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A Classification of Transitive Sofic Systems

Motosige Osikawa

1. Introduction.

We consider an infinite direct product space W^Z of copies of a finite set W equipped with the direct product topology of the discrete topology on W. Here Z is the set of all integers. We denote by ω_j the j-th coordinate of an element ω in W^Z for $i \in Z$. A shift σ is a homeomorphism from the compact space W^Z onto itself defined by $(\sigma\omega)_j = \omega_{j+1}$ for $j \in Z$ and a point ω in W^Z. A σ-invariant closed subset of W^Z is called a subshift. By a directed graph $G = (W, V, i, t)$ we mean that W and V are finite sets and i and t are mappings from W onto V. We call W an arc set, V a vertex set, i an initial map and t a terminal map. For a directed graph $G = (W, V, i, t)$ an element ω of W^Z is called a G-admissible path if $t(\omega_j) = i(\omega_{j+1})$ for all $j \in Z$, and we denote by $\Sigma(G)$ the set of all G-admissible paths. By a labeled graph $\Xi = (G, S, \lambda)$ we mean that $G = (W, V, i, t)$ is a directed graph, S is a finite set and λ is a mapping from W onto S. We call S a label set and λ a label map. For a labeled graph $\Xi = (G, S, \lambda)$ let Λ be a mapping from $\Sigma(G)$ into S^Z defined by $(\Lambda\omega)_j = \lambda(\omega_j)$ for $j \in Z$ and $\omega = (\omega_j)$ in $\Sigma(G)$. We denote by $\Omega(\Xi)$ the image set $\Lambda(\Sigma(G))$. It is easy to see that $\Sigma(G)$ and $\Omega(\Xi)$ are subshifts. A subshift Ω is called an arc-Markov subshift and a sofic system if $\Omega = \Sigma(G)$ for a directed graph G and $\Omega = \Omega(\Xi)$ for a labeled graph Ξ, respectively. An arc Markov subshift is a Markov subshift in the usual sense, but the converse is not true. Here we adopt the above definition of a sofic system though there are several equivalent definitions.

For a directed graph $G = (W, V, i, t)$ a finite sequence $(w_1, w_2, \cdots, w_n)$ of elements of W is called a G-admissible route if $t(w_j) = i(w_{j+1})$ for $j = 1, 2, \cdots, n\text{-}1$. We say that a directed graph $G = (W, V, \mathrm{i}, \mathrm{t})$ is irreducible if for any two vertices v and v' there exist a G-admissible route $(w_1, w_2, \cdots, w_n)$ such that $i(w_1) = v$ and $t(w_n) = v'$. A labeled graph $\Xi = (G, S, \lambda)$ is said to be irreducible if G is irreducible. A labeled graph $\Xi = (W, V, i, t, S, \lambda)$ is said to be right resolving if $i(w) = i(w')$ and $\lambda(w) = \lambda(w')$, $w, w' \in W$ imply $w = w'$. A labeled graph Ξ is said to be right reduced if $L(\Xi)_v = L(\Xi)_{v'}$, $v, v' \in V$ implies $v = v'$, where $L(\Xi)_v$ is the set of all sequences $(\lambda(w_1), \lambda(w_2), \cdots, \lambda(w_n))$ for a G-admissible route $(w_1, w_2, \cdots, w_n)$ with $i(w_1) = v$. A labeled graph Ξ is called a right Fischer graph if it is right resolving, right reduced and irreducible. A subshift Ω is said to be transitive if for any pair of non-empty open subsets A_1 and A_2 of Ω there exists an integer n such that $A_1 \cap \sigma^n A_2 \neq \phi$. For a transitive sofic system Ω there exists a unique right Fischer graph Ξ with $\Omega=\Omega(\Xi)$ (Fischer[3]).

For a labeled graph Ξ we denote by $\Phi_1(\Xi)$ the set of all G-admissible paths ω such that the inverse image $\Lambda^{-1}(\Lambda\omega)$ is a one point set $\{\omega\}$ itself and we denote by $\Phi_2(\Xi)$ the difference set $\Sigma(G)\setminus\Phi_1(\Xi)$. If a transitive sofic system $\Omega=\Omega(\Xi)$ is conjugate to a transitive sofic system $\Omega'=\Omega(\Xi')$ by a conjugacy map θ then there exists a conjugacy map ξ from $\Sigma(G)$ onto $\Sigma(G')$ such that $\theta\Lambda=\Lambda'\xi$ (Nasu[6]). Therefore the following Ψ_0, Ψ_1, Ψ_2 and Ψ_3 are subclasses of the all conjugate classes Ψ of transitive sofic systems;

Ψ_0 : $\Phi_2(\Xi)$ is empty

Ψ_1 : $\Phi_2(\Xi)$ is at most finite.

