly, we examine sign of

$$\lim_{u\to 1-0}\tilde{V}_u=\lim_{u\to 1-0}\frac{1}{\varphi_\alpha}\left(\varphi_\alpha V_u-\varphi_u V_\alpha\right)$$

in (R_-). From (79), (87), (78), (88),

$$\lim_{u\to 1-0}\frac{1}{2}\frac{\partial\varphi}{\partial\alpha}=\frac{1}{\left(1+\sqrt{1+\alpha}\right)^2}\left\{-\frac{1+\sqrt{1+\alpha}}{1+\alpha}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})+\frac{1}{\sqrt{1+\alpha}}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}<0$$

$$\lim_{u\to 1-0}\frac{3}{2\pi r^3}\frac{\partial V}{\partial u}=-\frac{3}{\sqrt{1+\alpha}}\lim_{u\to 1-0}\frac{1}{\sqrt{1-u^2}}\to-\infty$$

$$\lim_{u\to 1-0}\frac{1}{2}\frac{\partial\varphi}{\partial u}=-\frac{1}{\sqrt{1+\alpha}}\lim_{u\to 1-0}\frac{1}{\sqrt{1-u^2}}\to-\infty$$

$$\lim_{u\to 1-0}\frac{3}{2\pi r^3}\frac{\partial V}{\partial \alpha} = \frac{3}{\left(1+\sqrt{1+\alpha}\right)^4}\left[\left\{-\frac{\alpha\left(1+\alpha-3\sqrt{1+\alpha}\right)}{(1+\alpha)^{3/2}}\right\} F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha}) + \left(\frac{2+\alpha-6\sqrt{1+\alpha}}{\sqrt{1+\alpha}}\right) E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right] < 0$$

Now, we examine sign of

$$A(\alpha) := \lim_{u\to 1-0}\frac{3}{4\pi r^3}\left(\varphi_\alpha V_u - \varphi_u V_\alpha\right). \quad (\text{i.e. } \lim_{u\to 1-0}\tilde{V}_u = \lim_{u\to 1-0}\frac{1}{\varphi_\alpha}\cdot\frac{4\pi r^3}{3}\cdot A(\alpha)\,.)$$

Firstly, both "$\lim_{u\to 1-0}\varphi_\alpha V_u$" and "$\lim_{u\to 1-0}\varphi_u V_\alpha$" are positive. Thus, we can't find out sign of $A(\alpha)$ soon. Hence, we will try to compute $A(\alpha)$.

Remark 2.14.2. *Now, both $\lim_{u\to 1-0}\frac{3}{2\pi r^3}V_u$ and $\lim_{u\to 1-0}\frac{1}{2}\varphi_u$ diverges in the same order. Therefore, we set*

$$\lim_{u\to 1-0}\frac{3}{2\pi r^3}V_u := -\frac{3}{\sqrt{1+\alpha}}, \quad \lim_{u\to 1-0}\frac{1}{2}\varphi_u := -\frac{1}{\sqrt{1+\alpha}}.$$

$$\begin{aligned} A(\alpha) &= \frac{1}{\left(1+\sqrt{1+\alpha}\right)^2}\left\{-\frac{1+\sqrt{1+\alpha}}{1+\alpha} F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha}) + \frac{1}{\sqrt{1+\alpha}} E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}\cdot\left(-\frac{3}{\sqrt{1+\alpha}}\right) \\ &\quad -\left(-\frac{1}{\sqrt{1+\alpha}}\right)\cdot\frac{3}{\left(1+\sqrt{1+\alpha}\right)^4}\left[\left\{-\frac{\alpha\left(1+\alpha-3\sqrt{1+\alpha}\right)}{(1+\alpha)^{3/2}}\right\} F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha}) \right. \\ &\quad \left. + \left(\frac{2+\alpha-6\sqrt{1+\alpha}}{\sqrt{1+\alpha}}\right) E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right] \\ &= -\frac{6}{\sqrt{1+\alpha}\left(1+\sqrt{1+\alpha}\right)^3}\left\{\frac{1-3\sqrt{1+\alpha}}{1+\alpha} F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha}) + \frac{4}{1+\sqrt{1+\alpha}} E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\} \end{aligned}$$

Now,

$$-\frac{6}{\sqrt{1+\alpha}\left(1+\sqrt{1+\alpha}\right)^3} < 0$$

and from (91),

$$\frac{1-3\sqrt{1+\alpha}}{1+\alpha} F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha}) + \frac{4}{1+\sqrt{1+\alpha}} E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha}) =: f(\alpha)$$

Thus, $\exists 1\alpha_0 \in (-0.9,-0.8)\,,(\alpha_0 \fallingdotseq -0.83)$ such that when $-1<\alpha\le\alpha_0$, $A(\alpha)<0\,(\to -\infty)$. Hence, we concluded $\lim_{u\to 1-0}\tilde{V}_u > 0\,(\to +\infty)$ because $\lim_{u\to 1-0}\varphi_\alpha < 0$. Therefore, volume $\tilde{V}$ of unduloid fixed two boundary circles in (R_-) is monotone increasing because $\lim_{u\to 1-0}\tilde{V}_u > 0\,(\to +\infty)$. Conversely, when $\alpha_0<\alpha<0$, $A(\alpha)>0\,(\to +\infty)$. Hence, we concluded $\lim_{u\to 1-0}\tilde{V}_u < 0\,(\to -\infty)$ because $\lim_{u\to 1-0}\varphi_\alpha < 0$. Therefore, volume $\tilde{V}$ of unduloid fixed two boundary circles in (R_-) is monotone decreasing because $\lim_{u\to 1-0}\tilde{V}_u < 0\,(\to -\infty)$.

2.14.3 Partially differentiation $\tilde{V}_u$ in (B_-)

Firstly, we examine sign of

$$\lim_{u\to 1-0}\tilde{V}_u=\lim_{u\to 1-0}\frac{1}{\varphi_\alpha}\left(\varphi_\alpha V_u-\varphi_u V_\alpha\right)$$

in (B_-). From (81), (89), (80), (90),

$$\lim_{u\to 1-0}\frac{1}{4}\frac{\partial\varphi}{\partial\alpha}=\frac{1}{2\left(1+\sqrt{1+\alpha}\right)^2}\left\{-\frac{1+\sqrt{1+\alpha}}{1+\alpha}\;F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})+\frac{1}{\sqrt{1+\alpha}}\;E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}<0$$

$$\begin{aligned}\lim_{u\to 1-0}\frac{3}{4\pi r^3}\frac{\partial V}{\partial u}\quad=\quad&\frac{3}{\sqrt{1+\alpha}\left(1+\sqrt{1+\alpha}\right)^3}\left\{-\left(8+\alpha\right)\sqrt{1+\alpha}\;E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right.\\&\left.+\frac{\alpha\left(4+\alpha\right)}{\sqrt{1+\alpha}}\;F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}<0\end{aligned}$$

$$\lim_{u\to 1-0}\frac{1}{4}\frac{\partial\varphi}{\partial u}=\frac{1}{\sqrt{1+\alpha}\left(1+\sqrt{1+\alpha}\right)}\left\{\frac{\alpha}{\sqrt{1+\alpha}}\;F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})-\sqrt{1+\alpha}\;E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}<0$$

$$\begin{aligned}\lim_{u\to 1-0}\frac{3}{4\pi r^3}\frac{\partial V}{\partial\alpha}\quad=\quad&\frac{3}{2\left(1+\sqrt{1+\alpha}\right)^4}\left[\left\{-\frac{\alpha\left(1+\alpha-3\sqrt{1+\alpha}\right)}{\left(1+\alpha\right)^{3/2}}\right\}\;F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right.\\&\left.+\left(\frac{2+\alpha-6\sqrt{1+\alpha}}{\sqrt{1+\alpha}}\right)\;E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right]<0\end{aligned}$$

Now, we examine sign of

$$A\left(\alpha\right):=\lim_{u\to 1-0}\frac{3}{16\pi r^3}\left(\varphi_\alpha V_u-\varphi_u V_\alpha\right).\quad\text{(i.e.}\lim_{u\to 1-0}\tilde{V}_u=\lim_{u\to 1-0}\frac{1}{\varphi_\alpha}\cdot\frac{16\pi r^3}{3}\cdot A\left(\alpha\right).\text{)}$$

Firstly, both "$\lim_{u\to 1-0}\varphi_\alpha V_u$" and "$\lim_{u\to 1-0}\varphi_u V_\alpha$" are positive. Thus, we can't find out sign of $A\left(\alpha\right)$ soon. Hence, we will try to compute $A\left(\alpha\right)$.

$$
\begin{aligned}
A(\alpha) &= \frac{3}{2\sqrt{1+\alpha}\left(1+\sqrt{1+\alpha}\right)^5}\Bigg[-\frac{\alpha(4+\alpha)\left(1+\sqrt{1+\alpha}\right)}{(1+\alpha)^{3/2}}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\\
&+\left\{\frac{\left(1+\sqrt{1+\alpha}\right)(8+\alpha)\sqrt{1+\alpha}}{1+\alpha}+\frac{\alpha(4+\alpha)}{1+\alpha}\right\}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\\
&-\frac{(8+\alpha)\sqrt{1+\alpha}}{\sqrt{1+\alpha}}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\Bigg]\\
&-\frac{3}{2\sqrt{1+\alpha}\left(1+\sqrt{1+\alpha}\right)^5}\Bigg[-\frac{\alpha^2\left(1+\alpha-3\sqrt{1+\alpha}\right)}{(1+\alpha)^2}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\\
&+\left\{\frac{\alpha\left(2+\alpha-6\sqrt{1+\alpha}\right)}{1+\alpha}+\frac{\alpha\sqrt{1+\alpha}\left(1+\alpha-3\sqrt{1+\alpha}\right)}{(1+\alpha)^{3/2}}\right\}\\
&\quad\times F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\\
&-\frac{\left(2+\alpha-6\sqrt{1+\alpha}\right)\sqrt{1+\alpha}}{\sqrt{1+\alpha}}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\Bigg]\\
&= \frac{3}{2\sqrt{1+\alpha}\left(1+\sqrt{1+\alpha}\right)^5}\Bigg[\frac{-4\alpha\left(1+\sqrt{1+\alpha}\right)}{1+\alpha}\left\{F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}^2\\
&+\frac{2\left(1+\sqrt{1+\alpha}\right)(4+5\alpha)}{1+\alpha}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\\
&-6\left(1+\sqrt{1+\alpha}\right)\left\{E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}^2\Bigg]\\
&=: \frac{3}{2\sqrt{1+\alpha}\left(1+\sqrt{1+\alpha}\right)^5}\Bigg[a\left\{F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}^2\\
&+b\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})+c\left\{E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}^2\Bigg]\\
&=: \frac{3}{2\sqrt{1+\alpha}\left(1+\sqrt{1+\alpha}\right)^5}\cdot P
\end{aligned}
$$

Now,

$$
a:=\frac{-4\alpha\left(1+\sqrt{1+\alpha}\right)}{1+\alpha}>0,\quad b:=\frac{2\left(1+\sqrt{1+\alpha}\right)(4+5\alpha)}{1+\alpha},\quad c:=-6\left(1+\sqrt{1+\alpha}\right)<0.
$$

Thus,

$$
a+b+c=\frac{2}{1+\alpha}\left(1+\sqrt{1+\alpha}\right)>0. \tag{92}
$$

Now,

$$
\begin{aligned}
P \quad &:= \quad a\left\{F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}^2 \\
&\quad +b\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})+c\left\{E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}^2 \\
&= \quad a\left\{E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}^2\left[\left\{\frac{F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})}{E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})}\right\}^2+\frac{b}{a}\left\{\frac{F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})}{E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})}\right\}+\frac{c}{a}\right]
\end{aligned}
$$

