SPECULATIONES SUPER FORMULA INTEGRALI
$\int \frac{x^n dx}{\sqrt{(aa - 2bx + cxx)}}$
UBI SIMUL EGREGIAE OBSERVATIONES CIRCA FRACTIONES CONTINUAS OCCURRUNT

Commentatio 606 indicis ENESTROEMIANI
Acta academiae scientiarum Petropolitanae 1782: II, 1786, p. 62—84

1. Incipiamus a casu simplicissimo, quo $n = 0$, et quaeramus integrale formulae

$$\frac{dx}{\sqrt{(aa - 2bx + cxx)}},$$

quae posito $x = \frac{b+z}{c}$ transit in hanc

$$\frac{dz}{\sqrt{(aacc - bbc + czz)}},$$

ubi duo casus distingui convenit, prout c fuerit vel quantitas positiva vel negativa.

Sit igitur primo $c = +ff$ et formula nostra fiet

$$\frac{dz}{f\sqrt{(aaff - bb + zz)}},$$

cuius integrale est

$$\frac{1}{f}\, l \frac{z + \sqrt{(aaff - bb + zz)}}{C},$$

ideoque erit nostrum integrale

$$\frac{1}{\sqrt{c}}\, l \frac{cx - b + \sqrt{(aac - 2bcx + ccxx)}}{C},$$

quod ergo ita sumtum, ut evanescat posito $x = 0$, evadet

$$\frac{1}{\sqrt{c}}\, l \frac{cx - b + \sqrt{c(aa - 2bx + cxx)}}{-b + a\sqrt{c}}.$$

At vero si c fuerit quantitas negativa, puta $c = -gg$, formula differentialis per z expressa erit

$$\frac{dz}{g\sqrt{(aagg + bb - zz)}},$$

cuius integrale est

$$\frac{1}{g}\,\mathrm{A\,sin.} \frac{z}{\sqrt{(aagg + bb)}} + C,$$

quare integrale ita sumtum, ut evanescat posito $x = 0$, fiet

$$= \frac{1}{g}\,\mathrm{A\,sin.} \frac{cx - b}{\sqrt{(aagg + bb)}} + \frac{1}{g}\,\mathrm{A\,sin.} \frac{b}{\sqrt{(aagg + bb)}}.$$

2. Denotet nunc Π valorem formulae integralis $\int \frac{dx}{\sqrt{(aa - 2bx + cxx)}}$ ita sumtum, ut evanescat posito $x = 0$, sive c fuerit quantitas positiva sive negativa; ac si sit $c = +ff$, erit, uti vidimus,

$$\Pi = \frac{1}{f}\, l \frac{ffx - b + f\sqrt{(aa - 2bx + ffxx)}}{af - b};$$

altero vero casu, quo $c = -gg$, erit

$$\Pi = -\frac{1}{g}\,\mathrm{A\,sin.} \frac{ggx + b}{\sqrt{(aagg + bb)}} + \frac{1}{g}\,\mathrm{A\,sin.} \frac{b}{\sqrt{(aagg + bb)}}$$

sive ambobus arcubus contractis habebimus

$$\Pi = \frac{1}{g}\,\mathrm{A\,sin.} \frac{bg\sqrt{(aa - 2bx - ggxx)} - abg - ag^3x}{aagg + bb}.$$

Quoniam igitur mox ostendemus integrationem formulae generalis $\int\frac{x^n dx}{\sqrt{(aa-2bx+cxx)}}$ semper reduci posse ad casum $n=0$, si modo fuerit n numerus integer positivus, omnia haec integralia per istum valorem Π exprimi poterunt.

3. Iam post integrationem quantitati variabili x eiusmodi valorem constantem tribuamus, quo formula irrationalis $\sqrt{(aa-2bx+cxx)}$ ad nihilum redigatur, id quod fit, si sumatur

$$x=\frac{b\pm\sqrt{(bb-aac)}}{c},$$

ideoque duobus casibus. Ponamus pro utroque casu functionem Π abire in $\varDelta$, ita ut casu $c=ff$ sit

$$\varDelta=\frac{1}{f}\,l\,\frac{\sqrt{(bb-aaff)}}{af-b}=\frac{1}{f}\,l\sqrt{\frac{b+af}{b-af}},$$

pro altero autem casu, quo $c=-gg$,

$$\varDelta=\frac{1}{g}\,\text{A sin.}\frac{\pm ag\sqrt{(bb+aagg)}}{aagg+bb}=\frac{1}{g}\,\text{A sin.}\frac{ag}{\sqrt{(bb+aagg)}}.$$

Hos autem valores $\varDelta$ in sequentibus casibus, quibus ipsa formula radicalis $\sqrt{(aa-2bx+cxx)}$ evanescit, potissimum sumus contemplaturi.

4. Nunc ad sequentem casum progressuri consideremus formulam

$$s=\sqrt{(aa-2bx+cxx)}-a,$$

ut scilicet evanescat facto $x=0$, et quoniam est

$$ds=\frac{-bdx+cxdx}{\sqrt{(aa-2bx+cxx)}},$$

erit vicissim integrando

$$c\int\frac{xdx}{\sqrt{(aa-2bx+cxx)}}=b\int\frac{dx}{\sqrt{(aa-2bx+cxx)}}+s,$$

unde colligimus

$$\int\frac{xdx}{\sqrt{(aa-2bx+cxx)}}=\frac{b}{c}\Pi+\frac{\sqrt{(aa-2bx+cxx)}-a}{c};$$

quare si post integrationem statuamus $x=\frac{b\pm\sqrt{(bb-aac)}}{c}$, quippe quibus casibus fit $\sqrt{(aa-2bx+cxx)}=0$ et $\Pi=\varDelta$, fiet

$$\int\frac{xdx}{\sqrt{(aa-2bx+cxx)}}=\frac{b}{c}\varDelta-\frac{a}{c}.$$

5. Sumamus porro

$$s=x\sqrt{(aa-2bx+cxx)};$$

fiet

$$ds=\frac{aadx-3bxdx+2cxxdx}{\sqrt{(aa-2bx+cxx)}},$$

unde vicissim integrando colligitur

$$2c\int\frac{xxdx}{\sqrt{(aa-2bx+cxx)}}=3b\int\frac{xdx}{\sqrt{(aa-2bx+cxx)}}-aa\int\frac{dx}{\sqrt{(aa-2bx+cxx)}}+s,$$

unde statim pro casu $\sqrt{(aa-2bx+cxx)}=0$ deducimus

$$\int\frac{xxdx}{\sqrt{(aa-2bx+cxx)}}=\frac{3bb-aac}{2cc}\varDelta-\frac{3ab}{2cc}.$$

6. Iam ad altiores potestates ascensuri statuamus

$$s=xx\sqrt{(aa-2bx+cxx)},$$

et quia hinc fit

$$ds=\frac{2aaxdx-5bxxdx+3cx^3dx}{\sqrt{(aa-2bx+cxx)}},$$

erit

$$3c\int\frac{x^3dx}{\sqrt{(aa-2bx+cxx)}}=5b\int\frac{xxdx}{\sqrt{(aa-2bx+cxx)}}-2aa\int\frac{xdx}{\sqrt{(aa-2bx+cxx)}}+s$$

hincque porro pro casu, quo post integrationem statuitur $x=\frac{b\pm\sqrt{(bb-aac)}}{c}$, habebitur

$$\int\frac{x^3dx}{\sqrt{(aa-2bx+cxx)}}=\frac{5b^3-3aabc}{2c^3}\varDelta-\frac{15abb}{6c^3}+\frac{2a^3}{3cc}$$
$$=\left(\frac{5b^3}{2c^3}-\frac{3aab}{2cc}\right)\varDelta-\frac{5abb}{2c^3}+\frac{2a^3}{3cc}.$$

7. Simili modo sit

$$s = x^3\sqrt{(aa-2bx+cxx)},$$

et quia hinc fit

$$ds = \frac{3aaxxdx - 7bx^3dx + 4cx^4dx}{\sqrt{(aa-2bx+cxx)}},$$

erit vicissim integrando

$$4c\int\frac{x^4dx}{\sqrt{(aa-2bx+cxx)}} = 7b\int\frac{x^3dx}{\sqrt{(aa-2bx+cxx)}} - 3aa\int\frac{xxdx}{\sqrt{(aa-2bx+cxx)}} + s;$$

tum igitur pro casu, quo fit $\sqrt{(aa-2bx+cxx)} = 0$, habebimus

$$\int\frac{x^4dx}{\sqrt{(aa-2bx+cxx)}} = \left(\frac{35b^4}{8c^4} - \frac{15aabb}{4c^3} + \frac{3a^4}{8cc}\right)\varDelta - \frac{35ab^3}{8c^4} + \frac{55a^3b}{24c^3}.$$

8. Quo autem ordo in his formulis melius explorari possit, singulas exhibeamus per factores, quemadmodum ordine oriuntur, sine ulla abbreviatione atque hoc modo formulae integrales inventae ita repraesententur:

$$\int\frac{dx}{\sqrt{(aa-2bx+cxx)}} = \varDelta,$$

$$\int\frac{xdx}{\sqrt{(aa-2bx+cxx)}} = \frac{b}{c}\varDelta - \frac{a}{c},$$

$$\int\frac{xxdx}{\sqrt{(aa-2bx+cxx)}} = \left(\frac{1\cdot 3bb}{1\cdot 2cc} - \frac{aa}{1\cdot 2c}\right)\varDelta - \frac{1\cdot 3ab}{1\cdot 2cc},$$

$$\int\frac{x^3dx}{\sqrt{(aa-2bx+cxx)}} = \left(\frac{1\cdot 3\cdot 5b^3}{1\cdot 2\cdot 3c^3} - \frac{1\cdot 3\cdot 3aab}{1\cdot 2\cdot 3cc}\right)\varDelta - \frac{1\cdot 3\cdot 5abb}{1\cdot 2\cdot 3c^3} + \frac{1\cdot 2\cdot 2a^3}{1\cdot 2\cdot 3cc},$$

$$\int\frac{x^4dx}{\sqrt{(aa-2bx+cxx)}} = \left(\frac{1\cdot 3\cdot 5\cdot 7b^4}{1\cdot 2\cdot 3\cdot 4c^4} - \frac{1\cdot 3\cdot 5\cdot 6aabb}{1\cdot 2\cdot 3\cdot 4c^3} + \frac{1\cdot 3\cdot 3a^4}{1.2\cdot 3.4cc}\right)\varDelta$$

$$- \frac{1\cdot 3\cdot 5\cdot 7ab^3}{1\cdot 2\cdot 3\cdot 4c^4} + \frac{1\cdot 5\cdot 11a^3b}{1\cdot 2\cdot 3\cdot 4c^3}.$$

9. Instituamus nunc in genere istam evolutionem sumendo

$$s = x^n\sqrt{(aa-2bx+cxx)},$$

et quia hinc fit

$$ds = \frac{naax^{n-1}dx - (2n+1)bx^ndx + (n+1)cx^{n+1}dx}{\sqrt{(aa-2bx+cxx)}},$$

inde vicissim integrando colligitur

$$(n+1)c\int\frac{x^{n+1}dx}{\sqrt{(aa-2bx+cxx)}} = (2n+1)b\int\frac{x^ndx}{\sqrt{(aa-2bx+cxx)}}$$

$$- naa\int\frac{x^{n-1}dx}{\sqrt{(aa-2bx+cxx)}} + x^n\sqrt{(aa-2bx+cxx)}.$$

Quodsi vero iam ante elicuerimus

$$\int\frac{x^{n-1}dx}{\sqrt{(aa-2bx+cxx)}} = M\varDelta - \mathfrak{M} \quad \text{et} \quad \int\frac{x^ndx}{\sqrt{(aa-2bx+cxx)}} = N\varDelta - \mathfrak{N},$$

ita ut hae duae formulae sint cognitae, sequens ex iis ita determinabitur, ut sit

$$\int\frac{x^{n+1}dx}{\sqrt{(aa-2bx+cxx)}} = \left(\frac{(2n+1)bN}{(n+1)c} - \frac{naaM}{(n+1)c}\right)\varDelta - \frac{(2n+1)b\mathfrak{N}}{(n+1)c} + \frac{naa\mathfrak{M}}{(n+1)c}.$$

Hoc igitur modo has integrationes, quousque libuerit, continuare licet, dum ex binis quibusque sequens ope huius regulae formatur, ita ut omnia haec integralia vel a logarithmis vel ab arcubus circularibus pendeant, prouti coefficiens c fuerit vel positivus vel negativus. Manifestum autem est istos valores assignari non posse, nisi exponens n fuerit numerus integer positivus.

10. Ex forma integrali modo inventa, si post integrationem statuatur $x = \frac{b \pm \sqrt{(bb-aac)}}{c}$, unde fit $s = 0$, erit

$$naa\int\frac{x^{n-1}dx}{\sqrt{(aa-2bx+cxx)}}$$

$$= (2n+1)b\int\frac{x^ndx}{\sqrt{(aa-2bx+cxx)}} - (n+1)c\int\frac{x^{n+1}dx}{\sqrt{(aa-2bx+cxx)}};$$

unde si brevitatis gratia ponamus

$$\int\frac{x^{n-1}dx}{\sqrt{(aa-2bx+cxx)}} = P, \quad \int\frac{x^ndx}{\sqrt{(aa-2bx+cxx)}} = Q,$$

$$\int\frac{x^{n+1}dx}{\sqrt{(aa-2bx+cxx)}} = R, \quad \int\frac{x^{n+2}dx}{\sqrt{(aa-2bx+cxx)}} = S \text{ etc.},$$

hae quantitates P, Q, R, S etc. ita a se invicem pendent, ut sit

$$\begin{aligned} naaP &= (2n+1)bQ-(n+1)cR,\\ (n+1)aaQ &= (2n+3)bR-(n+2)cS,\\ (n+2)aaR &= (2n+5)bS-(n+3)cT,\\ (n+3)aaS &= (2n+7)bT-(n+4)cU,\\ (n+4)aaT &= (2n+9)bU-(n+5)cW\\ &\text{etc.} \end{aligned}$$

Ex his relationibus deducuntur sequentes determinationes

$$\begin{aligned} \frac{P}{Q} &= \frac{(2n+1)b}{naa}-\frac{(n+1)c}{naaQ:R},\\ \frac{Q}{R} &= \frac{(2n+3)b}{(n+1)aa}-\frac{(n+2)c}{(n+1)aaR:S},\\ \frac{R}{S} &= \frac{(2n+5)b}{(n+2)aa}-\frac{(n+3)c}{(n+2)aaS:T},\\ \frac{S}{T} &= \frac{(2n+7)b}{(n+3)aa}-\frac{(n+4)c}{(n+3)aaT:U}\\ &\text{etc.;} \end{aligned}$$

hinc igitur patet singulas has fractiones $\frac{P}{Q}$, $\frac{Q}{R}$, $\frac{R}{S}$ etc. per sequentes satis commode determinari.

11. Quodsi iam in qualibet harum expressionum valores modo exhibiti successive substituantur, pro fractione $\frac{P}{Q}$ impetrabimus fractionem continuam in infinitum progredientem, quae erit

$$naa\frac{P}{Q}=(2n+1)b-\cfrac{(n+1)^2aac}{(2n+3)b-\cfrac{(n+2)^2aac}{(2n+5)b-\cfrac{(n+3)^2aac}{(2n+7)b-\cfrac{(n+4)^2aac}{(2n+9)b-\text{etc.}}}}}$$

sicque pervenimus ad fractionem continuam satis concinnam et ordine perspicuo progredientem, cuius igitur valor semper vel per logarithmos (si fuerit $c>0$) vel per arcus circulares (si fuerit $c<0$) exprimi potest.

12. Sumamus nunc $n=1$ ac fiet

$$P=\int\frac{dx}{\sqrt{(aa-2bx+cxx)}}=\varDelta$$

et

$$Q=\int\frac{xdx}{\sqrt{(aa-2bx+cxx)}}=\frac{b}{c}\varDelta-\frac{a}{c},$$

qui casus nobis suppeditat sequentem fractionem continuam

$$\frac{aac\varDelta}{b\varDelta-a}=3b-\cfrac{4aac}{5b-\cfrac{9aac}{7b-\cfrac{16aac}{9b-\cfrac{25aac}{11b-\text{etc.}}}}}$$

quae ob elegantiam omni attentione digna est censenda. Hic autem notasse iuvabit, si c fuerit numerus negativus, tum omnes numeratores in hac fractione evadere positivos.

12[a][1]). Fractio autem haec continua capite quasi trunca videtur; unde si superne ei adiungatur membrum $b-aac$, ea adhuc concinnior eiusque valor simplicior reddetur. Si enim ista fractio brevitatis gratia designetur littera S, ita ut sit $S=\frac{aac\varDelta}{b\varDelta-a}$, adiecto isto membro eius valor erit $b-\frac{aac}{S}=\frac{a}{\varDelta}$ sicque habebimus

$$\frac{a}{\varDelta}=b-\cfrac{aac}{3b-\cfrac{4aac}{5b-\cfrac{9aac}{7b-\cfrac{16aac}{9b-\cfrac{25aac}{11b-\text{etc.}}}}}}$$

quae expressio eo magis est memorabilis, quod nulla adhuc via patet, qua talis fractionis continuae valor a priori inveniri potest.

1) In editione principe falso numerus 12 iteratur. A. L.

13. Evolvamus nunc seorsim binos casus supra memoratos, et quos sollicite a se invicem distingui convenit. Sit igitur primo $c = ff$ atque supra invenimus fore

$$\varDelta = \frac{1}{f}\, l\, \frac{\sqrt{(bb-aaff)}}{af-b},$$

ubi signum radicale ambigue accipi potest. Ante omnia igitur necesse est, ut sit $bb > aaff$, quia alioquin haec expressio evaderet imaginaria; duo ergo casus se offerunt, prouti b fuerit quantitas sive positiva sive negativa.

Priore casu, quo $b > 0$ atque adeo $b > af$, evidens est signo radicali tribui debere signum —, ut fiat

$$\varDelta = \frac{1}{f}\, l\, \frac{\sqrt{(bb-aaff)}}{b-af} = \frac{1}{2f}\, l\, \frac{b+af}{b-af},$$

et iam habebimus istam summationem

$$\frac{2af}{l\frac{b+af}{b-af}} = b - \cfrac{aaff}{3b - \cfrac{4aaff}{5b - \cfrac{9aaff}{7b - \cfrac{16aaff}{9b - \text{etc.}}}}}$$

unde, cum sit $\frac{b+af}{b-af} > 1$, patet valorem huius expressionis fore positivum.

14. Sin autem fuerit b numerus negativus sive si loco b scribatur $-b$, etiamnunc esse debet $b > af$; tum autem erit

$$\varDelta = \frac{1}{2f}\, l\, \frac{b-af}{b+af},$$

qui ergo logarithmus erit negativus, sive

$$\varDelta = -\frac{1}{2f}\, l\, \frac{b+af}{b-af},$$

unde obtinebitur sequens aequatio

$$\frac{-2af}{l\frac{b+af}{b-af}} = -b - \cfrac{aaff}{-3b - \cfrac{4aaff}{-5b - \cfrac{9aaff}{-7b - \cfrac{16aaff}{-9b - \text{etc.}}}}}$$

sive mutatis signis

$$\frac{2af}{l\frac{b+af}{b-af}} = b + \cfrac{aaff}{-3b + \cfrac{4aaff}{5b + \cfrac{9aaff}{-7b + \cfrac{16aaff}{9b + \text{etc.}}}}}$$

cuius ergo fractionis continuae summa aequalis est illi, quam in paragrapho praecedente invenimus. Ista autem aequalitas harum duarum expressionum calculum facienti mox fiet manifesta.

15. Eodem modo evolvamus casum, quo $c = -gg$, pro quo supra invenimus

$$\varDelta = \frac{1}{g}\, \text{A sin.} \frac{ag}{\sqrt{(bb+aagg)}},$$

qui valor per cosinum expressus dabit

$$\varDelta = \frac{1}{g}\, \text{A cos.} \frac{b}{\sqrt{(bb+aagg)}},$$

unde patet per tangentem istum valorem adhuc fore simpliciorem; fit scilicet

$$\varDelta = \frac{1}{g}\, \text{A tang.} \frac{ag}{b},$$

quamobrem pro hoc casu prodit ista summatio

$$\frac{ag}{\text{A tang.} \frac{ag}{b}} = b + \cfrac{aagg}{3b + \cfrac{4aagg}{5b + \cfrac{9aagg}{7b + \cfrac{16aagg}{9b + \text{etc.}}}}}$$

ubi nulla amplius limitatione est opus.

DE FRACTIONIBUS CONTINUIS A LOGARITHMIS PENDENTIBUS

16. Perpendamus nunc etiam aliquos casus speciales in utraque forma contentos, et quoniam iam observavimus binas formas in § 13 et 14 inter se congruere, utamur priori, qua erat

$$\frac{2af}{l\frac{b+af}{b-af}} = b - \cfrac{aaff}{3b - \cfrac{4aaff}{5b - \cfrac{9aaff}{7b - \text{etc.}}}}$$

ac primo consideremus casum, quo $b = af$, quippe quo evadit summa fractionis

$$\frac{2af}{l\frac{b+af}{b-af}} = 0 = b - \cfrac{bb}{3b - \cfrac{4bb}{5b - \cfrac{9bb}{7b - \text{etc.}}}}$$

quae per reductionem facile mutatur in hanc

$$0 = 1 - \cfrac{1}{3 - \cfrac{4}{5 - \cfrac{9}{7 - \cfrac{16}{9 - \text{etc.}}}}}$$

17. In ista igitur forma nihilo aequali necesse est, ut denominator primae fractionis sit $= 1$ ideoque

$$1 = 3 - \cfrac{4}{5 - \cfrac{9}{7 - \text{etc.}}} \qquad \text{sive} \qquad 0 = 2 - \cfrac{4}{5 - \cfrac{9}{7 - \text{etc.}}}$$

Hic igitur ob eandem rationem necesse est, ut prior denominator fiat $= 2$, ita ut

$$2 = 5 - \cfrac{9}{7 - \cfrac{16}{9 - \text{etc.}}} \qquad \text{sive} \qquad 0 = 3 - \cfrac{9}{7 - \cfrac{16}{9 - \text{etc.}}}$$

Hic iterum primus denominator debet esse $= 3$ ideoque

$$3 = 7 - \cfrac{16}{9 - \cfrac{25}{11 - \text{etc.}}} \qquad \text{sive} \qquad 0 = 4 - \cfrac{16}{9 - \cfrac{25}{11 - \text{etc.}}}$$

Denuo igitur primus denominator esse debet $= 4$, ita ut

$$4 = 9 - \cfrac{25}{11 - \text{etc.}},$$

atque hoc modo patet istam relationem eodem ordine in infinitum locum habere, in quo ipso criterium veritatis huius aequationis est situm.

18. Quoniam in hac forma numerus b maior esse debet quam af, statuamus nunc $b = 2af$ et nanciscemur sequentem summationem

$$\frac{2af}{l3} = 2af - \cfrac{aaff}{6af - \cfrac{4aaff}{10af - \cfrac{9aaff}{14af - \text{etc.}}}}$$

quae reducitur ad hanc formam mere numericam

$$\frac{2}{l3} = 2 - \cfrac{1}{6 - \cfrac{4}{10 - \cfrac{9}{14 - \cfrac{16}{18 - \text{etc.}}}}}$$

19. Simili modo omnes litterae ex calculo expelli possunt, si pro b accipiatur multiplum ipsius af. Sit enim in genere $b = naf$ ac prodit

$$\frac{2af}{l\frac{n+1}{n-1}} = naf - \cfrac{aaff}{3naf - \cfrac{4aaff}{5naf - \cfrac{9aaff}{7naf - \text{etc.}}}}$$

quae fractio reducitur ad formam sequentem

$$\frac{2}{l\frac{n+1}{n-1}} = n - \cfrac{1}{3n - \cfrac{4}{5n - \cfrac{9}{7n - \text{etc.}}}}$$

unde intelligitur, quemadmodum omnes logarithmos per fractiones continuas exprimi conveniat.

20. Possent hic pro n numeri fracti accipi, tum autem priores termini in singulis membris prodirent fracti, quas quidem per reductionem ad integros revocare liceret; verum huiusmodi casus facillime ex forma generali derivari possunt scribendo statim $b = n$ et $af = m$; tum enim habebimus

$$\frac{2m}{l\frac{n+m}{n-m}} = n - \cfrac{mm}{3n - \cfrac{4mm}{5n - \cfrac{9mm}{7n - \text{etc.}}}}$$

unde, si loco m scribatur $\sqrt{k}$, erit

$$\frac{2\sqrt{k}}{l\frac{n+\sqrt{k}}{n-\sqrt{k}}} = n - \cfrac{k}{3n - \cfrac{4k}{5n - \cfrac{9k}{7n - \text{etc.}}}}$$

21. Hinc igitur omnium numerorum integrorum logarithmos hyperbolicos per fractiones continuas exprimere poterimus. Propositus igitur sit in genere numerus integer i ac statuatur $\frac{n+m}{n-m} = i$; erit $\frac{n}{m} = \frac{i+1}{i-1}$. Capiatur ergo $n = i+1$ et $m = i-1$ atque habebimus

$$\frac{2(i-1)}{li} = i+1 - \cfrac{(i-1)^2}{3(i+1) - \cfrac{4(i-1)^2}{5(i+1) - \cfrac{9(i-1)^2}{7(i+1) - \cfrac{16(i-1)^2}{9(i+1) - \text{etc.}}}}}$$

unde colligimus

$$li = \cfrac{2(i-1)}{i+1 - \cfrac{(i-1)^2}{3(i+1) - \cfrac{4(i-1)^2}{5(i+1) - \cfrac{9(i-1)^2}{7(i+1) - \text{etc.}}}}}$$

22. Si huiusmodi fractiones desideremus pro logarithmis numerorum fractorum, statuamus $\frac{n+m}{n-m} = \frac{p}{q}$, unde fit $n = p+q$ et $m = p-q$, quamobrem habebimus

$$l\frac{p}{q} = \cfrac{2(p-q)}{1(p+q) - \cfrac{1(p-q)^2}{3(p+q) - \cfrac{4(p-q)^2}{5(p+q) - \cfrac{9(p-q)^2}{7(p+q) - \text{etc.}}}}}$$

quae forma eo magis est notatu digna, quod satis commode adhiberi potest ad logarithmos proxime investigandos. Eo magis autem istae fractiones continuae convergent, quo minor fuerit fractio $\frac{p-q}{p+q}$.

23. Quo hoc exemplo illustremus, sumamus $p = 2$ et $q = 1$, unde quidem non adeo vehemens convergentia est expectanda, eritque

$$l2 = \cfrac{2}{3 - \cfrac{1}{9 - \cfrac{4}{15 - \cfrac{9}{21 - \text{etc.}}}}}$$

unde sumendo tantum primum membrum $\frac{2}{3}$ in fractione decimali prodit 0,666666, dum ex tabulis habetur $l2 = 0{,}693147$, ubi error iam satis est exiguus. Capiamus iam bina membra priora

$$\cfrac{2}{3 - \cfrac{1}{9}} = \frac{9}{13} = 0{,}6923.$$

Sumendo autem tria membra habebimus

$$\cfrac{2}{3-\cfrac{1}{9-\cfrac{4}{15}}}=\cfrac{2}{3-\cfrac{15}{131}}=\frac{262}{378}=0{,}693121,$$

qui valor a veritate deficit quantitate 0,000026.

Multo promptior autem deprehendetur convergentia, si sumamus $p=3$ et $q=2$, ut habeamus

$$l\frac{3}{2}=\cfrac{2}{5-\cfrac{1}{15-\cfrac{4}{25-\cfrac{9}{35-\text{etc.}}}}}$$

cuius primum membrum dat $\frac{2}{5}=0{,}400000$; revera autem est $l\frac{3}{2}=0{,}405465108$. Sumtis autem duobus membris

$$\cfrac{2}{5-\cfrac{1}{15}}$$

colligitur $l\frac{3}{2}=0{,}40540$, ubi error tantum in quintam figuram irrepit. Sumantur tria membra

$$\cfrac{2}{5-\cfrac{1}{15-\cfrac{4}{25}}}=\cfrac{2}{5-\cfrac{25}{371}}=0{,}4054645,$$

ubi error demum in septima figura se manifestat.

24. Ob hunc insignem usum, qui se praeter expectationem obtulit, operae pretium erit talem investigationem in genere expedire; atque in hunc finem utamur formula inter litteras m et n supra § 20 data, ubi fit

$$l\frac{n+m}{n-m}=\cfrac{2m}{n-\cfrac{mm}{3n-\cfrac{4mm}{5n-\cfrac{9mm}{7n-\cfrac{16mm}{9n-\text{etc.}}}}}}$$

unde, si capiamus tantum primum membrum, fiet propemodum

$$l\frac{n+m}{n-m}=\frac{2m}{n};$$

sumtis autem binis prioribus membris

$$\cfrac{2m}{n-\cfrac{mm}{3n}}$$

erit iam propius

$$l\frac{n+m}{n-m}=\frac{6mn}{3nn-mm},$$

sumtis vero tribus membris erit

$$l\frac{n+m}{n-m}=\cfrac{2m}{n-\cfrac{mm}{3n-\cfrac{4mm}{5n}}}=\frac{30mnn-8m^3}{15n^3-9mmn}.$$

25. Non adeo autem operosum est has fractiones ulterius continuare; fractionibus enim iam inventis praefigamus fractionem $\frac{0}{1}$, ut obtineamus hanc fractionum progressionem

I	II	III	IV
$\frac{0}{1}$,	$\frac{2m}{n}$,	$\frac{6mn}{3nn-mm}$,	$\frac{30mnn-8m^3}{15n^3-9mmn}$,

cuius tam numeratores quam denominatores ex binis praecedentibus ad similitudinem serierum recurrentium formari possunt. Tertia scilicet ex prima et secunda formatur ope huius scalae relationis $3n$, $-mm$; quarta vero formatur ex binis praecedentibus ope huius scalae relationis $5n$, $-4mm$.

Pro quinta igitur utendum erit hac scala $7n$, $-9mm$, pro sexta hac $9n$, $-16mm$, et ita porro. Hoc igitur modo facile reperitur fractio quinta

$$\text{V} = \frac{210mn^3 - 110m^3n}{105n^4 - 90mmnn + 9m^4},$$

simili modo

$$\text{VI} = \frac{1890mn^4 - 1470m^3nn + 128m^5}{945n^5 - 1050mmn^3 + 225m^4n}$$

etc.

26. Hic autem imprimis notasse iuvabit has fractiones continuo augeri et per incrementa continuo minora ad veritatem accedere. Incrementa autem ista egregio ordine procedunt, uti videre hic licet:

$$\text{II} - \text{I} = \frac{2m}{n},$$

$$\text{III} - \text{II} = \frac{2m^3}{n(3nn - mm)},$$

$$\text{IV} - \text{III} = \frac{2 \cdot 4m^5}{(3nn - mm)(15n^3 - 9mmn)},$$

$$\text{V} - \text{IV} = \frac{2 \cdot 4 \cdot 9m^7}{(15n^3 - 9mmn)(105n^4 - 90mmnn + 9m^4)},$$

$$\text{VI} - \text{V} = \frac{2 \cdot 4 \cdot 9 \cdot 16m^9}{(105n^4 - 90mmnn + 9m^4)(945n^5 - 1050mmn^3 + 225m^4n)},$$

unde patet, quo maior fuerit numerus n prae m, eo citius has differentias tam fieri exiguas, ut sine errore negligi queant.

DE FRACTIONIBUS CONTINUIS AB ARCUBUS CIRCULARIBUS PENDENTIBUS

27. Ex § 15 arcus circuli, cuius tangens est $\frac{ag}{b}$, ita per fractionem exprimetur, ut sit

$$\text{A tang.} \frac{ag}{b} = \cfrac{ag}{b + \cfrac{aagg}{3b + \cfrac{4aagg}{5b + \cfrac{9aagg}{7b + \text{etc.}}}}}$$

Ponamus nunc ad similitudinem superiorum formarum $ag = m$ et $b = n$ atque habebimus

$$\text{A tang.} \frac{m}{n} = \cfrac{m}{n + \cfrac{mm}{3n + \cfrac{4mm}{5n + \cfrac{9mm}{7n + \text{etc.}}}}}$$

quae forma eo citius convergit, quo maior fuerit numerus n prae m; unde patet etiam hanc expressionem cum fructu ad calculum accommodari posse.

28. Incipiamus a casu, quo $m = 1$ et $n = 1$, quo fit

$$\text{A tang.} \frac{m}{n} = \frac{\pi}{4} = \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{4}{5 + \cfrac{9}{7 + \cfrac{16}{9 + \text{etc.}}}}}}$$

quae quidem fractio non adeo convergit; attamen videamus, quomodo paullatim ad veritatem accedat, quandoquidem novimus esse $\frac{\pi}{4} = 0{,}78539816339$. Ac primum quidem membrum dabit

$$\frac{\pi}{4} = \frac{1}{1} \quad \text{(nimis magnum)};$$

duo membra praebent

$$\frac{\pi}{4} = \cfrac{1}{1 + \cfrac{1}{3}} = \frac{3}{4} \quad \text{(nimis parvum)};$$

tria membra dant

$$\frac{\pi}{4} = \cfrac{1}{1+\cfrac{1}{3+\cfrac{4}{5}}} = \frac{19}{24} = 0{,}7916 \quad \text{(nimis magnum)}.$$

Sumantur quatuor membra, ut fiat

$$\frac{\pi}{4} = \cfrac{1}{1+\cfrac{1}{3+\cfrac{4}{5+\cfrac{9}{7}}}} = \frac{40}{51} = 0{,}7843 \quad \text{[nimis parvum]},$$

ubi error demum in tertia figura deprehenditur.

Ceterum haec fractio continua similis fere est illi, quam olim BROUNCKERUS[1]) in medium protulit, quae ita se habebat

$$\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\cfrac{49}{2+\text{etc.}}}}}$$

Manifestum autem est nostram fractionem multo magis convergere; neque minus concinna est censenda.

29. Quo autem fractionem continuam magis convergentem nanciscamur, statuamus A tang. $\frac{m}{n} = 30^\circ$; cuius tangens cum sit $\frac{1}{\sqrt{3}}$, ne numerus n fiat irrationalis, sumamus $m = \sqrt{3}$ et $n = 3$; hinc igitur fiet

$$\frac{\pi}{6} = \cfrac{\sqrt{3}}{3+\cfrac{3}{9+\cfrac{12}{15+\cfrac{27}{21+\cfrac{48}{27+\text{etc.}}}}}}$$

1) Vide notam p. 227. A. L.

quae forma reducitur ad sequentem

$$\frac{\pi}{6\sqrt{3}} = \cfrac{1}{3+\cfrac{1}{3+\cfrac{4}{15+\cfrac{9}{7+\cfrac{16}{27+\cfrac{25}{11+\text{etc.}}}}}}}$$

pro qua evolvenda quaeramus primo proxime valorem $\frac{\pi}{6\sqrt{3}}$, qui est 0,3022998. Nunc vero primum membrum praebet

$$\frac{\pi}{6\sqrt{3}} = 0{,}3333;$$

duo autem priora praebent

$$\frac{\pi}{6\sqrt{3}} = \cfrac{1}{3+\cfrac{1}{3}} = \frac{3}{10} = 0{,}3000;$$

tria membra dant

$$\frac{\pi}{6\sqrt{3}} = \cfrac{1}{3+\cfrac{1}{3+\cfrac{4}{15}}} = \frac{49}{162} = 0{,}30247,$$

ubi error quartam demum figuram afficit.

30. Multo promptior autem convergentia procurari potest, dum angulum rectum in duas partes secamus, quemadmodum olim[1]) ostendi esse

$$\text{A tang.}\,\frac{1}{2} + \text{A tang.}\,\frac{1}{3} = \text{A tang.}\,1 = \frac{\pi}{4}.$$

Sic igitur duas fractiones continuas reperiemus, quarum summa dabit valorem ipsius $\frac{\pi}{4}$, quae erunt

1) Vide *Introductionem in analysin infinitorum*, Lausannae 1748, t. I cap. VIII, § 142; LEONHARDI EULERI *Opera omnia*, series I, vol. 8. A. L.

$$\text{A tang.}\,\frac{1}{2}=\cfrac{1}{2+\cfrac{1}{6+\cfrac{4}{10+\cfrac{9}{14+\text{etc.}}}}}\qquad\text{et}\qquad \text{A tang.}\,\frac{1}{3}=\cfrac{1}{3+\cfrac{1}{9+\cfrac{4}{15+\cfrac{9}{21+\text{etc.}}}}}$$

Manifestum autem est has ambas fractiones et potissimum posteriorem vehementer convergere.

31. Convertamus vero etiam nostram fractionem continuam generalem in fractiones communes; ac ex primo membro solo reperimus

$$\text{A tang.}\,\frac{m}{n}=\frac{m}{n};$$

ex duobus membris prodit

$$\text{A tang.}\,\frac{m}{n}=\frac{3mn}{3nn+mm};$$

tria membra praebent

$$\text{A tang.}\,\frac{m}{n}=\frac{15mnn+4m^3}{15n^3+9mmn}.$$

Sumantur quatuor membra, unde fit

$$\text{A tang.}\,\frac{m}{n}=\frac{105mn^3+55m^3n}{105n^4+90mmnn+9m^4}.$$

Quodsi nunc his fractionibus praefigatur ut supra $\frac{0}{1}$, orietur haec progressio

I	II	III	IV	V
$\frac{0}{1}$,	$\frac{m}{n}$,	$\frac{3mn}{3nn+mm}$,	$\frac{15mnn+4m^3}{15n^3+9mmn}$,	$\frac{105mn^3+55m^3n}{105n^4+90mmnn+9m^4}$,

cuius singuli termini itidem ex praecedentibus binis secundum certam legem formari possunt, scilicet

pro III scala relationis est $3n, \; +mm$,

pro IV scala relationis est $5n, \; +4mm$,

pro V scala relationis est $7n, \; +9mm$

etc.

METHODUS FACILIS INVENIENDI INTEGRALE HUIUS FORMULAE

$$\int\frac{\partial x}{x}\cdot\frac{x^{n+p}-2x^n\cos.\zeta+x^{n-p}}{x^{2n}-2x^n\cos.\theta+1}$$

CASU QUO POST INTEGRATIONEM PONITUR VEL $x=1$ VEL $x=\infty$

Conventui exhibita die 18. Martii 1776

Commentatio 620 indicis Enestroemiani
Nova acta academiae scientiarum Petropolitanae 3 (1785), 1788, p. 3—24
Summarium ibidem p. 161—164

SUMMARIUM

La méthode que l'illustre Euler a mise en usage pour trouver l'intégrale de la formule annoncée dans le titre de ce mémoire, est celle qu'on employe ordinairement dans l'intégration de cette espèce de fractions, savoir de les transformer en fractions partielles, selon les facteurs du dénominateur, et d'intégrer chacune de ces fractions simples séparément. Mais pour peu qu'on considère avec attention la formule en question, quiconque connoît l'esprit de cette méthode, sera peut-être surpris de la voir mise en usage pour l'intégration d'une formule aussi compliquée, et s'attendra ou à des résultats plus compliqués encore, ou, s'il en apperçoit la simplicité, il s'attendra à des artifices de calcul peu communs et capables de répandre de l'intérêt sur un sujet qui en paroît d'abord peu susceptible; et c'est effectivement ce qui fait le prix de ce mémoire.

L'Auteur commence par donner à sa formule la forme

$$\int\frac{\partial x}{x}\cdot\frac{x^p-2\cos.\zeta+x^{-p}}{x^n-2\cos.\theta+x^{-n}},$$

qu'elle prend lorsqu'on divise le numérateur et le dénominateur par x^n, où le dénominateur est un produit de n facteurs simples trinomiaux de la forme $x^1 - 2\cos.\omega + x^{-1}$; et la forme de chaque fraction partielle qui naît de la résolution de la fraction proposée, devient

$$\frac{2(\cos. p\omega - \cos. \zeta)}{n \sin. \theta} \cdot \frac{\sin. \omega}{x^1 - 2\cos. \omega + x^{-1}},$$

les valeurs de l'angle ω étant au nombre de n, savoir

$$\frac{\theta}{n}, \ \frac{\theta + 2\pi}{n}, \ \frac{\theta + 4\pi}{n}, \ \ldots \ \frac{\theta + 2(n-1)\pi}{n}.$$

En multipliant donc chacune de ces fractions par $\frac{\partial x}{x}$, et mettant après l'intégration $x - 1$, l'intégrale de la formule proposée devient

$$\frac{Q}{n \sin. \theta} - \frac{R \cos. \zeta}{n \sin. \theta},$$

les valeurs de Q et R étant

$$R = \frac{n\pi - \theta}{n} + \frac{(n-2)\pi - \theta}{n} + \frac{(n-4)\pi - \theta}{n} + \text{etc.},$$
$$Q = \frac{n\pi - \theta}{n} \cos. \frac{p\theta}{n} + \frac{(n-2)\pi - \theta}{n} \cos. \frac{p}{n}(\theta + 2\pi) + \text{etc.};$$

où il est d'abord clair que $R = \pi - \theta$, de sorte que tout revient à déterminer la somme de la série Q.

Pour cet effet l'Auteur considère les deux séries suivantes de n termes

$$t = \cos.(\alpha + 2\beta) + \cos.(\alpha + 4\beta) + \cdots + \cos.(\alpha + 2n\beta),$$
$$u - \cos.(\alpha + 2\beta) + 2\cos.(\alpha + 4\beta) + \cdots + n\cos.(\alpha + 2n\beta),$$

dont il lui est facile de trouver les sommes par des transformations et combinaisons tirées des principes de son calcul des sinus. Ensuite il considère cette progression formée des deux précédentes

$$V = (a + b)\cos.(\alpha + 2\beta) + (a + 2b)\cos.(\alpha + 4\beta) + \cdots + (a + nb)\cos.(\alpha + 2n\beta)$$

de manière que $V - at + bu$, ce qui, les sommes des progressions t et u étant trouvées, lui fournit la somme V. Or en comparant entre elles les progressions V et Q, on peut déterminer les quantités a, b, α, β, et on parvient enfin à cette expression

$$Q - \frac{\pi \sin. \frac{p}{n}(\pi - \theta)}{\sin. \frac{p\pi}{n}},$$

de façon que l'intégrale cherchée sera pour $x = 1$

$$\frac{\pi \sin. \frac{p}{n}(\pi - \theta)}{n \sin. \theta \sin. \frac{\pi p}{n}} - \frac{(\pi - \theta)\cos. \zeta}{n \sin. \theta},$$

et deux fois plus grande, lorsqu'après l'intégration on met $x = \infty$.

Il y a à faire les remarques suivantes par rapport à cette intégration.

1. L'exposant p doit être plus petit que l'exposant n; car autrement la fraction proposée seroit une fraction impropre: elle contiendroit des parties entières dont il faudroit chercher séparément l'intégrale et l'ajouter à l'intégrale trouvée d'après la méthode exposée ici. (Cette opération se trouve détaillée dans le mémoire suivant.)

2. Les opérations par lesquelles on est parvenu à l'intégrale de la formule proposée, ne sauroient avoir lieu, à moins que l'exposant p ne soit un nombre entier. Cependant M. EULER fait voir que les cas où l'exposant p est une fraction quelconque, peuvent toujours être réduits à d'autres où les exposans sont des nombres entiers; et que par conséquent cette circonstance ne change rien aux résultats; mais que, quel que soit le nombre p, entier ou fractionnaire, pourvu que $p < n$, l'intégrale reste comme elle a été assignée.

3. On peut même donner à cet exposant p une valeur imaginaire, pourvu qu'elle soit telle que la formule différentielle reste réelle. Ainsi, en mettant $p = q\sqrt{-1}$, la formule à intégrer devient [pour $\zeta = 90^0$]

$$\int \frac{\partial x}{x} \cdot \frac{2\cos. qlx}{x^n - 2\cos. \theta + x^{-n}},$$

dont l'intégrale, prise depuis $x = 0$ jusqu'à $x = 1$, est

$$\frac{\pi}{n \sin. \theta} \cdot \frac{e^{-\frac{q}{n}(\pi - \theta)} - e^{+\frac{q}{n}(\pi - \theta)}}{e^{-\frac{q\pi}{n}} - e^{+\frac{q\pi}{n}}}$$

et deux fois plus grande pour les termes d'intégration $x - 0$ et $x - \infty$; vérité de laquelle, comme l'Auteur ajoute, il seroit difficile de donner une démonstration directe.

Ces remarques sont suivies de quelques autres très-propres à répandre du jour tant sur cette intégration que sur bien d'autres, et qui tendent toutes à démontrer que l'intégrale, telle qu'elle a été trouvée, est vraye, quelles que soyent les valeurs de p, n et θ, soit entières, rompues, ou même imaginaires, les seuls cas exceptés où $p - n$ est une quantité positive et réelle.

1. Denotet S integrale huius formulae generaliter sumtum, ita ut quaeri debeat valor ipsius S casu, quo statuitur $x = 1$; ubi primum observo formam propositam multo concinniorem reddi, si fractionis numerator et denominator per x^n dividantur; tum enim habebimus

$$S = \int \frac{\partial x}{x} \cdot \frac{x^p - 2\cos. \zeta + x^{-p}}{x^n - 2\cos. \theta + x^{-n}}.$$

Hic statim patet denominatorem $x^n - 2\cos.\theta + x^{-n}$ semper in n factores resolvi posse, qui singuli sint formae $x^1 - 2\cos.\omega + x^{-1}$, ubi angulum ω ita capi oportet, ut, dum iste evanescit, simul etiam ipse denominator ad nihilum redigatur.

2. Posito autem isto factore $x^1 - 2\cos.\omega + x^{-1} = 0$, unde fit

$$x = \cos.\omega + \sqrt{-1}\cdot\sin.\omega,$$

inde in genere colligitur

$$x^\lambda = \cos.\lambda\omega + \sqrt{-1}\cdot\sin.\lambda\omega \quad \text{et} \quad x^{-\lambda} = \cos.\lambda\omega - \sqrt{-1}\cdot\sin.\lambda\omega.$$

Hinc ergo denominator istum accipiet valorem $2\cos.n\omega - 2\cos.\theta$, qui igitur evanescet, si pro $n\omega$ sumatur aliquis ex his valoribus

$$\theta,\ \theta + 2\pi,\ \theta + 4\pi,\ \theta + 6\pi,\ \theta + 8\pi \ \text{etc.};$$

quare, cum numerus horum valorum debeat esse $= n$, omnes valores anguli ω erunt sequentes

$$\frac{\theta}{n},\ \frac{\theta + 2\pi}{n},\ \frac{\theta + 4\pi}{n},\ \frac{\theta + 6\pi}{n},\ \ldots\ \frac{\theta + 2(n-1)\pi}{n}.$$

Praeterea vero cum sit $\cos.n\omega = \cos.\theta$, erit etiam $\sin.n\omega = \sin.\theta$.

3. Cum igitur denominator habeat n factores huius formae

$$x^1 - 2\cos.\omega + x^{-1},$$

nostra fractio, quicunque fuerit eius numerator, in n fractiones simplices resolvi poterit, quarum denominatores erunt illi n factores denominatoris. Scribamus igitur brevitatis gratia Π loco numeratoris $x^p - 2\cos.\zeta + x^{-p}$ atque haec fractio

$$\frac{\Pi}{x^n - 2\cos.\theta + x^{-n}}$$

resolvetur in n fractiones simplices, quarum singulae hanc habebunt formam

$$\frac{P}{x^1 - 2\cos.\omega + x^{-1}};$$

quocirca statuamus

$$\frac{\Pi}{x^n - 2\cos.\theta + x^{-n}} = \frac{P}{x^1 - 2\cos.\omega + x^{-1}} + R,$$

ubi littera R omnes reliquas complectatur fractiones, unde statim habebimus

$$\frac{\Pi(x^1 - 2\cos.\omega + x^{-1})}{x^n - 2\cos.\theta + x^{-n}} = P + R(x^1 - 2\cos.\omega + x^{-1}).$$

Quodsi iam faciamus $x^1 - 2\cos.\omega + x^{-1} = 0$, hinc colligemus numeratorem P; erit enim

$$P = \frac{\Pi(x^1 - 2\cos.\omega + x^{-1})}{x^n - 2\cos.\theta + x^{-n}},$$

siquidem in hac aequatione ponatur $x = \cos.\omega + \sqrt{-1}\cdot\sin.\omega$.

4. At vero iam vidimus, si ipsi x hunc valorem tribuamus, illius fractionis tam numeratorem quam denominatorem evanescere, quamobrem secundum regulam notissimam loco numeratoris et denominatoris sua scribamus differentialia ac prodibit

$$P = \frac{\Pi(x^1 - x^{-1})}{nx^n - nx^{-n}}.$$

Nunc igitur si loco x valor assignatus scribatur, primo pro Π nanciscemur hunc valorem

$$\Pi = 2\cos.p\omega - 2\cos.\zeta;$$

ex fractione autem oritur iste valor $\frac{\sin.\omega}{n\sin.n\omega}$; quae ergo expressio cum sit realis, numerator quaesitus erit

$$P = \frac{2\sin.\omega(\cos.p\omega - \cos.\zeta)}{n\sin.n\omega}.$$

Iam autem vidimus esse $\sin.n\omega = \sin.\theta$, unde iste numerator erit

$$P = \frac{2\sin.\omega(\cos.p\omega - \cos.\zeta)}{n\sin.\theta}.$$

5. Quaelibet igitur fractio partialis ex resolutione fractionis propositae oriunda erit huiusmodi

$$\frac{2(\cos.p\omega - \cos.\zeta)}{n\sin.\theta}\cdot\frac{\sin.\omega}{x^1 - 2\cos.\omega + x^{-1}};$$

in qua forma si angulo ω successive tribuantur omnes valores supra assignati, qui erant

$$\frac{\theta}{n}, \quad \frac{\theta + 2\pi}{n}, \quad \frac{\theta + 4\pi}{n}, \quad \ldots \quad \frac{\theta + 2(n-1)\pi}{n},$$

orientur omnes fractiones partiales, quae in unam summam collectae ipsam formam propositam $\frac{x^p - 2 \cos. \zeta + x^{-p}}{x^n - 2 \cos. \theta + x^{-n}}$ producere debebunt, unde etiam singulae in $\frac{\partial x}{x}$ ductae et integratae, tum vero in unam summam collectae, exhibebunt integrale quaesitum S.

