M4 IS REGULAR-CLOSED

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$\mathcal{M}_4$ IS REGULAR-CLOSED

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Abstract. For each $n \geq 2$, we investigate a family of iterated function systems which is parameterized by a common contraction ratio $s \in \mathbb{D}^\times \equiv \{s \in \mathbb{C} : 0 < |s| < 1\}$ and possesses a rotational symmetry of order n. Let $\mathcal{M}_n$ be the locus of contraction ratio s for which the corresponding self-similar set is connected. The purpose of this paper is to show that $\mathcal{M}_n$ is regular-closed, i.e. $\overline{\mathrm{int}\,\mathcal{M}_n} = \mathcal{M}_n$ holds for $n \geq 4$. This gives a new result for $n = 4$ and a simple geometric proof of the previously known result by [BanHu] for $n \geq 5$.

1. Introduction

Below we fix $n \geq 2$. Let $\{\varphi_k\}_{k=0}^{n-1}$ be the family of similitudes $\varphi_k : \mathbb{C} \to \mathbb{C}$ with a common contraction ratio $s \in \mathbb{D}^\times \equiv \{s \in \mathbb{C} : 0 < |s| < 1\}$ where the fixed points p_k of φ_k form the vertices of a regular n-gon in the anti-clockwise order. Without loss of generality, we may assume $p_0 = 0$ and $p_1 = 1$ so that $\varphi_0(z) = sz$ and $\varphi_1(z) = sz + (1 - s)$. It then follows that $\varphi_{k+1}(z) - \varphi_k(z) = \xi^k(1 - s)$ and $p_{k+1} - p_k = \xi^k$, where $\xi = e^{2\pi i/n}$. Let Λ_s be the self-similar set associated to $\{\varphi_k\}_{k=0}^{n-1}$, i.e. Λ_s is the unique non-empty compact set satisfying $\Lambda_s = \Phi(\Lambda_s)$, where $\Phi(A) \equiv \bigcup_{k=0}^{n-1} \varphi_k(A)$ (see, e.g., [F]). Note that Λ_s has a rotational symmetry of order n. We also define X_s to be the union of Λ_s and the bounded components of $\mathbb{C} \setminus \Lambda_s$.

Let $\mathcal{M}_n \subset \mathbb{D}^\times$ be the connectedness locus for the family $\{\varphi_k\}_{k=0}^{n-1}$, i.e.

$$\mathcal{M}_n \equiv \big\{s \in \mathbb{D}^\times : \Lambda_s \text{ is connected}\big\}.$$

The locus $\mathcal{M}_2$ was first introduced in 1985 by [BarHa], and since then it has been investigated by several authors (see [Bo1, Bo2, Ban, CKW] for $n = 2$ and [BanHu] for general $n \geq 2$).

A subset $A \subset \mathbb{C}$ is called *regular-closed* if $A = \overline{\mathrm{int}\,A}$. The purpose of this paper is to show

Theorem 1.1. *$\mathcal{M}_n$ is regular-closed for $n \geq 4$.*

The case $n = 4$ of Theorem 1.1 is a new result (see Remark 6 in [BanHu] for a partial result) and the case $n \geq 5$ gives a simple and geometric proof of a previous result in [BanHu]. Note that $\mathcal{M}_3$ is known to be regular-closed [BanHu] (see also Remark 2.11 of this article). On the other hand, it was shown in [BarHa] that there is a neighborhood of $s = \pm 1/2$ in which $\mathcal{M}_2$ is contained in $\mathbb{R}$. Moreover, in the remarkable paper [CKW] Calegari et al solved a conjecture of Bandt [Ban]; the interior of $\mathcal{M}_2$ is dense away from $\mathcal{M}_2 \cap \mathbb{R}$, i.e. $\mathcal{M}_2 = \overline{\mathrm{int}\,\mathcal{M}_2} \cup (\mathcal{M}_2 \cap \mathbb{R})$. Their method is to decompose the locus $\mathcal{M}_2$ into two parts depending on the convexity of X_s, and investigate their structures separately. The proof of Theorem 1.1 follows this idea and is obtained by extending some arguments in [CKW] to general $n \geq 2$ in an appropriate way.

