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SOME REMARKS ON THE MAXIMUM EIGENVALUE OF 3RD ORDER PAIRWISE COMPARISON MATRICES IN AHP

By

Shunsuke Shiraishi* and **Tsuneshi Obata**†

Dedicated to the memory of Shiraishi's late father, Takuzo Shiraishi.

Abstract

In AHP, a number of consistency indices have been proposed. Saaty's $C.I.$ is a pioneer and generally adopted by users of AHP. We also proposed a new consistency index with the aid of the characteristic polynomial of the pairwise comparison matrix. Surprisingly, 3rd order random matrices make the completely same numerical order of two consistency indices, i.e. Saaty's $C.I.$ and our consistency index. In this short paper, we show this experimental result is theoretically correct.

Key Words and Phrases: AHP, characteristic polynomial, consistency index, Cardano's method.

1. Introduction

In AHP (Analytic Hierarchy Process), Saaty's $C.I.$ (Consistency Index) has been de facto standard of consistency for pairwise comparison matrices. Nevertheless there exist alternative approaches to measure inconsistency (see Brunelli (2018) and references therein). In Shiraishi et al. (1998), we also proposed a new consistency index, say, c_{mod}.[1] We think that the existing researches have given attention to the following two research questions, at least.

1. Are new $C.I.$'s better than Saaty's one in some sense?

2. Are new $C.I.$'s compatible with Saaty's one?

As for Question 1, from the computational viewpoint, c_{mod} showed best performance when one reconstructs an inconsistent matrix to be a better one (see Brunelli et al. (2007)). From the theoretical viewpoint, Saaty's $C.I.$ has a more desirable axiomatic feature than the others (see Brunelli et al. (2015)). Our proposed c_{mod} also has the same axiomatic feature as Saaty's one.

As for Question 2, in Brunelli et al. (2013), they have investigated the correlation coefficient computed on 10,000 randomly generated pairwise comparison matrices of order 6 comparing other several newly proposed indices. The correlation coefficient

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[1] We will explain in Section 2.

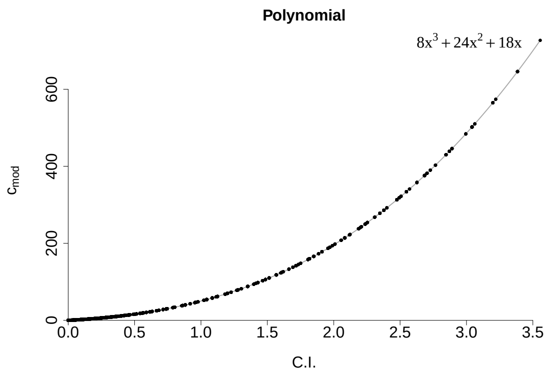

Figure 1: polynomial fitting N=5,000

between $C.I.$ and c_{mod} is 0.952, which reveals the best performance rather than other consistency indices proposed by several researchers. Hence, we really need a mathematical foundation of our consistency index. For this purpose, we started by computational experiment. Recently we found that the case of 3rd order matrices is special. With the aid of the computational experiment of producing $5,000$ number of 3rd order random matrices, the cubic regression says $C.I.$ and c_{mod} completely fits. See Fig. 1.

Here we generate a simple example of 5 random 3rd and 4th order pairwise comparison matrices, respectively. Then we computed Saaty's $C.I.$ and c_{mod}. From Table 1, it is easily seen that the numerical order of $C.I.$ and c_{mod} are completely the same when matrices are 3rd order. This is not the case when matrices are 4th order (See Table 2).

Table 1: 3rd order matrices

$C.I.$	c_{mod}	$C.I.$'s order	c_{mod}'s order
0.2603215	6.4533333	2	2
0.2178446	5.1428571	1	1
1.115187	61.015873	4	4
1.115187	61.015873	4	4
0.2804168	7.1111111	3	3

Table 2: 4th order matrices

$C.I.$	c_{mod}	$C.I.$'s order	c_{mod}'s order
0.2489949	16.8324515	1	2
0.9642403	146.2095238	3	3
2.076002	292.075586	4	4
0.2661867	16.7619048	2	1
2.468707	882.923457	5	5

The aim of the present paper is to show the observation from this toy experiment is mathematically true. We assert that $C.I.$ is a monotone function of c_{mod} and vise versa, when the size of matrices is 3. As a consequence, numerical order of the $C.I.$ and c_{mod} completely coincidence.

