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ON INVARIANTS OF SURFACES IN THE 3-SPHERE

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Abstract

In this paper we study isotopy classes of closed connected orientable surfaces in the standard 3-sphere. Such a surface splits the 3-sphere into two compact connected submanifolds, and by using their Heegaard splittings, we obtain a 2-component handlebody-link. In this paper, we first show that the equivalence class of such a 2-component handlebody-link up to attaching trivial 1-handles can recover the original surface. Therefore, we can reduce the study of surfaces in the 3-sphere to that of 2-component handlebody-links up to stabilizations. Then, by using G-families of quandles, we construct invariants of 2-component handlebody-links up to attaching trivial 1-handles, which lead to invariants of surfaces in the 3-sphere. In order to see the effectiveness of our invariants, we will also show that our invariants can distinguish certain explicit surfaces in the 3-sphere.

1 Introduction

Throughout this paper, we work in the PL category. In this paper, we will study closed connected orientable surfaces embedded in the standard 3-sphere S^3. Such embedded surfaces were extensively studied by Fox, Homma, Tsukui, and Suzuki around 1950s–1970s (e.g., see [3, 5, 15, 18]). In their studies, 3-manifold theory and fundamental group techniques were mainly used.

A closed connected orientable surface in S^3 splits S^3 into two compact connected submanifolds of codimension 0 with common boundary. Then, by considering their Heegaard splittings, we obtain a pair of a handlebody and a compression body from each submanifold. Moreover, it is known as the Reidemeister–Singer theorem that any two Heegaard splittings of a compact connected orientable 3-manifold become equivalent after finitely many stabilizations. Hence, given a surface embedded in S^3, we can associate a certain 2-component handlebody-link up to stabilizations. Then, we show that an isotopy class of the original surface is uniquely determined from the equivalence class of an associated 2-component handlebody-link Therefore, we can reduce the study of surfaces in S^3 to that of 2-component handlebody links in a certain sense.

Handlebody-links (-knots) have recently been studied quite actively (e.g., see [6, 7, 8]). For example, by using a G-family of quandles, which is an algebraic system with binary operations parametrized by the elements of a group G, we can construct invariants of handlebody-links (see [7]).

One of the main purposes of the present paper is to construct invariants of surfaces embedded in S^3 by means of the associated 2-component handlebody-links and G-families of quandles.

The present paper is organized as follows. In Section 2, we first recall the notion of compression bodies. Handlebodies can be regarded as a special class of compression bodies. We will also recall the notions of Heegaard splittings of 3-manifolds and their stabilizations. Then, we review the Reidemeister–Singer theorem, which will be used in Section 3. We will show that for two surfaces in S^3, if the associated 2-component handlebody-links are equivalent up to attaching 1-handles, then two surfaces are isotopic.

In Section 3, we construct invariants of surfaces embedded in S^3. Plenty of invariants of handlebody-links were given in [7]. We first use X-colorings for the construction of surfaces in S^3. We observe a variation of the cardinality of the set of X-coloring due to a stabilization. Then, we give a technique which cancels the variation of the cardinality of the set of X-colorings after a stabilization of 2-component handlebody-link. Hence, we obtain an invariant of handlebody-links up to stabilizations. We consider such an invariant for each connected component of a 2-component handlebody-link. Then, as one of the results, we obtain an invariant as the unordered pair of rational numbers. Moreover, we will show that, by using a similar argument, we can construct another invariant of surfaces in S^3 by using another quandle invariant which is defined by using cohomology theory of G-families of quandles [7].

In Section 4, we compute our invariants constructed in Theorem 3.1 for two explicit examples of surfaces in S^3. By using a G-family of quandles with a simple algebraic structure, we show that our invariant distinguish the two surfaces by a small amount of calculation.

In Section 5, we study surface in S^3 from geometric viewpoints. We construct a geometric invariant of surfaces, which is an analogy of the unknotting tunnel number of a knot. Then, we study the relationship between a surface in S^3 and the closures of the connected components of the exterior of the surface.

Throughout the paper, for a manifold M, we denote by ∂M and $\mathrm{int}(M)$ the boundary and the interior of M, respectively. For a subset N of M, denote by $\mathrm{cl}(N)$ the closure of N in M. Furthermore, for two sets M_1 and M_2, we denote by $M_1 \sqcup M_2$ the disjoint union of M_1 and M_2. For a set Z we denote by $\sharp Z$ the cardinality of Z. We also denote by id_Z the identity map of Z. For a group G, we denote by e_G the identity element of G. We denote by $g(F)$ the genus of a closed connected orientable surface F. For a compact connected orientable surface F in a compact connected orientable 3-manifold M, we denote by $N(F)$ a regular neighborhood of F in M. A multiset is a set whose elements allow duplications. A duplication of an element of a multiset is called the multiplicity of the element.

2 Preliminaries

In this section, we present definitions and known results that will be used when we construct invariants of embedded surfaces in S^3. The main theme of this paper is to study closed connected orientable surfaces in S^3 as we mentioned.

Let F be a closed connected orientable surface embedded in S^3. We denote by V_F and W_F the closures of the connected components of $S^3 \backslash F$. Then, by considering Heegaard splittings of V_F and W_F, we will obtain a 2-component handlebody-link from the original surface F. Moreover, we study the relationship between a surface in S^3 and the associated 2-component handlebody-link. We also recall definitions and related results of G-family of quandles. By the Alexander theorem ([1]), all 2-spheres in S^3 are isotopic, then we consider surfaces of genus greater than 0.

2.1 3-manifolds and Heegaard splittings

In this section, we recall terminologies of 3-manifolds and Heegaard splittings. We also introduce several known results related to Heegaard splittings. Originally, Heegaard splittings were introduced for closed 3-manifolds. After that, the definition of a Heegaard splitting was extended to compact 3-manifolds, and such a splitting is called a *generalized Heegaard splitting* (Definition 2.6). We start from the definition of *compression bodies.*

Definition 2.1 ([12, 13])**.** Let Σ be a closed connected orientable surface. Consider the product manifold $\Sigma \times [0, 1]$. Then, attach 2-handles along mutually disjoint simple closed curves on $\Sigma \times \{0\}$, and cap off every resulting 2-sphere component by a 3-handle. The resulting 3-manifold C is called a *compression body.* We denote $\Sigma \times \{1\}$ by ∂_+C and $\partial C \backslash \partial_+C$ by ∂_-C. If C is constructed without any 3-handle, then C can be obtained from $\partial_-C \times [0, 1]$ by attaching mutually disjoint 1-handles on $\partial_-C \times \{1\}$. A compression body C is said to be a *handlebody* if $\partial_-C = \varnothing$. We define the *genus* of a handlebody as the genus of its boundary.

Let us recall *handlebody-knots* (*-links*) and their related terminologies. A *handlebody-knot* of genus g is a handlebody of genus g embedded in S^3. Two handlebody-knots are *equivalent* if there exists an ambient isotopy of S^3 which maps one to the other. Let $H_1, H_2, \ldots, H_n$ be mutually disjoint handlebody-knots. Then, $L := H_1 \sqcup H_2 \sqcup \cdots \sqcup H_n$ is called a *handlebody-link*, or sometimes an *n-component handlebody-link.* Two handlebody-links are *equivalent* if there exists an ambient isotopy of S^3 which maps one to the other. A handlebody-knot H (-link L) is represented by a spatial trivalent graph K if a regular neighborhood of K is ambient isotopic to H (L). Here, a spatial trivalent graph is a finite graph embedded in S^3 or $\mathbb{R}^3$ such that each vertex is of valence three. It is known that any handlebody-knot (-link) can be represented by a spatial trivalent graph (see [6]). A *diagram* D of a handlebody-knot H (L) is a diagram of a spatial trivalent graph K of H (L) obtained by projecting K to the 2-sphere S^2 or the 2-plane $\mathbb{R}^2$. A diagram D of a handlebody-knot (-link) consists of arcs, which are parts of curves, and each of the endpoints of an arc are a vertex or an undercrossing. At each crossing of D, an arc is called an *under-arc* if one of the its endpoints is undercrossing. Otherwise, an arc is called an *over-arc.* An oriented diagram is a diagram of whose each of an arc is oriented.

Let M and N be a compact 3-manifold and a submanifold of M, respectively. Then, N is said to be *properly embedded* in M if $\partial N \subset \partial M$ and $\text{int}(N) \subset \text{int}(M)$.

A properly embedded 2-disk D in M is said to be *inessential* if there exists a 2-disk D' in ∂M such that $\partial D = \partial D'$ and $D \cup D'$ is the boundary of a 3-ball in M. A properly embedded 2-disk D is said to be *essential* if D is not inessential. Moreover, let M be a compact connected 3-manifold. Then, an essential 2-disk D in M is said to be *separating* if $M \backslash D$ consists of two connected components. Otherwise, D is said to be *non-separating.*

Definition 2.2. Let M be a compact connected orientable 3-manifold. Then, M is said to be *irreducible* if any 2-sphere S in M bounds a 3-ball in M.

We refer the reader to [2, 4] about the 3-manifold theory used in this paper.

Definition 2.3 ([20])**.** Let M be a compact orientable 3-manifold. Let F be a compact orientable surface in M or in ∂M. Then, F is said to be *compressible* if F satisfies either of the following conditions.

(1) There exists a properly embedded 2-disk D in M such that $D \cap \text{int}(F) = \partial D$ and ∂D is non-contractible in $\text{int}(F)$.

(2) There exists a 3-ball B such that $B \cap F = \partial B$.

The surface F is said to be *incompressible* if F is not compressible.

The following theorem and lemma given in [2] and [20], respectively, will be used to see a relationship between the set of equivalence classes of 2-component handlebody-links up to attaching 1-handles and the set of isotopy classes of embedded surfaces.

Theorem 2.4 ([2]). *Let M be an orientable irreducible 3-manifold. Let F be a compressible boundary component of M. Then, there exists an incompressible neighborhood V of F, which is unique up to isotopy of M.*

For the definition of an incompressible neighborhood of a compressible boundary component, the reader is referred to Definition 2.14.

Lemma 2.5 ([20]). *Let M be a compact connected orientable 3-manifold. Let F be the union of mutually disjoint incompressible surfaces in M. We set $M' = \mathrm{cl}(M \backslash N(F))$. Then M' is irreducible if and only if M is irreducible.*

We introduce the notion of Heegaard splittings of 3-manifolds. Originally, Heegaard splittings were introduced to represent a closed connected orientable 3-manifold by the union of two handlebodies along their common boundaries. In the context of knot theory, the Heegaard genus of a Heegaard splitting of the exterior $E(K)$ of a knot K is related to the tunnel number of K.

Definition 2.6 (Heegaard Splittings). Let M be a compact connected orientable 3-manifold possibly with boundary. Fix a partition of ∂M as $\partial M = \partial_1 M \sqcup \partial_2 M$. A *Heegaard splitting* of M is a decomposition of M into two compression bodies C_1 and C_2 such that $M = C_1 \cup C_2$, $\partial_+ C_1 = S = \partial_+ C_2$, $C_1 \cap C_2 = S$, $\partial_- C_1 = \partial_1 M$, and $\partial_- C_2 = \partial_2 M$. We call $\partial_+ C_1 = S = \partial_+ C_2$ a *Heegaard surface* of M. We denote by (M, S) a Heegaard splitting of M with a Heegaard surface S. We also denote by $C_1 \cup_S C_2$ a Heegaard splitting of M consisting of two compression bodies C_1 and C_2 with a Heegaard surface S. The minimal genus of Heegaard surfaces of M is called the *Heegaard genus* of M. In particular, a Heegaard splitting of M with the minimal genus is called a *minimal genus Heegaard splitting* of M.

Concerning Heegaard splittings of 3-manifolds, it is known as Moise's theorem that every compact connected orientable 3-manifold possibly with boundary admits a Heegaard splitting ([10]).

We define a parallel arc in a compression body. Parallel arcs will be used when we define the stabilization of a Heegaard splitting of a 3-manifold.

Definition 2.7 (Parallel arc). Let C be a compression body. A properly embedded arc α in C with $\partial C \cap \partial \alpha = \partial_+ C \cap \partial \alpha$ is *parallel* to an arc β in $\partial_+ C$ with $\partial \beta = \partial \alpha$ if there is an embedded disk D in C such that $\partial D = \alpha \cup \beta$.

Definition 2.8 (Stabilization). Let M be a compact connected orientable 3-manifold possibly with boundary. Let (M, S) be a Heegaard splitting of M with a Heegaard surface S. The following procedure to construct a new Heegaard splitting (M, S') from (M, S) is called a *stabilization*. Suppose that $M = C_1 \cup_S C_2$ is a Heegaard splitting of M consisting of two compression bodies C_1 and C_2 with $\partial_+ C_1 = S = \partial_+ C_2$. Take a parallel arc α in C_2 to an arc β in $\partial_+ C_2$. Then, we remove a tubular neighborhood $N(\alpha)$ of α from C_2, take the closure of $C_2 \backslash N(\alpha)$, and add $N(\alpha)$ to C_1, namely, we have $C_1' := C_1 \cup N(\alpha)$ and $C_2' := \mathrm{cl}(C_2 \backslash N(\alpha))$. We can show that C_1' and C_2' are also compression bodies satisfying $C_1' \cup_{S'} C_2' = M$, where $S' := \partial C_1' = \partial C_2'$.