Ψ_2 : $\Phi_2(\Xi)$ is at most countable

Ψ_3 : $\Phi_2(\Xi)$ is not dense in $\Sigma(G)$.

Here are inclusions $\Psi_0\subset\Psi_1\subset\Psi_2\subset\Psi_3\subset\Psi$. Ψ_0 is nothing but the subclass of all subshifts of finite type, Ψ_1 is the subclass of all near Markov subshifts ([2]) and Ψ_3 is the subclass of all subshifts almost of finite type ([1]). In this paper we give characterizations of such subclasses by Fischer graphs.

2. Theorem and proof.

Let G be a directed graph. A G-admissible path ω is called a G-cycle path if there is a positive integer p such that $\omega_{j+p}=\omega_j$ for all $j\in Z$. A G-admissible path ω is said to go forward (backward) into a cycle if there is a positive integer p and an integer N such that $\omega_{j+p}=\omega_j$ for $j\geqq N(j\leqq N)$. For a labeled graph $\Xi=(G, S, \lambda)$ G-admissible paths ω and ω' are called Ξ-admissible pair paths if $\lambda(\omega_j)=\lambda(\omega'_j)$ for all $j\in Z$ and $\omega_k\neq\omega'_k$ for some k. Ξ-admissible pair paths ω and ω' are said to cross each other if $t(\omega_k)=t(\omega'_k)$ for some k.

Lemma 1. Let $\Xi=(G, S, \lambda)$ be a right resolving labeled graph, and let ω and ω' Ξ-admissible pair paths. Then

(1) If ω is a G-cycle path then ω' is also G-cycle path.

(2) If ω goes forward into a cycle then ω' also goes forward into a cycle.

(3) If ω goes backward into a cycle then ω' also goes backward into a cycle.

Proof. We may only to prove (2). If ω goes forward into a cycle, that is, $\omega_{j+p}=\omega_j$, $j\geqq N$ for a positive integer p and an integer N, then, $\lambda(\omega_{j+p})=\lambda(\omega_j)$ for $j\geqq N$, and hence $\lambda(\omega'_{j+p})=\lambda(\omega'_j)$ for $j\geqq N$. Since a set $\{t(\omega'_{jp}): jp\geqq N\}$ of vertices is finite there exists integers m and k $(m<k)$ such that t

$(\omega'_{mp}) = t(\omega'_{kp}) = t(\omega'_{mp+(k-m)p})$. Since Ξ is right resolving $\omega'_{j+(k-m)p} = \omega'_j$ for $j \geqq mp$. This means that ω' goes forward into a cycle.

From Lemma 1 we see that for a right resolving labeled graph Ξ any Ξ-admissible pair paths ω and ω' take one of the following seven cases :

P-1 : Both are cycle paths and they do not cross each other.
P-2 : They go both forward and backward into cycles, but they do not cross each other.
P-3 : They go backward into cycles and cross each other.
P-4 : They go backward into cycles, but they do not go forward into cycles nor cross each other.
P-5 : They go forward into cycles, but they do not backward into cycles nor cross each other.
P-6 : They go neither backward nor forward into cycles and they do not cross each other.
P-7 : They cross each other but they do not go backward into cycles.

For a labeled graph $\Xi = (G, S, \lambda)$ G-admissible routes $(w_1, w_2, \cdots, w_n)$ and $(w'_1, w'_2, \cdots, w'_n)$ are called Ξ-admissible pair routes if $\lambda(w_j) = \lambda(w'_j)$ and $w_j \neq w'_j$ for all $j = 1, 2, \cdots, n$. For a right resolving labeled graph Ξ we consider the following four kinds of Ξ-admissible pair routes $(w_1, w_2, \cdots, w_n)$ and $(w'_1, w'_2, \cdots, w'_n)$:

C-1 : $i(w_1) = t(w_n)$ and $i(w'_1) = t(w_n)$.
C-2 : $(w_1, w_2, \cdots, w_p)$ and $(w'_1, w'_2, \cdots, w'_p)$ are of type C-1 and $(w_q, w_{q+1}, \cdots, w_n)$ and $(w'_q, w'_{q+1}, \cdots, w'_n)$ are of type C-1 for some p and q $(1 \leqq p < q \leqq n)$, and $\lambda(w_{p+1}) \neq \lambda(w_1)$.
C-3 : $(w_1, w_2, \cdots, w_p)$ and $(w'_1, w'_2, \cdots, w'_p)$ are of type C-1 for some $p\,(1 \leqq p < n)$ and $t(w_n) = t(w'_n)$.
C-4 : $(w_1, w_2, \cdots, w_p)$ and $(w'_1, w'_2, \cdots, w'_p)$ are of type C-1, and $(w_{p+1}, w_{p+2}, \cdots, w_n)$ and $(w'_{p+1}, w'_{p+2}, \cdots, w'_n)$ are of type C-1, and $\lambda(w_{p+1}) \neq \lambda(w_1)$.

It is easy to see that an existence of Ξ-admissible pair routes of type C-4 implies one of type C-2, and that an existence of Ξ-admissible pair routes of type C-2 or type C-3 implies one of type C-1.

Lemma 2. Let Ξ be a right resolving labeled graph.

(1) There exist Ξ-admissible pair paths of type P-1 if and only if there exist Ξ-admissible pair routes or type C-1.

(2) There exist Ξ-admissible pair paths of type P-2 if and only if there exist Ξ-admissible pair routes of type C-2.

(3) There exist Ξ-admissible pair paths of type P-3 if and only if there exist Ξ-admissible pair routes

of type C–3.

(4) If there exist Ξ-admissible pair paths of type P–7 there exist Ξ-admissible pair routes of type C–3.

(5) If there exist Ξ-admissible pair paths of type P–4 or P–5 or P–6 or P–7, there exist Ξ-admissible pair routes of type C–4.

(6) If there exist infinite number of Ξ-admissible pair routes of type C–1 there exist Ξ-admissible pair routes of type C–4.

Proof. (1), (2) and (3) of Lemma 2 follow from Lemma 1.

(4) : Let ω and ω' be Ξ-admissible pair paths such that $t(\omega_k) = t(\omega'_k)$ for some integer k and $\omega_j \neq \omega'_j$ for all $j \leqq k$. Since a subset $\{(i(\omega_j), i(\omega'_j)) : j \leqq k\}$ of $V \times V$ is finite there exist integers n and m $(n < m \leqq k)$ such that $i(\omega_n) = i(\omega_m)$ and $i(\omega'_n) = i(\omega'_m)$. Then $(\omega_n, \omega_{n+1}, \cdots, \omega_m)$ and $(\omega'_n, \omega'_{n+1}, \cdots, \omega'_m)$ are of type C–3.

(5) : Let ω and ω' be Ξ-admissible pair paths which do not go backward into cycles. From the similar reason as above there exists a decreasing sequence of integers $n(1), m(1), n(2), m(2), \cdots, n(k)$, m $(k), \cdots$ such that $i(\omega_{n(k)}) = i(\omega_{m(k)})$, $i(\omega'_{n(k)}) = i(\omega'_{m(k)})$ and $\lambda(\omega_{n(k)}) \neq \lambda(\omega_{m(k)})$ for all $k = 1, 2, \cdots$. Furthermore from the similar reason there exist positive integers k and k' $(k < k')$ such that $i(\omega_{n(k)}) = i(\omega_{n(k')})$ and $i(\omega'_{n(k)}) = i(\omega'_{n(k')})$. Then $(\omega_{n(k')}, \omega_{n(k')+1}, \cdots, \omega_{n(k)-1})$ and $(\omega'_{n(k')}, \omega'_{n(k')+1}, \cdots, \omega'_{n(k)-1})$ are of type C–4. By the same way we can find Ξ-admissible pair routes of type C–4 from Ξ-admissible pair paths which do not go forward into cycles and do not cross each other.