2. Proof of Theorem 1.1

2.1. **Case X_s is convex.** We first treat the case X_s is convex. The following proposition is an adaptation of Lemma 7.2.3 in [CKW] to general case $n \geq 2$.

Proposition 2.1. *Assume that X_s is convex. Then, there exist $q \in \mathbb{N}$ and $0 \leq k \leq n - 1$ so that $\xi^k s^q > 0$ and $2|s|^q \geq 1$ hold. In particular, we have $|s| \geq 1/2$.*

A *supporting line* to X_s is a line $\ell \subset \mathbb{C}$ so that $X_s \cap \ell \neq \emptyset$ and $X_s \setminus \ell$ is contained in one connected component of $\mathbb{C} \setminus \ell$. An *edge* of X_s is a maximal segment in ∂X_s with positive length. The *direction* of a supporting line to X_s (resp. a segment in ∂X_s) is defined by a complex number up to a positive constant multiple so that X_s is on the left-hand side of the line (resp. the segment in ∂X_s) with respect to the direction. Note that a supporting line to X_s of a given direction always exists uniquely due to the compactness of X_s. When we write $\sigma = \sigma'$ or $\sigma \supset \sigma'$ for two segments σ and σ' in ∂X_s, we require that their directions are the same.

Lemma 2.2. *Assume that X_s is convex. Then, X_s contains an edge whose direction is $\xi^k(1-s)$ for every $0 \leq k \leq n-1$.*

Proof. Let ℓ be the supporting line to X_s with direction $\xi^k(1-s)/s$. Since $\varphi_{k+1}(z) - \varphi_k(z) = \xi^k(1-s)$ holds, we have $L \equiv \varphi_k(\ell) = \varphi_{k+1}(\ell)$. Moreover, since φ_k is orientation preserving, $\varphi_k(X_s)$ and $\varphi_{k+1}(X_s)$ are on the left-hand side of L. It is easy to see from the formula of φ_i that $\varphi_i(X_s)$ is also contained in the left-hand side of L for $i \neq k, k+1$. This shows that L is the supporting line of X_s with the direction $\xi^k(1-s)$.

Let p be a point in the intersection of X_s and ℓ. Since $\varphi_k(p) \neq \varphi_{k+1}(p)$, the segment $[\varphi_k(p), \varphi_{k+1}(p)]$ has strictly positive length. Moreover, the segment is contained in L and in ∂X_s by the convexity of X_s, hence it is contained in an edge of X_s with its direction $\xi^k(1-s)$. □

We remark that the edges of X_s with directions $\xi^k(1-s)$ $(0 \leq k \leq n-1)$ also possess rotational symmetry of order n.

Lemma 2.3. *Assume that X_s is convex and let σ be an edge of X_s whose direction is $\xi^k(1-s)$ for some $0 \leq k \leq n-1$. Then, there exists a unique edge σ' of X_s so that $\sigma = \varphi_k(\sigma') \cup \varphi_{k+1}(\sigma')$.*

Proof. Since the edge with direction $\xi^k(1-s)$ which we found in Lemma 2.2 has positive length, there exists an edge σ' with direction $\xi^k(1-s)/s$ so that $\sigma \supset \varphi_k(\sigma')$. Since $\varphi_{k+1}(z) - \varphi_k(z) = \xi^k(1-s)$, it yields that $\sigma \supset \varphi_k(\sigma') \cup \varphi_{k+1}(\sigma')$.

