

## ON INVARIANTS OF SURFACES IN THE 3-SPHERE

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# ON INVARIANTS OF SURFACES IN THE 3-SPHERE

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## Abstract

In this paper we study isotopy classes of closed connected orientable surfaces in the standard 3-sphere. Such a surface splits the 3-sphere into two compact connected submanifolds, and by using their Heegaard splittings, we obtain a 2-component handlebody-link. In this paper, we first show that the equivalence class of such a 2-component handlebody-link up to attaching trivial 1-handles can recover the original surface. Therefore, we can reduce the study of surfaces in the 3-sphere to that of 2-component handlebody-links up to stabilizations. Then, by using  $G$ -families of quandles, we construct invariants of 2-component handlebody-links up to attaching trivial 1-handles, which lead to invariants of surfaces in the 3-sphere. In order to see the effectiveness of our invariants, we will also show that our invariants can distinguish certain explicit surfaces in the 3-sphere.

## 1 Introduction

Throughout this paper, we work in the PL category. In this paper, we will study closed connected orientable surfaces embedded in the standard 3-sphere  $S^3$ . Such embedded surfaces were extensively studied by Fox, Homma, Tsukui, and Suzuki around 1950s–1970s (e.g., see [3, 5, 15, 18]). In their studies, 3-manifold theory and fundamental group techniques were mainly used.

A closed connected orientable surface in  $S^3$  splits  $S^3$  into two compact connected submanifolds of codimension 0 with common boundary. Then, by considering their Heegaard splittings, we obtain a pair of a handlebody and a compression body from each submanifold. Moreover, it is known as the Reidemeister–Singer theorem that any two Heegaard splittings of a compact connected orientable 3-manifold become equivalent after finitely many stabilizations. Hence, given a surface embedded in  $S^3$ , we can associate a certain 2-component handlebody-link up to stabilizations. Then, we show that an isotopy class of the original surface is uniquely determined from the equivalence class of an associated 2-component handlebody-link. Therefore, we can reduce the study of surfaces in  $S^3$  to that of 2-component handlebody links in a certain sense.

Handlebody-links (-knots) have recently been studied quite actively (e.g., see [6, 7, 8]). For example, by using a  $G$ -family of quandles, which is an algebraic system with binary operations parametrized by the elements of a group  $G$ , we can construct invariants of handlebody-links (see [7]).

One of the main purposes of the present paper is to construct invariants of surfaces embedded in  $S^3$  by means of the associated 2-component handlebody-links and  $G$ -families of quandles.

The present paper is organized as follows. In Section 2, we first recall the notion of compression bodies. Handlebodies can be regarded as a special class of compression bodies. We will also

recall the notions of Heegaard splittings of 3-manifolds and their stabilizations. Then, we review the Reidemeister–Singer theorem, which will be used in Section 3. We will show that for two surfaces in  $S^3$ , if the associated 2-component handlebody-links are equivalent up to attaching 1-handles, then two surfaces are isotopic.

In Section 3, we construct invariants of surfaces embedded in  $S^3$ . Plenty of invariants of handlebody-links were given in [7]. We first use  $X$ -colorings for the construction of surfaces in  $S^3$ . We observe a variation of the cardinality of the set of  $X$ -coloring due to a stabilization. Then, we give a technique which cancels the variation of the cardinality of the set of  $X$ -colorings after a stabilization of 2-component handlebody-link. Hence, we obtain an invariant of handlebody-links up to stabilizations. We consider such an invariant for each connected component of a 2-component handlebody-link. Then, as one of the results, we obtain an invariant as the unordered pair of rational numbers. Moreover, we will show that, by using a similar argument, we can construct another invariant of surfaces in  $S^3$  by using another quandle invariant which is defined by using cohomology theory of  $G$ -families of quandles [7].

In Section 4, we compute our invariants constructed in Theorem 3.1 for two explicit examples of surfaces in  $S^3$ . By using a  $G$ -family of quandles with a simple algebraic structure, we show that our invariant distinguish the two surfaces by a small amount of calculation.

In Section 5, we study surface in  $S^3$  from geometric viewpoints. We construct a geometric invariant of surfaces, which is an analogy of the unknotting tunnel number of a knot. Then, we study the relationship between a surface in  $S^3$  and the closures of the connected components of the exterior of the surface.

Throughout the paper, for a manifold  $M$ , we denote by  $\partial M$  and  $\text{int}(M)$  the boundary and the interior of  $M$ , respectively. For a subset  $N$  of  $M$ , denote by  $\text{cl}(N)$  the closure of  $N$  in  $M$ . Furthermore, for two sets  $M_1$  and  $M_2$ , we denote by  $M_1 \sqcup M_2$  the disjoint union of  $M_1$  and  $M_2$ . For a set  $Z$  we denote by  $\sharp Z$  the cardinality of  $Z$ . We also denote by  $\text{id}_Z$  the identity map of  $Z$ . For a group  $G$ , we denote by  $e_G$  the identity element of  $G$ . We denote by  $g(F)$  the genus of a closed connected orientable surface  $F$ . For a compact connected orientable surface  $F$  in a compact connected orientable 3-manifold  $M$ , we denote by  $N(F)$  a regular neighborhood of  $F$  in  $M$ . A multiset is a set whose elements allow duplications. A duplication of an element of a multiset is called the multiplicity of the element.

## 2 Preliminaries

In this section, we present definitions and known results that will be used when we construct invariants of embedded surfaces in  $S^3$ . The main theme of this paper is to study closed connected orientable surfaces in  $S^3$  as we mentioned.

Let  $F$  be a closed connected orientable surface embedded in  $S^3$ . We denote by  $V_F$  and  $W_F$  the closures of the connected components of  $S^3 \setminus F$ . Then, by considering Heegaard splittings of  $V_F$  and  $W_F$ , we will obtain a 2-component handlebody-link from the original surface  $F$ . Moreover, we study the relationship between a surface in  $S^3$  and the associated 2-component handlebody-link. We also recall definitions and related results of  $G$ -family of quandles. By the Alexander theorem ([1]), all 2-spheres in  $S^3$  are isotopic, then we consider surfaces of genus greater than 0.

## 2.1 3-manifolds and Heegaard splittings

In this section, we recall terminologies of 3-manifolds and Heegaard splittings. We also introduce several known results related to Heegaard splittings. Originally, Heegaard splittings were introduced for closed 3-manifolds. After that, the definition of a Heegaard splitting was extended to compact 3-manifolds, and such a splitting is called a *generalized Heegaard splitting* (Definition 2.6). We start from the definition of *compression bodies*.

**Definition 2.1** ([12, 13]). Let  $\Sigma$  be a closed connected orientable surface. Consider the product manifold  $\Sigma \times [0, 1]$ . Then, attach 2-handles along mutually disjoint simple closed curves on  $\Sigma \times \{0\}$ , and cap off every resulting 2-sphere component by a 3-handle. The resulting 3-manifold  $C$  is called a *compression body*. We denote  $\Sigma \times \{1\}$  by  $\partial_+ C$  and  $\partial C \setminus \partial_+ C$  by  $\partial_- C$ . If  $C$  is constructed without any 3-handle, then  $C$  can be obtained from  $\partial_- C \times [0, 1]$  by attaching mutually disjoint 1-handles on  $\partial_- C \times \{1\}$ . A compression body  $C$  is said to be a *handlebody* if  $\partial_- C = \emptyset$ . We define the *genus* of a handlebody as the genus of its boundary.

Let us recall *handlebody-knots* (-links) and their related terminologies. A *handlebody-knot* of genus  $g$  is a handlebody of genus  $g$  embedded in  $S^3$ . Two handlebody-knots are *equivalent* if there exists an ambient isotopy of  $S^3$  which maps one to the other. Let  $H_1, H_2, \dots, H_n$  be mutually disjoint handlebody-knots. Then,  $L := H_1 \sqcup H_2 \sqcup \dots \sqcup H_n$  is called a *handlebody-link*, or sometimes an *n-component handlebody-link*. Two handlebody-links are *equivalent* if there exists an ambient isotopy of  $S^3$  which maps one to the other. A handlebody-knot  $H$  (-link  $L$ ) is represented by a spatial trivalent graph  $K$  if a regular neighborhood of  $K$  is ambient isotopic to  $H$  ( $L$ ). Here, a spatial trivalent graph is a finite graph embedded in  $S^3$  or  $\mathbb{R}^3$  such that each vertex is of valence three. It is known that any handlebody-knot (-link) can be represented by a spatial trivalent graph (see [6]). A *diagram*  $D$  of a handlebody-knot  $H$  ( $L$ ) is a diagram of a spatial trivalent graph  $K$  of  $H$  ( $L$ ) obtained by projecting  $K$  to the 2-sphere  $S^2$  or the 2-plane  $\mathbb{R}^2$ . A diagram  $D$  of a handlebody-knot (-link) consists of arcs, which are parts of curves, and each of the endpoints of an arc are a vertex or an undercrossing. At each crossing of  $D$ , an arc is called an *under-arc* if one of the its endpoints is undercrossing. Otherwise, an arc is called an *over-arc*. An oriented diagram is a diagram of whose each of an arc is oriented.

Let  $M$  and  $N$  be a compact 3-manifold and a submanifold of  $M$ , respectively. Then,  $N$  is said to be *properly embedded* in  $M$  if  $\partial N \subset \partial M$  and  $\text{int}(N) \subset \text{int}(M)$ .

A properly embedded 2-disk  $D$  in  $M$  is said to be *inessential* if there exists a 2-disk  $D'$  in  $\partial M$  such that  $\partial D = \partial D'$  and  $D \cup D'$  is the boundary of a 3-ball in  $M$ . A properly embedded 2-disk  $D$  is said to be *essential* if  $D$  is not inessential. Moreover, let  $M$  be a compact connected 3-manifold. Then, an essential 2-disk  $D$  in  $M$  is said to be *separating* if  $M \setminus D$  consists of two connected components. Otherwise,  $D$  is said to be *non-separating*.

**Definition 2.2.** Let  $M$  be a compact connected orientable 3-manifold. Then,  $M$  is said to be *irreducible* if any 2-sphere  $S$  in  $M$  bounds a 3-ball in  $M$ .

We refer the reader to [2, 4] about the 3-manifold theory used in this paper.

**Definition 2.3** ([20]). Let  $M$  be a compact orientable 3-manifold. Let  $F$  be a compact orientable surface in  $M$  or in  $\partial M$ . Then,  $F$  is said to be *compressible* if  $F$  satisfies either of the following conditions.

- (1) There exists a properly embedded 2-disk  $D$  in  $M$  such that  $D \cap \text{int}(F) = \partial D$  and  $\partial D$  is non-contractible in  $\text{int}(F)$ .

(2) There exists a 3-ball  $B$  such that  $B \cap F = \partial B$ .

The surface  $F$  is said to be *incompressible* if  $F$  is not compressible.

The following theorem and lemma given in [2] and [20], respectively, will be used to see a relationship between the set of equivalence classes of 2-component handlebody-links up to attaching 1-handles and the set of isotopy classes of embedded surfaces.

**Theorem 2.4** ([2]). *Let  $M$  be an orientable irreducible 3-manifold. Let  $F$  be a compressible boundary component of  $M$ . Then, there exists an incompressible neighborhood  $V$  of  $F$ , which is unique up to isotopy of  $M$ .*

For the definition of an incompressible neighborhood of a compressible boundary component, the reader is referred to Definition 2.14.

**Lemma 2.5** ([20]). *Let  $M$  be a compact connected orientable 3-manifold. Let  $F$  be the union of mutually disjoint incompressible surfaces in  $M$ . We set  $M' = \text{cl}(M \setminus N(F))$ . Then  $M'$  is irreducible if and only if  $M$  is irreducible.*

We introduce the notion of Heegaard splittings of 3-manifolds. Originally, Heegaard splittings were introduced to represent a closed connected orientable 3-manifold by the union of two handlebodies along their common boundaries. In the context of knot theory, the Heegaard genus of a Heegaard splitting of the exterior  $E(K)$  of a knot  $K$  is related to the tunnel number of  $K$ .

**Definition 2.6** (Heegaard Splittings). Let  $M$  be a compact connected orientable 3-manifold possibly with boundary. Fix a partition of  $\partial M$  as  $\partial M = \partial_1 M \sqcup \partial_2 M$ . A *Heegaard splitting* of  $M$  is a decomposition of  $M$  into two compression bodies  $C_1$  and  $C_2$  such that  $M = C_1 \cup C_2$ ,  $\partial_+ C_1 = S = \partial_+ C_2$ ,  $C_1 \cap C_2 = S$ ,  $\partial_- C_1 = \partial_1 M$ , and  $\partial_- C_2 = \partial_2 M$ . We call  $\partial_+ C_1 = S = \partial_+ C_2$  a *Heegaard surface* of  $M$ . We denote by  $(M, S)$  a Heegaard splitting of  $M$  with a Heegaard surface  $S$ . We also denote by  $C_1 \cup_S C_2$  a Heegaard splitting of  $M$  consisting of two compression bodies  $C_1$  and  $C_2$  with a Heegaard surface  $S$ . The minimal genus of Heegaard surfaces of  $M$  is called the *Heegaard genus* of  $M$ . In particular, a Heegaard splitting of  $M$  with the minimal genus is called a *minimal genus Heegaard splitting* of  $M$ .

Concerning Heegaard splittings of 3-manifolds, it is known as Moise's theorem that every compact connected orientable 3-manifold possibly with boundary admits a Heegaard splitting ([10]).

We define a parallel arc in a compression body. Parallel arcs will be used when we define the stabilization of a Heegaard splitting of a 3-manifold.

**Definition 2.7** (Parallel arc). Let  $C$  be a compression body. A properly embedded arc  $\alpha$  in  $C$  with  $\partial C \cap \partial \alpha = \partial_+ C \cap \partial \alpha$  is *parallel* to an arc  $\beta$  in  $\partial_+ C$  with  $\partial \beta = \partial \alpha$  if there is an embedded disk  $D$  in  $C$  such that  $\partial D = \alpha \cup \beta$ .

