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Abstract. Let $k_1, \ldots, k_r$ be positive integers. Let $q_1, \ldots, q_r$ be pairwise coprime positive integers with $q_i > 2$ $(i = 1, \ldots, r)$, and set $q = q_1 \cdots q_r$. For each $i = 1, \ldots, r$, let T_i be a set of $\varphi(q_i)/2$ representatives mod q_i such that the union $T_i \cup (-T_i)$ is a complete set of coprime residues mod q_i. Let K be an algebraic number field over which the qth cyclotomic polynomial Φ_q is irreducible. Then, $\varphi(q)/2^r$ numbers

$$\prod_{i=1}^{r} \frac{d^{k_i-1}}{dz_i^{k_i-1}}(\cot \pi z_i)|_{z_i=a_i/q_i} \quad (a_i \in T_i,\ i = 1, \ldots, r)$$

are linearly independent over K. As an application, a generalization of the Baker–Birch–Wirsing theorem on the non-vanishing of the multiple Dirichlet series $L(s_1, \ldots, s_r; f)$ with periodic coefficients at $(s_1, \ldots, s_r) = (k_1, \ldots, k_r)$ is proven under a parity condition.

1. Introduction

Sarvadaman Chowla [**7**] proved that *if p is an odd prime, then the $(p-1)/2$ real numbers* $\cot(2\pi a/p)$ $(a = 1, 2, \ldots, (p-1)/2)$ *are linearly independent over the field $\mathbb{Q}$ of rational numbers.* Other proofs were provided by [**2, 3, 10, 14**].

Let k, q be positive integers with $q > 2$, and let T be a set of $\varphi(q)/2$ representatives mod q such that the union $T \cup (-T)$ is a complete set of coprime residues mod q. Okada [**15**] and Wang [**16**] independently generalized the aforementioned Chowla's theorem as follows.

THEOREM 1.1. (Okada [**15**] and Wang [**16**]) *The $\varphi(q)/2$ numbers*

$$\frac{d^{k-1}}{dz^{k-1}}(\cot \pi z)|_{z=a/q} \quad (a \in T)$$

are linearly independent over $\mathbb{Q}$.

For a positive integer k and a periodic function $f : \mathbb{Z} \to \mathbb{C}$, we consider the value $L(k, f)$ of the Dirichlet series defined by

$$L(k, f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^k}.$$

In the early 1960s, Chowla made the following conjecture: Let $f : \mathbb{Z} \to \mathbb{Q}$ be a non-zero periodic function with prime period p satisfying $f(p) = 0$. Then, $L(1, f) \neq 0$ if this converges. Chowla [**6**] proved this conjecture in the case where f is an odd function. Baker, Birch, and Wirsing [**4**] proved Chowla's conjecture in a more general setting, as follows.

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THEOREM 1.2. (Baker, Birch, and Wirsing [**4**]) *Let q be a positive integer and let $f : \mathbb{Z} \to \overline{\mathbb{Q}}$ be a non-zero periodic function with algebraic values and period q such that*
(i) $f(r) = 0$ *if* $1 < \gcd(r, q) < q$, *and*
(ii) *the qth cyclotomic polynomial Φ_q is irreducible over $\mathbb{Q}(f(1), \ldots, f(q))$.*
Then, $L(1, f) \neq 0$ if this converges.

Under the same assumptions as Theorem 1.2, Adhikari, Saradha, Shorey, and Tijdeman [**1**] proved that $L(1, f)$ is a transcendental number if this converges. Concerning the values $L(k, f)$ $(k \geq 1)$, Okada [**15**] obtained the following theorem, which is a generalization of Theorem 1.2 under a parity condition.

THEOREM 1.3. (Okada [**15**]) *Let k be a positive integer, and let $f : \mathbb{Z} \to \overline{\mathbb{Q}}$ be a non-zero periodic function with algebraic values and period $q > 2$ such that*
(i) *f is even or odd based on whether k is even or odd,*
(ii) $f(n) = 0$ *if* $\gcd(n, q) > 1$, *and*
(iii) *the qth cyclotomic polynomial Φ_q is irreducible over $\mathbb{Q}(f(1), \ldots, f(q))$.*
Then, $L(k, f) \neq 0$ if this converges.

Let $k_1, \ldots, k_r$ be positive integers, and let $f : \mathbb{Z}^r \to \mathbb{C}$ be a function such that for each i, $f(\ldots, n_i, \ldots)$ is periodic. We investigate the value $L(k_1, \ldots, k_r; f)$ of the multiple Dirichlet series for f defined by

$$L(k_1, \ldots, k_r; f) = \sum_{n_1, \ldots, n_r = 1}^{\infty} \frac{f(n_1, \ldots, n_r)}{n_1^{k_1} \cdots n_r^{k_r}}.$$

Weatherby [**17**] proved the transcendence properties of the value $L(1, \ldots, 1; f)$.