Hence we obtained a new Heegaard splitting $(M, S^{'})$ of M from the given Heegaard splitting (M, S). We also have $g(S^{'}) = g(S) + 1$.

We give an example of a Heegaard splitting of a handlebody of genus g.

Example 2.9 (The trivial Heegaard splitting)**.** Let H_g be a handlebody of genus g. Set $\partial H_g = F$. Consider a parallel surface $F^{'}$ to F inside of H_g. Then, the closures of the connected components of the exterior of $F^{'}$ consists of a handlebody of genus g and a compression body with the same boundaries. These handlebody and compression body give a Heegaard splitting of H_g. Such a splitting is called the *trivial splitting* of H_g.

Concerning Heegaard splittings of the handlebody of any genus g, it is known that all Heegaard splittings of a handlebody of genus g are standard, that is, they are obtained from the trivial Heegaard splitting by applying a finite number of stabilizations ([13]).

We introduce the notion of the equivalence of two Heegaard splittings of a compact connected orientable 3-manifold M possibly with boundary. Let (M, S) and $(M, S^{'})$ be Heegaard splittings of M with Heegaard surfaces S and $S^{'}$. They are *equivalent* if there exists an ambient isotopy of M which maps S to $S^{'}$. The following theorem, known as the Reidemeister–Singer theorem, plays an important role when we construct invariants of embedded surfaces in S^3.

Theorem 2.10 (Reidemeister–Singer theorem [11, 14])**.** *Let M be a compact connected orientable 3-manifold possibly with boundary. We fix a partition of ∂M as $\partial M = \partial_1 M \sqcup \partial_2 M$. Then, any two Heegaard splittings of M become equivalent after finitely many stabilizations.*

The following is also known as Waldhausen's theorem.

Theorem 2.11 ([19])**.** *A Heegaard splitting of S^3 is unique up to isotopy in every genus of a Heegaard surface.*

In Definition 2.8, we introduced the notion of the stabilization of a Heegaard splitting of a compact conneccted orientable 3-manifold. Then, a *stabilization* of handlebody-links. This definition is induced from the definition of the stabilization of a Heegaard splitting of a compact connected orientable 3-manifold with connected boundary. Two handlebody-links L_1 and L_2 are said to be *separated* if there exists 3-balls B_1^3 and B_2^3 in S^3 such that $B_1^3 \cap B_2^3 = \partial B_1^3 \cap \partial B_2^3$, $L_1 \subset \mathrm{int}(B_1^3)$, and $L_2 \subset \mathrm{int}(B_2^3)$. Similarly, n handlebody-links $L_1, L_2, \ldots, L_n$ are said to be *separated* if there exists 3-balls $B_1^3, B_2^3, \ldots, B_n^3$ in S^3 such that $B_i^3 \cap B_j^3 = \partial B_i^3 \cap \partial B_j^3$ (whenever $i \neq j$) and $L_1 \subset \mathrm{int}(B_1^3), L_2 \subset \mathrm{int}(B_2^3), \ldots, L_n \subset \mathrm{int}(B_n^3)$.

Definition 2.12 (Stabilization of handlebody-links)**.** Let H be a handlebody-knot. Then, the *stabilization* of H is the disk sum of H and the standard solid torus T, denoted by $H \natural T$, by a 2-disk D on ∂H so that the connected components of $H \natural T \backslash N(D)$ are separated by a 3-ball. Let $L = H_1 \sqcup H_2$ be a 2-component handlebody-link. Let T^1 and T^2 be the standard solid tori. Then, the *stabilization* of L is stabilizations of H_1 with T^1 by a 2-disk D_1 on ∂H_1 or that of H_2 with T^2 by a 2-disk D_2 on ∂H_2 so that the connected components of $\mathrm{cl}(L \natural T^1 \natural T^2 \backslash \sqcup_{i=1}^{2} N(D_i))$ are separated by 3-balls in S^3. Two handlebody-links are *stably equivalent* if they are equivalent after finitely many stabilizations.

We note that if both H_1 and H_2 are stabilized, then the attached solid tori are separated by some 3-balls in S^3.

2.2 Surfaces in S^3 and handlebody-links

Let us now study the relationship between embedded surfaces and 2-component handlebody-links. Let us start with a classification of embedded surfaces into three classes. We denote by V_F and W_F the closures of the connected components of the exterior of a given embedded surface F.

Definition 2.13. Let F be a closed connected orientable surface in S^3.

(i) The surface F is said to be an *unknotted surface* if both V_F and W_F are homeomorphic to handlebodies.

(ii) The surface F is said to be a *knotted surface* if exactly one of V_F or W_F is homeomorphic to a handlebody.

(iii) The surface F is said to be a *bi-knotted surface* if neither V_F nor W_F is homeomorphic to a handlebody.

Let F_1 and F_2 be surfaces in S^3. Two surfaces F_1 and F_2 are said to be *isotopic*, denoted by $F_1 \cong F_2$, if there exists an isotopy $f_t : S^3 \to S^3$, $t \in [0,1]$ such that $f_0 = \mathrm{id}_{S^3}$ and $f_1(F_1) = F_2$. We denote by V_i and W_i the closures of the connected components of $S^3 \backslash F_i$, $(i = 1, 2)$. Let $V_i = H_{V_i} \cup C_{V_i}$ and $W_i = H_{W_i} \cup C_{W_i}$ be Heegaard splittings of V_i and W_i, respectively, consisting of handlebodies H_{V_i}, H_{W_i} and compression bodies C_{V_i}, C_{W_i}. We call the 2-component handlebody link $L_i := H_{V_i} \sqcup H_{W_i}$ an *associated 2-component handlebody-link* of F_i.

We now introduce the notion of *incompressible neighborhood.*

Definition 2.14 (Incompressible neighborhood, [2])**.** Let M be an orientable irreducible 3-manifold. Let F be a compressible boundary component of M. Then, a 3-dimensional submanifold V of M is said to be an *incompressible neighborhood* of F if V satisfies the following conditions.

(1) The 3-manifold V is a compact connected submanifold of M such that $F \subset V \subset M$ and $\partial V \backslash F \subset \mathrm{int}(M)$.

(2) The 2-manifold $\partial V \backslash F$ is incompressible in M.

(3) For some $x_0 \in F$,

$$\mathrm{Image}(\pi_1(V, x_0) \to \pi_1(M, x_0)) = \mathrm{Image}(\pi_1(F, x_0) \to \pi_1(M, x_0)).$$

We first give a necessary and sufficient condition of the existence of a closed connected orientable surface in S^3 corresponding to a given 2-component handlebody-link. We use Kneser's theorem and a characterization of 3-manifolds (with connected boundary) whose fundamental groups are free. We also use Lemma 2.5, then we have the following.

Proposition 2.15. *Let $L = H_1 \sqcup H_2$ be a 2-component handlebody-link. We denote by $E(L) := S^3 \backslash \mathrm{int}(L)$ the exterior of L. Then, there exists a closed connected orientable surface F in S^3 such that ∂H_1 and ∂H_2 are Heegaard surfaces of the connected components the exterior of F if and only if the fundamental group $\pi_1(E(L))$ is given by free products of the fundamental group of a closed connected orientable surface and some infinite cyclic groups.*

Generally, since a Heegaard splitting of a compact connected orientable 3-manifold is not unique, then a 2-component handlebody-link associated to a closed connected orientable surface in S^3 is not unique. However, by using the Reidemeister–Singer theorem, we have the following.

Proposition 2.16. *Let F be a closed connected orientable surface in S^3 and L be an associated 2-component handlebody-link of F, respectively. Then, L is unique up to stabilizations of Heegaard splittings of the closures of the connected components of $S^3 \backslash F$.*

Proposition 2.17. *Let F_1 and F_2 be closed connected orientable surfaces in S^3. We denote by V_i and W_i the closures of connected components of $S^3 \backslash F_i$ $(i = 1, 2)$. Let $V_i = H_{V_i} \cup C_{V_i}$ and $W_i = H_{W_i} \cup C_{W_i}$ be Heegaard splittings of V_i and W_i $(i = 1, 2)$, respectively, where H_{V_i} and H_{W_i} are handlebodies and C_{V_i} and C_{W_i} are compression bodies. Let $L_1 = H_{V_1} \sqcup H_{W_1}$ and $L_2 = H_{V_2} \sqcup H_{W_2}$ be associated 2-component handlebody-links of F_1 and F_2, respectively. If L_1 and L_2 are stably equivalent, then F_1 and F_2 are isotopic.*

Proof. We assume that H_{V_1} is mapped to H_{V_2} and H_{W_1} is mapped to H_{W_2} by an isotopy of S^3. We denote by $E(L_1)$ the exterior of L_1. Then, by Lemma 2.5, we can show that $E(L_1)$ is irreducible. If both C_{V_1} and C_{V_2} admit 1-handles, then we can show that ∂H_{V_1} and ∂H_{V_1} are compressible in $E(L_1)$. Then, we can assume that C_{V_1} and C_{V_2} are incompressible neighborhoods of ∂H_{V_1} and ∂H_{V_2}. By Theorem 2.4, C_{V_1} and C_{V_2} are unique up to ambient isotopy of $E(L_1)$. Therefore, by the use of isotopy extension theorem, we can show that F_1 and F_2 are isotopic. If exactly one of C_{V_1} or C_{V_2} admits 1-handles, let us suppose that C_{V_1} admits 1-handles. Then, by using similar argument, we can show that C_{V_1} is an incompressible neighborhood of ∂H_{V_1} and unique. On the other hand, since C_{V_2} does not admit any 1-handle, then by the triviality of a Heegaard splitting of a handlebody, C_{V_2} is in the form $\partial H_{V_2} \times [0, 1]$. Then we have the equivalence of F_1 and F_2. If neither C_{V_1} nor C_{V_2} admits any 1-handle, then by using Waldhausen's theorem (Theorem 2.11), we can show that F_1 and F_2 are isotopic. □

Proposition 2.17 implies that an isotopy class of a surface in S^3 is uniquely determined from the stably equivalence class of an associated 2-component handlebody-link of the surface.

2.3 G-family of quandles and handlebody-links

Let us now go on to introducing the notions of quandles and a G-families of quandles. A lot of invariants of links have been obtained by using quandles. Moreover, G-families of quandles can be used for studying handlebody-links and gives plenty of invariants(refer to [7]). We start by the definition of quandles. We refer the reader [7] about the theory of G-families of quandles used in the paper.

Definition 2.18. Let X be a non-empty set with a binary operation $* : X \times X \to X$. The pair $(X, *)$ is a *quandle* if $*$ satisfies the following conditions.

(i) $x * x = x$ for any $x \in X$.

(ii) The map $S_x : X \to X$ defined by $S_x(y) = y * x$ is bijective for any $x \in X$.

(iii) $(x * y) * z = (x * z) * (y * z)$ for any $x, y, z \in X$.

Definition 2.19. Let G be a group and X be a non-empty set with a family of binary operations $*_g : X \times X \to X$ parametrized by $g \in G$, respectively. The pair $(X, \{*_g\}_{g \in G})$ is a *G-family of quandles* if for any $x, y, z \in X$ and any $g, h \in G$, $*_g$ satisfies the following conditions (see [7]).

(i) $x *_g x = x$.

(ii) $x *_{gh} y = (x *_g y) *_h y$ and $x *_{e_G} y = x$.

(iii) $(x *_g y) *_h z = (x *_h z) *_{h^{-1}gh} (y *_h z)$.

We note that for a G-family of quandles, the pair $(X, *_g)$ is a quandle for each $g \in G$.

Definition 2.20. Let G and $(X, \{*_g\}_{g\in G})$ be a group and a G-family of quandles, respectively. Set $Q = X \times G$. We define the binary operation $* : Q \times Q \to Q$ by $(x, g) * (y, h) = (x *_h y, h^{-1}gh)$. Then, the pair $(Q, *)$ is a quandle called the *associated quandle* of X.

Using a G-family of quandles $(X, \{*_g\}_{g\in G})$, we can introduce the notion of an X-coloring for an oriented diagram D of a habdlebody-link. We denote by $\mathscr{A}(D)$ the set of arcs of D. The normal orientation of an oriented arc is given by rotating an orientation of the arc counterclockwise by $\pi/2$. The normal orientation of an oriented arc is represented by an arrow on the arc (see Figure 1). Using a G-family of quandles $(X, \{*_g\}_{g\in G})$, we can introduce the notion of an X-coloring for an oriented diagram D of a habdlebody-link.