We set the following.

$$
x:=\frac{F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})}{E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})}(>1)
$$

Then, we can express the following.

$$
P=a\left\{E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}^2\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)=:a\left\{E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}^2 f\left(x\right)
$$

Now, we set $f\left(x\right):=x^2+\frac{b}{a}x+\frac{c}{a}$. We remark that $a\left\{E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}^2>0$ because $a>0$. We define discriminant of $f\left(x\right)$ as D.

$$
D=\left(\frac{b}{a}\right)^2-4\cdot 1\cdot\left(\frac{c}{a}\right)=\frac{b^2-4ac}{a^2}
$$

Denominator of discriminant D is positive, thus we compute b^2-4ac,

$$
\begin{aligned}
b^2-4ac \quad &= \quad \left\{\frac{2\left(1+\sqrt{1+\alpha}\right)\left(4+5\alpha\right)}{1+\alpha}\right\}^2-4\cdot\left\{\frac{-4\alpha\left(1+\sqrt{1+\alpha}\right)}{1+\alpha}\right\}\cdot\left\{-6\left(1+\sqrt{1+\alpha}\right)\right\} \\
&= \quad \frac{4\left(1+\sqrt{1+\alpha}\right)^2}{1+\alpha}\left\{\frac{\alpha^2+16\left(1+\alpha\right)}{1+\alpha}\right\}>0
\end{aligned}
$$

We turned out that quadratic equation $f\left(x\right)$ has two different real solution because $D>0$. Now, we want to show $f\left(x\right)>0$ when $x>1$.

$$
f\left(x\right)=0\Leftrightarrow x=\frac{-\frac{b}{a}\pm\sqrt{\left(\frac{b}{a}\right)^2-4\cdot 1\cdot\frac{c}{a}}}{2\cdot 1}=\frac{b\mp\sqrt{b^2-4ac}}{-2a}=:x_{\mp}
$$

Now, $a>0\Leftrightarrow -2a<0$ and $\sqrt{b^2-4ac}>|b|$. Thus, $b+\sqrt{b^2-4ac}>0$ and $b-\sqrt{b^2-4ac}<0$. Moreover, from $a>0$ and (92), $f\left(1\right)=1+\frac{b}{a}+\frac{c}{a}=\frac{a+b+c}{a}>0$. Hence, we showed that

$f(x) > 0$ when $x > 1$. Therefore,

$$f(x) > 0 \quad \Leftrightarrow \quad a\left\{E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha})\right\}^2 f(x) > 0\,(\Leftrightarrow P > 0)$$
$$\Leftrightarrow \quad A(\alpha) = \frac{3}{2\sqrt{1+\alpha}\left(1+\sqrt{1+\alpha}\right)^5} \cdot P > 0$$

Now,

$$\frac{3}{2\sqrt{1+\alpha}\left(1+\sqrt{1+\alpha}\right)^5} > 0$$

Thus, $A(\alpha)\,(=\text{finite}) > 0$. Hence, we concluded $\lim_{u\to 1-0} \tilde{V}_u\,(=\text{finite}) < 0$ because $\lim_{u\to 1-0} \varphi_\alpha < 0$. Therefore, volume $\tilde{V}$ of unduloid fixed two boundary circles in (B_-) is monotone decreasing because $\lim_{u\to 1-0} \tilde{V}_u\,(=\text{finite}) < 0$.

2.14.4 Bifurcation around one period of an unduloid between two bulges

(i) In the case of $-1 < \alpha \leq \alpha_0$:
Firstly, (Y_0) is stable from previous result. Secondly, (R_-) is unstable because the profile curve of (R_-) is longer than one period. Thirdly, in the case of (G_-), we assume that $-1 < \alpha_1 \leq \alpha_0$ where α_1 is a parameter of (Y_0). We examine whether $(-1 <)\alpha_2 < \alpha_1$ is correct or not under the same condition where α_2 is a parameter of (G_-). Now, φ_u and φ_α in (G_-) are "$+\infty$" and "negative" respectively. Now,

$$\varphi(u,\alpha) \equiv \text{constant} = \frac{h}{r}.$$

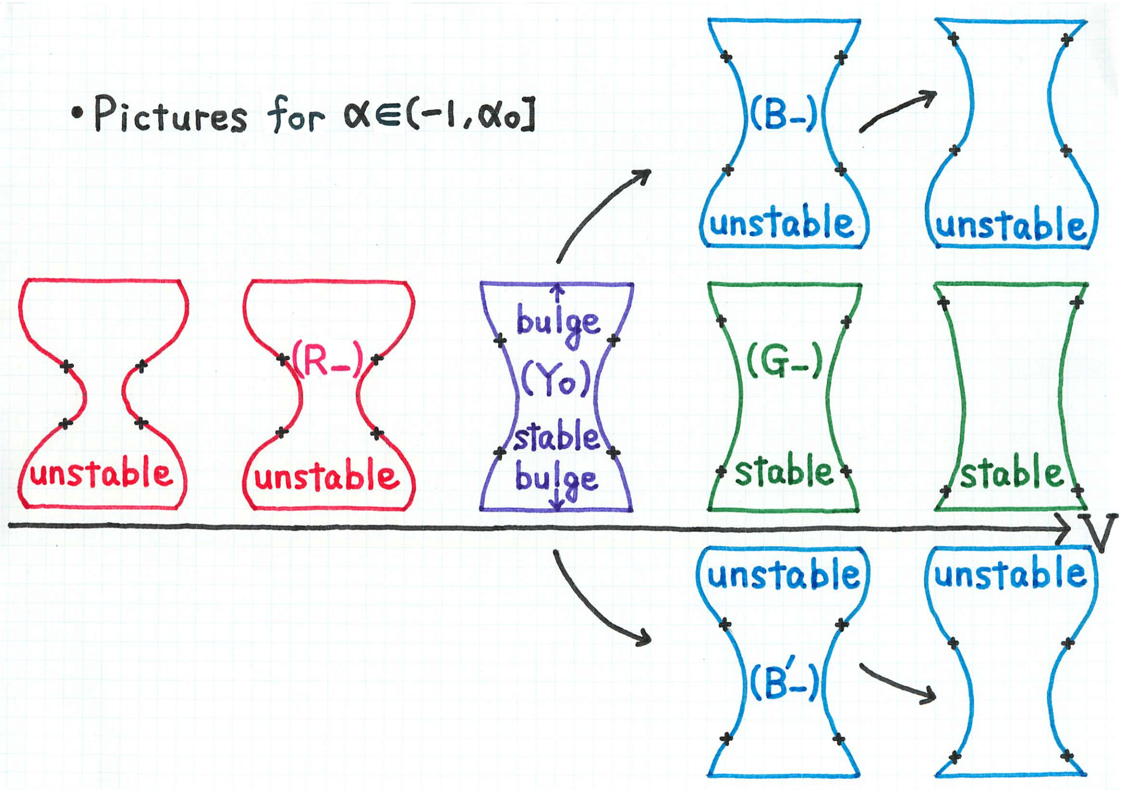
We set $\alpha = \alpha(u)$,

$$\varphi_\alpha(u,\alpha)\cdot\alpha_u + \varphi_u = 0 \Leftrightarrow \frac{d\alpha}{du} = -\frac{\varphi_u}{\varphi_\alpha} > 0$$

holds. Therefore, $(-1 <)\alpha_2 < \alpha_1$ is correct. Consequently, (G_-) is stable. Lastly, (B_-) is unstable from above results.

(ii) In the case of $\alpha_0 < \alpha < 0$:
Firstly, (Y_0) is unstable from previous result. Secondly, (R_-) is unstable because the profile curve of (R_-) is longer than one period. Thirdly, in the case of (G_-) and (B_-), now, $I(\alpha) < 0$. We fix α. We denote I by $\tilde{I}_\alpha(u)$. $\tilde{I}_\alpha(u)$ is continuous with respect to u. Hence, there exists $\varepsilon > 0$ such that when $u_1 \in (1-\varepsilon, 1)$, $\tilde{I}_\alpha(u_1) < 0$ holds. Therefore, (G_-) and (B_-) are unstable. Consequently, all surfaces are unstable.

Theorem 2.14.1. *We prove the following.*



$\exists 1 \alpha_0 \in (-0.9, -0.8), (\alpha_0 \fallingdotseq -0.83)$ *such that*

(i) If $-1 < \alpha \leq \alpha_0$*, a "subcritical pitchfork bifurcation" occurs.*

(ii) If $\alpha_0 < \alpha < 0$*, all surfaces are unstable.*

2.15 Partially differentiation $\hat{H}_u$ of mean curvature of unduloid fixed boundary circles with respect to u

$$r = \frac{u \pm \sqrt{u^2 + \alpha}}{-2H} \Leftrightarrow H(u, \alpha) = \frac{u \pm \sqrt{u^2 + \alpha}}{-2r} (< 0), \quad (r > 0, h > 0 : \text{fix})$$

We regard as $\alpha = \alpha(u)$, and can express $H(u, \alpha(u)) =: \hat{H}(u)$.

$$\hat{H}_u = \frac{d\hat{H}(u)}{du} = \frac{\partial H}{\partial u} + \frac{\partial H}{\partial \alpha}\frac{d\alpha}{du} = H_u + H_\alpha \cdot \alpha_u$$

Now, from $\varphi(u, \alpha) = \varphi(u, \alpha(u)) = h/r \equiv$ constant, the following holds.

$$\varphi_u + \varphi_\alpha \cdot \alpha_u = 0 \Leftrightarrow \alpha_u = -\frac{\varphi_u}{\varphi_\alpha}$$

Thus,

$$\hat{H}_u = H_u + H_\alpha \cdot \alpha_u = H_u + H_\alpha\left(-\frac{\varphi_u}{\varphi_\alpha}\right) = H_u - H_\alpha \cdot \frac{\varphi_u}{\varphi_\alpha} = \frac{1}{\varphi_\alpha}\left(\varphi_\alpha H_u - \varphi_u H_\alpha\right)$$

where

$$H_u = \frac{1}{-2r}\left(1 \pm \frac{u}{\sqrt{u^2+\alpha}}\right), \quad H_\alpha = \frac{1}{-2r}\left(\pm\frac{1}{2\sqrt{u^2+\alpha}}\right), \quad (r>0, -1<\alpha<0)$$

2.15.1 Partially differentiation $\hat{H}_u$ in (G_+)

$$r = \frac{u-\sqrt{u^2+\alpha}}{-2H} \Leftrightarrow H(u,\alpha) = \frac{u-\sqrt{u^2+\alpha}}{-2r}, \quad \hat{H}_u = \frac{1}{\varphi_\alpha}\left(\varphi_\alpha H_u - \varphi_u H_\alpha\right)$$

$$\lim_{u\to 1-0}\frac{\partial\varphi}{\partial\alpha} = \frac{2}{\left(1-\sqrt{1+\alpha}\right)^2}\left\{\frac{1-\sqrt{1+\alpha}}{1+\alpha}\ F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) + \frac{1}{\sqrt{1+\alpha}}\ E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha})\right\} > 0$$

$$\lim_{u\to 1-0}\frac{\partial H}{\partial u} = \frac{1}{-2r}\left(\frac{\sqrt{1+\alpha}-1}{\sqrt{1+\alpha}}\right) > 0$$

$$\lim_{u\to 1-0}\frac{\partial\varphi}{\partial u} = \frac{2}{\sqrt{1+\alpha}}\lim_{u\to 1-0}\frac{1}{\sqrt{1-u^2}} \to +\infty$$

$$\lim_{u\to 1-0}\frac{\partial H}{\partial\alpha} = \frac{1}{-2r}\left(-\frac{1}{2\sqrt{1+\alpha}}\right) > 0$$

Therefore, $\forall\alpha \in (-1,0)$

$$\lim_{u\to 1-0}\hat{H}_u = \lim_{u\to 1-0}\frac{1}{\varphi_\alpha}\left(\varphi_\alpha H_u - \varphi_u H_\alpha\right) < 0\,(\to -\infty)$$

holds.