**Definition 2.8** (Stabilization). Let  $M$  be a compact connected orientable 3-manifold possibly with boundary. Let  $(M, S)$  be a Heegaard splitting of  $M$  with a Heegaard surface  $S$ . The following procedure to construct a new Heegaard splitting  $(M, S')$  from  $(M, S)$  is called a *stabilization*. Suppose that  $M = C_1 \cup_S C_2$  is a Heegaard splitting of  $M$  consisting of two compression bodies  $C_1$  and  $C_2$  with  $\partial_+ C_1 = S = \partial_+ C_2$ . Take a parallel arc  $\alpha$  in  $C_2$  to an arc  $\beta$  in  $\partial_+ C_2$ . Then, we remove a tubular neighborhood  $N(\alpha)$  of  $\alpha$  from  $C_2$ , take the closure of  $C_2 \setminus N(\alpha)$ , and add  $N(\alpha)$  to  $C_1$ , namely, we have  $C'_1 := C_1 \cup N(\alpha)$  and  $C'_2 := \text{cl}(C_2 \setminus N(\alpha))$ . We can show that  $C'_1$  and  $C'_2$  are also compression bodies satisfying  $C'_1 \cup_{S'} C'_2 = M$ , where  $S' := \partial C'_1 = \partial C'_2$ .

Hence we obtained a new Heegaard splitting  $(M, S')$  of  $M$  from the given Heegaard splitting  $(M, S)$ . We also have  $g(S') = g(S) + 1$ .

We give an example of a Heegaard splitting of a handlebody of genus  $g$ .

**Example 2.9** (The trivial Heegaard splitting). Let  $H_g$  be a handlebody of genus  $g$ . Set  $\partial H_g = F$ . Consider a parallel surface  $F'$  to  $F$  inside of  $H_g$ . Then, the closures of the connected components of the exterior of  $F'$  consists of a handlebody of genus  $g$  and a compression body with the same boundaries. These handlebody and compression body give a Heegaard splitting of  $H_g$ . Such a splitting is called the *trivial splitting* of  $H_g$ .

Concerning Heegaard splittings of the handlebody of any genus  $g$ , it is known that all Heegaard splittings of a handlebody of genus  $g$  are standard, that is, they are obtained from the trivial Heegaard splitting by applying a finite number of stabilizations ([13]).

We introduce the notion of the equivalence of two Heegaard splittings of a compact connected orientable 3-manifold  $M$  possibly with boundary. Let  $(M, S)$  and  $(M, S')$  be Heegaard splittings of  $M$  with Heegaard surfaces  $S$  and  $S'$ . They are *equivalent* if there exists an ambient isotopy of  $M$  which maps  $S$  to  $S'$ . The following theorem, known as the Reidemeister–Singer theorem, plays an important role when we construct invariants of embedded surfaces in  $S^3$ .

**Theorem 2.10** (Reidemeister–Singer theorem [11, 14]). *Let  $M$  be a compact connected orientable 3-manifold possibly with boundary. We fix a partition of  $\partial M$  as  $\partial M = \partial_1 M \sqcup \partial_2 M$ . Then, any two Heegaard splittings of  $M$  become equivalent after finitely many stabilizations.*

The following is also known as Waldhausen’s theorem.

**Theorem 2.11** ([19]). *A Heegaard splitting of  $S^3$  is unique up to isotopy in every genus of a Heegaard surface.*

In Definition 2.8, we introduced the notion of the stabilization of a Heegaard splitting of a compact connected orientable 3-manifold. Then, a *stabilization* of handlebody-links. This definition is induced from the definition of the stabilization of a Heegaard splitting of a compact connected orientable 3-manifold with connected boundary. Two handlebody-links  $L_1$  and  $L_2$  are said to be *separated* if there exists 3-balls  $B_1^3$  and  $B_2^3$  in  $S^3$  such that  $B_1^3 \cap B_2^3 = \partial B_1^3 \cap \partial B_2^3$ ,  $L_1 \subset \text{int}(B_1^3)$ , and  $L_2 \subset \text{int}(B_2^3)$ . Similarly,  $n$  handlebody-links  $L_1, L_2, \dots, L_n$  are said to be *separated* if there exists 3-balls  $B_1^3, B_2^3, \dots, B_n^3$  in  $S^3$  such that  $B_i^3 \cap B_j^3 = \partial B_i^3 \cap \partial B_j^3$  (whenever  $i \neq j$ ) and  $L_1 \subset \text{int}(B_1^3), L_2 \subset \text{int}(B_2^3), \dots, L_n \subset \text{int}(B_n^3)$ .

**Definition 2.12** (Stabilization of handlebody-links). Let  $H$  be a handlebody-knot. Then, the *stabilization* of  $H$  is the disk sum of  $H$  and the standard solid torus  $T$ , denoted by  $H \natural T$ , by a 2-disk  $D$  on  $\partial H$  so that the connected components of  $H \natural T \setminus N(D)$  are separated by a 3-ball. Let  $L = H_1 \sqcup H_2$  be a 2-component handlebody-link. Let  $T^1$  and  $T^2$  be the standard solid tori. Then, the *stabilization* of  $L$  is stabilizations of  $H_1$  with  $T^1$  by a 2-disk  $D_1$  on  $\partial H_1$  or that of  $H_2$  with  $T^2$  by a 2-disk  $D_2$  on  $\partial H_2$  so that the connected components of  $\text{cl}(L \natural T^1 \natural T^2 \setminus \sqcup_{i=1}^2 N(D_i))$  are separated by 3-balls in  $S^3$ . Two handlebody-links are *stably equivalent* if they are equivalent after finitely many stabilizations.

We note that if both  $H_1$  and  $H_2$  are stabilized, then the attached solid tori are separated by some 3-balls in  $S^3$ .

## 2.2 Surfaces in $S^3$ and handlebody-links

Let us now study the relationship between embedded surfaces and 2-component handlebody-links. Let us start with a classification of embedded surfaces into three classes. We denote by  $V_F$  and  $W_F$  the closures of the connected components of the exterior of a given embedded surface  $F$ .

**Definition 2.13.** Let  $F$  be a closed connected orientable surface in  $S^3$ .

- (i) The surface  $F$  is said to be an *unknotted surface* if both  $V_F$  and  $W_F$  are homeomorphic to handlebodies.
- (ii) The surface  $F$  is said to be a *knotted surface* if exactly one of  $V_F$  or  $W_F$  is homeomorphic to a handlebody.
- (iii) The surface  $F$  is said to be a *bi-knotted surface* if neither  $V_F$  nor  $W_F$  is homeomorphic to a handlebody.

Let  $F_1$  and  $F_2$  be surfaces in  $S^3$ . Two surfaces  $F_1$  and  $F_2$  are said to be *isotopic*, denoted by  $F_1 \cong F_2$ , if there exists an isotopy  $f_t : S^3 \rightarrow S^3$ ,  $t \in [0, 1]$  such that  $f_0 = \text{id}_{S^3}$  and  $f_1(F_1) = F_2$ . We denote by  $V_i$  and  $W_i$  the closures of the connected components of  $S^3 \setminus F_i$ , ( $i = 1, 2$ ). Let  $V_i = H_{V_i} \cup C_{V_i}$  and  $W_i = H_{W_i} \cup C_{W_i}$  be Heegaard splittings of  $V_i$  and  $W_i$ , respectively, consisting of handlebodies  $H_{V_i}$ ,  $H_{W_i}$  and compression bodies  $C_{V_i}$ ,  $C_{W_i}$ . We call the 2-component handlebody link  $L_i := H_{V_i} \sqcup H_{W_i}$  an *associated 2-component handlebody-link* of  $F_i$ .

We now introduce the notion of *incompressible neighborhood*.

**Definition 2.14** (Incompressible neighborhood, [2]). Let  $M$  be an orientable irreducible 3-manifold. Let  $F$  be a compressible boundary component of  $M$ . Then, a 3-dimensional submanifold  $V$  of  $M$  is said to be an *incompressible neighborhood* of  $F$  if  $V$  satisfies the following conditions.

- (1) The 3-manifold  $V$  is a compact connected submanifold of  $M$  such that  $F \subset V \subset M$  and  $\partial V \setminus F \subset \text{int}(M)$ .
- (2) The 2-manifold  $\partial V \setminus F$  is incompressible in  $M$ .
- (3) For some  $x_0 \in F$ ,

$$\text{Image}(\pi_1(V, x_0) \rightarrow \pi_1(M, x_0)) = \text{Image}(\pi_1(F, x_0) \rightarrow \pi_1(M, x_0)).$$

We first give a necessary and sufficient condition of the existence of a closed connected orientable surface in  $S^3$  corresponding to a given 2-component handlebody-link. We use Kneser's theorem and a characterization of 3-manifolds (with connected boundary) whose fundamental groups are free. We also use Lemma 2.5, then we have the following.

**Proposition 2.15.** Let  $L = H_1 \sqcup H_2$  be a 2-component handlebody-link. We denote by  $E(L) := S^3 \setminus \text{int}(L)$  the exterior of  $L$ . Then, there exists a closed connected orientable surface  $F$  in  $S^3$  such that  $\partial H_1$  and  $\partial H_2$  are Heegaard surfaces of the connected components the exterior of  $F$  if and only if the fundamental group  $\pi_1(E(L))$  is given by free products of the fundamental group of a closed connected orientable surface and some infinite cyclic groups.

Generally, since a Heegaard splitting of a compact connected orientable 3-manifold is not unique, then a 2-component handlebody-link associated to a closed connected orientable surface in  $S^3$  is not unique. However, by using the Reidemeister–Singer theorem, we have the following.

**Proposition 2.16.** *Let  $F$  be a closed connected orientable surface in  $S^3$  and  $L$  be an associated 2-component handlebody-link of  $F$ , respectively. Then,  $L$  is unique up to stabilizations of Heegaard splittings of the closures of the connected components of  $S^3 \setminus F$ .*

**Proposition 2.17.** *Let  $F_1$  and  $F_2$  be closed connected orientable surfaces in  $S^3$ . We denote by  $V_i$  and  $W_i$  the closures of connected components of  $S^3 \setminus F_i$  ( $i = 1, 2$ ). Let  $V_i = H_{V_i} \cup C_{V_i}$  and  $W_i = H_{W_i} \cup C_{W_i}$  be Heegaard splittings of  $V_i$  and  $W_i$  ( $i = 1, 2$ ), respectively, where  $H_{V_i}$  and  $H_{W_i}$  are handlebodies and  $C_{V_i}$  and  $C_{W_i}$  are compression bodies. Let  $L_1 = H_{V_1} \sqcup H_{W_1}$  and  $L_2 = H_{V_2} \sqcup H_{W_2}$  be associated 2-component handlebody-links of  $F_1$  and  $F_2$ , respectively. If  $L_1$  and  $L_2$  are stably equivalent, then  $F_1$  and  $F_2$  are isotopic.*

*Proof.* We assume that  $H_{V_1}$  is mapped to  $H_{V_2}$  and  $H_{W_1}$  is mapped to  $H_{W_2}$  by an isotopy of  $S^3$ . We denote by  $E(L_1)$  the exterior of  $L_1$ . Then, by Lemma 2.5, we can show that  $E(L_1)$  is irreducible. If both  $C_{V_1}$  and  $C_{V_2}$  admit 1-handles, then we can show that  $\partial H_{V_1}$  and  $\partial H_{V_2}$  are compressible in  $E(L_1)$ . Then, we can assume that  $C_{V_1}$  and  $C_{V_2}$  are incompressible neighborhoods of  $\partial H_{V_1}$  and  $\partial H_{V_2}$ . By Theorem 2.4,  $C_{V_1}$  and  $C_{V_2}$  are unique up to ambient isotopy of  $E(L_1)$ . Therefore, by the use of isotopy extension theorem, we can show that  $F_1$  and  $F_2$  are isotopic. If exactly one of  $C_{V_1}$  or  $C_{V_2}$  admits 1-handles, let us suppose that  $C_{V_1}$  admits 1-handles. Then, by using similar argument, we can show that  $C_{V_1}$  is an incompressible neighborhood of  $\partial H_{V_1}$  and unique. On the other hand, since  $C_{V_2}$  does not admit any 1-handle, then by the triviality of a Heegaard splitting of a handlebody,  $C_{V_2}$  is in the form  $\partial H_{V_2} \times [0, 1]$ . Then we have the equivalence of  $F_1$  and  $F_2$ . If neither  $C_{V_1}$  nor  $C_{V_2}$  admits any 1-handle, then by using Waldhausen's theorem (Theorem 2.11), we can show that  $F_1$  and  $F_2$  are isotopic.  $\square$

Proposition 2.17 implies that an isotopy class of a surface in  $S^3$  is uniquely determined from the stably equivalence class of an associated 2-component handlebody-link of the surface.

### 2.3 $G$ -family of quandles and handlebody-links

Let us now go on to introducing the notions of quandles and a  $G$ -families of quandles. A lot of invariants of links have been obtained by using quandles. Moreover,  $G$ -families of quandles can be used for studying handlebody-links and gives plenty of invariants (refer to [7]). We start by the definition of quandles. We refer the reader [7] about the theory of  $G$ -families of quandles used in the paper.

**Definition 2.18.** Let  $X$  be a non-empty set with a binary operation  $*$  :  $X \times X \rightarrow X$ . The pair  $(X, *)$  is a *quandle* if  $*$  satisfies the following conditions.

- (i)  $x * x = x$  for any  $x \in X$ .
- (ii) The map  $S_x : X \rightarrow X$  defined by  $S_x(y) = y * x$  is bijective for any  $x \in X$ .
- (iii)  $(x * y) * z = (x * z) * (y * z)$  for any  $x, y, z \in X$ .

**Definition 2.19.** Let  $G$  be a group and  $X$  be a non-empty set with a family of binary operations  $*_g : X \times X \rightarrow X$  parametrized by  $g \in G$ , respectively. The pair  $(X, \{*_g\}_{g \in G})$  is a  *$G$ -family of quandles* if for any  $x, y, z \in X$  and any  $g, h \in G$ ,  $*_g$  satisfies the following conditions (see [7]).



- (i)  $x *_g x = x$ .
- (ii)  $x *_g y = (x *_g y) *_h y$  and  $x *_g y = x$ .
- (iii)  $(x *_g y) *_h z = (x *_h z) *_g y (y *_h z)$ .

