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Okabe, Tomonaga
Department of Aerospace Engineering, Tohoku University

Onodera, Sota
Department of Aerospace Engineering, Tohoku University

Kumagai, Yuta
Department of Aerospace Engineering, Tohoku University

Nagumo, Yoshiko
Department of Aerospace Engineering, Tohoku University

KYUSHU UNIVERSITY

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Tomonaga Okabe[1], Sota Onodera[1], Yuta Kumagai[1] and Yoshiko Nagumo[1]

[1]Department of Aerospace Engineering, Tohoku University

Corresponding author:

Tomonaga Okabe, Department of Aerospace Engineering, Tohoku University, 6-6-01,

Aramaki-Aza-Aoba, Aoba-ku, Sendai, 980-8579, JAPAN

Email: okabe@plum.mech.tohoku.ac.jp

Abstract

In this study, the continuum damage mechanics model for predicting the stiffness re-
duction of composite laminates including transverse cracks is formulated as a function
of crack density. To formulate the model, first the damage variable in the direction
normal to the fiber of a ply including transverse cracks is derived. The damage vari-
able is derived by the model assuming a plane strain field in the isotropic plane and

using the Gudmundson-Zang model for comparison. The effective compliance based on the strain equivalent principle proposed by Murakami and his colleagues and classical laminate theory are then used to formulate the elastic moduli of laminates of arbitrary lay-up configurations as a function of the damage variable. Finally, the results obtained from this model are compared to the finite element analysis (FEA) reported in previous studies. The model proposed in this paper can predict the stiffness of laminates containing damage due to transverse cracks (or surface crack) from just the mechanical properties of a ply and the lay-up configurations. Furthermore, this model can precisely predict the FEA results and experiment results for the elastic moduli of the laminate of arbitrary lay-up configuration, such as cross-ply, angle ply, and quasi-isotropic, including transverse cracks. This model only considers the damage of the transverse crack; it does not consider damage such as delamination. However, this model seems to be effective in the early stage of damage formation when transverse cracking mainly occurs. The model assuming plane strain field in the isotropic plane which is proposed in this paper can calculate the local stress distribution in a ply including transverse cracks as a function of crack density. The damage evolution of transverse cracks can thus be simulated by determining the fracture criterion.

Keywords

Damage, transverse, crack, analysis, laminates

Introduction

Recently, composite materials such as fiber-reinforced plastic (FRP) have been used as a main component in primary structures. Generally, laminates made by stacking unidirectional fiber-reinforced lamina, which have predefined mechanical properties, are commonly used because unidirectional fiber-reinforced plastic is highly anisotropic. There are some kinds of laminate, such as cross-ply laminate, which has fiber angles of $0°$ and $90°$ to the loading direction, angle-ply laminate, which has fiber angles of $\pm\theta°$ to the loading direction, and quasi-isotropic laminate, which has isotropic elastic moduli in the plane. To utilize these laminates of arbitrary lay-up configurations, it is essential to know the fracture process of the laminate. Fiber-reinforced laminate exhibits specific fracture processes such as transverse cracks (cracks in a direction parallel to the fiber), delamination, and breakage of the fiber. In these fracture processes, transverse cracking occurs in the earliest stage. The initiation of transverse cracking causes stress

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redistribution, resulting in delamination and breakage of the fiber due to the stress concentration at the crack tip. Therefore, it is important to clarify the mechanical behavior of laminate, including transverse cracking.

There have been various studies on laminate including transverse cracks.[1–22] The simplest method considering the effect of transverse cracking is called the ply discount method.[1,2] In this method, the stiffness of the ply including transverse cracks is ignored, resulting in a lower estimation of the stiffness of the laminate. Moreover, it is impossible to estimate the decrease in stiffness caused by changing the transverse crack density using this method. Therefore, continuum damage mechanics is a valid approach to this issue. In continuum damage mechanics, the internal state variable and damage variable are utilized to consider the effect of transverse cracks on the reduction of stiffness shown in previous papers.[3–9] This approach has been applied to the numerical modeling for the damage progress with finite-element analysis.[3] Here after, we briefly introduce the previous works in this field.

Lee, Allen and Harris[1] derived approximate relationship between the internal state variable (ISV) and the stiffness of the laminate. They derived the upper limit of the effective Young's modulus of the $[0_q/90_r]_s$ laminate including transverse cracking in

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the 90° ply by using the minimum potential energy principle.

Gudmundson and Zang[2] suggested a sophisticated model that predicts the thermo-elastic properties of the laminate including transverse cracks. In this model, the average crack opening displacement of a transverse crack is assumed to be that of an isotropic elastic medium, and the incremental strain of the ply in which the transverse crack is initiated is calculated. They embedded this model into classical laminate theory, so that the relationship between transverse crack density of laminates with arbitrary lay-up configurations and their mechanical properties could be estimated analytically. Kobayashi et al.[13] derived the energy release rate for transverse cracks with this model and predicted the formation of transverse cracks in quasi-isotropic laminates. They deemed that the local stress distribution has not been determined, although the average stress distribution in each ply can be derived. In contrast, as discussed later, the model proposed in this paper can represent the local stress distribution in the ply including transverse cracks as a function of crack density.

For the quasi-isotropic $[0/90/-45/+45]_s$ laminate, Tong et al.[14,15] observed the damage evolution experimentally under quasi-static and fatigue loadings and investigated the mechanical properties of laminate using a FEM simulation that takes account

of the experiment results. Based on the experiment results, when the laminate was subjected to quasi-static and fatigue loadings in the direction of $0°$, cracks were observed in the $90°$, $-45°$, and $+45°$ plies. The crack size was smaller and the fatigue crack growth rate was slower in the $-45°$ ply compared to that of the other plies.

The effective compliance of the laminate including transverse cracks was formulated using a damage variable of a second-rank symmetric tensor proposed by Murakami and his colleagues[11, 12] in terms of continuum damage mechanics. They formulated the effective compliance of laminate as a function of the damage variable by embedding the effective compliance of the plies into classical laminate theory. However, they assumed that the damage variable is proportional to the crack density.