Definition 2.21 (X-colorings)**.** Let G be a group and $(X, \{*_g\}_{g\in G})$ a G-family of quandles, respectively. Let D be an oriented diagram of a handlebody-link. A map $C : \mathscr{A}(D) \to Q = X \times G$ is an *X-coloring* of D if C satisfies the following conditions:

(i) at each crossing χ of D, the map C satisfies $C(\chi_2) = C(\chi_1) * C(\chi_3)$, and

(ii) at each vertex ω of D and for the two natural projections $p_X : Q \to X$ and $p_G : Q \to G$, the map C satisfies

$$\begin{cases} p_X \circ C(\alpha_1) = p_X \circ C(\alpha_2) = p_X \circ C(\alpha_3), \\ (p_G \circ C(\alpha_1))^{\varepsilon(\omega,\alpha_1)} \cdot (p_G \circ C(\alpha_2))^{\varepsilon(\omega,\alpha_2)} \cdot (p_G \circ C(\alpha_3))^{\varepsilon(\omega,\alpha_3)} = e_G, \end{cases}$$

where χ_1, χ_2 are under arcs and χ_3 is an over-arc of D (Figure 1). Furthermore, $\varepsilon(\omega, \alpha_i)$ is the sign of an arc α_i at ω which is defined as follows (see Figure 2) :

$$\varepsilon(\omega, \alpha_i) = \begin{cases} 1, & \text{if the orientation of } \alpha_i \text{ points into } \omega, \\ -1, & \text{otherwise.} \end{cases}$$

We denote by $\mathrm{Col}_X(D)$ the set of X-colorings of D.

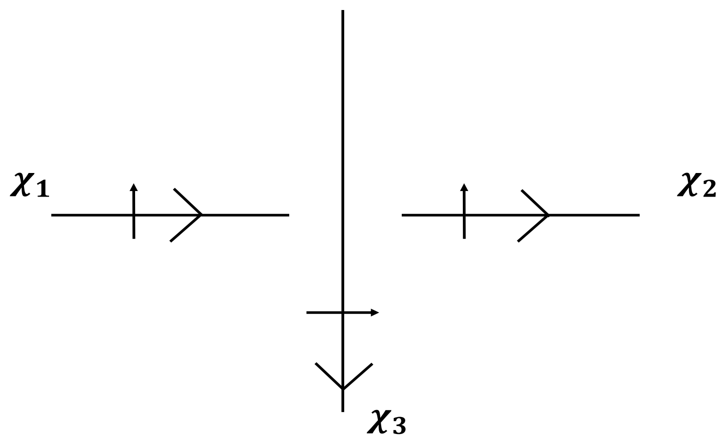

Figure 1: Crossing χ and normal orientations of χ_1, χ_2, and χ_3

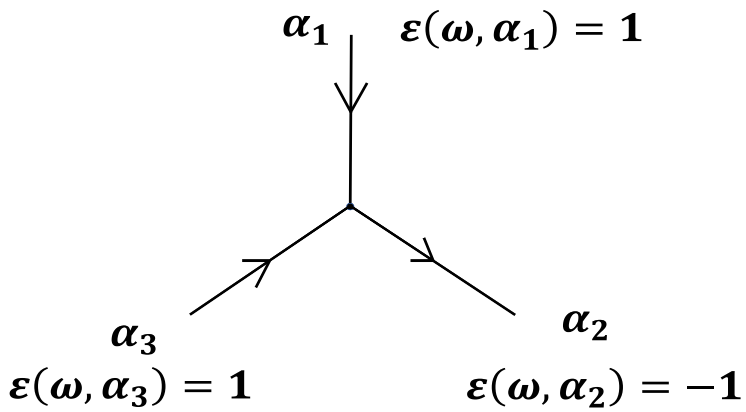

Figure 2: Vertex ω

Theorem 2.22. ([7]) *Let G be a group and $(X, \{*_g\}_{g \in G})$ be a G-family of quandles, respectively. Let D be a diagram of an oriented spatial trivalent graph of a handlebody-link. Then, the cardinality $\sharp \mathrm{Col}_X(D)$ is an invariant of the handlebody-link.*

Let D be an oriented diagram of a handlebody-knot H. Let $H^{'}$ be a handlebody-knot obtained from the disk sum of H and the standard solid torus. Then, a diagram $D^{'}$ of $H^{'}$ is obtained from D by attaching an edge and a circle component to an arc of D. Therefore, an orientation of $D^{'}$ is induced from that of D except for the edge and the circle component.

The following lemma will be used in Section 3 for constructing invariants of embedded surfaces.

Lemma 2.23. *Let G and X be a finite group and a finite set, respectively, and $(X, \{*_g\}_{g\in G})$ be a G-family of quandles. Let F be a closed connected orientable surface in S^3. We denote by V_F and W_F the closures of the connected components of $S^3\backslash F$. Let (V_F, F_V) be a Heegaard splitting of V_F. Let $(V_F, F_V^{'})$ be a Heegaard splitting obtained from (V_F, F_V) by applying a stabilization. Let H_{F_V} and $H_{F_V^{'}}$ be handlebody-knots obtained from the Heegaard splittings (V_F, F_V) and $(V_F, F_V^{'})$, respectively. Let D_{F_V} be an oriented diagram of H_{F_V} and $D_{F_V^{'}}$ be a diagram of $H_{F_V^{'}}$ with an orientation induced from that of D_{F_V}. Then, we have $\sharp\mathrm{Col}_X(D_{F_V^{'}}) = \sharp\mathrm{Col}_X(D_{F_V}) \cdot \sharp G$.*

Proof. By the definition of a stabilization and by using an isotopy S^3, we may assume that the diagram $D_{F_V^{'}}$ is obtained from D_{F_V} by attaching an edge e_0 and an S^1 component β to an arc α_0 of D_{F_V} so that e_0 and β do not admit crossings. Then, we give arbitrary orientations to e_0 and β. For the other arcs, orientations are induced from that of D_{F_V} as shown in Figure 3. Let C be an X-coloring of D_{F_V}. Suppose that $C(\alpha_0) = (x, g) \in X \times G$. Then, C is extended to an X-coloring of $D_{F_V^{'}}$ by defining $C(e_0) = (x, e_G)$ and $C(\beta) = (x, h)$. Hence, we obtain $\sharp\mathrm{Col}_X(D_{F_1^{'}}) \geqslant \sharp\mathrm{Col}_X(D_F) \cdot \sharp G$. Conversely, for any $C(\beta) = (x, h)$ we have $C(e_0) = (x, e_G)$ by the axiom of an X-coloring. Then we have $\sharp\mathrm{Col}_X(D_{F_V^{'}}) = \sharp\mathrm{Col}_X(D_{F_V}) \cdot \sharp G$. The same holds for those of W_F. □

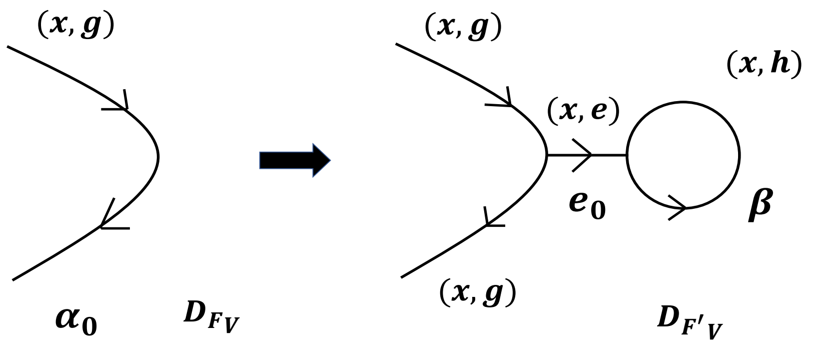

Figure 3: Attaching an edge e_0 and a circle component β to D_{F_V}

Let us introduce the notions of an X-set Y and an X_Y-coloring of a diagram of a handlebody-link (refer to [7]).

Definition 2.24. Let $(X, \{*_g\}_{g\in G})$ be a G-family of quandles, and let Y be a non-empty set with a family of maps $\bar{*}_g : Y \times X \to Y$ parametrized by $g \in G$. The pair $(Y, \{\bar{*}_g\}_{g\in G})$ is called an *X-set* if for any $y \in Y$, $x, x^{'} \in X$, and any $g, h \in G$, the following conditions are satisfied.

(i) $y\bar{*}_{gh}x = (y\bar{*}_g x)\bar{*}_h x$ and $y\bar{*}_{e_G}x = y$.

(ii) $(y\bar{*}_g x)\bar{*}_h x^{'} = (y\bar{*}_h x^{'})\bar{*}_{h^{-1}gh}(x *_h x^{'})$.

Let D be an oriented diagram of a handlebody-link. Let $D^{'}$ be a diagram obtained from D by connecting undercrossing arcs at each crossing of D. Then $D^{'}$ admits no crossing. We call a connected component of $\mathbb{R}^2\backslash D'$ a *complementary region* of D. We denote by $\mathscr{R}(D)$ the set of complementary regions of D. We set $y * (x, g) = y \bar{*}_g x$ for $y \in Y$ and $(x, g) \in X \times G = Q$. Then, we introduce the notion of X_Y*-colorings*.

Definition 2.25 (X_Y-coloring)**.** Let G be a group, and $(X, \{*_g\}_{g\in G})$ and $(Y, \{\bar{*}_g\}_{g\in G})$ be a G-family of quandles and an X-set, respectively. Let D be an oriented diagram of a handlebody-link. An X_Y*-coloring* of D is a map $C : \mathscr{A}(D) \cup \mathscr{R}(D) \to Q \cup Y$ satisfying the following conditions. We denote by $\mathrm{Col}_X(D)_Y$ the set of X_Y-colorings of the oriented diagram D.

C1. $C(\mathscr{A}(D)) \subset Q$ and $C(\mathscr{R}(D)) \subset Y$.

C2. The restriction of C on $\mathscr{A}(D)$ is an X-coloring of D.

C3. For an over-arc α and adjacent complementary regions α_1 and α_2, C satisfies $C(\alpha_2) = C(\alpha_1) * C(\alpha)$, where the normal orientation of α points from α_1 to α_2 (see Figure 4).

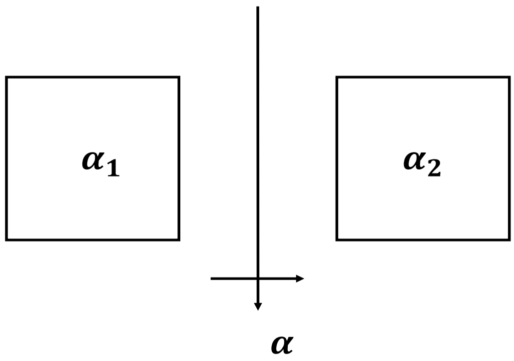

Figure 4: The coloring condition C3

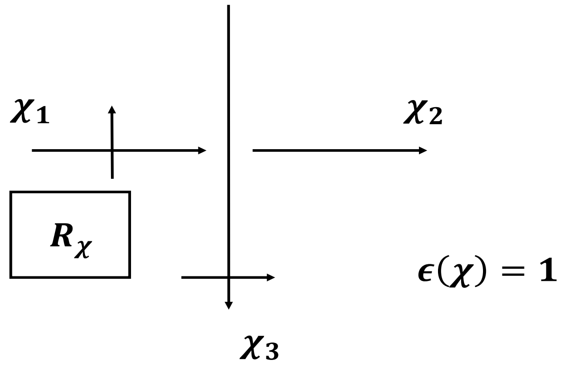

Figure 5: Weight of a crossing χ

Using a G-family of quandles and an Abelian group A, we define a chain complex, denoted by $C_*(X)_Y$, and the cochain complex, denoted by $C^*(X;A)_Y := \mathrm{Hom}(C_*(X)_Y, A)$. Then, we also define the associated homology groups and cohomology groups (refer to [7]). For an X_Y-coloring C and a crossing χ of a diagram D of an oriented spatial trivalent graph of a handlebody-link, we define the weight of the crossing χ by $w(\chi;C) := \epsilon(\chi)(C(R_\chi), C(\chi_1), C(\chi_3))$, where χ_1 is an under-arc such that the orientation of χ_1 points into the crossing χ, χ_3 is an over-arc whose normal orientation points from χ_1 to the other arc χ_2, R_χ is a complementary region such that normal orientations of χ_1 and χ_3 point to the opposite regions with respect to χ_1 and χ_3 (Figure 5), and $\epsilon(\chi)$ is the sign of the crossing χ.

Concerning homology theory of G-families of quandles, the following lemma is known (see [7]).

Lemma 2.26. *Let $(X, \{*_g\}_{g\in G})$ and $(Y, \{\bar{*}_g\}_{g\in G})$ be a G-family of quandles and an X-set, respectively. Let D be a diagram of an oriented spatial trivalent graph of a handlebody-link. Let C be an X_Y-coloring of D. Then, the sum of the weights $W(D;C) := \sum_{\chi\in D} w(\chi;C)$ is a 2-cycle of $C_*(X)_Y$.*

Let A be an Abelian group. Let θ be a 2-cocycle of the cochain complex $C^*(X;A)_Y$. We define the multiset $\Phi_\theta(D)$ as follows:

$$\Phi_\theta(D) := \{\theta(W(D;C)) \in A \mid C \in \mathrm{Col}_X(D)_Y\}.$$

Concerning cohomology theory of G-families of quandles, the following theorem is also known (see [7]).