2.15.2 Partially differentiation $\hat{H}_u$ in (R_+)

$$r = \frac{u-\sqrt{u^2+\alpha}}{-2H} \Leftrightarrow H(u,\alpha) = \frac{u-\sqrt{u^2+\alpha}}{-2r}, \quad \hat{H}_u = \frac{1}{\varphi_\alpha}\left(\varphi_\alpha H_u - \varphi_u H_\alpha\right)$$

$$\lim_{u\to 1-0}\frac{\partial\varphi}{\partial\alpha} = \frac{2}{\left(1-\sqrt{1+\alpha}\right)^2}\left\{\frac{1-\sqrt{1+\alpha}}{1+\alpha}\ F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) + \frac{1}{\sqrt{1+\alpha}}\ E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha})\right\} > 0$$

$$\lim_{u\to 1-0}\frac{\partial H}{\partial u} = \frac{1}{-2r}\left(\frac{\sqrt{1+\alpha}-1}{\sqrt{1+\alpha}}\right) > 0$$

$$\lim_{u\to 1-0}\frac{\partial\varphi}{\partial u} = -\frac{2}{\sqrt{1+\alpha}}\lim_{u\to 1-0}\frac{1}{\sqrt{1-u^2}} \to -\infty$$

$$\lim_{u\to 1-0}\frac{\partial H}{\partial \alpha}=\frac{1}{-2r}\left(-\frac{1}{2\sqrt{1+\alpha}}\right)>0$$

Therefore, $\forall\alpha\in(-1,0)$

$$\lim_{u\to 1-0}\hat{H}_u=\lim_{u\to 1-0}\frac{1}{\varphi_\alpha}\left(\varphi_\alpha H_u-\varphi_u H_\alpha\right)>0\,(\to+\infty)$$

holds.

2.15.3 Partially differentiation $\hat{H}_u$ in (B_+)

$$r=\frac{u-\sqrt{u^2+\alpha}}{-2H}\Leftrightarrow H\left(u,\alpha\right)=\frac{u-\sqrt{u^2+\alpha}}{-2r},\quad \hat{H}_u=\frac{1}{\varphi_\alpha}\left(\varphi_\alpha H_u-\varphi_u H_\alpha\right)$$

$$\lim_{u\to 1-0}\frac{\partial\varphi}{\partial\alpha}=\frac{2}{\left(1-\sqrt{1+\alpha}\right)^2}\left\{\frac{1-\sqrt{1+\alpha}}{1+\alpha}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})+\frac{1}{\sqrt{1+\alpha}}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}>0$$

$$\lim_{u\to 1-0}\frac{\partial H}{\partial u}=\frac{1}{-2r}\left(\frac{\sqrt{1+\alpha}-1}{\sqrt{1+\alpha}}\right)>0$$

$$\lim_{u\to 1-0}\frac{\partial\varphi}{\partial u}=\frac{4}{\sqrt{1+\alpha}\left(1-\sqrt{1+\alpha}\right)}\left\{\frac{-\alpha}{\sqrt{1+\alpha}}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})+\sqrt{1+\alpha}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}>0$$

$$\lim_{u\to 1-0}\frac{\partial H}{\partial \alpha}=\frac{1}{-2r}\left(-\frac{1}{2\sqrt{1+\alpha}}\right)>0$$

Now, we can't find out sign of $\hat{H}_u$ soon. Hence, we will try to compute $\hat{H}_u$.

$$\lim_{u\to 1-0}\varphi_\alpha H_u=\frac{-2}{(1+\alpha)\left(1-\sqrt{1+\alpha}\right)}\left\{\frac{1-\sqrt{1+\alpha}}{\sqrt{1+\alpha}}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})+\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}\cdot\frac{1}{-2r}$$

$$\lim_{u\to 1-0}\varphi_u H_\alpha=\frac{-2}{(1+\alpha)\left(1-\sqrt{1+\alpha}\right)}\left\{\frac{-\alpha}{\sqrt{1+\alpha}}\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})+\sqrt{1+\alpha}\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}\cdot\frac{1}{-2r}$$

Hence,

$$\begin{aligned}\lim_{u\to 1-0}\left(\varphi_\alpha H_u-\varphi_u H_\alpha\right) &= \frac{-2}{(1+\alpha)\left(1-\sqrt{1+\alpha}\right)}\cdot\frac{1}{-2r}\cdot\frac{1}{\sqrt{1+\alpha}}\left\{\sqrt{1+\alpha}\left(\sqrt{1+\alpha}-1\right)\ F(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right.\\ &\left.+\sqrt{1+\alpha}\left(1-\sqrt{1+\alpha}\right)\ E(\frac{1}{\sqrt{1+\alpha}},\arccos\sqrt{-\alpha})\right\}\end{aligned}$$

Now,

$$-\sqrt{1+\alpha}\left(\sqrt{1+\alpha}-1\right)=\sqrt{1+\alpha}\left(1-\sqrt{1+\alpha}\right)>0$$

and

$$F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) > \ E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) > 0$$

Therefore,

$$\begin{aligned}
& -\sqrt{1+\alpha}\left(\sqrt{1+\alpha}-1\right)\ F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) > \sqrt{1+\alpha}\left(1-\sqrt{1+\alpha}\right)\ E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) \\
\Leftrightarrow\ & \sqrt{1+\alpha}\left(\sqrt{1+\alpha}-1\right)\ F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) + \sqrt{1+\alpha}\left(1-\sqrt{1+\alpha}\right)\ E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) < 0
\end{aligned}$$

Thus,

$$\lim_{u\to 1-0}\left(\varphi_\alpha H_u - \varphi_u H_\alpha\right) < 0$$

Hence, $\forall\alpha \in (-1, 0)$

$$\lim_{u\to 1-0}\hat{H}_u = \lim_{u\to 1-0}\frac{1}{\varphi_\alpha}\left(\varphi_\alpha H_u - \varphi_u H_\alpha\right) < 0$$

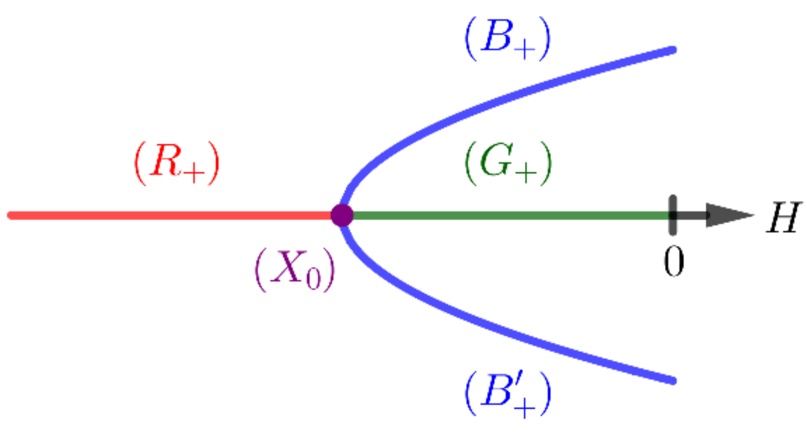
holds.

2.15.4 Bifurcation around one period of an unduloid between two necks (Mean curvature)

Theorem 2.15.1. *From*

$$\begin{cases}
\lim_{u\to 1-0}\hat{H}_u < 0\,(\to -\infty)\,, & (\mathbf{G}_+) \\
\lim_{u\to 1-0}\hat{H}_u > 0\,(\to +\infty)\,, & (\mathbf{R}_+) \\
\lim_{u\to 1-0}\hat{H}_u < 0\,(\text{finite})\,, & (\mathbf{B}_+)
\end{cases} \qquad (\forall\alpha \in (-1, 0))$$

the following bifurcation occurs.



2.15.5 Partially differentiation $\hat{H}_u$ in (G_-)

$$r = \frac{u+\sqrt{u^2+\alpha}}{-2H} \Leftrightarrow H(u,\alpha) = \frac{u+\sqrt{u^2+\alpha}}{-2r}, \quad \hat{H}_u = \frac{1}{\varphi_\alpha}\left(\varphi_\alpha H_u - \varphi_u H_\alpha\right)$$

$$\lim_{u\to 1-0}\frac{\partial\varphi}{\partial\alpha} = \frac{2}{\left(1+\sqrt{1+\alpha}\right)^2}\left\{\frac{-\left(1+\sqrt{1+\alpha}\right)}{1+\alpha}\ F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) + \frac{1}{\sqrt{1+\alpha}}\ E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha})\right\} < 0$$

$$\lim_{u\to 1-0}\frac{\partial H}{\partial u} = \frac{1}{-2r}\left(\frac{\sqrt{1+\alpha}+1}{\sqrt{1+\alpha}}\right) < 0$$

$$\lim_{u\to 1-0}\frac{\partial\varphi}{\partial u} = \frac{2}{\sqrt{1+\alpha}}\lim_{u\to 1-0}\frac{1}{\sqrt{1-u^2}} \to +\infty$$

$$\lim_{u\to 1-0}\frac{\partial H}{\partial\alpha} = \frac{1}{-2r}\left(+\frac{1}{2\sqrt{1+\alpha}}\right) < 0$$

Therefore, $\forall\alpha \in (-1,0)$

$$\lim_{u\to 1-0}\hat{H}_u = \lim_{u\to 1-0}\frac{1}{\varphi_\alpha}\left(\varphi_\alpha H_u - \varphi_u H_\alpha\right) < 0\,(\to -\infty)$$

holds.

2.15.6 Partially differentiation $\hat{H}_u$ in (R_-)

$$r = \frac{u+\sqrt{u^2+\alpha}}{-2H} \Leftrightarrow H(u,\alpha) = \frac{u+\sqrt{u^2+\alpha}}{-2r}, \quad \hat{H}_u = \frac{1}{\varphi_\alpha}\left(\varphi_\alpha H_u - \varphi_u H_\alpha\right)$$

$$\lim_{u\to 1-0}\frac{\partial\varphi}{\partial\alpha} = \frac{2}{\left(1+\sqrt{1+\alpha}\right)^2}\left\{\frac{-\left(1+\sqrt{1+\alpha}\right)}{1+\alpha}\ F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) + \frac{1}{\sqrt{1+\alpha}}\ E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha})\right\} < 0$$

$$\lim_{u\to 1-0}\frac{\partial H}{\partial u} = \frac{1}{-2r}\left(\frac{\sqrt{1+\alpha}+1}{\sqrt{1+\alpha}}\right) < 0$$

$$\lim_{u\to 1-0}\frac{\partial\varphi}{\partial u} = -\frac{2}{\sqrt{1+\alpha}}\lim_{u\to 1-0}\frac{1}{\sqrt{1-u^2}} \to -\infty$$

$$\lim_{u\to 1-0}\frac{\partial H}{\partial\alpha} = \frac{1}{-2r}\left(+\frac{1}{2\sqrt{1+\alpha}}\right) < 0$$

Therefore, $\forall\alpha \in (-1,0)$

$$\lim_{u\to 1-0}\hat{H}_u = \lim_{u\to 1-0}\frac{1}{\varphi_\alpha}\left(\varphi_\alpha H_u - \varphi_u H_\alpha\right) > 0\,(\to +\infty)$$

holds.