6. Consideremus igitur fractionem

$$\frac{\sin. \omega}{x^1 - 2 \cos. \omega + x^{-1}},$$

quae ducta in $\frac{\partial x}{x}$ praebet

$$\frac{\partial x \sin. \omega}{x^2 - 2x \cos. \omega + 1},$$

cuius integrale ita sumtum, ut evanescat posito $x = 0$, constat esse

$$= \text{A tang.} \frac{x \sin. \omega}{1 - x \cos. \omega}.$$

Hinc igitur ex hac fractione partiali oritur ista pars integralis

$$\frac{2(\cos. p\omega - \cos. \zeta)}{n \sin. \theta} \cdot \text{A tang.} \frac{x \sin. \omega}{1 - x \cos. \omega},$$

unde ergo facile deducuntur omnes n partes integralis quaesiti, si loco ω ordine omnes eius valores assignati substituantur atque in unam summam colligantur.

7. Quoniam autem hoc loco eum tantum integralis valorem postulamus, qui oritur posito $x = 1$, hoc casu fiet

$$\text{A tang.} \frac{x \sin. \omega}{1 - x \cos. \omega} = \text{A tang.} \frac{\sin. \omega}{1 - \cos. \omega}.$$

At vero ista formula $\frac{\sin. \omega}{1 - \cos. \omega}$ exprimit cotangentem anguli $\frac{1}{2}\omega$ ideoque tangentem anguli $\frac{\pi - \omega}{2}$, ita ut hoc casu pars integralis futura sit

$$\frac{\cos. p\omega - \cos. \zeta}{n \sin. \theta}(\pi - \omega).^{1)}$$

Hic autem in transitu notasse iuvabit, si desideretur integrale pro casu $x = \infty$, tum proditurum esse

$$\text{A tang.} - \frac{\sin. \omega}{\cos. \omega};$$

quia igitur est $-\frac{\sin. \omega}{\cos. \omega}$ tangens anguli $\pi - \omega$, cum ante habuissemus $\frac{\pi - \omega}{2}$, hinc patet casu $x = \infty$ etiam totum integrale duplo maius fore quam casu $x = 1$.

8. Tribuamus igitur angulo ω successive omnes eius valores eosque ordine hic sistamus eritque

$$S = \frac{\cos. \frac{p\theta}{n} - \cos. \zeta}{n \sin. \theta} \cdot \frac{n\pi - \theta}{n}$$
$$+ \frac{\cos. \frac{p}{n}(\theta + 2\pi) - \cos. \zeta}{n \sin. \theta} \cdot \frac{(n-2)\pi - \theta}{n}$$
$$+ \frac{\cos. \frac{p}{n}(\theta + 4\pi) - \cos. \zeta}{n \sin. \theta} \cdot \frac{(n-4)\pi - \theta}{n}$$
$$+ \frac{\cos. \frac{p}{n}(\theta + 6\pi) - \cos. \zeta}{n \sin. \theta} \cdot \frac{(n-6)\pi - \theta}{n}$$
$$+ \text{etc.},$$

quarum formularum numerus debet esse $= n$. Haec autem expressio statim in duas partes distinguatur hoc modo indicandas

$$S = \frac{Q}{n \sin. \theta} - \frac{R \cos. \zeta}{n \sin. \theta},$$

1) Quae formula non valet, nisi sit $0 < \omega < 2\pi$, unde sequitur haec conditio

$$0 < \theta < 2\pi.$$ A. L.

ita ut sit

$$Q = \frac{n\pi - \theta}{n} \cos. \frac{p\theta}{n} + \frac{(n-2)\pi - \theta}{n} \cos. \frac{p}{n}(\theta + 2\pi)$$
$$+ \frac{(n-4)\pi - \theta}{n} \cos. \frac{p}{n}(\theta + 4\pi) + \text{etc.},$$
$$R = \frac{n\pi - \theta}{n} + \frac{(n-2)\pi - \theta}{n} + \frac{(n-4)\pi - \theta}{n} + \frac{(n-6)\pi - \theta}{n} + \text{etc.},$$

ita ut iam nobis incumbat in valores litterarum Q et R inquirere.

9. Primo autem statim patet valorem ipsius R esse progressionem arithmeticam decrescentem differentia $\frac{2\pi}{n}$, unde summa n terminorum erit $= \pi - \theta$, ita ut sit $R = \pi - \theta$. At vero inventio progressionis Q maiorem requirit apparatum, quem in finem sequentes investigationes generaliores praemittamus.

10. Consideretur primo progressio ista cosinuum, quorum anguli in progressione arithmetica progrediantur et quorum numerus sit n,

$$t = \cos.(\alpha + 2\beta) + \cos.(\alpha + 4\beta) + \cos.(\alpha + 6\beta) + \cdots + \cos.(\alpha + 2n\beta).$$

Iam multiplicemus utrinque per $2 \sin. \beta$, et cum sit

$$2 \sin. \beta \cos. \gamma = \sin.(\beta + \gamma) - \sin.(\gamma - \beta),$$

proveniet sequens forma

$$2t \sin. \beta = -\sin.(\alpha + \beta) + \sin.(\alpha + 3\beta) + \sin.(\alpha + 5\beta) + \cdots + \sin.(\alpha + (2n+1)\beta),$$
$$- \sin.(\alpha + 3\beta) - \sin.(\alpha + 5\beta) - \cdots$$

ubi omnes termini intermedii manifesto se destruunt, ita ut soli extremi remaneant, hincque ergo fiet

$$t = \frac{\sin.(\alpha + (2n+1)\beta) - \sin.(\alpha + \beta)}{2 \sin. \beta}.$$

11. Deinde vero iidem cosinus combinentur cum numeris naturalibus 1, 2, 3, ... n ac statuatur

$$u = 1 \cos.(\alpha + 2\beta) + 2 \cos.(\alpha + 4\beta) + 3 \cos.(\alpha + 6\beta) + \cdots + n \cos.(\alpha + 2n\beta),$$

qua expressione ducta in $2 \sin. \beta$ adhibita resolutione, qua modo sumus usi, consequemur

$$2u \sin. \beta = -\sin.(\alpha + \beta) + \sin.(\alpha + 3\beta) + 2 \sin.(\alpha + 5\beta) + \cdots + n \sin.(\alpha + (2n+1)\beta),$$
$$- 2 \sin.(\alpha + 3\beta) - 3 \sin.(\alpha + 5\beta) - \cdots$$

quae forma reducitur ad istam

$$n \sin.(\alpha + (2n+1)\beta) - 2u \sin. \beta$$
$$= \sin.(\alpha + \beta) + \sin.(\alpha + 3\beta) + \cdots + \sin.(\alpha + (2n-1)\beta),$$

quae vocetur $= v$.

12. Nunc ista series denuo ducatur in $2 \sin. \beta$, et cum in genere sit

$$2 \sin. \beta \sin. \gamma = \cos.(\gamma - \beta) - \cos.(\gamma + \beta),$$

nanciscemur

$$2v \sin. \beta = \cos. \alpha - \cos.(\alpha + 2\beta) - \cos.(\alpha + 4\beta) - \cdots - \cos.(\alpha + 2n\beta),$$
$$+ \cos.(\alpha + 2\beta) + \cos.(\alpha + 4\beta) + \cdots$$

unde ob terminos medios omnes se destruentes colligitur

$$v = \frac{\cos. \alpha - \cos.(\alpha + 2n\beta)}{2 \sin. \beta};$$

quare cum sit

$$u = \frac{n \sin.(\alpha + (2n+1)\beta) - v}{2 \sin. \beta},$$

hinc obtinemus

$$u = \frac{n \sin.(\alpha + (2n+1)\beta)}{2 \sin. \beta} - \frac{\cos. \alpha - \cos.(\alpha + 2n\beta)}{4 (\sin. \beta)^2}.$$

13. Combinemus nunc ambas summationes modo traditas in genere ac statuamus

$$V = (a + b) \cos.(\alpha + 2\beta) + (a + 2b) \cos.(\alpha + 4\beta) + (a + 3b) \cos.(\alpha + 6\beta)$$
$$+ \cdots + (a + nb) \cos.(\alpha + 2n\beta)$$

atque evidens est fore $V=at+bu$, unde loco t et u valoribus substitutis erit

$$V=\frac{a\sin.(\alpha+(2n+1)\beta)-a\sin.(\alpha+\beta)}{2\sin.\beta}+\frac{bn\sin.(\alpha+(2n+1)\beta)}{2\sin.\beta}$$
$$-\frac{b\cos.\alpha-b\cos.(\alpha+2n\beta)}{4(\sin.\beta)^2}.$$

14. Iam satis perspicuum est progressionem, quam supra littera Q exhibuimus, in ista forma generali pro V inventa contineri, quandoquidem utrinque idem terminorum numerus n occurrit atque coefficientes cosinuum seriei Q etiam progressionem arithmeticam constituunt. Quamobrem pro coefficientibus primo faciamus

$$a+b=\frac{n\pi-\theta}{n}\quad\text{et}\quad a+2b=\frac{(n-2)\pi-\theta}{n},$$

unde deducimus

$$b=-\frac{2\pi}{n}\quad\text{et}\quad a=\frac{(n+2)\pi-\theta}{n}.$$

Nunc etiam angulos inter se coaequemus faciamusque

$$\alpha+2\beta=\frac{p\theta}{n}\quad\text{et}\quad\alpha+4\beta=\frac{p}{n}(\theta+2\pi),$$

unde colligimus

$$\beta=\frac{\pi p}{n}\quad\text{hincque porro}\quad\alpha=\frac{p}{n}(\theta-2\pi)=-\frac{p}{n}(2\pi-\theta),$$

hocque modo fiet

$$V=Q.$$

At anguli in expressione ipsius V occurrentes erunt primo

$$\alpha+(2n+1)\beta=-\frac{p}{n}(\pi-\theta)+2\pi p,$$

ubi, cum p sit numerus integer, postrema pars $2\pi p$ totam circumferentiam exprimens omissa est, ex quo habebimus

$$\sin.(\alpha+(2n+1)\beta)=-\sin.\frac{p}{n}(\pi-\theta).$$

Deinde occurrit angulus

$$\alpha+\beta=-\frac{p}{n}(\pi-\theta),$$

cuius sinus est

$$\sin.(\alpha+\beta)=-\sin.\frac{p}{n}(\pi-\theta).$$

Denique erit

$$\alpha+2n\beta=-\frac{p}{n}(2\pi-\theta)+2\pi p$$

et

$$\cos.(\alpha+2n\beta)=\cos.\frac{p}{n}(2\pi-\theta).$$

His igitur valoribus substitutis prodibit

$$Q=V=-\frac{\frac{(n+2)\pi-\theta}{n}\sin.\frac{p}{n}(\pi-\theta)-\frac{(n+2)\pi-\theta}{n}\sin.\frac{p}{n}(\pi-\theta)}{2\sin.\frac{\pi p}{n}}$$
$$+\frac{2\pi\sin.\frac{p}{n}(\pi-\theta)}{2\sin.\frac{\pi p}{n}}+\frac{\frac{2\pi}{n}\cos.\frac{p}{n}(2\pi-\theta)-\frac{2\pi}{n}\cos.\frac{p}{n}(2\pi-\theta)}{4\left(\sin.\frac{\pi p}{n}\right)^2},$$

quae expressio manifesto reducitur ad hanc

$$Q=V=\frac{\pi\sin.\frac{p}{n}(\pi-\theta)}{\sin.\frac{\pi p}{n}}.$$

15. Inventis igitur valoribus litterarum Q et R valor integralis, quem quaerimus, pro casu $x=1$ erit

$$S=\frac{\pi\sin.\frac{p}{n}(\pi-\theta)}{n\sin.\theta\sin.\frac{\pi p}{n}}-\frac{(\pi-\theta)\cos.\zeta}{n\sin.\theta}.$$

Sin autem integrale quaeratur a termino $x=0$ usque ad $x=\infty$, eius valor duplo maior evadet.

16. His iam in genere expeditis consideremus casum iam saepius tractatum, quo est $\zeta=90^\circ$ et $\theta=90^\circ$ haecque formula integranda proponitur

$$\int\frac{\partial x}{x}\cdot\frac{x^p+x^{-p}}{x^n+x^{-n}},$$

atque pro eius valore casu $x=1$ habebimus

$$S=\frac{\pi\sin.\frac{p\pi}{2n}}{n\sin.\frac{p\pi}{n}},$$

quae ob $\sin.\frac{p\pi}{n}=2\sin.\frac{p\pi}{2n}\cos.\frac{p\pi}{2n}$ abit in formulam illam notissimam

$$\frac{\pi}{2n\cos.\frac{p\pi}{2n}}=\frac{\pi}{2n}\sec.\frac{p\pi}{2n}.$$

Sin autem tantum sumamus $\zeta=90^0$, ut formula integranda sit

$$\int\frac{\partial x}{x}\cdot\frac{x^p+x^{-p}}{x^n-2\cos.\theta+x^{-n}},$$

eius valor ab $x=0$ usque ad $x=1$ extensus erit

$$\frac{\pi\sin.\frac{p}{n}(\pi-\theta)}{n\sin.\theta\sin.\frac{p\pi}{n}},$$

quae expressio reducitur ad hanc

$$\frac{\pi}{n\sin.\theta}\left(\cos.\frac{p\theta}{n}-\sin.\frac{p\theta}{n}\cot.\frac{p\pi}{n}\right).$$

OBSERVATIONES IN INTEGRATIONEM TRADITAM

I. Primum hic observo terminum medium in numeratore exhibitum nullo modo integrationem turbare, quoniam, si solus adesset, integratio nulla laboraret difficultate; tum enim formula

$$\int\frac{\partial x}{x}\cdot\frac{1}{x^n-2\cos.\theta+x^{-n}}$$

reducitur ad hanc formam

$$\int\frac{x^{n-1}\partial x}{x^{2n}-2x^n\cos.\theta+1},$$

quae posito $x^n=y$ abit in hanc

$$\frac{1}{n}\int\frac{\partial y}{y^2-2y\cos.\theta+1},$$

cuius integrale est

$$\frac{1}{n\sin.\theta}\,\mathrm{A\,tang.}\frac{y\sin.\theta}{1-y\cos.\theta},$$

cuius valor casu $x=1$ fit

$$\frac{1}{n\sin.\theta}\,\mathrm{A\,tang.}\frac{\sin.\theta}{1-\cos.\theta}=\frac{\pi-\theta}{2n\sin.\theta},$$

qui ductus in $-2\cos.\zeta$ praebet illam ipsam partem hinc oriundam in forma supra inventa, quamobrem superfluum foret hunc terminum in calculo retinere; unde hanc formam integralem sumus contemplaturi

$$\int\frac{\partial x}{x}\cdot\frac{x^p+x^{-p}}{x^n-2\cos.\theta+x^{-n}},$$

cuius valorem casu $x=1$ invenimus

$$=\frac{\pi\sin.\frac{p}{n}(\pi-\theta)}{n\sin.\theta\sin.\frac{p\pi}{n}},$$

quem brevitatis gratia littera P designemus, ita ut sit

$$P=\frac{\pi\sin.\frac{p}{n}(\pi-\theta)}{n\sin.\theta\sin.\frac{p\pi}{n}}.$$

Tum vero etiam invenimus casu $x=\infty$ valorem huius formulae esse $=2P$.

II. Secundo loco probe notari oportet exponentem p necessario minorem esse debere quam exponentem n, quia alioquin fractio foret spuria et variabilis x in numeratore tot vel plures dimensiones esset habitura quam in denominatore. Quoties autem hoc evenit, integrali praeter partes, quas per resolutionem in fractiones partiales sumus nacti, quantitas quaedam integra adiici debet, id quod in nostra solutione non est factum, quamobrem tales casus hinc prorsus excludi convenit. Ceterum quolibet casu has partes integras facile erit adiicere ad partes, quas nobis nostra methodus suppeditabit.

III. Ex ipsa solutione, quam dedimus, perspicuum est exponentem p necessario integrum statui debere, quia alias operationes ibi exhibitae locum habere non possent; unde eo magis mirum videbitur, quod conclusiones inventae subsistere queant, etiamsi iste exponens p fuerit numerus fractus quicunque, dummodo minor quam n, propterea quod hos casus semper ad exponentes integros reducere licet. Ad hoc ostendendum ponamus esse $p = \frac{q}{\lambda}$ atque forma nostra posito $x = z^\lambda$ reducetur ad hanc formam

$$\lambda \int \frac{\partial z}{z} \cdot \frac{z^q + z^{-q}}{z^{\lambda n} - 2 \cos. \theta + z^{-\lambda n}};$$

ubi cum omnes exponentes sint integri ac pro terminis integrationis, qui erant $x = 0$, $x = 1$ et $x = \infty$, etiam fiat $z = 0$, $z = 1$ et $z = \infty$, pro $z = 1$ valor integralis erit

$$\frac{\lambda \pi \sin. \frac{q}{\lambda n}(\pi - \theta)}{\lambda n \sin. \theta \sin. \frac{q\pi}{\lambda n}},$$

qui restituto loco $\frac{q}{\lambda}$ valore p abit in hunc

$$\frac{\pi \sin. \frac{p}{n}(\pi - \theta)}{n \sin. \theta \sin. \frac{p\pi}{n}},$$

quae expressio cum superiore prorsus congruit. Atque hinc intelligitur, etiamsi[1]) exponenti p valores irrationales tribuantur, dumne superent exponentem n, semper hoc evenire debere.

IV. Hic iam quaestio oritur maximi momenti, utrum etiam exponenti p dare liceat valores imaginarios necne. Hoc autem affirmandum videtur, quandoquidem imaginaria certe non sint maiora quam n; unde concludimus, dummodo valor ipsius p ita capiatur imaginarius, ut ipsa formula differentialis maneat realis, tum etiam conclusiones nostras veritati consentaneas esse mansuras.[2]) Hoc autem evenit, si statuamus $p = q\sqrt{-1}$; tum enim, cum in genere sit

$$e^{\varphi\sqrt{-1}} + e^{-\varphi\sqrt{-1}} = 2 \cos. \varphi,$$

1) Editio princeps: ... *intelligitur, quominus etiam exponenti* ... Correxit A. L.

2) Quamquam hoc non satis confirmatum est, tamen hic revera ponere licet $p = q\sqrt{-1}$, quia integrale valore absoluto ipsius p manente satis parvo in seriem integram secundum ipsius p potestates progredientem resolvi potest. A. L.

quia nostro casu est $\varphi = qlx$, ipsa formula integralis erit

$$\int \frac{\partial x}{x} \cdot \frac{2 \cos. qlx}{x^n - 2 \cos. \theta + x^{-n}}.$$

Nunc igitur videamus, quamnam formam nostrum integrale casu $x = 1$ sit recepturum, et quoniam sinus angulorum imaginariorum sunt etiam imaginarii, quandoquidem

$$e^{\varphi\sqrt{-1}} - e^{-\varphi\sqrt{-1}} = \frac{2}{\sqrt{-1}} \sin. \varphi,$$

loco φ scribamus $\psi\sqrt{-1}$ eritque

$$\sin. \psi\sqrt{-1} = \frac{\sqrt{-1}}{2}(e^{-\psi} - e^{+\psi}),$$

unde in forma integrali erit $\frac{p}{n} = \frac{q\sqrt{-1}}{n}$, ideoque loco ψ scribamus $\frac{q}{n}(\pi - \theta)$ pro numeratore, at $\frac{q\pi}{n}$ pro denominatore, ex quo valor integralis ab $x = 0$ ad $x = 1$ extensus erit

$$\frac{\pi}{n \sin. \theta} \cdot \frac{e^{-\frac{q}{n}(\pi - \theta)} - e^{+\frac{q}{n}(\pi - \theta)}}{e^{-\frac{q\pi}{n}} - e^{+\frac{q\pi}{n}}}.$$

Hinc igitur formemus sequens theorema notatu dignissimum.

THEOREMA

Quodsi ista formula integralis

$$\int \frac{\partial x}{x} \cdot \frac{\cos. qlx}{x^n - 2 \cos. \theta + x^{-n}}$$

a termino $x = 0$ *usque ad* $x = 1$ *extendatur, eius valor semper erit*

$$= \frac{\pi}{2n \sin. \theta} \cdot \frac{e^{-\frac{q}{n}(\pi - \theta)} - e^{+\frac{q}{n}(\pi - \theta)}}{e^{-\frac{q\pi}{n}} - e^{+\frac{q\pi}{n}}}.$$

Sin autem integrale extendatur ab $x = 0$ *ad* $x = \infty$, *valor prodibit duplo maior.*

Hoc theorema utique eo maiorem attentionem meretur, quod nulla via patet eius veritatem directe demonstrandi.

V. Revertamur autem ad formam integralem primo expositam, et quoniam numerator duabus constat partibus x^p et x^{-p}, unde summa integralium pro $x = 1$ inventa est $= P$, at pro casu $x = \infty$ duplo maior $= 2P$, hic maxime notatu dignum occurrit, quod pro termino $x = \infty$ utraque pars numeratoris eundem producat valorem $= P$. Semper enim erit integrale ab $x = 0$ ad $x = \infty$ extendendo

$$\int \frac{\partial x}{x} \cdot \frac{x^p}{x^n - 2 \cos. \theta + x^{-n}} = \int \frac{\partial x}{x} \cdot \frac{x^{-p}}{x^n - 2 \cos. \theta + x^{-n}} = P.$$

Ad hoc ostendendum ponamus pro posteriore formula $x = \frac{1}{z}$ eaque induet hanc formam

$$-\int \frac{\partial z}{z} \cdot \frac{z^{+p}}{z^{-n} - 2 \cos. \theta + z^n};$$

quae cum sit priori formae prorsus similis solo signo — excepto, eius valor a termino $z = 0$ usque ad $z = \infty$ negative sumtus primae formulae erit aequalis. Cum autem sit $z = \frac{1}{x}$, isti termini integralis erunt ab $x = \infty$ usque ad $x = 0$; qui ergo si invertantur, etiam signum integralis erit mutandum sicque ipsi priori formulae aequale evadet; quare cum ambae formulae coniunctae summam habeant $= 2P$, utriusque seorsim sumtae valor erit $= P$, unde deducitur sequens theorema notatu pariter dignissimum.

THEOREMA

Istius formulae integralis

$$\int \frac{\partial x}{x} \cdot \frac{x^{\pm p}}{x^n - 2 \cos. \theta + x^{-n}}$$

valor a termino $x = 0$ *usque ad* $x = \infty$ *extensus semper est*

$$= P = \frac{\pi \sin. \frac{p}{n} (\pi - \theta)}{n \sin. \theta \sin. \frac{p\pi}{n}}.$$

Evidens autem est hanc aequalitatem pro casu $x = 1$ neutiquam locum habere posse.

VI. Quoniam in nostra formula differentiali tantum occurrit terminus $2 \cos. \theta$, cuius valor idem manet, etiamsi pro θ sumeremus $\theta \pm 2i\pi$, maxime

hic mirum videri debet, quod tum valor integralis maxime diversus sit proditurus, scilicet

$$= \frac{\pi \sin. \frac{p}{n} (\pi - \theta \pm 2i\pi)}{n \sin. \theta \sin. \frac{p\pi}{n}},$$

unde merito quaeritur, quisnam horum valorum veritati sit conformis; ad quod certe nihil aliud responderi potest, nisi quod omnes veritati aeque consentanei sint censendi[1]), id quod eo minus mirum videri debet, quod omnes huiusmodi formulae integrales revera sunt functiones multiformes atque adeo infinitiformes, id quod ex hoc exemplo simplicissimo $\int \frac{\partial x}{1 + xx}$ intelligi potest. Cum enim eius integrale exhibeat arcum circuli, cuius tangens est x, tamen autem arcus innumerabiles dentur, quorum eadem sit tangens $= x$, necesse est, ut omnes aeque in hac forma integrali contineantur. Quin etiam in nostro valore invento P loco π quoque scribere licet $\pi + 2i\pi$ eiusque valor nihilominus cum veritate consistere poterit. Verum in huiusmodi integrationibus perpetuo valores minimi desiderari solent hocque modo omnis difficultas e medio est sublata.

VII. Deinde in analysi supra adhibita supposuimus omnes factores denominatoris inter se esse inaequales, id quod utique semper evenit, nisi sit $\cos. \theta = \pm 1$, quippe quibus casibus denominator quadratum involvit; fit enim is

$$= x^{-n} (x^n \pm 1)^2,$$

ex quo patet omnes factores $x^n \pm 1$ bis occurerre debere. Hoc incommodum etiam innuitur per ipsam nostram formulam P, quae casu $\theta = 0$ valorem indicat infinitum. Verum posito $\theta = \pi$ singulare phaenomenon se offert, dum formulae pro P inventae tam numerator quam denominator evanescunt atque adeo fractio determinatum nanciscitur valorem. Ponamus enim $\theta = \pi - \omega$ existente ω infinite parvo eritque $\sin. \theta = \sin. \omega = \omega$; at ob $\pi - \theta = \omega$ in numeratore habebimus $\sin. \frac{p\omega}{n} = \frac{p\omega}{n}$, unde valor ipsius P resultat $\frac{\pi p}{nn \sin. \frac{p\pi}{n}}$; qui cum penitus sit determinatus, nullum plane dubium superesse potest, quin cum veritate conspiret, unde sequens enascitur theorema maxime memorabile.

1) Formula inventa nonnisi hac conditione $0 < \theta < 2\pi$ valet (vide notam p. 271). A. L.

THEOREMA

Proposita formula differentiali

$$\frac{\partial x}{x} \cdot \frac{x^p + x^{-p}}{x^n + 2 + x^{-n}} = \frac{x^{n-1} \partial x (x^p + x^{-p})}{(x^n + 1)^2}$$

si eius integrale a termino $x = 0$ *usque ad* $x = 1$ *extendatur, eius valor semper erit*

$$\frac{\pi p}{nn \sin. \frac{p\pi}{n}};$$

sin autem usque ad terminum $x = \infty$ *extendatur, eius valor erit duplo maior*

$$= \frac{2\pi p}{nn \sin. \frac{p\pi}{n}}.$$

DEMONSTRATIO HUIUS THEOREMATIS DIRECTA

Formula ista integralis resolvatur sequenti modo

$$\int \frac{\partial x}{x} \cdot \frac{x^{n+p} + x^{n-p}}{(1 + x^n)^2} = \frac{Q}{1 + x^n} + \int \frac{\partial x}{x} \cdot \frac{R}{1 + x^n}.$$

Sumantur igitur differentialia simulque ducantur in $\frac{x}{\partial x}$ positoque $\partial Q = Q' \partial x$ orietur ista aequatio

$$\frac{x^{n+p} + x^{n-p}}{(1 + x^n)^2} = \frac{Q'x}{1 + x^n} - \frac{nQx^n}{(1 + x^n)^2} + \frac{R}{1 + x^n},$$

quae per $1 + x^n$ multiplicata hoc modo repraesentetur

$$\frac{x^{n+p} + x^{n-p} + nQx^n}{1 + x^n} = Q'x + R,$$

ubi iam Q ita accipi debet, ut illa fractio ad integrum revocetur. Facile autem patet hoc fieri statuendo

$$nQ = -x^p + x^{n-p};$$

tum enim illa fractio fiet

$$\frac{x^{n-p} + x^{2n-p}}{1 + x^n} = x^{n-p},$$

ita ut nunc habeamus

$$x^{n-p} = Q'x + R.$$

Cum igitur sit $Q = \frac{x^{n-p} - x^p}{n}$, erit

$$Q'x = \frac{(n-p)x^{n-p} - px^p}{n}$$

hincque colligitur

$$R = \frac{p}{n}(x^{n-p} + x^p),$$

quocirca formula integralis proposita reducta est ad hanc formam

$$\frac{x^{n-p} - x^p}{n(1 + x^n)} + \frac{p}{n} \int \frac{\partial x}{x} \cdot \frac{x^{n-p} + x^p}{1 + x^n},$$

quod integrale ita est sumendum, ut evanescat posito $x = 0$. Nunc igitur statuamus $x = 1$ ac prior pars absoluta evanescit, formulae autem integralis valor per ea, quae dudum[1] sunt inventa, prodit

$$\frac{p}{n} \cdot \frac{\pi}{n \sin. \frac{p\pi}{n}} = \frac{\pi p}{nn \sin. \frac{p\pi}{n}},$$

qui ergo cum ante invento perfecte congruit.

VIII. Tribuatur nunc etiam in hac postrema forma exponenti p valor imaginarius ponendo $p = q\sqrt{-1}$, et cum, ut ante vidimus, hinc fiat $x^p + x^{-p} = 2 \cos. qlx$, formula integralis proposita erit

$$= 2 \int \frac{\partial x}{x} \cdot \frac{x^n \cos. qlx}{(1 + x^n)^2}.$$

Pro eius valore autem iam ante vidimus fore

$$\sin. \frac{\pi q \sqrt{-1}}{n} = \frac{e^{-\frac{\pi q}{n}} - e^{+\frac{\pi q}{n}}}{2\sqrt{-1}},$$

1) Vide L. EULERI Commentationem 60 (indicis ENESTROEMIANI): *De inventione integralium, si post integrationem variabili quantitati determinatus valor tribuatur*, Miscellanea Berolin. 7, 1743, p. 129; *LEONHARDI EULERI Opera omnia*, series I, vol. 17, p. 58. A. L.

quamobrem valor nostrae formulae ab $x=0$ ad $x=1$ extensus erit

$$\frac{2\pi q}{nn\left(e^{\frac{\pi q}{n}}-e^{-\frac{\pi q}{n}}\right)},$$

unde deducimus sequens theorema omni attentione dignum.

THEOREMA

Si valor istius formulae integralis

$$\int\frac{x^{n-1}\partial x\cos.qlx}{(1+x^n)^2}$$

a termino $x=0$ usque ad $x=1$ extendatur, is semper aequabitur huic formulae

$$\frac{\pi q}{nn\left(e^{\frac{\pi q}{n}}-e^{-\frac{\pi q}{n}}\right)}.$$

Cuius autem theorematis demonstratio ex principiis iam cognitis vix elici posse videtur.

IX. Praeterea etiam perspicuum est methodum, qua usi sumus ad nostram formulam integrandam, subsistere non posse, nisi terminus medius denominatoris binario sit minor, quam ob causam eum hac forma $2\cos.\theta$ expressimus. Quamobrem hinc oritur quaestio maximi momenti, utrum nostrae conclusiones etiamnunc valeant, si terminus ille medius binario maior acciperetur sive si angulus θ foret imaginarius, necne. Verum etiam hoc casu nullum dubium superesse potest, quin formula nostra finalis etiamnunc veritati consentanea sit futura. Ante omnia autem hic est observandum illi termino medio $2\cos.\theta$ valorem negativum tribui convenire, quia alioquin ipse denominator in nihilum abiret, dum quantitas nostra variabilis x a termino 0 usque ad 1 augetur. Hanc ob rem statuamus angulum $\theta=\pi-\eta$ et valor noster integralis erit

$$\int\frac{\partial x}{x}\cdot\frac{x^p+x^{-p}}{x^n+2\cos.\eta+x^{-n}}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=\frac{\pi\sin.\frac{p\eta}{n}}{n\sin.\eta\sin.\frac{\pi p}{n}}.$$

In hac igitur forma faciamus angulum η imaginarium ponendo $\eta=\varphi\sqrt{-1}$

eritque per ea, quae iam supra observavimus, $2\cos.\varphi\sqrt{-1}=e^\varphi+e^{-\varphi}$, ita ut iam noster denominator sit

$$x^n+e^\varphi+e^{-\varphi}+x^{-n}=\frac{1}{x^n}(x^n+e^\varphi)(x^n+e^{-\varphi}),$$

quem idcirco statim in duos factores reales formae $x+k$ resolvere licet; tum vero fiet

$$\sin.\eta=\sin.\varphi\sqrt{-1}=\frac{e^{-\varphi}-e^{+\varphi}}{2\sqrt{-1}}$$

similique modo erit

$$\sin.\frac{p}{n}\eta=\sin.\frac{p}{n}\varphi\sqrt{-1}=\frac{e^{-\frac{p\varphi}{n}}-e^{+\frac{p\varphi}{n}}}{2\sqrt{-1}},$$

unde formula nostra integralis emergit realis

$$=\frac{\pi\left(e^{-\frac{p\varphi}{n}}-e^{+\frac{p\varphi}{n}}\right)}{n(e^{-\varphi}-e^{+\varphi})\sin.\frac{\pi p}{n}}.$$

Statuamus autem hic brevitatis gratia $e^\varphi=f$, ut sit $e^{-\varphi}=\frac{1}{f}$, atque nostra formula integralis sequentem induet formam

$$\int\frac{\partial x}{x}\cdot\frac{x^p+x^{-p}}{x^n+\left(f+\frac{1}{f}\right)+x^{-n}}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=\frac{\pi\left(f^{\frac{p}{n}}-f^{-\frac{p}{n}}\right)}{n(f-f^{-1})\sin.\frac{p\pi}{n}},$$

id quod tamquam theorema omni attentione dignum spectari potest; ubi per se intelligitur valorem eiusdem integralis usque ad $x=\infty$ extensum fore duplo maiorem.

X. Quodsi iam in hac forma etiam exponenti p valorem imaginarium tribuamus, pariter nullo modo dubitari poterit, quin conclusio nostra vera sit mansura. Ponamus igitur $p=q\sqrt{-1}$ eritque ut ante $x^p+x^{-p}=2\cos.qlx$; tum vero erit

$$\sin.\frac{p\pi}{n}=\frac{e^{-\frac{q\pi}{n}}-e^{+\frac{q\pi}{n}}}{2\sqrt{-1}},$$

pro integralis autem numeratore erit

$$f^{\frac{p}{n}}-f^{-\frac{p}{n}}=2\sqrt{-1}\cdot\sin.\frac{q}{n}lf.$$

His igitur valoribus substitutis sequens nanciscimur

THEOREMA

Valor istius formulae integralis

$$\int\frac{\partial x}{x}\cdot\frac{\cos.qlx}{x^n+\left(f+\frac{1}{f}\right)+x^{-n}}$$

a termino $x=0$ *usque ad* $x=1$ *extensus semper aequabitur formulae*

$$\frac{2\pi\sin.\frac{q}{n}lf}{n(f-f^{-1})\left(e^{\frac{q\pi}{n}}-e^{-\frac{q\pi}{n}}\right)}.$$

XI. Deinde iam pridem[1]) observavi omnia huiusmodi integralia satis commode per series infinitas exprimi posse. Cum enim ista fractio

$$\frac{x^p}{x^n-2\cos.\theta+x^{-n}}=\frac{x^{n+p}}{x^{2n}-2x^n\cos.\theta+1}$$

resolvatur in hanc seriem

$$\frac{1}{\sin.\theta}(x^{n+p}\sin.\theta+x^{2n+p}\sin.2\theta+x^{3n+p}\sin.3\theta+\text{etc.}),$$

integrale istud

$$\int\frac{\partial x}{x}\cdot\frac{x^p}{x^n-2\cos.\theta+x^{-n}}$$

a termino $x=0$ usque ad $x=1$ extensum aequabitur huic seriei infinitae

$$\frac{1}{\sin.\theta}\left(\frac{\sin.\theta}{n+p}+\frac{\sin.2\theta}{2n+p}+\frac{\sin.3\theta}{3n+p}+\frac{\sin.4\theta}{4n+p}+\text{etc.}\right).$$

1) Vide Commentationem 589 huius voluminis, imprimis p. 204. A. L.

Hinc ergo si p negative caperemus, tum formula nostra principalis

$$\int\frac{\partial x}{x}\cdot\frac{x^p+x^{-p}}{x^n-2\cos.\theta+x^{-n}}$$

ab $x=0$ ad $x=1$ extensa semper aequabitur huic seriei infinitae geminatae

$$\frac{1}{\sin.\theta}\left\{\begin{matrix}\frac{\sin.\theta}{n+p}+\frac{\sin.2\theta}{2n+p}+\frac{\sin.3\theta}{3n+p}+\frac{\sin.4\theta}{4n+p}+\text{etc.}\\+\frac{\sin.\theta}{n-p}+\frac{\sin.2\theta}{2n-p}+\frac{\sin.3\theta}{3n-p}+\frac{\sin.4\theta}{4n-p}+\text{etc.}\end{matrix}\right\},$$

quae binis homologis coniungendis contrahitur in hanc seriem

$$\frac{2n}{\sin.\theta}\left(\frac{\sin.\theta}{nn-pp}+\frac{2\sin.2\theta}{4nn-pp}+\frac{3\sin.3\theta}{9nn-pp}+\frac{4\sin.4\theta}{16nn-pp}+\text{etc.}\right).$$

XII. Hinc iam manifesto pro casu, quo ponitur $p=q\sqrt{-1}$, ista series infinita exoritur

$$\frac{2n}{\sin.\theta}\left(\frac{\sin.\theta}{nn+qq}+\frac{2\sin.2\theta}{4nn+qq}+\frac{3\sin.3\theta}{9nn+qq}+\frac{4\sin.4\theta}{16nn+qq}+\text{etc.}\right),$$

quae ergo exprimit valorem huius formulae integralis

$$\int\frac{\partial x}{x}\cdot\frac{2\cos.qlx}{x^n-2\cos.\theta+x^{-n}},$$

scilicet ab $x=0$ ad $x=1$ extensae, ita ut istius seriei summa finito modo expressa sit etiam

$$\frac{\pi}{n\sin.\theta}\cdot\frac{e^{-\frac{q}{n}(\pi-\theta)}-e^{+\frac{q}{n}(\pi-\theta)}}{e^{-\frac{q\pi}{n}}-e^{+\frac{q\pi}{n}}}.$$

Quin etiam facile intelligitur hic quoque angulum θ imaginarium accipi posse. Vidimus enim posito $\theta=\varphi\sqrt{-1}$ fore

$$\sin.\theta=\frac{e^{-\varphi}-e^{+\varphi}}{2\sqrt{-1}}$$

hincque in genere

$$\sin.\lambda\theta=\frac{e^{-\lambda\varphi}-e^{+\lambda\varphi}}{2\sqrt{-1}}.$$

Quare si statuamus $e^{\varphi}=f$, erit

$$\frac{\sin.\lambda\theta}{\sin.\theta}=\frac{f^{\lambda}-f^{-\lambda}}{f-\frac{1}{f}},$$

unde series illa satis concinnam formam accipiet.

XII[a].[1]) Denique operationes, quibus in integratione nostrae formulae sumus usi, consistere nequeunt, nisi exponens n fuerit numerus integer. Interim tamen valor integralis, quem invenimus pro casu vel $x=1$ vel $x=\infty$, veritati conformis deprehenditur, non solum quando pro n numerus fractus quicunque, sed etiam adeo imaginarius accipitur, quorum prius facile ostenditur. Sit enim $n=\frac{m}{\lambda}$ ac ponatur $x=y^{\lambda}$ atque ob $\frac{\partial x}{x}=\frac{\lambda\partial y}{y}$ orietur haec forma integralis exponentibus integris contenta

$$\int\frac{\lambda\partial y}{y}\cdot\frac{y^{\lambda p}+y^{-\lambda p}}{y^m-2\cos.\theta+y^{-m}},$$

cuius ergo valor casu $x=1$ debet esse secundum formulam inventam

$$\frac{\lambda\pi}{m}\cdot\frac{\sin.\frac{\lambda p}{m}(\pi-\theta)}{\sin.\theta\sin.\frac{\lambda p\pi}{m}},$$

qui, si iam loco m valor λn restituatur, manifesto abit in ipsam nostram formulam supra [V] inventam

$$\frac{\pi}{n}\cdot\frac{\sin.\frac{p}{n}(\pi-\theta)}{\sin.\theta\sin.\frac{p\pi}{n}}.$$

Hinc autem nulli amplius dubio relinquitur, quin veritas haec subsistat, etiamsi n fuerit numerus imaginarius.[2]) Ponamus igitur $n=m\sqrt{-1}$ et formula integralis reducetur ad hanc formam

$$\int\frac{\partial x}{x}\cdot\frac{x^p+x^{-p}}{2\cos.ml x-2\cos.\theta},$$

1) In editione principe falso numerus XII iteratur. A. L.

2) Manifestum est hanc conclusionem falsam esse, quotiescunque pro n numerus imaginarius formae $m\sqrt{-1}$ accipitur. A. L.

cuius ergo valor casu $x=1$ certe erit

$$\frac{p}{m\sqrt{-1}}\cdot\frac{e^{\frac{p}{m}(\pi-\theta)}-e^{-\frac{p}{m}(\pi-\theta)}}{\sin.\theta\left(e^{\frac{\pi p}{m}}-e^{-\frac{\pi p}{m}}\right)},$$

ubi mirum videbitur istum valorem semper esse imaginarium, licet ipsa formula differentialis, dum variabilis x a termino 0 usque ad terminum $x=1$ augetur, maneat realis, id quod merito maxime videtur paradoxum. Interim tamen non desunt casus, quibus valor integralis formulae differentialis realis manifesto evadit imaginarius, id quod in ista formula simpliciori

$$\int\frac{\partial x}{x\cos.ml x}$$

ostendisse sufficiet, quae utique, dum x a 0 ad 1 augetur, constanter manet realis. Ad hanc ergo formulam integrandam statuamus $lx=-z$, ubi notetur, dum x a 0 usque ad 1 progreditur, tum quantitatem z ab ∞ usque ad 0 decrescere. Nunc igitur formula nostra integralis erit

$$\int\frac{-\partial z}{\cos.mz};$$

cum vero constet esse

$$\int\frac{\partial\varphi}{\sin.\varphi}=l\,\mathrm{tang}.\frac{1}{2}\varphi,$$

sumamus $\varphi=90^{\circ}-mz$ eritque $\partial\varphi=-m\partial z$ hincque

$$\int\frac{-m\partial z}{\cos.mz}=+l\,\mathrm{tang}.\left(45^{\circ}-\frac{1}{2}mz\right),$$

quod integrale manifesto evanescit pro termino $z=0$; dum autem ab hoc termino quantitas z in infinitum usque augetur, infinities tangens huius anguli fiet negativa eiusque logarithmus propterea imaginarius, unde non amplius mirabimur, quod formulae differentialis realis integrale evadere possit certis casibus imaginarium.

XIII. Hoc igitur modo evictum est formulae nostrae differentialis propositae

$$\int\frac{\partial x}{x}\cdot\frac{x^p+x^{-p}}{x^n-2\cos.\theta+x^{-n}}$$

integrale assignatum a termino $x=0$ usque ad $x=1$ semper cum veritate consistere, quicunque valores ternis litteris n, p et θ tribuantur sive integri sive fracti sive etiam imaginarii. Interim tamen dantur casus iam initio indicati, quibus isti valores integrales a veritate aberrabunt, quippe quod semper usu venire debet, quoties exponens p maior est exponente n, quam ob causam sedulo excludere debemus omnes casus, quibus formula $p-n$ evadit realis et positiva. His autem exceptis variae formulae, ad quas hic sumus perducti, ita sunt comparatae, ut maxima attentione dignae videantur simulque non contemnenda incrementa scientiae analyticae promittant.

DE SUMMO USU CALCULI IMAGINARIORUM IN ANALYSI

Conventui exhibita die 18. Martii 1776

Commentatio 621 indicis ENESTROEMIANI
Nova acta academiae scientiarum Petropolitanae 3 (1785), 1788, p. 25—46
Summarium ibidem p. 164—167

SUMMARIUM

Les Géomètres de nos jours connoissent suffisamment la grande utilité du Calcul des Imaginaires; ils savent combien il a contribué à l'avancement de l'Analyse et que dans l'intégration des formules différentielles fractionnaires, qui se fait par la résolution en fractions partielles ayant des dénominateurs en partie imaginaires, on ne sauroit se passer de ce calcul. Feu M. EULER avoit fait, à la vérité, quelques tentatives de dégager l'intégration des formules rationnelles de tout emploi des Imaginaires, et quoiqu'il y ait réussi en partie, le succès n'avoit pas été parfaitement heureux dans les cas, où le dénominateur a deux ou plusieurs facteurs égaux. On ne sauroit donc se passer entièrement des Imaginaires et on rencontre parfois des formules intégrales qui paroissent se refuser à toute voye d'intégration, à moins qu'on n'ait recours aux Imaginaires. C'est ce que l'Auteur se propose de montrer par un nouvel exemple frappant dans le cours de ce mémoire.

Le cas que M. EULER a choisi pour cet effet lui a été fourni par le mémoire précédent. Car ayant trouvé

$$\frac{\pi\sin.\frac{\theta p}{n}}{n\sin.\theta\sin.\frac{\pi p}{n}}$$

pour l'intégrale

$$\int\frac{\partial x}{x}\cdot\frac{x^p+x^{-p}}{x^n+2\cos.\theta+x^{-n}},$$

prise depuis $x = 0$ jusqu'à $x = 1$, il observe qu'en gardant les mêmes termes d'intégration, on peut déduire de là

$$\int \frac{\partial x}{x\,lx} \cdot \frac{x^p - x^{-p}}{x^n + 2\cos.\theta + x^{-n}} = \frac{\pi}{n \sin.\theta} \int \frac{\partial p \sin.\frac{\theta p}{n}}{\sin.\frac{\pi p}{n}},$$

p étant regardé comme une quantité variable et l'intégrale prise de manière qu'elle évanouisse en mettant $p = 0$. Et c'est l'intégration de cette formule, ou bien, pour éviter les fractions, de celle-ci $\int \frac{\partial \varphi \sin. m\varphi}{\sin. n\varphi}$, qui fait le sujet de ce mémoire.

L'Auteur commence par dégager cette expression des quantités angulaires, en mettant $t = \cos.\varphi + \sqrt{-1} \cdot \sin.\varphi$ et $u = \cos.\varphi - \sqrt{-1} \cdot \sin.\varphi$, ce qui le conduit à cette expression purement algébrique

$$\frac{\partial t}{t\sqrt{-1}} \cdot \frac{t^m - t^{-m}}{t^n - t^{-n}},$$

pour l'intégration de laquelle il faut soigneusement distinguer les cas où l'exposant m surpasse n, des cas où le contraire arrive, vu que dans le premier cas la fraction est impropre et contient des entiers dont il faut avant tout déterminer les intégrales, ce qui étant fait, le reste se réduit à trouver l'intégrale de la formule proposée pour les cas où l'exposant n est plus grand que m.

Pour cette intégration, M. EULER met en usage la méthode qu'il a employée avec succès dans le mémoire précédent, savoir la résolution en fractions partielles par la décomposition du dénominateur en n facteurs simples trinomiaux de la forme $t^1 - 2\cos.\omega + t^{-1}$, qui donnent autant de fractions partielles à intégrer, chacune de la forme

$$-\frac{2\sin.\omega \sin. m\omega}{n \cos n\omega} \cdot \frac{\partial t}{t\sqrt{-1}} \cdot \frac{1}{t^1 - 2\cos.\omega + t^{-1}},$$

les n valeurs de l'angle ω étant $0, \frac{\pi}{n}, \frac{2\pi}{n}, \frac{3\pi}{n}, \dots \frac{(n-1)\pi}{n}$.

Mais comme l'intégration de chacune de ces fractions partielles, quoique très-facile, mène à un arc imaginaire qu'il faudroit réduire à des quantités réelles, pour s'épargner cet embarras, l'Auteur fait rentrer son angle φ dans le calcul, ce qui étant fait, l'intégrale partielle en question, à cause de $\frac{\partial t}{t\sqrt{-1}} = \partial\varphi$ et $t^1 + t^{-1} = 2\cos.\varphi$, prend cette forme

$$-\frac{\sin.\omega \sin. m\omega}{n \cos. n\omega} \cdot \frac{\partial \varphi}{\cos.\varphi - \cos.\omega},$$

de façon qu'il ne reste plus qu'à trouver l'intégrale de $\frac{\partial\varphi}{\cos.\varphi - \cos.\omega}$, que M. EULER trouve d'une manière très-aisée

$$= \frac{1}{\sin.\omega}\, l \frac{\sin.\frac{1}{2}(\omega + \varphi)}{\sin.\frac{1}{2}(\omega - \varphi)}.$$

La forme générale des intégrales partielles qui composent l'intégrale complète de la formule proposée sera donc

$$-\frac{\sin. m\omega}{n \cos. n\omega}\, l \frac{\sin.\frac{1}{2}(\omega + \varphi)}{\sin.\frac{1}{2}(\omega - \varphi)},$$

ou bien, ce qui revient au même, en mettant $\frac{\pi}{n} = 2\alpha$ et $\varphi = 2\psi$, on aura

$$\int \frac{2\partial\psi \sin. 2m\psi}{\sin. 2n\psi} = \frac{\sin. 2m\alpha}{n}\, l \frac{\sin.(\alpha + \psi)}{\sin.(\alpha - \psi)} - \frac{\sin. 4m\alpha}{n}\, l \frac{\sin.(2\alpha + \psi)}{\sin.(2\alpha - \psi)}$$
$$+ \frac{\sin. 6m\alpha}{n}\, l \frac{\sin.(3\alpha + \psi)}{\sin.(3\alpha - \psi)} - \text{etc.},$$

où le nombre des termes est $n - 1$ et $\psi < \alpha$.

Ayant donc trouvé une expression finie pour l'intégrale de $\frac{\partial p \sin.\frac{\theta p}{n}}{\sin.\frac{\pi p}{n}}$, on pourra aussi assigner, par une expression finie, l'intégrale de la formule

$$\int \frac{\partial x}{x\,lx} \cdot \frac{x^p - x^{-p}}{x^n + 2\cos.\theta + x^{-n}},$$

toutes les fois que l'angle θ est à π dans un rapport rationnel, c'est à dire $\theta : \pi = \mu : \nu$, ou bien $\nu = \frac{\mu\pi}{\theta}$. En mettant donc $\frac{p}{n} = r$ et $\frac{\pi}{2\nu} = \varrho$, la forme générale de toutes les parties dont l'intégrale de la formule proposée est composée, sera

$$\pm \frac{\sin. i\theta}{\sin.\theta}\, l \frac{\sin.\varrho(i + r)}{\sin.\varrho(i - r)}.$$

Mais comme de cette manière le nombre des termes peut être réduit à la moitié, pour faciliter cette contraction l'Auteur distingue quatre cas, selon que les nombres μ et ν sont tous les deux pairs, ou tous les deux impairs, ou l'un pair et l'autre impair, et il finit son mémoire par quelques exemples propres à éclaircir cette intégration remarquable.

Quanta incrementa Calculo Imaginariorum per universam Analysin accepta sint referenda, nunc quidem amplius nemo dubitabit. Nuper[1]) equidem conatus sum integrationem formularum rationalium a Calculo Imaginariorum penitus liberare; verumtamen hoc negotium in casibus, ubi denominator plures habet factores inter se aequales, minus feliciter successit. Quin etiam non ita pridem in tales formulas integrales incidi, quae quomodo sine subsidio imaginariorum tractari queant, nullo adhuc modo perspicio. Cum enim*)

*) Vide Dissertationem praecedentem p. 284.