Theorem 2.27. *Let $(X, \{*_g\}_{g\in G})$ and $(Y, \{\bar{*}_g\}_{g\in G})$ be a G-family of quandles and an X-set, respectively. Let H be a handlebody-link and D be a diagram of an oriented spatial trivalent graph of H. Let A and θ be an Abelian group and a 2-cocycle of the cochain complex $C^*(X;A)_Y$, respectively. Then, $\Phi_\theta(D)$ does not depend on the choice of D and is an invariant of the handlebody-link H.*

We can write the multiset $\Phi_\theta(D)$ in the form

$$\Phi_\theta(D) = \{(a_1)_{l_1}, \ldots, (a_m)_{l_m}\},$$

where l_j is the multiplicity of $a_j \in A$, and $(a_j)_{l_j}$ represents $\underbrace{a_j, \ldots, a_j}_{l_j\text{-times}}$. Using these notations and a natural number N, we define the set $\Phi_\theta(D)_N$ as follows:

$$\Phi_\theta(D)_N := \{(a_1, l_1/N), \ldots, (a_m, l_m/N) \mid (a_i, l_i/N) \in A \times \mathbb{Q}\}.$$

3 Main results

In this section, we construct algebraic invariants of embedded surfaces in S^3. We first use X-colorings combined with Lemma 2.23. Then, we will also use another quandle invariant given in Theorem 2.27.

Theorem 3.1. *Let G be a finite group and $(X, \{*_g\}_{g\in G})$ a G-family of quandles, where X is a finite set. Let F be a closed connected orientable surface in S^3. We denote by V_F and W_F the connected components of the exterior of F. Let F_V and F_W be Heegaard surfaces of V_F and W_F, and H_V and H_W the corresponding handlebodies, respectively. We denote by D_V and D_W diagrams of oriented spatial trivalent graphs representing H_V and H_W, respectively. Then, the unordered pair*

$$\left(\frac{\sharp\mathrm{Col}_X(D_V)}{(\sharp G)^{g(F_V)}}, \frac{\sharp\mathrm{Col}_X(D_W)}{(\sharp G)^{g(F_W)}}\right)$$

of rational numbers is an isotopy invariant of F.

Proof. Let (V_F, F_V) and $(V_F, F_V^{'})$ be Heegaard splittings of V_F. Then, we have the corresponding pairs of a handlebody-knot and a compression body in S^3, say (H_V, C_V) and $(H_V^{'}, C_V^{'})$, respectively. By the Reidemeister–Singer theorem, the two Heegaard splittings (V_F, F_V) and $(V_F, F_V^{'})$ become equivalent to the same Heegaard splitting $(V_F, \bar{F}_V)$ after m times and n times stabilizations, respectively, for some non-negative integers m and n. Let $\bar{H}$ be the handlebody-knot and $\bar{C}_V$ be the compression body corresponding to $(V_F, \bar{F}_V)$. Let $\bar{D}_V$ be a diagram of an oriented spatial trivalent graph representing $\bar{H}_V$. Then, by Lemma 2.23, we observed a variation of the cardinality of the set of X-coloring after applying a stabilization. Then, we have $\sharp\mathrm{Col}_X(\bar{D}_V) = \sharp\mathrm{Col}_X(D_V) \cdot (\sharp G)^m = \sharp\mathrm{Col}_X(D_V^{'}) \cdot (\sharp G)^n$. On the other hand, we have $g(\bar{F}_V) = g(F_V) + m = g(F_V^{'}) + n$. Hence we see

$$\left(\frac{\sharp\mathrm{Col}_X(\bar{D}_V)}{(\sharp G)^{g(\bar{F}_V)}}\right) = \left(\frac{\sharp\mathrm{Col}_X(D_V)}{(\sharp G)^{g(F_V)}}\right) = \left(\frac{\sharp\mathrm{Col}_X(D_V^{'})}{(\sharp G)^{g(F_V^{'})}}\right).$$

Combining Theorem 2.22, we have the desired conclusion. □

In Theorem 3.1, we used each connected component of a 2-component handlebody-link independently. Then we constructed an invariant of handlebody-knots up to stabilizations. However, we can show the following even if we use an invariant of 2-component handlebody-links up to stabilizations.

Theorem 3.2. *Let G be a finite group and $(X, \{*_g\}_{g\in G})$ a G-family of quandles, where X is a finite set. Let F be a closed connected orientable surface in S^3. We denote by V_F and W_F the connected components of the exterior of F. Let F_V and F_W be Heegaard surfaces of V_F and W_F, and H_V and H_W the corresponding handlebodies, respectively. Let $L = H_V \sqcup H_W$ be the 2-component handlebody-link. We denote by D a diagram of an oriented spatial trivalent graph representing L. Then, the rational number*

$$\frac{\sharp \mathrm{Col}_X(D)}{(\sharp G)^{g(F_V)+g(F_W)}}$$

is an isotopy invariant of F.

By using an argument similar to that used in Theorem 3.1, we can also construct another algebraic invariant of embedded surfaces in S^3. In order to construct such an invariant, we use Theorem 2.22 and 2.27.

Theorem 3.3. *Let G be a finite group and $(X, \{*_g\}_{g\in G})$ a G-family of quandles, where X is a finite set. Let Y be an X-set, where Y is a finite set. Let F be a closed connected orientable surface in S^3. We denote by V_F and W_F the connected components of the exterior of F. Let F_V and F_W be Heegaard surfaces of V_F and W_F, and H_V and H_W the corresponding handlebodies, respectively. Let $L = H_V \sqcup H_W$ be a 2-component handlebody-link. We denote by D a diagram of an oriented spatial trivalent graph representing L. Then, the multiset $\Phi_\theta(D)_{(\sharp G)^{g(F_V)+g(F_W)}}$ is an isotopy invariant of F.*

Proof. Let $L^{'}$ be a 2-component handlebody-link obtained from L by applying a stabilization, and we denote by $D^{'}$ a diagram of an oriented spatial trivalent graph representing $L^{'}$. By the definition of a stabilization of handlebody-links, attached solid tori are separated, then the condition of crossings of $D^{'}$ is same as that of D. Moreover, by using an argument similar to that used in the proof of Lemma 2.23, we can show that $\sharp \mathrm{Col}_X(D^{'})_Y = \sharp \mathrm{Col}_X(D_V)_Y \cdot \sharp G$. Then, we also have an invariance of the weight of each crossing. Therefore, we obtain the required conclusion. □

For a 2-component handlebody-link, its linking number is an invariant (refer to [9]). We have a 2-component handlebody-link from a closed connected orientable surface in S^3 by considering Heegaard splittings of the connected components of the exterior of the surface. By the definition of the stabilization of handlebody-links, we can show that linking number does not change after applying a stabilization. Then, the linking number of a 2-component handlebody-link obtained from F is also an isotopy invariant of F.

4 Examples

We compute our invariants given in Theorem 3.1 for the following examples of bi-knotted surfaces. Throughout this section, we set $X = \mathbb{Z}/3\mathbb{Z}$, $G = \mathbb{Z}/2\mathbb{Z}$, $g *_0 h := g$, $g *_1 h = 2h - g$ for any $g, h \in \mathbb{Z}/3\mathbb{Z}$ and $0, 1 \in \mathbb{Z}/2\mathbb{Z}$. Then, it is known that $(X, \{*_g\}_{g\in G})$ is a G-family of quandles ([7]).

In order to construct bi-knotted surfaces, we consider two mutually disjoint 3-balls B_1 and B_2, respectively. Then we take properly embedded arcs α_1 and α_2 from B_1 and B_2, respectively. We assume that α_1 is unknotted and α_2 is knotted corresponding to the trefoil. We denote by $N(\alpha_1)$ and $N(\alpha_2)$ regular neighborhoods of α_1 and α_2. Then we remove regular neighborhoods

$N(\alpha_1)$ and $N(\alpha_2)$ from B_1 and B_2. We set $B_1^{'} = \mathrm{cl}(B_1 \backslash N(\alpha_1))$ and $B_2^{'} = \mathrm{cl}(B_2 \backslash N(\alpha_2))$. Then we connect $B_1^{'}$ and $B_2^{'}$ by the 1-handle $D^1 \times D^2$ so that $\{-1\} \times D^2$ is attached to $\partial B_1^{'}$, $\{1\} \times D^2$ is attached to $\partial B_2^{'}$, and the attached 1-handle throughs straightforwardly both regions which are obtained from B_1 and B_2 by removing $N(\alpha_1)$ and $N(\alpha_2)$. We assume that the attached 1-handle admits a tangle induced from that of $N(\alpha_2)$. We set F as the boundary of the the resulting compact connected orientable 3-manifold M (see Figure 6). Similarly, we construct compact connected orientable 3-manifold $M^{'}$ by using an unknotted arc and a knotted arc corresponding to the figure eight knot. Then we set $F^{'}$ as the boundary of the 3-manifold (see Figure 7).

We assume that M is embedded in S^3. Then, M is one of the connected components of the exterior of $F = \partial M$, say, V_F (see Figure 8). Then, we obtain the other connected components of the exterior of F by the following procedures. We first consider the closure of the exterior of $B_1^{'}$ in S^3, which is obtained from a 3-ball B^3 by attaching $N(\alpha_1)$. Then, we remove the interior of B_2 from B^3 and attach $N(\alpha_2)$ to ∂B_2. Finally, we remove the 2-handle $D^2 \times D^1$ so that $D^2 \times \{-1\}$ is attached to ∂B^3, $D^2 \times \{1\}$ is attached to ∂B_2, and the 2-handle throughs both $N(\alpha_1)$ and $N(\alpha_2)$ straightforwardly. The resulting compact connected orientable 3-manifold, say W_F, is the other connected component of F (see Figure 9). We set $M^{'} = V_F^{'}$ (see Figure 14). Then, by a similar argument, we obtain the other connected component $W_F^{'}$ of $F^{'}$ (see Figure 15).

Moreover, by removing 2-handles from V_F and from W_F, respectively, we obtain two handlebody-knots H_V and H_W (Figures 10 and 11). Then, by projecting their oriented spatial trivalent graphs on $\mathbb{R}^2$, we obtain diagrams D_V and D_W of H_V and H_W (Figure 12 and Figure 13). By using a similar argument, we obtain handlebody-knots $H_F^{'}$ and $H_W^{'}$ (Figures 16 and 17). Similarly, by projecting their oriented spatial trivalent graphs on $\mathbb{R}^2$, we obtain diagrams $D_V^{'}$ and $D_W^{'}$ of $H_V^{'}$ and $H_W^{'}$ (Figures 18 and 19)

Example 4.1. For the surface F (Figure 6), we compute $\sharp\mathrm{Col}_X(D_V)$ and $\sharp\mathrm{Col}_X(D_W)$. Then we have $\sharp\mathrm{Col}_X(D_V) = 6$ and $\sharp\mathrm{Col}_X(D_W) = 48$. Therefore we obtain

$$\frac{\sharp\mathrm{Col}_X(D_V)}{(\sharp(\mathbb{Z}/2\mathbb{Z}))^3} = \frac{3}{4}, \ \frac{\sharp\mathrm{Col}_X(D_W)}{(\sharp(\mathbb{Z}/2\mathbb{Z}))^3} = 6.$$

Example 4.2. Let us now focus on the surface $F^{'}$ (Figure 7). We also compute $\sharp\mathrm{Col}_X(D_V^{'})$ and $\sharp\mathrm{Col}_X(D_W^{'})$. Then we have $\sharp\mathrm{Col}_X(D_V^{'}) = 6$ and $\sharp\mathrm{Col}_X(D_W^{'}) = 24$. Hence we have

$$\frac{\sharp\mathrm{Col}_X(D_{V'})}{(\sharp(\mathbb{Z}/2\mathbb{Z}))^3} = \frac{3}{4}, \ \frac{\sharp\mathrm{Col}_X(D_{W'})}{(\sharp(\mathbb{Z}/2\mathbb{Z}))^3} = 3.$$

We can show that two surfaces F and $F^{'}$ are not isotopic by using our invariant.

