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By

Tsuneshi Obata* and **Shunsuke Shiraishi†**

Abstract

Pairwise comparison matrix (PCM) and its maximum eigenvalue play key roles in analytic hierarchy process (AHP). We shed light on the characteristic polynomial of a PCM of 4th order. By computational simulation, we can confirm that the value of the characteristic polynomial for 4 is non-positive in case which Saaty's discrete scale will be used. Thus we can show the real-number solution of the characteristic equation exists and is greater than 4 in the practical use of AHP.

Key Words and Phrases: analytic hierarchy process, pairwise comparison matrix, remainder theorem, exhaustive enumeration.

1. Introduction

Analytic hierarchy process (AHP) established its wide popularity by employing the maximum eigenvalue $\lambda_{\max}$ of pairwise comparison matrix (PCM) as consistency index. See Saaty (1980), Tone (1986), Brunelli (2015), Kułakowski (2021). The consistency index is defined by C.I. $= \dfrac{\lambda_{\max} - n}{n-1}$ where n is the order of PCM. To calculate the maximum eigenvalue, one usually utilizes power method[1]. For the 3rd order PCM, Newton's method can be utilized to calculate the maximum eigenvalue. We have theoretically proven that a sequence generated by Newton's method from the initial value of 3 converges to the maximum eigenvalue (Shiraishi and Obata (2021a)). The result relies on one of the most favorable properties of the 3rd order matrix; uniqueness of the solution of the characteristic equation.

When we attempt to generalize the result for the 4th order matrix, we face a theoretical difficulty. Let A be 4th order PCM and $P_A(\lambda)$ be its characteristic polynomial. By the remainder theorem, the remainder of $P_A(\lambda)$ divided by $\lambda - 4$ is equal to $P_A(4)$. If we can verify $P_A(4) \leq 0$, we easily obtain that the maximum eigenvalue $\lambda_{\max}$ exists and is greater than or equal to 4.

This article verifies this fact by computational simulations in R[2]. The simulations are performed in two ways. To get started simulations, we generate $1,000$ random

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[1] `https://encyclopediaofmath.org/wiki/Iteration_methods`

[2] R is a language and environment for statistical computing and graphics. `https://www.r-project.org`.

samples in which elements of matrices are applied to Saaty's discrete scale of all 17 cases from 1 to 9 and their reciprocals. We also attempt to try exhaustive enumeration of Saaty's scale allocated to the upper six triangular elements of PCMs. The simulation shows $P_A(4) \leq 0$ holds for all cases. This counts $17^6 = 24,137,569$ cases. We can say that the assertion is true in the practical use of AHP.

As a byproduct, we count the number of appearance of consistency matrices under using Saaty's scale. It is $7^3 = 343$. The ratio of the appearance is 0.001421%. We should not expect that decision maker's pairwise comparison becomes to be completely consistent. On the other hand, the near consistent cases in which C.I. ≤ 0.1 counts $942,065$. The ratio of appearance is 3.902899%. According to Saaty (1980) p.51, if C.I. is less than 0.1, we may be satisfied with our judgments.

2. Characteristic polynomial of 4th order matrix

In AHP, the consistency index of n-th order PCM is defined by

$$\text{C.I.} = \frac{\lambda_{\max} - n}{n-1},$$

where $\lambda_{\max}$ is the maximum eigenvalue of the matrix. PCM has the following form:

$$A = \begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 1/a_{12} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 1/a_{1n} & 1/a_{2n} & \cdots & 1 \end{pmatrix},$$

where $a_{ij} > 0$. One says that A is *consistent* if $a_{ij}a_{jk} = a_{ik}$ for all i, j, k. In general, C.I. ≥ 0 and the consistency of A and C.I. $= 0$ are equivalent (Saaty (1980), Tone (1986), Brunelli (2015), Kułakowski (2021)).

From the general theory of characteristic polynomial[3], we have

$$P_A(\lambda) = \lambda^n - \operatorname{tr} A\lambda^{n-1} + c_2\lambda^{n-2} + c_3\lambda^{n-3} + \cdots + (-1)^n \det A.$$

For PCM, $\operatorname{tr} A = n$ and the following results are widely known now.