The purpose of this paper is to generalize the Okada–Wang theorem (Theorem 1.1) and apply our result to the multiple Dirichlet series to generalize Okada's theorem (Theorem 1.3). The remainder of this paper is organized as follows. In Section 2, we generalize the Okada–Wang theorem and apply our result to compute the dimension of the Chowla–Milnor spaces. In Section 3, we present the basic properties of the values of the multiple Dirichlet series to generalize Weatherby's results [**17**]. In Section 4, we state a generalization of Okada's theorem. Section 5 is devoted to the proof of this generalization.

2. Okada–Wang theorem

2.1. Main theorem

The following is a generalization of the Okada–Wang theorem.

THEOREM 2.1. *Let $k_1, \ldots, k_r$ be positive integers. Let $q_1, \ldots, q_r$ be pairwise coprime positive integers with $q_i > 2$ $(i = 1, \ldots, r)$, and set $q = q_1 \cdots q_r$. For each $i = 1, \ldots, r$, let T_i be a set of $\varphi(q_i)/2$ representatives* mod q_i *such that the union $T_i \cup (-T_i)$ is a complete set of coprime residues* mod q_i. *Let K be an algebraic number field over which the qth cyclotomic polynomial Φ_q is irreducible. Then, the $\varphi(q)/2^r$ numbers*

$$\prod_{i=1}^{r} \frac{d^{k_i - 1}}{dz_i^{k_i - 1}} (\cot \pi z_i)|_{z_i = a_i/q_i} \quad (a_i \in T_i,\ i = 1, \ldots, r)$$

are linearly independent over K.

Proof. We prove the theorem by induction on r. Murty and Saradha [**12**, Lemma 11] proved the case $r = 1$. Let $r > 1$, and we assume that the case $r - 1$ is correct. Let

$$F_k(z) = \frac{k}{(-2\pi i)^k} \frac{d^{k-1}}{dz^{k-1}}(\pi \cot \pi z).$$

We can observe that $F_{k_i}(a_i/q_i) \in \mathbb{Q}(\zeta_{q_i})$, where ζ_{q_i} is a primitive q_ith root of unity. We first prove that

$$F_{k_1}(a_1/q_1) \cdots F_{k_r}(a_r/q_r) \quad (a_i \in T_i,\, i = 1, \ldots, r) \tag{2.1}$$

are linearly independent over $\mathbb{Q}$. Assume that

$$\sum_{a_1 \in T_1} \cdots \sum_{a_r \in T_r} C_{a_1,\ldots,a_r} F_{k_1}(a_1/q_1) \cdots F_{k_r}(a_r/q_r) = 0,$$

with $C_{a_1,\ldots,a_r} \in \mathbb{Q}$. This equation can be written as

$$\sum_{a_r \in T_r} \left(\sum_{a_1 \in T_1} \cdots \sum_{a_{r-1} \in T_{r-1}} C_{a_1,\ldots,a_r} F_{k_1}(a_1/q_1) \cdots F_{k_{r-1}}(a_{r-1}/q_{r-1}) \right) F_{k_r}(a_r/q_r) = 0. \tag{2.2}$$

As $F_{k_r}(a_r/q_r)$ $(a_r \in T_r)$ are linearly independent over $\mathbb{Q}$, using $\mathbb{Q}(\zeta_{q_1\cdots q_{r-1}}) \cap \mathbb{Q}(\zeta_{q_r}) = \mathbb{Q}$, $F_{k_r}(a_r/q_r)$ $(a_r \in T_r)$ are linearly independent over $\mathbb{Q}(\zeta_{q_1\cdots q_{r-1}})$. Combining

$$\sum_{a_1 \in T_1} \cdots \sum_{a_{r-1} \in T_{r-1}} C_{a_1,\ldots,a_r} F_{k_1}(a_1/q_1) \cdots F_{k_{r-1}}(a_{r-1}/q_{r-1}) \in \mathbb{Q}(\zeta_{q_1\cdots q_{r-1}})$$

with (2.2), we obtain

$$\sum_{a_1 \in T_1} \cdots \sum_{a_{r-1} \in T_{r-1}} C_{a_1,\ldots,a_r} F_{k_1}(a_1/q_1) \cdots F_{k_{r-1}}(a_{r-1}/q_{r-1}) = 0 \quad (a_r \in T_r).$$