As a byproduct of the proof, in Section 3. we can show the representing formula of $C.I.$ by the element of the matrix $A = (a_{ij})$ (Morris (1979)). Using the result, in Section 4. we also show the upper and lower bounds of comparison elements according to $C.I.$ and c_{mod}.

2. The largest eigenvalue of 3rd order pairwise comparison matrix and c_{mod}

In general, the characteristic polynomial $P_A(\lambda)$ of n-th order matrix A has the following form (Saito (1966)):

$$P_A(\lambda) = \lambda^n - \operatorname{tr} A\ \lambda^{n-1} + c_2\lambda^{n-2} + c_3\lambda^{n-3} + \cdots + (-1)^n \det A. \tag{1}$$

In pairwise comparison matrix, $\operatorname{tr} A = n$. We showed $c_2 = 0$ (Shiraishi et al. (1998)). We also showed

$$c_3 = \sum_{i<j<k} \Big(2 - \big(\frac{a_{ij}a_{jk}}{a_{ik}} + \frac{a_{ik}}{a_{ij}a_{jk}}\big)\Big). \tag{2}$$

The well-known relationships between arithmetic mean and geometric mean implies $c_3 \leq 0$. In the sequel, we will set $c_{mod} = -c_3$. Computational experiments by several researchers suggest that c_{mod} can be used a new consistency index. See Obata et al. (1999), Brunelli et al. (2013) and Pelàez and Lamata (2003).

If $n = 3$,

$$P_A(\lambda) = \lambda^3 - 3\lambda^2 + c_3 = \lambda^3 - 3\lambda^2 - c_{mod} \tag{3}$$

and $c_{mod} = \det A$. Hence the maximum eigenvalue λ_{max} of A satisfies the following equation.

$$\lambda_{max}^3 - 3\lambda_{max}^2 = c_{mod}. \tag{4}$$

We consider the function $f(x) = x^3 - 3x^2 = x^2(x-3)$. Since $f'(x) = 3x(x-2)$, we see that this function is monotone increasing when $x > 2$. From the well-known results of pairwise comparison matrix, λ_{max} varies greater than 3 (see also Remark 2 below). Hence we have the followings.

Proposition 2.1. *c_{mod} is monotone increasing w.r.t. λ_{max}, i.e. $C.I.$*

The formula (4) can be rewritten as follows. Because $C.I. = \dfrac{\lambda_{max} - 3}{2}$, we have $\lambda_{max} = 2C.I. + 3$. By substituting to (4), it is easily seen that

$$c_{mod} = 2C.I.\Big(2C.I. + 3\Big)^2 = 8C.I.^3 + 24C.I.^2 + 18C.I. \tag{5}$$

So we can confirm Proposition 2.1 considering the other function $g(x) = 8x^3 + 24x^2 + 18x$. By differentiating g, we have $g'(x) = 24x^2 + 48x + 18 = 6\big(4(x+1)^2 - 1\big) = 6\big(2(x+1) - 1\big)\big(2(x+1) + 1\big)$, which shows $g(x)$ takes its local minimum at $\frac{-1}{2}$, and monotone increasing for $x \geq \frac{-1}{2}$.

Next, we represent $C.I.$ by c_{mod} explicitly through Cardano's method. As for Cardano's method, see Ueno (1996) and Encyclopedia of Mathematics (Hazewinkel (2001)).[2]

[2] `https://www.encyclopediaofmath.org/index.php/Cardano_formula`

In the sequel, we denote c_{mod} by c for simplicity. If we set $\lambda = x+1$ in (3), then we have

$$\begin{aligned} P_A(\lambda) &= \lambda^3 - 3\lambda^2 + c_3 \\ &= (x+1)^3 - 3(x+1)^2 - c \\ &= x^3 - 3x - 2 - c. \end{aligned} \tag{6}$$

Let $x = u + v$. The equation $P_A(\lambda) = 0$ is equivalent to

$$\begin{aligned} 0 &= x^3 - 3x - 2 - c \\ &= (u+v)^3 - 3(u+v) - 2 - c \\ &= \big(u^3 + v^3 - (c+2)\big) + 3(u+v)(uv-1). \end{aligned}$$