1) Vide Commentationem 572 huius voluminis p. 113. A. L.

ostendissem huius formulae integralis

$$\int \frac{\partial x}{x} \cdot \frac{x^p + x^{-p}}{x^n + 2\cos.\theta + x^{-n}}$$

valorem a termino $x = 0$ usque ad $x = 1$ extensum esse

$$\frac{\pi \sin.\frac{\theta p}{n}}{n \sin.\theta \sin.\frac{\pi p}{n}}$$

denotante π peripheriam circuli, cuius diameter $= 1$, inde facile deducitur haec conclusio maxime memorabilis, quod huius formulae integralis

$$\int \frac{\partial x}{x l x} \cdot \frac{x^p - x^{-p}}{x^n + 2\cos.\theta + x^{-n}}$$

valor pariter a termino $x = 0$ usque ad $x = 1$ extensus aequetur isti integrali

$$\frac{\pi}{n \sin.\theta} \int \frac{\partial p \sin.\frac{\theta p}{n}}{\sin.\frac{\pi p}{n}},$$

ubi scilicet quantitas p tamquam variabilis spectatur et integrale ita capitur, ut evanescat posito $p = 0$. Quodsi ergo nunc faciamus $\frac{p}{n} = \varphi$, integrari oportet huiusmodi formulam differentialem $\frac{\partial\varphi \sin.m\varphi}{\sin.n\varphi}$. Quemadmodum igitur ista integratio auxilio imaginariorum tractari debeat, hic sum ostensurus.

DE INTEGRATIONE FORMULAE

$$\int \frac{\partial\varphi \sin.m\varphi}{\sin.n\varphi}$$

1. Ante omnia hanc formulam ad quantitates algebraicas ordinarias revocari convenit, id quod commodius quam per imaginaria praestari nequit. Hunc in finem statuamus brevitatis gratia

$$t = \cos.\varphi + \sqrt{-1} \cdot \sin.\varphi \quad \text{et} \quad u = \cos.\varphi - \sqrt{-1} \cdot \sin.\varphi,$$

ita ut sit $tu = 1$; tum vero erit

$$\partial t = -\partial\varphi(\sin.\varphi - \sqrt{-1} \cdot \cos.\varphi)$$

ideoque

$$\partial t \sqrt{-1} = -\partial\varphi(\cos.\varphi + \sqrt{-1} \cdot \sin.\varphi) = -t\partial\varphi,$$

unde ergo fiet

$$\partial\varphi = -\frac{\partial t \sqrt{-1}}{t} = \frac{\partial t}{t\sqrt{-1}}.$$

2. His autem formulis constitutis ex elementis Calculi Imaginariorum constat esse

$$t^\lambda = \cos.\lambda\varphi + \sqrt{-1} \cdot \sin.\lambda\varphi \quad \text{et} \quad u^\lambda = \cos.\lambda\varphi - \sqrt{-1} \cdot \sin.\lambda\varphi;$$

unde ergo colligitur $t^\lambda - u^\lambda = 2\sqrt{-1} \cdot \sin.\lambda\varphi$ ideoque

$$\sin.\lambda\varphi = \frac{t^\lambda - u^\lambda}{2\sqrt{-1}}.$$

Hinc ergo, si loco λ scribamus numeros m et n, erit

$$\frac{\sin.m\varphi}{\sin.n\varphi} = \frac{t^m - u^m}{t^n - u^n},$$

quocirca, si integrale quaesitum littera S designemus, ut sit

$$S = \int \frac{\partial\varphi \sin.m\varphi}{\sin.n\varphi},$$

facta substitutione nunc habebimus

$$\partial S = \frac{\partial t}{t\sqrt{-1}} \cdot \frac{t^m - u^m}{t^n - u^n}.$$

Quia autem est $u = \frac{1}{t} = t^{-1}$, formula proposita ad speciem consuetam solam variabilem t involventem est reducta, cum sit

$$\partial S\sqrt{-1} = \frac{\partial t}{t} \cdot \frac{t^m - t^{-m}}{t^n - t^{-n}},$$

cuius formulae adeo integralis iam passim evoluta reperitur. Hic autem probe meminisse oportet ipsam quantitatem t non esse realem, cum sit $t = \cos.\varphi + \sqrt{-1} \cdot \sin.\varphi$.

3. Manifestum hic est ambos numeros m et n semper tamquam integros spectari posse, cum iis ratio indicetur, quam ambo anguli $m\varphi$ et $n\varphi$ inter se tenent. Hic igitur ante omnia dispiciendum erit, utrum exponens m

maior minorve sit exponente n, quandoquidem notum est, si fuerit $m > n$, fractionem nostram esse spuriam atque partes integras ante ex ea elici debere, quam integratio suscipiatur. Hos ergo casus hic primum evolvi conveniet.

Sit igitur primo $m = n + \lambda$, ita tamen, ut sit $\lambda < n$, ac facile patebit fractionem

$$\frac{t^{n+\lambda} - t^{-(n+\lambda)}}{t^n - t^{-n}}$$

continere partem integram $t^\lambda + t^{-\lambda}$, qua ab ista fractione sublata remanet

$$-\frac{t^{n-\lambda} - t^{-(n-\lambda)}}{t^n - t^{-n}},$$

quae fractio non amplius est spuria. Ex parte integra autem ducta in $\frac{\partial t}{t}$ oritur integrale $\frac{t^\lambda - t^{-\lambda}}{\lambda}$. At vero est $t^\lambda - t^{-\lambda} = t^\lambda - u^\lambda = 2\sqrt{-1} \cdot \sin.\lambda\varphi$, quod per $\sqrt{-1}$ divisum dat partem integralis hinc oriundam

$$= \frac{2 \sin.\lambda\varphi}{\lambda}.$$

4. Sin autem fuerit $m > 2n$ sive $m = 2n + \lambda$, tum fractio nostra

$$\frac{t^{2n+\lambda} - t^{-2n-\lambda}}{t^n - t^{-n}}$$

hanc continebit partem integram $t^{n+\lambda} + t^{-n-\lambda}$, qua ablata remanet adhuc ista fractio

$$\frac{t^\lambda - t^{-\lambda}}{t^n - t^{-n}},$$

quae iam est genuina ob $\lambda < n$. At vero ex parte integra ducta in $\frac{\partial t}{t}$ oritur integrando $\frac{t^{n+\lambda} - t^{-n-\lambda}}{n+\lambda} = \frac{t^{n+\lambda} - u^{n+\lambda}}{n+\lambda}$, cuius valor est $\frac{2\sqrt{-1} \cdot \sin.(n+\lambda)\varphi}{n+\lambda}$, qui per $\sqrt{-1}$ divisus praebet partem integralis hinc natam

$$= \frac{2 \sin.(n+\lambda)\varphi}{n+\lambda}.$$

5. Simili modo si fuerit $m > 3n$ ac ponatur $m = 3n + \lambda$, fractio nostra erit

$$\frac{t^{3n+\lambda} - t^{-3n-\lambda}}{t^n - t^{-n}},$$

quae continebit partem integram $t^{2n+\lambda} + t^{\,2n-\lambda}$; hac autem ablata remanebit adhuc fractio

$$\frac{t^{n+\lambda} - t^{-n-\lambda}}{t^n - t^{-n}},$$

quae etiamnunc est spuria et continet partem integram $t^\lambda + t^{-\lambda}$; qua ablata demum remanet fractio genuina

$$-\frac{t^{n-\lambda} - t^{-(n-\lambda)}}{t^n - t^{-n}}.$$

Ex partibus autem integris oriuntur hae partes integralis

$$\frac{2 \sin.(2n+\lambda)\varphi}{2n+\lambda} + \frac{2 \sin.\lambda\varphi}{\lambda}.$$

6. Ponamus quoque esse $m > 4n$ ideoque $m = 4n + \lambda$ et fractio nostra erit

$$\frac{t^{4n+\lambda} - t^{-4n-\lambda}}{t^n - t^{-n}},$$

quae statim continet partem integram $t^{3n+\lambda} + t^{-3n-\lambda}$; hac autem ablata remanet adhuc ista fractio

$$\frac{t^{2n+\lambda} - t^{-2n-\lambda}}{t^n - t^{-n}},$$

quae denuo continet partem integram $t^{n+\lambda} + t^{-n-2}$; qua subtracta tandem remanet ista fractio genuina

$$\frac{t^\lambda - t^{-\lambda}}{t^n - t^{-n}}.$$

Iam vero ex partibus integris obtinentur pro integrali S istae partes

$$\frac{2 \sin.(3n+\lambda)\varphi}{3n+\lambda} + \frac{2 \sin.(n+\lambda)\varphi}{n+\lambda}.$$

7. Sit porro etiam $m > 5n$ sive $m = 5n + \lambda$ ac nostra fractio

$$\frac{t^{5n+\lambda} - t^{-5n-\lambda}}{t^n - t^{-n}}$$

primo continebit partem integram $t^{4n+\lambda}+t^{-4n-\lambda}$; qua ablata remanet adhuc ista fractio

$$\frac{t^{3n+\lambda}-t^{-3n-\lambda}}{t^n-t^{-n}},$$

quae per antecedentia continet adhuc duas partes integras, scilicet $t^{2n+\lambda}+t^{-2n-\lambda}$ et $t^\lambda+t^{-\lambda}$; quibus ablatis remanet tandem ista fractio genuina

$$\frac{t^{n-\lambda}-t^{-(n-\lambda)}}{t^n-t^{-n}}.$$

8. Ex his casibus iam satis perspicitur, quomodo, si exponens n adhuc maior accipiatur, partes integrae in integrale S ingredientes se sint habiturae, quas idcirco hic coniunctim aspectui exponamus.

I. Si $m=n+\lambda$, erit

$$\int\frac{\partial\varphi\sin.(n+\lambda)\varphi}{\sin.n\varphi}$$
$$=\frac{2\sin.\lambda\varphi}{\lambda}-\int\frac{\partial t}{t\sqrt{-1}}\cdot\frac{t^{n-\lambda}-t^{-(n-\lambda)}}{t^n-t^{-n}}.$$

II. Si $m=2n+\lambda$, erit

$$\int\frac{\partial\varphi\sin.(2n+\lambda)\varphi}{\sin.n\varphi}$$
$$=\frac{2\sin.(n+\lambda)\varphi}{n+\lambda}+\int\frac{\partial t}{t\sqrt{-1}}\cdot\frac{t^\lambda-t^{-\lambda}}{t^n-t^{-n}}.$$

III. Si $m=3n+\lambda$, erit

$$\int\frac{\partial\varphi\sin.(3n+\lambda)\varphi}{\sin.n\varphi}$$
$$=\frac{2\sin.(2n+\lambda)\varphi}{2n+\lambda}+\frac{2\sin.\lambda\varphi}{\lambda}-\int\frac{\partial t}{t\sqrt{-1}}\cdot\frac{t^{n-\lambda}-t^{-(n-\lambda)}}{t^n-t^{-n}}.$$

IV. Si $m=4n+\lambda$, erit

$$\int\frac{\partial\varphi\sin.(4n+\lambda)\varphi}{\sin.n\varphi}$$
$$=\frac{2\sin.(3n+\lambda)\varphi}{3n+\lambda}+\frac{2\sin.(n+\lambda)\varphi}{n+\lambda}+\int\frac{\partial t}{t\sqrt{-1}}\cdot\frac{t^\lambda-t^{-\lambda}}{t^n-t^{-n}}.$$

V. Si $m=5n+\lambda$, erit

$$\int\frac{\partial\varphi\sin.(5n+\lambda)\varphi}{\sin.n\varphi}$$
$$=\frac{2\sin.(4n+\lambda)\varphi}{4n+\lambda}+\frac{2\sin.(2n+\lambda)\varphi}{2n+\lambda}+\frac{2\sin.\lambda\varphi}{\lambda}-\int\frac{\partial t}{t\sqrt{-1}}\cdot\frac{t^{n-\lambda}-t^{-(n-\lambda)}}{t^n-t^{-n}}.$$

VI. Si $m=6n+\lambda$, erit

$$\int\frac{\partial\varphi\sin.(6n+\lambda)\varphi}{\sin.n\varphi}$$
$$=\frac{2\sin.(5n+\lambda)\varphi}{5n+\lambda}+\frac{2\sin.(3n+\lambda)\varphi}{3n+\lambda}+\frac{2\sin.(n+\lambda)\varphi}{n+\lambda}+\int\frac{\partial t}{t\sqrt{-1}}\cdot\frac{t^\lambda-t^{-\lambda}}{t^n-t^{-n}}$$

etc.

9. His igitur casibus, quibus $m>n$, felicissimo cum successu expeditis totum negotium reducitur ad integrationem formulae $\frac{\partial\varphi\sin.m\varphi}{\sin.n\varphi}$ pro casibus, quibus est $m<n$, quandoquidem ex modo allatis manifestum est, quomodo illi casus ad hos facillime reducuntur. Tum igitur ope nostrae substitutionis $t=\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi$ pervenitur ad hanc formulam

$$S\sqrt{-1}=\int\frac{\partial t}{t}\cdot\frac{t^m-t^{-m}}{t^n-t^{-n}},$$

cuius ergo integrationem data opera instituamus.

INVESTIGATIO INTEGRALIS $\int\frac{\partial t}{t}\cdot\frac{t^m-t^{-m}}{t^n-t^{-n}}$ EXISTENTE $m<n$

10. Hic ante omnia cuncti factores trinomiales nostri denominatoris t^n-t^{-n} indagari debebunt, quorum singulorum forma ita exhiberi potest $t^1-2\cos.\omega+t^{-1}$, ubi angulum ω ita definiri oportet, ut posito

$$t^1-2\cos.\omega+t^{-1}=0$$

simul ipse denominator evanescat; tum autem exinde colligitur

$$t=\cos.\omega+\sqrt{-1}\cdot\sin.\omega,$$

unde statim patet fore

$$t^n = \cos. n\omega + \sqrt{-1} \cdot \sin. n\omega \quad \text{et} \quad t^{-n} = \cos. n\omega - \sqrt{-1} \cdot \sin. n\omega,$$

quamobrem noster denominator reducetur ad hanc formam $2\sqrt{-1} \cdot \sin. n\omega$, qui ergo valor nihilo debet aequari.

11. Cum igitur debeat esse $\sin. n\omega = 0$, omnes valores, quos pro $n\omega$ accipere licet, erunt 0π, π, 2π, 3π etc., unde ipsius anguli ω valores erunt $\frac{0\pi}{n}$, $\frac{1\pi}{n}$, $\frac{2\pi}{n}$, $\frac{3\pi}{n}$ etc. et in genere $\frac{i\pi}{n}$ denotante i numerum integrum quemcunque. Hinc igitur pro omnibus factoribus nostri denominatoris videntur capi debere n horum valorum; verum manifestum est, quotcunque tales formulae $t^1 - 2\cos. \omega + t^{-1}$ in se invicem multiplicentur, ultimum terminum nunquam prodire posse $-t^{-n}$. At vero hic meminisse oportet, quae circa huiusmodi integrationes in genere sunt praecepta, scilicet talem factorem trinomialem $tt - 2t\cos. \omega + 1$ casu, quo $\omega = 0$, non factorem quadratum $(t-1)^2$, sed tantum simplicem $t - 1$ innui, quod idem quoque evenit, si $\omega = \pi$; tum enim quoque non factor quadratus $(t+1)^2$, sed tantum simplex $t+1$ est sumendus; quare cum hi ipsi casus inter valores ipsius ω occurrant, necesse est, ut numerus horum factorum unitate augeatur. Hic autem commode usu venit, ut isti casus ex valoribus $\omega = 0$ et $\omega = \pi$ oriundi e medio tollantur.

12. Cum igitur fractionem nostram $\frac{t^m - t^{-m}}{t^n - t^{-n}}$ in meras fractiones simplices resolvi oporteat, quarum denominatores sint $t^1 - 2\cos. \omega + t^{-1}$, pro unaquaque harum fractionum statuamus

$$\frac{t^m - t^{-m}}{t^n - t^{-n}} = \frac{\varDelta}{t - 2\cos. \omega + t^{-1}} + R,$$

ubi R complectatur omnes reliquas fractiones, et nunc utrinque multiplicemus per $t - 2\cos. \omega + t^{-1}$, ut prodeat

$$\frac{(t^m - t^{-m})(t - 2\cos. \omega + t^{-1})}{t^n - t^{-n}} = \varDelta + R(t - 2\cos. \omega + t^{-1});$$

unde si iam ponamus $t - 2\cos. \omega + t^{-1} = 0$, quod fit sumendo

$$t = \cos. \omega + \sqrt{-1} \cdot \sin. \omega,$$

hinc colligitur numerator nostrae fractionis

$$\varDelta = (t^m - t^{-m})\frac{t - 2\cos. \omega + t^{-1}}{t^n - t^{-n}}.$$

Tum autem manifestum est in hac fractione, ad quam sumus deducti, hoc casu tam numeratorem quam denominatorem in nihilum abire, unde iuxta regulam notissimam eorum loco sua scribamus differentialia, atque ista fractio induet hanc formam $\frac{t^1 - t^{-1}}{n(t^n + t^{-n})}$, ubi manifesto erit $t^1 - t^{-1} = 2\sqrt{-1} \cdot \sin. \omega$, at $t^n + t^{-n} = 2\cos. n\omega$, ita ut nunc valor huius fractionis futurus sit $\frac{\sqrt{-1} \cdot \sin. \omega}{n \cos. n\omega}$, qui ductus in $t^m - t^{-m} = 2\sqrt{-1} \cdot \sin. m\omega$ dabit numeratorem nostrum quaesitum

$$\varDelta = -\frac{2\sin. \omega \sin. m\omega}{n\cos. n\omega}.$$

Quia autem est $\sin. n\omega = 0$, semper erit vel $\cos. n\omega = 1$ vel $\cos. n\omega = -1$, prouti statuendo in genere $\omega = \frac{i\pi}{n}$ numerus i fuerit vel par vel impar.

13. Inventa igitur hac fractione

$$-\frac{2\sin. \omega \sin. m\omega}{n\cos. n\omega} \cdot \frac{1}{t - 2\cos. \omega + t^{-1}}$$

ea in $\frac{\partial t}{t}$ multiplicetur et integretur sicque ad istam pertingimus formulam integralem

$$-\frac{2\sin. \omega \sin. m\omega}{n\cos. n\omega}\int\frac{\partial t}{t} \cdot \frac{1}{t - 2\cos. \omega + t^{-1}},$$

cuius quidem integratio nulla amplius laborat difficultate; perduceret enim ad arcum circuli, cuius tangens $= \frac{t\sin. \omega}{1 - t\cos. \omega}$; verum quia ipsa quantitas t iam est imaginaria, hinc parum lucraremur, quoniam necesse foret istum arcum imaginarium ad quantitates reales reducere, siquidem constat arcus imaginarios ad logarithmos reales reduci.

14. Ut igitur hunc laborem evitemus, loco nostrae variabilis t ipsum angulum φ rursus in calculum revocemus, et quia iam vidimus esse $\frac{\partial t}{t} = \partial\varphi\sqrt{-1}$, tum vero $t + u = 2\cos. \varphi$, hisce valoribus substitutis formula

integranda erit

$$-\frac{\sin.\omega \sin.m\omega}{n \cos.n\omega}\cdot\frac{\partial\varphi\sqrt{-1}}{\cos.\varphi-\cos.\omega},$$

quae formula per $\sqrt{-1}$ divisa praebet partem ipsius integralis quaesiti S, ita ut sit S aggregatum omnium harum formularum

$$-\frac{\sin.\omega \sin.m\omega}{n \cos.n\omega}\int\frac{\partial\varphi}{\cos.\varphi-\cos.\omega},\ ^{1)}$$

isquidem angulo ω successive omnes suos valores tribuamus; ubi per se manifestum est in hac integratione angulum ω esse constantem solumque φ variabilem.

15. Ex coefficiente huius formulae statim patet, quod iam supra innuimus, ex valoribus ipsius ω primo et extremo, scilicet $\omega=0$ et $\omega=\pi$, partes integralis sponte e medio tolli, ita ut nunc sufficiat loco ω successive substitui hos valores $\frac{\pi}{n}, \frac{2\pi}{n}, \frac{3\pi}{n}, \dots \frac{(n-1)\pi}{n}$. Ubi recordandum, dum statuitur $\omega=\frac{i\pi}{n}$, quoties i fuerit numerus par, fore $\cos.n\omega=+1$; sin autem sit i numerus impar, tum fore $\cos.n\omega=-1$. Quibus observatis totum negotium reductum est ad integrationem huius formulae satis memorabilis $\int\frac{\partial\varphi}{\cos.\varphi-\cos.\omega}$.

16. Facile quidem foret istam formulam ad quantitates reales consuetas nevocare; interim tamen sequenti modo haec integratio facilius et elegantius rabsolvi potest. Ponamus enim brevitatis gratia $\frac{\partial\varphi}{\cos.\varphi-\cos.\omega}=\partial s$ et secundum calculum angulorum iam satis vulgatum novimus esse

$$\cos.\varphi-\cos.\omega=2\sin.\frac{\omega+\varphi}{2}\sin.\frac{\omega-\varphi}{2}$$

sicque habebimus

$$\partial s=\frac{\partial\varphi}{2\sin.\frac{\omega+\varphi}{2}\sin.\frac{\omega-\varphi}{2}}$$

sive

$$\frac{2\partial s}{\partial\varphi}=\frac{1}{\sin.\frac{\omega+\varphi}{2}\sin.\frac{\omega-\varphi}{2}},$$

1) Editio princeps: . . . *quaesiti S, ita ut sit*

$$S=-\frac{\sin.\omega\sin.m\omega}{n\cos.n\omega}\int\frac{\partial\varphi}{\cos.\varphi-\cos.\omega}.$$

A. L.

quae fractio, quia denominator duobus constat factoribus, commode resolvi potest in duas fractiones huiusmodi

$$\frac{\alpha\cos.\frac{\omega+\varphi}{2}}{\sin.\frac{\omega+\varphi}{2}}+\frac{\beta\cos.\frac{\omega-\varphi}{2}}{\sin.\frac{\omega-\varphi}{2}},$$

ubi statim patet sumi debere $\beta=\alpha$; tum enim summa harum fractionum prodit

$$\frac{\alpha\sin.\omega}{\sin.\frac{\omega+\varphi}{2}\sin.\frac{\omega-\varphi}{2}},$$

unde

$$\alpha=\beta=\frac{1}{\sin.\omega}.$$

Hinc autem erit

$$\partial s=\frac{1}{2\sin.\omega}\left(\frac{\partial\varphi\cos.\frac{\omega+\varphi}{2}}{\sin.\frac{\omega+\varphi}{2}}+\frac{\partial\varphi\cos.\frac{\omega-\varphi}{2}}{\sin.\frac{\omega-\varphi}{2}}\right),$$

in quibus formulis numerator manifesto est differentiale denominatoris, unde concludimus fore

$$s=\frac{1}{\sin.\omega}\,l\frac{\sin.\frac{\omega+\varphi}{2}}{\sin.\frac{\omega-\varphi}{2}}.$$

17. Invento iam hoc integrali, in quo cardo totius investigationis versabatur, quilibet factor denominatoris in valorem integralem quaesitum S ductus suppeditat istam partem

$$-\frac{\sin.m\omega}{n\cos.n\omega}\,l\frac{\sin.\frac{\omega+\varphi}{2}}{\sin.\frac{\omega-\varphi}{2}},$$

ubi tantum opus est, ut loco anguli ω successive omnes eius valores debiti substituantur; tum enim aggregatum omnium harum formularum praebebit verum valorem integralis

$$S=\int\frac{\partial\varphi\sin.m\varphi}{\sin.n\varphi}.$$

18. Quo autem totum integrale succinctius repraesentare valeamus, ponamus brevitatis gratia $\frac{\pi}{n}=2\alpha$, ita ut valores ipsius ω futuri sint 2α, 4α,

$6\alpha, \ldots 2(n-1)\alpha$; tum vero sit $\varphi = 2\psi$ atque formulae integralis

$$\int \frac{2\partial\psi \sin. 2m\psi}{\sin. 2n\psi}$$

valor completus erit

$$S = \frac{\sin. 2m\alpha}{n} l \frac{\sin. (\alpha+\psi)}{\sin. (\alpha-\psi)} - \frac{\sin. 4m\alpha}{n} l \frac{\sin. (2\alpha+\psi)}{\sin. (2\alpha-\psi)}$$
$$+ \frac{\sin. 6m\alpha}{n} l \frac{\sin. (3\alpha+\psi)}{\sin. (3\alpha-\psi)} - \frac{\sin. 8m\alpha}{n} l \frac{\sin. (4\alpha+\psi)}{\sin. (4\alpha-\psi)}$$
$$+ \frac{\sin. 10m\alpha}{n} l \frac{\sin. (5\alpha+\psi)}{\sin. (5\alpha-\psi)} - \frac{\sin. 12m\alpha}{n} l \frac{\sin. (6\alpha+\psi)}{\sin. (6\alpha-\psi)}$$
etc.,

donec horum membrorum numerus sit $n-1$. Haec autem formula tantum valet, quando $m < n$; si enim fuerit $m > n$, iam ante ostendimus, cuiusmodi termini insuper debeant adiungi.

19. Hic observandum est haec integralia ita esse sumta, ut evanescant posito $\varphi = 0$, quoniam hoc casu omnes logarithmi ad unitatem referuntur. Deinde etiam evidens est, si angulus ψ augeatur usque ad α, tum integrale iam in infinitum excrescere; unde patet hunc angulum non ultra istum terminum augeri convenire. Verum etiam casus initio memoratus, qui ad hanc formulam integralem ducit, non postulat, ut iste angulus ultra hunc terminum augeatur, quamobrem operae pretium erit integrationem inventam ad hunc ipsum casum accommodare.

PROBLEMA

Valorem istius formulae integralis

$$\int \frac{\partial x}{x l x} \cdot \frac{x^p - x^{-p}}{x^n + 2\cos. \theta + x^{-n}}$$

a termino $x = 0$ ad $x = 1$ extensum per expressionem finitam assignare.

SOLUTIO

20. Quoniam istum valorem quaesitum reduxi ad hanc formulam integralem

$$\frac{\pi}{n \sin. \theta} \int \frac{\partial p \sin. \frac{\theta p}{n}}{\sin. \frac{\pi p}{n}},$$

primum tenendum est eum finite exprimi non posse, nisi angulus θ ad π habeat rationem rationalem. Ponamus ergo hanc rationem esse $\theta : \pi = \mu : \nu$, ita ut μ et ν sint numeri integri, quamobrem pro formula ante tractata statuamus $m = \mu$ et $n = \nu$, unde fiet angulus $\nu\varphi = \frac{\pi p}{n}$. Ponamus hic brevitatis gratia $\frac{p}{n} = r$, ut habeamus $\varphi = \frac{\pi r}{\nu}$, et valor, quem quaerimus, ob $p = nr$ erit

$$\frac{\pi}{\sin. \theta} \int \frac{\partial r \sin. \theta r}{\sin. \pi r};$$

quare, cum hinc fiat $\varphi = \frac{\pi r}{\nu}$, formula supra tractata $\int \frac{\partial\varphi \sin. m\varphi}{\sin. n\varphi}$ abibit in hanc

$$S = \frac{\pi}{\nu} \int \frac{\partial r \sin. \frac{\mu\pi r}{\nu}}{\sin. \pi r}$$

sicque valor, quem hic quaerimus, erit $\frac{\nu S}{\sin. \theta}$, ita ut tantum opus sit valorem ipsius S pro hoc casu evolvere.

21. Consideremus nunc primum valorem ipsius ω, qui erat $\omega = \frac{\pi}{n} = \frac{\pi}{\nu}$, qui pro S produxit partem integralem

$$-\frac{\sin. m\omega}{n \cos. n\omega} l \frac{\sin. \frac{\omega+\varphi}{2}}{\sin. \frac{\omega-\varphi}{2}};$$

erit hic $m\omega = \frac{\mu\pi}{\nu} = \theta$ et $\cos. n\omega = -1$; tum vero

$$\omega + \varphi = \frac{\pi}{\nu}(1+r) \quad \text{et} \quad \omega - \varphi = \frac{\pi}{\nu}(1-r).$$

Primum igitur hic sumi debet angulus $\frac{\pi}{2\nu}$, quem brevitatis gratia ponamus $= \varrho$, ut sit $\varrho = \frac{\pi}{2\nu}$, et prima pars nostrae formulae S erit

$$\frac{\sin. \theta}{\nu} l \frac{\sin. \varrho(1+r)}{\sin. \varrho(1-r)},$$

sequentes autem partes erunt

$$-\frac{\sin. 2\theta}{\nu}\, l\frac{\sin.\varrho(2+r)}{\sin.\varrho(2-r)} + \frac{\sin. 3\theta}{\nu}\, l\frac{\sin.\varrho(3+r)}{\sin.\varrho(3-r)} - \text{etc.},$$

quae partes ductae in $\frac{\nu}{\sin.\theta}$ praebent ipsum valorem, quem nostrum problema postulat, qui ergo erit

$$\frac{\sin.\theta}{\sin.\theta}\, l\frac{\sin.\varrho(1+r)}{\sin.\varrho(1-r)} - \frac{\sin. 2\theta}{\sin.\theta}\, l\frac{\sin.\varrho(2+r)}{\sin.\varrho(2-r)}$$
$$+\frac{\sin. 3\theta}{\sin.\theta}\, l\frac{\sin.\varrho(3+r)}{\sin.\varrho(3-r)} - \frac{\sin. 4\theta}{\sin.\theta}\, l\frac{\sin.\varrho(4+r)}{\sin.\varrho(4-r)}$$
$$\text{etc.};$$

quae membra eousque continuari debent, donec eorum numerus fiat $\nu - 1$, ubi pro nostro problemate tantum notetur esse $r = \frac{p}{n}$ et $\varrho = \frac{\pi}{2\nu}$ existente $\theta : \pi = \mu : \nu$ sive $\theta = \frac{\mu\pi}{\nu}$, ita ut μ sit numerus integer. Cum igitur in formula proposita exponens p necessario minor sit quam n, erit r unitate minor ideoque omnes istae formulae finitae.

22. Forma igitur generalis omnium partium, ex quibus hoc integrale constat, est

$$\pm\frac{\sin. i\theta}{\sin.\theta}\, l\frac{\sin.\varrho(i+r)}{\sin.\varrho(i-r)},$$

ubi signum superius $+$ valet, quoties i fuerit numerus impar, inferius vero $-$, si par. Pro ultima igitur harum partium erit $i = \nu - 1$. Ubi probe notetur, si sumeremus $i = \nu$, partem hinc resultantem sponte esse evanituram, propterea quod $i\varrho = \nu\varrho = \frac{\pi}{2}$ ideoque ambo sinus post logarithmum inter se aequales, ita ut perinde sit, sive membrorum numerus statuatur $= \nu - 1$ sive $= \nu$.

23. Consideremus nunc ultimum membrum nostri valoris integralis sumendo $i = \nu - 1$, unde fiet $\sin.(\nu-1)\theta = \sin.(\mu\pi - \theta)$, qui erit $= \sin.\theta$, si μ fuerit numerus impar; sin autem μ fuerit numerus par, is erit $= -\sin.\theta$. Tum vero erit

$$i\varrho = (\nu - 1)\varrho = \frac{\pi}{2} - \frac{\pi}{2\nu}$$

ideoque

$$\sin.\varrho(\nu - 1 + r) = \sin.\left(\frac{\pi}{2} - \varrho(1-r)\right) = \cos.\varrho(1-r).$$

Simili modo pro denominatore erit

$$\sin.\varrho(i - r) = \sin.\left(\frac{\pi}{2} - \varrho(1+r)\right) = \cos.\varrho(1+r),$$

ita ut in ultimo membro cosinus eorundem angulorum occurrant, quorum sinus occurrunt in primo membro, quae permutatio etiam reperietur in membro penultimo et secundo, tum vero etiam in antepenultimo et tertio, unde bina huiusmodi membra in unum coniungi poterunt.

24. Hic autem quatuor casus examinari convenit, prouti ambo numeri μ et ν fuerint numeri vel pares vel impares.

Sint igitur primo ambo pares, unde coefficiens ultimi membri erit $\frac{\sin.(\mu\pi - \theta)}{\sin.\theta} = -\frac{\sin.\theta}{\sin.\theta}$ ideoque totum membrum ultimum $= -\frac{\sin.\theta}{\sin.\theta}\, l\frac{\cos.\varrho(1-r)}{\cos.\varrho(1+r)}$, quamobrem primum membrum cum ultimo coniunctum dabit

$$\frac{\sin.\theta}{\sin.\theta}\, l\frac{\sin.\varrho(1+r)}{\sin.\varrho(1-r)}\cdot\frac{\cos.\varrho(1+r)}{\cos.\varrho(1-r)} = \frac{\sin.\theta}{\sin.\theta}\, l\frac{\sin. 2\varrho(1+r)}{\sin. 2\varrho(1-r)}.$$

Simili modo secundum membrum et penultimum coalescent in

$$-\frac{\sin. 2\theta}{\sin.\theta}\, l\frac{\sin. 2\varrho(2+r)}{\sin. 2\varrho(2-r)},$$

tum vero etiam membrum tertium cum antepenultimo dabit

$$\frac{\sin. 3\theta}{\sin.\theta}\, l\frac{\sin. 2\varrho(3+r)}{\sin. 2\varrho(3-r)}.$$

Sicque de ceteris, ita ut hoc modo numerus membrorum ad semissem reducatur.

25. Maneat nunc ν numerus par, sit vero μ numerus impar eritque coefficiens ultimi membri $\frac{\sin.\theta}{\sin.\theta}$, quod ergo cum primo coniunctum dabit

$$\frac{\sin.\theta}{\sin.\theta}\, l\frac{\sin.\varrho(1+r)}{\sin.\varrho(1-r)}\cdot\frac{\cos.\varrho(1-r)}{\cos.\varrho(1+r)} = \frac{\sin.\theta}{\sin.\theta}\, l\frac{\text{tang.}\varrho(1+r)}{\text{tang.}\varrho(1-r)}.$$

Eodem modo membrum secundum cum penultimo contrahetur in hanc formam

$$-\frac{\sin. 2\theta}{\sin. \theta} l \frac{\text{tang.}\, \varrho(2+r)}{\text{tang.}\, \varrho(2-r)};$$

at tertium membrum cum antepenultimo coniunctum dabit

$$\frac{\sin. 3\theta}{\sin. \theta} l \frac{\text{tang.}\, \varrho(3+r)}{\text{tang.}\, \varrho(3-r)}.$$

26. Sit nunc ν numerus impar, at μ numerus par et ob priorem conditionem coefficiens ultimi termini erit $-\frac{\sin. (\mu\pi-\theta)}{\sin. \theta}$, qui ob μ numerum parem fiet $\frac{\sin. \theta}{\sin. \theta}$ ideoque uti in casu secundo, unde etiam primum membrum cum ultimo iunctum dabit

$$\frac{\sin. \theta}{\sin. \theta} l \frac{\text{tang.}\, \varrho(1+r)}{\text{tang.}\, \varrho(1-r)};$$

secundum vero cum penultimo iunctum

$$-\frac{\sin. 2\theta}{\sin. \theta} l \frac{\text{tang.}\, \varrho(2+r)}{\text{tang.}\, \varrho(2-r)};$$

tum vero etiam tertium cum antepenultimo iunctum dat

$$\frac{\sin. 3\theta}{\sin. \theta} l \frac{\text{tang.}\, \varrho(3+r)}{\text{tang.}\, \varrho(3-r)}.$$

27. Sint denique ambo numeri μ et ν impares atque evidens est hunc casum ad primum esse rediturum ideoque primum et ultimum membrum contrahi in

$$\frac{\sin. \theta}{\sin. \theta} l \frac{\sin. 2\varrho(1+r)}{\sin. 2\varrho(1-r)},$$

secundum et penultimum in

$$-\frac{\sin. 2\theta}{\sin. \theta} l \frac{\sin. 2\varrho(2+r)}{\sin. 2\varrho(2-r)},$$

tertium et antepenultimum in

$$\frac{\sin. 3\theta}{\sin. \theta} l \frac{\sin. 2\varrho(3+r)}{\sin. 2\varrho(3-r)}.$$

Unde patet hos quatuor casus ad duos reduci posse, prouti ambo numeri μ et ν fuerint vel eiusdem indolis, scilicet ambo vel pares vel impares, vel diversae indolis, alter par, alter impar. Priore casu eadem contractio locum habebit, quam casu primo dedimus, posteriore vero, quam pro secundo dedimus.

28. Ex his intelligitur, si numerus ν fuerit impar ideoque numerus membrorum primum inventorum $\nu-1$ par, tum omnia illa membra contrahi in numerum duplo minorem, scilicet $\frac{\nu-1}{2}$. At vero si ν fuerit numerus par, ob $\nu-1$ imparem facta illa contractione remanebit unum membrum medium respondens valori $i=\frac{\nu}{2}$, pro quo iste reperietur logarithmus

$$l \frac{\sin. \varrho\left(\frac{\nu}{2}+r\right)}{\sin. \varrho\left(\frac{\nu}{2}-r\right)} = l \frac{\sin. \left(\frac{\pi}{4}+\varrho r\right)}{\sin. \left(\frac{\pi}{4}-\varrho r\right)}.$$

Quia igitur est $\sin. \left(\frac{\pi}{4}-\varrho r\right) = \cos. \left(\frac{\pi}{4}+\varrho r\right)$, evidens est hoc casu haberi

$$l\, \text{tang.} \left(\frac{\pi}{4}+\varrho r\right);$$

coefficiens autem erit

$$\pm \frac{\sin. \frac{\nu}{2}\theta}{\sin. \theta},$$

ubi signum superius valebit, si $\frac{\nu}{2}$ fuerit impar, inferius vero, si par. Est vero $\sin. \frac{\nu}{2}\theta = \sin. \frac{\mu\pi}{2}$, unde patet, si fuerit μ numerus par, hoc membrum penitus e medio tolli; sin autem μ fuerit numerus impar, tum $\sin. \frac{\mu\pi}{2}$ erit vel $+1$ vel -1. Ista ambiguitas autem iam ante est sublata.

His notatis sequentia exempla simpliciora percurramus; ubi notasse iuvabit numerum μ semper minorem esse debere quam ν neque tamen sumi posse $\mu = 0$.

29. Quo autem evolutionem casuum specialium faciliorem reddamus, denotet Σ formulam illam integralem, cuius valorem hactenus per partes evolvimus, ita ut sit

$$\Sigma = \frac{\pi}{\sin. \theta} \int \frac{\partial r \sin. \theta r}{\sin. \pi r};$$

tum igitur duos casus distingui conveniet, prouti ambo numeri μ et ν fuerint eiusdem vel diversae indolis.

I. Sint μ et ν eiusdem indolis eritque

$$\Sigma = \frac{\sin.\theta}{\sin.\theta}\, l\frac{\sin.2\varrho(1+r)}{\sin.2\varrho(1-r)} - \frac{\sin.2\theta}{\sin.\theta}\, l\frac{\sin.2\varrho(2+r)}{\sin.2\varrho(2-r)}$$
$$+ \frac{\sin.3\theta}{\sin.\theta}\, l\frac{\sin.2\varrho(3+r)}{\sin.2\varrho(3-r)} - \frac{\sin.4\theta}{\sin.\theta}\, l\frac{\sin.2\varrho(4+r)}{\sin.2\varrho(4-r)}$$
etc.,

quas formulas non ultra multitudinem $\frac{\nu-1}{2}$ continuari necesse est; neque enim hic terminus medius locum habet; si enim fuerit ν numerus par, erit etiam μ par ideoque termini medii coefficiens evanescit.

II. Sint numeri μ et ν diversae indolis vidimusque fore

$$\Sigma = \frac{\sin.\theta}{\sin.\theta}\, l\frac{\text{tang.}\,\varrho(1+r)}{\text{tang.}\,\varrho(1-r)} - \frac{\sin.2\theta}{\sin.\theta}\, l\frac{\text{tang.}\,\varrho(2+r)}{\text{tang.}\,\varrho(2-r)}$$
$$+ \frac{\sin.3\theta}{\sin.\theta}\, l\frac{\text{tang.}\,\varrho(3+r)}{\text{tang.}\,\varrho(3-r)} - \frac{\sin.4\theta}{\sin.\theta}\, l\frac{\text{tang.}\,\varrho(4+r)}{\text{tang.}\,\varrho(4-r)}$$
etc.,

quos terminos non ultra multitudinem $\frac{\nu-1}{2}$ continuari oportet. Hic autem, quoties ν numerus par ideoque μ impar, occurret terminus medius, qui nunc ultimum locum occupabit eritque

$$\pm \frac{1}{\sin.\theta}\, l\,\text{tang.}\left(\frac{\pi}{4} + \varrho r\right),$$

ubi signorum ambiguitas sequitur alternationem signorum. Ceterum hic ubique recordandum est esse $\varrho = \frac{\pi}{2\nu}$ et $\theta = \frac{\mu\pi}{\nu}$.

EXEMPLUM 1 QUO $\nu = 2$

30. Hic igitur erit $\varrho = \frac{\pi}{4} = 45^\circ$, at numerus μ necessario est $= 1$. Quia igitur $\frac{\nu-1}{2} = \frac{1}{2}$, hic solus terminus, quem medium vocamus, occurrit, ita ut nunc habeamus

$$\Sigma = l\,\text{tang.}\frac{\pi}{4}(1+r) = l\,\text{tang.}\,45^\circ(1+r),$$

qui valor sponte ex forma generali deducitur, cum sit

$$\Sigma = \pi\int\frac{\partial r\,\sin.\frac{\pi r}{2}}{\sin.\pi r};$$

est vero $\sin.\pi r = 2\sin.\frac{\pi r}{2}\cos.\frac{\pi r}{2}$, unde fit

$$\Sigma = \frac{\pi}{2}\int\frac{\partial r}{\cos.\frac{\pi r}{2}}.$$

Quodsi iam ponamus $\frac{\pi r}{2} = \varphi$, ob $\frac{\pi\partial r}{2} = \partial\varphi$ erit

$$\Sigma = \int\frac{\partial\varphi}{\cos.\varphi} = l\,\text{tang.}\left(45^\circ + \frac{1}{2}\varphi\right).$$

Restituto ergo pro φ valore assumto erit $\Sigma = l\,\text{tang.}\,45^\circ(1+r)$, uti invenimus.

EXEMPLUM 2 QUO $\nu = 3$

31. Hic ergo erit $\varrho = \frac{\pi}{6} = 30^\circ$, et quia $\frac{\nu-1}{2} = 1$, integrale nostrum unico constabit termino. Nunc autem numerus μ duos valores habere potest, 1 et 2. Sit primo $\mu = 1$ hincque $\theta = \frac{\pi}{3} = 60^\circ$, et quia ambo numeri sunt impares, ex casu primo colligimus

$$\Sigma = l\frac{\sin.60^\circ(1+r)}{\sin.60^\circ(1-r)}.$$

At si fuerit $\mu = 2$ ideoque $\theta = 120^\circ$, quia numeri μ et ν habent disparia signa, ex casu secundo habebimus

$$\Sigma = l\frac{\text{tang.}\,30^\circ(1+r)}{\text{tang.}\,30^\circ(1-r)}.$$

EXEMPLUM 3 QUO $\nu = 4$

32. Hic ergo erit $\varrho = \frac{\pi}{8} = 22\frac{1}{2}^\circ$, et quia $\frac{\nu-1}{2} = 1\frac{1}{2}$, integrale unico tantum membro integro constabit, nisi forte terminus medius accedat, quemadmodum singulis casibus pro μ assumtis videbimus.

1°. Sit igitur $\mu = 1$; erit $\theta = 45^\circ$ et $2\theta = 90^\circ$. Hinc ergo ob numeros μ et ν dispares ex casu secundo habebimus

$$\Sigma = l\frac{\text{tang.}\,22\frac{1}{2}^\circ(1+r)}{\text{tang.}\,22\frac{1}{2}^\circ(1-r)} - \sqrt{2}\cdot l\,\text{tang.}\,22\tfrac{1}{2}^\circ(2+r).$$

2°. Sit $\mu = 2$ eritque $\theta = 90°$ et $2\theta = 180°$. Hinc ex casu primo nanciscimur

$$\Sigma = l\,\frac{\sin.45°(1+r)}{\sin.45°(1-r)}.$$

Cum autem sit $\sin.45°(1-r) = \cos.45°(1+r)$, evidens est fore

$$\Sigma = l\,\mathrm{tang}.\,45°(1+r),$$

quia casus utique convenit cum ratione $\mu:\nu = 1:2$.

3°. At si $\mu = 3$ ideoque $\theta = 135°$ et $2\theta = 270°$, cuius anguli sinus est -1, ob signa disparia habebimus ex casu secundo

$$\Sigma = l\,\frac{\mathrm{tang}.\,22\tfrac{1}{2}°(1+r)}{\mathrm{tang}.\,22\tfrac{1}{2}°(1-r)} + \sqrt{2}\cdot l\,\mathrm{tang}.\,22\tfrac{1}{2}°(2+r).$$

EXEMPLUM 4 QUO $\nu = 5$

33. Hic ergo erit $\varrho = 18°$, et quia $\frac{\nu-1}{2} = 2$, integralia ex duobus membris integris constabunt, quia terminus medius, quem quasi dimidium spectamus, hic non occurrit.

1°. Sit $\mu = 1$ eritque $\theta = 36°$ et $2\theta = 72°$; hinc ob ambo signa eadem casus primus nobis dat

$$\Sigma = l\,\frac{\sin.36°(1+r)}{\sin.36°(1-r)} - \frac{\sin.72°}{\sin.36°}\,l\,\frac{\sin.36°(2+r)}{\sin.36°(2-r)}.$$

2°. Sit $\mu = 2$ eritque $\theta = 72°$ ideoque $\sin.2\theta = \sin.36°$; unde ob signa disparia casus secundus dat

$$\Sigma = l\,\frac{\mathrm{tang}.\,18°(1+r)}{\mathrm{tang}.\,18°(1-r)} - \frac{\sin.36°}{\sin.72°}\,l\,\frac{\mathrm{tang}.\,18°(2+r)}{\mathrm{tang}.\,18°(2-r)}.$$

3°. Sit $\mu = 3$ ideoque $\theta = 108°$ sive $\sin.\theta = \sin.72°$ et $\sin.2\theta = -\sin.36°$; unde ob signa paria casus primus dat

$$\Sigma = l\,\frac{\sin.36°(1+r)}{\sin.36°(1-r)} + \frac{\sin.36°}{\sin.72°}\,l\,\frac{\sin.36°(2+r)}{\sin.36°(2-r)}.$$

4°. Sit denique $\mu = 4$ et $\theta = 144°$ hincque $\sin.\theta = \sin.36°$ et $\sin.2\theta = -\sin.72°$; unde ob signa disparia casus secundus praebet

$$\Sigma = l\,\frac{\mathrm{tang}.\,18°(1+r)}{\mathrm{tang}.\,18°(1-r)} + \frac{\sin.72°}{\sin.36°}\,l\,\frac{\mathrm{tang}.\,18°(2+r)}{\mathrm{tang}.\,18°(2-r)}.$$

EXEMPLUM 5 QUO $\nu = 6$

34. Hic igitur est $\varrho = \frac{\pi}{12} = 15°$, et quia $\frac{\nu-1}{2} = \frac{5}{2}$, integralia duobus membris integris constabunt, quibus accedere potest terminus medius sive membrum dimidium, quando scilicet μ est numerus impar.

1°. Sit $\mu = 1$; erit $\theta = \frac{\pi}{6} = 30°$, hinc $\sin.\theta = \frac{1}{2}$, $\sin.2\theta = \frac{\sqrt{3}}{2}$ et $\sin.3\theta = 1$; quare ob signa disparia secundus casus nobis suppeditat

$$\Sigma = l\,\frac{\mathrm{tang}.\,15°(1+r)}{\mathrm{tang}.\,15°(1-r)} - \sqrt{3}\cdot l\,\frac{\mathrm{tang}.\,15°(2+r)}{\mathrm{tang}.\,15°(2-r)} + 2l\,\mathrm{tang}.\,15°(3+r).$$

2°. Sit $\mu = 2$ ideoque $\theta = 60°$, unde fit $\sin.\theta = \frac{\sqrt{3}}{2}$, $\sin.2\theta = \frac{\sqrt{3}}{2}$ et $\sin.3\theta = 0$; unde ob signa paria ex casu primo colligimus

$$\Sigma = l\,\frac{\sin.30°(1+r)}{\sin.30°(1-r)} - l\,\frac{\sin.30°(2+r)}{\sin.30°(2-r)},$$

quae expressio perfecte aequalis prodiit ei, quam supra invenimus pro casu $\nu = 3$ et $\mu = 1$.

3°. Sit $\mu = 3$ ideoque $\theta = 90°$, hinc $\sin.\theta = 1$, $\sin.2\theta = 0$ et $\sin.3\theta = -1$; unde ob signa disparia casus secundus nobis praebet

$$\Sigma = l\,\frac{\mathrm{tang}.\,15°(1+r)}{\mathrm{tang}.\,15°(1-r)} + * - l\,\mathrm{tang}.\,15°(3+r)$$

sive

$$\Sigma = l\,\frac{\mathrm{tang}.\,15°(1+r)}{\mathrm{tang}.\,15°(1-r)\,\mathrm{tang}.\,15°(3+r)},$$

quae expressio aequalis esse debet ei, quae in primo exemplo prodiit, quia utroque casu est $\mu:\nu = 1:2$.

4°. Sit $\mu = 4$ ideoque $\theta = 120°$, hinc $\sin.\theta = \frac{\sqrt{3}}{2}$, $\sin.2\theta = -\frac{\sqrt{3}}{2}$, $\sin.3\theta = 0$; unde ob signa paria casus primus praebet

$$\Sigma = l\,\frac{\sin.30^0(1+r)}{\sin.30^0(1-r)} + l\,\frac{\sin.30^0(2+r)}{\sin.30^0(2-r)},$$

quae convenire debet cum superiore pro casu, quo $\mu : \nu = 2:3$.