Theorem 2.1 Shiraishi et al. (1998).

$$\begin{aligned} &C.I. = 0 \iff P_A(\lambda) = \lambda^n - n\lambda^{n-1}, \\ &c_2 = 0, \\ &c_3 = \sum_{1 \leq i<j<k \leq n} \Big(2 - \big(\frac{a_{ij}a_{jk}}{a_{ik}} + \frac{a_{ik}}{a_{ij}a_{jk}}\big)\Big). \end{aligned} \tag{1}$$

By the well known relationships between the arithmetic mean and the geometric mean, we have $c_3 \leq 0$ and the equivalency of C.I. $= 0 \Leftrightarrow c_3 = 0$. We now have the following representation formula of the characteristic polynomial of the 4th order PCM:

$$P_A(\lambda) = \lambda^4 - 4\lambda^3 + c_3\lambda + \det A. \tag{2}$$

[3] `https://encyclopediaofmath.org/wiki/Characteristic_polynomial`

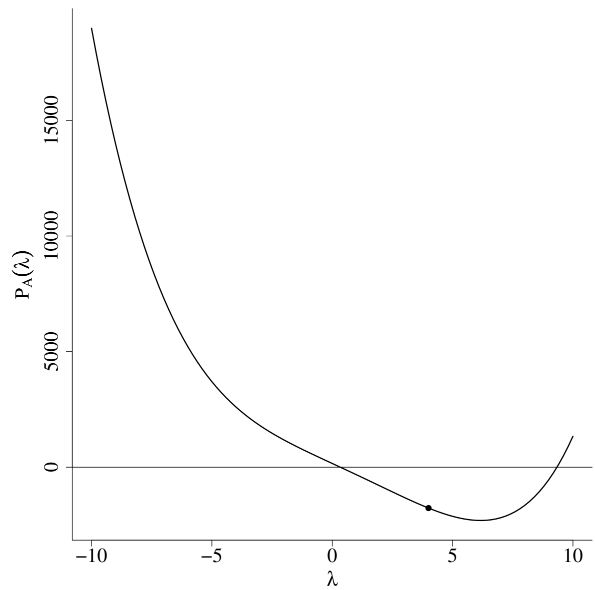

Figure 1: Graph of a characteristic polynomial

Let us demonstrate an example of 4th order PCM. If we set

$$A = \begin{pmatrix} 1 & 8 & \frac{1}{8} & 8 \\ \frac{1}{8} & 1 & 7 & 8 \\ 8 & \frac{1}{7} & 1 & 2 \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{2} & 1 \end{pmatrix},$$

we have $P_A(\lambda) \approx \lambda^4 - 4\lambda^3 - 482.4799\lambda + 162.3867$ and $P_A(4) \approx -1767.533$. Figure 1 shows the graph of $P_A(\lambda)$. For this matrix, the characteristic equation has two real-number solutions $\lambda \approx 0.3362$ and 9.3359, and two complex-number solutions $\lambda \approx -2.8361 \pm 6.6091i$. For this matrix, the consistency index C.I. $= 1.77863$ is far away from Saaty's desirable criteria C.I. ≤ 0.1. This inconsistency is a consequence of the existence of a contradictory triad of the objectives. Objective 1 is highly preferred to objective 2 ($a_{12} = 8$) and objective 2 is highly preferred to objective 3 ($a_{23} = 7$), but objective 3 is highly preferred to objective 1 ($a_{31} = 8$).

From the general discussion of the polynomial equation with real-number coefficient, the number of the real-number solution of the characteristic equation of the 4th order PCM must be even.

In this paper, we prove the the following fact on $P_A(\lambda)$ by computational simulation in Section 3.

(A) $$P_A(4) \leq 0.$$

$P_A(4) = 4c_3 + \det A$ is the *remainder* of $P_A(\lambda)$ dividing by $\lambda - 4$. Indeed,

$$\begin{aligned} P_A(\lambda) &= \lambda^4 - 4\lambda^3 + c_3\lambda + \det A \\ &= (\lambda - 4)(\lambda^3 + c_3) + 4c_3 + \det A. \end{aligned} \tag{3}$$

Since $\lim_{\lambda\to\infty} P_A(\lambda) = \infty$, under the fact (A), from the intermediate value theorem, we can say that there exists a real-number solution of the equation $P_A(\lambda_{\max}) = 0$ with $\lambda_{\max} \geq 4$[4]. In addition, we can state the following result.