Thus we reach the following system of equation.

$$0 = u^3 + v^3 - (c+2), \tag{7}$$
$$0 = (u+v)(uv-1). \tag{8}$$

If $u + v = 0$, (7) implies $c = -2$. It contradicts $c = c_{mod} = -c_3 \geq 0$. From (8), we have $v = \dfrac{1}{u}$. By substituting it to (7), we obtain

$$0 = u^3 + \frac{1}{u^3} - (c+2),$$
$$0 = (u^3)^2 - (c+2)u^3 + 1. \tag{9}$$

From (9), we get two solutions, by considering the symmetricity of u and v,

$$u = \sqrt[3]{\frac{(c+2) + \sqrt{(c+2)^2 - 4}}{2}},$$
$$v = \sqrt[3]{\frac{(c+2) - \sqrt{(c+2)^2 - 4}}{2}}.$$

Indeed,

$$\begin{aligned} v = \frac{1}{u} &= \sqrt[3]{\frac{2}{(c+2) + \sqrt{(c+2)^2 - 4}}} \\ &= \sqrt[3]{\frac{2\Big((c+2) - \sqrt{(c+2)^2 - 4}\Big)}{\Big((c+2) + \sqrt{(c+2)^2 - 4}\Big)\Big((c+2) - \sqrt{(c+2)^2 - 4}\Big)}} \\ &= \sqrt[3]{\frac{(c+2) - \sqrt{(c+2)^2 - 4}}{2}}. \end{aligned}$$

Thus, we get

$$x = \sqrt[3]{\frac{(c+2) + \sqrt{(c+2)^2 - 4}}{2}} + \sqrt[3]{\frac{(c+2) - \sqrt{(c+2)^2 - 4}}{2}}. \tag{10}$$

Since $\lambda = x + 1$, we finally get the followings.

$$\lambda = \sqrt[3]{\frac{(c+2) + \sqrt{(c+2)^2 - 4}}{2}} + \sqrt[3]{\frac{(c+2) - \sqrt{(c+2)^2 - 4}}{2}} + 1. \tag{11}$$

REMARK 1. The equation $x^3 - 3x - 2 - c = 0$ has generally three solutions. One is given in (10) which corresponds to the maximum eigenvalue. The other solutions are,

$$\omega \sqrt[3]{\frac{(c+2) + \sqrt{(c+2)^2 - 4}}{2}} + \omega^2 \sqrt[3]{\frac{(c+2) - \sqrt{(c+2)^2 - 4}}{2}},$$
$$\omega^2 \sqrt[3]{\frac{(c+2) + \sqrt{(c+2)^2 - 4}}{2}} + \omega \sqrt[3]{\frac{(c+2) - \sqrt{(c+2)^2 - 4}}{2}},$$

where ω is the primitive 3rd root of unity. The direct calculation shows (10) has maximum norm among the three solutions.

REMARK 2. As we noted before

$$\sqrt[3]{\frac{(c+2) - \sqrt{(c+2)^2 - 4}}{2}} = \sqrt[3]{\frac{2}{(c+2) + \sqrt{(c+2)^2 - 4}}},$$

which leads to

$$\begin{aligned} \lambda_{max} - 1 &\geq 2\sqrt{\sqrt[3]{\frac{(c+2) + \sqrt{(c+2)^2 - 4}}{2}} \sqrt[3]{\frac{2}{(c+2) + \sqrt{(c+2)^2 - 4}}}} \\ &= 2. \end{aligned}$$

Hence $\lambda_{max} \geq 3$.

PROPOSITION 2.2. *λ_{max}, hence C.I., is monotone increasing w.r.t. c_{mod}.*

PROOF. We consider the function

$$f(x) = \sqrt[3]{x + 2 + \sqrt{(x+2)^2 - 4}} + \sqrt[3]{x + 2 - \sqrt{(x+2)^2 - 4}},$$

and show its monotonicity.