5°. Sit $\mu = 5$ ideoque $\theta = 150°$, ergo $\sin.\theta = \frac{1}{2}$, $\sin.2\theta = -\frac{\sqrt{3}}{2}$, $\sin.3\theta = 1$; unde ob signa disparia secundus casus nobis dat

$$\Sigma = l\,\frac{\text{tang.}\,15^0(1+r)}{\text{tang.}\,15^0(1-r)} + \sqrt{3}\cdot l\,\frac{\text{tang.}\,15^0(2+r)}{\text{tang.}\,15^0(2-r)} + 2\,l\,\text{tang.}\,15^0\,(3+r).$$

EXEMPLUM 6 QUO $\nu = \infty$

35. Quia igitur fractio $\frac{\mu}{\nu}$ ut evanescens spectatur, ponamus $\mu = 1$ sicque angulus θr prae πr evanescet; unde cum loco sinuum angulorum θ et θr ipsos angulos ponere liceat, erit noster valor

$$\Sigma = \pi \int \frac{r\partial r}{\sin.\pi r}.$$

Deinde quia etiam angulus $\varrho = \frac{\pi}{2\nu}$ in nihilum abit, loco omnium sinuum in expressione pro Σ inventa occurrentium ipsos angulos scribere licebit, quo observato valor quantitatis Σ sequenti modo exprimetur

$$l\,\frac{1+r}{1-r} - 2\,l\,\frac{2+r}{2-r} + 3\,l\,\frac{3+r}{3-r} - 4\,l\,\frac{4+r}{4-r} + \text{etc.}\ ^{1)}$$

1) Haec quidem series divergens est, sed formula ab EULERO in § 37 data hoc modo demonstrari potest.

Ponendo $\mu = 1$, $\nu = 2n+1$, ita ut sit

$$\theta = 2\varrho = \frac{\pi}{2n+1},$$

ex casu primo nanciscimur

$$\Sigma = \frac{\pi}{\sin.\theta}\int_0^r \frac{\sin.\theta r\,dr}{\sin.\pi r} = \sum_{j=1}^{j=n}(-1)^{j-1}\frac{\sin.j\theta}{\sin.\theta}\log.\frac{\sin.(j+r)\theta}{\sin.(j-r)\theta},$$

quae expressio, si ponamus

$$\frac{\text{tang.}\,r\theta}{\text{tang.}\,j\theta} = x_j,$$

36. Singuli hi logarithmi commode in series resolvi possunt. Cum enim forma generalis omnium terminorum sit $i\,l\frac{i+r}{i-r}$, tum vero per notam resolutionem sit

$$l\,\frac{i+r}{i-r} = \frac{2r}{i} + \frac{2r^3}{3i^3} + \frac{2r^5}{5i^5} + \frac{2r^7}{7i^7} + \text{etc.},$$

erit totum membrum

$$= 2r\left(1 + \frac{rr}{3ii} + \frac{r^4}{5i^4} + \frac{r^6}{7i^6} + \text{etc.}\right);$$

quamobrem singulis partibus hoc modo evolutis fiet

hanc induet formam

$$\Sigma = \sum_{j=1}^{j=n}(-1)^{j-1}\frac{\sin.j\theta}{\sin.\theta}\log.\frac{1+x_j}{1-x_j}.$$

Cum autem sit $r<1$ et $n\theta < \frac{\pi}{2}$, erit x_j pro $j = 1, 2, \ldots n$ numerus positivus unitate minor, quamobrem hic scribere licet

$$\log.\frac{1+x_j}{1-x_j} = 2x_j\left(1 + \frac{1}{3}x_j^2 + \frac{1}{5}x_j^4 + \frac{1}{7}x_j^6 + \cdots\right).$$

Hoc modo ponendo

$$\sum_{j=1}^{j=n}(-1)^{j-1}x_j^{2i}\cos.j\theta = S_i$$

habebimus

$$\Sigma = 2\,\frac{\text{tang.}\,r\theta}{\sin.\theta}\left(S_0 + \frac{1}{3}S_1 + \frac{1}{5}S_2 + \frac{1}{7}S_3 + \cdots\right),$$

ubi, cum sit

$$x_{j+1} < x_j, \quad \cos.(j+1)\theta < \cos.j\theta, \quad x_1 < r,$$

erit

$$S_i > 0 \quad \text{et} \quad S_i < r^{2i}.$$

Ex his perspicitur numero n ad infinitum crescente fore

$$\lim \Sigma = \pi\int_0^r \frac{r\,dr}{\sin.\pi r} = 2r\left(\lim S_0 + \frac{1}{3}\lim S_1 + \frac{1}{5}\lim S_2 + \cdots\right).$$

Quae est EULERI formula, quia pro $i>0$ manifestum est fore

$$\lim S_i = r^{2i}\left(1 - \frac{1}{2^{2i}} + \frac{1}{3^{2i}} - \frac{1}{4^{2i}} + \cdots\right)$$

et ponendo $i = 0$ habemus

$$S_0 = \frac{1}{2} + \frac{(-1)^{n-1}\cos.\left(n+\frac{1}{2}\right)\theta}{2\cos.\frac{\theta}{2}}$$

sive, cum sit $\theta = \frac{\pi}{2n+1}$,

$$S_0 = \frac{1}{2}.$$

A. L

$$\begin{aligned}\frac{\Sigma}{2r} = &+1 + \frac{rr}{3} + \frac{r^4}{5} + \frac{r^6}{7} + \frac{r^8}{9} + \text{etc.}\\ &-1 - \frac{rr}{3\cdot 4} - \frac{r^4}{5\cdot 4^2} - \frac{r^6}{7\cdot 4^3} - \frac{r^8}{9\cdot 4^4} - \text{etc.}\\ &+1 + \frac{rr}{3\cdot 9} + \frac{r^4}{5\cdot 9^2} + \frac{r^6}{7\cdot 9^3} + \frac{r^8}{9\cdot 9^4} + \text{etc.}\\ &-1 - \frac{rr}{3\cdot 16} - \frac{r^4}{5\cdot 16^2} - \frac{r^6}{7\cdot 16^3} - \frac{r^8}{9\cdot 16^4} - \text{etc.}\\ &\qquad\qquad \text{etc.}\end{aligned}$$

37. Quodsi iam istas series secundum columnas verticales disponamus, quia prima columna dat

$$1 - 1 + 1 - 1 + 1 - 1 + \text{etc.} = \frac{1}{2},$$

prodibit haec expressio

$$\begin{aligned}\frac{\Sigma}{2r} = \frac{1}{2} &+ \frac{1}{3}rr\left(1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \text{etc.}\right)\\ &+ \frac{1}{5}r^4\left(1 - \frac{1}{4^2} + \frac{1}{9^2} - \frac{1}{16^2} + \frac{1}{25^2} - \text{etc.}\right)\\ &+ \frac{1}{7}r^6\left(1 - \frac{1}{4^3} + \frac{1}{9^3} - \frac{1}{16^3} + \frac{1}{25^3} - \text{etc.}\right)\\ &\qquad\qquad \text{etc.}\end{aligned}$$

Quoniam igitur harum serierum omnium summae sunt cognitae, hinc per approximationem eo facilius valor litterae Σ definiri poterit, quia littera r semper denotat fractionem unitate minorem.

38. Quodsi ergo in subsidium vocemus ea, quae olim[1]) circa summas harum potestatum erueram, atque iisdem denominationibus utamur ponendo

1) Vide EULERI Commentationes 41 et 61 (indicis ENESTROEMIANI): *De summis serierum reciprocarum*, Comment. acad. sc. Petrop. 7 (1734/5), 1740, p. 123, et *De summis serierum reciprocarum ex potestatibus numerorum naturalium ortarum dissertatio altera: in qua eaedem summationes ex fonte maxime diverso derivantur*, Miscellanea Berolin. 7, 1743, p. 172; *LEONHARDI EULERI Opera omnia*, series I, vol. 14. Vide etiam N. BERNOULLI, *Inquisitio in summam seriei*

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} \text{ etc.},$$

$$\begin{aligned}A\pi^2 &= 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \text{etc.},\\ B\pi^4 &= 1 + \frac{1}{4^2} + \frac{1}{9^2} + \frac{1}{16^2} + \frac{1}{25^2} + \text{etc.},\\ C\pi^6 &= 1 + \frac{1}{4^3} + \frac{1}{9^3} + \frac{1}{16^3} + \frac{1}{25^3} + \text{etc.}\\ &\qquad \text{etc.},\end{aligned}$$

quoniam hinc facile summae derivantur, quando terminorum signa alternantur, habebitur

$$\frac{\Sigma}{2r} = \frac{1}{2} + \frac{1}{3}\left(1 - \frac{1}{2}\right)A\pi\pi rr + \frac{1}{5}\left(1 - \frac{1}{8}\right)B\pi^4 r^4 + \frac{1}{7}\left(1 - \frac{1}{32}\right)C\pi^6 r^6 + \text{etc.}$$

Ubi meminisse convenit esse

$$A = \frac{1}{6},\quad B = \frac{1}{90},\quad C = \frac{1}{945},\quad D = \frac{1}{9450},\quad E = \frac{1}{93555}\quad \text{etc.}$$

Horum autem valorum ratio iam saepius abunde est exposita.

Comment. acad. sc. Petrop. **10** (1738), 1747, p. 19, nec non epistolas a N. BERNOULLI d. 13. Iulii et 24. Octobris 1742 ad EULERUM scriptas, *Correspondance math. et phys. publiée par P. H. Fuss*, St.-Pétersbourg 1843, t. II, p. 681 et 690; *LEONHARDI EULERI Opera omnia*, series III. Vide praeterea EULERI *Introductionem in analysin infinitorum*, Lausannae 1748, t. I cap. X; *LEONHARDI EULERI Opera omnia*, series I, vol. 8. A. L.

EVOLUTIO FORMULAE INTEGRALIS

$$\int \partial x \left(\frac{1}{1-x} + \frac{1}{lx} \right)$$

A TERMINO $x=0$ USQUE AD $x=1$ EXTENSAE

Conventui exhibita die 29. Februarii 1776

Commentatio 629 indicis ENESTROEMIANI
Nova acta academiae scientiarum Petropolitanae 4 (1786), 1789, p. 3—16
Summarium ibidem p. 109—110

SUMMARIUM

Dans un mémoire intitulé *Observationes de progressionibus harmonicis*[1]), qui se trouve dans le 7e volume des Commentaires de l'Académie pour les années 1734 et 1735, feu M. EULER avoit déjà fait des recherches sur la somme de la progression harmonique $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{i}$, et il l'avoit trouvée $= l(i+1) + C$, i étant un nombre infiniment grand et $C = 0{,}57721$ une constante introduite par l'intégration, et qui exprime par conséquent la différence entre la somme de la série $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{\infty}$ et $l\infty$.

M. EULER a poussé plus loin cette recherche dans la suite, dans son ouvrage *Institutiones Calculi Differentialis*[2]), où l'on trouve, dans le 6e chapitre de la seconde section, qui traite *De summatione progressionum per series infinitas*, le calcul de la constante C poussé jusqu'à 16 décimales, savoir $C = 0{,}5772156649015325$.[3]) (*Inst. Calc. Diff.*)

1) Voir la note 2 p. 319. A. L.

2) L. EULERI *Institutiones calculi differentialis*, Petropoli 1755; *LEONHARDI EULERI Opera omnia*, series I, vol. 10, p. 339. A. L.

3) Voir la note 3 p. 319. A. L

Ce nombre C, trouvé par approximation, avoit paru à M. EULER assez remarquable pour l'engager à en faire le sujet d'un mémoire plus récent[1]), inséré dans le second volume des Actes de l'Académie, pour l'année 1781, où notre Géomètre a tâché de réduire ce nombre à une expression connue. Il y a donné pour cet effet plusieurs séries assez convergentes et régulières dont ce nombre C est la somme, et entre autres il y a démontré que C est la valeur de l'intégrale $\int \partial x \left(\frac{1}{1-x} + \frac{1}{lx} \right)$, prise depuis $x = 0$ jusqu'à $x = 1$. Or n'ayant pu réussir, dans le mémoire mentionné, à réduire ce nombre C à une quantité transcendante déjà connue, son but est d'examiner ici, si la résolution de cette formule intégrale ne mènera pas à quelque résultat plus satisfaisant.

Il essaye premièrement la voye des Quadratures, en considérant une ligne courbe dont l'abscisse est x et l'ordonnée $y = \frac{1}{1-x} + \frac{1}{lx}$. Il en examine la figure singulière, ce qui, quoiqu'il ne contribue rien à la connoissance plus parfaite du nombre C représenté par un espace déterminé de cette courbe, n'est pas pourtant destitué de tout intérêt.

L'Auteur essaye, après cet examen, plusieurs transformations en séries, qui, quoique très-instructives et neuves en partie, conduisent à des séries ni de forme connue ni assez convergentes pour qu'on pût en tirer quelque avantage pour la connoissance plus parfaite du nombre en question.

1. Ista formula integralis eo magis est notatu digna, quod eius valorem ostendi convenire cum eo, quem praebet ista expressio

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - ln,$$

si numerus n sumatur infinite magnus, et quem per approximationem olim[2]) inveni esse $= 0{,}5772156649015325$[3]), cuius valorem nullo adhuc modo ad

1) Voir la note p. 326. A. L.

2) Vide L. EULERI Commentationem 43 (indicis ENESTROEMIANI): *De progressionibus harmonicis observationes*, Commment. acad. sc. Petrop. 7 (1734/5), 1740, p. 150; *LEONHARDI EULERI Opera omnia*, series I, vol. 14. Vide porro L. EULERI *Institutionum calculi integralis* vol. 1, Petropoli 1768, sectio 1, cap. IV; *LEONHARDI EULERI Opera omnia*, series I, vol. 11, p. 122. Imprimis autem vide L. MASCHERONII *Adnotationes ad calculum integralem EULERI*, Ticini 1790/2; *LEONHARDI EULERI Opera omnia*, series I, vol. 12, p. 423 et 502. A. L.

3) Secundum calculos recentiores ultima figura decimalis 5 mutanda est in 9; vide *LEONHARDI EULERI Opera omnia*, series I, vol. 11, notam p. 339, atque imprimis vol. 12, notam 2 p. 431. A. L.

mensuras transcendentes iam cognitas redigere potui; unde haud inutile erit resolutionem huius formulae propositae pluribus modis tentare. Ac primo quidem, quoniam duabus constat partibus

$$\int \frac{\partial x}{1-x} \quad \text{et} \quad \int \frac{\partial x}{lx},$$

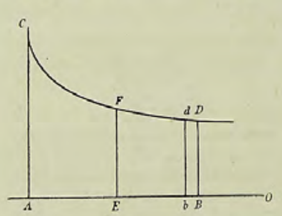
manifestum est prioris partis valorem $-l(1-x)$ posito $x=1$ fore $-l0$ ideoque $=\infty$; tum vero etiam facile perspicitur posterioris partis valorem quoque esse infinitum, sed signo contrario affectum, ita ut haud difficulter intelligatur aggregatum earum partium finitum habere valorem.

EVOLUTIO PRIMA GEOMETRICA

2. Primo igitur hanc formulam per quadraturas exhibeamus considerando lineam curvam, cuius abscissae x respondeat applicata

$$y = \frac{1}{1-x} + \frac{1}{lx};$$

tum vero eius area $\int y\partial x$ abscissae x insistens ipsum valorem quaesitum repraesentabit, quamobrem formam huius curvae accuratius perpendamus. Ac primo quidem evidens est hanc curvam neutiquam in regionem abscissarum negativarum porrigi, sed a termino $x=0$ incipere. Posito autem $x=0$ manifesto fit $y=1$ ob $lx=\infty$; at existente x infinite parvo fiet $y=1+x+\frac{1}{lx}$, ubi facile perspicitur postremum membrum $\frac{1}{lx}$ esse negativum et quasi infinities maius quam x, ita ut fiat $y=1-ix$ existente i numero maximo; unde patet, si curvam ad axem AO referamus in eoque abscissas x a puncto A capiamus, in ipso puncto A applicatam fore $AC=1$ et curvam in C hanc applicatam AC tangere, propterea quod decrementum applicatae infinities superat incrementum abscissae. Curva igitur originem ducet ab ipso puncto C hincque continuo propius ad axem inflectetur, quem tandem in distantia infinita attinget. Posito enim $x=\infty$ fit



$$y = -\frac{1}{\infty} + \frac{1}{l\infty};$$

ubi notetur prius membrum $\frac{1}{\infty}$ prae altero evanescere, ita ut iste valor sit positivus, unde patet hanc curvam a puncto C ad axem continuo propius esse accessuram.

3. Consideremus nunc abscissam $AB=1$, ubi sumto $x=1$ fit $y=\frac{1}{0}+\frac{1}{0}$, unde nihil plane concludere liceret; hanc ob causam statuamus $x=1-\omega$, ut fiat $y=\frac{1}{\omega}+\frac{1}{l(1-\omega)}$. Iam $l(1-\omega)$ in seriem evolvendo fiet

$$y = \frac{1}{\omega} - \frac{1}{\omega+\frac{1}{2}\omega^2+\frac{1}{3}\omega^3+\text{etc.}} = \frac{\frac{1}{2}+\frac{1}{3}\omega}{1+\frac{1}{2}\omega+\frac{1}{3}\omega^2}.$$

Fiat nunc $\omega=0$ ac manifestum est applicatam in puncto B fore $BD=\frac{1}{2}$, cum esset $AC=1$ et $AB=1$. Hinc simul patet sumto ω minimo, scilicet $Bb=\omega$, fore applicatam in hoc puncto

$$bd = \frac{\frac{1}{2}+\frac{1}{3}\omega}{1+\frac{1}{2}\omega} = \frac{1}{2}+\frac{1}{12}\omega$$

sicque elementum curvae Dd ad axem inclinatur sub angulo, cuius tangens est $\frac{1}{12}$, qui est propemodum $4°46'$.

4. Sumamus nunc abscissam $AE=\frac{1}{2}$ eique respondebit applicata $EF=2-\frac{1}{l2}=0{,}557$ propemodum atque hinc iam proxime aream $ABCD$ colligere licet. Namque si CF esset linea recta, foret area $ACFE=0{,}389$; quia autem versus axem incurvatur, haec area erit aliquanto minor. Pro altera parte, quia FD minus incurvatur, erit area $BDFE$ aliquantillum minor quam $\frac{1}{2}BE(BD+EF)=0{,}264$ propemodum, unde tota area $ACDB$ certe minor erit quam $0{,}653$, id quod iam satis convenit cum veritate, quandoquidem haec area esse debet $0{,}577$. At si abscissam $AB=1$ in plures partes dividere et areas singulis partibus respondentes indagare vellemus, earum summa eo propius ad valorem cognitum accessura foret, quo plures partes fuerint constitutae. Quia autem de vero valore huius formulae iam certi sumus, talis labor frustra susciperetur, sed hic sufficiat formam huius curvae prorsus singularis, quippe quae in puncto C subito incipit, expendisse.

EVOLUTIO SECUNDA

5. Evolvamus nunc lx in seriem, et quia est $x = 1 - (1 - x)$, erit

$$lx = -(1-x) - \frac{1}{2}(1-x)^2 - \frac{1}{3}(1-x)^3 - \frac{1}{4}(1-x)^4 - \text{etc.};$$

et quia est

$$y = \frac{1}{1-x} + \frac{1}{lx} = \frac{lx + 1 - x}{(1-x)lx},$$

istam seriem tantum in numeratore loco lx scribamus prodibitque

$$y = \frac{-\frac{1}{2}(1-x)^2 - \frac{1}{3}(1-x)^3 - \frac{1}{4}(1-x)^4 - \text{etc.}}{(1-x)lx}$$

atque hinc

$$y = \frac{-\frac{1}{2}(1-x) - \frac{1}{3}(1-x)^2 - \frac{1}{4}(1-x)^3 - \text{etc.}}{lx};$$

hinc igitur per partes integrando valor quaesitus erit

$$\int y \partial x = -\frac{1}{2}\int \frac{(1-x)\partial x}{lx} - \frac{1}{3}\int \frac{(1-x)^2 \partial x}{lx} - \frac{1}{4}\int \frac{(1-x)^3 \partial x}{lx} - \text{etc.},$$

quae formulae singulae facile ad formulam illam generalem reducuntur, qua ostendi esse

$$\int \frac{x^m - x^n}{lx} \partial x = l\frac{m+1}{n+1}.\text{*)}$$

Hinc enim statim erit

$$\int \frac{(1-x)\partial x}{lx} = l\frac{1}{2},$$

et quia est $(1-x)^2 = 1 - x - (x - xx)$, erit

$$\int \frac{(1-x)^2 \partial x}{lx} = l\frac{1}{2} - l\frac{2}{3} = l\frac{1 \cdot 3}{2^2}.$$

*) Hoc integrale duplici modo ab Ill. huius dissertationis Auctore fuit inventum in Tomo XIX Novorum Commentariorum, p. 70 et 79.[1])

1) Scilicet in Commentatione 464 (indicis ENESTROEMIANI): *Nova methodus quantitates integrales determinandi*, Novi comment. acad. sc. Petrop. 19 (1774), 1775, p. 66; *LEONHARDI EULERI Opera omnia*, series I, vol. 17, p. 421. Vide etiam Commentationem 587 huius voluminis, imprimis § 9. A. L.

Simili modo facile patebit fore

$$\int \frac{(1-x)^3 \partial x}{lx} = l\frac{1 \cdot 3^3}{2^3 \cdot 4},$$

$$\int \frac{(1-x)^4 \partial x}{lx} = l\frac{1 \cdot 3^6 \cdot 5}{2^4 \cdot 4^4},$$

$$\int \frac{(1-x)^5 \partial x}{lx} = l\frac{1 \cdot 3^{10} \cdot 5^5}{2^5 \cdot 4^{10} \cdot 6},$$

$$\int \frac{(1-x)^6 \partial x}{lx} = l\frac{1 \cdot 3^{15} \cdot 5^{15} \cdot 7}{2^6 \cdot 4^{20} \cdot 6^6}$$

etc.

6. Ex his igitur valor nostrae formulae $\int y \partial x$ per seriem logarithmicam prorsus singularem sequenti modo exprimetur

$$\int y \partial x = \frac{1}{2} l2 + \frac{1}{3} l\frac{2^2}{1 \cdot 3} + \frac{1}{4} l\frac{2^3 \cdot 4}{1 \cdot 3^3} + \frac{1}{5} l\frac{2^4 \cdot 4^4}{1 \cdot 3^6 \cdot 5} + \frac{1}{6} l\frac{2^5 \cdot 4^{10} \cdot 6}{1 \cdot 3^{10} \cdot 5^5}$$
$$+ \frac{1}{7} l\frac{2^6 \cdot 4^{20} \cdot 6^6}{1 \cdot 3^{15} \cdot 5^{15} \cdot 7} + \text{etc.}$$

Ubi probe notandum est omnes logarithmos capi debere hyperbolicos; facile autem intelligitur terminos huius seriei continuo prodire minores neque tamen hanc seriem tantopere convergere, ut ex ea valor quaesitus commode computari possit.

EVOLUTIO TERTIA

7. Utamur eadem resolutione logarithmi x in seriem infinitam ac ponamus brevitatis gratia $1 - x = t$, ut sit

$$lx = -t - \frac{1}{2}tt - \frac{1}{3}t^3 - \frac{1}{4}t^4 - \text{etc.},$$

eritque

$$\frac{1}{lx} = \frac{-1}{t(1 + \frac{1}{2}t + \frac{1}{3}tt + \frac{1}{4}t^3 + \frac{1}{5}t^4 + \text{etc.})}.$$

Iam fractionem $\frac{1}{1 + \frac{1}{2}t + \frac{1}{3}tt + \text{etc.}}$ convertamus more solito in seriem recurrentem, quae sit

$$1 + \alpha t + \beta tt + \gamma t^3 + \delta t^4 + \varepsilon t^5 + \zeta t^6 + \text{etc.},$$

ubi coefficientes α, β, γ, δ etc. ita erunt comparati, ut sit

$$\begin{aligned}
&\alpha+\frac{1}{2}=0, &&\text{hincque}\quad \alpha=-\frac{1}{2},\\
&\beta+\frac{1}{2}\alpha+\frac{1}{3}=0, &&\beta=-\frac{1}{12},\\
&\gamma+\frac{1}{2}\beta+\frac{1}{3}\alpha+\frac{1}{4}=0, &&\gamma=-\frac{1}{24},\\
&\delta+\frac{1}{2}\gamma+\frac{1}{3}\beta+\frac{1}{4}\alpha+\frac{1}{5}=0 &&\delta=-\frac{19}{720}\\
&\qquad\text{etc.} &&\qquad\text{etc.,}
\end{aligned}$$

unde hanc seriem tamquam cognitam spectare licet.

8. Hoc igitur valore substituto erit

$$\frac{1}{lx}=-\frac{1}{t}-\alpha-\beta t-\gamma tt-\delta t^3-\varepsilon t^4-\text{etc.},$$

quare, cum sit $\frac{1}{1-x}=\frac{1}{t}$, erit

$$y=-\alpha-\beta t-\gamma tt-\delta t^3-\varepsilon t^4-\text{etc.}$$

sive

$$y=-\alpha-\beta(1-x)-\gamma(1-x)^2-\delta(1-x)^3-\text{etc.}$$

Cum nunc in genere sit

$$\int\partial x(1-x)^n=C-\frac{(1-x)^{n+1}}{n+1}=\frac{1}{n+1}-\frac{(1-x)^{n+1}}{n+1},$$

posito $x=1$, quemadmodum assumsimus, erit

$$\int\partial x(1-x)^n=\frac{1}{n+1}.$$

Hinc igitur singulis integralibus collectis reperietur

$$\int y\partial x=-\frac{\alpha}{1}-\frac{\beta}{2}-\frac{\gamma}{3}-\frac{\delta}{4}-\frac{\varepsilon}{5}-\text{etc.,}\ ^{1)}$$

1) Editio princeps:

$$\int y\partial x=-\frac{\alpha}{2}-\frac{\beta}{3}-\frac{\gamma}{4}-\frac{\delta}{5}-\frac{\varepsilon}{6}-\text{etc.},$$

unde per valores ante evolutos fiet

$$\int y\partial x=\frac{1}{4}+\frac{1}{36}+\frac{1}{96}+\frac{19}{3600}+\text{etc.}$$

Correxit A. L.

unde per valores ante evolutos fiet

$$\int y\partial x=\frac{1}{2}+\frac{1}{24}+\frac{1}{72}+\frac{19}{2880}+\text{etc.},$$

quae series utique parum est convergens.

EVOLUTIO QUARTA

9. Cum habeamus $y=\frac{lx+1-x}{(1-x)lx}$, quemadmodum ante partem lx in seriem infinitam resolvimus, ita nunc vicissim ipsam quantitatem in seriem per logarithmos ipsius x procedentem evolvamus. Quia enim est $x=e^{lx}$, erit

$$x=1+lx+\frac{1}{2}(lx)^2+\frac{1}{6}(lx)^3+\frac{1}{24}(lx)^4+\text{etc.},$$

ubi loco lx brevitatis ergo scribamus u, atque hanc seriem tantum in numeratorem introducamus, ut fiat

$$y=\frac{-\frac{1}{2}uu-\frac{1}{6}u^3-\frac{1}{24}u^4-\frac{1}{120}u^5-\text{etc.}}{u(1-x)}$$

sive

$$y=\frac{-\frac{1}{2}u-\frac{1}{6}uu-\frac{1}{24}u^3-\frac{1}{120}u^4-\text{etc.}}{1-x}$$

ideoque

$$\int y\partial x=-\frac{1}{2}\int\frac{\partial x lx}{1-x}-\frac{1}{6}\int\frac{\partial x(lx)^2}{1-x}-\frac{1}{24}\int\frac{\partial x(lx)^3}{1-x}-\frac{1}{120}\int\frac{\partial x(lx)^4}{1-x}-\text{etc.}$$

10. Cum nunc in genere, sumto scilicet integrali ab $x=0$ ad $x=1$, sit

$$\int\partial x(lx)^n=\pm 1\cdot 2\cdot 3\cdot 4\cdot 5\cdots n,$$

ubi signum $+$ valet, quando n est numerus par, contra vero signum $-$, erit porro

$$\int x^{n-1}\partial x(lx)^\lambda=\pm\frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots \lambda}{n^{\lambda+1}},$$

ubi signum $+$ valet, si λ fuerit numerus par, inferius vero, si impar. Hinc igitur singulas nostras formulas per series integremus, dum loco $\frac{1}{1-x}$ seriem scribimus

$$1+x+xx+x^3+x^4+x^5+\text{etc.},$$

atque hinc primo nanciscemur

$$\int \frac{\partial x\, lx}{1-x} = -1\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\text{etc.}\right),$$

$$\int \frac{\partial x (lx)^2}{1-x} = 1\cdot 2\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\frac{1}{6^3}+\text{etc.}\right),$$

$$\int \frac{\partial x (lx)^3}{1-x} = -1\cdot 2\cdot 3\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\frac{1}{6^4}+\text{etc.}\right),$$

$$\int \frac{\partial x (lx)^4}{1-x} = 1\cdot 2\cdot 3\cdot 4\left(1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\frac{1}{5^5}+\frac{1}{6^5}+\text{etc.}\right)$$

etc.

His igitur seriebus substitutis reperiemus

$$\begin{aligned}\int y\,\partial x = &\ \frac{1}{2}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\text{etc.}\right)\\ &-\frac{1}{3}\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\text{etc.}\right)\\ &+\frac{1}{4}\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\text{etc.}\right)\\ &-\frac{1}{5}\left(1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\frac{1}{5^5}+\text{etc.}\right)\\ &\text{etc.,}\end{aligned}$$

cuius expressionis iam nuper ostendi valorem esse numerum illum memorabilem 0,5772156649015325.*)

EVOLUTIO QUINTA

11. Utamur hic eadem resolutione in seriem ipsius numeri x, sed eam alio modo adhibeamus. Scilicet cum posito $lx = u$ sit

$$x = 1 + u + \frac{1}{2}uu + \frac{1}{6}u^3 + \frac{1}{24}u^4 + \frac{1}{120}u^5 + \text{etc.},$$

*) Videatur Dissertatio *De numero memorabili in summatione progressionis harmonicae naturalis occurrente.* Acta Acad. pro Anno 1781. Pars posterior, p. 49 seq.[1]) F[uss].

1) Quae dissertatio est Commentatio 583 (indicis Enestroemiani), *Leonhardi Euleri Opera omnia*, series I, vol. 15. A. L.

erit formulae nostrae ipsa pars prior

$$\frac{1}{1-x} = \frac{-1}{u+\frac{1}{2}uu+\frac{1}{6}u^3+\frac{1}{24}u^4+\frac{1}{120}u^5+\text{etc.}} = -\frac{1}{u}\cdot\frac{1}{1+\frac{1}{2}u+\frac{1}{6}uu+\frac{1}{24}u^3+\text{etc.}}.$$

Hanc fractionem

$$\frac{1}{1+\frac{1}{2}u+\frac{1}{6}uu+\frac{1}{24}u^3+\text{etc.}}$$

more solito in seriem recurrentem convertamus, quae sit

$$1 - Au + Buu - Cu^3 + Du^4 - Eu^5 + \text{etc.},$$

eritque facta comparatione

$$\begin{array}{lll} A = \frac{1}{2}, & \text{ergo} & A = \frac{1}{2},\\ B = \frac{1}{2}A - \frac{1}{6}, & & B = \frac{1}{12},\\ C = \frac{1}{2}B - \frac{1}{6}A + \frac{1}{24}, & & C = 0,\\ D = \frac{1}{2}C - \frac{1}{6}B + \frac{1}{24}A - \frac{1}{120}, & & D = -\frac{1}{720},\\ E = \frac{1}{2}D - \frac{1}{6}C + \frac{1}{24}B - \frac{1}{120}A + \frac{1}{720} & & E = 0\\ \text{etc.,} & & \text{etc.}\end{array}$$

12. Hac igitur serie introducta et loco u restituto valore lx formula nostra $\frac{1}{1-x}+\frac{1}{lx} = y$ sequentem induet formam

$$y = -\frac{1}{u} + A - Bu + Cuu - Du^3 + \text{etc.} + \frac{1}{u}$$

sive

$$y = A - Blx + C(lx)^2 - D(lx)^3 + E(lx)^4 - \text{etc.},$$

unde, cum in genere sit

$$\int \partial x (lx)^n = \pm 1\cdot 2\cdot 3\cdot 4\cdots n,$$

postquam scilicet absoluta integratione positum fuerit $x = 1$, ubi signum superius valet, quando n est numerus par, inferius vero, si n impar, hoc observato nanciscimur valorem quaesitum $\int y \partial x$ sequenti modo expressum

$$\int y \partial x = A + 1B + 1\cdot 2C + 1\cdot 2\cdot 3D + 1\cdot 2\cdot 3\cdot 4E + 1\cdot 2\cdots 5F + \text{etc.}^{1)},$$

quae series utique parum convergit ob coefficientes litterarum A, B, C, D etc.; verum perpendendum est ipsos valores harum litterarum continuo magis decrescere, quandoquidem certum est seriei valorem esse debere 0,5772156649015325, quare operae pretium erit harum litterarum seriem accuratius evolvere.

TRANSFORMATIO FRACTIONIS

$$\frac{1}{1 + \frac{1}{2}u + \frac{1}{6}uu + \frac{1}{24}u^3 + \frac{1}{120}u^4 + \text{etc.}}$$

IN SERIEM

$$1 - Au + Buu - Cu^3 + Du^4 - Eu^5 + \text{etc.}$$

13. Designet littera s summam huius seriei eritque $s = \frac{u}{e^u - 1}$, unde fit $e^u = \frac{u+s}{s}$ hincque $u = l(u+s) - ls$, ergo differentiando erit

$$\partial u = \frac{\partial u + \partial s}{u+s} - \frac{\partial s}{s} = \frac{s\partial u - u\partial s}{s(u+s)};$$

sive statim ponatur $s = pu$, ut sit $u = l\frac{1+p}{p}$, unde fit $\partial u = -\frac{\partial p}{p(p+1)}$; quae expressio quo seriem praebeat concinniorem, statuamus $p = q - \frac{1}{2}$, ut iam sit $s = \left(q - \frac{1}{2}\right)u$; tum vero erit $\partial u = -\frac{\partial q}{qq - \frac{1}{4}}$, unde colligitur haec aequatio

$$qq - \frac{1}{4} + \frac{\partial q}{\partial u} = 0.$$

14. Ex hac igitur aequatione investigari debet series valorem ipsius q exhibens, ubi ante omnia principium huius seriei inde constitui oportet, quod

1) Haec evolutio fieri non potest, quia formula

$$\frac{u}{e^u - 1} = 1 - Au + Bu^2 - Cu^3 + \cdots$$

pro $u < -2\pi$ non valet. A. L.

posito $u = 0$ fieri debeat $s = 1$ et $q = \frac{1}{u} + \frac{1}{2}$; unde patet seriei pro q fingendae primum terminum esse debere $\frac{1}{u}$; tum vero facile perspicitur in hac serie potestates ipsius u tantum impares assumi debere. Quamobrem fingatur ista series

$$q = \frac{1}{u} + au + bu^3 + cu^5 + du^7 + eu^9 + \text{etc.}$$

eritque

$$\begin{aligned} qq = \frac{1}{uu} + 2a + 2bu^2 &+ 2cu^4 + 2du^6 + 2eu^8 + 2fu^{10} + \text{etc.},\\ &+ aa \quad + 2ab \quad + 2ac \quad + 2ad \quad + 2ae\\ &\qquad\qquad\qquad + bb \quad + 2bc \quad + 2bd\\ &\qquad\qquad\qquad\qquad\qquad\qquad + cc\\ \frac{\partial q}{\partial u} = -\frac{1}{uu} + a + 3bu^2 &+ 5cu^4 + 7du^6 + 9eu^8 + 11fu^{10} + \text{etc.};\end{aligned}$$

harum ergo serierum summa debet esse $\frac{1}{4}$, unde deducuntur sequentes determinationes

$$\begin{aligned} 3a &= \frac{1}{4}, & \text{ergo}\quad a &= \frac{1}{12},\\ 5b + aa &= 0, & b &= -\frac{aa}{5},\\ 7c + 2ab &= 0, & c &= -\frac{2ab}{7},\\ 9d + 2ac + bb &= 0, & d &= -\frac{2ac + bb}{9},\\ 11e + 2ad + 2bc &= 0, & e &= -\frac{2ad + 2bc}{11},\\ 13f + 2ae + 2bd + cc &= 0 & f &= -\frac{2ae + 2bd + cc}{13}\\ &\text{etc.}, & &\text{etc.},\end{aligned}$$

ex quibus formulis valores numerici litterarum a, b, c, d etc. computari poterunt.

15. His autem litteris a, b, c, d etc. definitis ipsa series pro s quaesita erit

$$s = 1 - \frac{1}{2}u + auu + bu^4 + cu^6 + du^8 + eu^{10} + fu^{12} + gu^{14} + \text{etc.},$$

quare, cum supra posuerimus

$$s = 1 - Au + Buu - Cu^3 + Du^4 - Eu^5 + Fu^6 - Gu^7 + Hu^8 - \text{etc.},$$

valores harum litterarum maiuscularum per minusculas sequenti modo definientur

$$A = \frac{1}{2},\quad B = a,\quad C = 0,\quad D = b,\quad E = 0,\quad F = c,\quad G = 0,\quad H = d \quad \text{etc.}$$

sicque potestatum imparium coefficientes per se evanescunt. Evidens autem est ope formularum hic inventarum valores litterarum a, b, c, d etc. multo facilius et promptius assignari posse quam per relationes supra allatas, scilicet erit

$$A = \frac{1}{2},\quad B = a = \frac{1}{12},\quad C = 0,\quad D = -\frac{1}{720},\quad E = 0,\quad F = \frac{1}{30240} \quad \text{etc.}$$

16. Quoniam hoc modo calculus istarum litterarum mox ad numeros vehementer magnos deduceret, loco litterarum a, b, c, d etc. quaeramus alias $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc., quarum signa iam alternentur et quarum valores ad illos sequenti modo referantur

$$a = \frac{\mathfrak{A}}{12},\quad b = -\frac{\mathfrak{B}}{12^2},\quad c = +\frac{\mathfrak{C}}{12^3},\quad d = -\frac{\mathfrak{D}}{12^4},\quad e = +\frac{\mathfrak{E}}{12^5} \quad \text{etc.},$$

ita ut iam sit nostra series

$$s = 1 - \frac{1}{2}u + \frac{\mathfrak{A}uu}{12} - \frac{\mathfrak{B}u^4}{12^2} + \frac{\mathfrak{C}u^6}{12^3} - \frac{\mathfrak{D}u^8}{12^4} + \text{etc.},$$

atque istae novae litterae $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc., sequenti modo determinabuntur

$$\mathfrak{A} = 1,\quad \mathfrak{B} = \frac{\mathfrak{A}\mathfrak{A}}{5},\quad \mathfrak{C} = \frac{2\mathfrak{A}\mathfrak{B}}{7},\quad \mathfrak{D} = \frac{2\mathfrak{A}\mathfrak{C} + \mathfrak{B}\mathfrak{B}}{9},\quad \mathfrak{E} = \frac{2\mathfrak{A}\mathfrak{D} + 2\mathfrak{B}\mathfrak{C}}{11},$$
$$\mathfrak{F} = \frac{2\mathfrak{A}\mathfrak{E} + 2\mathfrak{B}\mathfrak{D} + \mathfrak{C}\mathfrak{C}}{13} \quad \text{etc.},$$

qui valores nunc haud difficulter in numeris evolventur ac reperientur

$$\mathfrak{A} = 1,\quad \mathfrak{B} = \frac{1}{5},\quad \mathfrak{C} = \frac{2}{35},\quad \mathfrak{D} = \frac{3}{175},\quad \mathfrak{E} = \frac{2}{385},\quad \mathfrak{F} = \frac{1382}{875875} \quad \text{etc.}$$

17. Introducamus igitur istas novas litteras $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc. in seriem § 12 pro $\int y\partial x$ inventam eritque

$$\int y\partial x = \frac{1}{2} + \frac{1\cdot\mathfrak{A}}{12} - \frac{1\cdot2\cdot3\mathfrak{B}}{12^2} + \frac{1\cdot2\cdot3\cdot4\cdot5\mathfrak{C}}{12^3} - \frac{1\cdot2\cdot3\cdots7\mathfrak{D}}{12^4} + \text{etc.};$$

hanc autem seriem non satis convergere iam supra observavimus.

TRANSFORMATIO FRACTIONIS

$$\frac{1}{1 + \frac{1}{2}t + \frac{1}{3}tt + \frac{1}{4}t^3 + \frac{1}{5}t^4 + \text{etc.}}$$

IN SERIEM

$$1 + \alpha t + \beta tt + \gamma t^3 + \delta t^4 + \varepsilon t^5 + \zeta t^6 + \text{etc.}$$

17[a].[1]). Ad hanc transformationem perducti sumus supra in § 7, ubi evolutio litterarum $\alpha, \beta, \gamma, \delta$ etc. mox fiebat nimis molesta. Nunc igitur simili modo utamur quo ante positaque hac serie, quam quaerimus, $= s$ erit

$$s = \frac{-t}{l(1-t)} = tv \quad \text{ideoque} \quad l(1-t) = -\frac{1}{v},$$

ergo differentiando

$$-\frac{\partial t}{1-t} = \frac{\partial v}{vv} \quad \text{seu} \quad \frac{\partial v}{vv} + \frac{\partial t}{1-t} = 0,$$

cui hanc formam tribuamus

$$vv + \frac{\partial v}{\partial t}(1-t) = 0,$$

ex qua aequatione series idonea pro v elici debet.

1) In editione principe falso numerus 17 iteratur. A. L.

18. Cum igitur posito $t=0$ fiat $s=1$, hoc casu esse debebit $v=\frac{1}{t}$, quamobrem fingamus istam seriem

$$v=\frac{1}{t}+a+bt+ctt+dt^3+et^4+\text{etc.},$$

qui valor sequenti modo substituatur

$$\begin{aligned}
\frac{\partial v}{\partial t} &= -\frac{1}{tt}+*+b+2ct+3dtt+4et^3+5ft^4+6gt^5+\text{etc.},\\
-\frac{t\partial v}{\partial t} &= \phantom{-\frac{1}{tt}}+\frac{1}{t}-*-bt-2ctt-3dt^3-4et^4-5ft^5-\text{etc.},\\
+vv &= +\frac{1}{tt}+\frac{2a}{t}+2b+2ct+2dtt+2et^3+2ft^4+2gt^5+\text{etc.}\\
&\qquad +aa+2ab+2ac+2ad+2ae+2af\\
&\qquad\qquad +bb+2bc+2bd+2be\\
&\qquad\qquad\qquad +cc+2cd
\end{aligned}$$

Hinc igitur prodeunt sequentes determinationes

$$\begin{aligned}
&1+2a=0,\\
&3b+aa=0,\\
&4c+2ab-b=0,\\
&5d+2ac-2c+bb=0,\\
&6e+2ad+2bc-3d=0,\\
&7f+2ae+2bd-4e+cc=0\\
&\qquad\text{etc.},
\end{aligned}$$

quae formulae ob $a=-\frac{1}{2}$ contrahuntur in sequentes

$$\begin{aligned}
3b &= -\frac{1}{4}, &&\text{ergo}\quad a=-\frac{1}{2},\\
4c &= 2b, && b=-\frac{1}{12},\\
5d &= 3c-bb, && c=-\frac{1}{24},\\
6e &= 4d-2bc, && d=-\frac{19}{720},\\
7f &= 5e-2bd-cc, && e=-\frac{3}{160},\\
8g &= 6f-2be-2cd && f=-\frac{863}{32\cdot 270\cdot 7}\ ^{1)}\\
&\text{etc.}, && \text{etc.}
\end{aligned}$$

19. Hinc igitur erit series quaesita

$$s=1+at+btt+ct^3+dt^4+\text{etc.},$$

quae supra [§ 7] posita fuerat

$$s=1+\alpha t+\beta tt+\gamma t^3+\delta t^4+\text{etc.};$$

litterae igitur latinae et graecae prorsus conveniunt eritque ergo

$$\int y\partial x=-\frac{a}{1}-\frac{b}{2}-\frac{c}{3}-\frac{d}{4}-\frac{e}{5}-\text{etc.}$$

et valoribus modo inventis substitutis

$$\int y\partial x=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 12}+\frac{1}{3\cdot 24}+\frac{19}{4\cdot 720}+\frac{3}{5\cdot 160}+\text{etc.}\ ^{2)}$$

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1) Editio princeps: $f=-\frac{827}{32\cdot 270\cdot 7}$. Correxit A. L.

2) Editio princeps:

$$\int y\partial x=-\frac{a}{2}-\frac{b}{3}-\frac{c}{4}-\frac{d}{5}-\frac{e}{6}-\text{etc.}$$

et valoribus modo inventis substitutis

$$\int y\partial x=\frac{1}{2\cdot 2}+\frac{1}{3\cdot 12}+\frac{1}{4\cdot 24}+\frac{19}{5\cdot 720}+\frac{3}{6\cdot 160}+\text{etc.}$$

Correxit A. L.

20. Quo autem calculus harum litterarum expeditior reddatur, ponamus

$$a=-\frac{A}{2},\quad b=-\frac{B}{4},\quad c=-\frac{C}{8},\quad d=-\frac{D}{16},\quad e=-\frac{E}{32}\quad \text{etc.},$$

ut sit

$$\int y\partial x=\frac{A}{1\cdot 2}+\frac{B}{2\cdot 4}+\frac{C}{3\cdot 8}+\frac{D}{4\cdot 16}+\frac{E}{5\cdot 32}+\text{etc.};^{1})$$

pro his autem litteris habebimus sequentes determinationes

$$A=1,\quad B=\frac{1}{3},\quad C=\frac{4B}{4},\quad D=\frac{6C+BB}{5},\quad E=\frac{8D+2BC}{6},$$

$$F=\frac{10E+2BD+CC}{7},\quad G=\frac{12F+2BE+2CD}{8}\quad \text{etc.},$$

unde colligitur

$$A=1,\quad B=\frac{1}{3},\quad C=\frac{1}{3},\quad D=\frac{19}{45},\quad E=\frac{3}{5}{}^{2})\ \text{etc.}$$

Haec igitur ad calculos superiores sublevandos sufficere poterunt.

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1) Editio princeps:

$$\int y\partial x=\frac{A}{2\cdot 2}+\frac{B}{3\cdot 4}+\frac{C}{4\cdot 8}+\frac{D}{5\cdot 16}+\frac{E}{6\cdot 32}+\text{etc.}$$

Correxit A. L.

2) Editio princeps: $E=\frac{39}{50}$. Correxit A. L.

UBERIOR EXPLICATIO METHODI SINGULARIS NUPER EXPOSITAE INTEGRALIA ALIAS MAXIME ABSCONDITA INVESTIGANDI

Conventui exhibita die 29. Februarii 1776

Commentatio 630 indicis Enestroemiani
Nova acta academiae scientiarum Petropolitanae 4 (1786), 1789, p. 17—54
Summarium ibidem p. 110—111

SUMMARIUM

Il n'y a pas longtems que le calcul intégral a été enrichi de la méthode de traiter les formules différentielles qui sont divisées par le logarithme de la quantité variable. Feu M. Euler, à qui cette branche des sciences mathématiques est redevable de ses plus brillans progrès, dit dans un de ses mémoires: *Cum mihi saepius occurrissent formulae differentiales, quae per logarithmum quantitatis variabilis erant divisae, veluti* $\frac{Pdz}{lz}$, *nunquam perspicere potui, ad quodnam genus quantitatum earum integralia sint referenda.* (V. Nov. Comment. Tom. XIX p. 66.)[1]). Cependant il s'est frayé un nouveau chemin dans le même mémoire intitulé: *Nova methodus quantitates integrales determinandi*, et il y a donné l'intégrale de plusieurs pareilles formules, et entre autres de celle-ci $\int \frac{x^a-x^b}{lx}\partial x$, dont il a trouvé, par une méthode tout à fait nouvelle, l'intégrale, prise depuis $x=0$ jusqu'à $x=1$, égale à $l\frac{a+1}{b+1}$.

Cette nouvelle méthode de déterminer les intégrales de beaucoup de formules qui se refusent à toute autre voye d'intégration, a paru à M. Euler susceptible de bien plus de généralisation, et propre, par là même, à avancer les bornes du calcul intégral. C'est le but de ce mémoire, qui mérite à tous égards l'attention des Géomètres, mais qui n'est pas

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1) Voir la note p. 322 de ce volume et principalement p. 424 du volume précédent. A. L.

susceptible d'extrait, étant tout hérissé de calculs, et de calculs inintelligibles, sans une explication préalable des nouveaux caractères que l'Auteur s'est vu obligé d'introduire, afin de rendre ses recherches plus courtes et plus lumineuses. Explication que les connoisseurs aimeront mieux chercher, ainsi que l'esprit de la méthode, dans le mémoire même, où nous les renvoyons.

Methodus illa singularis, qua non ita pridem deductus sum ad integrationem formulae

$$\int \frac{x^a - x^b}{lx} \partial x,$$

cuius valorem a termino $x = 0$ usque ad $x = 1$ extensum inveni esse

$$l\frac{a+1}{b+1}\ ^{*)},$$

multo latius patet ideoque accuratiorem evolutionem meretur, quandoquidem multo maiora incrementa scientiae analyticae polliceri videtur. Quo autem hoc feliciori successu et sine ambagibus praestari possit, necesse erit peculiarem signandi modum usurpare, quem ergo ante omnia explicari conveniet.

EXPLICATIO CHARACTERUM IN SEQUENTIBUS ADHIBENDORUM

I. Si V denotet functionem quamcunque binarum variabilium x et p, tum iste character

$$\frac{\partial^\lambda}{x} \cdot V$$

mihi designabit eam quantitatem, quae oritur, si functio V solam x pro variabili sumendo toties successive differentietur, quot unitates in indice λ

*) Iam in dissertatione praecedente annotavimus Ill. Auctorem hanc integrationem exposuisse in Novorum Commentariorum Tomo XIX. pag. 70.[1])

1) Vide notam p. 322. A. L.

continentur, simulque ubique differentiale ∂x reiiciatur. Eodem modo iste character

$$\frac{\partial^\lambda}{p} \cdot V$$

designabit eam quantitatem, quae per totidem differentiationes resultat, dum sola p ut variabilis tractatur. Hinc igitur ista signandi ratio sequenti modo ad formulas usu receptas reducetur

$$\frac{\partial}{x} \cdot V = \left(\frac{\partial V}{\partial x}\right) \quad \text{et} \quad \frac{\partial}{p} \cdot V = \left(\frac{\partial V}{\partial p}\right),$$

$$\frac{\partial^2}{x} \cdot V = \left(\frac{\partial \partial V}{\partial x^2}\right) \quad \text{et} \quad \frac{\partial^2}{p} \cdot V = \left(\frac{\partial \partial V}{\partial p^2}\right),$$

$$\frac{\partial^3}{x} \cdot V = \left(\frac{\partial^3 V}{\partial x^3}\right) \quad \text{et} \quad \frac{\partial^3}{p} \cdot V = \left(\frac{\partial^3 V}{\partial p^3}\right).$$

II. Vicissim autem integrando iste character

$$\frac{\int^\lambda}{x} \cdot V$$

designabit eam quantitatem, quae ex continua integratione λ vicibus repetita oritur, dum sola x variabilis accipitur; et pariter hic character

$$\frac{\int^\lambda}{p} \cdot V$$

eam quantitatem significat, quae oritur per continuam integrationem λ vicibus repetitam, dum sola p variabilis accipitur. Haec ergo sequenti modo ad formas usu receptas revocabuntur

$$\frac{\int}{x} \cdot V = \int V \partial x \quad \text{et} \quad \frac{\int}{p} \cdot V = \int V \partial p,$$

$$\frac{\int^2}{x} \cdot V = \int \partial x \int V \partial x \quad \text{et} \quad \frac{\int^2}{p} \cdot V = \int \partial p \int V \partial p,$$

$$\frac{\int^3}{x} \cdot V = \int \partial x \int \partial x \int V \partial x \quad \text{et} \quad \frac{\int^3}{p} \cdot V = \int \partial p \int \partial p \int V \partial p.$$

III. At quoniam omnes quantitates per integrationem inventae per se sunt indeterminatae, in posterum perpetuo omnia integralia ita capi statuamus, ut evanescant posito vel $x=0$ vel $p=0$; prius scilicet, si sola x ut variabilis fuerit tractata, posterius vero, si sola p fuerit variabilis.