Applying the assumption of induction to these equations, we have

$$C_{a_1,\ldots,a_r} = 0 \quad (a_i \in T_i,\, i = 1, \ldots, r).$$

We prove that the numbers in (2.1) are linearly independent over K. By assumption, K and $\mathbb{Q}(\zeta_q)$ are linearly disjoint over $\mathbb{Q}$. As the numbers in (2.1), which belong to $\mathbb{Q}(\zeta_q)$, are linearly independent over $\mathbb{Q}$, they are linearly independent over K. □

2.2. *Application: the dimension of the Chowla–Milnor space*

For $x \in \mathbb{R}$ with $0 < x \le 1$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the Hurwitz zeta function is defined by

$$\zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(n + x)^s}.$$

Definition 2.2. Let $q_1, \ldots, q_r > 1$ be integers, and let K be an algebraic number field. For integers $k_1, \ldots, k_r > 1$, let $V_{k_1,\ldots,k_r}(q_1, \ldots, q_r;\, K)$ be the K-linear space defined by

$$\begin{aligned} &V_{k_1,\ldots,k_r}(q_1, \ldots, q_r;\, K) \\ &= K\text{-span of } \{\zeta(k_1, a_1/q_1) \cdots \zeta(k_r, a_r/q_r) \mid 1 \le a_i < q_i,\, \gcd(a_i, q_i) = 1,\, i = 1, \ldots, r\}, \end{aligned}$$

which is called the *Chowla–Milnor space*.

Using Theorem 2.1, we can provide the following non-trivial lower bound for the dimension of the Chowla–Milnor space.

THEOREM 2.3. *Let $q_1, \dots, q_r$ be integers with $q_i > 2$ $(i = 1, \dots, r)$, and set $q = q_1 \cdots q_r$. Let K be an algebraic number field over which the qth cyclotomic polynomial Φ_q is irreducible. Then,*

$$\dim_K V_{k_1,\dots,k_r}(q_1, \dots, q_r; K) \geq \frac{\varphi(q)}{2^r}.$$

Proof. According to Chatterjee [**5**, Lemma 3], for $a_i \in \mathbb{Z}$ with $1 \leq a_i < q_i$ and $\gcd(a_i, q_i) = 1$, it holds that

$$\zeta(k_i, a_i/q_i) + (-1)^{k_i}\zeta(k_i, 1 - a_i/q_i) = \frac{(-1)^{k_i-1}}{(k_i-1)!}\frac{d^{k_i-1}}{dz_i^{k_i-1}}(\pi \cot \pi z_i)|_{z_i=a_i/q_i}.$$

For each i, let

$$T_i = \{a_i \in \mathbb{Z} \mid 1 \leq a_i \leq q_i/2,\ \gcd(a_i, q_i) = 1\}.$$

The set $\{a_i, q_i - a_i \mid a_i \in T_i\}$ becomes a complete set of coprime residues mod q_i. Using Theorem 2.1, the $\varphi(q)/2^r$ numbers

$$\prod_{i=1}^{r}(\zeta(k_i, a_i/q_i) + (-1)^{k_i}\zeta(k_i, 1 - a_i/q_i)) \quad (a_i \in T_i,\ i = 1, \dots, r),$$

which belong to $V_{k_1,\dots,k_r}(q_1, \dots, q_r; K)$, are linearly independent over K. This completes the proof. □

Remark 2.4. Gun, Murty, and Rath [**8, 9**] formulated the following conjecture, which is called the *Chowla–Milnor conjecture*: Let k, q be positive integers with $q > 1$, and let K be an algebraic number field over which the qth cyclotomic polynomial Φ_q is irreducible. Then, it holds that $\dim_K V_k(q; K) = \varphi(q)$. They [**9**] proved the case $r = 1$ for Theorem 2.3.

3. Multiple Dirichlet series

We prove the basic properties of the values of the multiple Dirichlet series to generalize Weatherby's results [**17**].

Let $q_1, \dots, q_r$ be positive integers, and let K be a subfield of $\mathbb{C}$. We write $\mathcal{F}(q_1, \dots, q_r; K)$ for the set of all functions $f : \mathbb{Z}^r \to K$ such that, for each i, $f(\dots, n_i, \dots)$ is periodic with period q_i. The function $f \in \mathcal{F}(q_1, \dots, q_r; K)$ is called a *multiple periodic function with multiple period* $(q_1, \dots, q_r)$.

3.1. Convergence

Let $k_1, \dots, k_r$ be positive integers. For $f \in \mathcal{F}(q_1, \dots, q_r; \mathbb{C})$, we define the value $L(k_1, \dots, k_r; f)$ of the multiple Dirichlet series by

$$\begin{aligned} L(k_1, \dots, k_r; f) &= \sum_{n_1,\dots,n_r=1}^{\infty} \frac{f(n_1, \dots, n_r)}{n_1^{k_1} \cdots n_r^{k_r}} \\ &= \lim_{N_1,\dots,N_r \to \infty} \sum_{n_1=1}^{N_1} \cdots \sum_{n_r=1}^{N_r} \frac{f(n_1, \dots, n_r)}{n_1^{k_1} \cdots n_r^{k_r}}, \end{aligned}$$

where each N_i can go to infinity independent of the other N_j. That is, this series converges to the same value regardless of how $N_1, \ldots, N_r$ go to infinity.