We note that for a  $G$ -family of quandles, the pair  $(X, *_g)$  is a quandle for each  $g \in G$ .

**Definition 2.20.** Let  $G$  and  $(X, \{*_g\}_{g \in G})$  be a group and a  $G$ -family of quandles, respectively. Set  $Q = X \times G$ . We define the binary operation  $*$  :  $Q \times Q \rightarrow Q$  by  $(x, g) * (y, h) = (x *_h y, h^{-1}gh)$ . Then, the pair  $(Q, *)$  is a quandle called the *associated quandle* of  $X$ .

Using a  $G$ -family of quandles  $(X, \{*_g\}_{g \in G})$ , we can introduce the notion of an  $X$ -coloring for an oriented diagram  $D$  of a handlebody-link. We denote by  $\mathcal{A}(D)$  the set of arcs of  $D$ . The normal orientation of an oriented arc is given by rotating an orientation of the arc counterclockwise by  $\pi/2$ . The normal orientation of an oriented arc is represented by an arrow on the arc (see Figure 1). Using a  $G$ -family of quandles  $(X, \{*_g\}_{g \in G})$ , we can introduce the notion of an  $X$ -coloring for an oriented diagram  $D$  of a handlebody-link.

**Definition 2.21** ( $X$ -colorings). Let  $G$  be a group and  $(X, \{*_g\}_{g \in G})$  a  $G$ -family of quandles, respectively. Let  $D$  be an oriented diagram of a handlebody-link. A map  $C : \mathcal{A}(D) \rightarrow Q = X \times G$  is an  $X$ -coloring of  $D$  if  $C$  satisfies the following conditions:

- (i) at each crossing  $\chi$  of  $D$ , the map  $C$  satisfies  $C(\chi_2) = C(\chi_1) * C(\chi_3)$ , and
- (ii) at each vertex  $\omega$  of  $D$  and for the two natural projections  $p_X : Q \rightarrow X$  and  $p_G : Q \rightarrow G$ , the map  $C$  satisfies

$$\begin{cases} p_X \circ C(\alpha_1) = p_X \circ C(\alpha_2) = p_X \circ C(\alpha_3), \\ (p_G \circ C(\alpha_1))^{\varepsilon(\omega, \alpha_1)} \cdot (p_G \circ C(\alpha_2))^{\varepsilon(\omega, \alpha_2)} \cdot (p_G \circ C(\alpha_3))^{\varepsilon(\omega, \alpha_3)} = e_G, \end{cases}$$

where  $\chi_1, \chi_2$  are under arcs and  $\chi_3$  is an over-arc of  $D$  (Figure 1). Furthermore,  $\varepsilon(\omega, \alpha_i)$  is the sign of an arc  $\alpha_i$  at  $\omega$  which is defined as follows (see Figure 2) :

$$\varepsilon(\omega, \alpha_i) = \begin{cases} 1, & \text{if the orientation of } \alpha_i \text{ points into } \omega, \\ -1, & \text{otherwise.} \end{cases}$$

We denote by  $\text{Col}_X(D)$  the set of  $X$ -colorings of  $D$ .

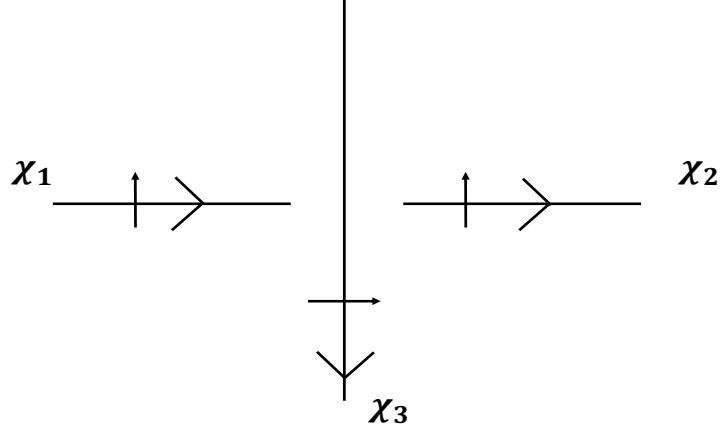


Figure 1: Crossing  $\chi$  and normal orientations of  $\chi_1$ ,  $\chi_2$ , and  $\chi_3$

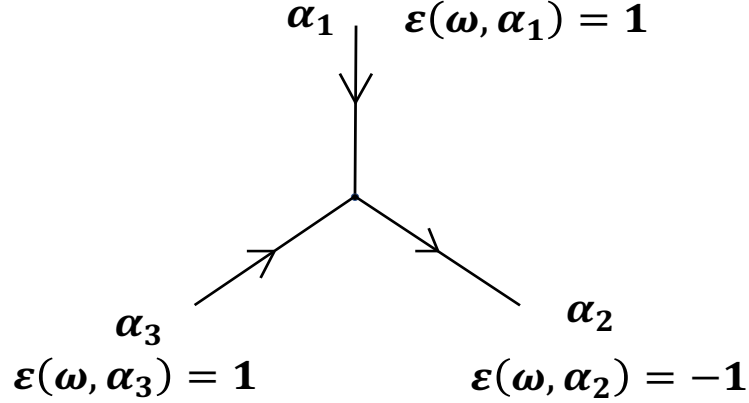


Figure 2: Vertex  $\omega$

**Theorem 2.22.** ([7]) *Let  $G$  be a group and  $(X, \{\ast_g\}_{g \in G})$  be a  $G$ -family of quandles, respectively. Let  $D$  be a diagram of an oriented spatial trivalent graph of a handlebody-link. Then, the cardinality  $\sharp \text{Col}_X(D)$  is an invariant of the handlebody-link.*

Let  $D$  be an oriented diagram of a handlebody-knot  $H$ . Let  $H'$  be a handlebody-knot obtained from the disk sum of  $H$  and the standard solid torus. Then, a diagram  $D'$  of  $H'$  is obtained from  $D$  by attaching an edge and a circle component to an arc of  $D$ . Therefore, an orientation of  $D'$  is induced from that of  $D$  except for the edge and the circle component.

The following lemma will be used in Section 3 for constructing invariants of embedded surfaces.

**Lemma 2.23.** *Let  $G$  and  $X$  be a finite group and a finite set, respectively, and  $(X, \{*_g\}_{g \in G})$  be a  $G$ -family of quandles. Let  $F$  be a closed connected orientable surface in  $S^3$ . We denote by  $V_F$  and  $W_F$  the closures of the connected components of  $S^3 \setminus F$ . Let  $(V_F, F_V)$  be a Heegaard splitting of  $V_F$ . Let  $(V_F, F'_V)$  be a Heegaard splitting obtained from  $(V_F, F_V)$  by applying a stabilization. Let  $H_{F_V}$  and  $H_{F'_V}$  be handlebody-knots obtained from the Heegaard splittings  $(V_F, F_V)$  and  $(V_F, F'_V)$ , respectively. Let  $D_{F_V}$  be an oriented diagram of  $H_{F_V}$  and  $D_{F'_V}$  be a diagram of  $H_{F'_V}$  with an orientation induced from that of  $D_{F_V}$ . Then, we have  $\sharp \text{Col}_X(D_{F'_V}) = \sharp \text{Col}_X(D_{F_V}) \cdot \sharp G$ .*

*Proof.* By the definition of a stabilization and by using an isotopy  $S^3$ , we may assume that the diagram  $D_{F'_V}$  is obtained from  $D_{F_V}$  by attaching an edge  $e_0$  and an  $S^1$  component  $\beta$  to an arc  $\alpha_0$  of  $D_{F_V}$  so that  $e_0$  and  $\beta$  do not admit crossings. Then, we give arbitrary orientations to  $e_0$  and  $\beta$ . For the other arcs, orientations are induced from that of  $D_{F_V}$  as shown in Figure 3. Let  $C$  be an  $X$ -coloring of  $D_{F_V}$ . Suppose that  $C(\alpha_0) = (x, g) \in X \times G$ . Then,  $C$  is extended to an  $X$ -coloring of  $D_{F'_V}$  by defining  $C(e_0) = (x, e_G)$  and  $C(\beta) = (x, h)$ . Hence, we obtain  $\sharp \text{Col}_X(D_{F'_V}) \geq \sharp \text{Col}_X(D_{F_V}) \cdot \sharp G$ . Conversely, for any  $C(\beta) = (x, h)$  we have  $C(e_0) = (x, e_G)$  by the axiom of an  $X$ -coloring. Then we have  $\sharp \text{Col}_X(D_{F'_V}) = \sharp \text{Col}_X(D_{F_V}) \cdot \sharp G$ . The same holds for those of  $W_F$ .  $\square$

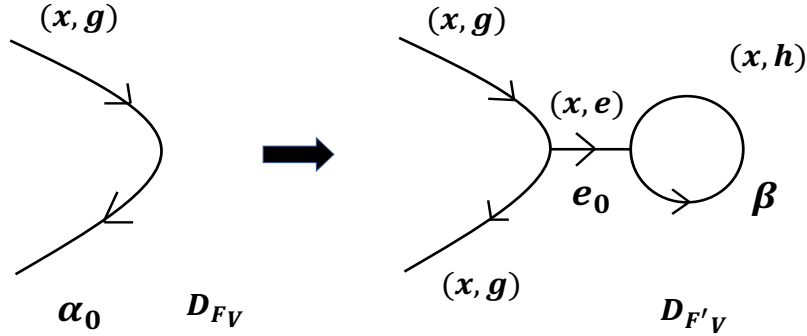


Figure 3: Attaching an edge  $e_0$  and a circle component  $\beta$  to  $D_{F_V}$

Let us introduce the notions of an  $X$ -set  $Y$  and an  $X_Y$ -coloring of a diagram of a handlebody-link (refer to [7]).

**Definition 2.24.** Let  $(X, \{*_g\}_{g \in G})$  be a  $G$ -family of quandles, and let  $Y$  be a non-empty set with a family of maps  $\bar{*}_g : Y \times X \rightarrow Y$  parametrized by  $g \in G$ . The pair  $(Y, \{\bar{*}_g\}_{g \in G})$  is called an  $X$ -set if for any  $y \in Y$ ,  $x, x' \in X$ , and any  $g, h \in G$ , the following conditions are satisfied.

- (i)  $y \bar{*}_{gh} x = (y \bar{*}_g x) \bar{*}_h x$  and  $y \bar{*}_{e_G} x = y$ .
- (ii)  $(y \bar{*}_g x) \bar{*}_h x' = (y \bar{*}_h x') \bar{*}_{h^{-1}gh} (x \bar{*}_h x')$ .

Let  $D$  be an oriented diagram of a handlebody-link. Let  $D'$  be a diagram obtained from  $D$  by connecting undercrossing arcs at each crossing of  $D$ . Then  $D'$  admits no crossing. We call

a connected component of  $\mathbb{R}^2 \setminus D'$  a *complementary region* of  $D$ . We denote by  $\mathcal{R}(D)$  the set of complementary regions of  $D$ . We set  $y * (x, g) = y \bar{*}_g x$  for  $y \in Y$  and  $(x, g) \in X \times G = Q$ . Then, we introduce the notion of  $X_Y$ -colorings.

**Definition 2.25** ( $X_Y$ -coloring). Let  $G$  be a group, and  $(X, \{*_g\}_{g \in G})$  and  $(Y, \{\bar{*}_g\}_{g \in G})$  be a  $G$ -family of quandles and an  $X$ -set, respectively. Let  $D$  be an oriented diagram of a handlebody-link. An  $X_Y$ -coloring of  $D$  is a map  $C : \mathcal{A}(D) \cup \mathcal{R}(D) \rightarrow Q \cup Y$  satisfying the following conditions. We denote by  $\text{Col}_X(D)_Y$  the set of  $X_Y$ -colorings of the oriented diagram  $D$ .

- C1.  $C(\mathcal{A}(D)) \subset Q$  and  $C(\mathcal{R}(D)) \subset Y$ .
- C2. The restriction of  $C$  on  $\mathcal{A}(D)$  is an  $X$ -coloring of  $D$ .
- C3. For an over-arc  $\alpha$  and adjacent complementary regions  $\alpha_1$  and  $\alpha_2$ ,  $C$  satisfies  $C(\alpha_2) = C(\alpha_1) * C(\alpha)$ , where the normal orientation of  $\alpha$  points from  $\alpha_1$  to  $\alpha_2$  (see Figure 4).

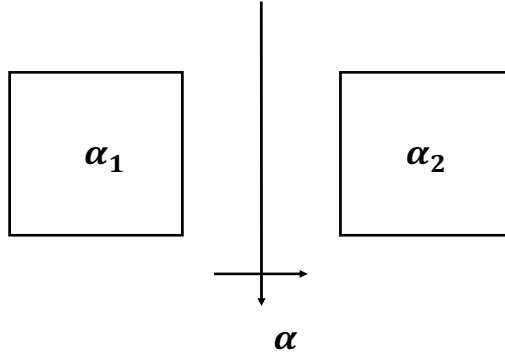


Figure 4: The coloring condition C3

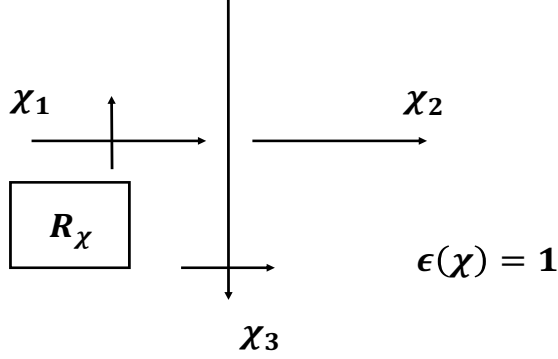


Figure 5: Weight of a crossing  $\chi$

Using a  $G$ -family of quandles and an Abelian group  $A$ , we define a chain complex, denoted by  $C_*(X)_Y$ , and the cochain complex, denoted by  $C^*(X; A)_Y := \text{Hom}(C_*(X)_Y, A)$ . Then, we also define the associated homology groups and cohomology groups (refer to [7]). For an  $X_Y$ -coloring  $C$  and a crossing  $\chi$  of a diagram  $D$  of an oriented spatial trivalent graph of a handlebody-link, we define the weight of the crossing  $\chi$  by  $w(\chi; C) := \epsilon(\chi)(C(R_\chi), C(\chi_1), C(\chi_3))$ , where  $\chi_1$  is an under-arc such that the orientation of  $\chi_1$  points into the crossing  $\chi$ ,  $\chi_3$  is an over-arc whose normal orientation points from  $\chi_1$  to the other arc  $\chi_2$ ,  $R_\chi$  is a complementary region such that normal orientations of  $\chi_1$  and  $\chi_3$  point to the opposite regions with respect to  $\chi_1$  and  $\chi_3$  (Figure 5), and  $\epsilon(\chi)$  is the sign of the crossing  $\chi$ .