PROPOSITION 2.2. *Under the fact (A), the number of distinct real-number solutions of the characteristic equation of the 4th order PCM is two.*

PROOF. Assume that there exist 4 real-number solutions $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4$. Since we have

$$\begin{aligned} P_A(\lambda) &= \lambda^4 - 4\lambda^3 + c_3\lambda + \det A \\ &= (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)(\lambda-\lambda_4), \end{aligned}$$

the relationship between the coefficients and the roots states that

$$\begin{aligned} 4 &= \lambda_1+\lambda_2+\lambda_3+\lambda_4, \\ 0 &= \lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4. \end{aligned}$$

Thus we have

$$\begin{aligned} 16 &= (\lambda_1+\lambda_2+\lambda_3+\lambda_4)^2 \\ &= \lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2+2(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4) \\ &= \lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2. \end{aligned}$$

Under the fact (A), $\lambda_4 \geq 4$ holds. This occurs only when $\lambda_1 = \lambda_2 = \lambda_3 = 0, \lambda_4 = 4$. This is a special case where the matrix is consistent. When the matrix is inconsistent, the equation has two real-number solutions and two complex-number solutions.

Since $c_3 \leq 0$, it follows from (3) that (A) holds if $\det A \leq 0$. We showed the existence of A such that $\det A > 0$ in Shiraishi and Obata (2021b). So the possibility of the case where $P_A(4) = 4c_3 + \det A > 0$ still remains. In the next section, we verify (A) by computer simulation in R. In addition, we will also observe that $P_A(4) = 4c_3 + \det A = 0$ implies $c_3 = 0$ (hence A is consistent) in the simulation.

3. Verification by computational simulation

3.1. Random samples

We execute Listing 3 below in Section 5.2. by R. We generate $1,000$ random matrices in which the integer from 1 to 9 and their reciprocals are allocated to the upper six triangular elements. Then we obtain the maximum eigenvalues and the graph of the characteristic polynomial. With one trial of random sampling, the maximum value of $4c_3 + \det A$, $\det A$ and c_3 become nearly equal to -3.4666, 377.0421, and -0.8666, respectively. The maximum value of $4c_3 + \det A$ is attained by the following matrix:

$$A = \begin{pmatrix} 1 & 9 & 5 & \frac{1}{2} \\ \frac{1}{9} & 1 & 1 & \frac{1}{9} \\ \frac{1}{5} & 1 & 1 & \frac{1}{9} \\ 2 & 9 & 9 & 1 \end{pmatrix}.$$

[4] In the context of AHP, the existence of the solution is assured by Perron-Frobenius theorem, see Saaty (1980).

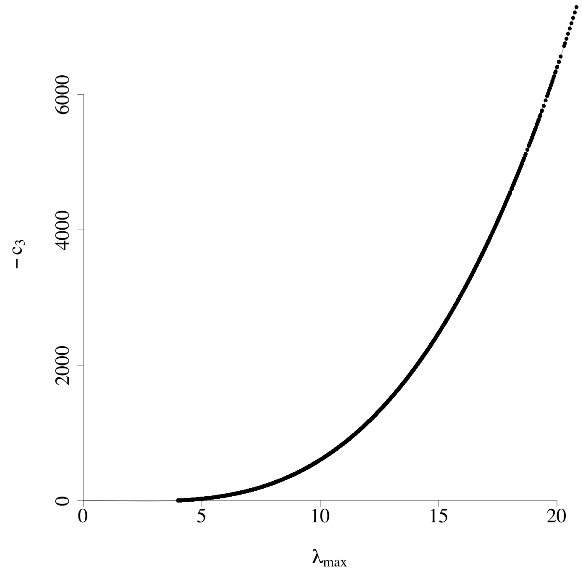

Figure 2: $-c_3$ vs $\lambda_{\max}$ when $\det A = 0$

This is an example in which $\det A = 0$, but $c_3 = \frac{-7}{8} < 0$. When $\det A = 0$, we generally have $c_3 = -3\text{C.I.}(3\text{C.I.} + 4)^2$, see Shiraishi and Obata (2021b) and Figure 2.

On the other hand, we could not observe any sample in which $c_3 = 0$ in this trial. Our exhaustive enumeration in the next section shows the number of consistency matrices is very scarce. It is $343 = 7^3$, although the number of possibilities on the whole is $17^6 = 24,137,569$. It is only 0.001421% of the total. As long as one uses Saaty's scale, one should not expect C.I. $= 0$. On the other hand, C.I. ≤ 0.1 occurs $942,065$ cases which is equal to 3.902899% of the total.