We next show $\sigma = \varphi_k(\sigma') \cup \varphi_{k+1}(\sigma')$. If σ'' is an edge of X_s whose direction is different from $\xi^k(1-s)/s$, the direction of $\varphi_i(\sigma'')$ is different from $\xi^k(1-s)$ for any $0 \leq i \leq n-1$. Since there is at most one edge with a given direction, we conclude that σ' is the only edge satisfying the property $\sigma \supset \varphi_i(\sigma')$ for some $0 \leq i \leq n-1$ and i should be equal to k. The uniqueness of σ' then implies $\sigma = \varphi_k(\sigma') \cup \varphi_{k+1}(\sigma')$. □

Lemma 2.4. *Assume that X_s is convex and let σ be an edge of X_s whose direction is not $\xi^k(1-s)$ for any $0 \leq k \leq n-1$. Then, there exist a unique edge σ' of X_s and a unique $0 \leq i \leq n-1$ so that $\sigma = \varphi_i(\sigma')$.*

Proof. Since σ has positive length, there exists an edge σ' of X_s and $0 \leq i \leq n-1$ so that $\sigma \supset \varphi_i(\sigma')$. Suppose that there exists another edge σ'' and $0 \leq j \leq n-1$ so that $\sigma \supset \varphi_j(\sigma'')$ and $\varphi_i(\sigma') \neq \varphi_j(\sigma')$ hold. Since the contraction ratio s of φ_k is independent of k, the direction of σ' and σ'' should be the same. It then follows from the uniqueness of an edge of X_s with a given direction that $\sigma' = \sigma''$. Since we assume $\varphi_i(\sigma') \neq \varphi_j(\sigma')$ belong to the same edge σ, it follows that $\{i, j\} = \{k, k+1\}$ and the direction of σ should be equal to $\xi^k(1-s)$, which contradicts to the assumption. Therefore, we conclude that an edge σ' of X_s and a number $0 \leq i \leq n-1$ satisfying $\sigma \supset \varphi_i(\sigma')$ are unique, hence $\sigma = \varphi_i(\sigma')$. □

Proof of Proposition 2.1. Let σ_0 be an edge with direction $1-s$ found in Lemma 2.2. By Lemma 2.3, there exists an edge σ_1 with direction $(1-s)/s$ so that $\sigma_0 = \varphi_0(\sigma_1) \cup \varphi_1(\sigma_1)$. If the direction $(1-s)/s$ is not $\xi^k(1-s)$ for any $0 \leq k \leq n-1$, there exist unique edge σ_2 of X_s and $0 \leq i \leq n-1$ so that $\sigma_1 = \varphi_i(\sigma_2)$. Note that the direction of σ_2 is $(1-s)/s^2$, and $|s||\sigma_2| = |\sigma_1|$ holds. When we repeat this procedure, it stops at finitely many times due to the boundedness of X_s. Therefore, there exist $q \in \mathbb{N}$ and $0 \leq k \leq n-1$ so that the direction $(1-s)/s^q$ of σ_q

coincides with $\xi^k(1-s)$ for some $0 \le k \le n-1$. This in particular implies $\xi^k s^q > 0$. Moreover, since σ_q coincides with σ_0 or its rotationally symmetric images, we have $|\sigma_q| = |\sigma_0|$. Lemma 2.3 implies $\sigma_0 = \varphi_0(\sigma_1) \cup \varphi_1(\sigma_1)$, hence by using $|s|^{q-1}|\sigma_q| = |\sigma_1|$, we obtain

$$2|s|^q|\sigma_0| = 2|s|^q|\sigma_q| = 2|s||\sigma_1| = |\varphi_0(\sigma_1)| + |\varphi_1(\sigma_1)| \ge |\sigma_0|,$$

which finishes the proof. □

Remark 2.5. *The coefficient* 2 *appearing in Proposition 2.1 does not depend on the number of similitudes.*

2.2. **Case X_s is non-convex.** We next treat a non-convex X_s.

Proposition 2.6. *Assume that X_s is connected but not convex. Then, $s \in \overline{\operatorname{int} \mathcal{M}_n}$.*

This proposition is also obtained by modifying some arguments in [CKW]. To clarify the points of modifications, let us recall some constructions from [CKW].