2.15.7 Partially differentiation $\hat{H}_u$ in (B_-)

$$r = \frac{u+\sqrt{u^2+\alpha}}{-2H} \Leftrightarrow H\left(u,\alpha\right) = \frac{u+\sqrt{u^2+\alpha}}{-2r}, \quad \hat{H}_u = \frac{1}{\varphi_\alpha}\left(\varphi_\alpha H_u - \varphi_u H_\alpha\right)$$

$$\lim_{u\to 1-0}\frac{\partial\varphi}{\partial\alpha} = \frac{2}{\left(1+\sqrt{1+\alpha}\right)^2}\left\{\frac{-\left(1+\sqrt{1+\alpha}\right)}{1+\alpha}\ F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) + \frac{1}{\sqrt{1+\alpha}}\ E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha})\right\} < 0$$

$$\lim_{u\to 1-0}\frac{\partial H}{\partial u} = \frac{1}{-2r}\left(\frac{\sqrt{1+\alpha}+1}{\sqrt{1+\alpha}}\right) < 0$$

$$\lim_{u\to 1-0}\frac{\partial\varphi}{\partial u} = \frac{4}{\sqrt{1+\alpha}\left(1+\sqrt{1+\alpha}\right)}\left\{\frac{\alpha}{\sqrt{1+\alpha}}\ F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) - \sqrt{1+\alpha}\ E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha})\right\} < 0$$

$$\lim_{u\to 1-0}\frac{\partial H}{\partial\alpha} = \frac{1}{-2r}\left(+\frac{1}{2\sqrt{1+\alpha}}\right) < 0$$

Now, we can't find out sign of $\hat{H}_u$ soon. Hence, we will try to compute $\hat{H}_u$.

$$\lim_{u\to 1-0}\varphi_\alpha H_u = \frac{2}{\left(1+\alpha\right)\left(1+\sqrt{1+\alpha}\right)}\left\{\frac{-\left(1+\sqrt{1+\alpha}\right)}{\sqrt{1+\alpha}}\ F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) +\ E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha})\right\}\cdot\frac{1}{-2r}$$

$$\lim_{u\to 1-0}\varphi_u H_\alpha = \frac{2}{\left(1+\alpha\right)\left(1+\sqrt{1+\alpha}\right)}\left\{\frac{\alpha}{\sqrt{1+\alpha}}\ F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) - \sqrt{1+\alpha}\ E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha})\right\}\cdot\frac{1}{-2r}$$

Hence,

$$\begin{aligned}\lim_{u\to 1-0}\left(\varphi_\alpha H_u - \varphi_u H_\alpha\right) \quad = \quad & \frac{2}{\left(1+\alpha\right)\left(1+\sqrt{1+\alpha}\right)}\cdot\frac{1}{-2r}\cdot\frac{1}{\sqrt{1+\alpha}}\left\{-\sqrt{1+\alpha}\left(\sqrt{1+\alpha}+1\right)\ F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha})\right. \\ & \left. +\sqrt{1+\alpha}\left(1+\sqrt{1+\alpha}\right)\ E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha})\right\}\end{aligned}$$

Now,

$$\sqrt{1+\alpha}\left(\sqrt{1+\alpha}+1\right) = \sqrt{1+\alpha}\left(1+\sqrt{1+\alpha}\right) > 0$$

and

$$F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) >\ E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) > 0$$

Therefore,

$$\begin{aligned} & \sqrt{1+\alpha}\left(\sqrt{1+\alpha}+1\right)\ F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) > \sqrt{1+\alpha}\left(1+\sqrt{1+\alpha}\right)\ E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) \\ \Leftrightarrow \quad & -\sqrt{1+\alpha}\left(\sqrt{1+\alpha}+1\right)\ F(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) + \sqrt{1+\alpha}\left(1+\sqrt{1+\alpha}\right)\ E(\frac{1}{\sqrt{1+\alpha}}, \arccos\sqrt{-\alpha}) < 0\end{aligned}$$

Thus,

$$\lim_{u\to 1-0}(\varphi_\alpha H_u - \varphi_u H_\alpha) > 0$$

Hence, $\forall \alpha \in (-1,0)$

$$\lim_{u\to 1-0}\hat{H}_u = \lim_{u\to 1-0}\frac{1}{\varphi_\alpha}(\varphi_\alpha H_u - \varphi_u H_\alpha) < 0$$

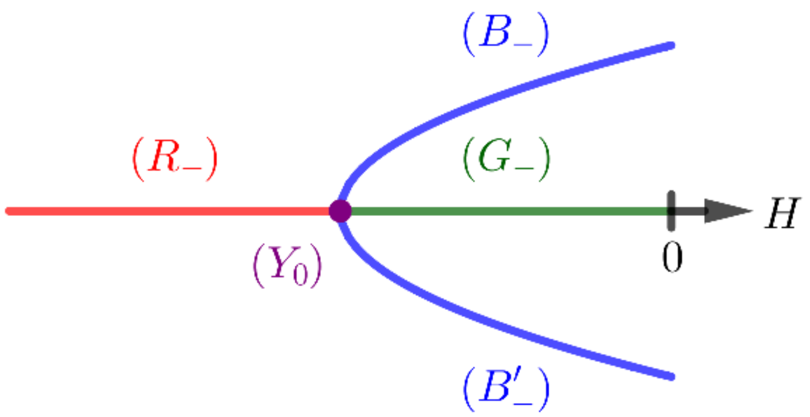
holds.

2.15.8 Bifurcation around one period of an unduloid between two bulges (Mean curvature)

Theorem 2.15.2. *From*

$$\begin{cases} \lim_{u\to 1-0}\hat{H}_u < 0\,(\to -\infty)\,, & (\mathbf{G}_-) \\ \lim_{u\to 1-0}\hat{H}_u > 0\,(\to +\infty)\,, & (\mathbf{R}_-) \\ \lim_{u\to 1-0}\hat{H}_u < 0\,(\text{finite})\,, & (\mathbf{B}_-) \end{cases} \qquad (\forall\alpha \in (-1,0))$$

the following bifurcation occurs.



2.A What is "α"?

2.A.1 In the case of unduloid

$$x = \frac{u \pm \sqrt{u^2+\alpha}}{-2H}, \quad H<0, \quad -1<\alpha<0.$$

$$N(\text{neck}) = \frac{1-\sqrt{1+\alpha}}{-2H}, \quad B(\text{bulge}) = \frac{1+\sqrt{1+\alpha}}{-2H}.$$

Thus,

$$\rho := \frac{N}{B} = \frac{1-\sqrt{1+\alpha}}{1+\sqrt{1+\alpha}} = \frac{(1-\sqrt{1+\alpha})^2}{1-(1+\alpha)} = \frac{1+1+\alpha-2\sqrt{1+\alpha}}{-\alpha} = \frac{2+\alpha-2\sqrt{1+\alpha}}{-\alpha}, \quad 0<\rho<1.$$

Hence,

$$\begin{aligned}-\alpha\rho = 2+\alpha-2\sqrt{1+\alpha} \quad &\Leftrightarrow \quad 2\sqrt{1+\alpha} = 2+\alpha+\alpha\rho \Leftrightarrow 4(1+\alpha) = (2+\alpha+\alpha\rho)^2 \\ &\Leftrightarrow \quad 4+4\alpha = 4+\alpha^2+\alpha^2\rho^2+4\alpha+4\alpha\rho+2\alpha^2\rho \\ &\Leftrightarrow \quad 4+4\alpha = 4+\alpha^2(1+\rho^2+2\rho)+4\alpha(1+\rho) \\ &\Leftrightarrow \quad 4+4\alpha = 4+\alpha^2(1+\rho)^2+4\alpha(1+\rho) \\ &\Leftrightarrow \quad \alpha^2(1+\rho)^2+4\alpha\rho = 0 \Leftrightarrow \alpha\{\alpha(1+\rho)^2+4\rho\} = 0\end{aligned}$$

Therefore,

$$\alpha = -\frac{4\rho}{(1+\rho)^2}. \quad (*)$$

Remark 2.A.1. (Cylinder) $N = B = \text{radius} > 0$. *Thus,* $\rho = 1$. *Hence,* $\alpha = -1$.
(Sphere) $N = 0,\ B = \text{radius} > 0$. *Thus,* $\rho = 0$. *Hence,* $\alpha = 0$.
Therefore, $(*)$ *holds for cylinder, sphere and unduloid.*

2.A.2 In the case of nodoid

$$x = \frac{\pm u + \sqrt{u^2+\alpha}}{-2H}, \quad H < 0, \quad \alpha > 0.$$

$$N(\text{neck}) = \frac{-1+\sqrt{1+\alpha}}{-2H}, \quad B(\text{bulge}) = \frac{1+\sqrt{1+\alpha}}{-2H}.$$

Thus,

$$\begin{aligned}\rho := \frac{N}{B} \quad &= \quad \frac{-1+\sqrt{1+\alpha}}{1+\sqrt{1+\alpha}} = \frac{-(1-\sqrt{1+\alpha})}{1+\sqrt{1+\alpha}} = \frac{-(1-\sqrt{1+\alpha})^2}{1-(1+\alpha)} \\ &= \quad \frac{-(1+1+\alpha-2\sqrt{1+\alpha})}{-\alpha} = \frac{2+\alpha-2\sqrt{1+\alpha}}{\alpha}, \quad 0 < \rho < 1.\end{aligned}$$

Hence,

$$\begin{aligned}\alpha\rho = 2+\alpha-2\sqrt{1+\alpha} \quad &\Leftrightarrow \quad 2\sqrt{1+\alpha} = 2+\alpha-\alpha\rho \Leftrightarrow 4(1+\alpha) = (2+\alpha-\alpha\rho)^2 \\ &\Leftrightarrow \quad 4+4\alpha = 4+\alpha^2+\alpha^2\rho^2+4\alpha-4\alpha\rho-2\alpha^2\rho \\ &\Leftrightarrow \quad 4+4\alpha = 4+\alpha^2(1+\rho^2-2\rho)+4\alpha(1-\rho) \\ &\Leftrightarrow \quad 4+4\alpha = 4+\alpha^2(1-\rho)^2+4\alpha(1-\rho) \\ &\Leftrightarrow \quad \alpha^2(1-\rho)^2-4\alpha\rho = 0 \Leftrightarrow \alpha\{\alpha(1-\rho)^2-4\rho\} = 0\end{aligned}$$

Therefore,

$$\alpha = \frac{4\rho}{(1-\rho)^2}. \quad (**)$$

Remark 2.A.2. (Sphere) $N = 0,\ B = \text{radius} > 0$. *Thus,* $\rho = 0$. *Hence,* $\alpha = 0$.
Therefore, $(**)$ *holds for sphere and nodoid.*

3 Construction of fundamental geometric theory that can be applied to all of smooth curves and surfaces, piecewise smooth curves and surfaces, discrete curves and surfaces

3.1 Introduction

We define "curvature" and "unit normal vector" of piecewise smooth curves and surfaces at each point in order to be capable of "application to variational problem". Firstly, we define "multi-valued unit normal vector" at each point. Secondly, we assume "integrating" and define the curvature at each point. Lastly, we want to be able to express variation of piecewise smooth curves and surfaces as variation of unit normal vector direction (normal variation).