Case 1 : When $x > 0$, calculation of the derivative of $f(x)$ shows that for $x > 0$,

$$
\begin{aligned}
f'(x) &= \frac{1}{3}\left[\frac{1+\dfrac{(x+2)}{\sqrt{(x+2)^2-4}}}{\left(\sqrt[3]{x+2+\sqrt{(x+2)^2-4}}\right)^2} + \frac{1-\dfrac{(x+2)}{\sqrt{(x+2)^2-4}}}{\left(\sqrt[3]{x+2-\sqrt{(x+2)^2-4}}\right)^2}\right] \\
&= \frac{1}{3\sqrt{(x+2)^2-4}}\left[\frac{(x+2)+\sqrt{(x+2)^2-4}}{\left(\sqrt[3]{(x+2)+\sqrt{(x+2)^2-4}}\right)^2}\right. \\
&\qquad\qquad \left. - \frac{(x+2)-\sqrt{(x+2)^2-4}}{\left(\sqrt[3]{(x+2)-\sqrt{(x+2)^2-4}}\right)^2}\right] \\
&= \frac{1}{3\sqrt{(x+2)^2-4}}\left[\sqrt[3]{(x+2)+\sqrt{(x+2)^2-4}}\right. \\
&\qquad\qquad \left. - \sqrt[3]{(x+2)-\sqrt{(x+2)^2-4}}\right] \\
&> 0.
\end{aligned}
$$

This means $f(x)$ is monotone increasing for $x > 0$.

Case 2 : When $x \geq 0$, the relationship between arithmetic and geometric means shows that

$$
\begin{aligned}
f(x) &= \sqrt[3]{x+2+\sqrt{(x+2)^2-4}} + \sqrt[3]{x-\sqrt{(x+2)^2-4}} \\
&\geq 2\sqrt{\sqrt[3]{x+2+\sqrt{(x+2)^2-4}}\,\sqrt[3]{x+2-\sqrt{(x+2)^2-4}}} \\
&= 2\sqrt[3]{2} = f(0).
\end{aligned}
$$

□

The following theorem is obvious from Propositions 2.1 and 2.2.

Theorem 2.3. *The numerical order of C.I. and that of c_{mod} are completely the same in all 3rd order pairwise comparison matrices.*

3. Byproduct results

Shiraishi et al. (1998) gave $c_3 = 2 - \left(\dfrac{a_{12}a_{23}}{a_{13}} + \dfrac{a_{13}}{a_{12}a_{23}}\right)$.

So $c = \left(\frac{a_{12}a_{23}}{a_{13}} + \frac{a_{13}}{a_{12}a_{23}}\right) - 2$. From this, we obtain:

$$\begin{aligned} c+2 &= \frac{a_{12}a_{23}}{a_{13}} + \frac{a_{13}}{a_{12}a_{23}}, \\ (c+2)^2 - 4 &= \left(\frac{a_{12}a_{23}}{a_{13}} + \frac{a_{13}}{a_{12}a_{23}}\right)^2 - 4 \\ &= \left(\frac{a_{12}a_{23}}{a_{13}}\right)^2 + 2 + \left(\frac{a_{13}}{a_{12}a_{23}}\right)^2 - 4 \\ &= \left(\frac{a_{12}a_{23}}{a_{13}} - \frac{a_{13}}{a_{12}a_{23}}\right)^2. \end{aligned}$$

Hence, one can transform (11) as follows.

Case 1 : When $\frac{a_{12}a_{23}}{a_{13}} - \frac{a_{13}}{a_{12}a_{23}} > 0$, one has

$$\begin{aligned} \lambda_{max} &= \sqrt[3]{\frac{1}{2}\left(\frac{a_{12}a_{23}}{a_{13}} + \frac{a_{13}}{a_{12}a_{23}} + \frac{a_{12}a_{23}}{a_{13}} - \frac{a_{13}}{a_{12}a_{23}}\right)} \\ &\quad + \sqrt[3]{\frac{1}{2}\left(\frac{a_{12}a_{23}}{a_{13}} + \frac{a_{13}}{a_{12}a_{23}} - \frac{a_{12}a_{23}}{a_{13}} + \frac{a_{13}}{a_{12}a_{23}}\right)} + 1 \\ &= \sqrt[3]{\frac{a_{12}a_{23}}{a_{13}}} + \sqrt[3]{\frac{a_{13}}{a_{12}a_{23}}} + 1. \end{aligned}$$