IV. Hos iam characteres pro lubitu inter se coniungere licet ac primo quidem haec formula

$$\frac{\partial^\mu}{x}\cdot\frac{\partial^\nu}{p}\cdot V$$

denotat functionem V primo μ vicibus differentiari debere sumta sola x variabili, tum vero quantitatem hinc oriundam denuo ν vicibus differentiari debere sumta sola p variabili. Hinc istos characteres ad morem solitum revocando erit

$$\frac{\partial}{x}\cdot\frac{\partial}{p}\cdot V=\left(\frac{\partial\partial V}{\partial x\partial p}\right),\qquad \frac{\partial}{p}\cdot\frac{\partial}{x}\cdot V=\left(\frac{\partial\partial V}{\partial p\partial x}\right),$$

$$\frac{\partial^2}{x}\cdot\frac{\partial}{p}\cdot V=\left(\frac{\partial^3 V}{\partial x^2\partial p}\right),\qquad \frac{\partial}{x}\cdot\frac{\partial^2}{p}\cdot V=\left(\frac{\partial^3 V}{\partial x\partial p^2}\right),$$

$$\frac{\partial^3}{x}\cdot\frac{\partial^2}{p}\cdot V=\left(\frac{\partial^5 V}{\partial x^3\partial p^2}\right)\qquad \frac{\partial^2}{x}\cdot\frac{\partial^3}{p}\cdot V=\left(\frac{\partial^5 V}{\partial x^2\partial p^3}\right)$$

etc. etc.

V. Ita formula

$$\frac{\partial^\mu}{x}\cdot\frac{\int^\nu}{p}\cdot V$$

denotat functionem V primo μ vicibus differentiari debere sumta sola x variabili, tum vero quantitatem hinc oriundam ν vicibus integrari debere sumta sola p variabili. Ita si fuerit $\mu=2$ et $\nu=1$, erit more solito

$$\frac{\partial^2}{x}\cdot\frac{\int^1}{p}\cdot V=\int\partial p\left(\frac{\partial\partial V}{\partial x^2}\right),$$

unde significatio aliorum huiusmodi characterum iam satis intelligi potest.

VI. Simili modo formula hoc charactere designata

$$\frac{\int^\mu}{x}\cdot\frac{\partial^\nu}{p}\cdot V$$

declarat functionem V primo μ vicibus integrari debere sumta sola x variabili, tum vero quantitatem hinc oriundam ν vicibus differentiari debere sumta sola p variabili. Quae ergo significatio satis clare perspicitur, etsi more solito non tam commode indicari posset. Si enim esset $\mu=2$ et $\nu=2$, valor huius formulae $\frac{\int^2}{x}\cdot\frac{\partial^2}{p}\cdot V$ hoc modo repraesentari deberet $\left(\frac{\partial\partial\int\partial x\int V\partial x}{\partial p^2}\right)$.

VII. Denique iste character

$$\frac{\int^\mu}{x}\cdot\frac{\int^\nu}{p}\cdot V$$

significat functionem V primo μ vicibus integrari debere sumta sola x pro variabili, tum vero quantitatem resultantem denuo ν vicibus integrari debere sumta sola p variabili. Ubi, quod in perpetuum est tenendum, priora integralia ita capi debent, ut evanescant posito $x=0$, posteriora vero posito $p=0$.

Hac characterum explicatione praemissa sequentia theoremata probe notentur, quorum veritas ex iis, quae de indole functionum duarum variabilium sunt exposita, satis clare perspicitur.

THEOREMA 1

Si V fuerit functio quaecunque duarum variabilium x et p, sequens aequalitas semper locum habebit

$$\frac{\partial^\mu}{x}\cdot\frac{\partial^\nu}{p}\cdot V=\frac{\partial^\nu}{p}\cdot\frac{\partial^\mu}{x}\cdot V.$$

Hinc ergo si ponamus

$$\frac{\partial^\mu}{x}\cdot V=Q\quad\text{et}\quad\frac{\partial^\nu}{p}\cdot V=R,$$

tum erit

$$\frac{\partial^\nu}{p}\cdot Q=\frac{\partial^\mu}{x}\cdot R.$$

THEOREMA 2

Si V fuerit functio quaecunque binarum variabilium x et p, tum sequens aequalitas semper locum habebit

$$\frac{\int^\mu}{x}\cdot\frac{\partial^\nu}{p}\cdot V=\frac{\partial^\nu}{p}\cdot\frac{\int^\mu}{x}\cdot V.$$

Hinc si ponamus

$$\frac{\int^\mu}{x}\cdot V = Q \quad \text{et} \quad \frac{\partial^\nu}{p}\cdot V = R,$$

erit

$$\frac{\partial^\nu}{p}\cdot Q = \frac{\int^\mu}{x}\cdot R.$$

THEOREMA 3

Si fuerit V functio quaecunque binarum variabilium x et p, tum sequens aequalitas semper locum habebit

$$\frac{\partial^\mu}{x}\cdot\frac{\int^\nu}{p}\cdot V = \frac{\int^\nu}{p}\cdot\frac{\partial^\mu}{x}\cdot V.$$

Hinc si ponamus

$$\frac{\partial^\mu}{x}\cdot V = Q \quad \text{et} \quad \frac{\int^\nu}{p}\cdot V = R,$$

erit

$$\frac{\int^\nu}{p}\cdot Q = \frac{\partial^\mu}{x}\cdot R.$$

THEOREMA 4

Si fuerit V functio quaecunque binarum variabilium x et p, tum sequens aequalitas semper locum habebit

$$\frac{\int^\mu}{x}\cdot\frac{\int^\nu}{p}\cdot V = \frac{\int^\nu}{p}\cdot\frac{\int^\mu}{x}\cdot V.$$

Hinc si ponamus

$$\frac{\int^\mu}{x}\cdot V = Q \quad \text{et} \quad \frac{\int^\nu}{p}\cdot V = R,$$

erit

$$\frac{\int^\nu}{p}\cdot Q = \frac{\int^\mu}{x}\cdot R.$$

SCHOLION

Hae aequalitates per se ita sunt manifestae, ut quovis casu evolutae evadant identicae. Ita si sumatur

$$V = x^m p^n,$$

ex theoremate primo sumto $\mu = 2$ et $\nu = 1$ reperietur

$$Q = \frac{\partial^2}{x}\cdot V = m(m-1)x^{m-2}p^n \quad \text{et} \quad R = \frac{\partial}{p}\cdot V = np^{n-1}x^m.$$

Hinc vero elicitur

$$\frac{\partial}{p}\cdot Q = mn(m-1)x^{m-2}p^{n-1} \quad \text{et} \quad \frac{\partial^2}{x}\cdot R = mn(m-1)x^{m-2}p^{n-1},$$

qui duo valores manifesto congruunt. Ex secundo autem theoremate [sumto] $\mu = 2$ et $\nu = 1$ fiet

$$Q = \frac{\int^2}{x}\cdot V = \frac{p^n x^{m+2}}{(m+1)(m+2)} \quad \text{et} \quad R = \frac{\partial}{p}\cdot V = np^{n-1}x^m.$$

Hinc ergo erit

$$\frac{\partial}{p}\cdot Q = \frac{np^{n-1}x^{m+2}}{(m+1)(m+2)} \quad \text{et} \quad \frac{\int^2}{x}\cdot R = \frac{np^{n-1}x^{m+2}}{(m+1)(m+2)}.$$

Ex tertio theoremate manente $\mu = 2$ et $\nu = 1$ erit

$$Q = \frac{\partial^2}{x}\cdot V = m(m-1)x^{m-2}p^n \quad \text{et} \quad R = \frac{\int}{p}\cdot V = \frac{p^{n+1}x^m}{n+1}.$$

Hinc igitur erit

$$\frac{\int}{p}\cdot Q = \frac{m(m-1)x^{m-2}p^{n+1}}{n+1} \quad \text{et} \quad \frac{\partial^2}{x}\cdot R = \frac{m(m-1)x^{m-2}p^{n+1}}{n+1}.$$

Ex quarto denique theoremate erit

$$Q = \frac{\int^2}{x}\cdot V = \frac{x^{m+2}p^n}{(m+1)(m+2)} \quad \text{et} \quad R = \frac{\int}{p}\cdot V = \frac{x^m p^{n+1}}{n+1}.$$

Hinc ergo colligitur

$$\frac{\int}{p}\cdot Q = \frac{x^{m+2}p^{n+1}}{(n+1)(m+1)(m+2)} \quad \text{et} \quad \frac{\int^2}{x}\cdot R = \frac{x^{m+2}p^{n+1}}{(n+1)(m+1)(m+2)}.$$

Ob has igitur aequalitates adeo identicas nullae conclusiones hinc deduci posse videbuntur. Verum longe aliter se res habere deprehenditur, si post omnes operationes institutas ipsi x determinatus valor, veluti $x = 1$, tribui debeat, quemadmodum in quatuor problematibus sequentibus ostendemus, quae se ad quatuor theoremata praecedentia referunt.

PROBLEMA 1

Si V fuerit functio quaecunque binarum variabilium x et p et omnes operationes in theoremate primo indicatae absolvantur, tum vero statuatur $x=1$, exhibere aequalitatem, ad quam hoc theorema perducit.

SOLUTIO

Quoniam in nostro primo theoremate posuimus $\frac{\partial^\mu}{x}\cdot V = Q$, deinde vero haec quantitas sola p variabili sumta differentiari debet, ita ut iam x pro constanti habeatur, statim loco x unitas scribi poterit, quo facto abeat Q in M, ita ut nunc M futura sit functio solius p. Manente igitur $R = \frac{\partial^\nu}{p}\cdot V$ consequemur hanc aequationem

$$\frac{\partial^\nu}{p}\cdot M = \frac{\partial^\mu}{x}\cdot R,$$

ubi plerumque eveniet, ut quantitas M multo promptius differentiari queat quam functio Q, unde aequalitas inventa plerumque non adeo erit obvia; id quod sequentibus exemplis illustrasse iuvabit, in quibus omnibus assumemus $V = x^{n+p}$, ita ut eius valor posito $x=1$ abeat in 1.

EXEMPLUM 1 QUO $\mu = 1$ ET $\nu = 1$

Hic ergo erit

$$Q = \frac{\partial}{x}\cdot x^{n+p} = (n+p)x^{n+p-1},$$

unde ergo posito $x=1$ fit

$$M = n+p;$$

quare cum sit

$$R = \frac{\partial}{p}\cdot x^{n+p} = x^{n+p}lx,$$

nanciscimur hanc aequationem $1 = \frac{\partial}{x}\cdot x^{n+p}lx$. Unde patet, si post differentiationem ponatur $x=1$, fore more exprimendi solito

$$\frac{1}{\partial x}\partial . x^{n+p}lx = 1,$$

id quod non amplius tam est obvium; est enim

$$\partial . x^{n+p}lx = (n+p)x^{n+p-1}\partial x lx + x^{n+p-1}\partial x,$$

quae expressio per ∂x divisa positoque $x=1$ abit in 1.

EXEMPLUM 2 QUO $\mu = 2$ ET $\nu = 1$

Hic igitur erit

$$Q = \frac{\partial^2}{x}\cdot x^{n+p} = (n+p)(n+p-1)x^{n+p-2};$$

posito ergo $x=1$ erit

$$M = (n+p)(n+p-1).$$

Quare cum sit

$$R = x^{n+p}lx,$$

erit

$$\frac{\partial}{p}\cdot(n+p)(n+p-1) = \frac{\partial^2}{x}\cdot x^{n+p}lx,$$

quamobrem per solitum exprimendi modum habebimus

$$\frac{\partial\partial . x^{n+p}lx}{\partial x^2} = 2(n+p)-1,$$

postquam scilicet gemina differentiatione absoluta ponitur $x=1$.

EXEMPLUM 3 QUO $\mu = 1$ ET $\nu = 2$

Hic igitur erit

$$Q = \frac{\partial}{x}\cdot x^{n+p} = (n+p)x^{n+p-1},$$

unde posito $x=1$ fit

$$M = n+p.$$

Quare cum sit

$$R = \frac{\partial^2}{p}\cdot x^{n+p} = x^{n+p}(lx)^2,$$

erit

$$\frac{\partial^2}{p}\cdot(n+p) = \frac{\partial}{x}\cdot x^{n+p}(lx)^2$$

sive solito exprimendi more

$$\frac{\partial . x^{n+p}(lx)^2}{\partial x} = 0,$$

postquam scilicet differentiatione absoluta ponitur $x=1$.

EXEMPLUM 4 QUO $\mu = 2$ ET $\nu = 2$

Cum igitur hoc casu sit

$$Q = \frac{\partial^2}{x} \cdot x^{n+p} = (n+p)(n+p-1)x^{n+p-2}$$

ideoque

$$M = (n+p)(n+p-1)$$

et

$$R = \frac{\partial^2}{p} \cdot x^{n+p} = x^{n+p}(lx)^2,$$

erit

$$\frac{\partial\partial . x^{n+p}(lx)^2}{\partial x^2} = \frac{\partial\partial M}{\partial p^2} = 2.$$

COROLLARIUM

Ex his exemplis iam abunde fit perspicuum, si exponentes μ et ν fuerint quicunque, tum posito $x = 1$ fore

$$M = (n+p)(n+p-1) \cdots (n+p-\mu+1)$$

ideoque functionem ipsius p tantum. Quare cum sit $R = x^{n+p}(lx)^\nu$, erit more solito

$$\frac{\partial^\mu . x^{n+p}(lx)^\nu}{\partial x^\mu} = \frac{\partial^\nu M}{\partial p^\nu},$$

quando scilicet omnibus operationibus peractis statuitur $x = 1$.

SCHOLION

Quemadmodum hic assumsimus $V = x^{n+p}$, ita eadem opera expedire licet hanc formam latius patentem

$$V = x^p X$$

denotante X functionem quamcunque ipsius x tantum, ita ut altera quantitas p non ingrediatur. Ponamus igitur sumto $x = 1$ fieri $X = A$, $\frac{\partial X}{\partial x} = A'$, $\frac{\partial\partial X}{\partial x^2} = A''$ etc., atque cum fiat

$$Q = \frac{\partial}{x} \cdot V = px^{p-1}X + x^p \frac{\partial X}{\partial x},$$

erit hoc casu

$$M = pA + A'.$$

Deinde vero habebimus

$$\frac{\partial^2}{x} \cdot V = p(p-1)x^{p-2}X + 2px^{p-1}\frac{\partial X}{\partial x} + x^p \frac{\partial\partial X}{\partial x^2} = Q;$$

hinc ergo colligitur

$$M = p(p-1)A + 2pA' + A''.$$

Prodit porro

$$\frac{\partial^3}{x} \cdot V = p(p-1)(p-2)x^{p-3}X + 3p(p-1)x^{p-2}\frac{\partial X}{\partial x} + 3px^{p-1}\frac{\partial\partial X}{\partial x^2} + x^p \frac{\partial^3 X}{\partial x^3};$$

hinc ergo erit

$$M = p(p-1)(p-2)A + 3p(p-1)A' + 3pA'' + A'''.$$

Hinc iam patet ex formula $\frac{\partial^4}{x} \cdot V$ oriturum esse valorem

$$M = p(p-1)(p-2)(p-3)A + 4p(p-1)(p-2)A' + 6p(p-1)A'' + 4pA''' + A'''',$$

unde lex progressionis satis est manifesta. At vero pro altera littera R habebimus

casu $\nu = 1$ $\quad R = x^p X lx$,

casu $\nu = 2$ $\quad R = x^p X (lx)^2$,

casu $\nu = 3$ $\quad R = x^p X (lx)^3$

atque adeo in genere casu $\nu = \nu$ erit

$$R = x^p X (lx)^\nu.$$

Ex his igitur formulis nanciscemur valores differentialium omnium ordinum formulae $x^p X(lx)^\nu$, postquam factis omnibus operationibus positum fuerit $x = 1$:

$$1. \quad \frac{1}{\partial x}\partial . x^p X(lx)^\nu = \frac{\partial^\nu (pA + A')}{\partial p^\nu},$$

qui valor semper erit $= 0$ excepto casu $\nu = 1$, quo prodit $= A$;

$$2. \quad \frac{1}{\partial x^2}\partial\partial . x^p X(lx)^\nu = \frac{\partial^\nu (p(p-1)A + 2pA' + A'')}{\partial p^\nu},$$

qui valor semper est 0, quando $\nu > 2$;

3. $\frac{1}{\partial x^3}\partial^3 . x^p X(lx)^\nu = \frac{\partial^\nu(p(p-1)(p-2)A + 3p(p-1)A' + 3pA'' + A''')}{\partial p^\nu}$,

qui valor semper evanescit exceptis casibus, quibus $\nu < 4$.

In his formulis notasse iuvabit esse:

Pro prima

$$\frac{\partial(pA + A')}{\partial p} = A;$$

pro secunda

$$\frac{\partial(p(p-1)A + 2pA' + A'')}{\partial p} = (2p-1)A + 2A'$$

et

$$\frac{\partial\partial(p(p-1)A + 2pA' + A'')}{p^2} = 2A;$$

sequentia autem sunt 0;

pro tertia

$$\frac{\partial(p(p-1)(p-2)A + 3p(p-1)A' + 3pA'' + A''')}{\partial p} = (3pp - 6p + 2)A + 3(2p-1)A' + 3A'',$$

$$\frac{\partial\partial(p(p-1)(p-2)A + 3p(p-1)A' + 3pA'' + A''')}{\partial p^2} = (6p-6)A + 6A'$$

et

$$\frac{\partial^3(p(p-1)(p-2)A + 3p(p-1)A' + 3pA'' + A''')}{\partial p^3} = 6A;$$

sequentia omnia evanescunt.

PROBLEMA 2

Si V fuerit functio quaecunque binarum variabilium x et p et omnes operationes in theoremate secundo indicatae absolvantur, tum vero statuatur $x = 1$, exhibere aequalitatem, ad quam hoc theorema perducit.

SOLUTIO

Quoniam in nostro secundo theoremate posuimus $\frac{\int^\mu}{x} \cdot V = Q$, deinde vero haec quantitas sumta sola p variabili ν vicibus differentiari debet posita x constante, iam ante has differentiationes ponere licet $x = 1$. Hoc

ergo facto abeat Q in M sicque habebitur $\frac{\partial^\nu}{p} \cdot Q = \frac{\partial^\nu M}{\partial p^\nu}$, quod iam est membrum primum aequalitatis quaesitae more solito expressum, quandoquidem M est sola functio ipsius p. Pro altero membro cum sit $R = \frac{\partial^\nu}{p} \cdot V$, erit hoc alterum membrum $\frac{\int^\mu}{x} \cdot R$. Quamobrem si post omnes has μ integrationes peractas (quae autem singula integralia semper ita sunt capienda, ut evanescant posito $x = 0$) statuatur $x = 1$, semper erit

$$\frac{\int^\mu}{x} \cdot R = \frac{\partial^\nu M}{\partial p^\nu},$$

de quo valore certi sumus, etiamsi forte integratio absolvi nequeat, quamobrem hanc veritatem exemplis illustremus, in quibus assumemus $V = x^{n+p}$.

EXEMPLUM 1 QUO $\mu = 1$ ET $\nu = 1$

Hoc ergo casu erit

$$Q = \int x^{n+p} \partial x = \frac{x^{n+p+1}}{n+p+1},$$

unde fit

$$M = \frac{1}{n+p+1}.$$

Deinde vero erit

$$R = \frac{\partial}{p} \cdot x^{n+p} = x^{n+p} lx,$$

ex quibus aequatio nostra fiet

$$\int x^{n+p} \partial x lx = \frac{\partial}{p} \cdot \frac{1}{n+p+1} = \frac{-1}{(n+p+1)^2}.$$

EXEMPLUM 2 QUO $\mu = 1$ ET $\nu = 2$

Hoc ergo casu erit

$$Q = \frac{\int}{x} \cdot x^{n+p} = \frac{x^{n+p+1}}{n+p+1}$$

ideoque M ut ante $\frac{1}{n+p+1}$. Deinde vero erit

$$R = \frac{\partial^2}{p} \cdot x^{n+p} = x^{n+p}(lx)^2,$$

quocirca posito $x=1$ habebitur ista aequatio

$$\int x^{n+p}\partial x(lx)^2=\frac{\partial^2}{p}\cdot\frac{1}{n+p+1}=\frac{+2}{(n+p+1)^3}.$$

EXEMPLUM 3 QUO $\mu=1$ ET $\nu=3$

Hoc igitur casu erit

$$Q=\frac{x^{n+p+1}}{n+p+1}\quad\text{et}\quad M=\frac{1}{n+p+1}.$$

Tum vero erit $R=x^{n+p}(lx)^3$, unde nascitur haec aequalitas

$$\int x^{n+p}\partial x(lx)^3=\frac{\partial^3}{p}\cdot\frac{1}{n+p+1}=\frac{-6}{(n+p+1)^4}.$$

EXEMPLUM 4 QUO $\mu=1$ ET $\nu=\nu$

Hic ex praecedentibus satis liquet aequationem hinc resultantem fore

$$\int x^{n+p}\partial x(lx)^\nu=\pm\frac{1\cdot2\cdot3\cdots\nu}{(n+p+1)^{\nu+1}},$$

ubi signum superius valet, si ν sit numerus par, inferius vero, si impar; quae reductio eo magis est notatu digna, quod alias per plures ambages ad eam perveniri solet.

EXEMPLUM 5 QUO $\mu=2$ ET $\nu=1$

Hoc ergo casu erit

$$Q=\frac{\int^2}{x}\cdot x^{n+p}=\frac{x^{n+p+2}}{(n+p+1)(n+p+2)},$$

quamobrem habebitur

$$M=\frac{1}{(n+p+1)(n+p+2)},$$

qui valor reducitur ad hunc

$$M=\frac{1}{n+p+1}-\frac{1}{n+p+2};$$

tum vero erit $R=x^{n+p}lx$, unde sequens aequalitas deducitur

$$\int\partial x\int x^{n+p}\partial x lx=\frac{-1}{(n+p+1)^2}+\frac{1}{(n+p+2)^2},$$

quae aequalitas more solito indagata iam satis molestos calculos postulat.

EXEMPLUM 6 QUO $\mu=2$ ET $\nu=2$

Hic ergo erit ut ante

$$M=\frac{1}{(n+p+1)(n+p+2)}=\frac{1}{n+p+1}-\frac{1}{n+p+2}\quad\text{et}\quad R=x^{n+p}(lx)^2$$

uti in Exemplo 2, unde statim colligitur ista aequatio

$$\int\partial x\int x^{n+p}\partial x(lx)^2=\frac{\partial\partial M}{\partial p^2}=\frac{2}{(n+p+1)^3}-\frac{2}{(n+p+2)^3}.$$

EXEMPLUM 7 QUO $\mu=2$ ET $\nu=\nu$

Hic ergo erit

$$M=\frac{1}{n+p+1}-\frac{1}{n+p+2}\quad\text{et}\quad R=x^{n+p}(lx)^\nu,$$

unde resultat aequatio

$$\int\partial x\int x^{n+p}\partial x(lx)^\nu=\pm\frac{1\cdot2\cdots\nu}{(n+p+1)^{\nu+1}}\mp\frac{1\cdot2\cdots\nu}{(n+p+2)^{\nu+1}},$$

ubi iterum signa superiora valent, si ν numerus par, inferiora vero, si impar.

EXEMPLUM 8 QUO $\mu=3$ ET $\nu=\nu$

Pro hoc casu ob

$$Q=\frac{\int^3}{x}\cdot x^{n+p}=\frac{x^{n+p+3}}{(n+p+1)(n+p+2)(n+p+3)}$$

posito $x=1$ fit

$$M=\frac{1}{(n+p+1)(n+p+2)(n+p+3)},$$

quae fractio resolvatur in suas simplices, fietque

$$M=\frac{1}{2(n+p+1)}-\frac{1}{n+p+2}+\frac{1}{2(n+p+3)},$$

unde facile patet sequentem prodituram esse aequalitatem

$$\int\partial x\int\partial x\int x^{n+p}\partial x(lx)^\nu=\pm\frac{3\cdot4\cdots\nu}{(n+p+1)^{\nu+1}}\mp\frac{2\cdot3\cdots\nu}{(n+p+2)^{\nu+1}}\pm\frac{3\cdot4\cdots\nu}{(n+p+3)^{\nu+1}}$$
$$=\pm3\cdot4\cdot5\cdots\nu\left(\frac{1}{(n+p+1)^{\nu+1}}-\frac{2}{(n+p+2)^{\nu+1}}+\frac{1}{(n+p+3)^{\nu+1}}\right),$$

ubi ratio signi ambigui est eadem ut ante. Facile autem intelligitur, si quis formulam illam integralem evolvere voluerit, eum in calculos valde molestos esse delapsurum.

SCHOLION

Superfluum foret indici μ maiores valores tribuere, siquidem evolutio simili modo expediri posset. Praecipuum autem negotium consistit in resolutione fractionis M in suas fractiones simplices, id quod necesse est, ut deinceps facilius omnes differentiationes atque adeo secundum indicem indefinitum ν institui queant. Hic autem labor subsidio sequentis propositionis promptissime absolvi poterit.

PROPOSITIO

Si X fuerit functio quaecunque ipsius x ac post integrationes statui debeat $x=1$, tum semper ista formula integralis complicata

$$\frac{\int^{\mu}}{x}\cdot X$$

reduci potest ad istam formulam integralem simplicem more solito expressam

$$\frac{\int X\partial x(1-x)^{\mu-1}}{1\cdot 2\cdot 3\cdots(\mu-1)}.$$

Hinc enim statim patet pro nostro casu, quo $X=x^{n+p}$, quantitatem M sequenti modo expressum iri

$$M=\frac{1}{1\cdot 2\cdots(\mu-1)}\left(\frac{1}{n+p+1}-\frac{\mu-1}{n+p+2}+\frac{(\mu-1)(\mu-2)}{1\cdot 2(n+p+3)}-\frac{(\mu-1)(\mu-2)(\mu-3)}{1\cdot 2\cdot 3\,(n+p+4)}+\text{etc.}\right),$$

unde iam facile differentialia omnium ordinum ipsius M derivari possunt. Ceterum hic adhuc observasse iuvabit loco functionis illius V vix alium valorem accipi posse praeter x^{n+p}, propterea quod hoc solo casu omnia $\frac{\int^{\mu}}{x}\cdot V$ actu expedire licet, id quod ad nostrum institutum imprimis requiritur, quia alioquin nullae aequationes memorabiles inde deduci possent.

PROBLEMA 3

Si V fuerit functio quaecunque binarum variabilium x et p et omnes operationes in theoremate tertio indicatae actu absolvantur, tum vero statuatur $x=1$, exhibere aequalitatem, ad quam hoc theorema perducit.

SOLUTIO

Quoniam in nostro tertio theoremate posuimus

$$\frac{\partial^{\mu}}{x}\cdot V=Q \quad \text{et} \quad \frac{\int^{\nu}}{p}\cdot V=R,$$

hinc deduximus sequentem aequalitatem $\frac{\int^{\nu}}{p}\cdot Q=\frac{\partial^{\mu}}{x}\cdot R$, ubi in valore pro Q invento loco x unitas scribi debet, unde resultet quantitas M, quae iam tantum erit functio ipsius p, ita ut nunc aequalitas nostra evadat

$$\frac{\partial^{\nu}}{p}\cdot M=\frac{\partial^{\mu}}{x}\cdot R.$$

Quodsi iam loco V hanc accipiamus functionem x^{n+p}, pro variis valoribus indicis μ littera M sequentes sortietur valores:

1. Si $\mu=1$, erit $M=n+p$,
2. si $\mu=2$, erit $M=(n+p)(n+p-1)$,
3. si $\mu=3$, erit $M=(n+p)(n+p-1)(n+p-2)$

etc.

hincque in genere

$$M=(n+p)(n+p-1)\cdots(n+p-\mu+1).$$

Pro littera autem R ex valoribus simplicioribus indicis ν colligetur:

$$1.\ \text{Si}\ \nu=1,\ \text{valor}\ R=\frac{x^{n+p}}{lx}+C;$$

quae constans C cum ita debeat accipi, ut integrale evanescat posito $p=0$, erit hac correctione adhibita

$$R=\frac{x^{n+p}}{lx}-\frac{x^{n}}{lx};$$

quae formula ducta in ∂p et denuo integrata adiectaque debita constante praebet,

2. si $\nu = 2$, valorem $R = \dfrac{x^{n+p} - x^n}{(lx)^2} - \dfrac{px^n}{lx}$,

3. si $\nu = 3$, " $R = \dfrac{x^{n+p} - x^n}{(lx)^3} - \dfrac{px^n}{(lx)^2} - \dfrac{ppx^n}{2lx}$,

4. si $\nu = 4$, " $R = \dfrac{x^{n+p} - x^n}{(lx)^4} - \dfrac{px^n}{(lx)^3} - \dfrac{ppx^n}{2(lx)^2} - \dfrac{p^3x^n}{6lx}$,

unde concluditur in genere esse proditurum

$$R = \frac{x^{n+p}}{(lx)^\nu} - x^n\left(\frac{1}{(lx)^\nu} + \frac{p}{(lx)^{\nu-1}} + \frac{pp}{1\cdot 2(lx)^{\nu-2}} + \cdots + \frac{p^{\nu-1}}{1\cdot 2\cdot 3\cdots(\nu-1)lx}\right).$$

His igitur valoribus evolutis sequentia exempla evolvamus.

EXEMPLUM 1 QUO $\mu = 1$ ET $\nu = 1$

Hoc ergo casu erit

$$M = n + p \quad \text{et} \quad R = \frac{x^{n+p} - x^n}{lx},$$

unde oritur haec aequalitas

$$\frac{1}{\partial x}\cdot\frac{\partial(x^{n+p} - x^n)}{lx} = \frac{\int}{p}\cdot(n+p) = np + \frac{pp}{2}$$

more solito expressa. Hic scilicet forma $\frac{x^{n+p} - x^n}{lx}$ per solam variabilem x differentiata et per ∂x divisa, si loco x scribatur 1, producet hunc valorem $np + \frac{1}{2}pp$, id quod neutiquam tam facile perspicitur. Si enim illa quantitas differentietur, omisso elemento ∂x pervenitur ad istam expressionem

$$\frac{(n+p)x^{n+p-1} - nx^{n-1}}{lx} - \frac{x^{n+p-1} - x^{n-1}}{(lx)^2},$$

ubi iam poni oportet $x = 1$; tum autem utrumque membrum evadit infinitum, quamobrem has duas fractiones ante omnia ad eundem denominatorem reduci convenit, ut habeatur ista fractio

$$\frac{(n+p)x^{n+p-1}lx - nx^{n-1}lx - x^{n+p-1} + x^{n-1}}{(lx)^2},$$

cuius tam numerator quam denominator evanescunt facto $x = 1$. Quamobrem secundum regulam cognitam loco tam numeratoris quam denominatoris eorum differentialia scribantur ac pro numeratore reperietur

$$(n+p)(n+p-1)x^{n+p-2}lx + (n+p)x^{n+p-2} - n(n-1)x^{n-2}lx - nx^{n-2}$$
$$-(n+p-1)x^{n+p-2} + (n-1)x^{n-2};$$

denominator vero erit $\frac{2lx}{x}$, ita ut iam tota fractio sit

$$\frac{(n+p)(n+p-1)x^{n+p-1}lx + (n+p)x^{n+p-1} - n(n-1)x^{n-1}lx - (n+p-1)x^{n+p-1} - x^{n-1}}{2lx},$$

ubi denuo posito $x = 1$ tam numerator quam denominator evanescunt; quamobrem eorum loco iterum differentialia substituamus, quo facto prodibit fractio, cuius numerator erit

$$(n+p-1)^2x^{n+p-2}((n+p)lx - 1) + 2(n+p)(n+p-1)x^{n+p-2}$$
$$- n(n-1)^2x^{n-2}lx - (nn-1)x^{n-2},$$

denominator vero erit $\frac{2}{x}$. Hic iam facto $x = 1$ numerator dabit

$$2(n+p)(n+p-1) - (n+p-1)^2 - (nn-1) = 2np + pp,$$

denominator vero 2, unde valor quaesitus resultat $np + \frac{1}{2}pp$, prorsus uti supra invenimus. Hinc igitur abunde patet egregius usus nostrae reductionis. Quin etiam casus adhuc simplicior, quo $\mu = 0$, haud exiguam moram creat.

EXEMPLUM 2 QUO $\mu = 0$ ET $\nu = 1$

Hic erit $M = 1$ ob $Q = x^{n+p}$ manente $R = \frac{x^{n+p} - x^n}{lx}$; tum erit $\frac{\int}{p}\cdot M = p$, unde aequatio more solito expressa fiet

$$\frac{x^{n+p} - x^n}{lx} = p.$$

Posito autem $x = 1$ in parte sinistra tam numerator quam denominator evanescunt, unde eorum differentialibus substitutis ista fractio evadet

$$\frac{(n+p)x^{n+p-1} - nx^{n-1}}{1:x},$$

quae fractio posito $x = 1$ praebet p.

EXEMPLUM 3 QUO $\mu = 0$ ET $\nu = 2$

Hic ergo erit $M = 1$ ideoque $\frac{\int^2}{p} \cdot M = \frac{1}{2} pp$, cui ergo ipsa quantitas R aequabitur; sicque orietur haec aequatio

$$\frac{x^{n+p} - x^n}{(lx)^2} - \frac{px^n}{lx} = \frac{1}{2} pp,$$

cuius veritas neutiquam in oculos incurrit; quamobrem quantitas R ad unicam fractionem reducatur, quae erit

$$\frac{x^{n+p} - x^n - px^n lx}{(lx)^2},$$

quae fractio, si loco numeratoris et denominatoris eorum differentialia substituantur, abit in sequentem

$$\frac{(n+p)x^{n+p} - nx^n - npx^n lx - px^n}{2lx};$$

haec vero fractio eadem operatione instituta reducitur ad hanc

$$\frac{(n+p)^2 x^{n+p} - nnx^n - nnpx^n lx - 2npx^n}{2},$$

quae expressio posito $x = 1$ manifesto abit in $\frac{1}{2} pp$.

EXEMPLUM 4 QUO $\mu = 0$ ET $\nu = \nu$

Hic ergo erit $M = 1$ ideoque

$$\frac{\int^\nu}{p} \cdot M = \frac{p^\nu}{1 \cdot 2 \cdot 3 \cdots \nu}.$$

Porro vero vidimus esse

$$R = \frac{x^{n+p}}{(lx)^\nu} - x^n \left(\frac{1}{(lx)^\nu} + \frac{p}{(lx)^{\nu-1}} + \cdots + \frac{p^{\nu-1}}{1 \cdot 2 \cdot 3 \cdots (\nu-1) lx} \right)$$

atque haec expressio R ita est comparata, ut posito $x = 1$ eius valor futurus sit $\frac{p^\nu}{1 \cdot 2 \cdot 3 \cdots \nu}$.

EXEMPLUM 5 QUO $\mu = 1$ ET $\nu = \nu$

Hic ergo erit $M = n + p$ ideoque

$$\frac{\int^\nu}{p} \cdot M = \frac{n(\nu+1)p^\nu + p^{\nu+1}}{1 \cdot 2 \cdot 3 \cdots (\nu+1)}.$$

Quodsi iam ponatur

$$R = \frac{x^{n+p}}{(lx)^\nu} - x^n \left(\frac{1}{(lx)^\nu} + \frac{p}{(lx)^{\nu-1}} + \frac{pp}{1 \cdot 2 (lx)^{\nu-2}} + \cdots + \frac{p^{\nu-1}}{1 \cdot 2 \cdots (\nu-1) lx} \right),$$

quae expressio ut functio solius x spectetur, tum posito $x = 1$ erit more solito

$$\left(\frac{\partial R}{\partial x} \right) = \frac{p^\nu (n(\nu+1) + p)}{1 \cdot 2 \cdot 3 \cdots (\nu+1)}.$$

Ubi facile intelligitur differentiale ipsius R formulam producere multo magis complicatam, cuius omnibus terminis ad communem denominatorem reductis, qui erit $(lx)^{\nu+1}$, si per regulam vulgarem istius fractionis valorem casu $x = 1$ explorare vellemus, tum tam numerator quam denominator $\nu + 1$ vicibus differentiari deberent, antequam eius verus valor definiri posset, quem tamen nunc certe novimus fore $\frac{p^\nu (n(\nu+1) + p)}{1 \cdot 2 \cdot 3 \cdots (\nu+1)}$.

EXEMPLUM 6 QUO $\mu = 2$ ET $\nu = \nu$

Hic ergo erit

$$M = (n+p)(n+p-1) = n(n-1) + (2n-1)p + pp$$

ideoque

$$\frac{\int^\nu}{p} \cdot M = \frac{n(n-1)p^\nu}{1 \cdot 2 \cdot 3 \cdots \nu} + \frac{(2n-1)p^{\nu+1}}{2 \cdot 3 \cdot 4 \cdots (\nu+1)} + \frac{p^{\nu+2}}{3 \cdot 4 \cdot 5 \cdots (\nu+2)};$$

tum igitur, si ut ante fuerit

$$R = \frac{x^{n+p}}{(lx)^\nu} - x^n \left(\frac{1}{(lx)^\nu} + \frac{p}{(lx)^{\nu-1}} + \frac{pp}{1 \cdot 2 (lx)^{\nu-2}} + \cdots + \frac{p^{\nu-1}}{1 \cdot 2 \cdots (\nu-1) lx} \right),$$

casu $x = 1$ erit

$$\left(\frac{\partial \partial R}{\partial x^2} \right) = \frac{n(n-1)p^\nu}{1 \cdot 2 \cdot 3 \cdots \nu} + \frac{(2n-1)p^{\nu+1}}{2 \cdot 3 \cdot 4 \cdots (\nu+1)} + \frac{p^{\nu+2}}{3 \cdot 4 \cdot 5 \cdots (\nu+2)},$$

quam veritatem more consueto evolvere nemo certe susceperit. Atque ex his iam facile apparet, quomodo has conclusiones pro maioribus valoribus indicis μ formari oporteat.

PROBLEMA 4

Si V fuerit functio quaecunque binarum variabilium x et p et omnes operationes in theoremate quarto indicatae absolvantur, tum vero statuatur $x=1$, exhibere aequalitatem, ad quam hoc theorema perducit.

SOLUTIO

Quoniam in nostro theoremate quarto posuimus $Q=\frac{\int^{\mu}}{x}\cdot V$, qui valor posito $x=1$ abeat in M, ita ut M futura sit sola functio ipsius p, tum vero $R=\frac{\int^{\nu}}{p}\cdot V$, vi nostri theorematis semper erit

$$\frac{\int^{\mu}}{x}\cdot R=\frac{\int^{\nu}}{p}\cdot M,$$

siquidem omnes integrationes ita absolvantur, ut singula integralia evanescant posito sive $x=0$ sive $p=0$, omnibus autem operationibus peractis statuatur $x=1$. Quodsi iam pro V accipiamus hanc functionem x^{n+p}, primo valores litterae M pro variis indicibus μ sequenti modo se habebunt:

1. Si $\mu=0$, erit $M=1$,

2. si $\mu=1$, erit $M=\frac{1}{n+p+1}$,

3. si $\mu=2$, erit $M=\frac{1}{(n+p+1)(n+p+2)}$,

4. si $\mu=3$, erit $M=\frac{1}{(n+p+1)(n+p+2)(n+p+3)}$

etc.

Hi autem valores ipsius M ope propositionis supra allegatae, quae erat

$$M=\frac{1}{1\cdot 2\cdot 3\cdots(\mu-1)}\left\{\begin{array}{c}\frac{1}{n+p+1}-\frac{\mu-1}{n+p+2}+\frac{(\mu-1)(\mu-2)}{1\cdot 2(n+p+3)}\\ -\frac{(\mu-1)(\mu-2)(\mu-3)}{1\cdot 2\cdot 3(n+p+4)}+\text{etc.}\end{array}\right\},$$

sequenti modo pro variis valoribus indicis μ se habebunt:

1. Si $\mu=0$, valor $M=1$,

2. si $\mu=1$, „ $M=\frac{1}{n+p+1}$,

3. si $\mu=2$, „ $M=\frac{1}{n+p+1}-\frac{1}{n+p+2}$,

4. si $\mu=3$, „ $M=\frac{1}{2}\left(\frac{1}{n+p+1}-\frac{2}{n+p+2}+\frac{1}{n+p+3}\right)$,

5. si $\mu=4$, „ $M=\frac{1}{6}\left(\frac{1}{n+p+1}-\frac{3}{n+p+2}+\frac{3}{n+p+3}-\frac{1}{n+p+4}\right)$,

6. si $\mu=5$, „ $M=\frac{1}{24}\left(\frac{1}{n+p+1}-\frac{4}{n+p+2}+\frac{6}{n+p+3}-\frac{4}{n+p+4}+\frac{1}{n+p+5}\right)$

etc.

Deinde pro littera R, si indici ν successive tribuantur valores 0, 1, 2, 3, 4 etc., reperietur,

1. si $\nu=0$, fore $R=x^{n+p}$,

2. si $\nu=1$, „ $R=\frac{x^{n+p}-x^n}{lx}$,

3. si $\nu=2$, „ $R=\frac{x^{n+p}-x^n}{(lx)^2}-\frac{px^n}{lx}$,

4. si $\nu=3$, „ $R=\frac{x^{n+p}-x^n}{(lx)^3}-\frac{px^n}{(lx)^2}-\frac{ppx^n}{2lx}$,

5. si $\nu=4$, „ $R=\frac{x^{n+p}-x^n}{(lx)^4}-\frac{px^n}{(lx)^3}-\frac{ppx^n}{2(lx)^2}-\frac{p^3x^n}{6lx}$

etc.

Hinc igitur sequentia exempla evolvamus.

EXEMPLUM 1 QUO $\mu=0$ ET $\nu=0$

Hoc casu erit

$$M=1 \quad \text{et} \quad R=x^{n+p},$$

unde facto $x=1$ erit utique

$$x^{n+p}=1.$$

EXEMPLUM 2 QUO $\mu = 0$ ET $\nu = 1$

Hoc ergo casu erit

$$M = 1 \quad \text{et} \quad R = \frac{x^{n+p} - x^n}{lx},$$

unde posito $x = 1$ fiet

$$\frac{x^{n+p} - x^n}{lx} = p.$$

EXEMPLUM 3 QUO $\mu = 0$ ET $\nu = 2$

Hoc ergo casu adhuc est

$$M = 1 \quad \text{et} \quad R = \frac{x^{n+p} - x^n}{(lx)^2} - \frac{px^n}{lx}.$$

Hinc ergo posito $x = 1$ prodibit ista aequalitas

$$\frac{x^{n+p}}{(lx)^2} - x^n\left(\frac{1}{(lx)^2} + \frac{p}{lx}\right) = \frac{pp}{2}.$$

EXEMPLUM 4 QUO $\mu = 0$ ET $\nu = 3$

Hic ergo manente $M = 1$ erit

$$R = \frac{x^{n+p}}{(lx)^3} - x^n\left(\frac{1}{(lx)^3} + \frac{p}{(lx)^2} + \frac{pp}{2lx}\right);$$

quare posito $x = 1$ habebitur ista aequatio

$$\frac{x^{n+p}}{(lx)^3} - x^n\left(\frac{1}{(lx)^3} + \frac{p}{(lx)^2} + \frac{pp}{2lx}\right) = \frac{p^3}{6}.$$

Haec autem exempla iam in praecedente problemate occurrunt, quia signa $\int^0$ et ∂^0 aequivalent.

EXEMPLUM 5 QUO $\mu = 1$ ET $\nu = 1$

Hoc casu erit

$$M = \frac{1}{n+p+1} \quad \text{et} \quad R = \frac{x^{n+p} - x^n}{lx},$$

unde, cum fiat $\int R\partial x = \int M\partial p$, erit

$$\int \frac{x^{n+p} - x^n}{lx} \partial x = l\frac{n+p+1}{n+1},$$

quod est illud ipsum theorema, quod non ita pridem inveneram et Geometris proposueram.[1])

EXEMPLUM 6 QUO $\mu = 2$ ET $\nu = 1$

Hoc casu erit

$$M = \frac{1}{n+p+1} - \frac{1}{n+p+2}$$

manente

$$R = \frac{x^{n+p} - x^n}{lx}.$$

Hinc igitur posito $x = 1$ oritur ista aequatio

$$\int \partial x \int \frac{x^{n+p} - x^n}{lx} \partial x = l\frac{n+p+1}{n+1} - l\frac{n+p+2}{n+2};$$

haec autem veritas haud difficulter ex praecedente exemplo deduci potest. Cum enim in genere sit

$$\int \partial x \int R\partial x = x\int R\partial x - \int Rx\partial x$$

ideoque casu $x = 1$

$$\int \partial x \int R\partial x = \int R\partial x - \int Rx\partial x,$$

ob $R = \frac{x^{n+p} - x^n}{lx}$ erit ex exemplo praecedente

$$\int R\partial x = l\frac{n+p+1}{n+1}$$

atque indidem loco n scribendo $n+1$ erit

$$\int Rx\partial x = l\frac{n+p+2}{n+2}$$

sicque ipse valor inventus prodit.

1) Vide notam p. 336. A. L.

EXEMPLUM 7 QUO $\mu = 3$ ET $\nu = 1$

Hoc ergo casu erit

$$M = \frac{1}{2}\left(\frac{1}{n+p+1} - \frac{2}{n+p+2} + \frac{1}{n+p+3}\right)$$

hincque

$$\int M\partial p = \frac{1}{2}\, l\frac{n+p+1}{n+1} - \frac{2}{2}\, l\frac{n+p+2}{n+2} + \frac{1}{2}\, l\frac{n+p+3}{n+3};$$

at pro R habetur adhuc valor praecedens $R = \frac{x^{n+p} - x^n}{lx}$. Quare cum per propositionem supra allatam sit

$$\int\partial x\int\partial x\int R\partial x = \int\frac{R\partial x(1-x)^2}{1\cdot 2},$$

habebimus per simplex signum summatorium

$$\int\frac{(1-x)^2(x^{n+p}-x^n)}{lx}\partial x = l\frac{n+p+1}{n+1} - 2l\frac{n+p+2}{n+2} + l\frac{n+p+3}{n+3}.$$

EXEMPLUM 8 QUO $\mu = 4$ ET $\nu = 1$

Hoc casu erit

$$M = \frac{1}{6}\left(\frac{1}{n+p+1} - \frac{3}{n+p+2} + \frac{3}{n+p+3} - \frac{1}{n+p+4}\right)$$

hincque

$$\int M\partial p = \frac{1}{6}\, l\frac{n+p+1}{n+1} - \frac{3}{6}\, l\frac{n+p+2}{n+2} + \frac{3}{6}\, l\frac{n+p+3}{n+3} - \frac{1}{6}\, l\frac{n+p+4}{n+4}.$$

Deinde, cum ut ante sit $R = \frac{x^{n+p}-x^n}{lx}$, ob

$$\int\partial x\int\partial x\int\partial x\int R\partial x = \frac{1}{6}\int R\partial x(1-x)^3$$

erit

$$\int\frac{(1-x)^3(x^{n+p}-x^n)}{lx}\partial x = l\frac{n+p+1}{n+1} - 3l\frac{n+p+2}{n+2} + 3l\frac{n+p+3}{n+3} - l\frac{n+p+4}{n+4}.$$

Superfluum autem foret indici μ maiores valores tribuere, cum facta evolutione formulae $(1-x)^{\mu-1}$ ex exemplo quinto iidem valores essent prodituri.