In this study, a model for predicting the stiffness reduction of composite laminates including transverse cracks is formulated as a function of crack density using two-dimensional elasticity. In particular, the damage variable in the direction normal to the fiber was derived for the ply including transverse cracking. Here, the damage variable in the direction normal to the fiber was formulated from the model assuming a plane strain field in the isotropic plane and the Gudmundson-Zang[2] model. Using the effective compliance based on the strain equivalent principle formulated by Murakami et

al.[11,12)], the effective compliance of a ply was calculated with the damage variable stated above. Next, the effective compliance of a ply was embedded in classical laminate theory to formulate the effective compliance of laminates with arbitrary lay-up configurations as a function of crack density. Finally, the validity of this model was estimated by comparing the results obtained from this model to those in previous studies.[2,14–17)]

Continuum damage mechanics model

Damage mechanics model of composite laminates

It is difficult to directly derive a damage model for laminate because of the difference in damage formation in each ply in the laminate. Therefore, this paper investigated the mechanical behavior of plies damaged by transverse cracks. A damage model of the laminate was then formulated with the help of classical laminate theory. When formulating the damage model, the following assumptions were made: (i) the ply is thin, (ii) damage is mainly according to cracking due to transverse cracks. Based on these assumptions, a plane stress condition without considering the bending of the ply was assumed, and the damage due to delamination was ignored. In this section, based on the study of Murakami et al.[11,12)], the effective compliance of the damaged ply is

shown, and the effective compliance of the laminate is derived.

First, the damage mechanics model of a ply is established. As seen in Fig. 1, the 1-axis is parallel to the fiber and the 2 and 3-axes are normal to the fiber. In the continuum damage mechanics model, the strain equivalent principle and the strain energy equivalent principle are known as a derivation method for determining the effective compliance. In this study, the effective compliance based on the strain equivalent principle was used. The strain equivalent principle is used as follows. The constitutive equation of damaged material under stress loading condition is written by substituting the effective stress for the stress term in the constitutive equation of undamaged material. By using the strain equivalent principle, the effective compliance of the ply damaged by transverse cracking seen in Fig. 1 is described as follows.

$$\boldsymbol{C} = \boldsymbol{C}_0 \boldsymbol{M} \tag{1}$$

Here, $\boldsymbol{C}_0$ is compliance under undamaged conditions and $\boldsymbol{M}$ is a damage effect tensor. In plane 1-2, the plane stress condition was assumed. When the in-plane stress and strain of a ply are defined as $\boldsymbol{\sigma} = [\sigma_1, \sigma_2, \sigma_{12}]^T$ and $\varepsilon = [\varepsilon_1, \varepsilon_2, 2\varepsilon_{12}]^T$, $\boldsymbol{C}_0$ and $\boldsymbol{M}$ on

the principal axis of the ply are specified as follows.

$$
\boldsymbol{C}_0 = \begin{bmatrix} \dfrac{1}{E_1} & -\dfrac{\nu_{12}}{E_1} & 0 \\[2ex] -\dfrac{\nu_{12}}{E_1} & \dfrac{1}{E_2} & 0 \\[2ex] 0 & 0 & \dfrac{1}{G_{12}} \end{bmatrix} \tag{2}
$$

$$
\boldsymbol{M} = \begin{bmatrix} 1 & 0 & 0 \\[2ex] 0 & \dfrac{1}{1-d_2} & 0 \\[2ex] 0 & 0 & \dfrac{1}{2}\left(1+\dfrac{1}{1-d_2}\right) \end{bmatrix} \tag{3}
$$

E_1 and E_2 are Young's modulus along the 1 and 2-axes respectively, G_{12} is a shear modulus in plane 1-2, ν_{12} is Poisson's ratio, and d_2 is the damage variable in the direction normal to the fiber. Damage variable d_2 is derived as a function of crack density ρ in the following sections. Thus, if the damage due to the transverse crack is ignored, d_2 will be 0.

Subsequently, the effective compliance of laminates of arbitrary lay-up configurations was derived based on the classical laminate theory. The coordinate system of the laminate is defined as $O\text{-}XYZ$. Here, let us assume that the X-Y plane is parallel to

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the 1-2 plane of the ply and the Z axis is in the same direction as the 3-axis. The effective compliance of the kth ply $(k = 1, 2, \cdots, N)$ is derived. The effective compliance of a ply $\boldsymbol{C}_k$ in the coordinate system of the principal axis of laminate $O\text{-}XYZ$ is given in Eq. (1). The direction of a fiber tilted at a α_k degree angle to the X-axis is indicated in Fig. 2.

$$\boldsymbol{C}_k = \boldsymbol{R}_k \boldsymbol{C} \boldsymbol{T}_k^{-1} = \boldsymbol{R}_k \boldsymbol{C}_0 \boldsymbol{M} \boldsymbol{T}_k^{-1} \tag{4}$$

Here, $\boldsymbol{R}_k$ and $\boldsymbol{T}_k$ are coordinate conversion matrices that are described as follows.

$$\boldsymbol{R}_k = \begin{bmatrix} \cos^2 \alpha_k & \sin^2 \alpha_k & -\sin \alpha_k \cos \alpha_k \\ \sin^2 \alpha_k & \cos^2 \alpha_k & \sin \alpha_k \cos \alpha_k \\ 2\sin \alpha_k \cos \alpha_k & -2\sin \alpha_k \cos \alpha_k & \cos^2 \alpha_k - \sin^2 \alpha_k \end{bmatrix} \tag{5}$$

$$\boldsymbol{T}_k = \begin{bmatrix} \cos^2 \alpha_k & \sin^2 \alpha_k & -2\sin \alpha_k \cos \alpha_k \\ \sin^2 \alpha_k & \cos^2 \alpha_k & 2\sin \alpha_k \cos \alpha_k \\ \sin \alpha_k \cos \alpha_k & -\sin \alpha_k \cos \alpha_k & \cos^2 \alpha_k - \sin^2 \alpha_k \end{bmatrix} \tag{6}$$

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In classical laminate theory, the effective compliance of laminate $\overline{C}$ is described as

$$\overline{C} = \left(\frac{1}{t_L} \sum_{k=1}^{N} t_k C_k^{-1} \right)^{-1} = \left(\frac{1}{t_L} \sum_{k=1}^{N} t_k (R_k C_0 M T_k^{-1})^{-1} \right)^{-1}, \tag{7}$$

where t_L is the thickness of the laminate and t_k is the thickness of the k-th ply. From Eq. (7), if the damage variable d_2, which is a component of the damage effect tensor M, can be formulated, the mechanical properties of the damaged laminate can be obtained. In the following sections, the formulation of d_2 of a ply using the model assuming a plane strain field in the isotropic plane is conducted using the Gudmundson-Zang model.