Concerning homology theory of  $G$ -families of quandles, the following lemma is known (see [7]).

**Lemma 2.26.** *Let  $(X, \{*_g\}_{g \in G})$  and  $(Y, \{\bar{*}_g\}_{g \in G})$  be a  $G$ -family of quandles and an  $X$ -set, respectively. Let  $D$  be a diagram of an oriented spatial trivalent graph of a handlebody-link. Let  $C$  be an  $X_Y$ -coloring of  $D$ . Then, the sum of the weights  $W(D; C) := \sum_{\chi \in D} w(\chi; C)$  is a 2-cycle of  $C_*(X)_Y$ .*

Let  $A$  be an Abelian group. Let  $\theta$  be a 2-cocycle of the cochain complex  $C^*(X; A)_Y$ . We define the multiset  $\Phi_\theta(D)$  as follows:

$$\Phi_\theta(D) := \{\theta(W(D; C)) \in A \mid C \in \text{Col}_X(D)_Y\}.$$

Concerning cohomology theory of  $G$ -families of quandles, the following theorem is also known (see [7]).

**Theorem 2.27.** *Let  $(X, \{*_g\}_{g \in G})$  and  $(Y, \{\bar{*}_g\}_{g \in G})$  be a  $G$ -family of quandles and an  $X$ -set, respectively. Let  $H$  be a handlebody-link and  $D$  be a diagram of an oriented spatial trivalent graph of  $H$ . Let  $A$  and  $\theta$  be an Abelian group and a 2-cocycle of the cochain complex  $C^*(X; A)_Y$ , respectively. Then,  $\Phi_\theta(D)$  does not depend on the choice of  $D$  and is an invariant of the handlebody-link  $H$ .*

We can write the multiset  $\Phi_\theta(D)$  in the form

$$\Phi_\theta(D) = \{(a_1)_{l_1}, \dots, (a_m)_{l_m}\},$$

where  $l_j$  is the multiplicity of  $a_j \in A$ , and  $(a_j)_{l_j}$  represents  $\underbrace{a_j, \dots, a_j}_{l_j\text{-times}}$ . Using these notations and a natural number  $N$ , we define the set  $\Phi_\theta(D)_N$  as follows:

$$\Phi_\theta(D)_N := \{(a_1, l_1/N), \dots, (a_m, l_m/N) \mid (a_i, l_i/N) \in A \times \mathbb{Q}\}.$$

### 3 Main results

In this section, we construct algebraic invariants of embedded surfaces in  $S^3$ . We first use  $X$ -colorings combined with Lemma 2.23. Then, we will also use another quandle invariant given in Theorem 2.27.

**Theorem 3.1.** *Let  $G$  be a finite group and  $(X, \{*_g\}_{g \in G})$  a  $G$ -family of quandles, where  $X$  is a finite set. Let  $F$  be a closed connected orientable surface in  $S^3$ . We denote by  $V_F$  and  $W_F$  the connected components of the exterior of  $F$ . Let  $F_V$  and  $F_W$  be Heegaard surfaces of  $V_F$  and  $W_F$ , and  $H_V$  and  $H_W$  the corresponding handlebodies, respectively. We denote by  $D_V$  and  $D_W$  diagrams of oriented spatial trivalent graphs representing  $H_V$  and  $H_W$ , respectively. Then, the unordered pair*

$$\left( \frac{\#\text{Col}_X(D_V)}{(\#G)^{g(F_V)}}, \frac{\#\text{Col}_X(D_W)}{(\#G)^{g(F_W)}} \right)$$

*of rational numbers is an isotopy invariant of  $F$ .*

*Proof.* Let  $(V_F, F_V)$  and  $(V_F, F'_V)$  be Heegaard splittings of  $V_F$ . Then, we have the corresponding pairs of a handlebody-knot and a compression body in  $S^3$ , say  $(H_V, C_V)$  and  $(H'_V, C'_V)$ , respectively. By the Reidemeister–Singer theorem, the two Heegaard splittings  $(V_F, F_V)$  and  $(V_F, F'_V)$  become equivalent to the same Heegaard splitting  $(V_F, \bar{F}_V)$  after  $m$  times and  $n$  times stabilizations, respectively, for some non-negative integers  $m$  and  $n$ . Let  $\bar{H}$  be the handlebody-knot and  $\bar{C}_V$  be the compression body corresponding to  $(V_F, \bar{F}_V)$ . Let  $\bar{D}_V$  be a diagram of an oriented spatial trivalent graph representing  $\bar{H}_V$ . Then, by Lemma 2.23, we observed a variation of the cardinality of the set of  $X$ -coloring after applying a stabilization. Then, we have  $\#\text{Col}_X(\bar{D}_V) = \#\text{Col}_X(D_V) \cdot (\#G)^m = \#\text{Col}_X(D'_V) \cdot (\#G)^n$ . On the other hand, we have  $g(\bar{F}_V) = g(F_V) + m = g(F'_V) + n$ . Hence we see

$$\left( \frac{\#\text{Col}_X(\bar{D}_V)}{(\#G)^{g(\bar{F}_V)}} \right) = \left( \frac{\#\text{Col}_X(D_V)}{(\#G)^{g(F_V)}} \right) = \left( \frac{\#\text{Col}_X(D'_V)}{(\#G)^{g(F'_V)}} \right).$$

Combining Theorem 2.22, we have the desired conclusion.  $\square$

In Theorem 3.1, we used each connected component of a 2-component handlebody-link independently. Then we constructed an invariant of handlebody-knots up to stabilizations. However, we can show the following even if we use an invariant of 2-component handlebody-links up to stabilizations.

**Theorem 3.2.** *Let  $G$  be a finite group and  $(X, \{*_g\}_{g \in G})$  a  $G$ -family of quandles, where  $X$  is a finite set. Let  $F$  be a closed connected orientable surface in  $S^3$ . We denote by  $V_F$  and  $W_F$  the connected components of the exterior of  $F$ . Let  $F_V$  and  $F_W$  be Heegaard surfaces of  $V_F$  and  $W_F$ , and  $H_V$  and  $H_W$  the corresponding handlebodies, respectively. Let  $L = H_V \sqcup H_W$  be the 2-component handlebody-link. We denote by  $D$  a diagram of an oriented spatial trivalent graph representing  $L$ . Then, the rational number*

$$\frac{\# \text{Col}_X(D)}{(\#G)^{g(F_V)+g(F_W)}}$$

*is an isotopy invariant of  $F$ .*

By using an argument similar to that used in Theorem 3.1, we can also construct another algebraic invariant of embedded surfaces in  $S^3$ . In order to construct such an invariant, we use Theorem 2.22 and 2.27.

**Theorem 3.3.** *Let  $G$  be a finite group and  $(X, \{*_g\}_{g \in G})$  a  $G$ -family of quandles, where  $X$  is a finite set. Let  $Y$  be an  $X$ -set, where  $Y$  is a finite set. Let  $F$  be a closed connected orientable surface in  $S^3$ . We denote by  $V_F$  and  $W_F$  the connected components of the exterior of  $F$ . Let  $F_V$  and  $F_W$  be Heegaard surfaces of  $V_F$  and  $W_F$ , and  $H_V$  and  $H_W$  the corresponding handlebodies, respectively. Let  $L = H_V \sqcup H_W$  be a 2-component handlebody-link. We denote by  $D$  a diagram of an oriented spatial trivalent graph representing  $L$ . Then, the multiset  $\Phi_\theta(D)_{(\#G)^{g(F_V)+g(F_W)}}$  is an isotopy invariant of  $F$ .*

*Proof.* Let  $L'$  be a 2-component handlebody-link obtained from  $L$  by applying a stabilization, and we denote by  $D'$  a diagram of an oriented spatial trivalent graph representing  $L'$ . By the definition of a stabilization of handlebody-links, attached solid tori are separated, then the condition of crossings of  $D'$  is same as that of  $D$ . Moreover, by using an argument similar to that used in the proof of Lemma 2.23, we can show that  $\# \text{Col}_X(D')_Y = \# \text{Col}_X(D)_Y \cdot \#G$ . Then, we also have an invariance of the weight of each crossing. Therefore, we obtain the required conclusion.  $\square$

For a 2-component handlebody-link, its linking number is an invariant (refer to [9]). We have a 2-component handlebody-link from a closed connected orientable surface in  $S^3$  by considering Heegaard splittings of the connected components of the exterior of the surface. By the definition of the stabilization of handlebody-links, we can show that linking number does not change after applying a stabilization. Then, the linking number of a 2-component handlebody-link obtained from  $F$  is also an isotopy invariant of  $F$ .

## 4 Examples

We compute our invariants given in Theorem 3.1 for the following examples of bi-knotted surfaces. Throughout this section, we set  $X = \mathbb{Z}/3\mathbb{Z}$ ,  $G = \mathbb{Z}/2\mathbb{Z}$ ,  $g *_0 h := g$ ,  $g *_1 h = 2h - g$  for any  $g, h \in \mathbb{Z}/3\mathbb{Z}$  and  $0, 1 \in \mathbb{Z}/2\mathbb{Z}$ . Then, it is known that  $(X, \{*_g\}_{g \in G})$  is a  $G$ -family of quandles ([7]).

In order to construct bi-knotted surfaces, we consider two mutually disjoint 3-balls  $B_1$  and  $B_2$ , respectively. Then we take properly embedded arcs  $\alpha_1$  and  $\alpha_2$  from  $B_1$  and  $B_2$ , respectively. We assume that  $\alpha_1$  is unknotted and  $\alpha_2$  is knotted corresponding to the trefoil. We denote by  $N(\alpha_1)$  and  $N(\alpha_2)$  regular neighborhoods of  $\alpha_1$  and  $\alpha_2$ . Then we remove regular neighborhoods

$N(\alpha_1)$  and  $N(\alpha_2)$  from  $B_1$  and  $B_2$ . We set  $B'_1 = \text{cl}(B_1 \setminus N(\alpha_1))$  and  $B'_2 = \text{cl}(B_2 \setminus N(\alpha_2))$ . Then we connect  $B'_1$  and  $B'_2$  by the 1-handle  $D^1 \times D^2$  so that  $\{-1\} \times D^2$  is attached to  $\partial B'_1$ ,  $\{1\} \times D^2$  is attached to  $\partial B'_2$ , and the attached 1-handle throughs straightforwardly both regions which are obtained from  $B_1$  and  $B_2$  by removing  $N(\alpha_1)$  and  $N(\alpha_2)$ . We assume that the attached 1-handle admits a tangle induced from that of  $N(\alpha_2)$ . We set  $F$  as the boundary of the the resulting compact connected orientable 3-manifold  $M$  (see Figure 6). Similarly, we construct compact connected orientable 3-manifold  $M'$  by using an unknotted arc and a knotted arc corresponding to the figure eight knot. Then we set  $F'$  as the boundary of the 3-manifold (see Figure 7).

We assume that  $M$  is embedded in  $S^3$ . Then,  $M$  is one of the connected components of the exterior of  $F = \partial M$ , say,  $V_F$  (see Figure 8). Then, we obtain the other connected components of the exterior of  $F$  by the following procedures. We first consider the closure of the exterior of  $B'_1$  in  $S^3$ , which is obtained from a 3-ball  $B^3$  by attaching  $N(\alpha_1)$ . Then, we remove the interior of  $B_2$  from  $B^3$  and attach  $N(\alpha_2)$  to  $\partial B_2$ . Finally, we remove the 2-handle  $D^2 \times D^1$  so that  $D^2 \times \{-1\}$  is attached to  $\partial B^3$ ,  $D^2 \times \{1\}$  is attached to  $\partial B_2$ , and the 2-handle throughs both  $N(\alpha_1)$  and  $N(\alpha_2)$  straightforwardly. The resulting compact connected orientable 3-manifold, say  $W_F$ , is the other connected component of  $F$  (see Figure 9). We set  $M' = V'_F$  (see Figure 14). Then, by a similar argument, we obtain the other connected component  $W'_F$  of  $F'$  (see Figure 15).

Moreover, by removing 2-handles from  $V_F$  and from  $W_F$ , respectively, we obtain two handlebody-knots  $H_V$  and  $H_W$  (Figures 10 and 11). Then, by projecting their oriented spatial trivalent graphs on  $\mathbb{R}^2$ , we obtain diagrams  $D_V$  and  $D_W$  of  $H_V$  and  $H_W$  (Figure 12 and Figure 13). By using a similar argument, we obtain handlebody-knots  $H'_F$  and  $H'_W$  (Figures 16 and 17). Similarly, by projecting their oriented spatial trivalent graphs on  $\mathbb{R}^2$ , we obtain diagrams  $D'_V$  and  $D'_W$  of  $H'_F$  and  $H'_W$  (Figures 18 and 19).