EXEMPLUM 9 QUO $\mu = 1$ ET $\nu = 2$

Hoc ergo casu erit

$$M = \frac{1}{n+p+1}$$

hincque

$$\int M\partial p = l\frac{n+p+1}{n+1} \quad \text{et} \quad \int\partial p\int M\partial p = (n+p+1)\,l\frac{n+p+1}{n+1} - p.$$

Facilius autem hic valor reperitur ope reductionis generalis

$$\int\partial p\int M\partial p = p\int M\partial p - \int Mp\partial p;$$

namque ob $M = \frac{1}{n+p+1}$ erit

$$\int M\partial p = l\frac{n+p+1}{n+1},$$

deinde vero ob

$$Mp = \frac{p}{n+p+1} = 1 - \frac{n+1}{n+p+1}$$

erit

$$\int Mp\partial p = p - (n+1)\,l\frac{n+p+1}{n+1},$$

unde colligitur

$$\int\partial p\int M\partial p = pl\frac{n+p+1}{n+1} + (n+1)\,l\frac{n+p+1}{n+1} - p$$

ut ante. Tum vero erit

$$R = \frac{x^{n+p}}{(lx)^2} - x^n\left(\frac{1}{(lx)^2} + \frac{p}{lx}\right);$$

hinc, cum sit $\int R\partial x = \int\partial p\int M\partial p$, erit

$$\int\frac{x^{n+p}}{(lx)^2}\partial x - \int x^n\left(\frac{1}{(lx)^2} + \frac{p}{lx}\right)\partial x = (n+p+1)\,l\frac{n+p+1}{n+1} - p.$$

EXEMPLUM 10 QUO $\mu = 2$ ET $\nu = 2$

Hoc ergo casu erit

$$M = \frac{1}{n+p+1} - \frac{1}{n+p+2}$$

hincque

$$\int M\partial p = l\frac{n+p+1}{n+1} - l\frac{n+p+2}{n+2}$$

et ob superiorem reductionem hinc fit

$$Mp = \frac{p}{n+p+1} - \frac{p}{n+p+2} = -\frac{n+1}{n+p+1} + \frac{n+2}{n+p+2}$$

ideoque

$$\int Mp\partial p = -(n+1)\,l\frac{n+p+1}{n+1} + (n+2)\,l\frac{n+p+2}{n+2},$$

ita ut iam sit

$$\int\partial p\int M\partial p = pl\frac{n+p+1}{n+1} - pl\frac{n+p+2}{n+2} + (n+1)\,l\frac{n+p+1}{n+1} - (n+2)\,l\frac{n+p+2}{n+2};$$

quare, cum sit

$$\int\partial x\int R\partial x = \int\partial p\int M\partial p,$$

ob

$$\int\partial x\int R\partial x = \int R\partial x - \int Rx\partial x$$

aequatio hinc oriunda fiet

$$\int\frac{(1-x)\,x^{n+p}\partial x}{(lx)^2} - \int(1-x)\,x^n\left(\frac{1}{(lx)^2} + \frac{p}{lx}\right)\partial x$$
$$= (n+p+1)\,l\frac{n+p+1}{n+1} - (n+p+2)\,l\frac{n+p+2}{n+2}.$$

EXEMPLUM 11 QUO $\mu = 3$ ET $\nu = 2$

Hoc ergo casu est

$$M = \frac{1}{2}\left(\frac{1}{n+p+1} - \frac{2}{n+p+2} + \frac{1}{n+p+3}\right),$$

hinc

$$\int M\partial p = \frac{1}{2}\,l\frac{n+p+1}{n+1} - \frac{2}{2}\,l\frac{n+p+2}{n+2} + \frac{1}{2}\,l\frac{n+p+3}{n+3},$$

tum vero

$$Mp = -\frac{\frac{1}{2}(n+1)}{n+p+1} + \frac{\frac{2}{2}(n+2)}{n+p+2} - \frac{\frac{1}{2}(n+3)}{n+p+3}$$

ideoque

$$\int Mp\partial p = -\frac{1}{2}(n+1)\,l\frac{n+p+1}{n+1} + \frac{2}{2}(n+2)\,l\frac{n+p+2}{n+2} - \frac{1}{2}(n+3)\,l\frac{n+p+3}{n+3};$$

consequenter

$$\int\partial p\int M\partial p = \left\{\begin{aligned} &+\frac{1}{2}(n+p+1)\,l\frac{n+p+1}{n+1}\\ &-\frac{2}{2}(n+p+2)\,l\frac{n+p+2}{n+2}\\ &+\frac{1}{2}(n+p+3)\,l\frac{n+p+3}{n+3}\end{aligned}\right\}.$$

Deinde vero manente R ut ante, quoniam sumto $x = 1$ in genere est

$$\int\partial x\int\partial x\int R\partial x = \tfrac{1}{2}\int R\partial x(1-x)^2,$$

hinc resultabit sequens aequatio

$$\int\frac{(1-x)^2x^{n+p}\partial x}{(lx)^2} - \int(1-x)^2x^n\left(\frac{1}{(lx)^2} + \frac{p}{lx}\right)\partial x$$
$$= \left\{\begin{aligned} &+\frac{1}{2}(n+p+1)\,l\frac{n+p+1}{n+1}\\ &-\frac{2}{2}(n+p+2)\,l\frac{n+p+2}{n+2}\\ &+\frac{1}{2}(n+p+3)\,l\frac{n+p+3}{n+3}\end{aligned}\right\}.$$

EXEMPLUM 12 QUO $\mu = 1$ ET $\nu = 3$

Hoc igitur casu erit

$$M = \frac{1}{n+p+1},$$

et quia in genere est

$$\int\partial p\int\partial p\int M\partial p = \tfrac{1}{2}pp\int M\partial p - \tfrac{2}{2}p\int Mp\partial p + \tfrac{1}{2}\int Mpp\partial p,$$

habebimus

$$\int M\partial p = l\frac{n+p+1}{n+1},$$

$$\int Mp\partial p = p-(n+1)\,l\frac{n+p+1}{n+1}$$

et

$$\int Mpp\partial p = \frac{1}{2}pp-(n+1)p+(n+1)^2\,l\frac{n+p+1}{n+1};$$

ex his colligitur

$$\int\partial p\int\partial p\int M\partial p = \frac{1}{2}(n+p+1)^2\,l\frac{n+p+1}{n+1}-\frac{3}{4}pp-\frac{1}{2}(n+1)p.$$

Deinde erit hic

$$R = \frac{x^{n+p}}{(lx)^3}-x^n\left(\frac{1}{(lx)^3}+\frac{p}{(lx)^2}+\frac{pp}{2lx}\right).$$

Hinc igitur resultat sequens aequatio

$$\int\frac{x^{n+p}\partial x}{(lx)^3}-\int x^n\left(\frac{1}{(lx)^3}+\frac{p}{(lx)^2}+\frac{pp}{2lx}\right)\partial x$$
$$= \frac{1}{2}(n+p+1)^2\,l\frac{n+p+1}{n+1}-\frac{3}{4}pp-\frac{1}{2}(n+1)p.$$

EXEMPLUM 13 QUO $\mu = 2$ ET $\nu = 3$

Cum hoc casu sit

$$M = \frac{1}{n+p+1}-\frac{1}{n+p+2},$$

ob

$$\int\partial p\int\partial p\int M\partial p = \frac{1}{2}pp\int M\partial p-\frac{2}{2}p\int Mp\partial p+\frac{1}{2}\int Mpp\partial p$$

quaeratur

$$\int M\partial p = l\frac{n+p+1}{n+1}-l\frac{n+p+2}{n+2}.$$

Porro ob

$$Mp = -\frac{n+1}{n+p+1}+\frac{n+2}{n+p+2}$$

erit

$$\int Mp\partial p = -(n+1)\,l\frac{n+p+1}{n+1}+(n+2)\,l\frac{n+p+2}{n+2}$$

et

$$\int Mpp\partial p = -(n+1)p+(n+1)^2\,l\frac{n+p+1}{n+1}+(n+2)p-(n+2)^2\,l\frac{n+p+2}{n+2},$$

unde fit

$$\int\partial p\int\partial p\int M\partial p = \frac{1}{2}(n+p+1)^2\,l\frac{n+p+1}{n+1}-\frac{1}{2}(n+p+2)^2\,l\frac{n+p+2}{n+2}+\frac{1}{2}p.$$

Deinde manente R ut supra erit $\int\partial x\int R\partial x = \int R\partial x(1-x)$, unde colligimus

$$\int\frac{(1-x)x^{n+p}\partial x}{(lx)^3}-\int(1-x)x^n\left(\frac{1}{(lx)^3}+\frac{p}{(lx)^2}+\frac{pp}{2lx}\right)\partial x$$
$$= \frac{1}{2}(n+p+1)^2\,l\frac{n+p+1}{n+1}-\frac{1}{2}(n+p+2)^2\,l\frac{n+p+2}{n+2}+\frac{1}{2}p.$$

SCHOLION

Ad illustranda haec problemata loco V alia functione determinata praeter $V = x^{n+p}$ uti non licuit, propterea quod alia huiusmodi forma non constat, cuius omnium ordinum integralia ex variabilitate ipsius x oriunda re ipsa exhiberi eorumque valores casu $x = 1$ dari queant. Hic enim ob nullum plane usum memorabilem reiici convenit tales formas $V = X + P$ et $V = XP$, ubi X significaret functionem ipsius X tantum, P vero ipsius p tantum. Sin autem in unica integratione ex sola variabili x nata acquiescere velimus, praeter formulam hactenus tractatam x^{n+p} etiam duae sequentes in usum vocari possunt

$$V = \frac{x^{n+p-1}+x^{n-p-1}}{1+x^{2n}} \quad \text{et} \quad V = \frac{x^{n+p-1}-x^{n-p-1}}{1-x^{2n}},$$

quandoquidem ostendi utroque casu valorem integralis $\frac{\int}{x}\cdot V$ sive $\int V\partial x$ casu, quo ponitur $x = 1$, admodum commode per functionem solius p exprimi posse, postquam scilicet integrale ita fuerit sumtum, ut evanescat posito $x = 0$. Iam dudum enim demonstravi*) sub his conditionibus fore

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*) Videatur Dissertatio Ill. EULERI: *De valore formulae integralis*

$$\int\frac{z^{m-1}\pm z^{n-m-1}}{1\pm z^n}\partial z$$

casu, quo post integrationem ponitur $z = 1$. Nov. Comment. T. XIX.[1])

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1) Quae dissertatio est Commentatio 462 (indicis ENESTROEMIANI), *LEONHARDI EULERI Opera omnia,* series I, vol. 17, p. 358. A. L.

$$\text{I.}\quad \int \frac{x^{n+p-1}+x^{n-p-1}}{1+x^{2n}}\partial x = \frac{\pi}{2n\cos.\frac{\pi p}{2n}},$$

$$\text{II.}\quad \int \frac{x^{n+p-1}-x^{n-p-1}}{1-x^{2n}}\partial x = -\frac{\pi}{2n}\operatorname{tang}.\frac{\pi p}{2n}.$$

Quamobrem operae pretium erit bina problemata II et IV etiam per has formulas illustrare. Ex utroque scilicet problemate sumto indice $\mu=1$ primo deduximus $Q=\frac{\int}{x}\cdot V$, tum vero posito $x=1$ fecimus $Q=M$, unde casu formulae prioris perpetuo erit

$$M=\frac{\pi}{2n\cos.\frac{\pi p}{2n}},$$

casu posterioris formulae

$$M=-\frac{\pi}{2n}\operatorname{tang}.\frac{\pi p}{2n}.$$

Pro altera autem littera R in problemate secundo erat $R=\frac{\partial^\nu}{p}\cdot V$, unde pro formula prima casu $\nu=1$ erit

$$R=\frac{x^{n+p-1}-x^{n-p-1}}{1+x^{2n}}lx$$

et pro posteriore

$$R=\frac{x^{n+p-1}+x^{n-p-1}}{1-x^{2n}}lx.$$

Deinde vero sumto $\nu=2$ erit pro priore formula

$$R=\frac{x^{n+p-1}+x^{n-p-1}}{1+x^{2n}}(lx)^2$$

et pro posteriore

$$R=\frac{x^{n+p-1}-x^{n-p-1}}{1-x^{2n}}(lx)^2.$$

Simili modo sumto $\nu=3$ erit pro priore formula

$$R=\frac{x^{n+p-1}-x^{n-p-1}}{1+x^{2n}}(lx)^3,$$

pro posteriore vero

$$R=\frac{x^{n+p-1}+x^{n-p-1}}{1-x^{2n}}(lx)^3.$$

Atque adeo in genere pro omni indice ν erit pro priore forma

$$R=\frac{x^{n+p-1}\pm x^{n-p-1}}{1+x^{2n}}(lx)^\nu,$$

pro posteriore vero

$$R=\frac{x^{n+p-1}\mp x^{n-p-1}}{1-x^{2n}}(lx)^\nu.$$

Ubi signa superiora valent, si ν numerus par, inferiora vero, si impar.

Pro quarto autem problemate, ubi quantitas R per integrationes definiri debet, cum sit $R=\frac{\int^\nu}{p}\cdot V$, reperimus sumto $\nu=1$ pro priore formula

$$R=\frac{x^{n+p-1}-x^{n-p-1}}{(1+x^{2n})lx},$$

pro posteriore vero formula reperitur

$$R=\frac{x^{n+p-1}+x^{n-p-1}-2x^{n-1}}{(1-x^{2n})lx}.$$

Sumto autem $\nu=2$ habebimus pro formula priore

$$R=\frac{x^{n+p-1}+x^{n-p-1}-2x^{n-1}}{(1+x^{2n})(lx)^2},$$

pro posteriore vero

$$R=\frac{x^{n+p-1}-x^{n-p-1}}{(1-x^{2n})(lx)^2}-\frac{2x^{n-1}p}{(1-x^{2n})lx}$$

sive

$$R=\frac{x^{n+p-1}-x^{n-p-1}-2x^{n-1}plx}{(1-x^{2n})(lx)^2}.$$

Deinde vero sumto $\nu=3$ erit pro priore formula

$$R=\frac{x^{n+p-1}-x^{n-p-1}-2px^{n-1}lx}{(1+x^{2n})(lx)^3}$$

et pro posteriore formula

$$R=\frac{x^{n+p-1}+x^{n-p-1}-2x^{n-1}-p^2x^{n-1}(lx)^2}{(1-x^{2n})(lx)^3}.$$

Sumatur porro $\nu=4$ ac reperiemus pro formula priore

$$R=\frac{x^{n+p-1}-x^{n-p-1}-2x^{n-1}-ppx^{n-1}(lx)^2}{(1+x^{2n})(lx)^4},$$

pro posteriore vero

$$R = \frac{x^{n+p-1} + x^{n-p-1} - 2px^{n-1}lx - \frac{1}{3}p^3x^{n-1}(lx)^3}{(1-x^{2n})(lx)^4}.$$

Sumatur porro $\nu = 5$ ac habebimus pro priore formula

$$R = \frac{x^{n+p-1} - x^{n-p-1} - 2px^{n-1}lx - \frac{1}{3}p^3x^{n-1}(lx)^3}{(1+x^{2n})(lx)^5}$$

et [pro posteriore]

$$R = \frac{x^{n+p-1} + x^{n-p-1} - 2x^{n-1} - ppx^{n-1}(lx)^2 - \frac{1}{12}p^4x^{n-1}(lx)^4}{(1-x^{2n})(lx)^5}.$$

Sit $\nu = 6$ eritque

$$R = \frac{x^{n+p-1} + x^{n-p-1} - 2x^{n-1}\left(1 + \frac{1}{2}p^2(lx)^2 + \frac{1}{24}p^4(lx)^4\right)}{(1+x^{2n})(lx)^6},$$

$$R = \frac{x^{n+p-1} - x^{n-p-1} - 2x^{n-1}\left(plx + \frac{1}{6}p^3(lx)^3 + \frac{1}{120}p^5(lx)^5\right)}{(1-x^{2n})(lx)^6}$$

et hinc lex iam satis elucet, qua sequentes valores progrediuntur.

CONSIDERATIO AEQUATIONIS

$$\int \frac{x^{n+p} + x^{n-p}}{1+x^{2n}} \cdot \frac{\partial x}{x} = \frac{\pi}{2n} \sec. \frac{\pi p}{2n}$$

Quodsi hic brevitatis gratia ponamus $M = \frac{\pi}{2n} \sec. \frac{\pi p}{2n}$, primo casu $x = 1$ ex problemate secundo derivantur sequentes aequalitates

$$\text{I.} \quad \int \frac{x^{n+p} - x^{n-p}}{1+x^{2n}} \cdot \frac{\partial x}{x}\, lx = \frac{\partial M}{\partial p},$$

$$\text{II.} \quad \int \frac{x^{n+p} + x^{n-p}}{1+x^{2n}} \cdot \frac{\partial x}{x}(lx)^2 = \frac{\partial\partial M}{\partial p^2},$$

$$\text{III.} \quad \int \frac{x^{n+p} - x^{n-p}}{1+x^{2n}} \cdot \frac{\partial x}{x}(lx)^3 = \frac{\partial^3 M}{\partial p^3},$$

$$\text{IV.} \quad \int \frac{x^{n+p} + x^{n-p}}{1+x^{2n}} \cdot \frac{\partial x}{x}(lx)^4 = \frac{\partial^4 M}{\partial p^4}$$

etc.

At vero ex problemate quarto prodeunt sequentes aequalitates

$$\text{I.} \quad \int \frac{x^{n+p} - x^{n-p}}{1+x^{2n}} \cdot \frac{\partial x}{xlx} = \int M\partial p,$$

$$\text{II.} \quad \int \frac{x^{n+p} + x^{n-p} - 2x^n}{1+x^{2n}} \cdot \frac{\partial x}{x(lx)^2} = \int \partial p \int M\partial p,$$

$$\text{III.} \quad \int \frac{x^{n+p} - x^{n-p} - 2x^n plx}{1+x^{2n}} \cdot \frac{\partial x}{x(lx)^3} = \int \partial p \int \partial p \int M\partial p,$$

$$\text{IV.} \quad \int \frac{x^{n+p} + x^{n-p} - 2x^n\left(1 + \frac{1}{2}pp(lx)^2\right)}{1+x^{2n}} \cdot \frac{\partial x}{x(lx)^4} = \int \partial p \int \partial p \int \partial p \int M\partial p,$$

$$\text{V.} \quad \int \frac{x^{n+p} - x^{n-p} - 2x^n\left(plx + \frac{1}{6}p^3(lx)^3\right)}{1+x^{2n}} \cdot \frac{\partial x}{x(lx)^5} = \int \partial p \int \partial p \int \partial p \int \partial p \int M\partial p,$$

$$\text{VI.} \quad \int \frac{x^{n+p} + x^{n-p} - 2x^n\left(1 + \frac{1}{2}p^2(lx)^2 + \frac{1}{24}p^4(lx)^4\right)}{1+x^{2n}} \cdot \frac{\partial x}{x(lx)^6} = \int \partial p \int \partial p \int \partial p \int \partial p \int \partial p \int M\partial p$$

etc.

CONSIDERATIO AEQUATIONIS

$$\int \frac{x^{n+p} - x^{n-p}}{1-x^{2n}} \cdot \frac{\partial x}{x} = -\frac{\pi}{2n} \text{tang.} \frac{\pi p}{2n}$$

Ponamus hic distinctionis gratia $N = -\frac{\pi}{2n} \text{tang.} \frac{\pi p}{2n}$ atque ex problemate secundo nascuntur sequentes aequalitates

$$\text{I.} \quad \int \frac{x^{n+p} + x^{n-p}}{1-x^{2n}} \cdot \frac{\partial x}{x}\, lx = \frac{\partial N}{\partial p},$$

$$\text{II.} \quad \int \frac{x^{n+p} - x^{n-p}}{1-x^{2n}} \cdot \frac{\partial x}{x}(lx)^2 = \frac{\partial\partial N}{\partial p^2},$$

$$\text{III.} \quad \int \frac{x^{n+p} + x^{n-p}}{1-x^{2n}} \cdot \frac{\partial x}{x}(lx)^3 = \frac{\partial^3 N}{\partial p^3},$$

$$\text{IV.} \quad \int \frac{x^{n+p} - x^{n-p}}{1-x^{2n}} \cdot \frac{\partial x}{x}(lx)^4 = \frac{\partial^4 N}{\partial p^4}$$

etc.

Verum ex theoremate quarto sequentes resultant aequalitates

$$\text{I. } \int \frac{x^{n+p}+x^{n-p}-2x^n}{1-x^{2n}} \cdot \frac{\partial x}{xlx} = \int N\partial p,$$

$$\text{II. } \int \frac{x^{n+p}-x^{n-p}-2x^n plx}{1-x^{2n}} \cdot \frac{\partial x}{x(lx)^2} = \int \partial p \int N\partial p,$$

$$\text{III. } \int \frac{x^{n+p}+x^{n-p}-2x^n\left(1+\frac{1}{2}pp(lx)^2\right)}{1-x^{2n}} \cdot \frac{\partial x}{x(lx)^3} = \int \partial p \int \partial p \int N\partial p,$$

$$\text{IV. } \int \frac{x^{n+p}-x^{n-p}-2x^n\left(plx+\frac{1}{6}p^3(lx)^3\right)}{1-x^{2n}} \cdot \frac{\partial x}{x(lx)^4} = \int \partial p \int \partial p \int \partial p \int N\partial p,$$

$$\text{V. } \int \frac{x^{n+p}+x^{n-p}-2x^n\left(1+\frac{1}{2}pp(lx)^2+\frac{1}{24}p^4(lx)^4\right)}{1-x^{2n}} \cdot \frac{\partial x}{x(lx)^5} = \int \partial p \int \partial p \int \partial p \int \partial p \int N\partial p$$

etc.

In his scilicet formulis quantitates M et N spectantur ut functiones ipsius p atque ex eius variabilitate tam differentiantur quam integrantur.

Ex his igitur abunde intelligitur omnia, quae super hoc argumento a me non ita pridem[1]) sunt prolata, tamquam casus valde particulares in praesenti tractatione contineri.

SCHOLION

Formulae autem istae sequenti modo succinctius exhiberi possunt, ad quas intelligendas notetur in formulis ad sinistram positis valores integralium esse extendendas ab $x=0$ ad $x=1$, in formulis autem ad dextram positis quantitatem p spectari ut variabilem et integralia ita capi, ut evanescant posito $p=0$, tum vero loco $\frac{\pi}{2}$ hic litteram ϱ scribi, ita ut ϱ sit character anguli recti. His igitur praenotatis ex integrali priori

$$\int \frac{x^p+x^{-p}}{x^n+x^{-n}} \cdot \frac{\partial x}{x} = \frac{\varrho}{n} \sec. \frac{p\varrho}{n}$$

per differentiationem sequentia deducuntur

1) Confer Commentationes 463 et 464 voluminis praecedentis. A. L.

$$\text{I. } \int \frac{x^p-x^{-p}}{x^n+x^{-n}} \cdot \frac{\partial x}{x}\, lx = \frac{\varrho}{n\partial p}\, \partial \sec. \frac{p\varrho}{n},$$

$$\text{II. } \int \frac{x^p+x^{-p}}{x^n+x^{-n}} \cdot \frac{\partial x}{x}(lx)^2 = \frac{\varrho}{n\partial p^2}\, \partial\partial \sec. \frac{p\varrho}{n},$$

$$\text{III. } \int \frac{x^p-x^{-p}}{x^n+x^{-n}} \cdot \frac{\partial x}{x}(lx)^3 = \frac{\varrho}{n\partial p^3}\, \partial^3 \sec. \frac{p\varrho}{n},$$

$$\text{IV. } \int \frac{x^p+x^{-p}}{x^n+x^{-n}} \cdot \frac{\partial x}{x}(lx)^4 = \frac{\varrho}{n\partial p^4}\, \partial^4 \sec. \frac{p\varrho}{n}$$

etc.,

per integrationem vero sequentes aequalitates oriuntur

$$\text{I. } \int \frac{x^p-x^{-p}}{x^n+x^{-n}} \cdot \frac{\partial x}{xlx} = \frac{\varrho}{n} \int \partial p \sec. \frac{p\varrho}{n},$$

$$\text{II. } \int \frac{x^p+x^{-p}-2}{x^n+x^{-n}} \cdot \frac{\partial x}{x(lx)^2} = \frac{\varrho}{n} \int \partial p \int \partial p \sec. \frac{p\varrho}{n},$$

$$\text{III. } \int \frac{x^p-x^{-p}-2plx}{x^n+x^{-n}} \cdot \frac{\partial x}{x(lx)^3} = \frac{\varrho}{n} \int \partial p \int \partial p \int \partial p \sec. \frac{p\varrho}{n},$$

$$\text{IV. } \int \frac{x^p+x^{-p}-2\left(1+\frac{1}{2}p^2(lx)^2\right)}{x^n+x^{-n}} \cdot \frac{\partial x}{x(lx)^4} = \frac{\varrho}{n} \int \partial p \int \partial p \int \partial p \int \partial p \sec. \frac{p\varrho}{n},$$

$$\text{V. } \int \frac{x^p-x^{-p}-2\left(plx+\frac{1}{6}p^3(lx)^3\right)}{x^n+x^{-n}} \cdot \frac{\partial x}{x(lx)^5} = \frac{\varrho}{n} \int \partial p \int \partial p \int \partial p \int \partial p \int \partial p \sec. \frac{p\varrho}{n}$$

etc.

Ex altera autem formula integrali principali

$$\int \frac{x^p-x^{-p}}{x^n-x^{-n}} \cdot \frac{\partial x}{x} = \frac{\varrho}{n} \operatorname{tang.} \frac{p\varrho}{n}$$

per differentiationem nascuntur sequentes aequationes

$$\text{I. } \int \frac{x^p+x^{-p}}{x^n-x^{-n}} \cdot \frac{\partial x}{x}\, lx = \frac{\varrho}{n\partial p}\, \partial \operatorname{tang.} \frac{p\varrho}{n},$$

$$\text{II. } \int \frac{x^p-x^{-p}}{x^n-x^{-n}} \cdot \frac{\partial x}{x}(lx)^2 = \frac{\varrho}{n\partial p^2}\, \partial\partial \operatorname{tang.} \frac{p\varrho}{n},$$

$$\text{III. } \int \frac{x^n+x^{-p}}{x^n-x^{-n}} \cdot \frac{\partial x}{x}(lx)^3 = \frac{\varrho}{n\partial p^3}\, \partial^3 \operatorname{tang.} \frac{p\varrho}{n},$$

$$\text{IV. } \int \frac{x^p-x^{-p}}{x^n-x^{-n}} \cdot \frac{\partial x}{x}(lx)^4 = \frac{\varrho}{n\partial p^4}\, \partial^4 \operatorname{tang.} \frac{p\varrho}{n}$$

etc.,

per integrationem vero colliguntur sequentes

$$\text{I. } \int \frac{x^p + x^{-p} - 2}{x^n - x^{-n}} \cdot \frac{\partial x}{x\,lx} = \frac{\varrho}{n} \int \partial p \,\text{tang.}\, \frac{p\varrho}{n},$$

$$\text{II. } \int \frac{x^p - x^{-p} - 2plx}{x^n - x^{-n}} \cdot \frac{\partial x}{x(lx)^2} = \frac{\varrho}{n} \int \partial p \int \partial p \,\text{tang.}\, \frac{p\varrho}{n},$$

$$\text{III. } \int \frac{x^p + x^{-p} - 2\left(1 + \frac{1}{2}p^2(lx)^2\right)}{x^n - x^{-n}} \cdot \frac{\partial x}{x(lx)^3} = \frac{\varrho}{n} \int \partial p \int \partial p \int \partial p \,\text{tang.}\, \frac{p\varrho}{n},$$

$$\text{IV. } \int \frac{x^p - x^{-p} - 2\left(plx + \frac{1}{6}p^3(lx)^3\right)}{x^n - x^{-n}} \cdot \frac{\partial x}{x(lx)^4} = \frac{\varrho}{n} \int \partial p \int \partial p \int \partial p \int \partial p \,\text{tang.}\, \frac{p\varrho}{n},$$

$$\text{V. } \int \frac{x^p + x^{-p} - 2\left(1 + \frac{1}{2}p^2(lx)^2 + \frac{1}{24}p^4(lx)^4\right)}{x^n - x^{-n}} \cdot \frac{\partial x}{x(lx)^5} = \frac{\varrho}{n} \int \partial p \int \partial p \int \partial p \int \partial p \int \partial p \,\text{tang.}\, \frac{p\varrho}{n}$$

etc.

Denique circa omnes has integrationes notari operae erit pretium, si integralia ad sinistram posita a termino $x = 0$ usque ad $x = \infty$ extendantur, tum eorum valores duplo fieri maiores.

INNUMERA THEOREMATA CIRCA FORMULAS INTEGRALES QUORUM DEMONSTRATIO VIRES ANALYSEOS SUPERARE VIDEATUR

Conventui exhibita die 18. Martii 1776

Commentatio 635 indicis ENESTROEMIANI
Nova acta academiae scientiarum Petropolitanae 5 (1787), 1789, p. 3—26
Summarium ibidem p. 61—62

SUMMARIUM

Comme ce mémoire n'est pas susceptible d'extrait, vu qu'il ne contient que des formules intégrales, nous nous contenterons d'indiquer la source, où l'immortel Auteur de ce mémoire a puisé le grand nombre de Théorèmes qu'on y trouve exposés et rédigés en quatorze ordres ou classes. Ces Théorèmes sont tous déduits de la considération: que si l'on désigne par la lettre P l'intégrale de la formule $\int V\partial x$, prise depuis le terme $x = 0$ jusqu'à un certain terme déterminé $x = k$, comme la variable x n'est plus contenue dans la quantité P, elle peut être regardée comme fonction d'une autre quantité p qui est renfermée en même tems dans la fonction V, et que de là on peut déduire, tant par la différentiation que par l'intégration, une infinité de formules intégrales, toutes comprises entre les mêmes termes d'intégration; par exemple:

Par la différentiation:

$$\int \partial x \left(\frac{\partial V}{\partial p}\right) = \frac{\partial P}{\partial p},$$

$$\int \partial x \left(\frac{\partial\partial V}{\partial p^2}\right) = \frac{\partial\partial P}{\partial p^2},$$

$$\int \partial x \left(\frac{\partial^3 V}{\partial p^3}\right) = \frac{\partial^3 P}{\partial p^3}$$

etc.

Par l'intégration:

$$\int \partial x \int V \partial p = \int P \partial p,$$

$$\int \partial x \int \partial p \int V \partial p = \int \partial p \int P \partial p,$$

$$\int \partial x \int \partial p \int \partial p \int V \partial p = \int \partial p \int \partial p \int P \partial p$$

etc.

en prenant les intégrales

$$\int V \partial p \quad \text{et} \quad \int P \partial p$$

de façon qu'elles évanouissent, en mettant $p = 0$.

C'est d'après ce principe que M. EULER traite, dans ce mémoire, quatorze formules dont il avoit déterminé autrefois les intégrales contenues entre les deux termes d'intégration $x = 0$ et $x = 1$ ou $x = \infty$, dans un mémoire inséré dans le volume précédent des Nouveaux Actes[1]); et il en déduit autant de classes de Théorèmes remarquables dont la démonstration directe paroît être effectivement au dessus des forces de l'Analyse.

Fundamentum horum theorematum in eiusmodi formulis integralibus $\int V \partial x$ est constitutum, quarum valor a termino $x = 0$ usque ad certum terminum definitum $x = k$ per expressionem finitam assignari queat. Quodsi enim istum valorem littera P designemus, ita ut sit

$$\int V \partial x \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = k \end{bmatrix} = P,$$

quoniam ipsa variabilis x in P non amplius inest, ea tamquam functio alius cuiuspiam quantitatis p, quae simul in functione V contineatur, spectari poterit; tum autem sub iisdem integrationis terminis innumerabiles aliae formulae integrales tam per differentiationem quam per integrationem, quemadmodum iam aliquoties fusius exposui[2]), derivari possunt, quae sunt:

1) Voir la note p. 375. A. L.

2) Vide imprimis Commentationem 464 (indicis ENESTROEMIANI): *Nova methodus quantitates integrales determinandi*, Novi comment. acad. sc. Petrop. 19 (1774), 1775, p. 66; *LEONHARDI EULERI Opera omnia*, series I, vol. 17, p. 421. A. L.

Per differentiationem:

$$\int \partial x \left(\frac{\partial V}{\partial p}\right) = \frac{\partial P}{\partial p},$$

$$\int \partial x \left(\frac{\partial\partial V}{\partial p^2}\right) = \frac{\partial\partial P}{\partial p^2},$$

$$\int \partial x \left(\frac{\partial^3 V}{\partial p^3}\right) = \frac{\partial^3 P}{\partial p^3},$$

$$\int \partial x \left(\frac{\partial^4 V}{\partial p^4}\right) = \frac{\partial^4 P}{\partial p^4}$$

etc.

Per integrationem:

$$\int \partial x \int V \partial p = \int P \partial p,$$

$$\int \partial x \int \partial p \int V \partial p = \int \partial p \int P \partial p,$$

$$\int \partial x \int \partial p \int \partial p \int V \partial p = \int \partial p \int \partial p \int P \partial p,$$

$$\int \partial x \int \partial p \int \partial p \int \partial p \int V \partial p = \int \partial p \int \partial p \int \partial p \int P \partial p$$

etc.,

ubi circa integralia

$$\int V \partial p \quad \text{et} \quad \int P \partial p$$

probe notandum est ea ita capi debere, ut evanescant posito $p = 0$, quod etiam de integrationibus repetitis perpetuo est tenendum.

Cum igitur nuper*) plures huiusmodi generis formulas integrales $\int V \partial x$ in medium attulerim, quarum valores a termino $x = 0$ usque ad terminum vel $x = 1$ vel $x = \infty$ expressione finita assignare licuit, ex qualibet earum formulas integrales tam per differentiationem quam per integrationem inde derivatas conspectui exponam, quas ergo secundum ordinem formularum principalium, ex quibus sunt deductae, hic distinguam.

*) Nova Acta Academiae Sc. Tom. III in Dissertatione: *Methodus facilis inveniendi integrale etc.*[1])

1) Quae dissertatio est Commentatio 620 huius voluminis. A. L.

ORDO PRIMUS
THEOREMATUM EX HAC FORMA PRINCIPALI DEDUCTORUM

$$\int \frac{\partial x}{x}\cdot\frac{x^p}{x^n+2\cos.\theta+x^{-n}}\left[\begin{matrix}\text{ab} & x=0\\ \text{ad} & x=\infty\end{matrix}\right]=\frac{\pi\sin.\frac{p}{n}\theta}{n\sin.\theta\sin.\frac{p}{n}\pi}$$

Haec integratio semper locum habet, nisi sit $p-n>0$; his igitur casibus exceptis ponamus brevitatis gratia

$$P=\frac{\pi\sin.\frac{p}{n}\theta}{n\sin.\theta\sin.\frac{p}{n}\pi};$$

tum vero etiam loco denominatoris $x^n+2\cos.\theta+x^{-n}$ scribamus Δ, ita ut iam P spectari possit tamquam functio ipsius p; quibus observatis per differentiationem hinc sequentia theoremata derivantur

$$\text{I. } \int \frac{x^p}{\Delta}\cdot\frac{\partial x lx}{x}\left[\begin{matrix}\text{ab} & x=0\\ \text{ad} & x=\infty\end{matrix}\right]=\frac{\partial P}{\partial p},$$

$$\text{II. } \int \frac{x^p}{\Delta}\cdot\frac{\partial x(lx)^2}{x}\left[\begin{matrix}\text{ab} & x=0\\ \text{ad} & x=\infty\end{matrix}\right]=\frac{\partial\partial P}{\partial p^2},$$

$$\text{III. } \int \frac{x^p}{\Delta}\cdot\frac{\partial x(lx)^3}{x}\left[\begin{matrix}\text{ab} & x=0\\ \text{ad} & x=\infty\end{matrix}\right]=\frac{\partial^3 P}{\partial p^3},$$

$$\text{IV. } \int \frac{x^p}{\Delta}\cdot\frac{\partial x(lx)^4}{x}\left[\begin{matrix}\text{ab} & x=0\\ \text{ad} & x=\infty\end{matrix}\right]=\frac{\partial^4 P}{\partial p^4}$$

etc.

Per integrationem autem inde sequentia theoremata oriuntur

$$\text{I. } \int \frac{x^p-1}{\Delta}\cdot\frac{\partial x}{xlx}\left[\begin{matrix}\text{ab} & x=0\\ \text{ad} & x=\infty\end{matrix}\right]=\int P\partial p,$$

$$\text{II. } \int \frac{x^p-1-plx}{\Delta}\cdot\frac{\partial x}{x(lx)^2}\left[\begin{matrix}\text{ab} & x=0\\ \text{ad} & x=\infty\end{matrix}\right]=\int\partial p\int P\partial p,$$

$$\text{III. } \int \frac{x^p-1-plx-\frac{1}{2}pp(lx)^2}{\Delta}\cdot\frac{\partial x}{x(lx)^3}\left[\begin{matrix}\text{ab} & x=0\\ \text{ad} & x=\infty\end{matrix}\right]=\int\partial p\int\partial p\int P\partial p,$$

$$\text{IV. } \int \frac{x^p-1-plx-\frac{1}{2}pp(lx)^2-\frac{1}{6}p^3(lx)^3}{\Delta}\cdot\frac{\partial x}{x(lx)^4}\left[\begin{matrix}\text{ab} & x=0\\ \text{ad} & x=\infty\end{matrix}\right]=\int\partial p\int\partial p\int\partial p\int P\partial p$$

etc.

Haecque theoremata aeque subsistunt, sive p sit numerus positivus sive negativus sive etiam integer sive fractus, dumne sit $p-n>0$ et integralia $\int P\partial p$, $\int\partial p\int P\partial p$, $\int\partial p\int\partial p\int P\partial p$ omniaque hinc deducta ita capiantur, ut evanescant posito $p=0$.

ORDO SECUNDUS
THEOREMATUM EX HAC FORMA PRINCIPALI DEDUCTORUM

$$\int \frac{x^p}{x^{-n}(1+x^n)^2}\cdot\frac{\partial x}{x}\left[\begin{matrix}\text{ab} & x=0\\ \text{ad} & x=\infty\end{matrix}\right]=\frac{\pi p}{nn\sin.\frac{p}{n}\pi}$$

Ponamus hic iterum denominatorem $x^{-n}(1+x^n)^2=\Delta$ sitque

$$P=\frac{\pi p}{nn\sin.\frac{p}{n}\pi},$$

ita ut P iterum sit functio ipsius p, ac primo per differentiationem hinc deducentur sequentia theoremata

$$\text{I. } \int \frac{x^p}{\Delta}\cdot\frac{\partial x lx}{x}\left[\begin{matrix}\text{ab} & x=0\\ \text{ad} & x=\infty\end{matrix}\right]=\frac{\partial P}{\partial p},$$

$$\text{II. } \int \frac{x^p}{\Delta}\cdot\frac{\partial x(lx)^2}{x}\left[\begin{matrix}\text{ab} & x=0\\ \text{ad} & x=\infty\end{matrix}\right]=\frac{\partial\partial P}{\partial p^2},$$

$$\text{III. } \int \frac{x^p}{\Delta}\cdot\frac{\partial x(lx)^3}{x}\left[\begin{matrix}\text{ab} & x=0\\ \text{ad} & x=\infty\end{matrix}\right]=\frac{\partial^3 P}{\partial p^3},$$

$$\text{IV. } \int \frac{x^p}{\Delta}\cdot\frac{\partial x(lx)^4}{x}\left[\begin{matrix}\text{ab} & x=0\\ \text{ad} & x=\infty\end{matrix}\right]=\frac{\partial^4 P}{\partial p^4}$$

etc.

Per integrationem autem inde sequentia theoremata oriuntur

$$\text{I. } \int \frac{x^p - 1}{\varDelta} \cdot \frac{\partial x}{x l x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \int P \partial p,$$

$$\text{II. } \int \frac{x^p - 1 - p l x}{\varDelta} \cdot \frac{\partial x}{x (l x)^2} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \int \partial p \int P \partial p,$$

$$\text{III. } \int \frac{x^p - 1 - p l x - \frac{1}{2} pp (l x)^2}{\varDelta} \cdot \frac{\partial x}{x (l x)^3} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \int \partial p \int \partial p \int P \partial p,$$

$$\text{IV. } \int \frac{x^p - 1 - p l x - \frac{1}{2} pp (l x)^2 - \frac{1}{6} p^3 (l x)^3}{\varDelta} \cdot \frac{\partial x}{x (l x)^4} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \int \partial p \int \partial p \int \partial p \int P \partial p$$

etc.,

ubi circa integrationes eadem sunt observanda, quae ante fuerant praecepta.

ORDO TERTIUS
THEOREMATUM EX HAC FORMA PRINCIPALI DEDUCTORUM

$$\int \frac{\partial x}{x} \cdot \frac{x^p}{x^n + \left(f + \frac{1}{f}\right) + x^{-n}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\pi \left(f^{\frac{p}{n}} - f^{-\frac{p}{n}}\right)}{n (f^1 - f^{-1}) \sin. \frac{p}{n} \pi}$$

Ponamus hic iterum pro denominatore

$$\varDelta = x^n + \left(f + \frac{1}{f}\right) + x^{-n};$$

tum vero sit

$$P = \frac{\pi \left(f^{\frac{p}{n}} - f^{-\frac{p}{n}}\right)}{n (f^1 - f^{-1}) \sin. \frac{p}{n} \pi} = \frac{\pi \left(f^{1 + \frac{p}{n}} - f^{1 - \frac{p}{n}}\right)}{n (ff - 1) \sin. \frac{p}{n} \pi}.$$

His positis ut ante per differentiationem sequentia theoremata deducuntur

$$\text{I. } \int \frac{x^p}{\varDelta} \cdot \frac{\partial x l x}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\partial P}{\partial p},$$

$$\text{II. } \int \frac{x^p}{\varDelta} \cdot \frac{\partial x (l x)^2}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\partial \partial P}{\partial p^2},$$

$$\text{III. } \int \frac{x^p}{\varDelta} \cdot \frac{\partial x (l x)^3}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\partial^3 P}{\partial p^3},$$

$$\text{IV. } \int \frac{x^p}{\varDelta} \cdot \frac{\partial x (l x)^4}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\partial^4 P}{\partial p^4}$$

etc.

Per integrationem autem eliciuntur sequentia

$$\text{I. } \int \frac{x^p - 1}{\varDelta} \cdot \frac{\partial x}{x l x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \int P \partial p,$$

$$\text{II. } \int \frac{x^p - 1 - p l x}{\varDelta} \cdot \frac{\partial x}{x (l x)^2} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \int \partial p \int P \partial p,$$

$$\text{III. } \int \frac{x^p - 1 - p l x - \frac{1}{2} pp (l x)^2}{\varDelta} \cdot \frac{\partial x}{x (l x)^3} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \int \partial p \int \partial p \int P \partial p,$$

$$\text{IV. } \int \frac{x^p - 1 - p l x - \frac{1}{2} pp (l x)^2 - \frac{1}{6} p^3 (l x)^3}{\varDelta} \cdot \frac{\partial x}{x (l x)^4} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \int \partial p \int \partial p \int \partial p \int P \partial p$$

etc.

Ubi denuo eadem sunt observanda, quae supra sunt praecepta.

ORDO QUARTUS
THEOREMATUM EX HAC FORMA PRINCIPALI DEDUCTORUM

$$\int \frac{\partial x}{x} \cdot \frac{x^p + x^{-p}}{x^n + 2 \cos. \theta + x^{-n}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\pi \sin. \frac{p}{n} \theta}{n \sin. \theta \sin. \frac{p}{n} \pi}$$

Statuamus hic iterum

$$\varDelta = x^n + 2 \cos. \theta + x^{-n}$$

sitque

$$P = \frac{\pi \sin. \frac{p}{n} \theta}{n \sin. \theta \sin. \frac{p}{n} \pi},$$

ita ut P tamquam functio ipsius p spectari possit; ubi probe notandum est hunc valorem integralem subsistere non posse, nisi sit $p < n$ ideoque fractio

$\frac{p}{n}$ unitate minor; atque sub iisdem conditionibus per differentiationem sequentia hinc deducuntur theoremata

$$\text{I. } \int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x lx}{x} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \frac{\partial P}{\partial p},$$

$$\text{II. } \int \frac{x^p + x^{-p}}{\Delta} \cdot \frac{\partial x (lx)^2}{x} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \frac{\partial \partial P}{\partial p^2},$$

$$\text{III. } \int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x (lx)^3}{x} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \frac{\partial^3 P}{\partial p^3},$$

$$\text{IV. } \int \frac{x^p + x^{-p}}{\Delta} \cdot \frac{\partial x (lx)^4}{x} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \frac{\partial^4 P}{\partial p^4}$$

etc.

Per integrationem autem colliguntur sequentia

$$\text{I. } \int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x}{x lx} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \int P \partial p,$$

$$\text{II. } \int \frac{x^p + x^{-p} - 2}{\Delta} \cdot \frac{\partial x}{x (lx)^2} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \int \partial p \int P \partial p,$$

$$\text{III. } \int \frac{x^p - x^{-p} - 2plx}{\Delta} \cdot \frac{\partial x}{x (lx)^3} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \int \partial p \int \partial p \int P \partial p,$$

$$\text{IV. } \int \frac{x^p + x^{-p} - 2 - \frac{3}{2} pp (lx)^2}{\Delta} \cdot \frac{\partial x}{x (lx)^4} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \int \partial p \int \partial p \int \partial p \int P \partial p$$

etc.

Quodsi eadem integralia extendantur ab $x=0$ ad $x=\infty$, eorum valores duplo evadent maiores.

ORDO QUINTUS
THEOREMATUM EX HAC FORMA PRINCIPALI DEDUCTORUM

$$\int \frac{\partial x}{x} \cdot \frac{x^p + x^{-p}}{x^{-n}(1+x^n)^2} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \frac{\pi p}{nn \sin. \frac{p}{n}\pi}$$

Statuamus igitur hic pro denominatore

$$\Delta = x^{-n}(1+x^n)^2$$

sitque

$$P = \frac{\pi p}{nn \sin. \frac{p}{n}\pi},$$

ita ut P spectari possit tamquam functio ipsius p, ubi perpetuo fractio $\frac{p}{n}$ unitate minor supponitur; quibus positis per differentiationem sequentia nascuntur theoremata

$$\text{I. } \int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x lx}{x} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \frac{\partial P}{\partial p},$$

$$\text{II. } \int \frac{x^p + x^{-p}}{\Delta} \cdot \frac{\partial x (lx)^2}{x} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \frac{\partial \partial P}{\partial p^2},$$

$$\text{III. } \int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x (lx)^3}{x} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \frac{\partial^3 P}{\partial p^3},$$

$$\text{IV. } \int \frac{x^p + x^{-p}}{\Delta} \cdot \frac{\partial x (lx)^4}{x} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \frac{\partial^4 P}{\partial p^4}$$

etc.

Per integrationem vero sequentia deducuntur

$$\text{I. } \int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x}{x lx} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \int P \partial p,$$

$$\text{II. } \int \frac{x^p + x^{-p} - 2}{\Delta} \cdot \frac{\partial x}{x (lx)^2} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \int \partial p \int P \partial p,$$

$$\text{III. } \int \frac{x^p - x^{-p} - 2plx}{\Delta} \cdot \frac{\partial x}{x (lx)^3} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \int \partial p \int \partial p \int P \partial p,$$

$$\text{IV. } \int \frac{x^p + x^{-p} - 2 - pp (lx)^2}{\Delta} \cdot \frac{\partial x}{x (lx)^4} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \int \partial p \int \partial p \int \partial p \int P \partial p$$

etc.

At si haec integralia ab $x=0$ ad $x=\infty$ capiantur, eorum valores evadent duplo maiores.

ORDO SEXTUS
THEOREMATUM EX HAC FORMA PRINCIPALI DEDUCTORUM

$$\int \frac{\partial x}{x} \cdot \frac{x^p + x^{-p}}{x^n + \left(f + \frac{1}{f}\right) + x^{-n}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\pi\left(f^{\frac{p}{n}} - f^{-\frac{p}{n}}\right)}{n(f^1 - f^{-1}) \sin. \frac{p}{n}\pi}$$

Statuamus

$$\Delta = x^n + \left(f + \frac{1}{f}\right) + x^{-n} - \frac{1}{x^n}(x^n + f)\left(x^n + \frac{1}{f}\right)$$

et sit

$$P = \frac{\pi\left(f^{\frac{p}{n}} - f^{-\frac{p}{n}}\right)}{n(f^1 - f^{-1}) \sin. \frac{p}{n}\pi},$$

ubi iterum fractio $\frac{p}{n}$ unitate minor supponitur. His observatis per differentiationem colligimus

$$\text{I. } \int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x l x}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial P}{\partial p},$$

$$\text{II. } \int \frac{x^p + x^{-p}}{\Delta} \cdot \frac{\partial x (lx)^2}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial\partial P}{\partial p^2},$$

$$\text{III. } \int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x (lx)^3}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial^3 P}{\partial p^3},$$

$$\text{IV. } \int \frac{x^p + x^{-p}}{\Delta} \cdot \frac{\partial x (lx)^4}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial^4 P}{\partial p^4}$$

etc.

Per integrationem autem sequentia theoremata nascuntur

$$\text{I. } \int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x}{x l x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int P \partial p,$$

$$\text{II. } \int \frac{x^p + x^{-p} - 2}{\Delta} \cdot \frac{\partial x}{x (lx)^2} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int P \partial p,$$

$$\text{III. } \int \frac{x^p - x^{-p} - 2plx}{\Delta} \cdot \frac{\partial x}{x (lx)^3} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int \partial p \int P \partial p,$$

$$\text{IV. } \int \frac{x^p + x^{-p} - 2 - pp(lx)^2}{\Delta} \cdot \frac{\partial x}{x (lx)^4} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int \partial p \int \partial p \int P \partial p$$

etc.

Quodsi haec integralia ab $x = 0$ ad $x = \infty$ extendantur, eorum valores erunt duplo maiores. Ceterum hic perspicuum est quantitatem f esse debere positivam, quia alias[1]) potestates $f^{\pm \frac{p}{n}}$ fieri possent imaginariae.

ORDO SEPTIMUS
THEOREMATUM EX HAC FORMA PRINCIPALI DEDUCTORUM

$$\int \frac{\partial x}{x} \cdot \frac{\cos. plx}{x^n + 2\cos.\theta + x^{-n}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\pi}{2n \sin.\theta} \cdot \frac{e^{\frac{p}{n}\theta} - e^{-\frac{p}{n}\theta}}{e^{\frac{p}{n}\pi} - e^{-\frac{p}{n}\pi}}$$

Statuamus hic iterum pro denominatore

$$\Delta = x^n + 2\cos.\theta + x^{-n}$$

sitque

$$P = \frac{\pi}{2n \sin.\theta} \cdot \frac{e^{\frac{p}{n}\theta} - e^{-\frac{p}{n}\theta}}{e^{\frac{p}{n}\pi} - e^{-\frac{p}{n}\pi}},$$

quae ergo quantitas iterum ut functio ipsius p spectari potest; ubi autem non amplius necesse est, ut fractio $\frac{p}{n}$ sit unitate minor. Hinc igitur per differentiationem sequentia derivantur theoremata

$$\text{I. } \int \frac{\sin. plx}{\Delta} \cdot \frac{\partial x l x}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = -\frac{\partial P}{\partial p},$$

$$\text{II. } \int \frac{\cos. plx}{\Delta} \cdot \frac{\partial x (lx)^2}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = -\frac{\partial\partial P}{\partial p^2},$$

$$\text{III. } \int \frac{\sin. plx}{\Delta} \cdot \frac{\partial x (lx)^3}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = +\frac{\partial^3 P}{\partial p^3},$$

$$\text{IV. } \int \frac{\cos. plx}{\Delta} \cdot \frac{\partial x (lx)^4}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = +\frac{\partial^4 P}{\partial p^4}$$

etc.,

1) Si quantitas f est negativa, integralia non valent. A. L.

per integrationem vero

$$\text{I. } \int \frac{\sin. plx}{\varDelta} \cdot \frac{\partial x}{xlx} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \int P \partial p,$$

$$\text{II. } \int \frac{1-\cos. plx}{\varDelta} \cdot \frac{\partial x}{x(lx)^2} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \int \partial p \int P \partial p,$$

$$\text{III. } \int \frac{plx-\sin. plx}{\varDelta} \cdot \frac{\partial x}{x(lx)^3} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \int \partial p \int \partial p \int P \partial p,$$

$$\text{IV. } \int \frac{\frac{1}{2}pp(lx)^2-1+\cos. plx}{\varDelta} \cdot \frac{\partial x}{x(lx)^4} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \int \partial p \int \partial p \int \partial p \int P \partial p,$$

$$\text{V. } \int \frac{\frac{1}{6}p^3(lx)^3-plx+\sin. plx}{\varDelta} \cdot \frac{\partial x}{x(lx)^5} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \int \partial p \int \partial p \int \partial p \int \partial p \int P \partial p$$

etc.