(6) : Let $(w_1{}^{(n)}, w_2{}^{(n)}, \cdots, w_{p(n)}{}^{(n)})$ and $(w'_1{}^{(n)}, w'_2{}^{(n)}, \cdots, w'_{p(n)}{}^{(n)})$, $n=1, 2, \cdots$ be infinite sequence of different Ξ-admissible pair routes of type C–1. Since a subset $\{(i(w_1{}^{(n)}), i(w'_1{}^{(n)}) : n=1, 2, \cdots\}$ is finite there exist positive integers n and n' $(n < n')$ such that $i(w_1{}^{(n)}) = i(w_1{}^{(n')})$ and $i(w'_1{}^{(n)}) = i(w'_1{}^{(n')})$. Then $(w_1{}^{(n)}, w_2{}^{(n)}, \cdots, w_{p(n)}{}^{(n)}, w_1{}^{(n')}, w_2{}^{(n')}, \cdots, w_{p(n')}{}^{(n')})$ and $(w'_1{}^{(n)}, w'_2{}^{(n)}, \cdots, w'_{p(n)}{}^{(n)}, w'_1{}^{(n')}, w'_2{}^{(n')}, w'_{p(n')}{}^{(n')})$ are of type C–4.

From the following theorem we obtain characterizations of each subclasses Ψ_0, Ψ_1, Ψ_2 and Ψ_3 by Fischer graphs.

Theorem. Let $\Xi = (G, S, \lambda)$ be a Fischer graph.

(1) $\Phi_1(\Xi)$ is uncountable and dense in $\Sigma(G)$.

(2) $\Phi_2(\Xi)$ is empty if and only if there do not exist Ξ-admissible pair route of type C–1.

(3) $\Phi_2(\Xi)$ is at most finite if and only if there do not exist Ξ-admissible pair routes of type C–2 nor C–3.

(4) $\Phi_2(\Xi)$ is at most countable if and only if there do not exist Ξ-admissible pair routes of type C–3 or C–4.

(5) $\Phi_2(\Xi)$ is dense in $\Sigma(G)$ if and only if there exist Ξ-admissible pair routes of type C–3.

Proof. We need to assume only right resolvingness of Ξ to prove (2), (3) and (4), but right resolvingness and irreducibility of Ξ to prove (5). Though (2) and (5) have been proved in [5] we prove them here again for the completeness.

(1) : For a right Fischer graph Ξ there exists a magic word (see[4]). G-admissible paths which go through a magic word are in $\Phi_1(\Xi)$ and dense in $\Sigma(G)$.

(2) : If $\Phi_2(\Xi)$ is not empty, by (2), (3), (4) and (5) of Lemma 2 there exist Ξ-admissible pair routes of type C–1. Conversely, if there exist Ξ-admissible pair routes of type C–1, by (1) of Lemma 2, $\Phi_2(\Xi)$ is not empty.

(3) : If there do not exist Ξ-admissible pair routes of type C–2 nor C–3, by (2), (3) and (4) of Lemma 2 there exist Ξ-admissible pair paths only of type P–1. By (5) of Lemma 2 there exist at most finite number of Ξ-admissible pair routes of type C–1, and hence $\Phi_2(\Xi)$ is infinite. Conversely, if there exist Ξ-admissible pair routes of type C–2 or C–3 there exist infinite number of Ξ-admissible pair paths of type P–2 or P–3, and hence, $\Phi_2(\Xi)$ is infinite.

(4) : If there do not exist Ξ-admissible pair routes of type C–3 nor C–4 there exist Ξ-admissible pair paths only of type P–1 or P–2 by (3) and (5) of Lemma 2. Because there exist a finite number of Ξ-admissible pair routes of type C–1 by (6) of Lemma 2 $\Phi_2(\Xi)$ is at most countable. The converse follows from that there exist uncountably many Ξ-admissible pair paths each of which comes from Ξ-admissible pair routes of type C–3 and that there exist uncountably many Ξ-admissible pair paths each of which moves in Ξ-admissible pair routes of type C–4.

(5) : If $\Phi_2(\Xi)$ is dense in $\Sigma(G)$ then for an arc w in W there exist Ξ-admissible pair paths which go through w, and they must be of type P–3 or P–7. By (3) and (4) of Lemma 2 there exist Ξ-admissible pair routes of type C–3. Conversely if there exist Ξ-admissible pair routes of type C–3 then from the irreducibility of G there exist Ξ-admissible pair paths which come from the Ξ-admissible pair routes of type C–3 and go through any given G-admissible route. This means that $\Phi_2(\Xi)$ is dense in $\Sigma(G)$.

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