Formulation of the damage variable in the direction normal to the fiber using the model assuming a plane strain field in the isotropic plane

This section derives the damage variable d_2 normal to the fiber by considering the damage due to transverse crack parallels to the fiber. Fig. 3 depicts a representative volume element including a transverse crack on both sides, which is a part of the ply seen in Fig. 1. The ply strain ε_y^p is applied to the corresponding ply. $2l$ is the crack distance and $2t\,(=t_k)$ is the thickness of the ply. The transverse crack is assumed to have a

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tunnel-like crack surface which is symmetrical about the y axis. The transverse crack does not propagate into the neighboring ply. It should be noted that the coordinate system differs from that in Fig. 1 in the previous section. The coordinates x, y, and z in Fig. 3 correspond to the 3-, 2- and 1-axes in Fig. 1, respectively. Due to the symmetric model, the range is limited to $0 \le x \le t$ and $0 \le y \le l$. A plane strain condition in the isotropic plane (x-y plane) in Fig. 3 is assumed. The deformations in directions x, y, and z are defined as u, v, and w. The stress-strain relation is expressed as follows.

$$\varepsilon_x(x, y) = \frac{\partial u}{\partial x}(x, y) = C_1 \sigma_x(x, y) - C_2 \sigma_y(x, y) \tag{8}$$

$$\varepsilon_y(x, y) = \frac{\partial v}{\partial y}(x, y) = C_1 \sigma_y(x, y) - C_2 \sigma_x(x, y) \tag{9}$$

$$\gamma_{xy} = \frac{\sigma_{xy}(x, y)}{G_{23}} = \frac{\partial v}{\partial x}(x, y) + \frac{\partial u}{\partial y}(x, y) \approx \frac{\partial v}{\partial x}(x, y) \tag{10}$$

The constants C_1 and C_2 are described as

$$C_1 = \frac{1 - \nu_{12}\nu_{21}}{E_2}, \tag{11}$$

$$C_2 = \frac{\nu_{23} + \nu_{12}\nu_{21}}{E_2}, \tag{12}$$

where ν_{23} and G_{23} are the Poisson' s ratio and shear modulus in the 2-3 plane in Fig.

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1. For the engineering shear strain γ_{xy} in Eq. (10), the gradient in the y direction of displacement u is assumed to be quite small. The relationship between the strain in the x direction ε_x and the strain in the y direction ε_y is expressed as

$$\varepsilon_x(x, y) = a\varepsilon_y(x, y), \tag{13}$$

where a is a proportionality constant defined to satisfy the equilibrium equation of stress. From Eqs. (8) and (9), Eq. (13) can be described as follows.

$$a\frac{\partial v}{\partial y}(x, y) = C_1\sigma_x(x, y) - C_2\sigma_y(x, y) \tag{14}$$

When $\sigma_y(x, y)$ in Eq. (9) is eliminated by using Eq. (14), $\sigma_x(x, y)$ is described as follows.

$$\sigma_x(x, y) = \frac{aC_1 + C_2}{C_1^2 - C_2^2}\frac{\partial v}{\partial y}(x, y) \tag{15}$$

In turn, when $\sigma_x(x, y)$ in Eq. (9) is eliminated using Eq. (14), $\sigma_y(x, y)$ is described as

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follows.

$$\sigma_y(x, y) = \frac{C_1 + aC_2}{C_1^2 - C_2^2} \frac{\partial v}{\partial y}(x, y) \tag{16}$$

Additionally, using Eq. (10), $\sigma_{xy}(x, y)$ is described as follows.

$$\sigma_{xy} = G_{23} \frac{\partial v}{\partial x}(x, y) \tag{17}$$

The equilibrium equation of stress is expressed as follows.

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \tag{18}$$

$$\frac{\sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \tag{19}$$

When $\partial^2 v / \partial x \partial y \neq 0$ is postulated and Eqs. (15) and (17) are substituted into Eq.

(18), the proportionality constant a is defined as follows.

$$\begin{aligned} a &= -\frac{C_2 + G_{23}(C_1^2 - C_2^2)}{C_1} \\ &= -\frac{E_2(\nu_{23} + \nu_{12}\nu_{21}) + G_{23}\{(1 - \nu_{12}\nu_{21})^2 - (\nu_{23} + \nu_{12}\nu_{21})^2\}}{E_2(1 - \nu_{12}\nu_{21})} \end{aligned} \tag{20}$$

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By substituting Eqs. (16) and (17) for the stress term in Eq. (19), the Laplace equation

for displacement v is obtained as

$$\frac{\partial^2 v}{\partial x^2} + \lambda^2 \frac{\partial^2 v}{\partial y^2} = 0, \tag{21}$$

where λ is a constant value determined as follows.

$$\lambda = \sqrt{\frac{C_1 + aC_2}{G_{23}(C_1^2 - C_2^2)}} = \sqrt{\frac{E_2 - G_{23}(\nu_{23} + \nu_{12}\nu_{21})}{G_{23}(1 - \nu_{12}\nu_{21})}} \tag{22}$$

To determine displacement v, the boundary conditions of the Laplace equation in Eq.

(21) are given as follows.

$$v(x, 0) = 0 \tag{23}$$

$$\frac{\partial v}{\partial y}(x, l) = 0 \tag{24}$$

$$\frac{\partial v}{\partial x}(0, y) = 0 \tag{25}$$

$$v(t, y) = \varepsilon_y^p y \tag{26}$$

At $y = 0$, there is no displacement v considered in Eq. (23). Equation (24) was

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determined from Eq. (16) considering the stress $\sigma_y = 0$ at the crack surface $(y = l)$. Equation (25) was defined based on Eq. (17), considering no shear deformation $\sigma_{xy} = 0$ on the center line of the ply $(x = 0)$. Additionally, the displacement distribution shown as Eq. (26) in the interface was assumed according to the previous study.[1] Therefore, the neighboring ply is thought to be deformed uniformly by ε_y^p regardless of the transverse cracks. Using separation of variables and assigning the boundary conditions shown above to Eq. (21), the solution $v(x, y)$ of the Laplace equation that satisfies the boundary conditions in Eqs. (23) to (26) can be described as follows.

$$v(x,y) = \frac{8l\varepsilon_y^p}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \frac{\cosh\left(\frac{2n-1}{2l}\pi\lambda x\right)}{\cosh\left(\frac{2n-1}{2l}\pi\lambda t\right)} \sin\left(\frac{2n-1}{2l}\pi y\right) \tag{27}$$

Damage variable d_2 in the direction normal to the fiber is represented by the proportion of the average ply strain ε_y^a, which ignores the crack opening displacement and the ply strain ε_y^p, which considers the crack opening displacement, $\varepsilon_y^a/\varepsilon_y^p$. Therefore, the proportion of average ply strain ε_y^a and ply strain ε_y^p is calculated using Eq. (27). At

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first, the average ply strain ε_y^a was defined as follows.

$$\varepsilon_y^a = \frac{1}{lt} \int_0^t v(x, l) dx \tag{28}$$

By substituting Eq. (27) into Eq. (28), $\varepsilon_y^a / \varepsilon_y^p$ is derived as follows.

$$\frac{\varepsilon_y^a}{\varepsilon_y^p} = \sum_{n=1}^{\infty} \frac{16}{(2n-1)^3 \pi^3 \lambda t_k} \frac{\tanh\left(\frac{2n-1}{2l} \pi \lambda t_k \rho\right)}{\rho} \tag{29}$$

where $\rho = 1/(2l)$ is the crack density and the thickness of the ply was defined as $t_k = 2t$.