Haec igitur integralia si ab $x=0$ ad $x=\infty$ extendantur, iterum duplo fiunt maiora.

ORDO OCTAVUS
THEOREMATUM EX HAC FORMA PRINCIPALI DEDUCTORUM

$$\int \frac{\partial x}{x} \cdot \frac{\cos. plx}{x^{-n}(x^n+1)^2} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \frac{\pi}{nn} \cdot \frac{p}{e^{\frac{p}{n}\pi}-e^{-\frac{p}{n}\pi}}$$

Statuamus hic pro denominatore

$$\varDelta = x^{-n}(x^n+1)^2$$

sitque

$$P = \frac{\pi}{nn} \cdot \frac{p}{e^{\frac{p}{n}\pi}-e^{-\frac{p}{n}\pi}}$$

atque per differentiationem hinc deducentur sequentia theoremata

$$\text{I. } \int \frac{\sin. plx}{\varDelta} \cdot \frac{\partial x lx}{x} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = -\frac{\partial P}{\partial p},$$

$$\text{II. } \int \frac{\cos. plx}{\varDelta} \cdot \frac{\partial x(lx)^2}{x} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = -\frac{\partial \partial P}{\partial p^2},$$

$$\text{III. } \int \frac{\sin. plx}{\varDelta} \cdot \frac{\partial x(lx)^3}{x} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = +\frac{\partial^3 P}{\partial p^3},$$

$$\text{IV. } \int \frac{\cos. plx}{\varDelta} \cdot \frac{\partial x(lx)^4}{x} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = +\frac{\partial^4 P}{\partial p^4}$$

etc.

Per integrationem vero elicitur

$$\text{I. } \int \frac{\sin. plx}{\varDelta} \cdot \frac{\partial x}{xlx} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \int P \partial p,$$

$$\text{II. } \int \frac{1-\cos. plx}{\varDelta} \cdot \frac{\partial x}{x(lx)^2} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \int \partial p \int P \partial p,$$

$$\text{III. } \int \frac{plx-\sin. plx}{\varDelta} \cdot \frac{\partial x}{x(lx)^3} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \int \partial p \int \partial p \int P \partial p,$$

$$\text{IV. } \int \frac{\frac{1}{2}pp(lx)^2-1+\cos. plx}{\varDelta} \cdot \frac{\partial x}{x(lx)^4} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \int \partial p \int \partial p \int \partial p \int P \partial p,$$

$$\text{V. } \int \frac{\frac{1}{6}p^3(lx)^3-plx+\sin. plx}{\varDelta} \cdot \frac{\partial x}{x(lx)^5} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \int \partial p \int \partial p \int \partial p \int \partial p \int P \partial p$$

etc.

ORDO NONUS
THEOREMATUM EX HAC FORMA PRINCIPALI DEDUCTORUM

$$\int \frac{\partial x}{x} \cdot \frac{\cos. plx}{x^n+\left(f+\frac{1}{f}\right)+x^{-n}} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \frac{2\pi \sin. \frac{p}{n} lf}{n\left(f-\frac{1}{f}\right)\left(e^{\frac{p}{n}\pi}-e^{-\frac{p}{n}\pi}\right)}$$

Statuatur

$$\varDelta = x^n+\left(f+\frac{1}{f}\right)+x^{-n}$$

sitque

$$P = \frac{2\pi \sin. \frac{p}{n} lf}{n\left(f-\frac{1}{f}\right)\left(e^{\frac{p}{n}\pi}-e^{-\frac{p}{n}\pi}\right)}$$

atque hinc per differentiationem sequentia prodeunt theoremata

$$\text{I. } \int \frac{\sin. plx}{\varDelta} \cdot \frac{\partial x lx}{x} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = -\frac{\partial P}{\partial p},$$

$$\text{II. } \int \frac{\cos. plx}{\varDelta} \cdot \frac{\partial x (lx)^2}{x} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = -\frac{\partial\partial P}{\partial p^2},$$

$$\text{III. } \int \frac{\sin. plx}{\varDelta} \cdot \frac{\partial x (lx)^3}{x} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = +\frac{\partial^3 P}{\partial p^3},$$

$$\text{IV. } \int \frac{\cos. plx}{\varDelta} \cdot \frac{\partial x (lx)^4}{x} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = +\frac{\partial^4 P}{\partial p^4}$$

etc.,

per integrationem vero

$$\text{I. } \int \frac{\sin. plx}{\varDelta} \cdot \frac{\partial x}{xlx} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \int P \partial p,$$

$$\text{II. } \int \frac{1-\cos. plx}{\varDelta} \cdot \frac{\partial x}{x(lx)^2} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \int \partial p \int P \partial p,$$

$$\text{III. } \int \frac{plx-\sin. plx}{\varDelta} \cdot \frac{\partial x}{x(lx)^3} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \int \partial p \int \partial p \int P \partial p,$$

$$\text{IV. } \int \frac{\frac{1}{2}pp(lx)^2-1+\cos. plx}{\varDelta} \cdot \frac{\partial x}{x(lx)^4} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \int \partial p \int \partial p \int \partial p \int P \partial p$$

etc.

Hic manifestum est quantitatem f negativam accipi non posse, quia alias[1]) iam ipsa functio P fieret imaginaria.

Adiungamus his theoremata simpliciora, quae ex hactenus allatis nascuntur, dum angulus θ sumitur rectus, ut sit $\cos.\theta = 0$ et $\sin.\theta = 1$. Hinc ergo sequentes ordines adiiciamus.

1) Vide notam p. 383. A. L.

ORDO DECIMUS
THEOREMATUM EX HAC FORMA PRINCIPALI DEDUCTORUM

$$\int \frac{\partial x}{x} \cdot \frac{x^p}{x^n + x^{-n}} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=\infty \end{bmatrix} = \frac{\pi}{2n \cos. \frac{\pi p}{2n}}$$

Haec forma scilicet nata est ex prima sumendo $\theta = \frac{\pi}{2}$, unde posito

$$\varDelta = x^n + x^{-n}$$

et

$$P = \frac{\pi}{2n \cos. \frac{\pi p}{2n}}$$

nascuntur eaedem formulae, quae in ordine primo sunt allatae. Hic autem imprimis notari meretur, quod integrale $\int P\partial p$ per logarithmos exhiberi potest; erit enim

$$\int P \partial p = \int \frac{\pi \partial p}{2n \cos. \frac{\pi p}{2n}} = l \operatorname{tang.} \left(45^\circ + \frac{\pi p}{4n}\right),$$

quod integrale ita est sumtum, ut evanescat facto $p = 0$.

ORDO UNDECIMUS
THEOREMATUM EX HAC FORMA PRINCIPALI DEDUCTORUM

$$\int \frac{\partial x}{x} \cdot \frac{x^p + x^{-p}}{x^n + x^{-n}} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \frac{\pi}{2n \cos. \frac{\pi p}{2n}}$$

Hic scilicet ordo natus est ex quarto ponendo $\theta = \frac{\pi}{2}$; quamobrem statuamus

$$\varDelta = x^n + x^{-n}$$

et

$$P = \frac{\pi}{2n \cos. \frac{\pi p}{2n}}$$

eademque theoremata inde nascuntur, quae supra pro ordine quarto sunt allata, ubi ergo iterum commode usu venit, ut sit

$$\int P \partial p = l \operatorname{tang.} \left(45^\circ + \frac{\pi p}{4n}\right).$$

ORDO DUODECIMUS
THEOREMATUM EX HAC FORMA PRINCIPALI DEDUCTORUM

$$\int \frac{\partial x}{x} \cdot \frac{\cos. plx}{x^n + x^{-n}} \begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = 1 \end{bmatrix} = \frac{\pi}{2n} \cdot \frac{1}{e^{\frac{p\pi}{2n}} + e^{-\frac{p\pi}{2n}}}$$

Quodsi ergo statuamus

$$\varDelta = x^n + x^{-n}$$

et

$$P = \frac{\pi}{2n\left(e^{\frac{p\pi}{2n}} + e^{-\frac{p\pi}{2n}}\right)},$$

eadem plane theoremata hinc oriuntur, quae supra pro casu septimo sunt allata. Hic autem iterum notasse iuvabit integrale $\int P\partial p$ revera exhiberi posse. Cum enim sit

$$\int P\partial p = \int \frac{\pi \partial p}{2n\left(e^{\frac{p\pi}{2n}} + e^{-\frac{p\pi}{2n}}\right)},$$

ponatur $\frac{\pi p}{2n} = z$ eritque

$$\int P\partial p = \int \frac{\partial z}{e^z + e^{-z}} = \int \frac{e^z \partial z}{e^{2z} + 1}.$$

Sit porro $e^z = v$; erit $\partial v = e^z \partial z$ hincque fiet

$$\int P\partial p = \int \frac{\partial v}{1 + vv} = \text{A tang.}\, v;$$

quare retro substituendo habebimus

$$\int P\partial p = \text{A tang.}\, e^{\frac{\pi p}{2n}}.$$

Denique adhuc referamus formulas illas integrales, in quarum denominatore erat $1 - x^{2n}$, quas quidem iam olim[1]) breviter tetigi, nunc autem uberius evolvam.

1) Vide L. Euleri Commentationem 462 (indicis Eneströmiani): *De valore formulae integralis* $\int^1 \frac{z^{m-1} \pm z^{n-m-1}}{1 \pm z^n} dz$ *casu, quo post integrationem ponitur* $z = 1$, Novi comment. acad. sc. Petrop. 19 (1774), 1775, p. 3; *Leonhardi Euleri Opera omnia*, series I, vol. 17, p. 358. A. L.

ORDO TERTIUS DECIMUS
THEOREMATUM EX HAC FORMA PRINCIPALI DEDUCTORUM

$$\int \frac{\partial x}{x} \cdot \frac{x^p - x^{-p}}{x^n - x^{-n}} \begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = 1 \end{bmatrix} = \frac{\pi}{2n} \,\text{tang.}\, \frac{\pi p}{2n}$$

Hic igitur iterum statuamus

$$\varDelta = x^n - x^{-n}$$

et

$$P = \frac{\pi}{2n} \,\text{tang.}\, \frac{\pi p}{2n}$$

atque per differentiationem nanciscemur sequentia theoremata

$$\text{I. } \int \frac{x^p + x^{-p}}{\varDelta} \cdot \frac{\partial x lx}{x} \begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = 1 \end{bmatrix} = \frac{\partial P}{\partial p},$$

$$\text{II. } \int \frac{x^p - x^{-p}}{\varDelta} \cdot \frac{\partial x (lx)^2}{x} \begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = 1 \end{bmatrix} = \frac{\partial\partial P}{\partial p^2},$$

$$\text{III. } \int \frac{x^p + x^{-p}}{\varDelta} \cdot \frac{\partial x (lx)^3}{x} \begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = 1 \end{bmatrix} = \frac{\partial^3 P}{\partial p^3},$$

$$\text{IV. } \int \frac{x^p - x^{-p}}{\varDelta} \cdot \frac{\partial x (lx)^4}{x} \begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = 1 \end{bmatrix} = \frac{\partial^4 P}{\partial p^4}$$

etc.

Integratio autem sequentia suppeditat

$$\text{I. } \int \frac{x^p + x^{-p} - 2}{\varDelta} \cdot \frac{\partial x}{x lx} \begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = 1 \end{bmatrix} = \int P\partial p,$$

$$\text{II } \int \frac{x^p - x^{-p} - 2plx}{\varDelta} \cdot \frac{\partial x}{x(lx)^2} \begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = 1 \end{bmatrix} = \int \partial p \int P\partial p,$$

$$\text{III. } \int \frac{x^p + x^{-p} - 2 - \frac{2}{2}pp(lx)^2}{\varDelta} \cdot \frac{\partial x}{x(lx)^3} \begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = 1 \end{bmatrix} = \int \partial p \int \partial p \int P\partial p,$$

$$\text{IV. } \int \frac{x^p - x^{-p} - 2plx - \frac{2}{6}p^3(lx)^3}{\varDelta} \cdot \frac{\partial x}{x(lx)^4} \begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = 1 \end{bmatrix} = \int \partial p \int \partial p \int \partial p \int P\partial p,$$

$$\text{V. } \int \frac{x^p + x^{-p} - 2 - \frac{2}{2}pp(lx)^2 - \frac{2}{24}p^4(lx)^4}{\varDelta} \cdot \frac{\partial x}{x(lx)^5} \begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = 1 \end{bmatrix} = \int \partial p \int \partial p \int \partial p \int \partial p \int P\partial p$$

etc.,

ubi iterum notetur formulam integralem $\int P\partial p$ actu exhiberi posse; erit enim

$$\int P\partial p = \int \frac{\pi\partial p}{2n}\,\text{tang.}\,\frac{\pi p}{2n} = -\,l\cos.\frac{\pi p}{2n} = +\,l\sec.\frac{\pi p}{2n}.$$

Hic probe notandum est fractionem $\frac{p}{n}$ semper esse debere unitate minorem.

ORDO QUARTUS DECIMUS
THEOREMATUM EX HAC FORMA GENERALI DEDUCTORUM

$$\int \frac{\partial x}{x}\cdot\frac{\sin.plx}{x^{-n}\mp x^{+n}}\begin{bmatrix}\text{ab} & x=0\\ \text{ad} & x=1\end{bmatrix} = \frac{\pi}{4n}\cdot\frac{e^{-\frac{p\pi}{2n}}-e^{+\frac{p\pi}{2n}}}{e^{+\frac{p\pi}{2n}}+e^{-\frac{p\pi}{2n}}}$$

Statuatur igitur ut hactenus

$$\varDelta = x^{-n} - x^{n}$$

et

$$P = \frac{\pi}{4n}\cdot\frac{e^{-\frac{p\pi}{2n}}-e^{+\frac{p\pi}{2n}}}{e^{\frac{p\pi}{2n}}+e^{-\frac{p\pi}{2n}}}$$

atque differentiatio nobis praebebit sequentia theoremata

$$\text{I. } \int \frac{\cos.plx}{\varDelta}\cdot\frac{\partial x lx}{x}\begin{bmatrix}\text{ab} & x=0\\ \text{ad} & x=1\end{bmatrix} = +\frac{\partial P}{\partial p},$$

$$\text{II. } \int \frac{\sin.plx}{\varDelta}\cdot\frac{\partial x(lx)^2}{x}\begin{bmatrix}\text{ab} & x=0\\ \text{ad} & x=1\end{bmatrix} = -\frac{\partial\partial P}{\partial p^2},$$

$$\text{III. } \int \frac{\cos.plx}{\varDelta}\cdot\frac{\partial x(lx)^3}{x}\begin{bmatrix}\text{ab} & x=0\\ \text{ad} & x=1\end{bmatrix} = -\frac{\partial^3 P}{\partial p^3},$$

$$\text{IV. } \int \frac{\sin.plx}{\varDelta}\cdot\frac{\partial x(lx)^4}{x}\begin{bmatrix}\text{ab} & x=0\\ \text{ad} & x=1\end{bmatrix} = +\frac{\partial^4 P}{\partial p^4}$$

etc.

Per integrationem autem impetramus sequentia

$$\text{I. } \int \frac{1-\cos.plx}{\varDelta}\cdot\frac{\partial x}{xlx}\begin{bmatrix}\text{ab} & x=0\\ \text{ad} & x=1\end{bmatrix} = \int P\partial p,$$

$$\text{II. } \int \frac{plx-\sin.plx}{\varDelta}\cdot\frac{\partial x}{x(lx)^2}\begin{bmatrix}\text{ab} & x=0\\ \text{ad} & x=1\end{bmatrix} = \int\partial p\int P\partial p,$$

$$\text{III. } \int \frac{\frac{1}{2}pp(lx)^2-1+\cos.plx}{\varDelta}\cdot\frac{\partial x}{x(lx)^3}\begin{bmatrix}\text{ab} & x=0\\ \text{ad} & x=1\end{bmatrix} = \int\partial p\int\partial p\int P\partial p,$$

$$\text{IV. } \int \frac{\frac{1}{6}p^3(lx)^3-plx+\sin.plx}{\varDelta}\cdot\frac{\partial x}{x(lx)^4}\begin{bmatrix}\text{ab} & x=0\\ \text{ad} & x=1\end{bmatrix} = \int\partial p\int\partial p\int\partial p\int P\partial p$$

etc.,

ubi iterum commode evenit, ut $\int P\partial p$ exhiberi possit, siquidem habemus

$$\int P\partial p = \int \frac{\pi\partial p}{4n}\cdot\frac{e^{-\frac{p\pi}{2n}}-e^{+\frac{p\pi}{2n}}}{e^{\frac{p\pi}{2n}}+e^{-\frac{p\pi}{2n}}}.$$

Ponatur enim $\frac{p\pi}{2n} = \varphi$ eritque

$$\int P\partial p = \int \frac{1}{2}\partial\varphi\cdot\frac{e^{-\varphi}-e^{+\varphi}}{e^{\varphi}+e^{-\varphi}},$$

ubi denominatoris differentiale est $e^{\varphi}\partial\varphi - e^{-\varphi}\partial\varphi$, unde concluditur

$$\int P\partial p = -\,l\sqrt{(e^{\varphi}+e^{-\varphi})} + C,$$

quae constans C ita assumi debet, ut integrale evanescat posito $\varphi = 0$, unde fit

$$\int P\partial p = \frac{1}{2}\,l\,\frac{2}{e^{\frac{p\pi}{2n}}+e^{-\frac{p\pi}{2n}}}.$$

Hic autem perinde est, utrum fractio $\frac{p}{n}$ maior sit minorve unitate.

COMPARATIO VALORUM FORMULAE INTEGRALIS

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}$$

A TERMINO $x=0$ USQUE AD $x=1$ EXTENSAE

Conventui exhibita die 10. Octobris 1776

Commentatio 640 indicis ENESTROEMIANI
Nova acta academiae scientiarum Petropolitanae 5 (1787), 1789, p. 86—117
Summarium ibidem p. 69—72

SUMMARIUM

Si dans la formule exposée dans le titre de ce mémoire, où n, p, q sont des nombres entiers positifs, on donne, pour chaque exposant n, aux p et q toutes les valeurs possibles, il en naît des formules intégrales dont les valeurs ont entre elles des rapports remarquables et tels que, si quelques-unes de ces formules sont connues, on en peut déduire les valeurs de toutes les autres. Feu M. EULER avoit déjà démontré dans son *Calcul Intégral*, Tom. I. Chap. VIII.[1]) plusieurs de ces rapports, mais d'une manière bien éloignée d'épuiser le sujet; il se propose donc ici d'employer une méthode plus féconde, moyennant laquelle on puisse assigner tous les rapports de ce genre et enrichir l'Analyse d'une infinité de nouveaux Théorèmes.

Quelque innombrable que soit la multitude des cas qui paroissent devoir naitre, lorsque, pour chaque exposant n, on donne aux p et q toutes les valeurs possibles, on pourra pourtant, quelque grandes que soyent les valeurs de p et q, réduire tous ces cas à d'autres, où les p et q sont diminués de la quantité n, et continuer cette réduction jusqu'à ce que tant p que q soit plus petit que n. Ainsi cette multitude de cas se réduira pour chaque exposant n à un nombre fort modique et déterminé.

1) Voir la note p. 394. A. L.

Le principe qui sert de fondement à toutes ces comparaisons, et que M. EULER n'a pas manqué de démontrer solidement, est que

$$\left[\int \frac{x^{a-1}\partial x}{\sqrt[n]{(1-x^n)^{n-b}}} \text{ ou } \int \frac{x^{b-1}\partial x}{\sqrt[n]{(1-x^n)^{n-a}}}\right] \times \left[\int \frac{x^{a+b-1}\partial x}{\sqrt[n]{(1-x^n)^{n-c}}} \text{ ou } \int \frac{x^{c-1}\partial x}{\sqrt[n]{(1-x^n)^{n-a-b}}}\right]$$
$$= \left[\int \frac{x^{a-1}\partial x}{\sqrt[n]{(1-x^n)^{n-c}}} \text{ ou } \int \frac{x^{c-1}\partial x}{\sqrt[n]{(1-x^n)^{n-a}}}\right] \times \left[\int \frac{x^{a+c-1}\partial x}{\sqrt[n]{(1-x^n)^{n-b}}} \text{ ou } \int \frac{x^{b-1}\partial x}{\sqrt[n]{(1-x^n)^{n-a-c}}}\right],$$

rapport que l'Auteur représente ainsi:

$$(a, b)\,(a+b, c) = (a, c)\,(a+c, b),$$

et ce symbolisme lui facilite infiniment les comparaisons à faire.

M. EULER parcourt donc successivement sept cas différens, en commençant par $a+b=3$ et finissant par $a+b=9$, ce qui lui donne sept classes de valeurs, qui comprennent en tout 39 formules. Ensuite il examine la formule générale proposée selon les différentes valeurs de l'exposant n, depuis $n=3$ jusqu'à $n=7$, ce qui forme cinq ordres différens de rapports, qui fournissent en tout 50 formules intégrales, parmi lesquelles il y a de fort remarquables. On y trouve par exemple les égalités suivantes:[1])

$$\int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}} \cdot \int \frac{x\partial x}{\sqrt[3]{(1-x^3)}} = \frac{2\pi}{3\sqrt{3}},$$
$$\int \frac{\partial x}{\sqrt[4]{(1-x^4)^3}} : \int \frac{\partial x}{\sqrt{(1-x^4)}} = \sqrt{2},$$
$$\int \frac{x\partial x}{\sqrt[6]{(1-x^6)^5}} : \int \frac{xx\partial x}{\sqrt[6]{(1-x^6)^4}} = 2.$$

Et quantité d'autres non moins remarquables, toutes pour les termes d'intégration $x=0$ et $x=1$.

Dans chaque ordre il y a donc un certain nombre de formules intégrales qui peuvent être exprimées ou par des quantités circulaires, ou par des quantités transcendantes qui dans un des ordres précédens ont été circulaires, ou bien leurs valeurs seront composées de

1) Editio princeps: *On y trouve par exemple les valeurs des produits suivans:*

$$\int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}} \times \int \frac{x\partial x}{\sqrt[3]{(1-x^3)}} = \frac{2\pi}{3\sqrt{3}};$$
$$\int \frac{\partial x}{\sqrt[4]{(1-x^4)^3}} \times \int \frac{\partial x}{\sqrt[4]{(1-x^4)}} = \sqrt{2};$$
$$\int \frac{x\partial x}{\sqrt[6]{(1-x^6)^5}} \times \int \frac{xx\partial x}{\sqrt[6]{(1-x^6)^4}} = 2.$$

Corrigé par A. L.

quantités circulaires et transcendantes. Et en regardant comme connues les formules transcendantes qui dans les ordres précédens ont eu des valeurs circulaires, on sera en état de déterminer à leur aide toutes les autres formules contenues dans l'ordre qu'on traite.

Afin de ne rien laisser à désirer dans ces recherches, l'Auteur donne à la fin de son Mémoire une méthode de déterminer, à peu près, les valeurs transcendantes des formules intégrales qu'on est obligé de regarder comme connues dans chaque ordre. Il faut pour cet effet, comme chacun sait, intégrer par série, et pour les mêmes termes d'intégration, la formule intégrale générale annoncée dans le titre du présent mémoire. Mais comme la méthode ordinaire ne fournit pas de série assez convergente, M. Euler s'occupe à remédier à ce défaut en exprimant par deux séries l'intégrale de la formule proposée, l'une depuis $x=0$ jusqu'à $x^n=\frac{1}{2}$, l'autre depuis $x^n=\frac{1}{2}$ jusqu'à $x=1$, dont chacune est très convergente et dont la somme donne la valeur requise pour les termes prescrits d'intégration.

1. In hac formula litterae n, p et q perpetuo designant numeros integros positivos et pro quolibet numero n binis litteris p et q omnes valores tribui concipiuntur, ita ut hinc pro quovis numero n innumerae nascantur huiusmodi formulae integrales, quarum valores plurimas egregias relationes inter se servant; unde si eorum aliquot fuerint cogniti, reliquae omnes ex iis definiri queant. Iam dudum[1]) equidem plures huiusmodi relationes demonstravi; cum autem hoc argumentum tum temporis neutiquam exhausissem, nunc accuratius in istas relationes inquirere constitui et eiusmodi methodum adhibebo, quae omnes plane huius generis relationes sit exhibitura; his enim inventis innumerabilia theoremata condi poterunt, quibus universa Analysis non mediocriter locupletari erit censenda.

2. Quoniam igitur hoc modo pro quolibet numero n ambae litterae p et q infinitos valores recipere possunt, ante omnia hic observari convenit omnes hos innumerabiles casus semper ad numerum finitum revocari posse. Quantumvis enim magni numeri pro litteris p et q accipiantur, eos casus semper

1) Vide L. Euleri Commentationem 321 (indicis Enestroemiani): *Observationes circa integralia formularum* $\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1}$ *posito post integrationem* $x=1$, Mélanges de phil. et de mathém. de la société de Turin 3_2 (1762/5), 1766, p. 156; *Leonhardi Euleri Opera omnia*, series I, vol. 17, p. 268. Vide porro L. Euleri *Institutionum calculi integralis* vol. 1, Petropoli 1768, sectio 1, cap. VIII; *Leonhardi Euleri Opera omnia*, series I, vol. 11, p. 208. A. L.

ad alios reducere licet, in quibus numeri p et q quantitate n futuri sint diminuti. Hoc igitur modo omnes huiusmodi casus tandem eo redigi poterunt, ut ambo numeri p et q infra exponentem n deprimantur; unde pro quolibet numero n eos tantum casus considerasse sufficiet, quibus litterae p et q minores valores recipiant quam n vel saltem hunc limitem non superent. Hoc igitur modo pro quovis numero n multitudo casuum, qui in computum veniunt et quos inter se comparari oportet, prorsus erit determinata.

3. Quemadmodum autem ista reductio litterarum p et q ad numeros continuo minores institui debeat, quamquam id satis in vulgus est notum, tamen ad formulam praesentem accommodasse iuvabit. Statuatur scilicet haec formula algebraica

$$x^p(1-x^n)^{\frac{q}{n}}=V$$

eritque

$$lV=plx+\frac{q}{n}l(1-x^n),$$

hinc differentiando

$$\frac{\partial V}{V}=\frac{p\partial x}{x}-\frac{qx^{n-1}\partial x}{1-x^n}=\frac{p\partial x-(p+q)x^n\partial x}{x(1-x^n)};$$

ubi si per V multiplicemus ac per partes integremus, orietur ista aequatio

$$V=p\int x^{p-1}\partial x(1-x^n)^{\frac{q-n}{n}}-(p+q)\int x^{p+n-1}\partial x(1-x^n)^{\frac{q-n}{n}}.$$

Quoniam igitur quantitas V pro utroque integrationis termino evanescit, hinc adipiscimur istam reductionem

$$\int x^{p+n-1}\partial x(1-x^n)^{\frac{q-n}{n}}=\frac{p}{p+q}\int x^{p-1}\partial x(1-x^n)^{\frac{q-n}{n}},$$

cuius ergo reductionis ope exponens ipsius x continuo quantitate n diminui poterit, donec tandem infra n deprimatur.

4. Deinde formula pro

$$\frac{\partial V}{V}=\frac{p\partial x-(p+q)x^n\partial x}{x(1-x^n)}$$

inventa hoc modo referri poterit

$$\frac{\partial V}{V} = \frac{(p+q)\partial x(1-x^n) - q\partial x}{x(1-x^n)},$$

quae forma per V multiplicata ac denuo per partes integrata dabit

$$V = (p+q)\int x^{p-1}\partial x(1-x^n)^{\frac{q}{n}} - q\int x^{p-1}\partial x(1-x^n)^{\frac{q-n}{n}},$$

unde, quia posito [$x=0$ et] $x=1$ fit $V=0$, oritur haec reductio

$$\int x^{p-1}\partial x(1-x^n)^{\frac{q}{n}} = \frac{q}{p+q}\int x^{p-1}\partial x(1-x^n)^{\frac{q-n}{n}},$$

cuius reductionis ope exponens binomii $1-x^n$ unitate minuitur sive, quod eodem redit, numerus q numero n imminuitur. Tali igitur reductione, quoties opus fuerit, repetita exponens q tandem infra n deprimi poterit.

5. Quoniam igitur pro quovis numero n ambos exponentes p et q tamquam minores quam n spectare licet, formulam propositam hoc modo expressam repraesentemus

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}.$$

Hic scilicet pro quovis numero n sufficiet litteris p et q omnes valores ipso n minores tribuisse, quo pacto multitudo omnium casuum ad quemlibet exponentem n pertinentium ad numerum satis modicum reducetur, qui tamen eo maior evadit, quo maior fuerit exponens n.

6. Multo magis autem numerus casuum diversorum diminuetur, si perpendamus ambas litteras p et q inter se permutari posse, ita ut huius formulae

$$\frac{x^{q-1}\partial x}{\sqrt[n]{(1-x^n)^{n-p}}}$$

valor ab illo prorsus non discrepet. Ad quod ostendendum ponamus

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}} = S,$$

si scilicet ista formula integralis ab $x=0$ usque ad $x=1$ extendatur. Iam faciamus

$$1-x^n = y^n,$$

ut formula sit

$$S = \int \frac{x^{p-1}\partial x}{y^{n-q}};$$

tum vero, quia $x^n = 1-y^n$, erit $x = (1-y^n)^{\frac{1}{n}}$ hincque

$$x^p = (1-y^n)^{\frac{p}{n}},$$

unde differentiando fit

$$px^{p-1}\partial x = -py^{n-1}\partial y(1-y^n)^{\frac{p-n}{n}},$$

quo valore substituto erit

$$S = -\int y^{q-1}\partial y(1-y^n)^{\frac{p-n}{n}},$$

quam formulam ab $x=0$ usque ad $x=1$, hoc est ab $y=1$ usque ad $y=0$, extendi oportet; permutatis igitur his terminis erit

$$S = \int \frac{y^{q-1}\partial y}{\sqrt[n]{(1-y^n)^{n-p}}}\begin{bmatrix}\text{ab } y=0\\ \text{ad } y=1\end{bmatrix}.$$

Sicque demonstratum est ambas litteras p et q semper inter se esse permutabiles.

7. His praemissis, quo calculos sequentes magis in compendium redigere liceat, loco formulae huius integralis

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}} = \int \frac{x^{q-1}\partial x}{\sqrt[n]{(1-x^n)^{n-p}}}$$

scribamus hunc characterem

$$(p, q),$$

ubi perinde est, sive p ante q sive q ante p collocetur; semper autem hic certus exponens n subintelligi debet. Hic autem duo casus prae reliquis maxime memorabiles occurrunt. Prior casus est, quo numerorum p et q

alteruter ipsi exponenti n est aequalis; si enim fuerit $q=n$, erit ex priore formula

$$(p,n)=\int x^{p-1}\partial x=\frac{1}{p}$$

sicque perpetuo habebimus

$$(p,n)=\frac{1}{p}$$

hincque etiam

$$(n,q)=\frac{1}{q}.$$

Alter casus notatu dignissimus locum habet, quando $p+q=n$, quo casu semper est

$$(p,q)=\frac{\pi}{n\sin.\frac{p\pi}{n}}=\frac{\pi}{n\sin.\frac{q\pi}{n}}.$$

Ad hoc ostendendum sit $q=n-p$ hincque formula proposita

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^p}};$$

tum ponatur

$$\frac{x}{\sqrt[n]{(1-x^n)}}=z,$$

et quia $\frac{x^p}{\sqrt[n]{(1-x^n)^p}}=z^p$, erit

$$S=\int \frac{z^p\partial x}{x}.$$

Ex facta autem positione sequitur

$$x^n=\frac{z^n}{1+z^n}$$

hincque

$$nlx=nlz-l(1+z^n),$$

ergo differentiando

$$\frac{\partial x}{x}=\frac{\partial z}{z}-\frac{z^{n-1}\partial z}{1+z^n}=\frac{\partial z}{z(1+z^n)},$$

ita ut iam sit

$$S=\int \frac{z^{p-1}\partial z}{1+z^n}.$$

Quia autem sumpto $x=0$ fit etiam $z=0$, at vero sumpto $x=1$ prodit $z=\infty$,

hoc integrale a termino $z=0$ usque ad $z=\infty$ extendi debet. Notum[1]) autem est valorem hoc modo resultantem esse $\frac{\pi}{n\sin.\frac{p\pi}{n}}$.

8. Progrediamur nunc ad ipsum fundamentum, unde omnes relationes, quas quaerimus, derivari convenit et quod reductioni priori innititur; unde fit

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}=\frac{p+q}{p}\int \frac{x^{n+p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}},$$

ubi loco $\sqrt[n]{(1-x^n)^{n-q}}$ scribamus X, ut sit

$$\int \frac{x^{p-1}\partial x}{X}=\frac{p+q}{p}\int \frac{x^{n+p-1}\partial x}{X};$$

hinc iam simili modo, si loco p scribamus $n+p$, erit

$$\int \frac{x^{n+p-1}\partial x}{X}=\frac{n+p+q}{n+p}\int \frac{x^{2n+p-1}\partial x}{X}$$

hincque sequitur fore

$$\int \frac{x^{p-1}\partial x}{X}=\frac{p+q}{p}\cdot\frac{n+p+q}{n+p}\int \frac{x^{2n+p-1}\partial x}{X}.$$

Quodsi simili modo ulterius progrediamur, perveniemus ad hanc aequationem

$$\int \frac{x^{p-1}\partial x}{X}=\frac{p+q}{p}\cdot\frac{n+p+q}{n+p}\cdot\frac{2n+p+q}{2n+p}\int \frac{x^{3n+p-1}\partial x}{X}.$$

Quare si hoc modo in infinitum progrediamur, habebimus

$$\int \frac{x^{p-1}\partial x}{X}=\frac{p+q}{p}\cdot\frac{n+p+q}{n+p}\cdot\frac{2n+p+q}{2n+p}\cdots\frac{in+p+q}{in+p}\int \frac{x^{(i+1)n+p-1}\partial x}{X},$$

ubi i denotat numerum infinite magnum.

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1) Vide L. EULERI Commentationem 60 (indicis ENESTROEMIANI): *De inventione integralium, si post integrationem variabili quantitati determinatus valor tribuatur*, Miscellanea Berolin. 7, 1743, p. 129, imprimis § 32; *LEONHARDI EULERI Opera omnia*, series I, vol. 17, p. 35. Vide etiam L. EULERI *Institutionum calculi integralis* vol. 1, Petropoli 1768, sectio 1, cap. VIII; *LEONHARDI EULERI Opera omnia*, series I, vol. 11, p. 208, imprimis p. 225. A. L.

9. Quodsi iam loco p alium quemcumque numerum r pariter ipso n minorem assumamus, erit simili modo

$$\int \frac{x^{r-1}\partial x}{X} = \frac{r+q}{r} \cdot \frac{n+r+q}{n+r} \cdot \frac{2n+r+q}{2n+r} \cdots \frac{in+r+q}{in+r} \int \frac{x^{(i+1)n+r-1}\partial x}{X},$$

ubi littera i eundem numerum infinitum designat, ita ut utrimque idem factorum numerus adsit. Dividamus iam priorem expressionem per istam, et quoniam extremae formulae integrales ob litteras p et r prae $(i+1)n$ evanescentes pro aequalibus inter se sunt habendae, facta divisione per singulos factores reperiemus hanc aequationem

$$\frac{\int x^{p-1}\partial x : X}{\int x^{r-1}\partial x : X}$$
$$= \frac{r(p+q)}{p(r+q)} \cdot \frac{(n+r)(n+p+q)}{(n+p)(n+r+q)} \cdot \frac{(2n+r)(2n+p+q)}{(2n+p)(2n+r+q)} \cdot \frac{(3n+r)(3n+p+q)}{(3n+p)(3n+r+q)} \cdot \text{etc.}$$

Restituamus iam loco harum formularum integralium characteres ante [§ 7] stabilitos atque adipiscemur istam relationem notatu dignissimam

$$\frac{(p,q)}{(r,q)} = \frac{r(p+q)}{p(r+q)} \cdot \frac{(n+r)(n+p+q)}{(n+p)(n+r+q)} \cdot \frac{(2n+r)(2n+p+q)}{(2n+p)(2n+r+q)} \cdot \text{etc.},$$

quod productum ex infinitis membris componitur, quorum singula sunt fractiones, quarum tam numeratores quam denominatores ex binis factoribus constant. Hos factores singulos eodem numero n augeri oportet, dum a quovis membro ad sequens progredimur, unde sufficiet solum primum productum nosse, quod ergo ita repraesentabimus

$$\frac{(p,q)}{(r,q)} = \frac{r(p+q)}{p(r+q)} \cdot \text{etc.}$$

10. Quoniam litterae p et q nobis numeros quasi indefinitos significant, utamur litteris alphabeti initialibus ad numeros determinatos designandos eritque eodem modo

$$\frac{(a,b)}{(\alpha,b)} = \frac{\alpha(a+b)}{a(\alpha+b)} \cdot \frac{(n+\alpha)(n+a+b)}{(n+a)(n+\alpha+b)} \cdot \text{etc.}$$

Hic iam loco α scribamus $a+c$ et productum infinitum hanc induet formam

$$\frac{(a,b)}{(a+c,b)} = \frac{(a+c)(a+b)}{a(a+c+b)} \cdot \frac{(n+a+c)(n+a+b)}{(n+a)(n+a+c+b)} \cdot \text{etc.},$$

in quo producto ambae litterae b et c manifesto permutari possunt, unde idem productum infinitum etiam exprimet valorem huius formae $\frac{(a,c)}{(a+b,c)}$, unde sequitur ista aequalitas maxime memorabilis

$$\frac{(a,b)}{(a+c,b)} = \frac{(a,c)}{(a+b,c)};$$

fractionibus igitur sublatis habebimus istud insigne theorema

$$(a,b)\ (a+b,c) = (a,c)\ (a+c,b)$$

huicque theoremati universa analysis, qua utemur, erit superstructa.

11. Cum ob rationes supra allegatas numeri p et q exponentem n superare non debeant, etiam in forma theorematis modo allati singuli termini ibi occurrentes, qui sunt a, b, c, $a+b$ et $a+c$, quovis casu exponentem n superare non debent sicque nec $a+b$ neque $a+c$ maior capi poterit quam n. Hic autem primo observo litteras b et c inter se inaequales statui debere; si enim esset $c=b$, aequalitas in theoremate expressa foret identica; hanc ob rem perpetuo assumemus $b>c$, ita ut maximus terminus in theoremate sit $a+b$, quem ergo exponentem n quovis casu excedere non oportet, quamobrem evolutionem formae generalis in theoremate contentae ita in classes distribuamus, quae inter se per maximum valorem termini $a+b$ distinguantur. Cum igitur nulla litterarum a, b, c nihilo aequalis sumi queat ac esse debeat $b>c$, minimus valor, quem terminus $a+b$ recipere potest, erit 3, in quo ergo primam classem constituemus; sequentes vero classes constituentur, dum termino $a+b$ valores 4, 5, 6, 7 etc. tribuantur.

I. EVOLUTIO CLASSIS QUA $a+b=3$

12. Hic ergo necessario erit $a=1$, $b=2$ et $c=1$, ita ut hic nulla varietas locum inveniat, unde theorema nostrum suppeditat hanc unicam relationem

$$(1,2)\,(3,1) = (1,1)\,(2,2).$$

Dummodo igitur exponens n non fuerit minor quam 3[1]), semper haec insignis

1) Manifestum est hanc conditionem necessariam non esse. A. L.

relatio locum habet

$$\int \frac{\partial x}{\sqrt[n]{(1-x^n)^{n-2}}} \cdot \int \frac{xx\partial x}{\sqrt[n]{(1-x^n)^{n-1}}} = \int \frac{\partial x}{\sqrt[n]{(1-x^n)^{n-1}}} \cdot \int \frac{x\partial x}{\sqrt[n]{(1-x^n)^{n-2}}},$$

quae forma, quia in quolibet charactere terminos inter se permutare licet, etiam hoc modo repraesentari poterit

$$\int \frac{x\partial x}{\sqrt[n]{(1-x^n)^{n-1}}} \cdot \int \frac{\partial x}{\sqrt[n]{(1-x^n)^{n-3}}} = \int \frac{\partial x}{\sqrt[n]{(1-x^n)^{n-1}}} \cdot \int \frac{x\partial x}{\sqrt[n]{(1-x^n)^{n-2}}}.$$

II. EVOLUTIO CLASSIS QUA $a+b=4$

13. Quoniam b binario minor esse nequit, hic erit vel $b=2$ vel $b=3$. Sit igitur primo $b=2$ eritque $a=2$ et $c=1$; unde ex nostro theoremate sequitur haec relatio

$$(2, 2)\,(4, 1) = (2, 1)\,(3, 2),$$

quae forma manifesto oritur ex classe prima, si ibi termini priores cuiusque characteris unitate augeantur; id quod etiam inde intelligere licet, quod omnes termini priores litteram a continent, qua unitate aucta processus semper fit ad classem sequentem.

14. Deinde vero hic quoque statui potest $b=3$, unde fit $a=1$; at vero littera c iam duos valores, vel 1 vel 2, sortiri poterit; priore casu, quo $c=1$, prodibit ista aequatio

$$(1, 3)\,(4, 1) = (1, 1)\,(2, 3);$$

alter vero casus, quo $c=2$, praebet hanc aequationem

$$(1, 3)\,(4, 2) = (1, 2)\,(3, 3).$$

Sicque haec classis omnino sequentes tres relationes continebit

1. $(2, 2)\,(4, 1) = (2, 1)\,(3, 2)$,
2. $(1, 3)\,(4, 1) = (1, 1)\,(2, 3)$,
3. $(1, 3)\,(4, 2) = (1, 2)\,(3, 3)$.

III. EVOLUTIO CLASSIS QUA $a+b=5$

15. In hac igitur classe primo occurrent tres relationes praecedentes, si modo termini priores cuiusque characteris unitate augeantur; hinc enim casus exsurgent, quibus est vel $b=2$ vel $b=3$. De novo igitur hic accedent casus, quibus $b=4$ et $a=1$, ubi ergo erit vel $c=1$ vel $c=2$ vel $c=3$, quibus ergo tribus casibus evolutis omnino in hac classe sex continebuntur relationes, quae erunt

1. $(3, 2)\,(5, 1) = (3, 1)\,(4, 2)$,
2. $(2, 3)\,(5, 1) = (2, 1)\,(3, 3)$,
3. $(2, 3)\,(5, 2) = (2, 2)\,(4, 3)$,
4. $(1, 4)\,(5, 1) = (1, 1)\,(2, 4)$,
5. $(1, 4)\,(5, 2) = (1, 2)\,(3, 4)$,
6. $(1, 4)\,(5, 3) = (1, 3)\,(4, 4)$.

IV. EVOLUTIO CLASSIS QUA $a+b=6$

16. Hic igitur primum occurrent omnes relationes proxime praecedentes, si modo termini priores cuiusque characteris unitate augeantur; hi scilicet nascuntur, si fuerit vel $b=2$ vel $b=3$ vel $b=4$. Praeterea vero insuper accedent casus $b=5$ et $a=1$, ubi littera c recipere poterit valores 1, 2, 3, 4, sicque omnino in hac classe occurrent decem relationes sequentes

1. $(4, 2)\,(6, 1) = (4, 1)\,(5, 2)$,
2. $(3, 3)\,(6, 1) = (3, 1)\,(4, 3)$,
3. $(3, 3)\,(6, 2) = (3, 2)\,(5, 3)$,
4. $(2, 4)\,(6, 1) = (2, 1)\,(3, 4)$,
5. $(2, 4)\,(6, 2) = (2, 2)\,(4, 4)$,
6. $(2, 4)\,(6, 3) = (2, 3)\,(5, 4)$,
7. $(1, 5)\,(6, 1) = (1, 1)\,(2, 5)$,
8. $(1, 5)\,(6, 2) = (1, 2)\,(3, 5)$,
9. $(1, 5)\,(6, 3) = (1, 3)\,(4, 5)$,
10. $(1, 5)\,(6, 4) = (1, 4)\,(5, 5)$.

V. EVOLUTIO CLASSIS QUA $a+b=7$

17. Hic igitur primo occurrent omnes relationes classis IV, postquam scilicet omnes terminos priores singulorum characterum unitate auxerimus, quos igitur hic apposuisse non erit necesse, ac sufficiet eas tantum relationes hic exponere, quae de novo accedunt et ex valore $b=6$ oriuntur existente $a=1$; ubi pro c sumi poterunt numeri 1, 2, 3, 4, 5, ita ut harum numerus sit quinque. Hae ergo relationes sunt

1. (1, 6) (7, 1) = (1, 1) (2, 6),
2. (1, 6) (7, 2) = (1, 2) (3, 6),
3. (1, 6) (7, 3) = (1, 3) (4, 6),
4. (1, 6) (7, 4) = (1, 4) (5, 6),
5. (1, 6) (7, 5) = (1, 5) (6, 6).

VI. EVOLUTIO CLASSIS QUA $a+b=8$

18. In hac iam classe primo occurrent omnes decem relationes classis IV, dum scilicet omnes termini priores binario augentur; praeterea quoque accedent quinque relationes in classe V allatae, dum partes priores unitate augebuntur; praeter has vero de novo accedent sex sequentes relationes ex valoribus $a=1$ et $b=7$ oriundae, dum litterae c valores 1, 2, 3, 4, 5, 6 ordine tribuuntur, quae ergo erunt

1. (1, 7) (8, 1) = (1, 1) (2, 7),
2. (1, 7) (8, 2) = (1, 2) (3, 7),
3. (1, 7) (8, 3) = (1, 3) (4, 7),
4. (1, 7) (8, 4) = (1, 4) (5, 7),
5. (1, 7) (8, 5) = (1, 5) (6, 7),
6. (1, 7) (8, 6) = (1, 6) (7, 7).

VII. EVOLUTIO CLASSIS QUA $a+b=9$

19. Ut omnes relationes ad hanc classem pertinentes adipiscamur, notandum est primo hic occurrere decem relationes classis IV, dum partes priores ternario augentur. Secundo adiici oportet quinque relationes in classe V exhibitas, ubi partes priores binario augeri debent. Tertio huc referri debent sex relationes classis VI partes priores unitate augendo. Insuper vero de novo accedent septem relationes ex valoribus $a=1$ et $b=8$ natae, dum litterae c tribuuntur ordine valores 1, 2, 3, 4, 5, 6, 7. Hae relationes sunt

1. (1, 8) (9, 1) = (1, 1) (2, 8),
2. (1, 8) (9, 2) = (1, 2) (3, 8),
3. (1, 8) (9, 3) = (1, 3) (4, 8),
4. (1, 8) (9, 4) = (1, 4) (5, 8),
5. (1, 8) (9, 5) = (1, 5) (6, 8),
6. (1, 8) (9, 6) = (1, 6) (7, 8),
7. (1, 8) (9, 7) = (1, 7) (8, 8).

20. Hinc iam ordo progressionis tam clare perspicitur, ut superfluum foret has evolutiones ulterius prosequi; quandoquidem ob ingentem multitudinem relationum, quae in sequentibus classibus occurrerent, nimis molestum foret omnes percurrere. Quin etiam nostrum institutum vix permittere videtur, ut in nostra formula generali exponentem n ultra sex vel septem augeamus, siquidem omnes relationes ad eum pertinentes enumerare voluerimus. Sin autem animus sit aliquas tantum expendere, classes allatae abunde sufficiunt, dum termini priores cuiusque classis quovis numero augebuntur.

21. His iam classibus expeditis formulam integralem propositam

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}$$

secundum diversos valores exponentis n pertractemus, dum scilicet successive assumemus $n=3$, $n=4$, $n=5$ etc., et pro quolibet ordine omnes relationes, quae in eo occurrere possunt, expendamus. Evidens autem est, quicumque numerus exponenti n tribuatur, formulas omnium classium inferiorum, in quibus scilicet terminus $a+b$ non superet n, in usum vocari posse. Ex quo intelligitur, si fuerit $n=3$, unicam relationem locum invenire; statim autem ac n magis augetur, numerus omnium relationum mox ita increscit, ut nimis molestum foret omnes recensere. Hos igitur diversos ordines ex exponente n constituendos a primo incipiendo ordine evolvamus.

ORDO I QUO $n=3$ ET FORMULA

$$(p,q)=\int\frac{x^{p-1}\partial x}{\sqrt[3]{(1-x^3)^{3-q}}}=\int\frac{x^{q-1}\partial x}{\sqrt[3]{(1-x^3)^{3-p}}}$$

22. Cum hic sit $n=3$, erit [§ 7]

$$(3,1)=1;$$

formulae autem integrales huius ordinis erunt tres, scilicet

$$1.\ (1,1),\quad 2.\ (1,2),\quad 3.\ (2,2),$$

quarum media ob $1+2=3$ a circulo pendet; quae ergo quia est cognita, ponatur

$$(1,2)=\frac{\pi}{3\sin.\frac{\pi}{3}}=\frac{2\pi}{3\sqrt{3}}=A.$$

Hic igitur tantum classis prima locum habet, quae nobis hanc unicam aequationem suppeditat

$$A=(1,1)(2,2).$$

23. Hinc ergo patet productum ex binis formulis transcendentibus $(1,1)$ et $(2,2)$ aequari quantitati circulari $A=\frac{2\pi}{3\sqrt{3}}$, ita ut pro ipsis formulis integralibus habeamus hanc relationem

$$\int\frac{\partial x}{\sqrt[3]{(1-x^3)^2}}\cdot\int\frac{x\partial x}{\sqrt[3]{(1-x^3)}}=\frac{2\pi}{3\sqrt{3}};$$

unde si altera harum duarum formularum fuerit cognita, etiam valor alterius assignari potest. Spectemus ergo priorem, quasi nobis esset cognita, etiamsi

sit transcendens, eamque ponamus

$$(1,1)=\int\frac{\partial x}{\sqrt[3]{(1-x^3)^2}}=P$$

eritque

$$(2,2)=\frac{A}{P}.$$

Sicque nihil praeterea in hoc ordine notandum relinquitur.