The following proposition is a criterion for the convergence of $L(k_1, \ldots, k_r; f)$.

PROPOSITION 3.1. *For $f \in \mathcal{F}(q_1, \ldots, q_r; \mathbb{C})$, the series $L(k_1, \ldots, k_r; f)$ converges if, and only if, for each i with $k_i = 1$,*

$$\sum_{n_i=1}^{q_i} f(a_1, \ldots, a_{i-1}, n_i, a_{i+1}, \ldots, a_r) = 0 \tag{3.1}$$

for all $a_j \in \{1, \ldots, q_j\}$ $(j \neq i)$.

Proof. Assume that $L(k_1, \ldots, k_r; f)$ converges. For $r = 1$, we refer the reader to Murty and Saradha [**11**, Theorem 16]. Let $r > 1$. When $k_i = 1$, we have

$$\sum_{n_1=1}^{N_1} \cdots \sum_{n_r=1}^{N_r} \frac{f(n_1, \ldots, n_r)}{n_1^{k_1} \cdots n_r^{k_r}}$$
$$= \sum_{a_1=1}^{N_1} \frac{1}{a_1^{k_1}} \cdots \sum_{a_{i-1}=1}^{N_{i-1}} \frac{1}{a_{i-1}^{k_{i-1}}} \sum_{a_{i+1}=1}^{N_{i+1}} \frac{1}{a_{i+1}^{k_{i+1}}} \cdots \sum_{a_r=1}^{N_r} \frac{1}{a_r^{k_r}} \sum_{n_i=1}^{N_i} \frac{f(a_1, \ldots, n_i, \ldots, a_r)}{n_i},$$

which converges as $N_i \to \infty$. Hence, $\sum_{n_i=1}^{\infty} f(a_1, \ldots, n_i, \ldots, a_r) n_i^{-1}$ converges. This yields (3.1).

Conversely, assume that (3.1) holds for each i with $k_i = 1$. We prove the claim by induction on r. For $r = 1$, we refer the reader to Murty and Saradha [**11**, Theorem 16]. Let $r > 1$, and we assume that the case $r - 1$ is correct. For $n_1 \in \mathbb{N}$, let

$$F(n_1) = \sum_{n_2, \ldots, n_r=1}^{\infty} \frac{f(n_1, n_2, \ldots, n_r)}{n_2^{k_2} \cdots n_r^{k_r}}, \tag{3.2}$$

which converges by the assumption of induction. We observe that $F : \mathbb{N} \to \mathbb{C}$ is a periodic function with period q_1. When $k_1 > 1$, we observe that $L(k_1, \ldots, k_r; f) = \sum_{n_1=1}^{\infty} F(n_1) n_1^{-k_1}$ converges. When $k_1 = 1$, using (3.1),

$$\sum_{n_1=1}^{q_1} F(n_1) = \sum_{n_2, \ldots, n_r=1}^{\infty} \sum_{n_1=1}^{q_1} f(n_1, n_2, \ldots, n_r) = 0,$$

which shows that $L(k_1, \ldots, k_r; f) = \sum_{n_1=1}^{\infty} F(n_1) n_1^{-k_1}$ converges. □

3.2. *Fourier analysis*

We recall the Fourier analysis for the multiple periodic functions discussed by [**17**]. For positive integers $q_1, \ldots, q_r$, let $\zeta_{q_i} = \exp(2\pi i/q_i)$ $(i = 1, \ldots, r)$ and

$$G(q_1, \ldots, q_r) = \mathbb{Z}/q_1\mathbb{Z} \times \cdots \times \mathbb{Z}/q_r\mathbb{Z}.$$

For $f \in \mathcal{F}(q_1, \ldots, q_r; \mathbb{C})$,

$$\widehat{f}(n_1, \ldots, n_r) = \frac{1}{q_1 \cdots q_r} \sum_{(a_1, \ldots, a_r) \in G(q_1, \ldots, q_r)} f(a_1, \ldots, a_r) \zeta_{q_1}^{-a_1 n_1} \cdots \zeta_{q_r}^{-a_r n_r}$$

is called the *Fourier transform* of f. If f is an algebraic-valued function, then $\widehat{f}$ is also an algebraic-valued function. Using $\widehat{f}$, f can be written as

$$f(n_1, \ldots, n_r) = \sum_{(a_1,\ldots,a_r)\in G(q_1,\ldots,q_r)} \widehat{f}(a_1, \ldots, a_r)\zeta_{q_1}^{a_1 n_1} \cdots \zeta_{q_r}^{a_r n_r},$$

which is called the *Fourier inversion formula*. According to [**17**, Lemma 8], (3.1) holds for all $a_j \in \{1, \ldots, q_j\}$ $(j \neq i)$ if and only if

$$\widehat{f}(a_1, \ldots, a_{i-1}, q_i, a_{i+1}, \ldots, a_r) = 0 \tag{3.3}$$

for all $a_j \in \{1, \ldots, q_j\}$ $(j \neq i)$.