3.2. Exhaustive enumeration

We execute Listing 4 below in Section 5.3. by R. We generate all matrices in which the integer from 1 to 9 and their reciprocals are allocated to the upper six triangular elements. The number of all cases counts $17^6 = 24,137,569$. We found $343 = 7^3$ examples in which both $4c_3 + \det A$ and c_3 become 0 simultaneously. We list up three such examples.

$$\begin{pmatrix} 1 & 8 & 2 & 8 \\ \frac{1}{8} & 1 & \frac{1}{4} & 1 \\ \frac{1}{2} & 4 & 1 & 4 \\ \frac{1}{8} & 1 & \frac{1}{4} & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \frac{1}{2} & 1 & \frac{1}{6} \\ 2 & 1 & 2 & \frac{1}{3} \\ 1 & \frac{1}{2} & 1 & \frac{1}{6} \\ 6 & 3 & 6 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \frac{1}{2} & 3 & \frac{1}{2} \\ 2 & 1 & 6 & 1 \\ \frac{1}{3} & \frac{1}{6} & 1 & \frac{1}{6} \\ 2 & 1 & 6 & 1 \end{pmatrix}.$$

Although it seems to be meaningful that the number of consistent matrices is $343 = 7^3$, the reason is not apparent.

On the other hand, the number of cases in which $\det A = 0$ is not few. It counts at least $471,648$ within Saaty's scale. It is because PCM, which has the following form, becomes $\det A = 0$, see Shiraishi and Obata (2021b).

$$A = \begin{pmatrix} 1 & a_{12} & a_{13} & a_{12}a_{24} \\ \frac{1}{a_{12}} & 1 & a_{23} & a_{24} \\ \frac{1}{a_{13}} & \frac{1}{a_{23}} & 1 & a_{34} \\ \frac{1}{a_{12}a_{24}} & \frac{1}{a_{24}} & \frac{1}{a_{34}} & 1 \end{pmatrix}.$$

4. Concluding remarks

As a byproduct of our computer simulation, we confirm that it is very rare case in which the pairwise comparison becomes completely consistent as long as the decision maker uses Saaty's discrete scale. It is also confirmed that the number of near consistent case is not few. So the decision maker's pairwise comparison is expected to be near consistent.

The theoretical issues to prove the followings are left to future research.

- $4c_3 + \det A \leq 0$
- $4c_3 + \det A = 0 \Rightarrow c_3 = 0$

In general, the convergence of Newton's method depends on the initial value of the iteration. As we noted in introduction, for the 3rd order PCM, Newton's method always converges to the maximum eigenvalue from the initial value of 3 which is equal to the order of the matrix. For the 4th order PCM, Newton's method sometimes converges to the minimum eigenvalue from the initial value of 4, see Shiraishi and Obata (2021c). So the choice of the initial value is interesting to study.

5. Appendices

5.1. Consistency index

Listing 1 is used to calculate c_3. We used this function both in the random sampling and the exhaustive enumeration.

Listing 1: `consistency_indices.R`

```
### calculate -c3 of pcm A
c3.minus <- function(A) {
  if (nrow(A) != ncol(A)) stop("not square matrix")
  n <- nrow(A)
  s <- 0
  k <- 1
  while (k <= n) {
    j <- 1
    while (j < k) {
      i <- 1
      while (i < j) {
        s <- s + A[i,j] * A[j,k] / A[i,k] + A[i,k] / (A[i,j] * A[j,k]) - 2
        i <- i+1
      }
      j <- j+1
    }
    k <- k+1
  }
```

```
  return(s)
}
```

5.2. Random sampling

We used Listing 2 and Listing 3 to obtain Figure 1. Listing 2 generate randomly $1,000$ discrete number of integer from 1 to 9 and their reciprocals to the upper-triangular six elements of the matrix. Listing 3 gives the eigenvalues of PCM and the graphs of their characteristic polynomials. We used MacBook Air 2020, 1.2GHz i7, 16GB. The running time was 39 seconds.