Case 2 : When $\frac{a_{12}a_{23}}{a_{13}} - \frac{a_{13}}{a_{12}a_{23}} < 0$, one has

$$\begin{aligned} \lambda_{max} &= \sqrt[3]{\frac{1}{2}\left(\frac{a_{12}a_{23}}{a_{13}} + \frac{a_{13}}{a_{12}a_{23}} - \frac{a_{12}a_{23}}{a_{13}} + \frac{a_{13}}{a_{12}a_{23}}\right)} \\ &\quad + \sqrt[3]{\frac{1}{2}\left(\frac{a_{12}a_{23}}{a_{13}} + \frac{a_{13}}{a_{12}a_{23}} + \frac{a_{12}a_{23}}{a_{13}} - \frac{a_{13}}{a_{12}a_{23}}\right)} + 1 \\ &= \sqrt[3]{\frac{a_{13}}{a_{12}a_{23}}} + \sqrt[3]{\frac{a_{12}a_{23}}{a_{13}}} + 1. \end{aligned}$$

In any cases, the following result holds.

Proposition 3.1 (Morris (1979), Crowford and Williams (1985)).

$$\lambda_{max} = \sqrt[3]{\frac{a_{12}a_{23}}{a_{13}}} + \sqrt[3]{\frac{a_{13}}{a_{12}a_{23}}} + 1. \tag{12}$$

Corollary 3.2. *One has*

$$C.I. \le \frac{c_{mod}}{2}.$$

The equality holds if and only if $a_{13} = a_{12}a_{23}$.

Proof. It is obvious from the inequality $a^{\frac{1}{3}} + a^{-\frac{1}{3}} \le a + a^{-1}$. As for the proof of the inequality, see Proposition 6.1. □

From (12), one has

$$\begin{aligned}\lambda_{max} &= \sqrt[3]{\frac{a_{12}a_{23}}{a_{13}}}\left(1+\left(\sqrt[3]{\left(\frac{a_{13}}{a_{12}a_{23}}\right)^2}+\sqrt[3]{\frac{a_{13}}{a_{12}a_{23}}}\right)\right. \\ &= \sqrt[3]{\frac{a_{12}a_{23}}{a_{13}}}\left(1+\left(\sqrt[3]{\left(\frac{a_{13}}{a_{12}a_{23}}\right)^2}+2\sqrt[3]{\frac{a_{13}}{a_{12}a_{23}}}\right)-1\right. \\ &= \sqrt[3]{\frac{a_{12}a_{23}}{a_{13}}}\left(\sqrt[3]{\frac{a_{13}}{a_{12}a_{23}}}+1\right)^2-1.\end{aligned}$$

PROPOSITION 3.3. *The following formula holds.*

$$\begin{aligned}\lambda_{max}+1 &= \sqrt[3]{\frac{a_{12}a_{23}}{a_{13}}}\left(\sqrt[3]{\frac{a_{13}}{a_{12}a_{23}}}+1\right)^2 \\ &= \sqrt[3]{\frac{a_{13}}{a_{12}a_{23}}}\left(\sqrt[3]{\frac{a_{12}a_{23}}{a_{13}}}+1\right)^2.\end{aligned}$$

4. Range according to the threshold

From (12), one can calculate $C.I.$ as follows.

$$C.I. = \frac{\sqrt[3]{\frac{a_{12}a_{23}}{a_{13}}}+\sqrt[3]{\frac{a_{13}}{a_{12}a_{23}}}-2}{2}. \tag{13}$$

If we set $x = \dfrac{a_{12}a_{23}}{a_{13}}$, (13) becomes

$$C.I. = \frac{x^{\frac{1}{3}}+x^{-\frac{1}{3}}-2}{2}.$$

So we can find the range of x which makes $C.I.$ be less than Saaty's standard criterion value 0.1, by solving the following equation

$$x^{\frac{1}{3}}+x^{-\frac{1}{3}} = 2.2.$$

If we set again $x^{\frac{1}{3}} = t$, we have

$$t^{-1}+t = 2.2,$$

which is equivalent to

$$t^2-2.2t+1 = 0.$$

Solve this quadratic equation, we have $t = 1.1 \pm \sqrt{(1.1)^2-1} \approx 0.64, 1.55$. So $x \approx 0.26, 3.78$.