3.2 Piecewise continuous (smooth) curve

Definition 3.2.1 (Multi-valued unit normal vector). *Let M be a piecewise-C^2 curve (resp. surface) in $\mathbb{R}^2$ (resp. $\mathbb{R}^3$). Denote by Ω the closed domain bounded by M. For a point p in M, a vector n is called an outer normal at p if n satisfies $\langle p-z, n\rangle \geq 0$ for any point z on Ω. The set of all outer normals at p is called the outer normal cone at p and is denoted by $C(p)$. $\nu(p) = n(p)/\|n(p)\|$ is called an outer unit normal at p. If $q \in M$ is a regular point in M, $C(q)$ is a half line. When $C(a) = \{\emptyset\}$, we consider inner normal cone $\tilde{C}(a)$ and inner normal $\tilde{n}$. We set $\nu(a) = -\tilde{n}(a)/\|\tilde{n}(a)\|$.*

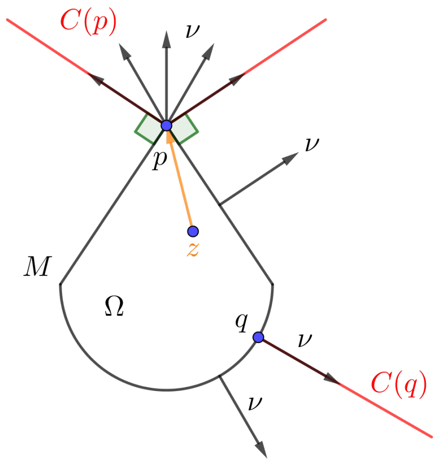


Figure 28: Multi-valued unit normal vector

Definition 3.2.2 (Curvature of piecewise smooth plane curve M). *Firstly, the curvature at smooth point is defined as usual. $P_1, \cdots, P_k$ suppose singular points in M. For multi-valued unit normal $\nu_j(\theta)$ at P_j ($\nu_j(\theta)$ moves from θ_j^1 to θ_j^2), set*

$$\kappa \frac{ds}{d\theta} := 1 \tag{93}$$

Now,

$$\int_{P_j} \kappa \; ds = \theta_j^2 - \theta_j^1 \tag{94}$$

holds. For a length of a parallel curve $M_t := M + t\nu$ *in* M*, the following Steiner's formula holds.*

$$L\left(M_t\right) = L\left(M\right) - t\int_M \kappa \; ds \tag{95}$$

Moreover, the following Minkowski's formula also holds.

$$L\left(M\right) = -\int_M \kappa\langle M, \nu\rangle \; ds + \langle Q_1, \eta_1\rangle + \langle Q_2, \eta_2\rangle. \tag{96}$$

where Q_1, Q_2 *are end points in* M*, and* η_i *is a outward unit conormal vector in* M *at* Q_i*.*

3.2.1 Example

- Rectangle $P_1P_2P_3P_4$ ($a :=$width, $b :=$height where $a, b > 0,\ a, b \in \mathbb{R}$)

$$P_1 = \left(\frac{a}{2}, \frac{b}{2}\right), \quad P_2 = \left(\frac{a}{2}, -\frac{b}{2}\right), \quad P_3 = \left(-\frac{a}{2}, -\frac{b}{2}\right), \quad P_4 = \left(-\frac{a}{2}, \frac{b}{2}\right), \quad O = (0, 0)$$

  - Steiner's formula:

$$\begin{aligned}
& L\left(M\right) - t\int_M \kappa \; ds = 2\left(a+b\right) - t\sum_{i=1}^{4}\int_{P_i} \kappa \; ds \\
= \quad & 2\left(a+b\right) - t\left(\int_{P_1} \kappa \; ds + \int_{P_2} \kappa \; ds + \int_{P_3} \kappa \; ds + \int_{P_4} \kappa \; ds\right) \\
= \quad & 2\left(a+b\right) - t\left(\int_{\frac{\pi}{2}}^{0} \kappa\frac{ds}{d\theta} \; d\theta + \int_{0}^{-\frac{\pi}{2}} \kappa\frac{ds}{d\theta} \; d\theta + \int_{-\frac{\pi}{2}}^{-\pi} \kappa\frac{ds}{d\theta} \; d\theta + \int_{-\pi}^{-\frac{3}{2}\pi} \kappa\frac{ds}{d\theta} \; d\theta\right) \\
= \quad & 2\left(a+b\right) - t\left(\int_{\frac{\pi}{2}}^{0} \; d\theta + \int_{0}^{-\frac{\pi}{2}} \; d\theta + \int_{-\frac{\pi}{2}}^{-\pi} \; d\theta + \int_{-\pi}^{-\frac{3}{2}\pi} \; d\theta\right) \\
= \quad & 2\left(a+b\right) - t\left(-\frac{\pi}{2} - \frac{\pi}{2} - \frac{\pi}{2} - \frac{\pi}{2}\right) = 2\left(a+b\right) + 2\pi t = L\left(M_t\right)
\end{aligned}$$

  - Minkowski's formula:

$$\begin{aligned}
& -\int_M \kappa\langle M, \nu\rangle \; ds = -\sum_{i=1}^{4}\int_{P_i} \kappa\langle P_i, \nu\left(P_i\right)\rangle \; ds \\
= \quad & -\left(\int_{P_1} \kappa\langle P_1, \nu\left(P_1\right)\rangle \; ds + \int_{P_2} \kappa\langle P_2, \nu\left(P_2\right)\rangle \; ds + \int_{P_3} \kappa\langle P_3, \nu\left(P_3\right)\rangle \; ds + \int_{P_4} \kappa\langle P_4, \nu\left(P_4\right)\rangle \; ds\right) \\
= \quad & -\left(\int_{\frac{\pi}{2}}^{0} \langle P_1, \nu\left(P_1\right)\rangle\kappa\frac{ds}{d\theta} \; d\theta + \int_{0}^{-\frac{\pi}{2}} \langle P_2, \nu\left(P_2\right)\rangle\kappa\frac{ds}{d\theta} \; d\theta\right.
\end{aligned}$$

$$
\begin{aligned}
& + \int_{-\frac{\pi}{2}}^{-\pi} \langle P_3, \nu(P_3) \rangle \kappa \frac{ds}{d\theta}\, d\theta + \int_{-\pi}^{-\frac{3}{2}\pi} \langle P_4, \nu(P_4) \rangle \kappa \frac{ds}{d\theta}\, d\theta \Biggr) \\
= \;& -\left(\int_{\frac{\pi}{2}}^{0} \langle P_1, \nu(P_1) \rangle\, d\theta + \int_{0}^{-\frac{\pi}{2}} \langle P_2, \nu(P_2) \rangle\, d\theta + \int_{-\frac{\pi}{2}}^{-\pi} \langle P_3, \nu(P_3) \rangle\, d\theta + \int_{-\pi}^{-\frac{3}{2}\pi} \langle P_4, \nu(P_4) \rangle\, d\theta \right) \\
= \;& -\Biggl\{ \int_{\frac{\pi}{2}}^{0} \frac{1}{2}(a\cos\theta + b\sin\theta)\, d\theta + \int_{0}^{-\frac{\pi}{2}} \frac{1}{2}(a\cos\theta - b\sin\theta)\, d\theta \\
& + \int_{-\frac{\pi}{2}}^{-\pi} \frac{1}{2}(-a\cos\theta - b\sin\theta)\, d\theta + \int_{-\pi}^{-\frac{3}{2}\pi} \frac{1}{2}(-a\cos\theta + b\sin\theta)\, d\theta \Biggr\} \\
= \;& -\frac{1}{2}\left([a\sin\theta - b\cos\theta]_{\frac{\pi}{2}}^{0} + [a\sin\theta + b\cos\theta]_{0}^{-\frac{\pi}{2}} + [-a\sin\theta + b\cos\theta]_{-\frac{\pi}{2}}^{-\pi} + [-a\sin\theta - b\cos\theta]_{-\pi}^{-\frac{3}{2}\pi} \right) \\
= \;& -\frac{1}{2}\{(-b-a) + (-a-b) + (-b-a) + (-a-b)\} = -\frac{1}{2}\cdot 4(-a-b) = 2(a+b) = L(M)
\end{aligned}
$$

3.3 Principal curvature, mean curvature, and Gaussian curvature of piecewise smooth (continuous) surface

Let M be a surface (a two dimensional manifold) which is connected with finite smooth surface patch continuously. Let E be one of the edges in M. If p is a interior point of E, smooth surfaces F_1, F_2 determine such that p is one of the boundary points. When we denote a outward unit normal vector of F_i at p by $\nu_i := \nu_i(p)$, we assume that ν_1 and ν_2 are always linearly independent.

We define the curvature (principal curvature, mean curvature, and Gaussian curvature) of piecewise smooth closed surface M.

The curvature at smooth point is defined as usual. Let E be one of the edges in M. If p is a interior point of E, smooth surfaces F_1, F_2 determine such that p is one of the boundary points. We set a outward unit normal vector of F_i at p as $\nu_i := \nu_i(p)$. ν_1, ν_2 determines a plane. We define outer normal cone at p that is a cone made by ν_1 and ν_2 on this plane. A outer unit normal is also defined as in the case of a plane curve. If we cut M at a plane including each outer unit normal, only some section is smooth curve C. The curvature of the curve C at p is defined as k_1 which is one of the principal curvatures at p in M. The other principal curvature k_2 is the curvature of section when we cut M at a orthogonal plane for Π. We define mean curvature and Gaussian curvature at p in M as "a method as usual" from k_1, k_2.

Below, let M be a polyhedron. Firstly, we consider edges E in M. We set $d\tau$ as a line element of E. When we consider principal curvatures k_1, k_2 in M, k_1 is a curvature of E because E is a segment. Thus, $k_1 = 0$. k_2 satisfies the following.

$$
K\, dA = k_1 k_2\, dA = 0, \quad 2H\, dA = (k_1 + k_2)\, dA = k_2\, dA = k_2 \frac{ds}{d\theta}\, d\theta d\tau = \, d\theta d\tau.
$$

Secondly, let q be a vertex in M, let ν be a outer unit normal in M at q. We remark that ν is a multi-valued map from M to $S^2 \cup \{\emptyset\}$, we set an area element of ν as dA_ν (dA_ν is defined integral value). Then, Gaussian curvature K, mean curvature H, and area element dA in M at q are defined as follows.

$$
K \frac{dA}{dA_\nu} = 1, \quad H\, dA = 0
$$

where dA_ν is given a positive or negative sign according to the direction of ν. Now, K is non-negative when M is a convex or concave point, however, K is non-positive when M is a saddle point.

Then, the following Steiner's formula and Minkowski's formula hold.

- Steiner's formula: For the area $A(M_t)$ of the parallel surface $M_t := M + t\nu$ in M,
$$A(M_t) = A(M) - 2t \int_M H \; dA + t^2 \int_M K \; dA. \tag{97}$$
- Minkowski's formula: let ∂M be a boundary in M, let η be an outward unit conormal of M in ∂M, and let $d\sigma$ be a line element of ∂M,
$$A(M) = -\int_M H\langle x, \nu(x)\rangle \; dA + \frac{1}{2}\int_{\partial M} \langle x, \eta(x)\rangle d\sigma. \tag{98}$$

Therefore, we are ready to expand the variational method for piecewise smooth curves and surfaces.