Finally, the damage variable d_2 in the direction normal to the fiber was formulated. Due to the plane stress condition in plane 1-2 in Fig. 1, the plane stress condition is also postulated in the z-y plane. The relationship between the ply strain $[\varepsilon_z^p, \varepsilon_y^p, 2\varepsilon_{zy}^p]^T$ and average ply stress $[\sigma_z^a, \sigma_y^a, \sigma_{zy}^a]^T$ in the z-y plane is expressed as follows when using

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the global effective elastic compliance given in Eq. (1).

$$
\begin{bmatrix} \varepsilon_z^p \\ \varepsilon_y^p \\ 2\varepsilon_{zy}^p \end{bmatrix} = \begin{bmatrix} \dfrac{1}{E_1} & -\dfrac{\nu_{12}}{(1-d_2)E_1} & 0 \\ -\dfrac{\nu_{12}}{E_1} & \dfrac{1}{(1-d_2)E_2} & 0 \\ 0 & 0 & \dfrac{1}{2G_{12}}\left(1+\dfrac{1}{1-d_2}\right) \end{bmatrix} \begin{bmatrix} \sigma_z^a \\ \sigma_y^a \\ \sigma_{zy}^a \end{bmatrix} \tag{30}
$$

Here, considering the loading state as $[\sigma_z^a, \sigma_y^a, \sigma_{zy}^a]^T = [0, \sigma_y, 0]^T$, the following relationship is obtained from Eq. (30).

$$
\varepsilon_y^p = \frac{\sigma_y}{(1-d_2)E_2} \tag{31}
$$

On the contrary, when $\sigma_y^a \gg \nu_{23}\sigma_x^a$ is assumed, ε_y^a is expressed as follows from Hooke's law.

$$
\varepsilon_y^a = \frac{\sigma_y^a}{E_2} - \frac{\nu_{23}}{E_2}\sigma_x^a \approx \frac{\sigma_y^a}{E_2} \tag{32}
$$

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Using Eq. (32), the damage variable d_2 is derived as follows from Eq. (31).

$$d_2 = 1 - \frac{\varepsilon_y^a}{\varepsilon_y^p} \tag{33}$$

Substituting Eq. (29) into Eq. (33), damage variable d_2 in the direction normal to the fiber is described as a function of the crack density ρ as follows.

$$d_2 = 1 - \sum_{n=1}^{\infty} \frac{16}{(2n-1)^3 \pi^3 \lambda t_k} \frac{\tanh\left(\frac{2n-1}{2}\pi\lambda t_k \rho\right)}{\rho} \tag{34}$$

In the equation above, t_k is the thickness of the ply in the laminate, and λ is the constant derived from the elastic moduli of a ply in Eq. (22). The damage effect tensor $\boldsymbol{M}$ seen in Fig. 3 can be calculated using Eq. (34). The change in the elastic moduli of the laminate due to a change in the transverse crack density is obtained by determining the elastic moduli of a ply and the laminated constitution. According to classical laminate theory, this method can apply to laminates with arbitrary lay-up configurations.

As seen in Fig. 4 (a), a crack may occur in the laminate surface. In thin laminated, cracks in the surface seriously affect the mechanical properties. Fig. 4 (b) presents a

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representative volume element of a ply that has a crack in the surface. In this case, the area of $0 \leq x \leq 2t$ and $0 \leq y \leq l$ should be considered as pointed out by Gudmundson and Zang. The surface crack can be considered by replacing the thickness of a ply t_k with $2t_k$ in Fig. 3. By replacing t_k with $2t_k$, in Eq. (34), the damage variable d_2 in the direction normal to the fiber considering the surface crack may be represented as follows.

$$
d_2 = 1 - \sum_{n=1}^{\infty} \frac{8}{(2n-1)^3 \pi^3 \lambda t_k} \frac{\tanh\{(2n-1)\pi\lambda t_k \rho\}}{\rho} \tag{35}
$$

When transverse cracks in a ply is considered, Eq. (34) is appropriate, as Eq. (35) is appropriate for a surface crack. When Eq. (27) is substituted into Eqs. (15) to (17), the stress at $P(x, y)$ is derived as follows.

$$
\sigma_x(x, y) = \frac{4}{\pi} \frac{aC_1 + C_2}{C_1^2 - C_2^2} \varepsilon_y^p \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \frac{\cosh\left(\frac{2n-1}{2l}\pi\lambda x\right)}{\cosh\left(\frac{2n-1}{2l}\pi\lambda t\right)} \cos\left(\frac{2n-1}{2l}\pi y\right) \tag{36}
$$

$$
\sigma_y(x, y) = \frac{4}{\pi} \frac{C_1 + aC_2}{C_1^2 - C_2^2} \varepsilon_y^p \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \frac{\cosh\left(\frac{2n-1}{2l}\pi\lambda x\right)}{\cosh\left(\frac{2n-1}{2l}\pi\lambda t\right)} \cos\left(\frac{2n-1}{2l}\pi y\right) \tag{37}
$$

$$
\sigma_{xy}(x, y) = \frac{4\lambda G_{23}}{\pi} \varepsilon_y^p \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \frac{\sinh\left(\frac{2n-1}{2l}\pi\lambda x\right)}{\cosh\left(\frac{2n-1}{2l}\pi\lambda t\right)} \sin\left(\frac{2n-1}{2l}\pi y\right) \tag{38}
$$

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The stress distribution in a ply including transverse cracks is obtained from Eqs. (36) to (38). As seen in Eqs. (36) to (38), the local stress distribution in the ply including transverse cracks can be calculated by using our model. In contrast, only the average stress in each ply in the laminate can be calculated by the Gudmundson-Zang model. This is the difference from the Gudmundson-Zang model.

Formulation of damage variables in the direction normal to the fiber using the Gudmundson-Zang model

As seen in Eq. (33), damage variable d_2 can be represented by the portion of $\varepsilon_y^a/\varepsilon_y^p$. Therefore, damage variable d_2 can be represented in various manners by deriving $\varepsilon_y^a/\varepsilon_y^p$ in some way. Here, by applying the Gudmundson-Zang model, which can predict the thermo-elastic properties of a laminate including transverse cracks, the damage variable d_2 was formulated.