Let $\Sigma \equiv \{0, \dots, n-1\}$. For a word $u = u_1 \cdots u_m \in \Sigma^m$ of length m, we write $\varphi_u(z) \equiv \varphi_{u_1} \circ \cdots \circ \varphi_{u_m}(z)$. Let $\pi_m : \Sigma^m \times \mathbb{D}^\times \times \mathbb{C} \to \mathbb{C}$ be the map defined by $\pi_m(u, s, z) \equiv \varphi_u(z)$. Given an infinite sequence $\underline{u} = u_1 u_2 \cdots \in \Sigma^{\mathbb{N}}$, it is easy to see that the limit $\lim_{m\to\infty} \pi_m(u_1 \cdots u_m, s, z)$ exists and is independent of the choice of $z \in \mathbb{C}$. Therefore, this defines a map $\pi : \Sigma^{\mathbb{N}} \times \mathbb{D}^\times \to \mathbb{C}$ given by $\pi(\underline{u}, s) = \lim_{m\to\infty} \pi_m(u_1 \cdots u_m, s, z)$. We also have $\pi(\Sigma^{\mathbb{N}}, s) = \Lambda_s$.

For $p, q \in \Lambda_s$, a constant $\varepsilon > 0$ and a disk D containing p and q, an (ε, D)-*short hop path* from p to q is a sequence $\underline{e}^0, \dots, \underline{e}^m \in \Sigma^{\mathbb{N}}$ with $p = \pi(\underline{e}^0, s)$ and $q = \pi(\underline{e}^m, s)$ so that $d(\pi(\underline{e}^i, s), \pi(\underline{e}^{i+1}, s)) < \varepsilon$ holds for $0 \le i \le m-1$ and $\pi(\underline{e}^j, s) \in D$ holds for $0 \le j \le m$.

The key idea to prove Proposition 2.6 is the notion of a *trap* (see Definition 7.1.3 in [CKW]). Here we adapt the original definition to our setting as follows.

Definition 2.7. *Let u (resp v) be a word over Σ starting with* 0 *(resp. with* 1*). Let D be a closed disk so that* $\operatorname{int} D \supset \Lambda_s$*. The pair u and v is called a trap for (s, D) if*

1. *there are points $p^\pm \in \varphi_u(\Lambda_s) \setminus \varphi_v(D)$ and $q^\pm \in \varphi_v(\Lambda_s) \setminus \varphi_u(D)$ so that for some paths $\alpha \subset \varphi_u(D)$ with endpoints $p^\pm$ and $\beta \subset \varphi_v(D)$ with endpoints $q^\pm$ their algebraic intersection number is non-zero,*
2. *$d(\varphi_0(\Lambda_s), \varphi_1(\Lambda_s)) \le \varepsilon$ holds, whenever the $\varepsilon/2$-neighborhood of Λ_s is contained in D.*

With this notion we obtain the following claim which is a modification of Proposition 7.1.6 in [CKW]. The crucial difference in the argument is that the non-empty intersection of particular two pieces $\varphi_0(\Lambda_s)$ and $\varphi_1(\Lambda_s)$ implies the connectivity of the whole Λ_s in our setting.

Lemma 2.8. *If there exists a trap u and v for (s, D), then $s \in \operatorname{int} \mathcal{M}_n$.*

Proof. In the proof of Proposition 7.1.6 in [CKW] we replace f by φ_0 and g by φ_1. Then, the proof implies that $\varphi_0(\Lambda_{s'}) \cap \varphi_1(\Lambda_{s'}) \ne \emptyset$ for $s' \in \mathbb{D}^\times$ sufficiently close to s. By the rotational symmetry of $\varphi_k(\Lambda_{s'})$, this yields that $\varphi_k(\Lambda_{s'}) \cap \varphi_{k+1}(\Lambda_{s'}) \ne \emptyset$ for all $0 \le k \le n-1$ (where we set $\varphi_n(z) \equiv \varphi_0(z)$). It then follows from a result of Hata [H] that $\Lambda_{s'}$ is connected for s' sufficiently close to s. Hence $s \in \operatorname{int} \mathcal{M}_n$. □

To finish the proof of Proposition 2.6 we need one more notion.