PROPOSITION 3.2. *For $f \in \mathcal{F}(q_1, \ldots, q_r; \mathbb{C}) \setminus \{0\}$, the following properties hold:*

(i) $\widehat{f} \in \mathcal{F}(q_1, \ldots, q_r; \mathbb{C})$*;*

(ii) *f and $\widehat{f}$ have the same parity, i.e. for each i, $f(\ldots, n_i, \ldots)$ and $\widehat{f}(\ldots, n_i, \ldots)$ are even or $f(\ldots, n_i, \ldots)$ and $\widehat{f}(\ldots, n_i, \ldots)$ are odd.*

Proof. The proof of (i) is easy. We prove (ii). Let $q = q_1 \cdots q_r$. For each i, let

$$\epsilon_i = \begin{cases} 1, & \text{if } f(\ldots, n_i, \ldots) \text{ is even,} \\ -1, & \text{if } f(\ldots, n_i, \ldots) \text{ is odd.} \end{cases} \tag{3.4}$$

We have

$$\begin{aligned}
&\widehat{f}(n_1, \ldots, n_i, \ldots, n_r) \\
&= \frac{1}{q} \sum_{\substack{a_j\in G(q_j)\\ j\neq i}} \prod_{j\neq i} \zeta_{q_j}^{-a_j n_j} \sum_{a_i=0}^{q_i-1} f(a_1, \ldots, a_{i-1}, q_i - a_i, a_{i+1}, \ldots, a_r)\zeta_{q_i}^{-(q_i-a_i)n_i} \\
&= \frac{1}{q} \sum_{\substack{a_j\in G(q_j)\\ j\neq i}} \prod_{j\neq i} \zeta_{q_j}^{-a_j n_j} \sum_{a_i=1}^{q_i} f(a_1, \ldots, a_{i-1}, -a_i, a_{i+1}, \ldots, a_r)\zeta_{q_i}^{-a_i(-n_i)} \\
&= \epsilon_i \widehat{f}(n_1, \ldots, n_{i-1}, -n_i, n_{i+1}, \ldots, n_r).
\end{aligned}$$

□

3.3. *Some formulas*

We recall polygamma functions to describe the values of multiple Dirichlet series. The digamma function $\psi(x) = \psi_0(x)$ is defined by

$$-\psi(x) - \gamma = \frac{1}{x} + \sum_{n=1}^{\infty}\left(\frac{1}{n+x} - \frac{1}{n}\right),$$

where γ is the Euler constant. For a positive integer k, the kth polygamma function $\psi_k(x)$ is defined by

$$\psi_k(x) = \frac{d^k}{dz^k}\psi(x) = (-1)^{k-1}k! \sum_{n=0}^{\infty} \frac{1}{(n+z)^{k+1}}.$$

Let $k_1, \ldots, k_r$ be positive integers. We set

$$c(k_1, \ldots, k_r) = \frac{(-1)^{k_1+\cdots+k_r}}{(k_1-1)!\cdots(k_r-1)!q^{k_1+\cdots+k_r}}$$

to simplify the notation.

PROPOSITION 3.3. *We assume that $L(k_1, \dots, k_r;\, f)$ converges for $f \in \mathcal{F}(q_1, \dots, q_r; \mathbb{C})$. Then, it holds that*

$$L(k_1, \dots, k_r;\, f) = c(k_1, \dots, k_r) \times \sum_{(a_1,\dots,a_r)\in G(q_1,\dots,q_r)} f(a_1, \dots, a_r)\psi_{k_1-1}(a_1/q_1)\cdots\psi_{k_r-1}(a_r/q_r). \tag{3.5}$$