According to the Gudmundson-Zang model, the increment of the in-plane strain due to the crack opening in the z-y plain in Fig. 3 can be represented as

$$\Delta\boldsymbol{\varepsilon}_I = t_k\rho(\boldsymbol{N}_I)^T\boldsymbol{\beta}\boldsymbol{N}_I(\boldsymbol{C}_0)^{-1}\boldsymbol{\varepsilon}_I^p, \tag{39}$$

where t_k is the thickness of the ply, ρ is the crack density in the corresponding ply, $\varepsilon_I^p = [\varepsilon_z^p, \varepsilon_y^p, 2\varepsilon_{zy}^p]^T$ is the ply strain in the z-y plane, and $\boldsymbol{C}_0$ is the compliance of the undamaged ply derived from Eq. (2). Additionally, $\boldsymbol{N}_I$ is a matrix determined from the normal unit vector defined as $\boldsymbol{n} = [n_x, n_y, n_z]^T$ that is perpendicular to the transverse crack in the ply.

$$
\boldsymbol{N}_I = \begin{bmatrix} n_z & 0 & n_y \\ 0 & n_y & n_z \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{40}
$$

where a normal unit vector is defined as $\boldsymbol{n} = [0, 1, 0]^T$, as seen in Fig. 3. Moreover, $\boldsymbol{\beta}$ is a matrix that connects the average crack opening displacement due to the transverse crack with the vector of the surface force affecting the surface of the transverse crack, which can be represented as follows.

$$
\boldsymbol{\beta} = \begin{bmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{bmatrix} \tag{41}
$$

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The ply strain $\boldsymbol{\varepsilon}_I^p$ in the z-y plane is represented as the sum of the average ply strain $\boldsymbol{\varepsilon}_I^a = [\varepsilon_z^a, \varepsilon_y^a, \varepsilon_{zy}^a]$ and the increment of the ply strain $\Delta\boldsymbol{\varepsilon}_I$.

$$\boldsymbol{\varepsilon}_I^p = \boldsymbol{\varepsilon}_I^a + \Delta\boldsymbol{\varepsilon}_I \tag{42}$$

When Eq. (39) is substituted into Eq. (42), the following equation is derived.

$$\boldsymbol{\varepsilon}_I^a = (\boldsymbol{I} - t_k\rho(\boldsymbol{N}_I)^T\boldsymbol{\beta}\boldsymbol{N}_I(\boldsymbol{C}_0)^{-1})\boldsymbol{\varepsilon}_I^p, \tag{43}$$

where $\boldsymbol{I}$ is the identity matrix. Here, we consider the following strain state $\boldsymbol{\varepsilon}_I^p$ given by

$$\boldsymbol{\varepsilon}_I^p = [0, \varepsilon_y^p, 0]^T. \tag{44}$$

After Eqs. (2), (40), (41), and (44) are substituted into Eq. (43) and the components related to the second line are calculated, $\varepsilon_y^a/\varepsilon_y^p$ is derived as follows.

$$\frac{\varepsilon_y^a}{\varepsilon_y^p} = 1 - \frac{E_1 E_2 t_k}{E_1 - E_2\nu_{12}^2}\beta_2\rho \tag{45}$$

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By substituting Eq. (45) for Eq. (33), damage variable d_2 in the direction normal to the fiber is represented as follows.

$$d_2 = \frac{E_1 E_2 t_k}{E_1 - E_2 \nu_{12}^2} \beta_2 \rho \tag{46}$$

In the equation above, β_2 is a function of the crack density; for a transverse crack initiated in the ply it is represented as follows.

$$\beta_2 = \frac{\pi}{2} \frac{1 - \nu_{12}\nu_{21}}{E_2} \sum_{j=1}^{10} \frac{a_j}{(1 + t_k \rho)^j} \tag{47}$$

When surface cracks are considered, β_2 is given as

$$\beta_2 = 2(1.12)^2 \left(\frac{\pi}{2} \frac{1 - \nu_{12}\nu_{21}}{E_2} \sum_{j=1}^{10} \frac{c_j}{(1 + t_k \rho)^j} \right) \tag{48}$$

where the constant sequence a_j and c_j is given in Table 1. As seen in Eq. (46), damage variable d_2 in the direction normal to the fiber can be represented as a function of the crack density ρ when the thickness of the ply t_k and the elastic modulus of the ply are given. The elastic moduli of the laminates of arbitrary lay-up configurations

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that are changed by the transverse crack density are derived from Eq. (7) by giving the elastic modulus of the ply and the laminated constitution as discusses in previous section. However, as stated above and pointed out by Kobayashi[13], the Gudmundson-Zang model does not give stress and strain fields around the transverse crack.

Results and discussion

By using the two models formulated in this paper (the model assuming a plane strain field in the isotropic plane and the Gudmundson-Zang model), the elastic moduli of the laminate were calculated as a function of crack density to compare with the experiment[2, 14–17] and with the finite-element analytical results in previous works on GFRP and CFRP laminates.[2] The validity of this model was evaluated for cross-ply, angle-ply, and quasi-isotropic laminates. When considering the damage due to transverse cracks in the laminate, damage variable d_2 was calculated using Eq. (34) or Eqs. (46) and (47). In the case of considering surface cracks, damage variable d_2 was calculated using Eq. (35) or Eqs. (46) and (48). The mechanical properties of the ply of GFRP and CFRP used in the calculation of the elastic moduli of laminate are listed in Table 2. Indexes 1, 2, and 3 in Table 2 represent the principal axes of the ply defined

in Fig. 1. Additionally, it is assumed that the thickness of the plies t_k is assigned as t_0 listed in Table 2 for all plies.

Cross-ply laminate

Here, the elastic modulus of cross-ply laminate is discussed. To calculate elastic moduli, the damage due to transverse cracks in the 90° ply was considered.

First, a cross-ply $[90/0]_s$ GFRP laminate with a surface crack in the 90° plies was considered. A representative periodic cell of the laminate is depicted in Fig. 5 (a). The material properties of GFRP-1 listed in Table 2 were used for the calculation. Figures 5 (b) and (c) give comparisons of the results for Young's moduli and Poisson's ratio for the laminate obtained from the Gudmundson-Zang model and the present model assuming a plane strain field in the isotropic plane, respectively. E_{L1} is the elastic modulus in the X direction of the principal axis of the laminate, ν_{L12} is the Poisson's ratio in the X-Y plane of the laminate, and ρ_N is the normalized crack density, which is the product of the thickness of each ply t_0 and the crack density ρ. Based on Figs. 5 (b) and (c), the results of the two models agree well with the analytical results obtained from the finite-element method. From this result, the elastic modulus and Poisson's

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ratio using damage variable d_2 derived from Eqs. (46) and (48) are lower than the values obtained from Eq. (35). Moreover, greater effects of the damage are obtained from the Gudmundson-Zang model than from the present model assuming a plane strain field in the isotropic plane.