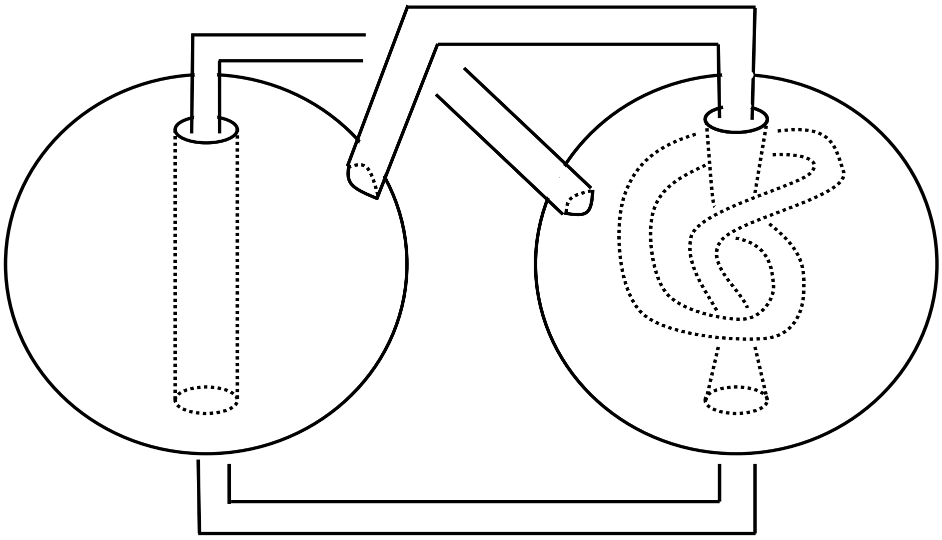

Figure 6: Bi-knotted surface F

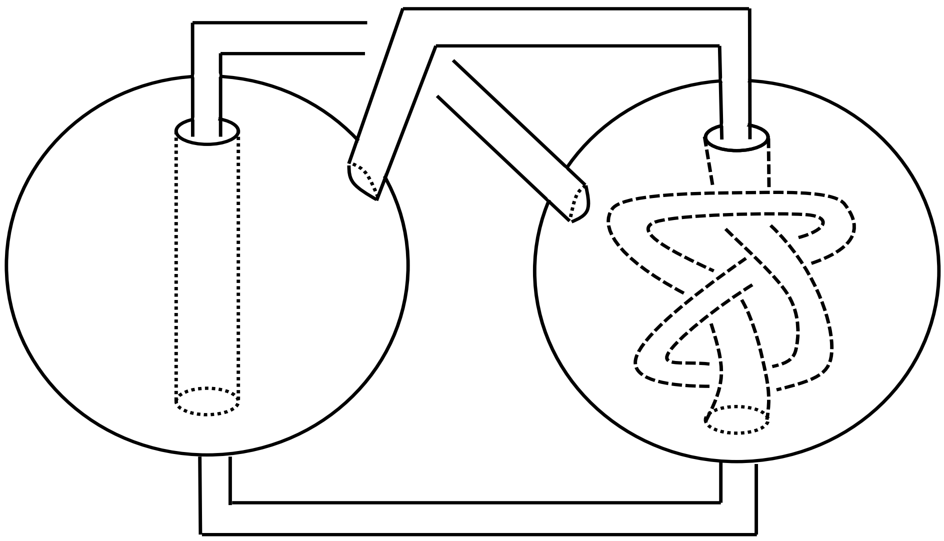

Figure 7: Bi-knotted surface $F^{'}$

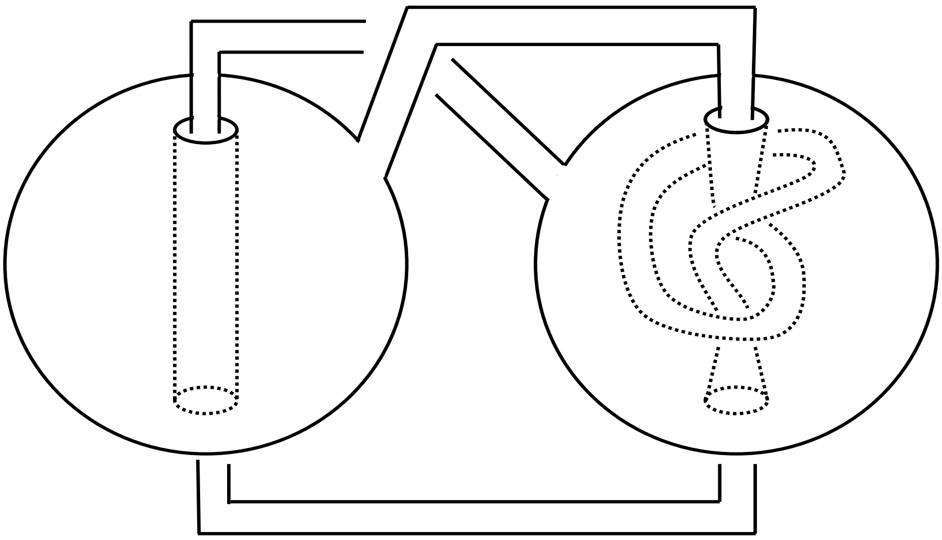

Figure 8: Exterior component V_F

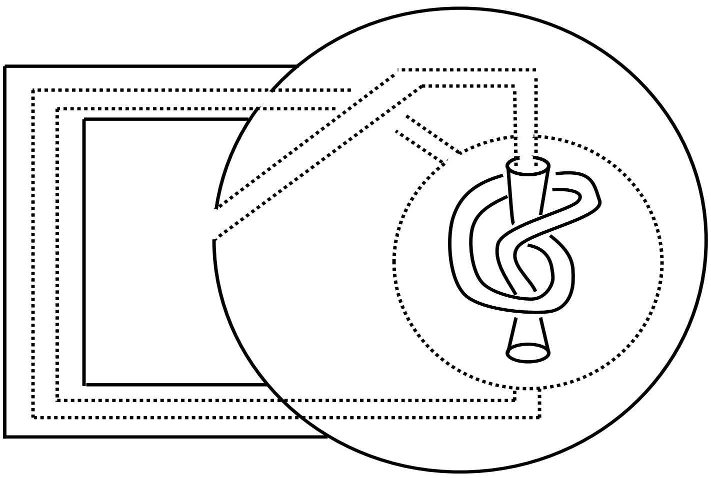

Figure 9: Exterior component W_F

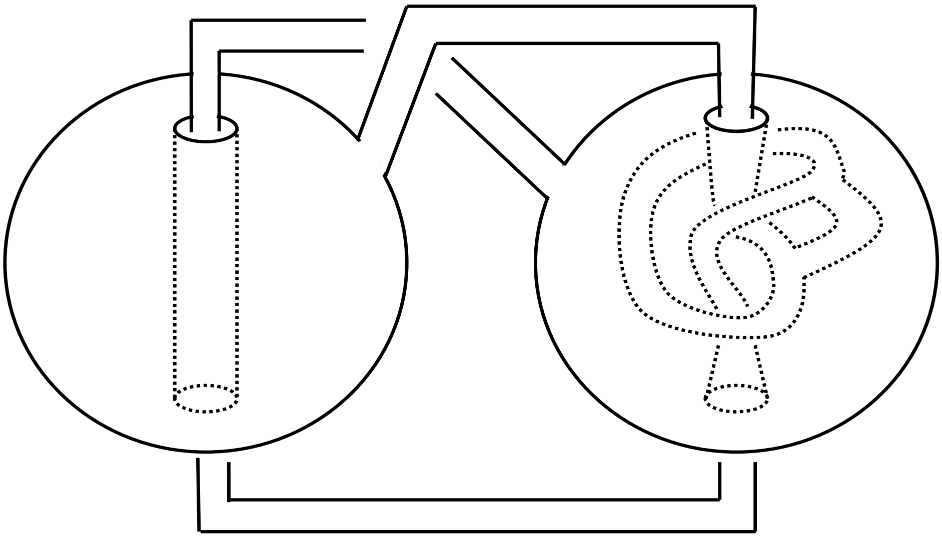

Figure 10: Handlebody-knot H_V

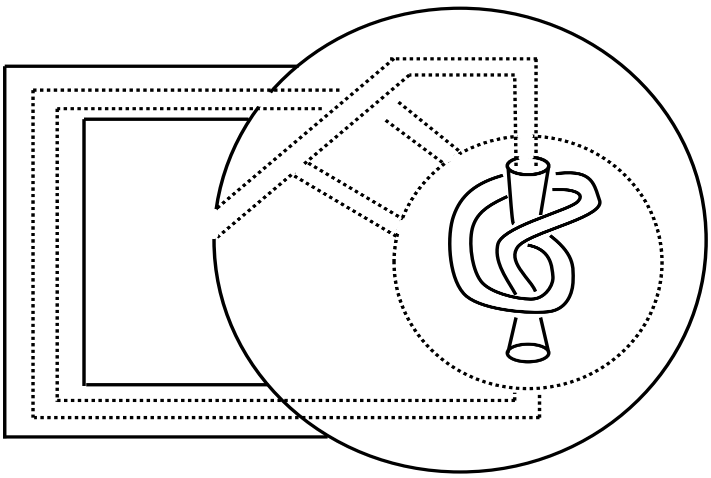

Figure 11: Handlebody-knot H_W

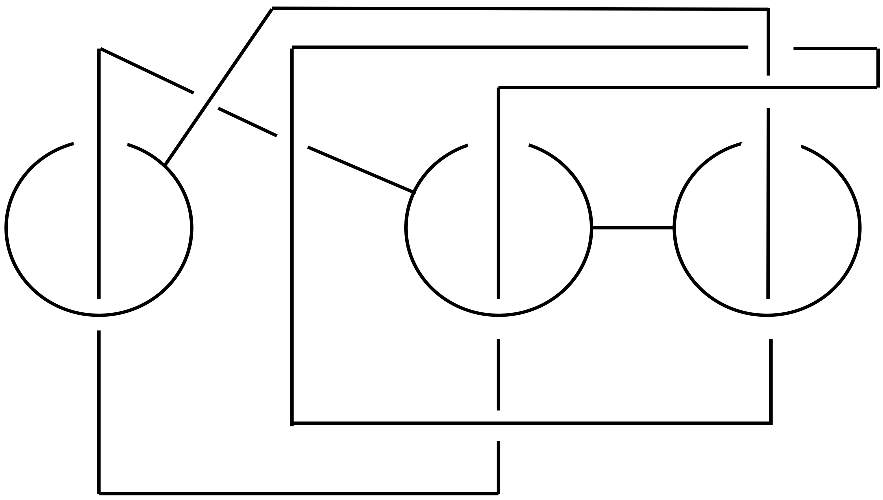

Figure 12: Diagram D_V of the handlebody-knot H_V

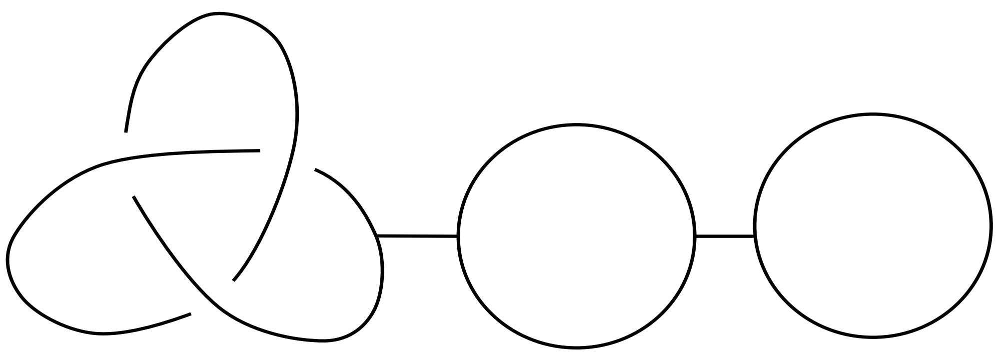

Figure 13: Diagram D_W of the handlebody-knot H_W

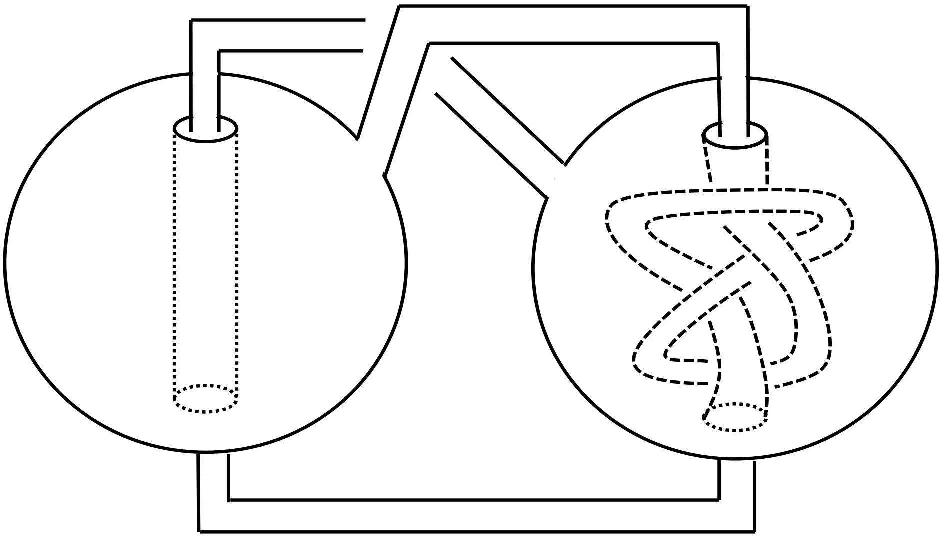

Figure 14: Exterior component $V_{F'}$

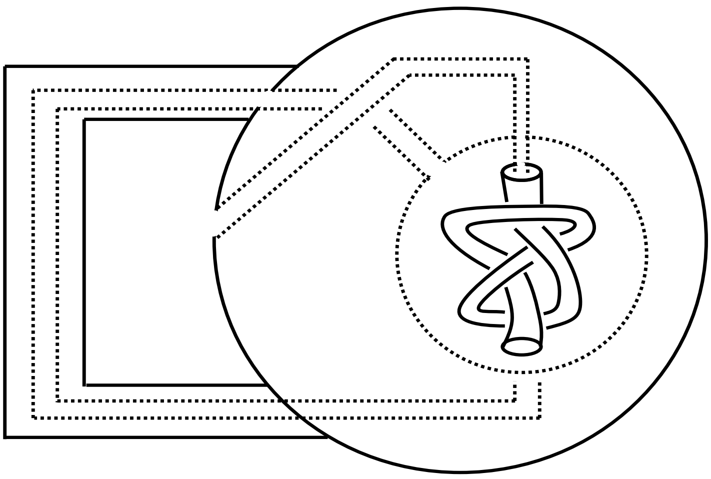

Figure 15: Exterior component $W_{F'}$

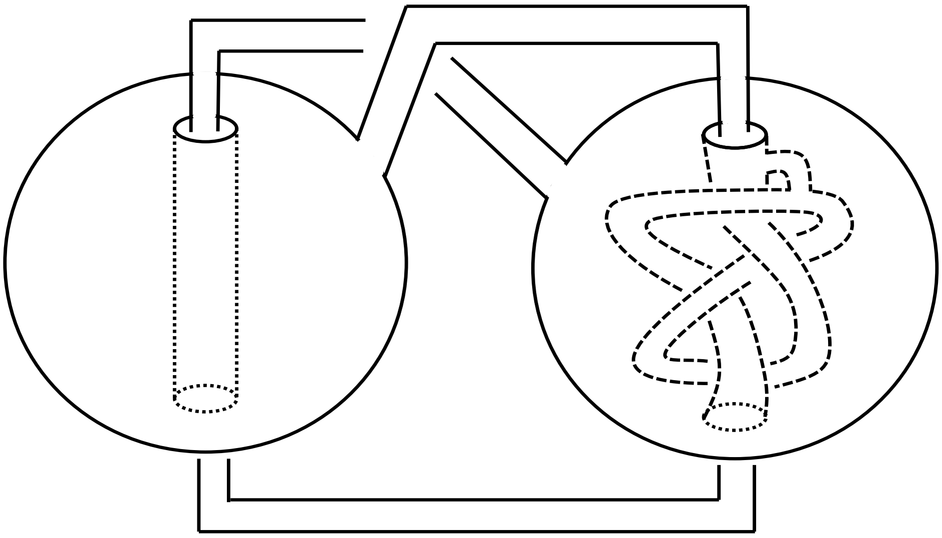

Figure 16: Handlebody-knot $H_{V'}$

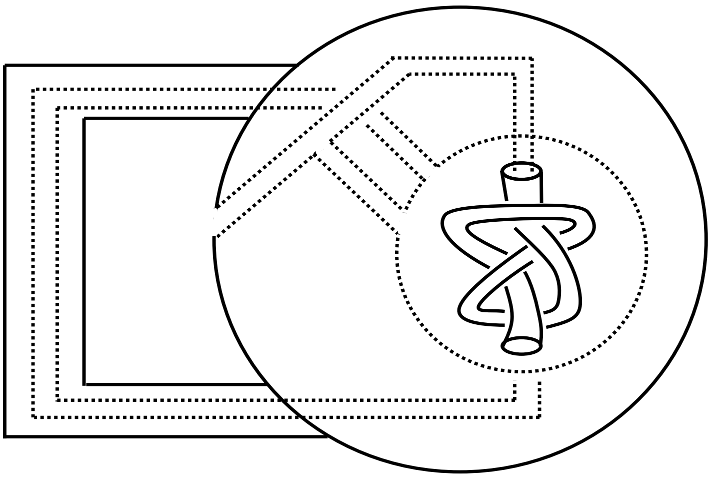

Figure 17: Handlebody-knot $H_{W'}$

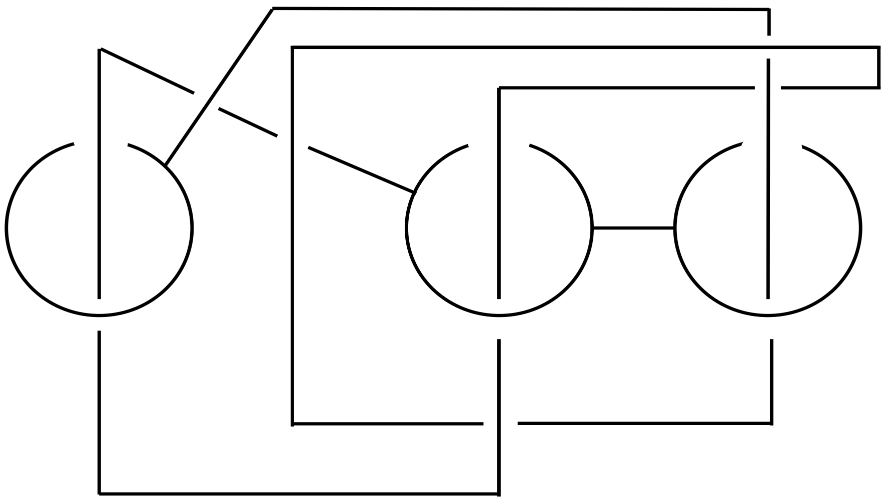

Figure 18: Diagram $D_{V'}$ of the handlebody-knot $H_{V'}$

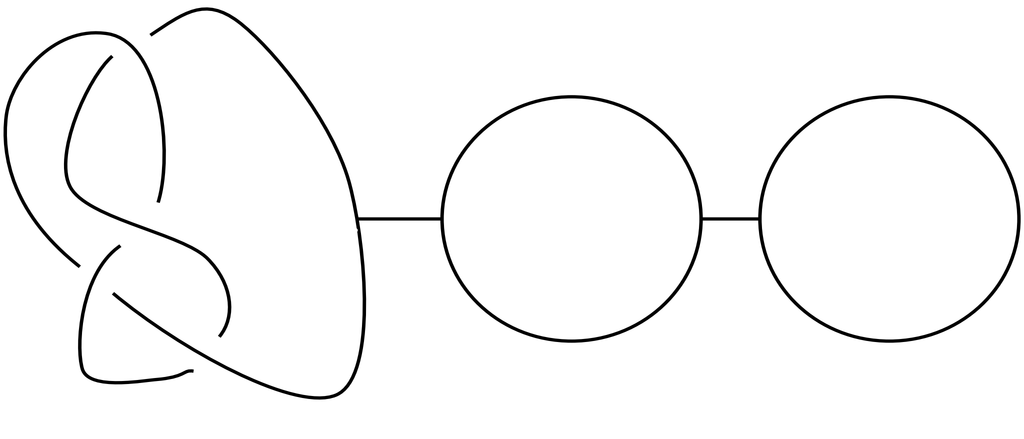

Figure 19: Diagram $D_{W'}$ of the handlebody-knot $H_{W'}$

5 Further results

In this section, we study closed connected orientable surfaces in S^3 from geometric viewpoints. Then, we construct a geometric invariant of surfaces in S^3. We also give a necessary condition of the existence of a surface which is obtained from a 2-component handlebody-link corresponding to a minimal genus Heegaard splittings of the closures of the connected components of the exterior of the surface. We denote the interval $[-1, 1]$ and the unit 2-disk by D^1 and D^2, respectively.

Definition 5.1 (Handle attaching [15]). Let F be a closed connected orientable surface embedded in S^3. Let $h_i : D^1 \times D^2 \to S^3$ $(i = 1, 2, \ldots, n)$ be embeddings of the 1-handle such that $h_i(D^1 \times D^2) \cap F = h_i(\partial D^1 \times D^2)$ and $h_i(D^1 \times D^2) \cap h_j(D^1 \times D^2) = \varnothing$ (whenever $i \neq j$). We call $h_i(D^1 \times \partial D^2)$ a *handle*. We denote by $F(h_1, h_2 \ldots, h_n) := F \cup (\sqcup_{i=1}^n h_i(D^1 \times \partial D^2)) \backslash \sqcup_{1=1}^n h_i(\partial D^1 \times \mathrm{int}(D^2))$ the surface obtained from F by attaching n handles. Note that $F(h_1, h_2 \ldots, h_n)$ is again a closed connected orientable surface embedded in S^3 and that its genus is equal to $g(F) + n$.

Two closed connected orientable surfaces F_1 and F_2 in S^3 are said to be *separated* if there exists 3-balls B_1^3 and B_2^3 such that $F_1 \subset \mathrm{int}(B_1^3)$ and $F_2 \subset \mathrm{int}(B_2^3)$. Two closed connected orientable surfaces F_1 and F_2 are also said to be *separated* if there exists a 3-ball B^3 such that $F_1 \subset \mathrm{int}(B^3)$ and $F_2 \subset S^3 \backslash B^3$. The 3-ball B^3 is called the *associated ball*. Similarly, n closed connected orientable surfaces $F_1, F_2, \ldots F_n$ in S^3 are said to be *separated* if there exists n 3-balls B_i^3 $(i = 1, 2, \ldots, n)$ such that $F_i \subset \mathrm{int}(B_i^3)$ for each $i = 1, 2, \ldots, n$.