Proof. We prove the proposition by induction on r. The case $r = 1$ was proved by Murty and Saradha [**12**, (6)]. Let $r > 1$, and assume that the case $r - 1$ is correct. Let $F : \mathbb{Z} \to \mathbb{C}$ be the periodic function with period q_1 defined by (3.2). Using this, it holds that

$$L(k_1, \dots, k_r;\, f) = \sum_{n_1=1}^{\infty} F(n_1)n_1^{-k_1}. \tag{3.6}$$

By the assumption of induction, $F(n_1)$ can be written as

$$F(n_1) = c(k_2, \dots, k_r) \times \sum_{(a_2,\dots,a_r)\in G(q_2,\dots,q_r)} f(n_1, a_2, \dots, a_r)\psi_{k_2-1}(a_2/q_2)\cdots\psi_{k_r-1}(a_r/q_r). \tag{3.7}$$

Applying the case $r = 1$ to (3.6), we have

$$L(k_1, \dots, k_r;\, f) = c(k_1) \sum_{a_1\in G(q_1)} F(a_1)\psi_{k_1-1}(a_1/q_1),$$

which becomes the right-hand side of (3.5). □

For a positive integer k and $z \in \mathbb{C}$ with $|z| \le 1$, the kth polylogarithm function $\mathrm{Li}_k(z)$ is defined by

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}.$$

Note that for $k = 1$, this series becomes $-\log(1 - z)$ for $|z| < 1$. We describe the values of multiple Dirichlet series in terms of polylogarithm functions.

PROPOSITION 3.4. *We assume that $L(k_1, \dots, k_r;\, f)$ converges for $f \in \mathcal{F}(q_1, \dots, q_r; \mathbb{C})$. Then, it holds that*

$$L(k_1, \dots, k_r;\, f) = \sum_{a_1=1}^{Q_1}\cdots\sum_{a_r=1}^{Q_r} \widehat{f}(a_1, \dots, a_r)\mathrm{Li}_{k_1}(\zeta_{q_1}^{a_1})\cdots\mathrm{Li}_{k_r}(\zeta_{q_r}^{a_r}), \tag{3.8}$$

where

$$Q_j = \begin{cases} q_j, & \text{if } k_j \ge 2,\\ q_j - 1, & \text{if } k_j = 1.\end{cases}$$

Proof. Using the Fourier inversion formula, we have

$$L(k_1, \dots, k_r;\, f) = \sum_{n_1,\dots,n_r=1}^{\infty} \frac{1}{n_1^{k_1}\cdots n_r^{k_r}} \sum_{a_1=1}^{q_1}\cdots\sum_{a_r=1}^{q_r} \widehat{f}(a_1, \dots, a_r)\zeta_{q_1}^{a_1n_1}\cdots\zeta_{q_r}^{a_rn_r}.$$

Applying Proposition 3.1 and (3.3) to this equation, we obtain the right-hand side of (3.8). □

4. Okada's theorem

In this section, we generalize Okada's theorem (Theorem 1.3). We provide some definitions to state our result.

Definition 4.1. Let K be a subfield of $\mathbb{C}$. A function $f \in \mathcal{F}(q_1, \ldots, q_r; K)$ is said to be of *Dirichlet type* if $f(n_1, \ldots, n_r) = 0$ whenever there exists i such that $\gcd(n_i, q_i) > 1$. We set

$$\mathcal{FD}(q_1, \ldots, q_r; K) = \{f \in \mathcal{F}(q_1, \ldots, q_r; K) \mid f \text{ is of Dirichlet type}\}.$$

Definition 4.2. Let $f \in \mathcal{F}(q_1, \ldots, q_r; \mathbb{C})$ and $(k_1, \ldots, k_r) \in \mathbb{N}^r$. We say that f and $(k_1, \ldots, k_r)$ have the *same parity* if, for each i, $f(\ldots, n_i, \ldots)$ and k_i are even or $f(\ldots, n_i, \ldots)$ and k_i are odd.

The following is a generalization of Theorem 1.3, which is one of the main theorems in this paper.

THEOREM 4.3. *Let $k_1, \ldots, k_r$ be positive integers. Let $q_1, \ldots, q_r$ be pairwise coprime positive integers with $q_i > 2$ $(i = 1, \ldots, r)$, and set $q = q_1 \cdots q_r$ Let K be an algebraic number field over which the qth cyclotomic polynomial Φ_q is irreducible. We assume that $f \in \mathcal{FD}(q_1, \ldots, q_r; K)\backslash\{0\}$ and $(k_1, \ldots, k_r)$ have the same parity. If $L(k_1, \ldots, k_r; f)$ converges, then $L(k_1, \ldots, k_r; f)$ is a transcendental number.*

It is interesting to study certain infinite series in [**13, 17**] to generalize this theorem.

As a corollary to Theorem 4.3, we have the following result on the linear independence of the values of the multiple Dirichlet L-functions.