The relationship between the normalized elastic moduli and normalized crack density of $[0/90]_s$ and $[0_2/90_2]_s$ CFRP laminates with transverse cracks in the $90°$ plies is illustrated in Fig. 6. E_{L10} is the elastic modulus in the X direction of the undamaged laminate. The experiment results obtained by Groves et al.[16] are also shown in Fig. 6. From Fig. 6, the results obtained from the two models shows good agreement with the results obtained by Groves et al. Moreover, the results obtained from the present model assuming a plane strain field in the isotropic plane shows better agreement than that of the Gudmundson-Zang model.

The relationship between normalized elastic moduli and normalized crack density of $[0/90_3]_s$ GFRP laminate with transverse cracks in the $90°$ plies is illustrated in Fig. 7. The material properties of GFRP-1 listed in Table 2 are used in the calculation. The experiment results of Highsmith and Reifsnider et al.[17] also appear in Fig. 7. From Fig. 7, as the crack density increases, the difference between the results obtained

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from the calculation using two models and those from the experiment increases. The reason for the difference between the results obtained from calculation and experiment involves damage formation such as local delamination and damage in $0°$ plies. For Fig. 7, the result obtained from the Gudmundson-Zang model that show a greater effect of damage is in good agreement with the experiment results.

The normalized elastic moduli and normalized crack density as a function of crack density of $[0/90]_s$ GFRP laminate including transverse cracks in the $90°$ plies are presented in Figs. 8 (a) and (b), respectively. ν_{L120} is Poisson's ratio in X-Y plane of an undamaged laminate. The material properties of GFRP-2 listed in Table 2 are used for the calculation. Figs. 8 (a) and (b) present the experiment results of Tong et al.[14] Based on Figs. 8 (a) and (b), the result calculated from the present model assuming a plane strain field in the isotropic plane is in good agreement with the experiment results in the region under 0.3mm^{-1} of crack density, but the experiment results dramatically decrease above 0.3mm^{-1}. The difference between the results obtained from calculation and from the experiment is due to damage formation such as local delamination and damage in the $0°$ plies.

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Angle-ply laminate

This section discusses angle-ply GFRP laminate exhibiting damage due to transverse cracks in each ply. The analysis was conducted relative to $[\pm 55]$ and $[\pm 67.5]$ angle-ply laminate. The material properties of GFRP-1 listed in Table 2 are used in the calculation. The calculation was conducted with the elastic modulus, shear moduli and Poisson's ratio as a function of normalized crack density. The analytical results for the $[\pm 55]$ laminate are presented in Figs. 9 (a)-(c) and that for the $[\pm 67.5]$ laminate is presented in Figs. 10 (a)-(c). E_{L1} and E_{L2} are the elastic moduli of the laminate in the X and Y directions of the principal axis of the laminate, G_{L12} is the shear modulus in the X-Y plane and ν_{12} is Poisson's ratio in the X-Y plane. Figs. 9 (a)-(c) and 10 (a)-(c) also provide the analytical results for the Gudmundson-Zang model. Based on Figs. 9 and 10, the results obtained from these analytical models are in good agreement with the FEA result of the Gudmundson-Zang model, although that obtained from the Gudmundson-Zang model underestimate the elastic moduli compared to that obtained from the present model assuming a plane strain field in the isotropic plane.

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Quasi-isotropic laminate

This section analyzes the quasi-isotropic $[0/90/-45/+45]_s$ laminate and compares it to experiment results obtained by Tong et al.[14, 15] The material used is GFRP and the material properties of GFRP-2 listed in Table 2 are used. The calculation was conducted assuming approximately the same damage occurrence due to transverse cracking in the $90°$ and $\pm 45°$ plies. The normalized elastic moduli and normalized Poisson's ratio obtained from the calculation using the model assuming a plane strain field in the isotropic plane and the Gudmundson-Zang model are presented in Figs. 11 (a) and (b) respectively. Based on Figs. 11 (a), the results obtained from the present model are in good agreement with the experiment results, although those obtained from the Gudmundson-Zang model underestimate the elastic moduli. In Fig. 11(b), the results obtained from the present model show the deviations at high crack densities that were attributed to delamination and damage in 0 plies.

The model assuming a plane strain field in the isotropic plane proposed in this paper can calculate the local stress distribution in a ply including transverse cracking as a function of crack density. Therefore, the damage evolution of the transverse crack can be simulated by determining the fracture criterion. Such a simulation based on the

present model is our future task.

Conclusions

In this paper, the damage variable is formulated by a model assuming a plane strain field in the isotropic plane and the Gudmundson-Zang model. The effective compliance based on the strain equivalent principle proposed by Murakami et al. and classical laminate theory is used to formulate the elastic moduli of laminates of arbitrary lay-up configurations as a function of crack density. The model proposed in this paper considers the effect of damage due to transverse cracks (or surface cracks) based only on the mechanical properties of the ply and the laminate constitution. Furthermore, this model can precisely predict the FEA results and experiment results for the elastic moduli of a laminate of arbitrary lay-up configuration, such as cross-ply, angle ply, and quasi-isotropic, including transverse cracks. Since this model only considers the damage from transverse cracks, it is impossible to consider the effect of other damage such as delamination. However, this model seems to be effective in the early stage of the damage formation during which transverse crack mainly occurs.

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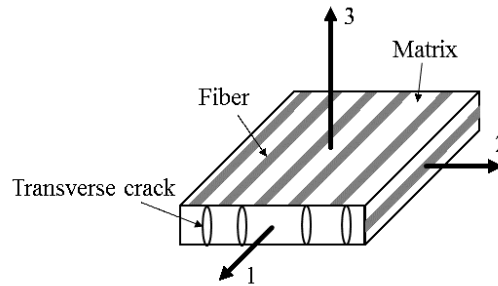
Figure 1 Ply including transverse crack
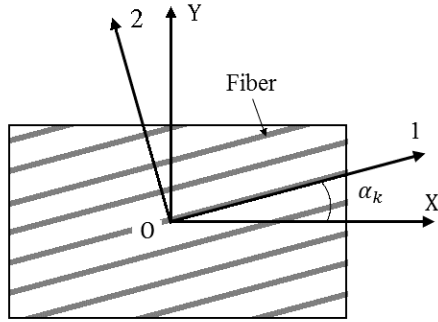
Figure 2 In-plane coordinate conversion
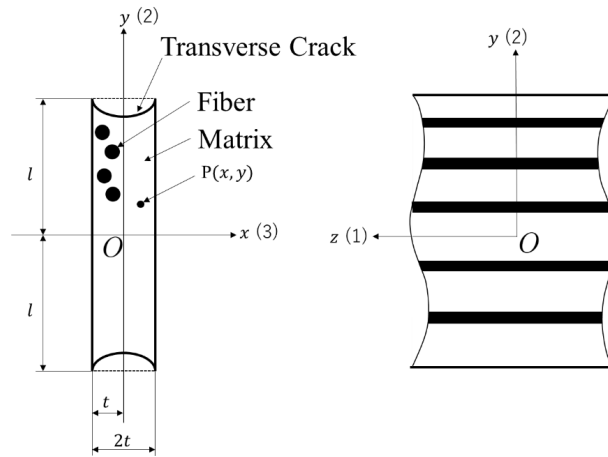
Figure 3 Transverse crack model