Given two separated embedded surfaces in S^3, we can construct a new embedded surface by using the following way, which is called the *isotopy sum*.

Definition 5.2 (Isotopy sum [18]). Let F_1 and F_2 be closed connected orientable separated surfaces embedded in S^3 and B^3 be the associated ball as above, respectively. Let $\varphi : D^1 \times D^2 \to S^3$ be an embedding of the 1-handle such that $\varphi(D^1 \times D^2) \cap F_1 = \varphi(\{-1\} \times D^2)$, $\varphi(D^1 \times D^2) \cap F_2 = \varphi(\{1\} \times D^2)$, and $\varphi(D^1 \times D^2) \cap \partial B^3 = \varphi(\{0\} \times D^2)$. We define the *isotopy sum* of F_1 and F_2, denoted by $F_1 \sharp_{\mathrm{iso}} F_2$, by $F_1 \sharp_{\mathrm{iso}} F_2 := F_1 \cup F_2 \cup \varphi(D^1 \times \partial D^2) \backslash \varphi(\partial D^1 \times \mathrm{int}(D^2))$.

We note that the isotopy sum of two separated embedded surfaces does not depend on the the choice of the associated ball and the choice of an embedding φ up to an isotopy of S^3. Moreover, it does not depend on the order of F_1 and F_2. We also note that $(F_1 \sharp_{\mathrm{iso}} F_2) \sharp_{\mathrm{iso}} F_3 \cong F_1 \sharp_{\mathrm{iso}} (F_2 \sharp_{\mathrm{iso}} F_3)$ for separated closed connected orientable surfaces F_1, F_2 and F_3 in S^3. In this sense the isotopy sum is well-defined. For an embedded surface F, if F is isotopic to the isotopy sum $F_1 \sharp_{\mathrm{iso}} F_2 \sharp_{\mathrm{iso}} \cdots \sharp_{\mathrm{iso}} F_k$ for separated surfaces $F_1, F_2, \ldots, F_k$, we call it a *decomposition of F into factors F_i, $i = 1, 2, \ldots, k$.*

In order to construct a geometric invariant of closed connected orientable surfaces in S^3, we introduce the notion of the *tunnel number* of a compact connected orientable 3-manifold with connected boundary embedded in S^3.

Definition 5.3 (Tunnel number). Let V be a compact connected orientable 3-manifold with connected boundary embedded in S^3. The *tunnel number* of V is the minimal number n of mutually disjoint 1-handles such that $\mathrm{cl}(S^3 \backslash (V \bigcup \sqcup_{i=1}^n h_i(D^1 \times D^2)))$ is homeomorphic to a handlebody and $V \cap h_i(D^1 \times D^2) = h_i(\partial D^1 \times D^2)$, where h_i $(i = 1, 2, \ldots n)$ are embeddings of the 1-handle $D^1 \times D^2$ into S^3. We denote by $t(V)$ the tunnel number of V_F.

It is not hard to see that since every compact connected orientable 3-manifold possibly with boundary admits a Heegaard splitting, $t(V)$ is always finite. Let F be a closed connected orientable surface in S^3. We denote by V_F and W_F the closures of the connected components of $S^3\backslash F$. By Definition 5.3, we also see that the Heegaard genus of W_F is given by $g(F) + t(V_F)$. Similarly, the Heegaard genus of V_F is given by $g(F) + t(W_F)$.

We note that if V_F is a handlebody-knot, then the tunnel number of V_F coincides with that of a handlebody-knot [8].