**Example 4.1.** For the surface  $F$  (Figure 6), we compute  $\sharp\text{Col}_X(D_V)$  and  $\sharp\text{Col}_X(D_W)$ . Then we have  $\sharp\text{Col}_X(D_V) = 6$  and  $\sharp\text{Col}_X(D_W) = 48$ . Therefore we obtain

$$\frac{\sharp\text{Col}_X(D_V)}{(\sharp(\mathbb{Z}/2\mathbb{Z}))^3} = \frac{3}{4}, \quad \frac{\sharp\text{Col}_X(D_W)}{(\sharp(\mathbb{Z}/2\mathbb{Z}))^3} = 6.$$

**Example 4.2.** Let us now focus on the surface  $F'$  (Figure 7). We also compute  $\sharp\text{Col}_X(D'_V)$  and  $\sharp\text{Col}_X(D'_W)$ . Then we have  $\sharp\text{Col}_X(D'_V) = 6$  and  $\sharp\text{Col}_X(D'_W) = 24$ . Hence we have

$$\frac{\sharp\text{Col}_X(D_{V'})}{(\sharp(\mathbb{Z}/2\mathbb{Z}))^3} = \frac{3}{4}, \quad \frac{\sharp\text{Col}_X(D_{W'})}{(\sharp(\mathbb{Z}/2\mathbb{Z}))^3} = 3.$$

We can show that two surfaces  $F$  and  $F'$  are not isotopic by using our invariant.