(a) Surface crack   (b) Representative Volume Element
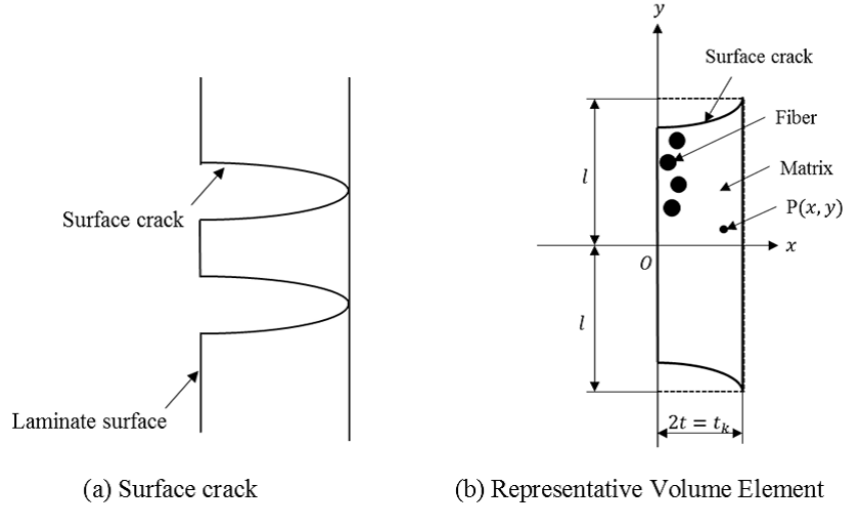
Figure 4 Surface crack model
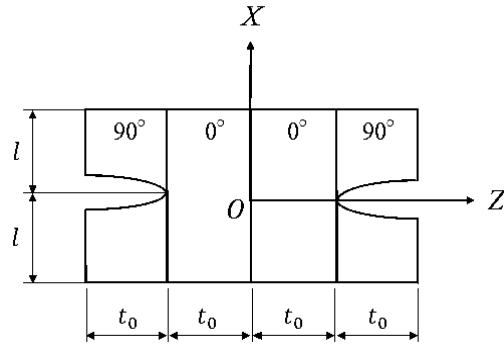
Figure 5 (a) Representative periodic cell for a cross-ply laminate with surface cracks.
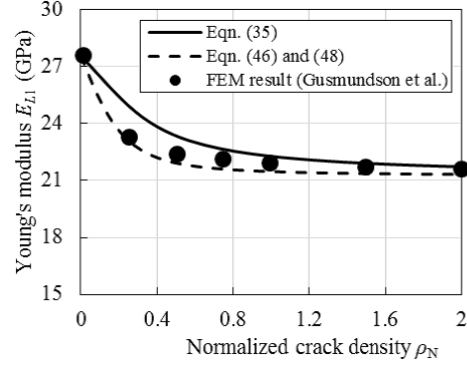
Figure 5 (b) Young's modulus as a function of normalized crack density in the 90°

plies for $[90/0]_s$ GFRP laminate with surface cracks.
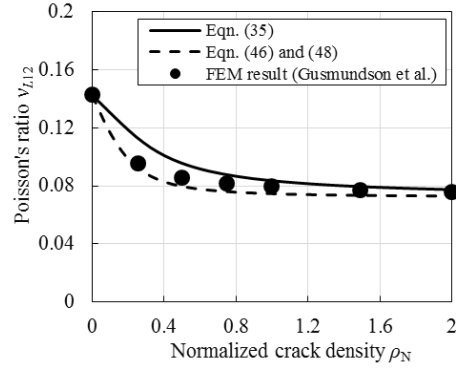
Figure 5 (c) Poisson's ratio as a function of normalized crack density in the 90° plies

for $[90/0]_s$ GFRP laminate with surface cracks.
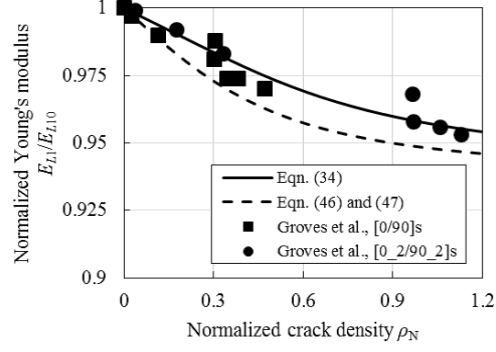
Figure 6 Normalized Young's modulus as a function of normalized crack density in the 90°plies for $[0/90]_s$ and $[0_2/90_2]_s$ CFRP laminate.
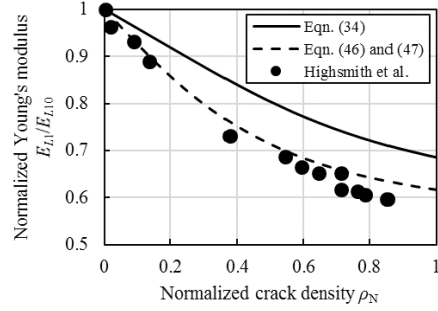
Figure 7 Normalized Young's modulus as a function of normalized crack density in the 90° plies for $[0/90_3]_s$ GFRP laminate.

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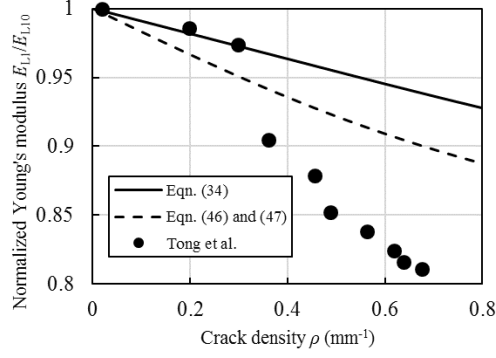
Figure 8 (a) Normalized Young's modulus as a function of crack density in the 90°

plies for $[0/90]_s$ GFRP laminate.
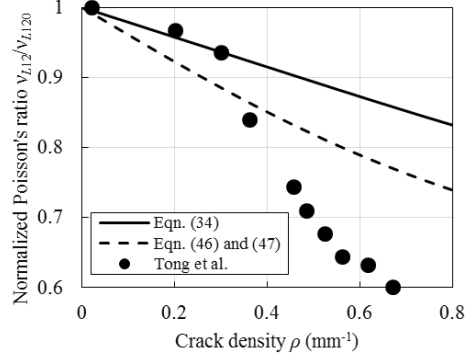
Figure 8 (b) Normalized Poisson's ratio as a function of crack density in the 90° plies