Proposition 5.4. *Let $h_i : D^1 \times D^2 \to S^3$ $(i = 1, 2, \dots, n)$ be embeddings of the 1-handle such that $h_i(D^1 \times D^2) \cap F = h_i(\partial D^1 \times D^2)$ and $h_i(D^1 \times D^2) \cap h_j(D^1 \times D^2) = \varnothing$ (whenever $i \neq j$). For any closed connected orientable surface F in S^3, there exists a finite numbers of handles such that $F(h_1, \dots, h_n)$, which is a surface obtained from F by attaching n handles to F, is an unknotted surface.*

Proof. Let V_F and W_F be the closures of the connected components of $S^3\backslash F$. Then, we have $S^3 = V_F \cup W_F$. Let $t(V_F)$ and $t(W_F)$ be the tunnel numbers of V_F and W_F. Then, by the definition of the tunnel number, we obtain two handlebody-knots H_1 and H_2 such that $H_1 \cup H_2 = S^3$ and $\partial H_1 = \partial H_2$ by attaching $t(V_F) + t(W_F)$ 1-handles to the common boundary $F = V_F \cap W_F$ of V_F and W_F. Since the union of H_1 and H_2 along their boundaries gives a Heegaard splitting of S^3. Then, combining Waldhausen's theorem 2.11, the common boundary of H_1 and H_2 is an unknotted surface. □

We introduce the definition of the handle number of embedded surface.

Definition 5.5 (Handle number)**.** Let F be a closed connected orientable surface in S^3. Let $h_i : D^1 \times D^2 \to S^3$ $(i = 1, 2, \dots, n)$ be embeddings of the 1-handle such that $h_i(D^1 \times D^2) \cap F = h_i(\partial D^1 \times D^2)$ and $h_i(D^1 \times D^2) \cap h_j(D^1 \times D^2) = \varnothing$ (whenever $i \neq j$). Let $F(h_1, h_2, \dots, h_n)$ be a surface obtained from F by attaching n handles to F. The *handle number* of F, denoted by $h(F)$, is the minimal number n of handles such that $F(h_1, \dots, h_n)$ is an unknotted surface.

Let F be a closed connected orientable surface in S^3. We denote by V_F and W_F the closures of the connected components of $S^3\backslash F$. We show the relationship between the handle number of F and the tunnel numbers of V_F and W_F.

Proposition 5.6. *Let F be a closed connected orientable surface in S^3. We denote by V_F and W_F the closures of the connected components of $S^3\backslash F$. Let $t(V_F)$ and $t(W_F)$ be the tunnel numbers of V_F and W_F. Then, for the handle number of F, we have $h(F) = t(V_F) + t(W_F)$.*

Proof. Let n be the handle number of F. Let $h_i : D^1 \times D^2 \to S^3$ $(i = 1, 2, \dots, n)$ and $\{h_1, \dots, h_n\}$ be embeddings of the 1handle satisfying the conditions given in Definition 5.1 and the set of attached handles to F such that $F(h_1, \dots, h_n)$ is an unknotted surface, respectively. We divide the set of handles $\{h_1, \dots, h_n\}$ into the disjoint union of the sets of handles $\{h_1^V, \dots, h_k^V\}$ and $\{h_1^W, \dots, h_l^W\}$, where $h_i^V(D^1 \times D^2)$ is attached to V_F, and $h_j^W(D^1 \times D^2)$ is attached to W_F. Then, by Proposition 5.4, we have $n = k + l \leqslant t(V_F) + t(W_F)$. On the other hand, by the definition of the tunnel number, we have $t(V_F) \leqslant k$ and $t(W_F) \leqslant l$. Then, we obtain $t(V_F) + t(W_F) \leqslant k + l$. Therefore, we have $h(F) = k + l = t(V_F) + t(W_F)$. □

As an immediate sequence, we have that the unordered pair $(t(V_F), t(W_F))$ of non-negative integers is an invariant of F. Furthermore, it is not hard to see the following. Let $V_F = H_V \cup C_V$ and $W_F = H_W \cup C_W$ be minimal genus Heegaard splittings of V_F and W_F consisting of pairs of a handlebody H_V and a compression body C_V and a handlebody H_W and a compression body C_W, respectively. If F is an unknotted surface, then both H_V and H_W are trivial handlebody-knots. Hence $t(V_F) = 0$ and $t(W_F) = 0$. If F is a knotted surface, then exactly one of V_F or

W_F is homeomorphic to a handlebody, and the other is not homeomorphic to a handlebody. Hence exactly one of $t(V_F)$ or $t(W_F)$ is equal to zero, and the other is a positive integer. If F is a bi-knotted surface, then we have that neither V_F nor W_F is homeomorphic to a handlebody. Hence both $t(V_F)$ and $t(W_F)$ are positive integers.

It is not hard to see that since the unordered pair $(t(V_F), t(W_F))$ of non-negative integers is an invariant of F, then the handle number $h(F)$ is also an invariant of F. We prove the following corollary.

Corollary 5.7. *Let F be a closed connected orientable surface in S^3. We denote by V_F and W_F the closures of the connected components of $S^3\backslash F$. Let n be the handle number of F. Let $h_i : D^1 \times D^2 \to S^3$ $(i = 1, 2, \ldots, n)$ be embeddings of the 1handle satisfying the conditions given in Definition 5.1. Let $\{h_1, h_2, \ldots, h_n\} = \{h^V_1, \ldots, h^V_k\} \sqcup \{h^W_1, \ldots, h^W_l\}$ be the set of attached handles, where $h^V_i(D^1 \times D^2)$ is attached to V_F, and $h^W_j(D^1 \times D^2)$ is attached to W_F. Let $t(V_F)$ and W_F be the tunnel numbers of V_F and W_F. Then, $k = t(V_F)$ and $l = t(W_F)$.*

Proof. In the proof of Theorem 5.6, we saw that $k + l = t(V_F) + t(W_F)$, $t(V_F) \leqslant k$, and $t(W_F) \leqslant l$. Let us assume that $t(V_F) < k$. Then, we have $t(W_F) > l$. This is a contradiction. Hence $k = t(V_F)$. The same holds for l, that is, $l = t(W_F)$. □

Proposition 5.8. *Let F be a closed connected orientable surface in S^3. We denote by V_F and W_F the closures of the connected components of $S^3\backslash F$. For any minimal genus Heegaard splitting of V_F, say $V_F = H_V \cup C_V$, the tunnel number $t(H_V)$ of the handlebody-knot H_V does not depend on the choice of minimal genus Heegaard splittings. The same holds for any minimal genus Heegaard splitting of W_F.*

Proof. Let $V_F = H_V \cup C_V$ and $V_F = H^{'}_V \cup C^{'}_V$ be minimal genus Heegaard splittings of V_F. Let $n = k + l$ be the handle number of the surface F, where k and l are the numbers of 1-handles attached to V_F and W_F. Let $h_i : D^1 \times D^2 \to S^3$ $(i = 1, 2, \ldots, n)$ and $\{h_1, \ldots, h_n\}$ be embeddings of the 1handle satisfying the conditions given in Definition 5.1 and the set of attached handles to F such that $F(h_1, \ldots, h_n)$ is an unknotted surface, respectively. Then, we divide the set of handles $\{h_1, \ldots, h_n\}$ into the disjoint union of two sets $\{h^V_1, \ldots, h^V_k\}$ and $\{h^W_1, \ldots, h^W_l\}$, where h^V_i is a handle attached to V_F, and h^{W_j} is a handle attached on W_F. By the definition of the tunnel number of W_F, $\mathrm{cl}(S^3\backslash(W_F \bigcup \sqcup_{i=1}^{t(W_F)} h_i(D^1 \times D_2)))$ is isotopic to a handlebody. In particular, since $V_F = H_V \cup C_V$ is a minimal genus Heegaard splitting, we can assume that $\mathrm{cl}(S^3\backslash(W_F \bigcup \sqcup_{i=1}^{t(W_F)} h_i(D^1 \times D^2)))$ is homeomorphic to H_V. Hence, ∂H_V is obtained from F by attaching $t(W_F)$ handles. Then, by the definition of the handle number of F, we can also assume that we obtain an unknotted surface from ∂H_V by attaching k handles. Moreover, by the definition of the tunnel number of H_V, we obtain an unknotted surface from H_V. Then, we have $k = t(H_V)$. By using a similar argument, we also have $k = t(H^{'}_V)$. The same holds for minimal genus Heegaard splittings of W_F. □

Let F be a closed connected orientable surface embedded in S^3. We denote by V_F and W_F the closures of the connected components of $S^3\backslash F$. By considering Heegaard splittings V_F and W_F, we obtain an associated 2-component handlebody-link L of F. By applying the Reidemeister–Singer theorem, we showed that L is uniquely determined from F up to stabilizations of Heegaard splittings of V_F and W_F. Conversely, let us consider a 2-component handlebody-link $L^{'}$. Then, does there always exist a closed connected orientable surface $F^{'}$ embedded in S^3 such that the connected components of $\partial L^{'}$ are Heegaard surfaces of $V_{F^{'}}$ and $W_{F^{'}}$? A necessary and sufficient condition was given in Proposition 2.15. We consider such a problem from the viewpoints of the handle number and the tunnel number.

By Proposition 5.8, we have the following corollary.

Corollary 5.9. *Let $L = H_1 \sqcup H_2$ be a 2-component handlebody-link. Let us assume that $t(H_1) - t(H_2) \neq g(\partial H_2) - g(\partial H_1)$. Then, there does not exist a closed connected orientable surface in S^3 such that L is obtained from minimal Heegaard splittings of the closures of the exterior of the surface.*

Let M be a compact connected orientable 3-manifold whose boundary is not homeomorphic to a 2-sphere. Then, M is said to be *∂-irreducible* if for every properly embedded 2-disk D in M, ∂D bounds a 2-disk on ∂M. We assume that the boundary of M is connected, and that M is not homeomorphic to a 3-ball. Then, M is said to be *∂-prime* if for any decomposition $M_1 \natural M_2$, where M_1 and M_2 are compact connected orientable 3-manifolds, either M_1 or M_2 is homeomorphic to a 3-ball, where $M_1 \natural M_2$ is the disk sum, or the boundary connected sum, of M_1 and M_2. Then, the following is known.

Proposition 5.10 ([16])**.** *Let M be a compact connected orientable 3-manifold embeddable in S^3 with connected boundary with $g(\partial M) \geqslant 2$. Then, the following conditions are equivalent.*

- (i) *The 3-manifold M is ∂-prime.*
- (ii) *The 3-manifold M is ∂-irreducible.*
- (iii) *The fundamental group $\pi_1(M)$ of M is indecomposable with respect to free products.*

We now present the notion of *prime surfaces* in S^3.

Definition 5.11 (Prime surface [18])**.** Let F be a closed connected orientable surface in S^3 with $g(F) \geqslant 1$. Then, F is said to be *prime* if for any decomposition $F \cong F_1 \sharp_{\mathrm{iso}} F_2$, either F_1 or F_2 is a surface of genus 0. We note that every closed connected orientable surface of genus 1 is prime.

About closed connected orientable prime surfaces in S^3, the following is known.

Theorem 5.12 ([17])**.** *Let F be a closed connected orientable surface of genus 2 in S^3. Then, F is prime if and only if either $\pi_1(V_F)$ or $\pi_1(W_F)$ is indecomposable with respect to free products.*

Let M be a compact connected orientable 3-manifold with connected boundary embedded in S^3. Then, M is said to be *reducibly embedded* if there exists a 2-sphere S in S^3 such that $M \cap S$ is a properly embedded 2-disk D in M, and that D separates M into two parts which are not homeomorphic to 3-balls. Otherwise, M is said to be *irreducibly embedded.* We call a reducibly embedded handlebody-knot a *reducible handlebody-knot.* Similarly, we call an irreducibly embedded handlebody-knot an *irreducible handlebody-knot.*

Let us now focus on prime surfaces of genus 2 embedded in S^3. Let F be a prime surface of genus 2 in S^3. We denote by V_F and W_F the closures of the connected components of $S^3 \backslash F$. By virtue of Theorem 5.12 and Proposition 5.10, we know that either V_F or W_F is ∂-prime. Then, let us assume that exactly one of V_F or W_F is ∂-prime. On that assumption, we have the following proposition.