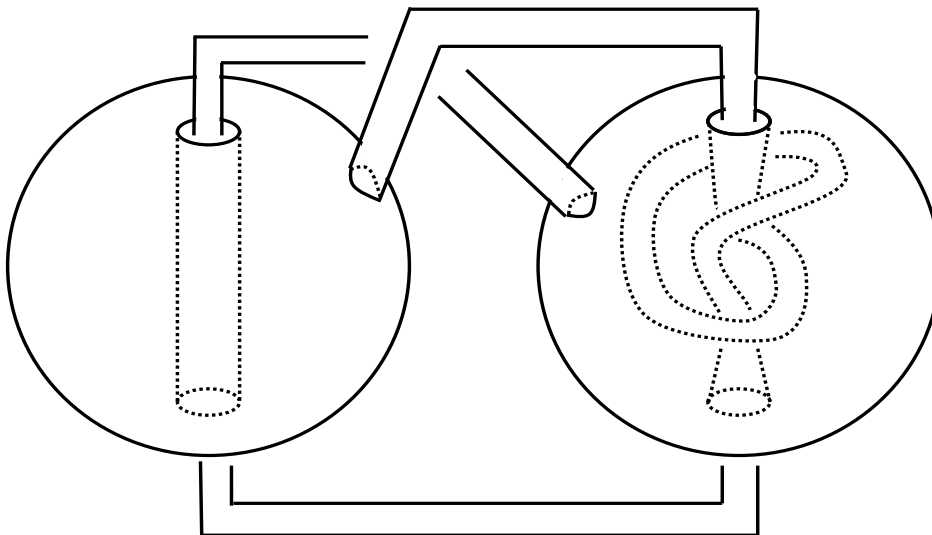


Figure 6: Bi-knotted surface  $F$

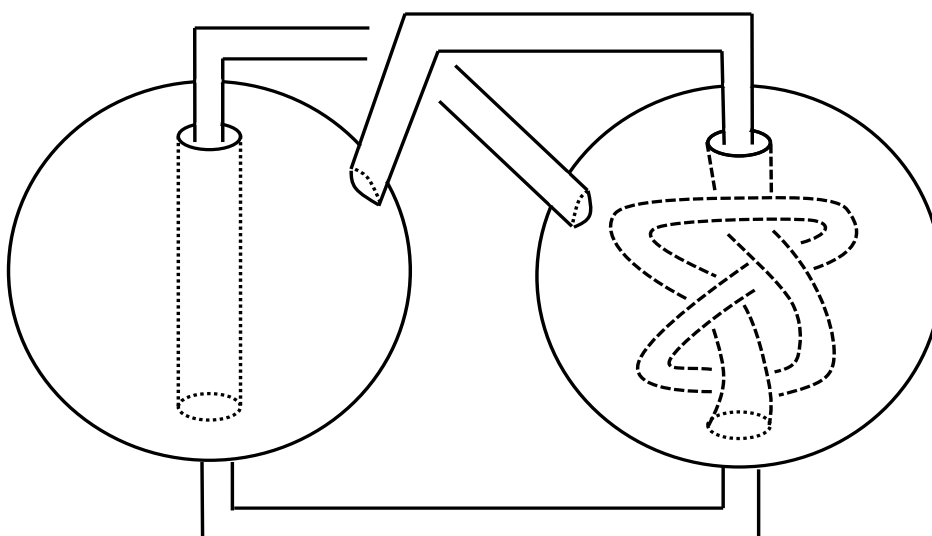


Figure 7: Bi-knotted surface  $F'$

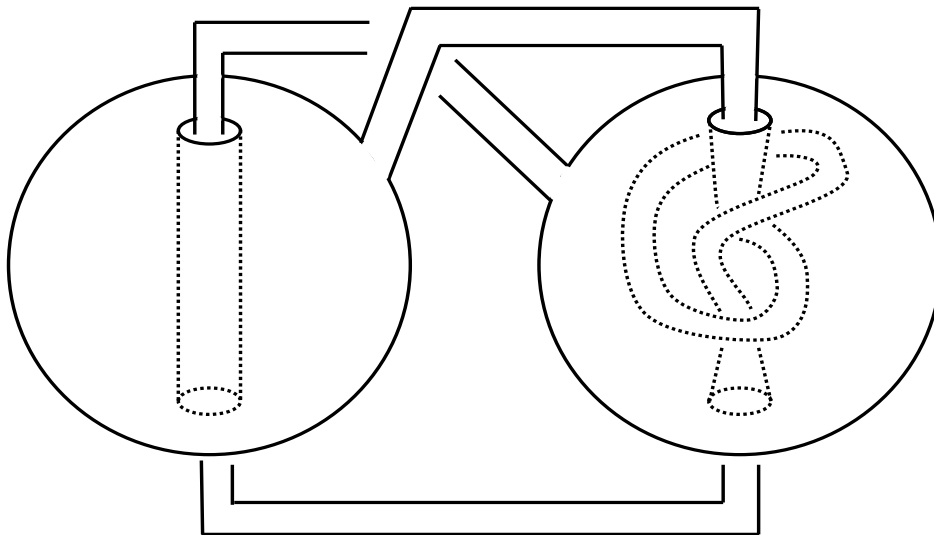


Figure 8: Exterior component  $V_F$

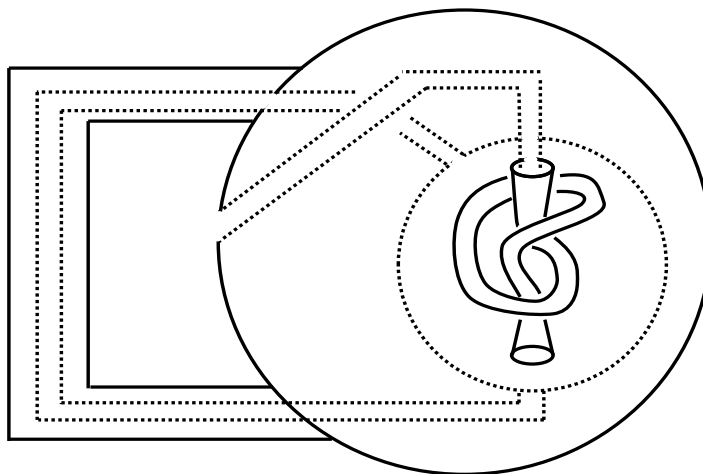


Figure 9: Exterior component  $W_F$

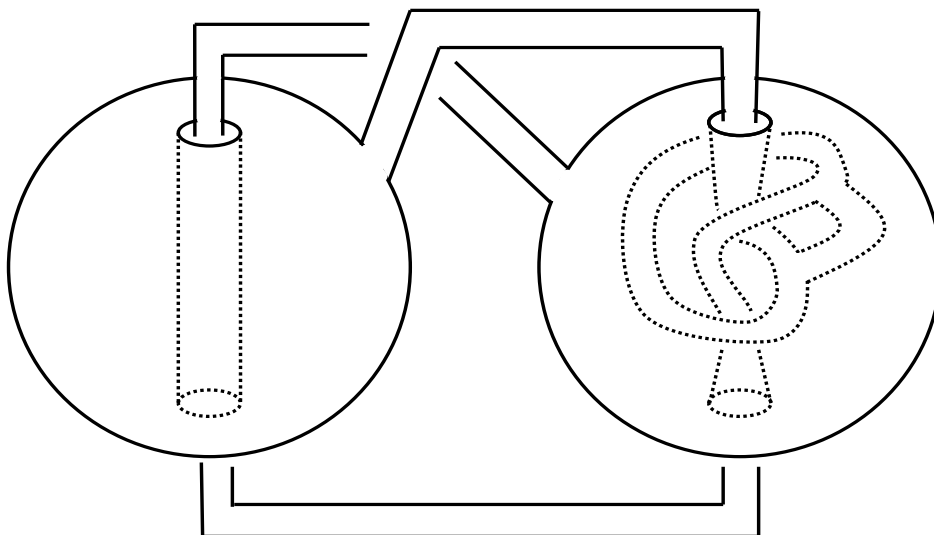


Figure 10: Handlebody-knot  $H_V$

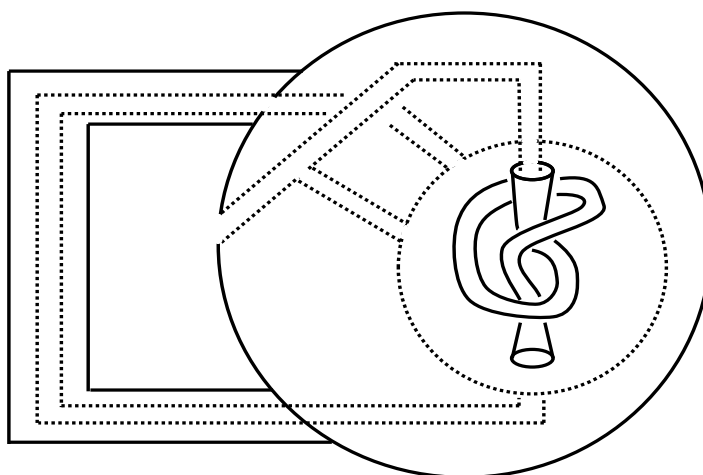


Figure 11: Handlebody-knot  $H_W$

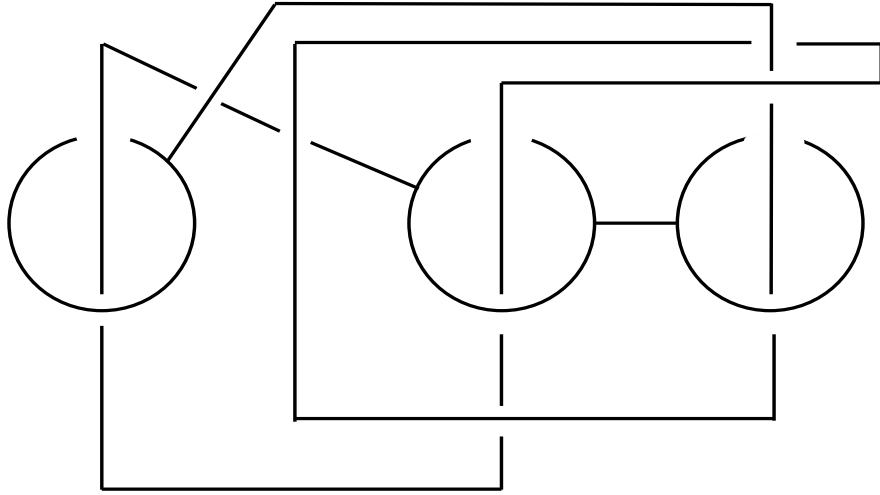


Figure 12: Diagram  $D_V$  of the handlebody-knot  $H_V$

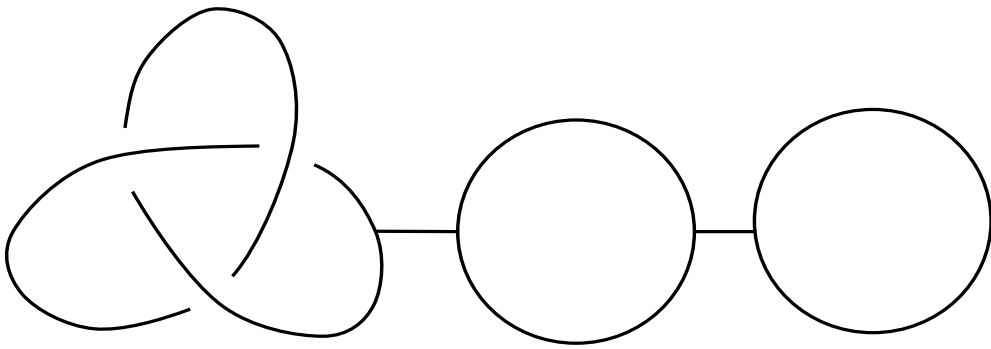


Figure 13: Diagram  $D_W$  of the handlebody-knot  $H_W$

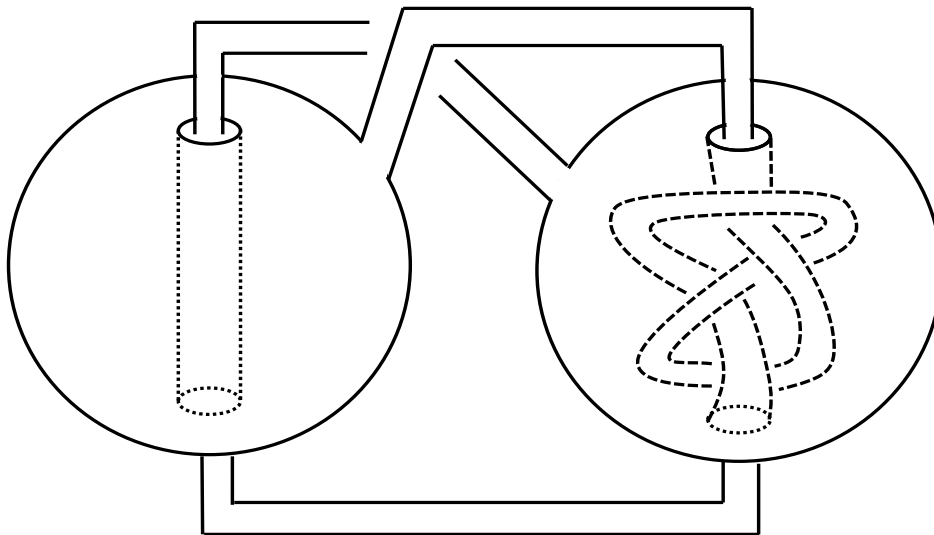


Figure 14: Exterior component  $V_{F'}$

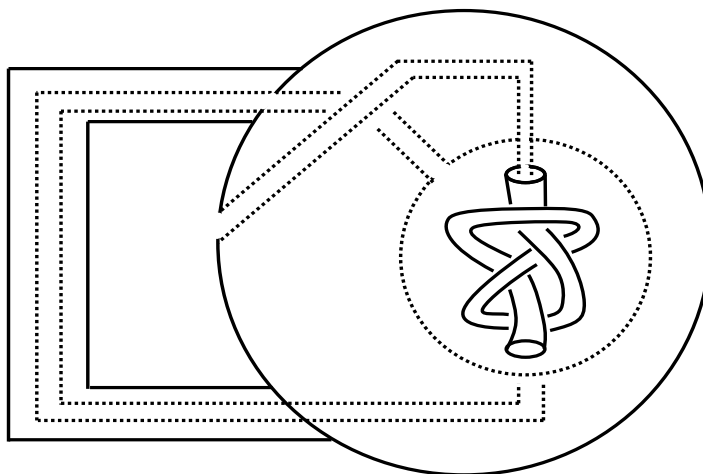


Figure 15: Exterior component  $W_{F'}$

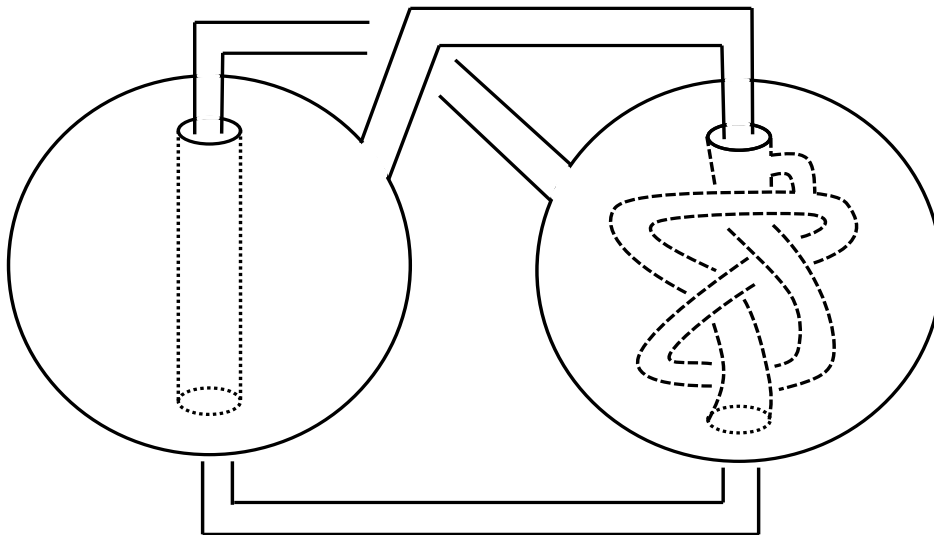


Figure 16: Handlebody-knot  $H_{V'}$

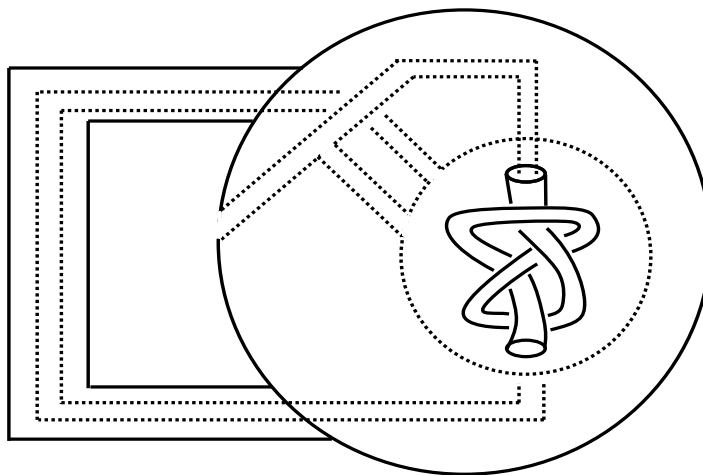


Figure 17: Handlebody-knot  $H_{W'}$

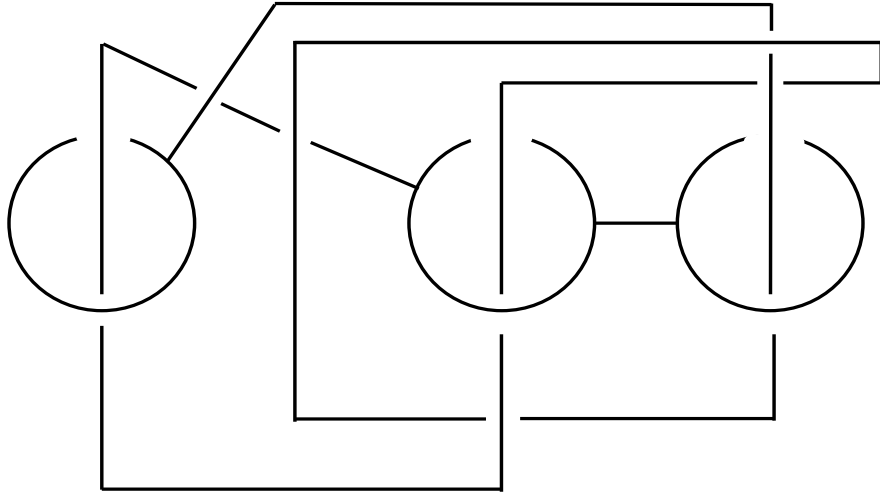


Figure 18: Diagram  $D_{V'}$  of the handlebody-knot  $H_{V'}$

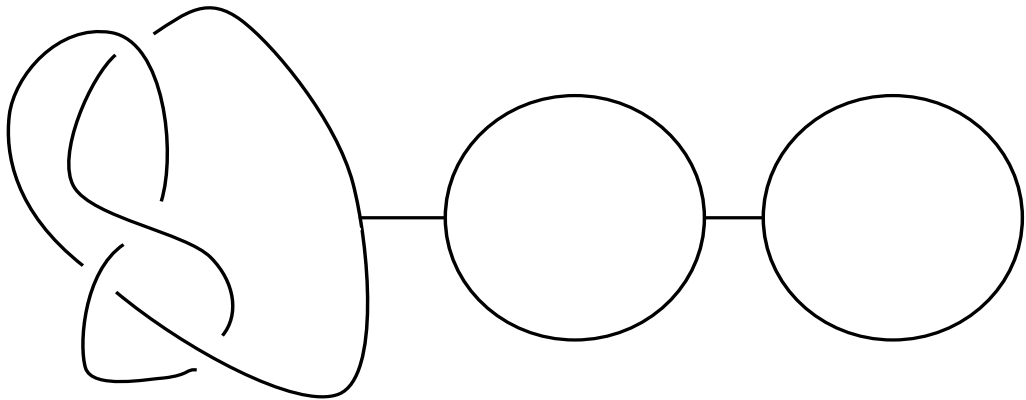


Figure 19: Diagram  $D_{W'}$  of the handlebody-knot  $H_{W'}$

## 5 Further results

In this section, we study closed connected orientable surfaces in  $S^3$  from geometric viewpoints. Then, we construct a geometric invariant of surfaces in  $S^3$ . We also give a necessary condition of the existence of a surface which is obtained from a 2-component handlebody-link corresponding to a minimal genus Heegaard splittings of the closures of the connected components of the exterior of the surface. We denote the interval  $[-1, 1]$  and the unit 2-disk by  $D^1$  and  $D^2$ , respectively.

**Definition 5.1** (Handle attaching [15]). Let  $F$  be a closed connected orientable surface embedded in  $S^3$ . Let  $h_i : D^1 \times D^2 \rightarrow S^3$  ( $i = 1, 2, \dots, n$ ) be embeddings of the 1-handle such that  $h_i(D^1 \times D^2) \cap F = h_i(\partial D^1 \times D^2)$  and  $h_i(D^1 \times D^2) \cap h_j(D^1 \times D^2) = \emptyset$  (whenever  $i \neq j$ ). We call  $h_i(D^1 \times \partial D^2)$  a *handle*. We denote by  $F(h_1, h_2, \dots, h_n) := F \cup (\sqcup_{i=1}^n h_i(D^1 \times \partial D^2)) \setminus \sqcup_{i=1}^n h_i(\partial D^1 \times \text{int}(D^2))$  the surface obtained from  $F$  by attaching  $n$  handles. Note that  $F(h_1, h_2, \dots, h_n)$  is again a closed connected orientable surface embedded in  $S^3$  and that its genus is equal to  $g(F) + n$ .

Two closed connected orientable surfaces  $F_1$  and  $F_2$  in  $S^3$  are said to be *separated* if there exists 3-balls  $B_1^3$  and  $B_2^3$  such that  $F_1 \subset \text{int}(B_1^3)$  and  $F_2 \subset \text{int}(B_2^3)$ . Two closed connected orientable surfaces  $F_1$  and  $F_2$  are also said to be *separated* if there exists a 3-ball  $B^3$  such that  $F_1 \subset \text{int}(B^3)$  and  $F_2 \subset S^3 \setminus B^3$ . The 3-ball  $B^3$  is called the *associated ball*. Similarly,  $n$  closed connected orientable surfaces  $F_1, F_2, \dots, F_n$  in  $S^3$  are said to be *separated* if there exists  $n$  3-balls  $B_i^3$  ( $i = 1, 2, \dots, n$ ) such that  $F_i \subset \text{int}(B_i^3)$  for each  $i = 1, 2, \dots, n$ .

Given two separated embedded surfaces in  $S^3$ , we can construct a new embedded surface by using the following way, which is called the *isotopy sum*.

**Definition 5.2** (Isotopy sum [18]). Let  $F_1$  and  $F_2$  be closed connected orientable separated surfaces embedded in  $S^3$  and  $B^3$  be the associated ball as above, respectively. Let  $\varphi : D^1 \times D^2 \rightarrow S^3$  be an embedding of the 1-handle such that  $\varphi(D^1 \times D^2) \cap F_1 = \varphi(\{-1\} \times D^2)$ ,  $\varphi(D^1 \times D^2) \cap F_2 = \varphi(\{1\} \times D^2)$ , and  $\varphi(D^1 \times D^2) \cap \partial B^3 = \varphi(\{0\} \times D^2)$ . We define the *isotopy sum* of  $F_1$  and  $F_2$ , denoted by  $F_1 \#_{\text{iso}} F_2$ , by  $F_1 \#_{\text{iso}} F_2 := F_1 \cup F_2 \cup \varphi(D^1 \times \partial D^2) \setminus \varphi(\partial D^1 \times \text{int}(D^2))$ .

We note that the isotopy sum of two separated embedded surfaces does not depend on the choice of the associated ball and the choice of an embedding  $\varphi$  up to an isotopy of  $S^3$ . Moreover, it does not depend on the order of  $F_1$  and  $F_2$ . We also note that  $(F_1 \#_{\text{iso}} F_2) \#_{\text{iso}} F_3 \cong F_1 \#_{\text{iso}} (F_2 \#_{\text{iso}} F_3)$  for separated closed connected orientable surfaces  $F_1, F_2$  and  $F_3$  in  $S^3$ . In this sense the isotopy sum is well-defined. For an embedded surface  $F$ , if  $F$  is isotopic to the isotopy sum  $F_1 \#_{\text{iso}} F_2 \#_{\text{iso}} \dots \#_{\text{iso}} F_k$  for separated surfaces  $F_1, F_2, \dots, F_k$ , we call it a *decomposition of  $F$  into factors  $F_i$ ,  $i = 1, 2, \dots, k$* .

In order to construct a geometric invariant of closed connected orientable surfaces in  $S^3$ , we introduce the notion of the *tunnel number* of a compact connected orientable 3-manifold with connected boundary embedded in  $S^3$ .

**Definition 5.3** (Tunnel number). Let  $V$  be a compact connected orientable 3-manifold with connected boundary embedded in  $S^3$ . The *tunnel number* of  $V$  is the minimal number  $n$  of mutually disjoint 1-handles such that  $\text{cl}(S^3 \setminus (V \cup \sqcup_{i=1}^n h_i(D^1 \times D^2)))$  is homeomorphic to a handlebody and  $V \cap h_i(D^1 \times D^2) = h_i(\partial D^1 \times D^2)$ , where  $h_i$  ( $i = 1, 2, \dots, n$ ) are embeddings of the 1-handle  $D^1 \times D^2$  into  $S^3$ . We denote by  $t(V)$  the tunnel number of  $V_F$ .

It is not hard to see that since every compact connected orientable 3-manifold possibly with boundary admits a Heegaard splitting,  $t(V)$  is always finite. Let  $F$  be a closed connected



orientable surface in  $S^3$ . We denote by  $V_F$  and  $W_F$  the closures of the connected components of  $S^3 \setminus F$ . By Definition 5.3, we also see that the Heegaard genus of  $W_F$  is given by  $g(F) + t(V_F)$ . Similarly, the Heegaard genus of  $V_F$  is given by  $g(F) + t(W_F)$ .

We note that if  $V_F$  is a handlebody-knot, then the tunnel number of  $V_F$  coincides with that of a handlebody-knot [8].