for $[0/90]_s$ GFRP laminate.
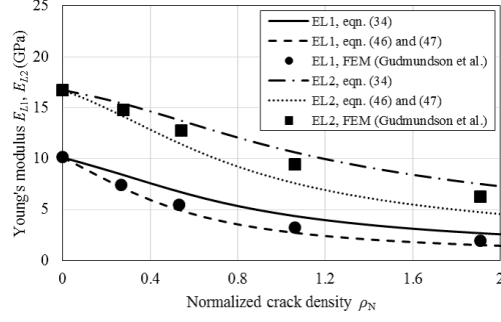
Figure 9 (a) Young's modulus as a function of normalized crack density for [±55] GFRP laminate.
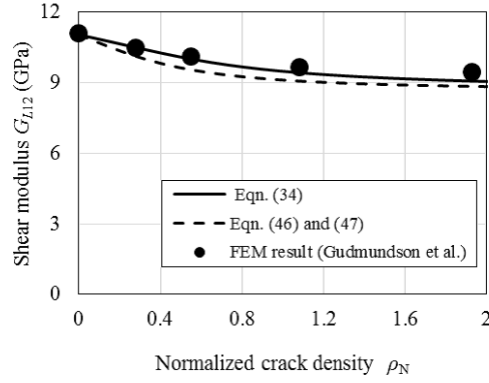
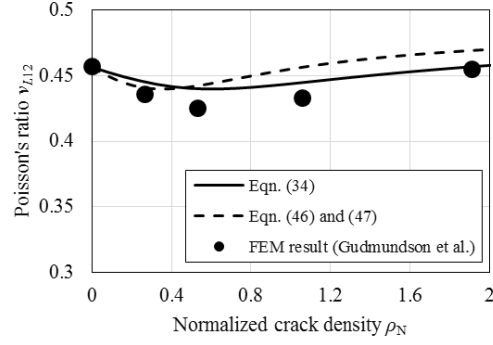Figure 9 (b) Shear modulus as a function of normalized crack density for [±55] GFRP laminate.

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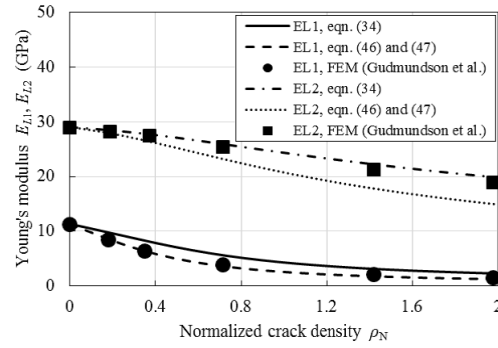Figure 9 (c) Poisson's ratio as a function of normalized crack density for [±55] GFRP laminate.

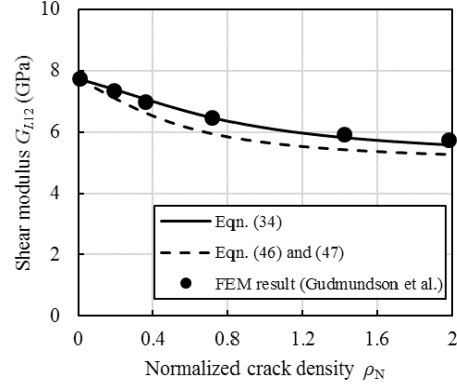Figure 10 (a) Young's modulus as a function of normalized crack density for [±67.5] GFRP laminate.

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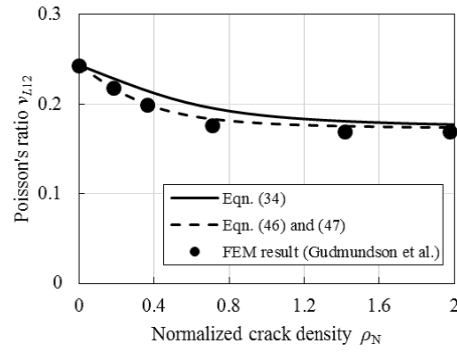Figure 10 (b) Shear modulus as a function of normalized crack density for [±67.5] GFRP laminate.

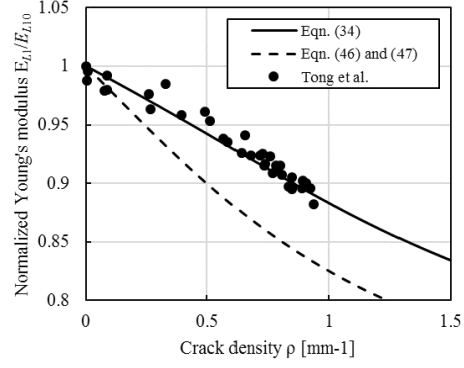Figure 10 (c) Poisson's ratio as a function of normalized crack density for [±67.5] GFRP laminate.

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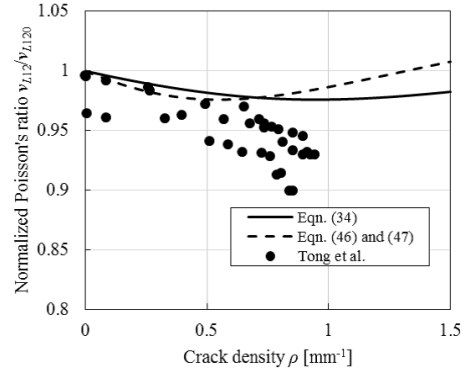Figure 11 (a) Normalized Young's modulus as a function of crack density for $[0/90/-45/+45]_s$ GFRP laminate.

Figure 11 (b) Normalized Poisson's ratio as a function of crack density for $[0/90/-45/+45]_s$ GFRP laminate.

Table 1 Numerical parameters used in Eqs. (47) and (48)[2].

j	a_j	c_j
1	0.63666	0.25256
2	0.51806	0.27079
3	0.51695	−0.49814
4	−1.04897	8.62962
5	8.95572	−51.24655
6	−33.09444	180.96305
7	74.32002	−374.29813
8	−103.06411	449.59474
9	73.60337	−286.51016
10	−20.34326	73.84223

Table 2 Material properties for GFRP and CFRP unidirectional plies.

Type	E_1 (GPa)	E_2 (GPa)	ν_{12}	ν_{23}	G_{12} (GPa)	G_{23} (GPa)	Ply thickness t_0 (mm)
GFRP-1	41.7	13	0.3	0.42	3.4	4.58	0.203
GFRP-2	46	13	0.3	0.42	5	4.6	0.5
CFRP	142	9.85	0.3	0.46	4.48	3.37	0.127