Definition 2.9. *Let $X \subset \mathbb{C}$ be full. A vector $w \in \mathbb{C}$ is called trap-like for X if*

1. *$X \cup (X + w)$ is connected,*
2. *there are* 4 *points in the outer-most boundary of $X \cup (X + w)$ that alternate between points in $X \setminus (X + w)$ and points in $(X + w) \setminus X$.*

Proof of Proposition 2.6. Let $s_0 \in \mathcal{M}_n$ and assume that X_{s_0} is not convex. Then, by Lemma 7.2.2 of [CKW] there is a vector w which is trap-like for X_{s_0}, i.e. one can find points $p_1, p_2 \in X_{s_0}$

and $q_1, q_2 \in X_{s_0}$ so that (2) of Definition 2.9 holds. Since $\partial X_{s_0} \subset \Lambda_{s_0}$, the points p_1, p_2 and q_1, q_2 lie in Λ_{s_0}. There exists $\varepsilon > 0$ so that $p_1, p_2 \in \Lambda_{s_0} \setminus \overline{N_\varepsilon(X_{s_0} + w)}$ and $q_1, q_2 \in (\Lambda_{s_0} + w) \setminus \overline{N_\varepsilon(X_{s_0})}$, where $N_\varepsilon(A)$ is the ε-neighborhood of $A \subset \mathbb{C}$. Since these conditions are open and since $\pi(\cdot, s) : \Sigma^{\mathbb{N}} \to \Lambda_s$ is a surjection, there exists $\delta > 0$ so that they hold for s with $|s - s_0| < \delta$.

Since $s_0 \in \mathcal{M}_n$ there exist $\underline{u} = u_1 u_2 \cdots, \underline{v} = v_1 v_2 \cdots \in \Sigma^{\mathbb{N}}$ so that $\pi(\underline{u}, s_0) = \pi(\underline{v}, s_0)$. Again, thanks to the rotational symmetry of $\varphi_k(\Lambda_s)$ and a theorem of [H], we may assume that $u_1 = 0$ and $v_1 = 1$. By Corollary 7.2.6 of [CKW], for any $\delta' > 0$ with $\delta \geq \delta'$ one can find $m \geq 1$ and s_1 with $|s_1 - s_0| < \delta'$ so that $s_1^{-m}(\pi_m(u, s_1, c) - \pi_m(v, s_1, c)) = w$, where $u = u_1 \cdots u_m$ and $v = v_1 \cdots v_m$ and c is the center of the regular n-gon formed by the fixed points of φ_k. Let $D \equiv \overline{N_\varepsilon(X_{s_1})}$. Then, as in Lemma 7.2.2 of [CKW] one can show that u and v form a trap for (s_1, D). By Lemma 2.8 we have $s_1 \in \operatorname{int} \mathcal{M}_n$. Since $|s_1 - s_0| < \delta'$ and $\delta' > 0$ is arbitrary, we conclude that $s_0 \in \overline{\operatorname{int} \mathcal{M}_n}$. □

2.3. **End of the proof.** To complete the proof of Theorem 1.1, we need the following a priori bound for $\mathcal{M}_n$.

Lemma 2.10. *If $1 > |s| > 1/\sqrt{n}$, then $s \in \mathcal{M}_n$. In particular, if $1 > |s| \geq 1/\sqrt{n}$, then $s \in \overline{\operatorname{int} \mathcal{M}_n}$.*

Proof. The first statement was proved in [Bo1] for $n = 2$ and later in [BanHu] for general case. This obviously implies $s \in \operatorname{int} \mathcal{M}_n$ for $1 > |s| > 1/\sqrt{n}$, hence the second conclusion follows. □