THEOREM 4.4. *Let $k_1, \ldots, k_r$ be positive integers. Let $q_1, \ldots, q_r$ be pairwise coprime positive integers with $q_i > 2$ $(i = 1, \ldots, r)$, and set $q = q_1 \cdots q_r$. Let K be an algebraic number field, and suppose that $K(e^{2\pi i/\varphi(q)}) \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. Let $\Delta(k_1, \ldots, k_r)$ be the set of $\chi \in \mathcal{FD}(q_1, \ldots, q_r; \mathbb{C})$ such that χ becomes a character of the unit group $(\mathbb{Z}/q_1\mathbb{Z} \times \cdots \times \mathbb{Z}/q_r\mathbb{Z})^\times$ and such that χ and $(k_1, \ldots, k_r)$ have the same parity. Then, the numbers $L(k_1, \ldots, k_r; \chi)$ $(\chi \in \Delta(k_1, \ldots, k_r))$ are linearly independent over $K(e^{2\pi i/\varphi(q)})$.*

Proof. We assume that

$$\sum_{\chi \in \Delta(k_1,\ldots,k_r)} c_\chi L(k_1, \ldots, k_r; \chi) = 0 \quad (c_\chi \in K(e^{2\pi i/\varphi(q)})).$$

Let $f = \sum_{\chi \in \Delta(k_1,\ldots,k_r)} c_\chi \chi$. As $L(k_1, \ldots, k_r; f) = 0$, using Theorem 4.3, one has $\sum_{\chi \in \Delta(k_1,\ldots,k_r)} c_\chi \chi = 0$. Thus, the orthogonality of characters yields $c_\chi = 0$ for any $\chi \in \Delta(k_1, \ldots, k_r)$. □

Remark 4.5. Theorem 4.4 is a generalization of Okada [**15**, Corollary 2] and Murty and Saradha [**12**, Theorem 13].

5. Proof of Theorem 4.3

In this section, we prove Theorem 4.3.

5.1. Non-vanishing of $L(k_1, \dots, k_r; f)$

In this subsection, we prove that $L(k_1, \dots, k_r; f)$ is not zero. As f is of Dirichlet type, using Proposition 3.3, it holds that

$$L(k_1, \dots, k_r; f)$$
$$= c(k_1, \dots, k_r) \sum_{(a_1,\dots,a_r)\in(\mathbb{Z}/q_1\mathbb{Z})^\times\times\cdots\times(\mathbb{Z}/q_r\mathbb{Z})^\times} f(a_1, \dots, a_r)\psi_{k_1-1}(a_1/q_1)\cdots\psi_{k_r-1}(a_r/q_r),$$

where each $(\mathbb{Z}/q_i\mathbb{Z})^\times$ is the unit group of $\mathbb{Z}/q_i\mathbb{Z}$. For each i, let

$$T_i = \{a_i \in \mathbb{Z} \mid 1 \le a_i \le q_i/2,\ \gcd(a_i, q_i) = 1\}.$$

The set $\{a_i, q_i - a_i \mid a_i \in T_i\}$ becomes a complete set of coprime residues mod q_i. Observing the parity of f, $L(k_1, \dots, k_r; f)$ becomes

$$c(k_1, \dots, k_r) \sum_{i=1}^{r} \sum_{\substack{b_i\in\{a_i,q_i-a_i\}\\ a_i\in T_i}} f(b_1, \dots, b_r)\psi_{k_1-1}(b_1/q_1)\cdots\psi_{k_r-1}(b_r/q_r)$$
$$= c(k_1, \dots, k_r) \sum_{(a_1,\dots,a_r)\in T_1\times\cdots\times T_r} f(a_1, \dots, a_r) \prod_{i=1}^{r}(\psi_{k_i-1}(a_i/q_i) + \epsilon_i\psi_{k_i-1}(1 - a_i/q_i)),$$

where ϵ_i is the number defined by (3.4). Murty and Saradha [**12**, (7)] proved that

$$-\frac{d^{k_i-1}}{dz^{k_i-1}}(\pi \cot \pi z) = \psi_{k_i-1}(z) + (-1)^{k_i}\psi_{k_i-1}(1 - z).$$

As f and $(k_1, \dots, k_r)$ have the same parity, $L(k_1, \dots, k_r; f)$ becomes

$$(-1)^r c(k_1, \dots, k_r) \sum_{(a_1,\dots,a_r)\in T_1\times\cdots\times T_r} f(a_1, \dots, a_r) \prod_{i=1}^{r}\left(\frac{d^{k_i-1}}{dz_i^{k_i-1}}(\pi \cot \pi z_i)|_{z_i=a_i/q_i}\right),$$

which is not zero from Theorem 2.1.

5.2. Algebraicity of $L(k_1, \dots, k_r; f)/\pi^{k_1+\cdots+k_r}$

In this subsection, we prove that $L(k_1, \dots, k_r; f)/\pi^{k_1+\cdots+k_r}$ is an algebraic number. Let $B_k(x)$ be the kth Bernoulli polynomial. We need the following lemma.