3.4 Example of variational problem for piecewise smooth curve

Question 3.4.1. *In $\mathbb{R}^2$, find the critical point of length (not always minimum) in piecewise-C^2 closed curve that enclose same area.*

$X = (x, y) : S^1 \to \mathbb{R}^2$ is a piecewise-C^2 closed curve. $S(X) \subset S^1$ is a singularity set of X. The variation $X(\varepsilon) : S^1 \to \mathbb{R}^2$ of X can represent the form of normal variation as follows.

$$X(\varepsilon) = X + \varepsilon\psi\nu + \mathcal{O}\left(\varepsilon^2\right)$$

Let $s(\varepsilon)$ be an arc length parameter of $X(\varepsilon)$. By computation,

$$A(\varepsilon) := A(X(\varepsilon)) = \frac{1}{2}\int_{S^1} \langle X(\varepsilon), \nu(\varepsilon)\rangle \; ds(\varepsilon), \quad A'(0) = \int_{S^1\setminus S(X)} \psi \; ds(0),$$

$$L(\varepsilon) := \text{the length of } X(\varepsilon), \quad L'(0) = -\int_{\tilde{S}^1} \psi\kappa \; ds(0).$$

where $\tilde{S}^1 = S^1\cup$ (width of normal cones) $= S^1 \cup \left(\cup_{j=1}^k \left[\theta_j^1, \theta_j^2\right]\right)$. Therefore, for $c \in \mathbb{R}$, for $\forall\psi : \tilde{S}^1 \to \mathbb{R}$, $(L + cA)'(0) = 0$. $\Leftrightarrow$ $\kappa \equiv c$ and $\tilde{S}^1 = S^1$. $\Leftrightarrow$ $X\left(S^1\right)$ is a circle. Consequently, a solution of variational problem (Question3.4.1) is a circle.

3.5 Summary

1. For singular point of piecewise smooth curves and surfaces, we consider outer normal or inner normal instead of normal as usual, defined a unit normal vector as multi-valued.
2. Importance of thinking about outer normal, inner normal:
   (a) A bending degree of curves and surfaces should be determined by rate of change of unit normal vector.

(b) It is convenient to consider normal variation as variation of curves and surfaces. However, if we consider normal vector as usual, curves and surfaces tear at singular points. If we take variation of multi-valued normal direction defined in this study, the connectivity of curves and surfaces can be maintained.

3. We defined the curvature of curves and surfaces which have singular points. These corresponds with the curvature as usual at smooth point.

4. "Steiner's formula" and "Minkowski's formula" are fundamental integral formulas for smooth curves and surfaces. These also hold for piecewise smooth curves and surfaces.

5. As an application, we showed that the critical point of length is a circle among piecewise-C^2 closed curve that enclose same area in plane.

4 Uniqueness of local minimizers for crystalline variational problems

4.1 Introduction

Motivation: A single crystal forms various polyhedron, for example, cube, regular octahedron and so on. We encountered cubic and truncated octahedral CeO_2 nanoparticles comprising smaller primary nanocrystals when we discussed with Professor Seiichi Takami of the Graduate School of Engineering, Nagoya University. A single crystal of CeO_2 usually forms a regular octahedron. Since this crystal is very tiny that is about dozens of nanometers, even if we observe it with an electron microscope, it was said that it was not known for certain how these were assembled. In our study, we tried clarifying the structure of this crystal mathematically.

How is the shape of the crystal determined?: It is considered to form the energy that depends on the direction of the surface is a minimizer or a local minimizer in the closed surfaces enclosing a given volume.

Main result: When a single crystal with the least energy is a regular polyhedron, a local minimizer of convex anisotropic energy is unique (up to homothety an

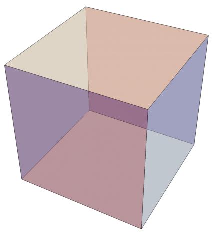


Figure 29: salt crystal (cube)

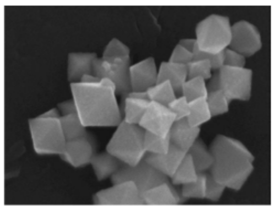


Figure 30: CeO_2 crystal (regular octahedron), (Asahina, Takami, et al., 2011, [2])

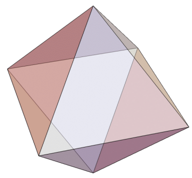


Figure 31: regular octahedron

Remark 4.1.1. *When the energy minimizer for given energy density function has singular points like a polyhedron, the energy density function has non-differentiable points and the classical variational method cannot be used. Thus, mathematical study about the energy local minimizer has not progressed. Now, it is called the crystalline variational problem that the problem of discussing the energy local minimizer when the energy minimizer is a polyhedron.*

4.2 Anisotropic surface energy and the Wulff shape

Let $\gamma : S^2 \to \mathbb{R}_{>0} := \{x \in \mathbb{R}; x > 0\}$ be a positive continuous function defined on a unit sphere $S^2 = \left\{\nu \in \mathbb{R}^3; \|\nu\| = 1\right\}$ in the 3-dimensional Euclidean space $\mathbb{R}^3$. This γ is a mathematical model of anisotropic surface energy density.

Let M be a piecewise smooth surface in $\mathbb{R}^3$, we assume that $\nu : M \setminus S(M) \to S^2$ is a unit normal vector field on $M \setminus S(M)$ where $S(M)$ is a singularity (not smooth point, the point that unit normal vector is not unique) set of M.

For this surface M, the integral on M of the anisotropic energy density $\mathcal{F}_\gamma(M) := \int_{M\setminus S(M)} \gamma(\nu)\, dA$ called the anisotropic energy of M where dA is the area element of M. In particular, when $\gamma \equiv 1$, $\mathcal{F}_\gamma(M)$ is the area of the surface M.

The following has already known:

Fact 4.2.1 (J. E. Taylor, 1978, [23]). *There is a unique minimizer of $\mathcal{F}_\gamma$ among closed surfaces enclosing the same volume in $\mathbb{R}^3$ (up to translation). It is the closed surface which is called the Wulff shape or its homothety.*

The Wulff shape W for γ is the boundary of the convex set

$$\bigcap_{\nu\in S^2} \left\{x \in \mathbb{R}^3; \langle x,\nu\rangle \le \gamma(\nu)\right\}.$$

In particular, when $\gamma \equiv 1$, W coincides with S^2.

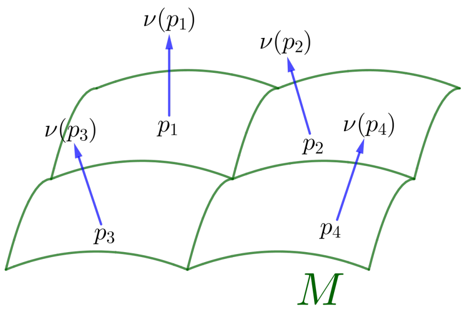


Figure 32: piecewise smooth surface M

4.3 Convex energy density function γ

For any convex closed surface W in $\mathbb{R}^3$, there exists the energy density function γ such that W is the Wulff shape. (γ is not necessarily unique.) However, there is a unique "convex" energy density function γ for each W. In particular, when W is smooth and strictly convex, γ is unique and the local minimizer of the anisotropic energy $\mathcal{F}_\gamma$ for all variations that preserve the enclosed volume is W or its homothety (Palmer, 1998, [18]).

Now, we explain the construction of convex γ. Firstly, for $\nu \in S^2$, we denote by $P(\nu)$ the plane that is perpendicular to ν and passes the origin in $\mathbb{R}^3$. The function $\gamma : S^2 \to \mathbb{R}_{>0}$ is defined as follows:

$$\gamma(\nu) := \max\left\{t \in \mathbb{R}; (t\nu + P(\nu)) \cap W \ne \emptyset\right\}.$$

We call this γ the support function of W.

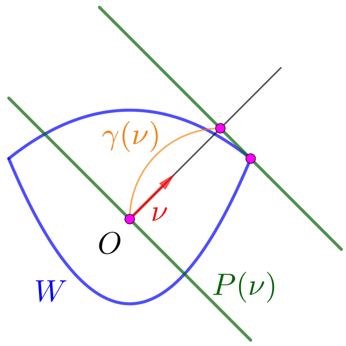


Figure 33: The construction of the convex anisotropic energy density function γ for a convex closed surface W

4.4 Main theorem

In the following, we consider only convex closed surfaces. Even when W is not smooth, the uniqueness of local minimizer is obtained as follows.

Theorem 4.4.1 (Uniqueness of local minimizers for crystalline variational problems)**.** *Let W be a regular polyhedron, let γ be the support function of W and let M be a piecewise-C^1 convex closed surface. Then, M is a local minimizer of $\mathcal{F}_\gamma(M) = \int_{M\setminus S(M)} \gamma(\nu)\ dA$ for all variations that preserve the enclosed volume if and only if $M = W$ (up to homothety and translation).*

The outline of the proof of main theorem consists of 3 steps as follows.

Step1. We assume that $\gamma(\nu_k) = 1 (k = 1, \cdots, m)$ for the outer unit normals $\nu_1, \cdots, \nu_m$ of each face $F_1, \cdots, F_m$ in W. On the other hand, we can assume without loss of generality that $\gamma(\nu) > 1$ for the others outer unit normal ν.

Step2. If M is a local minimizer, M is a polyhedron which has only faces in the same direction as W.

Step3. We consider that the Minkowski sum $M_t := M + tW := \{p + tq;\ p \in M,\ q \in W\}$ for non-negative number t. We compute (anisotropic) isoperimetric quotient $Q(t)$

$$Q(t) := \frac{(\mathcal{F}_\gamma(W))^3}{(V(W))^2} \cdot \frac{(V(M_t))^2}{(\mathcal{F}_\gamma(M_t))^3} = \frac{(A(W))^3}{(V(W))^2} \cdot \frac{(V(M_t))^2}{(A(M_t))^3}$$

where $A(M_t)$ and $V(M_t)$ are the area of M_t and the volume enclosed M_t respectively. If M is a local minimizer, $Q(t)$ must be a local maximum at $t = 0$. However, we can show that $Q(t)$ is not a local maximum at $t = 0$. Therefore, M is not a local minimizer.

4.5 Uniqueness of local minimizers for crystalline variational problems in the plane

Let X be a embedded closed curve. We denote the domain enclosed X by Ω. Below, we assume that for any θ,

$$\gamma(\theta+\pi)=\gamma(\theta)$$

hold. We assume that X is not convex ($X : S^1 \to \mathbb{R}^2 \in C^0$, piecewise-$C^2$). There exists $\eta_1, \eta_2\,(0 \leq \eta_1 < \eta_2 < 2\pi)$ such that segment $X(\eta_1)\,X(\eta_2) \not\supset \bar{\Omega}$. Now, there exists $\eta_0 \in (\eta_1, \eta_2)$ and $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$, open segment $\left(X(\eta_0-\varepsilon)\,X(\eta_0+\varepsilon)\right)^\circ \subset \bar{\Omega}^c$ where $\bar{\Omega}^c = \mathbb{R}^2 \setminus \bar{\Omega}$. Moreover, there exists $\varepsilon_1 \in (0, \varepsilon_0)$ such that for any $\varepsilon \in (0, \varepsilon_1)$, ($\pi$ rotation with respect to the midpoint of segment $X(\eta_0-\varepsilon)\,X(\eta_0+\varepsilon)$ in arc $\{X(\eta_0+t)\,; -\varepsilon < t < \varepsilon\}) \cap \bar{\Omega} = \{X(\eta_0-\varepsilon), X(\eta_0+\varepsilon)\}$. We set

$$C(\varepsilon) := \{X(\eta_0+t)\,; -\varepsilon < t < \varepsilon\},$$

$$\tilde{C}(\varepsilon) := \pi \text{ rotation with respect to the midpoint of } X(\eta_0-\varepsilon)\,X(\eta_0+\varepsilon) \text{ in } C(\varepsilon).$$

Now, we consider the following embedded closed curves.

$$\Gamma := \{X(\theta)\,; \theta \in \mathbb{R}\},$$

$$\tilde{\Gamma}(\varepsilon) := (\Gamma \setminus C(\varepsilon)) \cup \tilde{C}(\varepsilon).$$

We denote the domain enclosed Γ by $A(\Gamma)$,

$$A(\Gamma) < A\left(\tilde{\Gamma}(\varepsilon)\right), \quad \text{and} \quad \mathcal{F}_\gamma(\Gamma) = \mathcal{F}_\gamma\left(\tilde{\Gamma}(\varepsilon)\right), \quad \forall \varepsilon \in (0, \varepsilon_1)$$

hold. Now, we take $r(\varepsilon) \in (0,1)$ where

$$A(\Gamma) = A\left(r(\varepsilon)\,\tilde{\Gamma}(\varepsilon)\right), \quad 1 > r(\varepsilon) > 0$$

Now, we can take $r(\varepsilon)$ uniquely. Then,

$$\mathcal{F}_\gamma\left(r(\varepsilon)\,\tilde{\Gamma}(\varepsilon)\right) < \mathcal{F}_\gamma(\Gamma), \quad \forall \varepsilon \in (0, \varepsilon_1)$$

holds. Moreover, we consider the Hausdorff measure,

$$\tilde{\Gamma}(0) = \Gamma, \quad \lim_{\varepsilon \searrow +0} r(\varepsilon)\,\tilde{\Gamma}(\varepsilon) = \Gamma$$

hold. Consequently, Γ is not a local minimizer of $\mathcal{F}_\gamma$. This explanation can be extended in the case of surface in $\mathbb{R}^3$.