We can easily generalize this observation.

PROPOSITION 4.1. *$C.I. \leq k$ if and only if*

$$\frac{1}{\left((k+1)+\sqrt{(k+1)^2-1}\right)^3} \leq \frac{a_{12}a_{23}}{a_{13}} \leq \left((k+1)+\sqrt{(k+1)^2-1}\right)^3, \tag{14}$$

or its equivalent

$$\frac{1}{\left((k+1)+\sqrt{(k+1)^2-1}\right)^3} \leq \frac{a_{13}}{a_{12}a_{23}} \leq \left((k+1)+\sqrt{(k+1)^2-1}\right)^3.$$

PROOF. If we set $x = \frac{a_{12}a_{23}}{a_{13}}$, we have

$$\frac{x^{\frac{1}{3}} + x^{-\frac{1}{3}} - 2}{2} \leq k,$$

which leads to the equation

$$\frac{x^{\frac{1}{3}} + x^{-\frac{1}{3}} - 2}{2} = k. \tag{15}$$

Thus it is enough to solve the following equations to obtain the range of $x^{\frac{1}{3}}$.

$$(x^{\frac{1}{3}})^2 - 2(k+1)x^{\frac{1}{3}} + 1 = 0. \tag{16}$$

□

Since $c_{mod} = \frac{a_{12}a_{23}}{a_{13}} + \frac{a_{13}}{a_{12}a_{23}} - 2$, we can show the same result.

PROPOSITION 4.2. *$c_{mod} \leq k$ if and only if*

$$\frac{2}{(k+2)+\sqrt{(k+2)^2-4}} \leq \frac{a_{12}a_{23}}{a_{13}} \leq \frac{(k+2)+\sqrt{(k+2)^2-4}}{2}, \tag{17}$$

or its equivalent

$$\frac{2}{(k+2)+\sqrt{(k+2)^2-4}} \leq \frac{a_{13}}{a_{12}a_{23}} \leq \frac{(k+2)+\sqrt{(k+2)^2-4}}{2}.$$

PROOF. If we set $x = \frac{a_{12}a_{23}}{a_{13}}$, we have

$$x + x^{-1} - 2 \leq k,$$

which leads to the equation

$$x + x^{-1} - 2 = k. \tag{18}$$

Thus it is enough to solve the following equations to obtain the range of x.

$$x^2 - (k+2)x + 1 = 0. \tag{19}$$

□

There exist the relationships between (14) and (17).

PROPOSITION 4.3. *We have*

1. *If* $C.I. \leq k$, *then* $c_{mod} \leq \tilde{k}$, *where* $\tilde{k} = 2k(2k+3)^2$,
2. *If* $c_{mod} \leq k$, *then* $C.I. \leq \bar{k}$, *where*

$$\bar{k} = \frac{1}{2}\left(\sqrt[3]{\frac{(k+2)+\sqrt{(k+2)^2-4}}{2}} + \sqrt[3]{\frac{(k+2)-\sqrt{(k+2)^2-4}}{2}} - 2\right).$$

PROOF. Let x be a solution of (15). Then we have $2k \geq x^{\frac{1}{3}} + x^{-\frac{1}{3}} - 2$. Hence we have

$$\begin{aligned}\tilde{k} = 2k(2k+3)^2 &= \left(x^{\frac{1}{3}} + x^{-\frac{1}{3}} - 2\right)\left(x^{\frac{1}{3}} + x^{-\frac{1}{3}} + 1\right)^2 \\ &= x + x^{-1} - 2.\end{aligned}$$

This means x is a solution of (18) for $k = \tilde{k}$.
Conversely let x be a solution of (18). Then we have,

$$\begin{aligned}\bar{k} &= \frac{1}{2}\left(\sqrt[3]{\frac{(k+2)+\sqrt{(k+2)^2-4}}{2}} + \sqrt[3]{\frac{(k+2)-\sqrt{(k+2)^2-4}}{2}} - 2\right) \\ &= \frac{1}{2}\left(\sqrt[3]{\frac{(x+x^{-1})+\sqrt{(x+x^{-1})^2-4}}{2}}\right. \\ &\qquad\qquad \left. + \sqrt[3]{\frac{(x+x^{-1})-\sqrt{(x+x^{-1})^2-4}}{2}} - 2\right) \\ &= \frac{x^{\frac{1}{3}} + x^{-\frac{1}{3}} - 2}{2}.\end{aligned}$$