LEMMA 5.1. *Let $f \in \mathcal{F}(q_1, \dots, q_r; \mathbb{C}) \setminus \{0\}$ and $(k_1, \dots, k_r) \in \mathbb{N}^r$. We assume that f and $(k_1, \dots, k_r)$ have the same parity. If $L(k_1, \dots, k_r; f)$ converges, then*

$$2^r L(k_1, \dots, k_r; f)$$
$$= \frac{(-1)^r(2\pi i)^{k_1+\cdots+k_r}}{k_1!\cdots k_r!} \sum_{a_1=1}^{Q_1}\cdots\sum_{a_r=1}^{Q_r} \widehat{f}(a_1, \dots, a_r)B_{k_1}(a_1/q_1)\cdots B_{k_r}(a_r/q_r), \quad (5.1)$$

where each Q_j is the number defined in Proposition 3.4. In particular, if f is an algebraic-valued function, then $L(k_1, \dots, k_r; f)/\pi^{k_1+\cdots+k_r}$ is an algebraic number.

Proof. For each i, let

$$\delta_i = \begin{cases} 0, & \text{if } k_i \geq 2, \\ 1, & \text{if } k_i = 1. \end{cases}$$

Using (3.3) and Proposition 3.4, it holds that

$$\begin{aligned} &L(k_1, \ldots, k_r; f) \\ &= \sum_{a_1=\delta_1}^{q_1-1} \cdots \sum_{a_r=\delta_r}^{q_r-1} \widehat{f}(q_1 - a_1, \ldots, q_r - a_r) \mathrm{Li}_{k_1}(\zeta_{q_1}^{q_1-a_1}) \cdots \mathrm{Li}_{k_r}(\zeta_{q_r}^{q_r-a_r}) \\ &= \sum_{a_1=1}^{Q_1} \cdots \sum_{a_r=1}^{Q_r} \widehat{f}(-a_1, \ldots, -a_r) \mathrm{Li}_{k_1}(\zeta_{q_1}^{-a_1}) \cdots \mathrm{Li}_{k_r}(\zeta_{q_r}^{-a_r}) \\ &= \epsilon_1 \cdots \epsilon_r \sum_{a_1=1}^{Q_1} \cdots \sum_{a_r=1}^{Q_r} \widehat{f}(a_1, \ldots, a_r) \mathrm{Li}_{k_1}(\zeta_{q_1}^{-a_1}) \cdots \mathrm{Li}_{k_r}(\zeta_{q_r}^{-a_r}). \end{aligned}$$

Hence, we have

$$\begin{aligned} &2^r L(k_1, \ldots, k_r; f) \\ &= \sum_{a_1=1}^{Q_1} \cdots \sum_{a_r=1}^{Q_r} \widehat{f}(a_1, \ldots, a_r) \prod_{j=1}^{r} (\mathrm{Li}_{k_j}(\zeta_{q_j}^{a_j}) + \epsilon_j \mathrm{Li}_{k_j}(\zeta_{q_j}^{-a_j})). \end{aligned}$$

According to Murty and Saradha [**12**, Section 2], it holds that

$$-B_k(x) = \frac{k!}{(2\pi i)^k} (\mathrm{Li}_k(e^{2\pi i k}) + (-1)^k \mathrm{Li}_k(e^{-2\pi i k})).$$

As f and $(k_1, \ldots, k_r)$ have the same parity, (5.1) is obtained.

We prove the latter part of the lemma. By assumption, $\widehat{f}$ is an algebraic-valued function. As each $B_k(x)$ has rational coefficients, $L(k_1, \ldots, k_r; f)/\pi^{k_1+\cdots+k_r}$ is an algebraic number.

5.3. *Conclusion of the proof*

Finally, using Sections 5.1 and 5.2, we conclude that $L(k_1, \ldots, k_r; f)$ is a transcendental number. This completes the proof of Theorem 4.3.

From the discussion in Section 5.1, we can formulate the following conjecture, which is a generalization of [**12**, Conjecture 1].

CONJECTURE 5.2. *Let $k_1, \ldots, k_r$ be positive integers, and let $q_1, \ldots, q_r$ be pairwise coprime positive integers with $q_i > 2$ ($i = 1, \ldots, r$). Let K be an algebraic number field over which the qth cyclotomic polynomial Φ_q is irreducible. Then, the $\varphi(q)$ numbers*

$$\psi_{k_1}(a_1/q_1) \cdots \psi_{k_r}(a_r/q_r) \quad (1 \leq a_i \leq q_i,\ \gcd(a_i, q_i) = 1,\ i = 1, \ldots, r)$$

are linearly independent over K.

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