In $\mathbb{R}^2$, firstly, we denote vertices of W by $V = \{v_1, \cdots, v_k\}$ and edges of W by $E = \{e_1, \cdots, e_k\}$. We set $\eta_i := \eta(e_i) = (\cos\theta_i, \sin\theta_i)$. Let M be a piecewise-smooth convex closed curve.

Secondly, we denote vertices of M by $V^M = \left\{{v_1}^M, \cdots, {v_k}^M\right\}$ and edges of M by $E^M = \left\{{e_1}^M, \cdots, {e_k}^M\right\}$. Let ν be outward-pointing unit normal to M (multi-valued). A mapping $\xi : M \to W$, (multi-valued) anisotropic Gauss map of M is defined as follows: For $p \in M$, $\xi(p) := \eta^{-1}(\nu(p))$.

Case(a) : When

$$p \in M, \quad \nu(p) \in \{(\cos\theta, \sin\theta); \theta_{i-1} < \theta < \theta_i\},$$

$\xi(p) = v_i$. Thus,

$$M_t(p) := p + t\xi(p) = p + tv_i, \quad p \in M.$$

Hence,

$$M_t\left(\nu^{-1}\left(\{(\cos\theta, \sin\theta); \theta_{i-1} < \theta < \theta_i\}\right)\right) = \nu^{-1}\left(\{(\cos\theta, \sin\theta); \theta_{i-1} < \theta < \theta_i\}\right) + tv_i =: \sigma_i + tv_i$$

That is, $M_t(\sigma_i) = \sigma_i$ is the translation of σ_i by tv_i.

Case(b) :

$$p \in M, \ \nu(p) = (\cos\theta_i, \sin\theta_i)$$

$$\xi(p) = \eta^{-1}(\cos\theta_i, \sin\theta_i) = \eta^{-1}(\eta_i) = e_i \quad \text{(segment)}.$$

Therefore, $M_t(p) = p + te_i$ is a segment.

Now, we separate M as follows: $M = M^1 \cup M^2$ where M^1 consists only of edges parallel to the edges of W, M^2 consists of the others.

$$\begin{aligned} A(M_t) &= A(M) + t \cdot L\left(M^1\right) + t\int_{M^2} \langle v, \nu\rangle \, ds + t^2 A(W) \\ &= A(M) + t \cdot L\left(M^1\right) + t\sum_i \int_{M^2 \cap \xi^{-1}(v_i)} \langle v_i, \nu\rangle \, ds + t^2 A(W) \end{aligned}$$

where ds is a line element of M.

$$\mathcal{F}_\gamma(M_t) = \mathcal{F}_\gamma(M) + t\mathcal{F}_\gamma(W)$$

Now, we take $a(t) > 0$ where $\tilde{M}_t := a(t) M_t$ satisfies $A\left(\tilde{M}_t\right) = A(M)$.

$$\mathcal{F}_\gamma\left(\tilde{M}_t\right) = a(t)\mathcal{F}_\gamma(M_t), \quad A(M) = A\left(\tilde{M}_t\right) = (a(t))^2 A(M_t)$$

Thus,

$$\begin{aligned} f(t) &:= \left(\mathcal{F}_\gamma\left(\tilde{M}_t\right)\right)^2 = \left(a(t)\mathcal{F}_\gamma(M_t)\right)^2 = \frac{A(M)}{A(M_t)}\left(\mathcal{F}_\gamma(M_t)\right)^2 \\ &= \frac{A(M)\left(\mathcal{F}_\gamma(M) + t\mathcal{F}_\gamma(W)\right)^2}{A(M) + t\left(L\left(M^1\right) + \int_{M^2}\langle v, \nu\rangle \, ds\right) + t^2 A(W)} \end{aligned}$$

Hence,

$$\begin{aligned} f'(0) &= \frac{A(M)}{(A(M))^2} \cdot \left\{2\mathcal{F}_\gamma(M)\mathcal{F}_\gamma(W)A(M) - (\mathcal{F}_\gamma(M))^2\left(L\left(M^1\right) + \int_{M^2}\langle v, \nu\rangle \, ds\right)\right\} \\ &= \frac{\mathcal{F}_\gamma(M)}{A(M)} \cdot \left\{2\mathcal{F}_\gamma(W)A(M) - \mathcal{F}_\gamma(M)\left(L\left(M^1\right) + \int_{M^2}\langle v, \nu\rangle \, ds\right)\right\} \\ &=: \frac{\mathcal{F}_\gamma(M)}{A(M)} \cdot B \end{aligned}$$

We may assume without loss of generality that $A(M) = A(W)$.

$$\begin{aligned} B &= 2\mathcal{F}_\gamma(W)A(M) - \mathcal{F}_\gamma(M)\left(L\left(M^1\right) + \int_{M^2} \langle v,\nu\rangle\ ds\right) \\ &= 2\mathcal{F}_\gamma(W)A(W) - \mathcal{F}_\gamma(M)\left(\mathcal{F}_\gamma\left(M^1\right) + \int_{M^2} \gamma(\nu)\ ds\right) \\ &= 2\mathcal{F}_\gamma(W)A(W) - \mathcal{F}_\gamma(M)\left(\mathcal{F}_\gamma\left(M^1\right) + \mathcal{F}_\gamma\left(M^2\right)\right) \\ &= \left(\mathcal{F}_\gamma(W)\right)^2 - \left(\mathcal{F}_\gamma(M)\right)^2 \le 0 \end{aligned}$$

The equality holds if and only if $M \equiv W$ (up to translation). Therefore, $f'(0) \le 0$. The equality holds if and only if $M \equiv W$. If M is a local minimizer, it must be $f'(0) \ge 0$. Thus, $f'(0) = 0$. Hence, $M \equiv W$.

From the above, we obtain the following theorem. We remark that this result is the corollary of the theorem for F. Morgan (2005, [17]). However, his proof is only valid in $\mathbb{R}^2$. On the other hand, our proof can be extended in the case of higher dimension.

Theorem 4.5.1. *Let W be a regular polygon, let γ be the support function of W and let M be a piecewise smooth closed curve. Assume that the following (i) or (ii) hold.*

(i) W is a regular $2n$-sided polygon.

(ii) M is convex.

Then, M is a local minimizer of $\mathcal{F}_\gamma$ for all variations that preserve the enclosed area if and only if M is homothety of W.

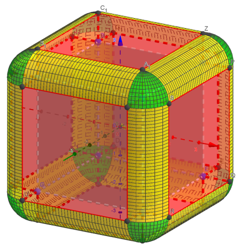
4.6 Uniqueness of local minimizers for crystalline variational problems in the space

In the case of a surface in $\mathbb{R}^3$, let W be a regular polyhedron. Assume that $\gamma : S^2 \to \mathbb{R}_{>0}$ is the support function of W. We denote vertices of W by $V := \{v_1, \cdots, v_l\}$, edges of W by $E := \{e_1, \cdots, e_m\}$ and faces of W by $F := \{f_1, \cdots, f_n\}$. The unit vector which is parallel to e_j is represented as t_j. We denote the outer unit normals of W by η. We assume that $\eta = \eta_k$ on f_k. η is multi-valued on v_i, e_j where we assume that when it is multi-valued, η is the interior point set.

Example 4.6.1. *In the case of a cube, for each vertex v_i, $\eta(v_i)$ is 1/8 of the sphere.*

The following equality holds.

$$\left(\bigcup_{i=1}^{l} \eta(v_i)\right) \cup \left(\bigcup_{j=1}^{m} \eta(e_j)\right) \cup \left(\bigcup_{k=1}^{n} \eta(f_k)\right) = S^2$$



where the left-hand side represents disjoint union. Now, let M be a piecewise smooth convex closed surface in $\mathbb{R}^3$. Let ν be the outer (multi-valued) unit normals of M where we assume that when it is multi-valued, ν is the interior point set. Let M be a local minimizer of $\mathcal{F}_\gamma$ for all variations that preserve the enclosed volume. We show that "$\exists r > 0$ such that $M = rW$ (up to translation)". In order to prove it, we will define a special variation M_t of M.

Remark 4.6.1. *(i)* $\forall p \in W,\ \exists \eta_p \subset S^2$ *(multi-valued), such that* $\eta(p) = \eta_p$.

(ii) $\forall p \in M,\ \exists \nu_p \subset S^2$, *such that* $\nu(p) = \nu_p$.

(iii) *(I)* $\forall \tilde{\eta} \in S^2,\ \exists p \in W$, *such that* $\eta(p) \ni \tilde{\eta}$.

(II) Assume that $p, q \in W,\ \tilde{\eta} \in S^2,\ \eta(p) \ni \tilde{\eta}$ *and* $\eta(q) \ni \tilde{\eta}$. *Then,*

$$\begin{cases} \exists 1 f_i \in F, \ \textit{such that } p, q \in f_i, \ \textit{or} \\ \exists 1 e_i \in E, \ \textit{such that } p, q \in e_i. \end{cases}$$

(iv) *(I)* $\forall \tilde{\nu} \in S^2,\ \exists p \in M$, *such that* $\nu(p) \ni \tilde{\nu}$.

(II) Assume that $p, q \in M,\ \tilde{\nu} \in S^2,\ \nu(p) \ni \tilde{\nu}$ *and* $\nu(q) \ni \tilde{\nu}$. *Then,*

$$\begin{cases} \exists 1 f : \textit{flat face of } M, \ \textit{such that } p, q \in f, \ \textit{or} \\ \exists 1 e : \textit{edge of } M, \ \textit{such that } p, q \in e. \end{cases}$$

Let σ_j be a section of S^2 in the plane perpendicular to t_j in the great circle of S^2. A mapping

$$\xi : M \to W \quad \text{(multi-valued) anisotropic Gauss map of } M$$

is defined as follows:

$$\xi(p) := \eta^{-1}(\nu(p)), \quad \forall p \in M.$$

The variation M_t of M is defined as $M_t(p) := p + t\xi(p),\ p \in M$.

Case(A) : When

$$\nu(p) \in \eta(v_i), \quad \exists i \in \{1, \cdots, l\},$$

$\xi(p) = v_i$ (one point). Therefore,

$$M_t(p) = p + t\xi(p) = p + tv_i.$$

That is, $M_t(p)$ is the translation of p by tv_i.