Listing 2: `generate_matrix.R`

```
### generate random pcm of degree n
generate.random.matrix <- function(n) {
  a <- c(1/(9:2), 1, 2:9)
  A <- matrix(nrow = n, ncol = n)
  m <- n * (n-1) / 2
  x <- sample(a, size = m, replace = TRUE)
  k <- 1
  i <- 1
  while (i < n) {
    j <- i+1
    while (j <= n) {
      A[i,j] <- x[k]
      A[j,i] <- 1/x[k]
      j <- j+1
      k <- k+1
    }
    i <- i+1
  }
  for (i in 1:n) {
    A[i,i] <- 1
  }

  return(A)
}
```

Listing 3: `pcm4rand.R`

```
### generage random pcm of degree 4
### calculate c3, det A, PA(4)
### plot graph of characteristic polynomial

source("generate_matrix.R")
source("consistency_indices.R")

elems <- function(A) {
  if (nrow(A) != ncol(A)) stop("not square matrix")
  n <- nrow(A)
  st <- ""
  for (i in 1:(n-1)) {
    for (j in (i+1):n) {
      st <- paste(st, A[i,j], sep = "_")
    }
  }
  return(substr(st,2,nchar(st)))
}
```

```
char.poly4 <- function(x, A) {
  return(x^4 - 4*x^3 - c3.minus(A)*x + det(A))
}

save.dir <- paste("results/",
                  format(Sys.time(), "%Y%m%d%H%M%S"),
                  "/", sep = "")
if (!file.exists(save.dir)) {
  dir.create(save.dir, recursive = TRUE)
}

sink(file = paste(save.dir, "log.txt", sep = ""), split = TRUE)

starttime <- Sys.time()

N <- 1000
digits <- log10(N) + 1
elements <- character(N)
c3s <- numeric(N)
determinants <- numeric(N)
eigenvalues1 <- complex(N)
eigenvalues2 <- complex(N)
eigenvalues3 <- complex(N)
eigenvalues4 <- complex(N)
charpoly4s <- numeric(N)
gtzeros <- character(N)

for (k in 1:N) {
  A <- generate.random.matrix(4)

  elements[k] <- elems(A)
  c3s[k] <- -1 * c3.minus(A)
  determinants[k] <- det(A)
  eigenvalues1[k] <- as.complex(eigen(A, symmetric = FALSE)$values[1])
  eigenvalues2[k] <- as.complex(eigen(A, symmetric = FALSE)$values[2])
  eigenvalues3[k] <- as.complex(eigen(A, symmetric = FALSE)$values[3])
  eigenvalues4[k] <- as.complex(eigen(A, symmetric = FALSE)$values[4])
  cp4 <- char.poly4(4, A)
  charpoly4s[k] <- cp4
  if (cp4 >= 0) {
    gtzeros[k] <- "GT_0"
  } else if (cp4 >= -0.0001) {
    gtzeros[k] <- "NR_0"
  } else {
    gtzeros[k] <- ""
  }

  filename <- paste(save.dir,
                    formatC(k, width = digits, flag = "0"),
                    "_", elems(A), ".png", sep = "")
  png(filename = filename, width = 720, height = 720)
  par(family = "Helvetica", mar = c(5,6,1,1))
  curve(char.poly4(x,A),
        xlim = c(-10,10),
        cex = 2, lwd = 2,
        xlab = parse(text = "lambda"),
        ylab = parse(text = "P[A](lambda)"),
        main = "",
        bty = "l",
        cex.main = 2, cex.lab = 2, cex.axis = 2)
  abline(h = 0)
  points(4, char.poly4(4,A), pch = 20, lwd = 5)
```

```
  dev.off()

}

endtime <- Sys.time()
print(starttime)
print(endtime)
print(endtime - starttime)

df.values <- data.frame(
  elem = elements,
  c3 = c3s,
  det = determinants,
  eig1 = eigenvalues1,
  eig2 = eigenvalues2,
  eig3 = eigenvalues3,
  eig4 = eigenvalues4,
  cp4 = charpoly4s,
  pos = gtzeros)

write.csv(df.values, file = paste(save.dir, "values.csv", sep = ""))

sink()
```

5.3. Exhaustive enumeration

Listing 4 allocates the integer from 1 to 9 and their reciprocals to the upper six triangular elements of the matrix in all cases, which counts $17^6 = 24,137,569$. We used MacBook Air 2020, 1.2GHz i7, 16GB. The running time was 2 hours 25 minutes 40 seconds.