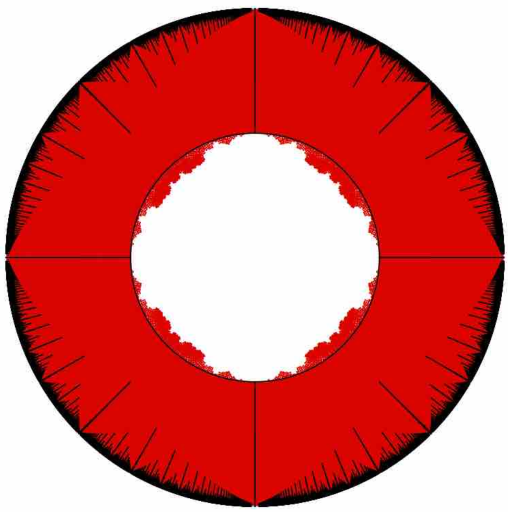

FIGURE 1. $\mathcal{M}_4$ (red), its "spikes" and the inner circle $|s| = 1/\sqrt{4}$ (black).

Now we are ready to prove our main result.

Proof of Theorem 1.1. Since $\mathcal{M}_n$ is closed [Bo1, BanHu], the inclusion $\mathcal{M}_n \supset \overline{\operatorname{int} \mathcal{M}_n}$ is obvious. Take $s \in \mathcal{M}_n$. If X_s is not convex, Proposition 2.6 yields $s \in \overline{\operatorname{int} \mathcal{M}_n}$. Therefore, we may assume that X_s is convex. Let us consider the condition:

$$\frac{1}{\sqrt[q]{2}} \geq \frac{1}{\sqrt{n}}. \tag{2.1}$$

This condition holds for all $q \geq 1$ if $n \geq 4$. It follows from Proposition 2.1 that $1 > |s| \geq 1/\sqrt{n}$, hence $s \in \overline{\operatorname{int} \mathcal{M}_n}$ by Lemma 2.10 (see Figure 1 where we observe that all "spikes" described in Proposition 2.1 for $n = 4$ are contained in the annulus $1 > |s| \geq 1/\sqrt{4}$). This shows that $\mathcal{M}_n \subset \overline{\operatorname{int} \mathcal{M}_n}$ and finishes the proof of Theorem 1.1. □

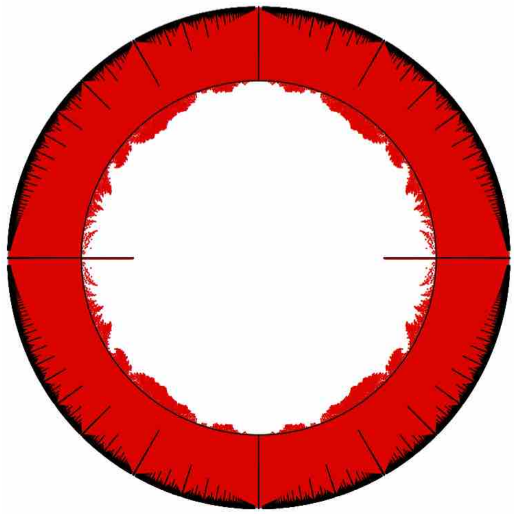

FIGURE 2. $\mathcal{M}_2$ (red), its "spikes" and the inner circle $|s| = 1/\sqrt{2}$ (black).

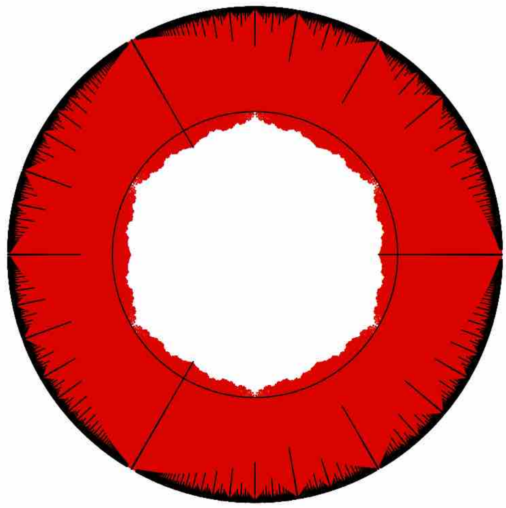

FIGURE 3. $\mathcal{M}_3$ (red), its "spikes" and the inner circle $|s| = 1/\sqrt{3}$ (black).