This means x is a solution of (16) for $k = \bar{k}$. □

5. Conclusion

Saaty's $C.I.$ is de facto standard in AHP. So it is desirable to be harmonized to it if one wishes to propose a new consistency index. We show, in this paper, $C.I.$ and c_{mod} are compatible completely when the matrices are 3rd order. By using our new result, we can determine the relationships between $C.I.$ and c_{mod} exactly. Since $c_{mod} = 2C.I.\left(2C.I. + 3\right)^2$, one can compute the value of c_{mod} according to $C.I.$ If $C.I. = 0.1$, one has $c_{mod} = 2.048$. So, we can say that if c_{mod} is less than 2.048 , the pairwise comparison matrix is consistent in Saaty's sense [3].

The results of the presenting paper are mathematically true only when the size of the matrices is 3rd order. This is because the characteristic polynomial has very

[3] Saaty (1980) says that if this number is less than 0.1, we may be satisfied with our judgements.

treatable form. Thus Crowford and Williams (1985), Morris (1979)'s simple formula of λ_{max} follows (Proposition 3.1).

As we see in the introduction, in the cases of the sizes of matrices are larger than 4, the numerical order between $C.I.$ and c_{mod} cannot be coincident. However in several articles, the relationship between them has been investigated (see Brunelli et al. (2013), Obata et al. (1999)).

Saaty's $C.I.$ has great advantage. It is easily understandable because thresholds are allowed to use the value 0.1 regardless of the sizes of matrices. In this sense, c_{mod} has some shortcomings. First, the value of c_{mod} becomes rather large although $C.I.$ has an upper limit (see Sekitani and Niina (2010)). In the case of the size is 4, computation on 500 randomly generated pairwise comparison matrices gives the maximum value of c_{mod} to be 727.0014. So one may hesitate use of c_{mod}. Some normalization should be needed.

Second, in general, one can observe c_{mod} gets greater as the size of matrices gets greater. This also occurs on $C.I.$[4] So one needs a new consistency index whose value is independent of the size of matrices. Dividing c_{mod} by some cubic polynomial of the size of matrices may be hopeful. We have proposed a method in Obata and Shiraishi (2016).

We believe that more modification of c_{mod} may solve these shortcomings. If we conquer this hurdle, a new useful consistency index comes to be available. The mathematics nature of 3rd order matrices is the starting point of future research.

6. Appendix

PROPOSITION 6.1. *Let $a > 0$. and n be a natural number. Then,*

$$a^{\frac{1}{n}} + a^{-\frac{1}{n}} \leq a + a^{-1}.$$

The equality holds if and only if $a = 1$.

PROOF. Let $t = a^{\frac{1}{n}}$. We consider the function $f(t) = t^n + t^{-n} - (t + t^{-1})$. By differentiate $f(t)$, we have

$$\begin{aligned} f'(t) &= nt^{n-1} - nt^{-(n+1)} - (1 - t^{-2}) \\ &= t^{-(n+1)}(nt^{2n} - n - t^{n+1} + t^{n-1}) \\ &= t^{-(n+1)}\big(n(t^{2n} - 1) - (t^{n+1} - t^{n-1})\big). \end{aligned}$$

Since

$$\begin{aligned} t^{2n} - 1 &= (t-1)(t^{2n-1} + t^{2n-2} + \cdots + 1) \\ t^{n+1} - t^{n-1} &= (t-1)(t^n + t^{n-1}), \end{aligned}$$

we have

$$\begin{aligned} f'(t) = t^{-(n+1)}(t-1)\Big(&nt^{2n-1} + nt^{2n-2} + \cdots \\ &+ (n-1)t^n + (n-1)t^{n-1} + nt^{n-2} + \cdots + n\Big), \end{aligned}$$

which implies $f(t)$ takes its minimum at $t = 1$ and $f(1) = 0$. Hence the assertion of the proposition immediately follows. □

[4] To overcome this shortcoming, in AHP, the consistency ratio $C.R.$ is also used. See Brunelli (2015).

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