Listing 4: `pcm4all.R`

```
### calculate c3, det A, PA(4) of all pcm of degree 4

source("consistency_indices.R")

pcm4 <- function(a12, a13, a14, a23, a24, a34) {
  A <- matrix(1, nrow = 4, ncol = 4)
  A[1, 2] <- a12
  A[2, 1] <- 1 / a12
  A[1, 3] <- a13
  A[3, 1] <- 1 / a13
  A[1, 4] <- a14
  A[4, 1] <- 1 / a14
  A[2, 3] <- a23
  A[3, 2] <- 1 / a23
  A[2, 4] <- a24
  A[4, 2] <- 1 / a24
  A[3, 4] <- a34
  A[4, 3] <- 1 / a34
  return(A)
}

arr1 <- 2:9
arr2 <- 1 / arr1
values <- c(1, arr1, arr2)

elems <- function(A) {
  if (nrow(A) != ncol(A))
```

```
    stop("not␣square␣matrix")
  n <- nrow(A)
  st <- ""
  for (i in 1:(n - 1)) {
    for (j in (i + 1):n) {
      st <- paste(st, round(A[i, j], 3), sep = "_")
    }
  }
  return(substr(st, 2, nchar(st)))
}

char.poly4 <- function(x, A) {
  return(x ^ 4 - 4 * x ^ 3 - c3.minus(A) * x + det(A))
}


save.dir <- paste("results/",
                  format(Sys.time(), "%Y%m%d%H%M%S"),
                  "/", sep = "")
if (!file.exists(save.dir)) {
  dir.create(save.dir, recursive = TRUE)
}

sink(file = paste(save.dir, "log.txt", sep = ""), split = TRUE)

starttime <- Sys.time()

df.zero <- data.frame()

k <- as.integer(1)
for (a12 in values) {
  for (a13 in values) {
    for (a14 in values) {

      k0 <- k
      iterations <- integer()
      elements <- character()
      c3s <- numeric()
      determinants <- numeric()
      eigenvalues1 <- complex()
      eigenvalues2 <- complex()
      eigenvalues3 <- complex()
      eigenvalues4 <- complex()
      charpoly4s <- numeric()
      gtzeros <- character()

      for (a23 in values) {
        for (a24 in values) {
          for (a34 in values) {
            A <- pcm4(a12, a13, a14, a23, a24, a34)

            iterations[length(iterations)+1] <- k
            elements[length(elements)+1] <- elems(A)
            c3s[length(c3s)+1] <- -1 * c3.minus(A)
            determinants[length(determinants)+1] <- det(A)
            eigenvalues1[length(eigenvalues1)+1] <-
              as.complex(eigen(A, symmetric = FALSE)$values[1])
            eigenvalues2[length(eigenvalues2)+1] <-
              as.complex(eigen(A, symmetric = FALSE)$values[2])
            eigenvalues3[length(eigenvalues3)+1] <-
              as.complex(eigen(A, symmetric = FALSE)$values[3])
            eigenvalues4[length(eigenvalues4)+1] <-
              as.complex(eigen(A, symmetric = FALSE)$values[4])
```

```
          cp4 <- char.poly4(4, A)
          charpoly4s[length(charpoly4s)+1] <- cp4
          if (cp4 > 0) {
            gtzeros[length(gtzeros)+1] <- "GT_0"
          } else if (cp4 == 0) {
            gtzeros[length(gtzeros)+1] <- "EQ_0"
          } else if (cp4 >= -0.0001) {
            gtzeros[length(gtzeros)+1] <- "NR_0"
          } else {
            gtzeros[length(gtzeros)+1] <- ""
          }

          k <- k + 1
        }
      }
    }

    toStr <- formatC(k, width = 10, flag = "0")
    df.values <- data.frame(
      iter = iterations,
      elem = elements,
      c3 = c3s,
      det = determinants,
      eig1 = eigenvalues1,
      eig2 = eigenvalues2,
      eig3 = eigenvalues3,
      eig4 = eigenvalues4,
      cp4 = charpoly4s,
      pos = gtzeros
    )
    write.csv(df.values,
              file = paste(save.dir,
                           "values_",
                           k0,
                           ".csv", sep = ""))

    df.zero <- rbind(df.zero,
                     df.values[df.values$pos != "",])

  }
 }
}

endtime <- Sys.time()
print(starttime)
print(endtime)
print(endtime - starttime)

write.csv(df.zero,
          file = paste(save.dir, "zero.csv", sep = ""))

sink()
```

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