**Proposition 5.4.** *Let  $h_i : D^1 \times D^2 \rightarrow S^3$  ( $i = 1, 2, \dots, n$ ) be embeddings of the 1-handle such that  $h_i(D^1 \times D^2) \cap F = h_i(\partial D^1 \times D^2)$  and  $h_i(D^1 \times D^2) \cap h_j(D^1 \times D^2) = \emptyset$  (whenever  $i \neq j$ ). For any closed connected orientable surface  $F$  in  $S^3$ , there exists a finite numbers of handles such that  $F(h_1, \dots, h_n)$ , which is a surface obtained from  $F$  by attaching  $n$  handles to  $F$ , is an unknotted surface.*

*Proof.* Let  $V_F$  and  $W_F$  be the closures of the connected components of  $S^3 \setminus F$ . Then, we have  $S^3 = V_F \cup W_F$ . Let  $t(V_F)$  and  $t(W_F)$  be the tunnel numbers of  $V_F$  and  $W_F$ . Then, by the definition of the tunnel number, we obtain two handlebody-knots  $H_1$  and  $H_2$  such that  $H_1 \cup H_2 = S^3$  and  $\partial H_1 = \partial H_2$  by attaching  $t(V_F) + t(W_F)$  1-handles to the common boundary  $F = V_F \cap W_F$  of  $V_F$  and  $W_F$ . Since the union of  $H_1$  and  $H_2$  along their boundaries gives a Heegaard splitting of  $S^3$ . Then, combining Waldhausen's theorem 2.11, the common boundary of  $H_1$  and  $H_2$  is an unknotted surface.  $\square$

We introduce the definition of the handle number of embedded surface.

**Definition 5.5** (Handle number). Let  $F$  be a closed connected orientable surface in  $S^3$ . Let  $h_i : D^1 \times D^2 \rightarrow S^3$  ( $i = 1, 2, \dots, n$ ) be embeddings of the 1-handle such that  $h_i(D^1 \times D^2) \cap F = h_i(\partial D^1 \times D^2)$  and  $h_i(D^1 \times D^2) \cap h_j(D^1 \times D^2) = \emptyset$  (whenever  $i \neq j$ ). Let  $F(h_1, h_2, \dots, h_n)$  be a surface obtained from  $F$  by attaching  $n$  handles to  $F$ . The *handle number* of  $F$ , denoted by  $h(F)$ , is the minimal number  $n$  of handles such that  $F(h_1, \dots, h_n)$  is an unknotted surface.

Let  $F$  be a closed connected orientable surface in  $S^3$ . We denote by  $V_F$  and  $W_F$  the closures of the connected components of  $S^3 \setminus F$ . We show the relationship between the handle number of  $F$  and the tunnel numbers of  $V_F$  and  $W_F$ .

**Proposition 5.6.** *Let  $F$  be a closed connected orientable surface in  $S^3$ . We denote by  $V_F$  and  $W_F$  the closures of the connected components of  $S^3 \setminus F$ . Let  $t(V_F)$  and  $t(W_F)$  be the tunnel numbers of  $V_F$  and  $W_F$ . Then, for the handle number of  $F$ , we have  $h(F) = t(V_F) + t(W_F)$ .*

*Proof.* Let  $n$  be the handle number of  $F$ . Let  $h_i : D^1 \times D^2 \rightarrow S^3$  ( $i = 1, 2, \dots, n$ ) and  $\{h_1, \dots, h_n\}$  be embeddings of the 1handle satisfying the conditions given in Definition 5.1 and the set of attached handles to  $F$  such that  $F(h_1, \dots, h_n)$  is an unknotted surface, respectively. We divide the set of handles  $\{h_1, \dots, h_n\}$  into the disjoint union of the sets of handles  $\{h_1^V, \dots, h_k^V\}$  and  $\{h_1^W, \dots, h_l^W\}$ , where  $h_i^V(D^1 \times D^2)$  is attached to  $V_F$ , and  $h_j^W(D^1 \times D^2)$  is attached to  $W_F$ . Then, by Proposition 5.4, we have  $n = k + l \leq t(V_F) + t(W_F)$ . On the other hand, by the definition of the tunnel number, we have  $t(V_F) \leq k$  and  $t(W_F) \leq l$ . Then, we obtain  $t(V_F) + t(W_F) \leq k + l$ . Therefore, we have  $h(F) = k + l = t(V_F) + t(W_F)$ .  $\square$

As an immediate sequence, we have that the unordered pair  $(t(V_F), t(W_F))$  of non-negative integers is an invariant of  $F$ . Furthermore, it is not hard to see the following. Let  $V_F = H_V \cup C_V$  and  $W_F = H_W \cup C_W$  be minimal genus Heegaard splittings of  $V_F$  and  $W_F$  consisting of pairs of a handlebody  $H_V$  and a compression body  $C_V$  and a handlebody  $H_W$  and a compression body  $C_W$ , respectively. If  $F$  is an unknotted surface, then both  $H_V$  and  $H_W$  are trivial handlebody-knots. Hence  $t(V_F) = 0$  and  $t(W_F) = 0$ . If  $F$  is a knotted surface, then exactly one of  $V_F$  or

$W_F$  is homeomorphic to a handlebody, and the other is not homeomorphic to a handlebody. Hence exactly one of  $t(V_F)$  or  $t(W_F)$  is equal to zero, and the other is a positive integer. If  $F$  is a bi-knotted surface, then we have that neither  $V_F$  nor  $W_F$  is homeomorphic to a handlebody. Hence both  $t(V_F)$  and  $t(W_F)$  are positive integers.

It is not hard to see that since the unordered pair  $(t(V_F), t(W_F))$  of non-negative integers is an invariant of  $F$ , then the handle number  $h(F)$  is also an invariant of  $F$ . We prove the following corollary.

**Corollary 5.7.** *Let  $F$  be a closed connected orientable surface in  $S^3$ . We denote by  $V_F$  and  $W_F$  the closures of the connected components of  $S^3 \setminus F$ . Let  $n$  be the handle number of  $F$ . Let  $h_i : D^1 \times D^2 \rightarrow S^3$  ( $i = 1, 2, \dots, n$ ) be embeddings of the 1-handle satisfying the conditions given in Definition 5.1. Let  $\{h_1, h_2, \dots, h_n\} = \{h_1^V, \dots, h_k^V\} \sqcup \{h_1^W, \dots, h_l^W\}$  be the set of attached handles, where  $h_i^V(D^1 \times D^2)$  is attached to  $V_F$ , and  $h_j^W(D^1 \times D^2)$  is attached to  $W_F$ . Let  $t(V_F)$  and  $t(W_F)$  be the tunnel numbers of  $V_F$  and  $W_F$ . Then,  $k = t(V_F)$  and  $l = t(W_F)$ .*

*Proof.* In the proof of Theorem 5.6, we saw that  $k + l = t(V_F) + t(W_F)$ ,  $t(V_F) \leq k$ , and  $t(W_F) \leq l$ . Let us assume that  $t(V_F) < k$ . Then, we have  $t(W_F) > l$ . This is a contradiction. Hence  $k = t(V_F)$ . The same holds for  $l$ , that is,  $l = t(W_F)$ .  $\square$

**Proposition 5.8.** *Let  $F$  be a closed connected orientable surface in  $S^3$ . We denote by  $V_F$  and  $W_F$  the closures of the connected components of  $S^3 \setminus F$ . For any minimal genus Heegaard splitting of  $V_F$ , say  $V_F = H_V \cup C_V$ , the tunnel number  $t(H_V)$  of the handlebody-knot  $H_V$  does not depend on the choice of minimal genus Heegaard splittings. The same holds for any minimal genus Heegaard splitting of  $W_F$ .*

*Proof.* Let  $V_F = H_V \cup C_V$  and  $V_F = H'_V \cup C'_V$  be minimal genus Heegaard splittings of  $V_F$ . Let  $n = k + l$  be the handle number of the surface  $F$ , where  $k$  and  $l$  are the numbers of 1-handles attached to  $V_F$  and  $W_F$ . Let  $h_i : D^1 \times D^2 \rightarrow S^3$  ( $i = 1, 2, \dots, n$ ) and  $\{h_1, \dots, h_n\}$  be embeddings of the 1-handle satisfying the conditions given in Definition 5.1 and the set of attached handles to  $F$  such that  $F(h_1, \dots, h_n)$  is an unknotted surface, respectively. Then, we divide the set of handles  $\{h_1, \dots, h_n\}$  into the disjoint union of two sets  $\{h_1^V, \dots, h_k^V\}$  and  $\{h_1^W, \dots, h_l^W\}$ , where  $h_i^V$  is a handle attached to  $V_F$ , and  $h_j^W$  is a handle attached on  $W_F$ . By the definition of the tunnel number of  $W_F$ ,  $\text{cl}(S^3 \setminus (W_F \cup \bigcup_{i=1}^{t(W_F)} h_i(D^1 \times D_2)))$  is isotopic to a handlebody. In particular, since  $V_F = H_V \cup C_V$  is a minimal genus Heegaard splitting, we can assume that  $\text{cl}(S^3 \setminus (W_F \cup \bigcup_{i=1}^{t(W_F)} h_i(D^1 \times D^2)))$  is homeomorphic to  $H_V$ . Hence,  $\partial H_V$  is obtained from  $F$  by attaching  $t(W_F)$  handles. Then, by the definition of the handle number of  $F$ , we can also assume that we obtain an unknotted surface from  $\partial H_V$  by attaching  $k$  handles. Moreover, by the definition of the tunnel number of  $H_V$ , we obtain an unknotted surface from  $H_V$ . Then, we have  $k = t(H_V)$ . By using a similar argument, we also have  $k = t(H'_V)$ . The same holds for minimal genus Heegaard splittings of  $W_F$ .  $\square$

Let  $F$  be a closed connected orientable surface embedded in  $S^3$ . We denote by  $V_F$  and  $W_F$  the closures of the connected components of  $S^3 \setminus F$ . By considering Heegaard splittings  $V_F$  and  $W_F$ , we obtain an associated 2-component handlebody-link  $L$  of  $F$ . By applying the Reidemeister–Singer theorem, we showed that  $L$  is uniquely determined from  $F$  up to stabilizations of Heegaard splittings of  $V_F$  and  $W_F$ . Conversely, let us consider a 2-component handlebody-link  $L'$ . Then, does there always exist a closed connected orientable surface  $F'$  embedded in  $S^3$  such that the connected components of  $\partial L'$  are Heegaard surfaces of  $V_{F'}$  and  $W_{F'}$ ? A necessary and sufficient

condition was given in Proposition 2.15. We consider such a problem from the viewpoints of the handle number and the tunnel number.

By Proposition 5.8, we have the following corollary.

**Corollary 5.9.** *Let  $L = H_1 \sqcup H_2$  be a 2-component handlebody-link. Let us assume that  $t(H_1) - t(H_2) \neq g(\partial H_2) - g(\partial H_1)$ . Then, there does not exist a closed connected orientable surface in  $S^3$  such that  $L$  is obtained from minimal Heegaard splittings of the closures of the exterior of the surface.*

Let  $M$  be a compact connected orientable 3-manifold whose boundary is not homeomorphic to a 2-sphere. Then,  $M$  is said to be  $\partial$ -irreducible if for every properly embedded 2-disk  $D$  in  $M$ ,  $\partial D$  bounds a 2-disk on  $\partial M$ . We assume that the boundary of  $M$  is connected, and that  $M$  is not homeomorphic to a 3-ball. Then,  $M$  is said to be  $\partial$ -prime if for any decomposition  $M_1 \natural M_2$ , where  $M_1$  and  $M_2$  are compact connected orientable 3-manifolds, either  $M_1$  or  $M_2$  is homeomorphic to a 3-ball, where  $M_1 \natural M_2$  is the disk sum, or the boundary connected sum, of  $M_1$  and  $M_2$ . Then, the following is known.

**Proposition 5.10** ([16]). *Let  $M$  be a compact connected orientable 3-manifold embeddable in  $S^3$  with connected boundary with  $g(\partial M) \geq 2$ . Then, the following conditions are equivalent.*

- (i) *The 3-manifold  $M$  is  $\partial$ -prime.*
- (ii) *The 3-manifold  $M$  is  $\partial$ -irreducible.*
- (iii) *The fundamental group  $\pi_1(M)$  of  $M$  is indecomposable with respect to free products.*

We now present the notion of *prime surfaces* in  $S^3$ .

**Definition 5.11** (Prime surface [18]). *Let  $F$  be a closed connected orientable surface in  $S^3$  with  $g(F) \geq 1$ . Then,  $F$  is said to be *prime* if for any decomposition  $F \cong F_1 \#_{\text{iso}} F_2$ , either  $F_1$  or  $F_2$  is a surface of genus 0. We note that every closed connected orientable surface of genus 1 is prime.*

About closed connected orientable prime surfaces in  $S^3$ , the following is known.

**Theorem 5.12** ([17]). *Let  $F$  be a closed connected orientable surface of genus 2 in  $S^3$ . Then,  $F$  is prime if and only if either  $\pi_1(V_F)$  or  $\pi_1(W_F)$  is indecomposable with respect to free products.*

Let  $M$  be a compact connected orientable 3-manifold with connected boundary embedded in  $S^3$ . Then,  $M$  is said to be *reducibly embedded* if there exists a 2-sphere  $S$  in  $S^3$  such that  $M \cap S$  is a properly embedded 2-disk  $D$  in  $M$ , and that  $D$  separates  $M$  into two parts which are not homeomorphic to 3-balls. Otherwise,  $M$  is said to be *irreducibly embedded*. We call a reducibly embedded handlebody-knot a *reducible handlebody-knot*. Similarly, we call an irreducibly embedded handlebody-knot an *irreducible handlebody-knot*.

Let us now focus on prime surfaces of genus 2 embedded in  $S^3$ . Let  $F$  be a prime surface of genus 2 in  $S^3$ . We denote by  $V_F$  and  $W_F$  the closures of the connected components of  $S^3 \setminus F$ . By virtue of Theorem 5.12 and Proposition 5.10, we know that either  $V_F$  or  $W_F$  is  $\partial$ -prime. Then, let us assume that exactly one of  $V_F$  or  $W_F$  is  $\partial$ -prime. On that assumption, we have the following proposition.