Case(B) : When

$$\nu(p) \in \eta(e_j), \quad \exists j \in \{1, \cdots, m\},$$

$$\xi(p) = \eta^{-1}(\nu(p)) = e_j \quad \text{(segment)}.$$

Therefore, $M_t(p) = p + t\xi(p) = p + te_j$, $\quad te_j$: segment.

Case(C) : When

$$\nu(p) = \eta(f_k), \quad \exists k \in \{1, \cdots, n\},$$

$$\xi(p) = \eta^{-1}(\nu(p)) = \eta^{-1}(\eta(f_k)) = f_k \quad \text{(polygon)}.$$

Therefore, $M_t(p) = p + t\xi(p) = p + tf_k$, $\quad tf_k$: polygon.

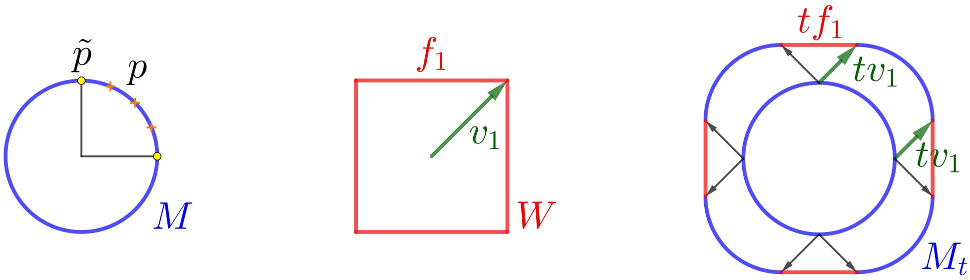


Figure 34: $\xi(p) = v_1, \quad \xi(\tilde{p}) = f_1$.

Now, we denote the volume enclosed M_t by $y(t)$, the anisotropic energy of M_t by $\varepsilon(t)$. We can write as follows:

$$\begin{aligned} y(t) &= y(0) + at + bt^2 + ct^3 && (99) \\ \varepsilon(t) &= \varepsilon(0) + \alpha t + \beta t^2 && (100) \end{aligned}$$

We set $r(t) > 0$ so that $\tilde{M}_t := r(t) M_t$ satisfies $V\left(\tilde{M}_t\right) = V(M)$.

$$\begin{aligned} y(0) &= V(M) = V\left(\tilde{M}_t\right) = V(r(t) M_t) = (r(t))^3 V(M_t) = (r(t))^3 y(t) && (101) \\ f(t) &:= \mathcal{F}_\gamma\left(\tilde{M}_t\right) = \mathcal{F}_\gamma(r(t) M_t) = (r(t))^2 \mathcal{F}_\gamma(M_t) = (r(t))^2 \varepsilon(t) && (102) \end{aligned}$$

From (101), (102),

$$f(t) = (r(t))^2 \varepsilon(t) = \left(\frac{y(0)}{y(t)}\right)^{2/3} \cdot \varepsilon(t)$$

$$F(t) := (f(t))^3 = (y(0))^2 \cdot \frac{(\varepsilon(t))^3}{(y(t))^2}$$

$\dfrac{(\mathcal{F}_\gamma(W))^3}{(V(W))^2} \cdot \dfrac{(y(t))^2}{(\varepsilon(t))^3}$ is called the anisotropic isoperimetric quotient of $\tilde{M}_t$.

$$\begin{aligned} F'(0) &= (y(0))^2 \cdot \frac{1}{(y(0))^4} \cdot \left\{3(\varepsilon(0))^2 \varepsilon'(0)(y(0))^2 - 2(\varepsilon(0))^3 y(0) y'(0)\right\} \\ &= \frac{(\varepsilon(0))^2}{y(0)} \cdot \{3\varepsilon'(0) y(0) - 2\varepsilon(0) y'(0)\} \\ &= \frac{(\mathcal{F}_\gamma(M))^2}{V(M)} \cdot \{3\alpha \cdot V(M) - 2a \cdot \mathcal{F}_\gamma(M)\} && (103) \end{aligned}$$

Now, We examine a and α. Obviously, a and α are ascribable to a part of corresponding to Case(A) and Case(B) respectively. We may assume without loss of generality that $V(M) = V(W)$. Then,

$$\begin{aligned} Q &:= 3\alpha V(M) - 2a\mathcal{F}_\gamma(M) = 3\alpha V(W) - 2a\mathcal{F}_\gamma(M) \\ &= \alpha \mathcal{F}_\gamma(W) - 2a\mathcal{F}_\gamma(M) && (104) \end{aligned}$$

holds because

$$\mathcal{F}_\gamma(W) = \int_W \gamma(\nu_W)\ dA_W = 3V(W)$$

Now, $\mathcal{F}_\gamma(W) \leq \mathcal{F}_\gamma(M)$ holds. The equality holds if and only if $W = M$. We separate M as follows:

$$M = M^1 \cup M^2, \quad \text{disjoint union}$$

where M^1 consists only of flat faces parallel to the faces of W,

$$M^2 = M \setminus M^1.$$

Then,

$$\begin{aligned} a &= A\left(M^1\right) + \sum_{i=1}^{l} \int_{M^2 \cap \xi^{-1}(v_i)} \langle v_i, \nu \rangle\ dA \\ &= A\left(M^1\right) + \sum_{i=1}^{l} \int_{M^2 \cap \xi^{-1}(v_i)} \gamma(\nu)\ dA \\ &= \mathcal{F}_\gamma\left(M^1\right) + \mathcal{F}_\gamma\left(M^2\right) = \mathcal{F}_\gamma(M) \end{aligned} \tag{105}$$

Now, since M is convex and α comes from bulge of parts (edges) of M which has the same normal as edges of W,

$$\alpha = 2\mathcal{F}_\gamma(W) \tag{106}$$

holds. From (104), (105) and (106),

$$Q = 2\left\{(\mathcal{F}_\gamma(W))^2 - (\mathcal{F}_\gamma(M))^2\right\} \leq 0$$

The equality holds if and only if $W = M$. Therefore, if M is a local minimizer of $\mathcal{F}_\gamma$, then M must coincide with W (up to translation and homothety). From the above, the following is proved.

Theorem 4.6.1 (Uniqueness of local minimizers for crystalline variational problems)**.** *Let W be a regular polyhedron, let γ be the support function of W and let M be a piecewise-C^1 convex closed surface. Then, M is a local minimizer of $\mathcal{F}_\gamma(M) = \int_{M \setminus S(M)} \gamma(\nu)\ dA$ for all variations that preserve the enclosed volume if and only if $M = W$ (up to homothety and translation).*

4.7 Example using main theorem

A regular tetrahedron is a polyhedron which has only faces parallel to the faces of the regular octahedron. Now, let W be a regular octahedron, and M be a regular tetrahedron. A length of edge denote by a respectively. When we consider Minkowski sum $M_t := M + \tau W$ $(\tau := t/(1-2t) \geq 0)$, M_t is a truncated regular tetrahedron. The following M_t set a truncated regular tetrahedron. Now, a vertex is truncated a length $tb\,(0 \leq t \leq 1/2)$ for a regular tetrahedron that a length of edge is equal to b.

Now, the anisotropic isoperimetric quotient $Q(t)$ corresponds with the isoperimetric quotient. By computation,

$$Q(t) := \frac{(\mathcal{F}_\gamma(W))^3}{(V(W))^2} \cdot \frac{(V(M_t))^2}{(\mathcal{F}_\gamma(M_t))^3} = \frac{(A(W))^3}{(V(W))^2} \cdot \frac{(V(M_t))^2}{(A(M_t))^3} = 108\sqrt{3} \cdot \frac{(V(M_t))^2}{(A(M_t))^3} = \frac{\left(1-4t^3\right)^2}{2\left(1-2t^2\right)^3}$$

and

$$\begin{aligned} Q'(t) &= \frac{1}{2} \cdot \frac{1}{(1-2t^2)^6} \cdot \left\{ -24t^2 \left(1-4t^3\right) \left(1-2t^2\right)^3 + 12t \left(1-4t^3\right)^2 \left(1-2t^2\right)^2 \right\} \\ &= 6 \cdot \frac{t(1-2t)\left(1-4t^3\right)}{(1-2t^2)^4} > 0, \quad \forall t \in \left(0, \frac{1}{2}\right) \end{aligned}$$

hold. If M is a local minimizer, $Q(t)$ must be a local maximum at $t = 0$. However, $Q(t)$ is strictly monotone increasing between $0 < t < 1/2$. Hence, $Q(t)$ is not a local maximum at $t = 0$. Therefore, M is not a local minimizer.

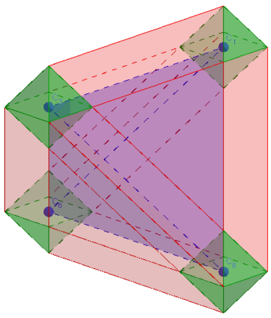


Figure 35: W: a regular octahedron, M: a regular tetrahedron, M_t: a truncated regular tetrahedron ($\tau = 1/4, t = 1/6$)

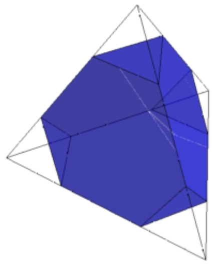


Figure 36: a truncated regular tetrahedron ($t = 0.3$)

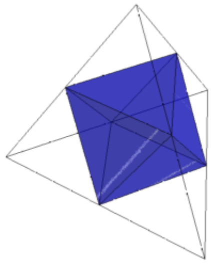


Figure 37: a truncated regular tetrahedron ($t = 1/2$)

4.8 Application to material science

A single crystal of CeO_2 usually forms regular octahedron. Only regular octahedra cannot fill the space. Inner structure of nanocrystals of CeO_2 in the water consists of regular octahedra and regular tetrahedra (Asahina, Takami, et al., 2011, [2]). If the energy density of CeO_2 is convex, from the main theorem, this regular tetrahedron is not a single crystal of CeO_2. Therefore, it is expected to be air or water. This observation matches the expectation by Professor Seiichi Takami (Graduate School of Engineering, Nagoya University).

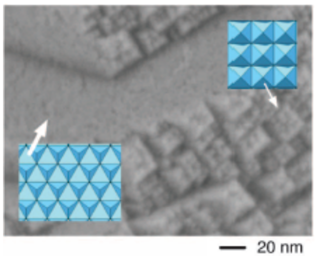


Figure 38: nanocrystals of CeO_2, (Asahina, Takami, et al., 2011, [2])

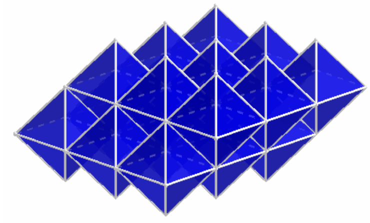


Figure 39: regular octahedron: a single crystal of CeO_2

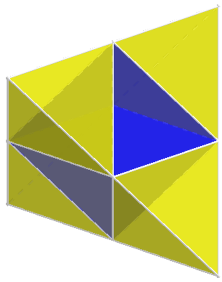


Figure 40: regular tetrahedron: air or water

4.9 Summary

We studied the crystalline variational problem that has as the solution mathematical model that express a shape of crystal. In closed surface that enclose same volume, when the energy minimizer is regular polyhedron, we showed that the energy local minimizer is the energy minimizer under the assumption that the energy density function is convex. That is, we showed that the uniqueness of a mathematical model of a single crystal.

We showed that the example of nanocrystals of CeO_2 in the water as application example to material science. It is considered that inner structure consists of CeO_2 of a regular octahedron and something of a regular tetrahedron. From main theorem, since this regular tetrahedron is not a single crystal of CeO_2, we obtained the mathematical conclusion that it is expected to be air or water.

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