Compare Figure 2 where we observe that $\mathcal{M}_2$ is not regular-closed; there are some portions of two "spikes" which are not contained in $\overline{\operatorname{int} \mathcal{M}_2}$.

Remark 2.11. *The condition (2.1) is satisfied for* $q \geq 2$ *in the case* $n = 3$ *as well. Therefore, to prove regular-closedness of* $\mathcal{M}_3$ *along a similar line to Theorem 1.1, the only remaining case is* $n = 3$ *and* $q = 1$. *Proposition 2.1 tells that the parameters which violate the condition (2.1) are of the form* $s = r, re^{2\pi i/3}, re^{4\pi i/3}$ *and* $1/2 \leq r < 1/\sqrt{3}$. *See Figure 3 where we observe that some portions of three "spikes" are contained in* $\overline{\operatorname{int} \mathcal{M}_3}$ *but not contained in the annulus* $1 > |s| \geq 1/\sqrt{3}$. *Therefore, if we are only able to prove that these particular parameters belong to* $\overline{\operatorname{int} \mathcal{M}_3}$, *it would give a relatively short proof of the regular-closedness of* $\mathcal{M}_3$ *compared to the one in* [BanHu]. *See Figures 4, 5 and 6 which describe the self-similar sets* Λ_s *for* $s = 0.5$, 0.51 *and* $0.51 + 0.01i$, *respectively. These figures suggest that the parameter* $s = 0.51$, *which is one of the particular parameters mentioned above, belongs to* $\overline{\operatorname{int} \mathcal{M}_3}$.

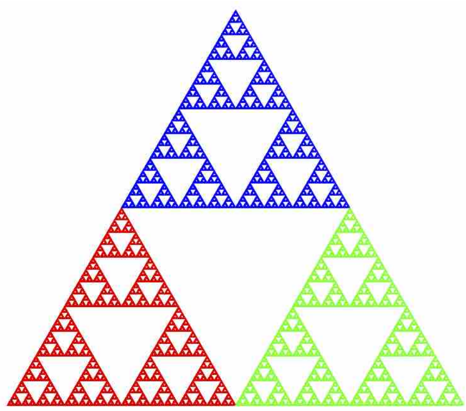

Figure 4. The self-similar set Λ_s for $n = 3$ and $s = 0.5$.

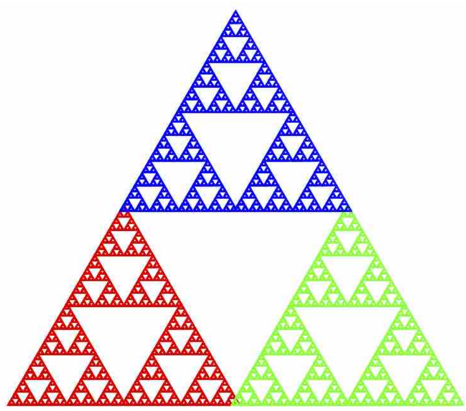

Figure 5. The self-similar set Λ_s for $n = 3$ and $s = 0.51$.

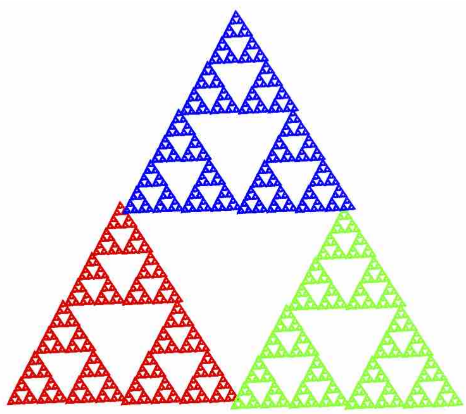

Figure 6. The self-similar set Λ_s for $n = 3$ and $s = 0.51 + 0.01i$.

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