Proposition 5.13. *Let F be a closed connected orientable prime surface of genus 2 in S^3. We denote by V_F and W_F the closures of the connected components of $S^3 \backslash F$. We assume that exactly one of V_F or W_F is ∂-prime. Let $V_F = H_V \cup C_V$ and $W_F = H_W \cup C_W$ be minimal genus Heegaard splittings of V_F and W_F, respectively. Then, exactly one of H_V or H_W is a reducible handlebody-knot, and the other is an irreducible handlebody-knot.*

Proof. By the assumption, we assume that V_F is not ∂-prime and W_F is ∂-prime. Since V_F is not ∂-prime, there exists a decomposition $V_F = V_F^1 \natural V_F^2$ such that neither V_F^1 nor V_F^2 is homeomorphic to a 3-ball B^3. Then, by the use of the solid torus theorem, V_F^i $(i = 1, 2)$ are either a handlebody-knot of genus 1 or the exterior of a hanlebody-knot of genus 1.
(1) In case of both V_F^1 and V_F^2 are handlebody-knots of genus1, since F is a prime surface, then $V_F = V_F^1 \natural V_F^2$ is an irreducible handlebody-knot. Since the Heegaard splittings of a handlebody-knot are standard, H_V is an irreducible handlebody-knot. On the other hand, W_F the exterior of the handlebody-knot V_F. Then the Heegaard surfaces of W_F are unknotted surfaces. Therefore H_W is a reducible handlebody-knot.
(2) In case of V_F^1 is a handlebody-knot of genus 1 and V_F^2 is the exterior of a handlebody-knot of genus 1, then V_F^2 is obtained from a 3-ball B^3 by removing a 2-handle (see Figure 20). Then, by connecting V_F^1 and V_F^2 by a 1-handle, we obtain V_F. It can be also assumed to be that V_F is obtained from V_F^2 by attaching a 1-handle which throughs the removed 2-handle from B^3 at least one time up to isotopy. Since V_F^1 is a handlebody-knot, by removing 2-handles from V_F^2, the minimal genus Heegaard splitting of V_F is obtained (see Figure 21). By considering a minimal genus Heegaard splitting of V_F^2, we obtain the trivial handlebody-knot of some genus $\geqslant 2$. Such a handlebody-knot is obtained from B^3 by removing 2-handles. However, by the assumption of the 1-handle attached to V_F^2 and applying a deformation depicted in Figure 22, the attached 1-handle throughs every removed 2-handle of the trivial handlebody-knot (Figure 23). Hence H_V is an irreducible handlebody-knot. On the other hand, $W_F = \mathrm{cl}(S^3 \backslash (V_F^1 \cup V_F^2 \cup \{\text{2-handle}\})) = \mathrm{cl}((B^3 \cup \{\text{1-handle}\}) \backslash (V_F^2 \cup \{\text{2-handle}\}))$. We note that the removed 2-handle throughs both V_F^2 and the attached 1-handle at least one time up to isotopy (see Figure 24). To obtain a Heegaard splitting of W_F, we need to move the 2-handle so as not to it throughs the attached 1-handle by removing 2-handles (Figures 25 and Figure 26). Then, after removing 2-handles from B^3 we obtain a minimal genus Heegaard splitting of W_F. By the above procedure, we can see that H_W is a reducible handlebody-knot. The same holds for in case of V_F^2 is a handlebody-knot of genus 1 and V_F^1 is the exterior of a handlebody-knot of genus 1.
(3) In case of both V_F^1 and V_F^2 are the exterior of handlebody-knots of genus 1, by applying the same procedure given in (2) for V_F, we can show that V_F is an irreducible handlebody-knot. On the other hand, by applying the same procedure given in (2) for W_F, we can show that W_F is a reducible handlebody-knot. □

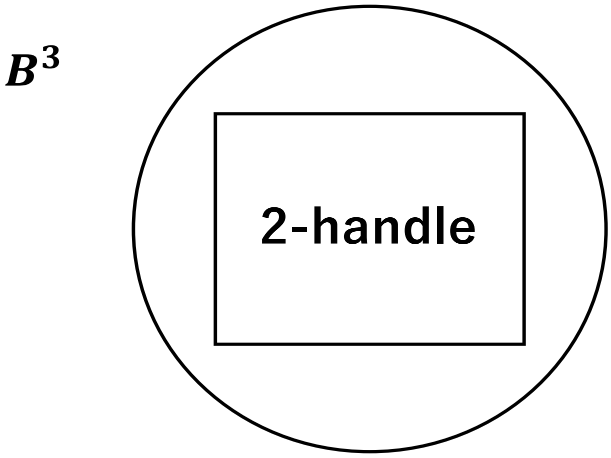

Figure 20: Construction of V_F^2

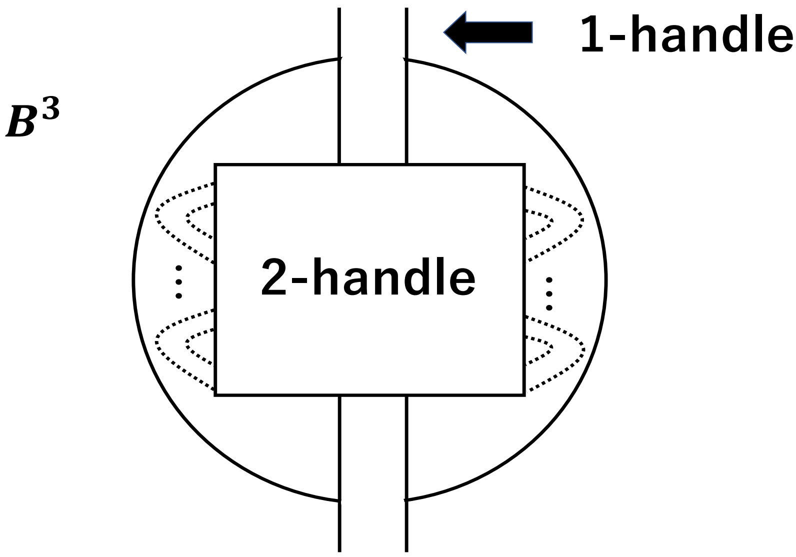

Figure 21: Removing of 2-handles

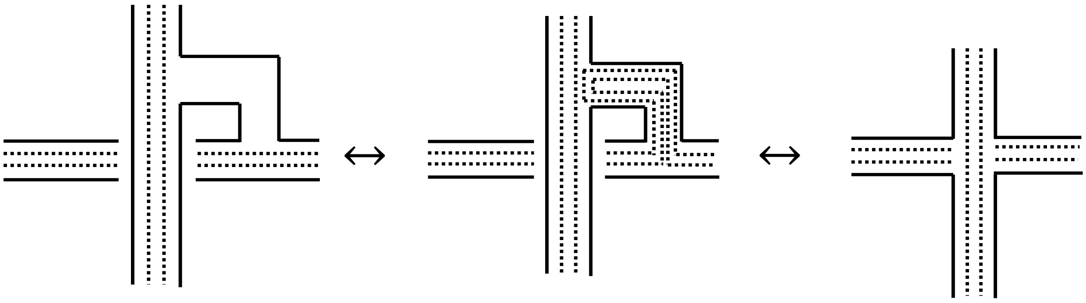

Figure 22: Deformation

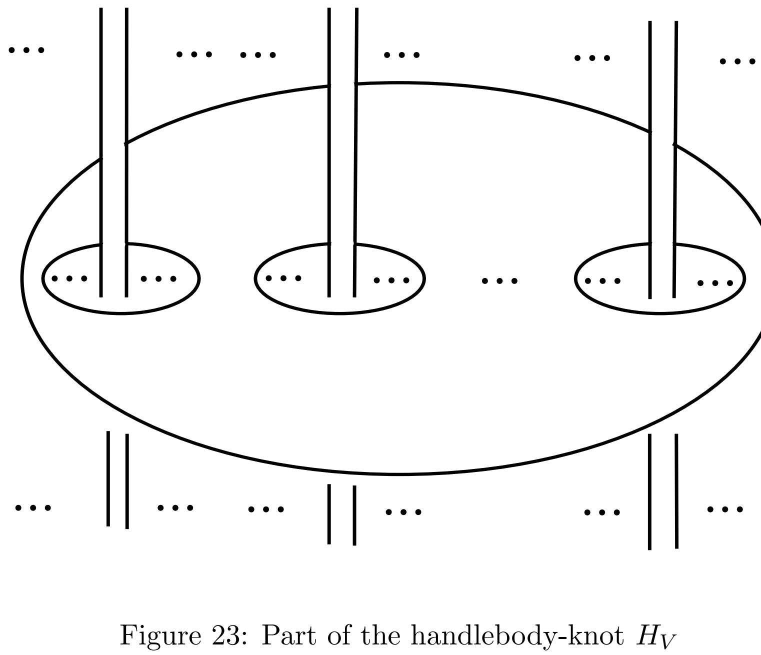

Figure 23: Part of the handlebody-knot H_V

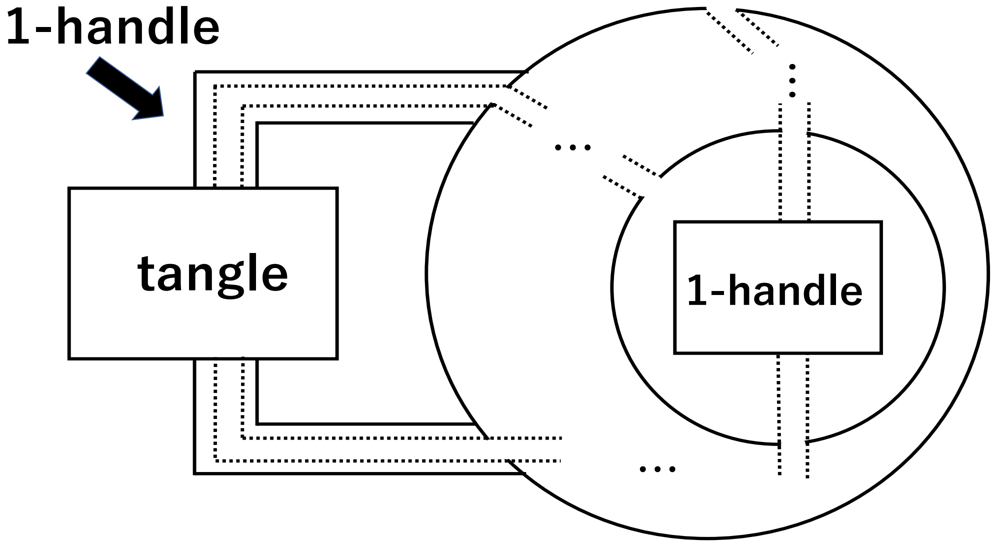

Figure 24: Exterior component W_F

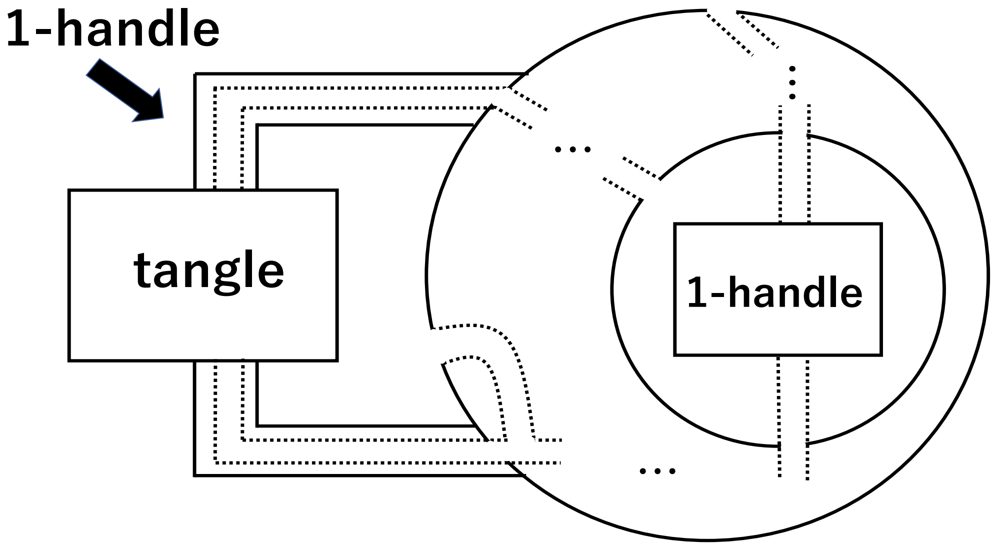

Figure 25: Removing of a 2-handle

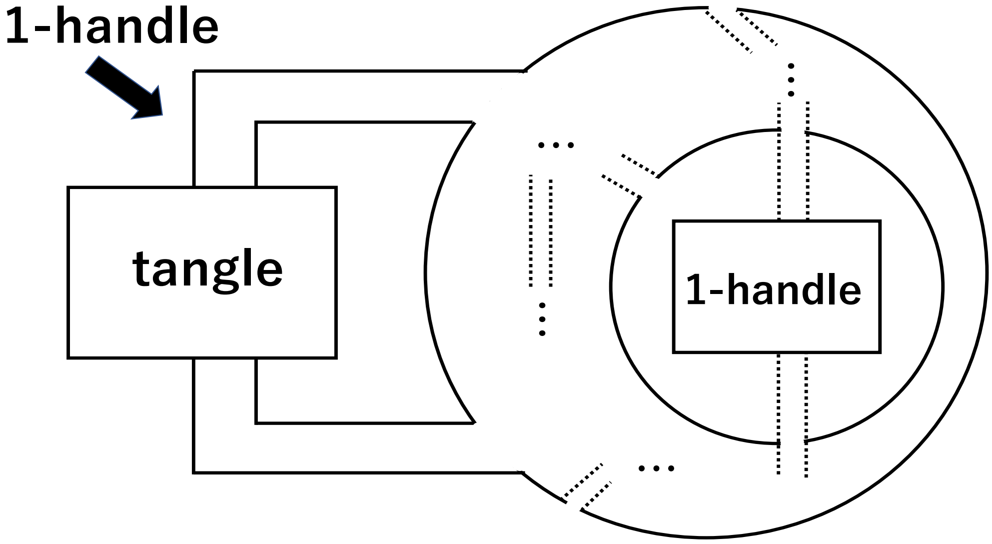

Figure 26: Procedure of a Heegaard splitting of W_F

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