**Proposition 5.13.** *Let  $F$  be a closed connected orientable prime surface of genus 2 in  $S^3$ . We denote by  $V_F$  and  $W_F$  the closures of the connected components of  $S^3 \setminus F$ . We assume that exactly one of  $V_F$  or  $W_F$  is  $\partial$ -prime. Let  $V_F = H_V \cup C_V$  and  $W_F = H_W \cup C_W$  be minimal genus Heegaard splittings of  $V_F$  and  $W_F$ , respectively. Then, exactly one of  $H_V$  or  $H_W$  is a reducible handlebody-knot, and the other is an irreducible handlebody-knot.*

*Proof.* By the assumption, we assume that  $V_F$  is not  $\partial$ -prime and  $W_F$  is  $\partial$ -prime. Since  $V_F$  is not  $\partial$ -prime, there exists a decomposition  $V_F = V_F^1 \natural V_F^2$  such that neither  $V_F^1$  nor  $V_F^2$  is homeomorphic to a 3-ball  $B^3$ . Then, by the use of the solid torus theorem,  $V_F^i$  ( $i = 1, 2$ ) are either a handlebody-knot of genus 1 or the exterior of a handlebody-knot of genus 1.

(1) In case of both  $V_F^1$  and  $V_F^2$  are handlebody-knots of genus 1, since  $F$  is a prime surface, then  $V_F = V_F^1 \natural V_F^2$  is an irreducible handlebody-knot. Since the Heegaard splittings of a handlebody-knot are standard,  $H_V$  is an irreducible handlebody-knot. On the other hand,  $W_F$  the exterior of the handlebody-knot  $V_F$ . Then the Heegaard surfaces of  $W_F$  are unknotted surfaces. Therefore  $H_W$  is a reducible handlebody-knot.

(2) In case of  $V_F^1$  is a handlebody-knot of genus 1 and  $V_F^2$  is the exterior of a handlebody-knot of genus 1, then  $V_F^2$  is obtained from a 3-ball  $B^3$  by removing a 2-handle (see Figure 20). Then, by connecting  $V_F^1$  and  $V_F^2$  by a 1-handle, we obtain  $V_F$ . It can be also assumed to be that  $V_F$  is obtained from  $V_F^2$  by attaching a 1-handle which throughs the removed 2-handle from  $B^3$  at least one time up to isotopy. Since  $V_F^1$  is a handlebody-knot, by removing 2-handles from  $V_F^2$ , the minimal genus Heegaard splitting of  $V_F$  is obtained (see Figure 21). By considering a minimal genus Heegaard splitting of  $V_F^2$ , we obtain the trivial handlebody-knot of some genus  $\geq 2$ . Such a handlebody-knot is obtained from  $B^3$  by removing 2-handles. However, by the assumption of the 1-handle attached to  $V_F^2$  and applying a deformation depicted in Figure 22, the attached 1-handle throughs every removed 2-handle of the trivial handlebody-knot (Figure 23). Hence  $H_V$  is an irreducible handlebody-knot. On the other hand,  $W_F = \text{cl}(S^3 \setminus (V_F^1 \cup V_F^2 \cup \{2\text{-handle}\})) = \text{cl}((B^3 \cup \{1\text{-handle}\}) \setminus (V_F^2 \cup \{2\text{-handle}\}))$ . We note that the removed 2-handle throughs both  $V_F^1$  and the attached 1-handle at least one time up to isotopy (see Figure 24). To obtain a Heegaard splitting of  $W_F$ , we need to move the 2-handle so as not to it throughs the attached 1-handle by removing 2-handles (Figures 25 and Figure 26). Then, after removing 2-handles from  $B^3$  we obtain a minimal genus Heegaard splitting of  $W_F$ . By the above procedure, we can see that  $H_W$  is a reducible handlebody-knot. The same holds for in case of  $V_F^2$  is a handlebody-knot of genus 1 and  $V_F^1$  is the exterior of a handlebody-knot of genus 1.

(3) In case of both  $V_F^1$  and  $V_F^2$  are the exterior of handlebody-knots of genus 1, by applying the same procedure given in (2) for  $V_F$ , we can show that  $V_F$  is an irreducible handlebody-knot. On the other hand, by applying the same procedure given in (2) for  $W_F$ , we can show that  $W_F$  is a reducible handlebody-knot.  $\square$

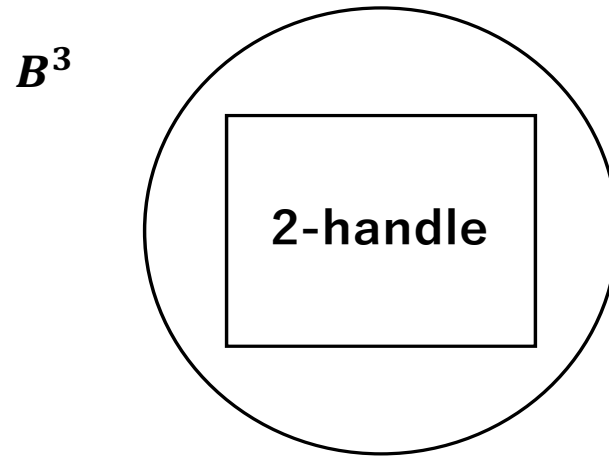


Figure 20: Construction of  $V_F^2$

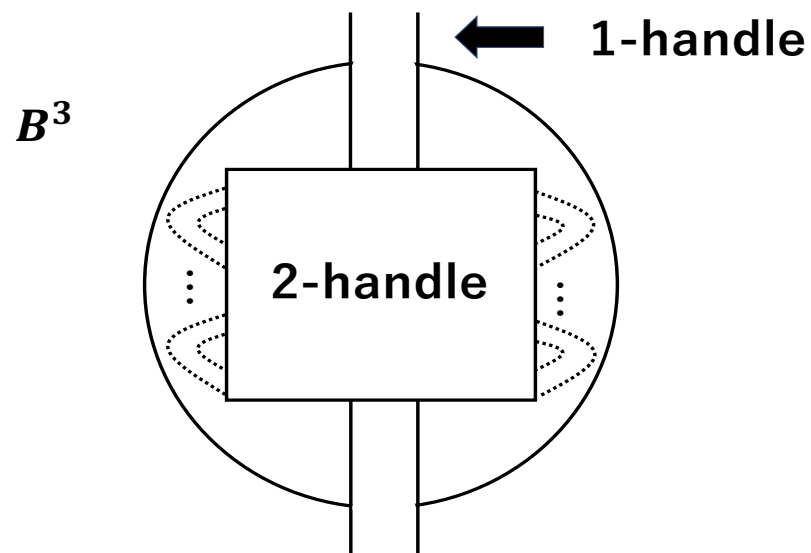


Figure 21: Removing of 2-handles

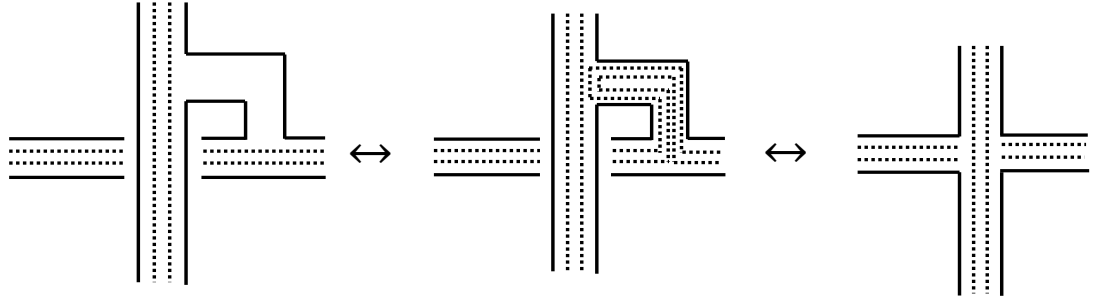


Figure 22: Deformation

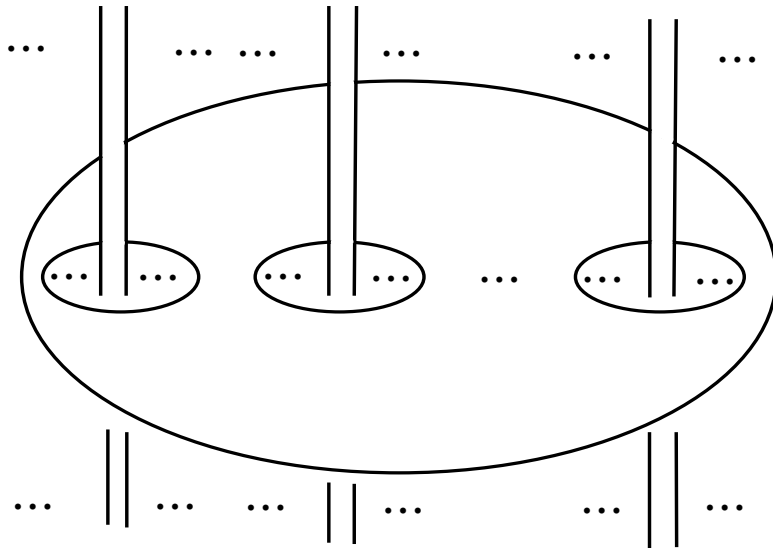


Figure 23: Part of the handlebody-knot  $H_V$

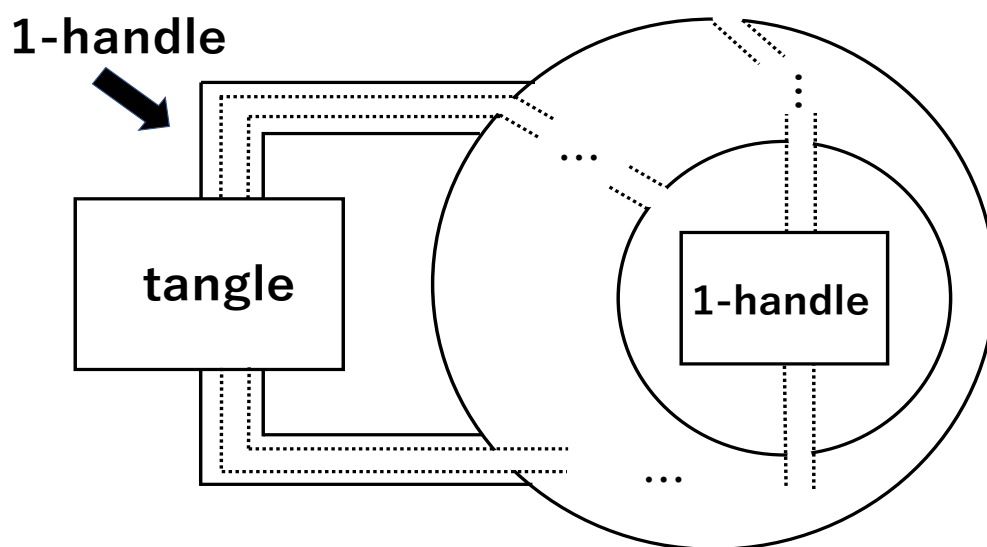


Figure 24: Exterior component  $W_F$

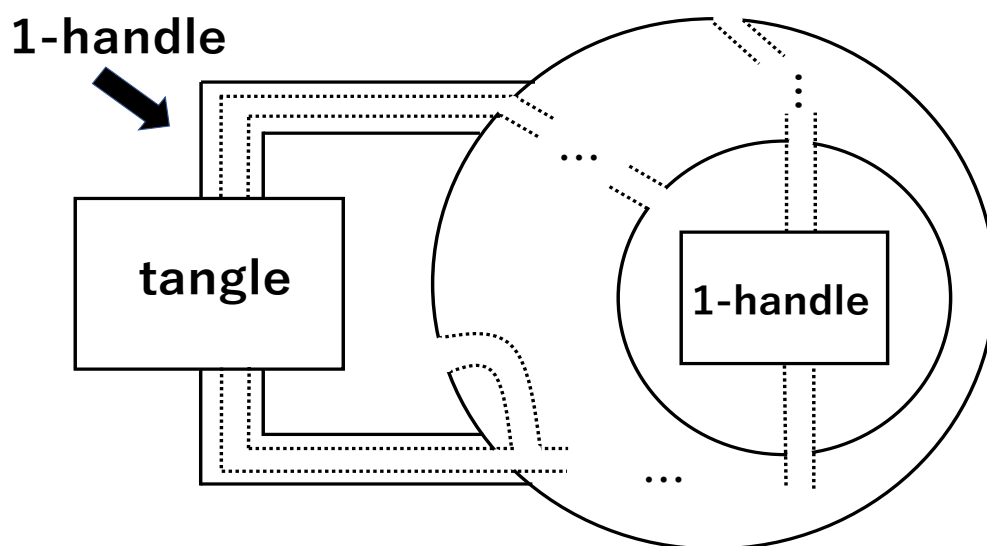


Figure 25: Removing of a 2-handle

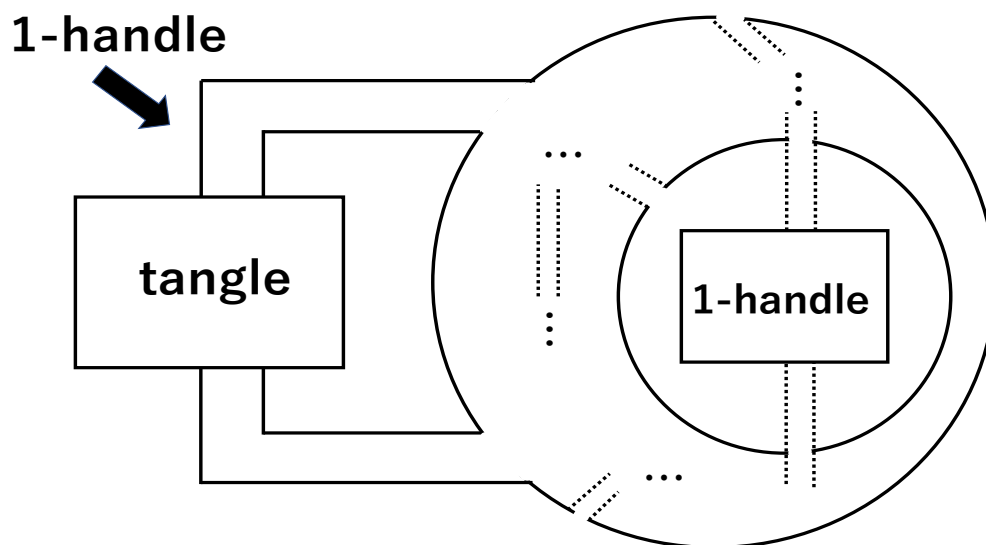


Figure 26: Procedure of a Heegaard splitting of  $W_F$

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