ORDO II QUO $n=4$ ET FORMULA

$$(p,q)=\int\frac{x^{p-1}\partial x}{\sqrt[4]{(1-x^4)^{4-q}}}=\int\frac{x^{q-1}\partial x}{\sqrt[4]{(1-x^4)^{4-p}}}$$

24. Cum igitur hic sit $n=4$, erit

$$(4,1)=1\quad\text{et}\quad(4,2)=\tfrac{1}{2};$$

formulae autem integrales ad hunc ordinem pertinentes erunt sex sequentes

$$1.\ (1,1),\quad 2.\ (1,2),\quad 3.\ (1,3),\quad 4.\ (2,2),\quad 5.\ (2,3),\quad 6.\ (3,3),$$

inter quas ergo reperiuntur duae formulae circulares $(1,3)$ et $(2,2)$, quas propterea litteris A et B designemus ponendo

$$(1,3)=\frac{\pi}{4\sin.\frac{\pi}{4}}=\frac{\pi}{2\sqrt{2}}=A$$

et

$$(2,2)=\frac{\pi}{4\sin.\frac{2\pi}{4}}=\frac{\pi}{4}=B,$$

ita ut sit

$$\frac{A}{B}=\sqrt{2}.$$

25. In hoc ergo ordine aequationes tam primae quam secundae classis locum habere possunt; secunda autem classis nobis has tres praebet aequationes

$$1.\ B=(2,1)(3,2),\quad 2.\ A=(1,1)(2,3),\quad 3.\ A=2(1,2)(3,3),$$

classis vero prima insuper dat hanc aequationem

$$A(1, 2) = (1, 1)B$$

sive $\frac{A}{B} = \frac{(1,1)}{(1,2)}$, quae autem aequatio iam ex duabus prioribus deducitur; namque ob $(3, 2) = (2, 3)$ secunda per primam divisa dabit $\frac{A}{B} = \frac{(1,1)}{(1,2)} = \sqrt{2}$, ita ut ratio inter has duas formulas sit algebraica, quae ergo imprimis notari meretur

$$\int \frac{\partial x}{\sqrt[4]{(1-x^4)^3}} : \int \frac{\partial x}{\sqrt{(1-x^4)}} = \sqrt{2}.$$

26. Iam in hoc ordine praeter binas formulas circulares $(1, 3) = A$ et $(2, 2) = B$ tamquam cognitam etiam introducamus formulam $(1, 2)$, quae in ordine praecedente erat circularis, nunc autem est transcendens, eamque ponamus

$$(1, 2) = \int \frac{\partial x}{\sqrt{(1-x^4)}} = P;$$

ubi caveatur, ne litterae A et P cum iis confundantur, quibus in formulis praecedentibus sumus usi, id quod etiam de ordinibus sequentibus est tenendum. His igitur litteris introductis aequationes nostrae erunt sequentes tres

$$1.\ B = P(3, 2), \quad 2.\ A = (1, 1)(2, 3), \quad 3.\ A = 2P(3, 3),$$

quandoquidem vidimus quartam in praecedentibus iam contineri.

27. Ope harum trium aequationum ergo ternas formulas integrales etiamnunc incognitas per ternas A, B et P, quas ut datas spectamus, determinare licebit. Ex prima enim fit $(3, 2) = \frac{B}{P}$; ex tertia autem fit $(3, 3) = \frac{A}{2P}$; tum vero ex secunda colligitur $(1, 1) = \frac{A}{(3,2)} = \frac{AP}{B}$. Cum igitur in hoc ordine omnino sint sex formulae integrales, earum ternae per tres reliquas definiri possunt, quas determinationes igitur ob oculos posuisse iuvabit:

$$1.\ (1, 3) = A = \frac{\pi}{2\sqrt{2}}, \qquad 4.\ (1, 1) = \frac{AP}{B},$$

$$2.\ (2, 2) = B = \frac{\pi}{4}, \qquad 5.\ (2, 3) = \frac{B}{P},$$

$$3.\ (1, 2) = P = \int \frac{\partial x}{\sqrt{(1-x^4)}}, \qquad 6.\ (3, 3) = \frac{A}{2P}.$$

Ex postremis ergo erit

$$(2, 3) : (3, 3) = 2B : A = \sqrt{2} : 1,$$

ita ut etiam hae duae formulae inter se habeant rationem algebraicam, qua est

$$\int \frac{xx\partial x}{\sqrt{(1-x^4)}} = \sqrt{2} \int \frac{xx\partial x}{\sqrt[4]{(1-x^4)}}.$$

Aliis insignibus relationibus utpote satis cognitis hic non immoramur.

ORDO III QUO $n = 5$ ET FORMULA

$$(p, q) = \int \frac{x^{p-1}\partial x}{\sqrt[5]{(1-x^5)^{5-q}}} = \int \frac{x^{q-1}\partial x}{\sqrt[5]{(1-x^5)^{5-p}}}$$

28. Hic igitur ob $n = 5$ ante omnia erit

$$(5, 1) = 1, \quad (5, 2) = \tfrac{1}{2}, \quad (5, 3) = \tfrac{1}{3};$$

formulae autem integrales huius ordinis erunt hae decem

1. $(1, 1)$, 2. $(1, 2)$, 3. $(1, 3)$, 4. $(1, 4)$, 5. $(2, 2)$,
6. $(2, 3)$, 7. $(2, 4)$, 8. $(3, 3)$, 9. $(3, 4)$, 10. $(4, 4)$,

inter quas quarta et sexta sunt circulares, quas ergo ita designemus

$$(1, 4) = \frac{\pi}{5 \sin. \frac{1}{5}\pi} = A$$

et

$$(2, 3) = \frac{\pi}{5 \sin. \frac{2\pi}{5}} = B.$$

Praeterea vero binas formulas, quae in ordine praecedenti erant circulares, nunc autem sunt transcendentes, etiam peculiaribus litteris notemus, scilicet

$$(1, 3) = P \quad \text{et} \quad (2, 2) = Q.$$

Mox enim patebit, dummodo etiam istae formulae tamquam cognitae spectentur, reliquas sex omnes per has quatuor determinari posse.

29. Quoniam hic tres classes priores locum habere possunt, consideremus primo aequationes, quas tertia classis suppeditat et quae introductis his valoribus erunt

1. $B = P(4, 2)$,	4. $A = (1, 1)(2, 4)$,
2. $B = (2, 1)(3, 3)$,	5. $A = 2(1, 2)(3, 4)$,
3. $B = 2Q(4, 3)$,	6. $A = 3P(4, 4)$.

Quas hoc modo succinctius repraesentare licet

$$A = (1, 1)(2, 4) = 2(1, 2)(3, 4) = 3P(4, 4),$$
$$B = P(4, 2) = (2, 1)(3, 3) = 2Q(4, 3),$$

ubi sex occurrunt producta ex binis formulis integralibus, quae singula quantitati circulari aequantur, unde totidem egregia theoremata formari possent, nisi hinc iam clare in oculos incurrerent.

30. Iam videamus, quot formulas integrales incognitas ex quatuor cognitis A, B, P et Q definire queamus; at vero prima dat $(4, 2) = \frac{B}{P}$, tertia praebet $(4, 3) = \frac{B}{2Q}$, sexta dat $(4, 4) = \frac{A}{3P}$; hinc autem porro ex quarta deducimus $(1, 1) = \frac{A}{(2, 4)} = \frac{AP}{B}$, ex quinta vero deducimus $(1, 2) = \frac{A}{2(3, 4)} = \frac{AQ}{B}$. Denique ex secunda elicimus $(3, 3) = \frac{B}{(2, 1)} = \frac{BB}{AQ}$ sicque ex his sex aequationibus sex determinationes sumus adepti; atque adeo per litteras A, B, P et Q valores omnium reliquarum litterarum assignavimus.

31. Quoniam igitur hactenus tantum classe tertia sumus usi, consideremus etiam aequationes secundae classis, quae sunt

1. $AQ = B(2, 1)$,

2. $AP = B(1, 1)$

et

3. $P(4, 2) = (1, 2)(3, 3)$;

verum si hic valores modo inventos substituamus, aequationes mere identicae resultant, ita ut hinc nulla nova determinatio sequatur. Idem usu venit ex

aequatione primae classis, quae erat $(2, 1)(3, 1) = (1, 1)(2, 2)$, quae facta substitutione quoque fit identica, ita ut duae priores classes nihil novi involvant. Neque tamen hinc concludere licet etiam in sequentibus ordinibus classes praecedentes praetermitti posse, siquidem in ordine sequente statim contrarium se manifestabit.

32. Cum igitur hic ordo complectatur decem formulas integrales, earum valores per quatuor litteras A, B, P et Q ordine ita aspectui exponamus:

1. $(1, 1) = \frac{AP}{B}$,	6. $(2, 3) = B$,
2. $(1, 2) = \frac{AQ}{B}$,	7. $(2, 4) = \frac{B}{P}$,
3. $(1, 3) = P$,	8. $(3, 3) = \frac{BB}{AQ}$,
4. $(1, 4) = A$,	9. $(3, 4) = \frac{B}{2Q}$,
5. $(2, 2) = Q$,	10. $(4, 4) = \frac{A}{3P}$.

33. Cum sit

$$\frac{A}{B} = \frac{\sin.\frac{2}{5}\pi}{\sin.\frac{1}{5}\pi} = 2\cos.\frac{1}{5}\pi,$$

tum vero

$$\cos.\frac{1}{5}\pi = \frac{1+\sqrt{5}}{4},$$

erit

$$\frac{A}{B} = \frac{1+\sqrt{5}}{2}$$

ideoque quantitas algebraica. Hinc igitur aliquot paria formularum integralium exhiberi poterunt, quae inter se teneant rationem algebraicam; erit enim

$$\frac{(1, 1)}{(1, 3)} = \frac{1+\sqrt{5}}{2}, \quad \frac{(1, 2)}{(2, 2)} = \frac{1+\sqrt{5}}{2}, \quad \frac{(3, 4)}{(3, 3)} = \frac{1+\sqrt{5}}{4}, \quad \frac{(4, 4)}{(2, 4)} = \frac{1+\sqrt{5}}{6},$$

unde totidem egregia theoremata condi possent, nisi ex his formulis manifesto elucerent.

ORDO IV QUO $n=6$ ET FORMULA

$$(p, q)=\int \frac{x^{p-1}\partial x}{\sqrt[6]{(1-x^6)^{6-q}}}=\int \frac{x^{q-1}\partial x}{\sqrt[6]{(1-x^6)^{6-p}}}$$

34. Quoniam hic est $n=6$, habebimus ante omnia

$$(6,1)=1,\quad (6,2)=\tfrac{1}{2},\quad (6,3)=\tfrac{1}{3},\quad (6,4)=\tfrac{1}{4};$$

formularum autem integralium in hoc ordine occurrentium numerus est quindecim, quae sunt

1. $(1,1)$, 2. $(1,2)$, 3. $(1,3)$, 4. $(1,4)$, 5. $(1,5)$,
6. $(2,2)$, 7. $(2,3)$, 8. $(2,4)$, 9. $(2,5)$, 10. $(3,3)$,
11. $(3,4)$, 12. $(3,5)$, 13. $(4,4)$, 14. $(4,5)$, 15. $(5,5)$,

inter quas reperiuntur tres circulares, quas singulari modo designemus, scilicet

$$(1,5)=\frac{\pi}{6\sin.\frac{1}{6}\pi}=\frac{\pi}{3}=A,$$

$$(2,4)=\frac{\pi}{6\sin.\frac{2\pi}{6}}=\frac{\pi}{3\sqrt{3}}=B$$

et

$$(3,3)=\frac{\pi}{6\sin.\frac{3\pi}{6}}=\frac{\pi}{6}=C,$$

ita ut sit

$$A=2C.$$

Praeterea vero ambas formulas, quae in ordine praecedente erant circulares, nunc vero sunt transcendentes, statuamus

$$(1,4)=P \quad \text{et} \quad (2,3)=Q.$$

His factis denominationibus evolvamus decem aequationes classis quartae, quae sunt

1. $B=P(5,2)$,
2. $C=(3,1)(4,3)$,
3. $C=2Q(5,3)$,
4. $B=(2,1)(3,4)$,
5. $B=2(2,2)(4,4)$,
6. $B=3Q(5,4)$,
7. $A=(1,1)(5,2)$,
8. $A=2(1,2)(3,5)$,
9. $A=3(1,3)(4,5)$,
10. $A=4P(5,5)$,

quas ita succinctius referre licet

$$A=(1,1)(5,2)=2(1,2)(3,5)=3(1,3)(4,5)=4P(5,5),$$

$$B=P(5,2)=(2,1)(3,4)=2(2,2)(4,4)=3Q(4,5),$$

$$C=(3,1)(4,3)=2Q(5,3).$$ [1]

Ecce ergo decem producta ex binis formulis integralibus, quorum singula quantitati circulari aequantur.

35. Cum deinde sit $\frac{A}{B}=\sqrt{3}$ et $\frac{A}{C}=2$, tum vero etiam $\frac{B}{C}=\frac{2}{\sqrt{3}}$, plura paria binarum formularum integralium exhiberi possunt, quae inter se teneant rationem algebraicam; erit enim

$$\frac{A}{B}=\sqrt{3}=\frac{(1,1)}{(1,4)}=\frac{2(3,5)}{(3,4)}=\frac{(1,3)}{(2,3)}=\frac{4(5,5)}{(5,2)},$$

$$\frac{A}{C}=2=\frac{(1,2)}{(2,3)}=\frac{3(4,5)}{(4,3)},$$

$$\frac{B}{C}=\frac{2}{\sqrt{3}}=\frac{(1,2)}{(1,3)}=\frac{3(4,5)}{2(3,5)}.$$ [2]

1) Editio princeps:

$$C=(3,1)(5,2)=2Q(5,3),$$

unde errores in formulis sequentibus nati sunt. Correxit A. L.

2) Editio princeps:

$$\frac{A}{C}=2=\frac{(1,1)}{(1,3)}=\frac{(1,2)}{(2,3)}=\frac{3(4,5)}{(2,5)},$$

$$\frac{B}{C}=\frac{2}{\sqrt{3}}=\frac{(1,4)}{(1,3)}=\frac{3(4,5)}{2(3,5)}.$$

Correxit A. L.

36. Quodsi iam quinque formulas litteris A, B, C, P et Q designatas tamquam cognitas spectemus, videamus, quomodo reliquae formulae per eas definiri queant. Ac primo quidem percurramus decem aequationes classis quartae supra allatas, quarum prima dabit $(5,2)=\frac{B}{P}$, tertia dat $(5,3)=\frac{C}{2Q}$, sexta praebet $(5,4)=\frac{B}{3Q}$, decima dat $(5,5)=\frac{A}{4P}$. Quodsi iam hos valores in reliquis surrogemus, septima praebet $(1,1)=\frac{A}{(5,2)}=\frac{AP}{B}$, octava dat $(1,2)=\frac{A}{2(3,5)}=\frac{AQ}{C}$, nona dat $(3,1)=\frac{A}{3(4,5)}=\frac{AQ}{B}$. Porro vero quarta dat $(3,4)=\frac{B}{(2,1)}=\frac{BC}{AQ}$, quem valorem etiam secunda praebet.[1]) At vero ex aequatione quinta nullum valorem elicere possumus, quia neque formula $(2,2)$ nec $(4,4)$ etiamnunc constat. Causa est, quia duae reliquarum aequationum eandem determinationem produxerunt.

37. Coacti igitur sumus ad aequationes praecedentium classium confugere atque adeo ex prima classe

$$(1,2)(3,1)=(1,1)(2,2)$$

statim colligimus

$$(2,2)=\frac{(1,2)(3,1)}{(1,1)}=\frac{AQQ}{CP},$$

qui valor in quinta aequatione substitutus suppeditat postremam aequationem, nempe

$$(4,4)=\frac{B}{2(2,2)}=\frac{BCP}{2AQQ}.$$

Omnes igitur hos valores hic ordine referemus:

1. $(1,1)=\frac{AP}{B}$,	4. $(1,4)=P$,	7. $(2,3)=Q$,
2. $(1,2)=\frac{AQ}{C}$,	5. $(1,5)=A$,	8. $(2,4)=B$,
3. $(1,3)=\frac{AQ}{B}$,	6. $(2,2)=\frac{AQQ}{CP}$,	9. $(2,5)=\frac{B}{P}$,

1) Editio princeps: *Quodsi iam hos valores in reliquis surrogemus, secunda dabit* $(3,1)=\frac{C}{(4,3)}=\frac{AQ}{B}$, *septima praebet ... nona dat* $(3,1)=\frac{A}{3(4,5)}=\frac{AQ}{B}$, *quem valorem etiam secunda praebuit. Porro vero quarta dat* $(3,4)=\frac{B}{(2,1)}=\frac{BC}{AQ}$. Correxit A. L.

10. $(3,3)=C$,	12. $(3,5)=\frac{C}{2Q}$,	14. $(4,5)=\frac{B}{3Q}$,
11. $(3,4)=\frac{BC}{AQ}$,	13. $(4,4)=\frac{BCP}{2AQQ}$,	15. $(5,5)=\frac{A}{4P}$.

38. Cum autem in hoc ordine etiam aequationes tam classis secundae quam tertiae valere debeant, videamus, utrum valores inventi his classibus conveniant an vero forte novam determinationem suppeditent? Facta autem substitutione in tribus aequationibus secundae classis ad identitatem pervenitur, quod idem quoque in aequationibus tertiae classis contingere debet, id quod evolventi mox patebit. Unde memorabile est omnes aequationes in quatuor primis classibus contentas, quarum numerus est 20, tantum decem determinationes in se complecti.

ORDO V QUO $n=7$ ET FORMULA

$$(p,q)=\int \frac{x^{p-1}\partial x}{\sqrt[7]{(1-x^7)^{7-q}}}=\int \frac{x^{q-1}\partial x}{\sqrt[7]{(1-x^7)^{7-p}}}$$

39. Quia hic $n=7$, ante omnia habebimus valores absolutos

$$(7,1)=1,\quad (7,2)=\tfrac{1}{2},\quad (7,3)=\tfrac{1}{3},\quad (7,4)=\tfrac{1}{4}\quad \text{et}\quad (7,5)=\tfrac{1}{5},$$

deinde inter formulas integrales huius ordinis imprimis notari debent circulares, quas hoc modo designemus:

$$(1,6)=\frac{\pi}{7\sin.\frac{\pi}{7}}=A,$$

$$(2,5)=\frac{\pi}{7\sin.\frac{2\pi}{7}}=B,$$

$$(3,4)=\frac{\pi}{7\sin.\frac{3\pi}{7}}=C.$$

Praeterea vero peculiaribus litteris notentur eae formulae, quae in ordine praecedenti erant circulares, hic autem valores transcendentes sortiuntur, qui sint

$$(1,5)=P,\quad (2,4)=Q \quad \text{et} \quad (3,3)=R;$$

per has enim sex litteras videbimus omnes reliquas formulas huius ordinis determinari posse.

40. Quoniam supra non omnes aequationes quintae classis expressimus, eas hic coniunctim exhibeamus et ad nostrum casum accommodemus:

I.	$(1,6)(7,1)=(1,1)(2,6)$	$A=\ (1,1)(2,6)$,
II.	$(1,6)(7,2)=(1,2)(3,6)$	$A=2\,(1,2)(3,6)$,
III.	$(1,6)(7,3)=(1,3)(4,6)$	$A=3\,(1,3)(4,6)$,
IV.	$(1,6)(7,4)=(1,4)(5,6)$	$A=4\,(1,4)(5,6)$,
V.	$(1,6)(7,5)=(1,5)(6,6)$	$A=5\ \ P\ \ (6,6)$,
VI.	$(2,5)(7,1)=(2,1)(3,5)$	$B=\ (2,1)(3,5)$,
VII.	$(2,5)(7,2)=(2,2)(4,5)$	$B=2\,(2,2)(4,5)$,
VIII.	$(2,5)(7,3)=(2,3)(5,5)$	$B=3\,(2,3)(5,5)$,
IX.	$(2,5)(7,4)=(2,4)(6,5)$	$B=4\ \ Q\ \ (6,5)$,
X.	$(3,4)(7,1)=(3,1)(4,4)$	$C=\ (3,1)(4,4)$,
XI.	$(3,4)(7,2)=(3,2)(5,4)$	$C=2\,(3,2)(5,4)$,
XII.	$(3,4)(7,3)=(3,3)(6,4)$	$C=3\ \ R\ \ (6,4)$,
XIII.	$(4,3)(7,1)=(4,1)(5,3)$	$C=\ (4,1)(5,3)$,
XIV.	$(4,3)(7,2)=(4,2)(6,3)$	$C=2\ \ Q\ \ (6,3)$,
XV.	$(5,2)(7,1)=(5,1)(6,2)$	$B=\ \ P\ \ (6,2)$.

Hic igitur habemus quina producta formulae A aequalia totidemque formulis B et C aequalia.

41. Omnino autem in hoc ordine occurrunt 21 formulae integrales, ex quibus sex litteris A, B, C, P, Q et R designavimus, per quas igitur reliquas quindecim formulas integrales definiri oportet, quae sunt

1. $(1,1)$, 2. $(1,2)$, 3. $(1,3)$, 4. $(2,2)$, 5. $(1,4)$,
6. $(2,3)$, 7. $(2,6)$, 8. $(3,5)$, 9. $(4,4)$, 10. $(3,6)$,
11. $(4,5)$, 12. $(4,6)$, 13. $(5,5)$, 14. $(5,6)$, 15. $(6,6)$.

42. Videamus igitur, quot harum formularum ex superioribus quindecim aequationibus determinare liceat; ac primo quidem ex aequationibus V, IX, XII, XIV et XV immediate deducuntur sequentes formulae

$$(6,6)=\frac{A}{5P},\quad (6,5)=\frac{B}{4Q},\quad (6,4)=\frac{C}{3R},\quad (6,3)=\frac{C}{2Q},\quad (6,2)=\frac{B}{P}.$$

His iam inventis ex aequationibus I, II, III et IV derivamus has formulas

$$(1,1)=\frac{AP}{B},\quad (1,2)=\frac{AQ}{C},\quad (1,3)=\frac{AR}{C},\quad (1,4)=\frac{AQ}{B}.$$

Ex his vero valoribus per aequationes VI, X et XIII colligimus

$$(3,5)=\frac{BC}{AQ},\quad (4,4)=\frac{CC}{AR}\quad \text{et}\quad (5,3)=\frac{BC}{AQ},$$

ubi notasse iuvabit eundem valorem pro $(5,3)$ prodiisse ex aequationibus VI et XIII. Ex reliquis autem aequationibus VII, VIII et XI nihil concludere licet, unde istae quatuor formulae $(2,2)$, $(2,3)$, $(5,4)$ et $(5,5)$ nobis etiamnunc manent incognitae.

43. Recurrere ergo coacti sumus ad aequationes praecedentium classium, quippe quae aeque ad nostrum ordinem pertinent atque aequationes classis quintae; hanc ob rem simili modo aequationes classis quartae hic apponamus et ad nostrum casum applicemus:

I.	$(1,5)(6,1)=(1,1)(2,5)$	$PA\ \ \ =(1,1)B$,
II.	$(1,5)(6,2)=(1,2)(3,5)$	$P(6,2)=(1,2)(3,5)$,
III.	$(1,5)(6,3)=(1,3)(4,5)$	$P(6,3)=(1,3)(4,5)$,
IV.	$(1,5)(6,4)=(1,4)(5,5)$	$P(6,4)=(1,4)(5,5)$,
V.	$(2,4)(6,1)=(2,1)(3,4)$	$QA\ \ \ =(2,1)C$,
VI.	$(2,4)(6,2)=(2,2)(4,4)$	$Q(6,2)=(2,2)(4,4)$,
VII.	$(2,4)(6,3)=(2,3)(5,4)$	$Q(6,3)=(2,3)(5,4)$,
VIII.	$(3,3)(6,1)=(3,1)(4,3)$	$RA\ \ \ =(3,1)C$,
IX.	$(3,3)(6,2)=(3,2)(5,3)$	$R(6,2)=(3,2)(5,3)$,
X.	$(4,2)(6,1)=(4,1)(5,2)$	$QA\ \ \ =(4,1)B$.

44. Ex aequationibus I, V, VIII et X immediate concludimus has formulas

$$(1, 1) = \frac{PA}{B}, \quad (2, 1) = \frac{QA}{C}, \quad (3, 1) = \frac{AR}{C}, \quad (4, 1) = \frac{AQ}{B},$$

quos autem valores iam ante adepti sumus. Secunda aequatio, si formulae iam inventae substituantur, praebet aequationem identicam. Ex tertia autem poterimus definire formulam (4, 5), cuius valor hinc colligitur

$$(4, 5) = \frac{CCP}{2AQR}.$$

Simili modo ex quarta elicimus

$$(5, 5) = \frac{BCP}{3AQR}.$$

Porro ex aequatione sexta concludimus fore

$$(2, 2) = \frac{ABQR}{CCP}.$$

Deinde septima aequatio dat

$$(2, 3) = \frac{AQR}{CP}.$$

Nona vero aequatio etiam praebet $(3, 2) = \frac{AQR}{CP}$. Sicque omnes quindecim formulas incognitas determinavimus per sex litteras cognitas A, B, C, P, Q et R.

45. Valores igitur omnium formularum huius ordinis hic aspectui coniunctim exponamus:

$(1, 6) = A$	$(6, 2) = \frac{B}{P}$	$(1, 1) = \frac{AP}{B}$	$(3, 5) = \frac{BC}{AQ}$	$(2, 3) = \frac{AQR}{CP}$,
$(2, 5) = B$	$(6, 3) = \frac{C}{2Q}$	$(1, 2) = \frac{AQ}{C}$	$(4, 4) = \frac{CC}{AR}$	$(4, 5) = \frac{CCP}{2AQR}$,
$(3, 4) = C$	$(6, 4) = \frac{C}{3R}$	$(1, 3) = \frac{AR}{C}$		$(5, 5) = \frac{BCP}{3AQR}$,
$(1, 5) = P$	$(6, 5) = \frac{B}{4Q}$	$(1, 4) = \frac{AQ}{B}$		$(2, 2) = \frac{ABQR}{CCP}$.
$(2, 4) = Q$	$(6, 6) = \frac{A}{5P}$			
$(3, 3) = R$				

46. Quoniam autem aequationes primae, secundae ac tertiae classis etiam in hoc ordine valent, si in iis valores hic inventos substituamus, perpetuo in aequationes identicas incidemus. Ita, cum aequatio primae classis sit

$$(1, 2)\,(3, 1) = (1, 1)\,(2, 2),$$

facta substitutione reperietur

$$(1, 2)\,(3, 1) = \frac{AAQR}{CC};$$

at vero $(1, 1)\,(2, 2)$ fit $= \frac{AAQR}{CC}$ haecque identitas etiam deprehendetur in tribus aequationibus secundae classis atque etiam in sex aequationibus tertiae classis, quemadmodum calculum instituenti mox patebit.

47. Simili modo haud difficile erit hanc investigationem ad ordines superiores extendere, neque tamen legem observare licet, secundum quam determinationes singularum formularum cuiusque ordinis progrediuntur. Interim tamen observasse iuvabit in ordine sequente sexto, ubi $n = 8$ et formulae occurrunt 28, eas omnes primo per quatuor formulas circulares

$$(1, 7) = A, \quad (2, 6) = B, \quad (3, 5) = C, \quad (4, 4) = D,$$

praeterea vero per has tres transcendentes

$$(1, 6) = P, \quad (2, 5) = Q \quad \text{et} \quad (3, 4) = R$$

determinari posse. Cum igitur quovis ordine determinatio singularum formularum praeter formulas circulares, quae utique pro cognitis haberi possunt, etiam aliquot formulas transcendentes postulat, si saltem valores harum formularum vero proxime cognoscere voluerimus, methodus adhuc desideratur istos valores proxime, veluti in fractionibus decimalibus, definiendi. Talem igitur methodum hic coronidis loco subiungemus.

PROBLEMA

48. *Proposita formula integrali cuiusque ordinis*

$$S = \int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}$$

a termino $x = 0$ usque ad $x = 1$ extendenda investigare seriem convergentem, quae istum valorem S exprimat.

SOLUTIO

Cum sit

$$\frac{1}{\sqrt[n]{(1-x^n)^{n-q}}} = (1-x^n)^{-\frac{n-q}{n}},$$

facta evolutione huius potestatis binomii more solito reperietur

$$\frac{1}{\sqrt[n]{(1-x^n)^{n-q}}} = 1 + \frac{n-q}{n}x^n + \frac{n-q}{n}\cdot\frac{2n-q}{2n}x^{2n} + \frac{n-q}{n}\cdot\frac{2n-q}{2n}\cdot\frac{3n-q}{3n}x^{3n} + \text{etc.}$$

Si haec series ducatur in $x^{p-1}\partial x$ et integretur, prodibit

$$S = \frac{x^p}{p} + \frac{n-q}{n}\cdot\frac{x^{n+p}}{n+p} + \frac{n-q}{n}\cdot\frac{2n-q}{2n}\cdot\frac{x^{2n+p}}{2n+p} + \frac{n-q}{n}\cdot\frac{2n-q}{2n}\cdot\frac{3n-q}{3n}\cdot\frac{x^{3n+p}}{3n+p} + \text{etc.},$$

quae series iam evanescit posito $x=0$; unde si ponamus $x=1$, valor quaesitus nostrae formulae fiet

$$S = \frac{1}{p} + \frac{n-q}{n}\cdot\frac{1}{n+p} + \frac{n-q}{n}\cdot\frac{2n-q}{n}\cdot\frac{1}{2n+p} + \frac{n-q}{n}\cdot\frac{2n-q}{2n}\cdot\frac{3n-q}{3n}\cdot\frac{1}{3n+p} + \text{etc.}$$

49. Verum ista series, quicumque numeri pro litteris n, p et q accipiantur, nimis lente convergit, quam ut ex ea valores ipsius S saltem ad tres quatuorve figuras decimales satis exacte definiri queant; quamobrem aliam evolutionem institui conveniet, dum scilicet valorem quaesitum in duas partes resolvemus. Statuamus igitur

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x^n = \frac{1}{2} \end{bmatrix} = P$$

et

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}} \begin{bmatrix} \text{ab } x^n = \frac{1}{2} \\ \text{ad } x = 1 \end{bmatrix} = Q$$

atque evidens est fore

$$S = P + Q.$$

Nunc autem tam pro P quam pro Q haud difficulter series satis convergentes exhiberi poterunt.

50. Quod primum ad valorem P attinet, eum ex valore generali, quem supra pro S invenimus, facile derivabimus ponendo $x^n = \frac{1}{2}$, ita ut sit $x = \sqrt[n]{\frac{1}{2}}$ et $x^p = \frac{1}{\sqrt[n]{2^p}}$, quo facto pro P obtinebimus hanc seriem

$$P = \frac{1}{\sqrt[n]{2^p}} \left\{ \begin{aligned} &\frac{1}{p} + \frac{n-q}{2n}\cdot\frac{1}{n+p} + \frac{n-q}{2n}\cdot\frac{2n-q}{4n}\cdot\frac{1}{2n+p} \\ &+ \frac{n-q}{2n}\cdot\frac{2n-q}{4n}\cdot\frac{3n-q}{6n}\cdot\frac{1}{3n+p} + \text{etc.} \end{aligned} \right\}.$$

In qua serie singuli termini plus quam in ratione dupla decrescunt, ita ut verbi gratia terminus decimus iam multo minor futurus sit quam $\frac{1}{1024}$, unde, si ad partes millionesimas certi esse velimus, sufficeret calculum ne quidem ad vicesimum usque terminum extendere.

51. Cum deinde posuerimus

$$Q = \int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}} \begin{bmatrix} \text{ab } x^n = \frac{1}{2} \\ \text{ad } x = 1 \end{bmatrix},$$

statuamus $1 - x^n = y^n$, ut sit

$$Q = \int \frac{x^{p-1}\partial x}{y^{n-q}};$$

tum vero erit $x^n = 1 - y^n$ ideoque $x^p = \sqrt[n]{(1-y^n)^p}$, unde differentiando colligitur

$$x^{p-1}\partial x = -y^{n-1}\partial y(1-y^n)^{\frac{p-n}{n}},$$

quo valore substituto erit

$$Q = -\int y^{q-1}\partial y(1-y^n)^{\frac{p-n}{n}} \begin{bmatrix} \text{ab } y^n = \frac{1}{2} \\ \text{ad } y = 0 \end{bmatrix}.$$

Quando enim fit $x^n = \frac{1}{2}$, tum etiam erit $y^n = \frac{1}{2}$, at facto $x = 1$ manifesto fit $y = 0$; quare si terminos integrationis permutemus, etiam signum ipsius formulae immutari debet sicque fiet

$$Q = \int y^{q-1}\partial y(1-y^n)^{\frac{p-n}{n}} \begin{bmatrix} \text{ab } y = 0 \\ \text{ad } y^n = \frac{1}{2} \end{bmatrix}.$$

52. Haec autem formula pro Q inventa omnino similis est illi, quam pro P invenimus, hoc tantum discrimine, quod litterae p et q inter se sunt permutatae; quocirca, si integratio per seriem instituatur, proveniet sequens

$$Q = \frac{1}{\sqrt[n]{2^q}} \left\{ \begin{aligned} &\frac{1}{q} + \frac{n-p}{2n} \cdot \frac{1}{n+q} + \frac{n-p}{2n} \cdot \frac{2n-p}{4n} \cdot \frac{1}{2n+q} \\ &+ \frac{n-p}{2n} \cdot \frac{2n-p}{4n} \cdot \frac{3n-p}{6n} \cdot \frac{1}{3n+q} + \text{etc.} \end{aligned} \right\},$$

quae series aeque converget ac praecedens pro P inventa. His autem duabus seriebus ad calculum revocatis semper erit valor quaesitus

$$S = P + Q.$$

COROLLARIUM 1

53. Iste calculus plurimum contrahetur iis casibus, quibus est $p = q$; tum enim fiet $P = Q$ hisque casibus, quibus

$$S = \int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-p}}},$$

valor istius formulae ab $x = 0$ ad $x = 1$ extensae erit

$$S = \frac{2}{\sqrt[n]{2^p}} \left\{ \begin{aligned} &\frac{1}{p} + \frac{n-p}{2n} \cdot \frac{1}{n+p} + \frac{n-p}{2n} \cdot \frac{2n-p}{4n} \cdot \frac{1}{2n+p} \\ &+ \frac{n-p}{2n} \cdot \frac{2n-p}{4n} \cdot \frac{3n-p}{6n} \cdot \frac{1}{3n+p} + \text{etc.} \end{aligned} \right\}.$$

COROLLARIUM 2

54. Quoniam igitur in singulis ordinibus nonnullae huiusmodi formulae (p, p) occurrunt, statim atque valores aliquot huiusmodi formularum fuerint ad calculum decimalem revocati, quoniam formulae circulares per se sunt notae, ex iis valores omnium reliquarum formularum eiusdem ordinis assignare licebit.

EXEMPLUM

55. Proposita sit formula ordinis primi, ubi $p = q = 2$ et

$$S = \int \frac{x\partial x}{\sqrt[3]{(1-x^3)}}.$$

Series igitur pro S inventa erit

$$S = \sqrt[3]{2}\left(\frac{1}{2} + \frac{1}{6} \cdot \frac{1}{5} + \frac{1}{6} \cdot \frac{4}{12} \cdot \frac{1}{8} + \frac{1}{6} \cdot \frac{4}{12} \cdot \frac{7}{18} \cdot \frac{1}{11} + \frac{1}{6} \cdot \frac{4}{12} \cdot \frac{7}{18} \cdot \frac{10}{24} \cdot \frac{1}{14} + \text{etc.}\right).$$

Subducto autem calculo reperitur

$$S = 0{,}54326 \sqrt[3]{2} = 0{,}68446,$$

qui ergo est valor formulae $(2, 2)$ in ordine primo (§ 22), ubi invenimus $(2, 2) = \frac{A}{P}$, ita ut iam sit $P = \frac{A}{(2, 2)}$. Est vero

$$A = \frac{2\pi}{3\sqrt{3}} = 1{,}20920,$$

hinc erit

$$P = 1{,}76664 = (1, 1),$$

unde in fractionibus decimalibus ternae formulae ordinis primi erunt

$$(1, 1) = 1{,}76664, \quad (1, 2) = 1{,}20920, \quad (2, 2) = 0{,}68446.$$ [1]

Hocque modo etiam omnes formulas sequentium ordinum evolvere licebit.

1) Editio princeps: *Subducto autem calculo reperitur*

$$S = 0{,}54325 \sqrt[3]{2} = 0{,}68445,$$

qui ergo ... Est vero $A = \frac{2\pi}{3\sqrt{3}} = 1{,}20918$, *hinc erit* $P = 2{,}22582 = (1, 1)$, *unde in ... primi erunt*

$$(1, 1) = 2{,}22582, \quad (1, 2) = 1{,}20918, \quad (2, 2) = 0{,}68445.$$ Correxit A. L.

ADDITAMENTUM AD DISSERTATIONEM DE VALORIBUS FORMULAE INTEGRALIS

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}$$

AB $x=0$ AD $x=1$ EXTENSAE[1])

Conventui exhibitum die 17. Octobris 1776

Nova acta academiae scientiarum Petropolitanae 5 (1787), 1789, p. 118—129
Summarium ibidem p. 72—73

SUMMARIUM

Les Géomètres ne méconnoîtront pas dans ce supplément un trait caractéristique du génie de feu M. EULER. Ils savent qu'il est peu de sujets pour lesquels il ne soit revenu sur ses traces, en donnant toujours un plus haut degré de perfection à tout ce qu'il avoit fait antérieurement. Nos Extraits mêmes en fournissent plus d'une preuve, où nous nous attachons principalement à indiquer, souvent à la vérité par peu de traits, ce que l'illustre Auteur avoit fait autrefois dans la même matière.

M. EULER s'étoit vu arrêté dans le cours de son Mémoire précédent par les difficultés que le grand nombre d'équations fait naître, dès que l'on veut donner à l'exposant n une valeur qui surpasse 7; c'est pourquoi il n'avoit poursuivi ses recherches que jusqu'au cinquième ordre. Ayant vu cependant que de ce grand nombre d'équations, qui résultent dans chaque ordre, toutes ne sont pas nécessaires à la détermination des formules contenues dans cet ordre, il a voulu examiner le huitième, où $n=10$, en ne tenant compte que des équations qui concourent à la détermination des formules de cet ordre. De cette façon l'Auteur trouve dans cet ordre 45 formules dont neuf, savoir cinq circulaires et quatre transcendantes, servent à déterminer les 36 autres; et cette méthode peut être employée avec le même avantage pour les ordres supérieurs.

Le Mémoire est terminé par une méthode générale de traiter l'ordre n.

1) Vide Commentationem 640, p. 392. Indicis ENESTROEMIANI numerus 640 etiam pro hoc Additamento valet. A. L.

1. Si methodum in praecedente dissertatione traditam ad altiores ordines quam $n=7$ transferre vellemus, ob ingentem aequationum considerandarum numerum labor fieret nimis molestus. Quoniam autem vidimus non omnes istas aequationes concurrere ad valores singularum formularum determinandos, opus non mediocriter sublevabitur, si quovis casu eas tantum aequationes in computum ducamus, quae immediate ad determinationes formularum perducant, quemadmodum hic pro casu $n=10$ sum ostensurus.

DETERMINATIO HARUM FORMULARUM PRO CASU $n=10$ UBI FORMULA

$$(p,q) = \int \frac{x^{p-1}\partial x}{\sqrt[10]{(1-x^{10})^{10-q}}} = \int \frac{x^{q-1}\partial x}{\sqrt[10]{(1-x^{10})^{10-p}}}$$

2. Hoc casu ergo formulae valorem absolutum recipientes sunt

$$(10,1)=1,\quad (10,2)=\frac{1}{2},\quad (10,3)=\frac{1}{3}\quad \text{et in genere}\quad (10,\alpha)=\frac{1}{\alpha}.$$

Deinde omnes formulae, in quibus est $p+q=10$, a circulo pendent ideoque pro cognitis haberi possunt, quas ergo propriis litteris designemus:

$$(1,9)=\frac{\pi}{10\sin.\frac{1}{10}\pi}=A,\qquad (6,4)=\frac{\pi}{10\sin.\frac{6}{10}\pi}=D,$$

$$(2,8)=\frac{\pi}{10\sin.\frac{2}{10}\pi}=B,\qquad (7,3)=\frac{\pi}{10\sin.\frac{7}{10}\pi}=C,$$

$$(3,7)=\frac{\pi}{10\sin.\frac{3}{10}\pi}=C,\qquad (8,2)=\frac{\pi}{10\sin.\frac{8}{10}\pi}=B,$$

$$(4,6)=\frac{\pi}{10\sin.\frac{4}{10}\pi}=D,\qquad (9,1)=\frac{\pi}{10\sin.\frac{9}{10}\pi}=A.$$

$$(5,5)=\frac{\pi}{10\sin.\frac{5}{10}\pi}=E,$$

3. Per has autem formulas circulares reliquas in forma generali contentas neutiquam determinare licet, sed insuper aliquot formulas transcendentes in subsidium vocari oportet, ex quibus cum circularibus illis coniunctis

reliquarum omnium valores assignare licebit. Nostro autem casu, quo $n=10$, sequentes formulas tamquam cognitas spectari conveniet, quae in ordine praecedenti, ubi $n=9$, erant circulares, nunc autem in ordinem transcendentium transeunt. Eas igitur sequenti modo designemus

$$(1,8)=P,\quad (2,7)=Q,\quad (3,6)=R,\quad (4,5)=S,$$
$$(5,4)=S,\quad (6,3)=R,\quad (7,2)=Q,\quad (8,1)=P.$$

Scilicet si valores harum litterarum quoque tamquam cognitos spectemus, per eos cum circularibus iunctos reliquas formulas omnes in hoc ordine contentas determinare poterimus. Cum igitur numerus omnium formularum integralium in hoc ordine $n=10$ contentarum sit 45, ex iis autem novem ut cognitae spectentur, reliquae 36 per has litteras maiusculas determinari debebunt.

4. Istas autem determinationes ex aequatione generali supra [§ 10] demonstrata peti oportet, quae hac forma continetur

$$(a,b)\,(a+b,c)=(a,c)\,(a+c,b),$$

ubi assumere licebit semper esse $b>c$, quoniam, si foret $c=b$, aequatio foret identica. Primo igitur, ut hinc aequationes, quae immediate determinationes praebeant, nanciscamur, sumamus $a+b=10$, ut sit $(10,c)=\frac{1}{c}$; tum vero capiatur $c=b-1$, quo facto pro a ordine scribendo numeros 1, 2, 3 etc. sequentes prodibunt determinationes

$$\begin{array}{llll}
(1,9)\,(10,8)=(1,8)\,(9,9) & \text{sive} \quad \frac{1}{8}A=P(9,9), & \text{ergo} & (9,9)=\frac{A}{8P},\\
(2,8)\,(10,7)=(2,7)\,(9,8) & \text{sive} \quad \frac{1}{7}B=Q(9,8), & \text{ergo} & (9,8)=\frac{B}{7Q},\\
(3,7)\,(10,6)=(3,6)\,(9,7) & \text{sive} \quad \frac{1}{6}C=R(9,7), & \text{ergo} & (9,7)=\frac{C}{6R},\\
(4,6)\,(10,5)=(4,5)\,(9,6) & \text{sive} \quad \frac{1}{5}D=S(9,6), & \text{ergo} & (9,6)=\frac{D}{5S},\\
(5,5)\,(10,4)=(5,4)\,(9,5) & \text{sive} \quad \frac{1}{4}E=S(9,5), & \text{ergo} & (9,5)=\frac{E}{4S},\\
(6,4)\,(10,3)=(6,3)\,(9,4) & \text{sive} \quad \frac{1}{3}D=R(9,4), & \text{ergo} & (9,4)=\frac{D}{3R},\\
(7,3)\,(10,2)=(7,2)\,(9,3) & \text{sive} \quad \frac{1}{2}C=Q(9,3), & \text{ergo} & (9,3)=\frac{C}{2Q},\\
(8,2)\,(10,1)=(8,1)\,(9,2) & \text{sive} \quad B=P(9,2), & \text{ergo} & (9,2)=\frac{B}{P}.
\end{array}$$

5. Ex formulis igitur incognitis illis numero 36 iam octo determinavimus quae nobis viam sternent ad novas determinationes, quas primo derivabimus ex aequatione generali sumendo $a=1$, $b=9$ et pro c scribendo ordine numeros 1, 2, 3, ... 8, unde calculus ita se habebit:

$$\begin{array}{l|lll}
(1,9)\,(10,1)=(1,1)\,(2,9) & A=(1,1)\frac{B}{P}, & \text{ergo} & (1,1)=\frac{AP}{B},\\
(1,9)\,(10,2)=(1,2)\,(3,9) & \frac{1}{2}A=(1,2)\frac{C}{2Q}, & \text{ergo} & (1,2)=\frac{AQ}{C},\\
(1,9)\,(10,3)=(1,3)\,(4,9) & \frac{1}{3}A=(1,3)\frac{D}{3R}, & \text{ergo} & (1,3)=\frac{AR}{D},\\
(1,9)\,(10,4)=(1,4)\,(5,9) & \frac{1}{4}A=(1,4)\frac{E}{4S}, & \text{ergo} & (1,4)=\frac{AS}{E},\\
(1,9)\,(10,5)=(1,5)\,(6,9) & \frac{1}{5}A=(1,5)\frac{D}{5S}, & \text{ergo} & (1,5)=\frac{AS}{D},\\
(1,9)\,(10,6)=(1,6)\,(7,9) & \frac{1}{6}A=(1,6)\frac{C}{6R}, & \text{ergo} & (1,6)=\frac{AR}{C},\\
(1,9)\,(10,7)=(1,7)\,(8,9) & \frac{1}{7}A=(1,7)\frac{B}{7Q}, & \text{ergo} & (1,7)=\frac{AQ}{B},\\
(1,9)\,(10,8)=(1,8)\,(9,9) & \frac{1}{8}A=(1,8)\frac{A}{8P}, & \text{ergo} & (1,8)=\frac{AP}{A};
\end{array}$$

hocque modo septem novas determinationes sumus adepti.

6. His autem inventis consideremus aequationes ex valoribus $a=1$ $b=8$, $c=1,2,3,\ldots 7$ ortas eritque

$$\begin{array}{l|l|l}
(1,8)\,(9,1)=(1,1)\,(2,8) & AP=(1,1)B & \text{identica},\\
(1,8)\,(9,2)=(1,2)\,(3,8) & B=(3,8)\frac{AQ}{C} & (3,8)=\frac{BC}{AQ},\\
(1,8)\,(9,3)=(1,3)\,(4,8) & \frac{CP}{2Q}=(4,8)\frac{AR}{D} & (4,8)=\frac{CDP}{2AQR},\\
(1,8)\,(9,4)=(1,4)\,(5,8) & \frac{DP}{3R}=(5,8)\frac{AS}{E} & (5,8)=\frac{DEP}{3ARS},\\
(1,8)\,(9,5)=(1,5)\,(6,8) & \frac{EP}{4S}=(6,8)\frac{AS}{D} & (6,8)=\frac{DEP}{4ASS},\\
(1,8)\,(9,6)=(1,6)\,(7,8) & \frac{DP}{5S}=(7,8)\frac{AR}{C} & (7,8)=\frac{CDP}{5ARS},\\
(1,8)\,(9,7)=(1,7)\,(8,8) & \frac{CP}{6R}=(8,8)\frac{AQ}{B} & (8,8)=\frac{BCP}{6AQR}.
\end{array}$$

7. Novas determinationes reperiemus ponendo $a=1$, $b=7$, $c=3, 4, 5, 6$; hinc enim nanciscimur sequentes determinationes

$$\begin{array}{l|r|l}
(1, 7)(8, 3) = (1, 3)(4, 7) & D = (4, 7)\frac{AR}{D} & (4, 7) = \frac{CD}{AR}, \\
(1, 7)(8, 4) = (1, 4)(5, 7) & \frac{CDP}{2BR} = (5, 7)\frac{AS}{E} & (5, 7) = \frac{CDEP}{2ABRS}, \\
(1, 7)(8, 5) = (1, 5)(6, 7) & \frac{DEPQ}{3BRS} = (6, 7)\frac{AS}{D} & (6, 7) = \frac{DDEPQ}{3ABRSS}, \\
(1, 7)(8, 6) = (1, 6)(7, 7) & \frac{DEPQ}{4BSS} = (7, 7)\frac{AR}{C} & (7, 7) = \frac{CDEPQ}{4ABRSS}.
\end{array}$$

8. Sumamus nunc $a=1$, $b=6$, $c=4, 5$ eritque

$$\begin{array}{l|r|l}
(1, 6)(7, 4) = (1, 4)(5, 6) & D = (5, 6)\frac{AS}{E} & (5, 6) = \frac{DE}{AS}, \\
(1, 6)(7, 5) = (1, 5)(6, 6) & \frac{DEP}{2BS} = (6, 6)\frac{AS}{D} & (6, 6) = \frac{DDEP}{2ABSS}.
\end{array}$$

Hactenus igitur omnes formulas (p, q) determinavimus, in quibus $p+q>10$. Ex reliquis autem, ubi $p+q<9$, iam nacti sumus istas

$$(1, 1),\ (1, 2),\ (1, 3),\ (1, 4),\ (1, 5),\ (1, 6),\ (1, 7),$$

ita ut adhuc determinandae relinquantur istae

$$\begin{array}{l}
(2, 2),\ (2, 3),\ (2, 4),\ (2, 5),\ (2, 6), \\
(3, 3),\ (3, 4),\ (3, 5), \\
(4, 4).
\end{array}$$

9. Pro his inveniendis sumamus $a=1$ et $c=1$, pro b autem ordine capiamus numeros 2, 3 etc. atque consequemur has aequationes

$$\begin{array}{l|r|l}
(1, 2)(3, 1) = (1, 1)(2, 2) & \frac{AAQR}{CD} = (2, 2)\frac{AP}{B} & (2, 2) = \frac{ABQR}{CDP}, \\
(1, 3)(4, 1) = (1, 1)(2, 3) & \frac{AARS}{DE} = (2, 3)\frac{AP}{B} & (2, 3) = \frac{ABRS}{DEP}, \\
(1, 4)(5, 1) = (1, 1)(2, 4) & \frac{AASS}{DE} = (2, 4)\frac{AP}{B} & (2, 4) = \frac{ABSS}{DEP}, \\
(1, 5)(6, 1) = (1, 1)(2, 5) & \frac{AARS}{CD} = (2, 5)\frac{AP}{B} & (2, 5) = \frac{ABRS}{CDP}, \\
(1, 6)(7, 1) = (1, 1)(2, 6) & \frac{AAQR}{BC} = (2, 6)\frac{AP}{B} & (2, 6) = \frac{ABQR}{BCP}
\end{array}$$

sicque etiamnunc determinandae restant formulae (3, 3), (3, 4), (3, 5) et (4, 4).

10. Pro his sumatur $a=1$, $c=2$ et $b=3, 4, 5$ etc.; tum enim prodibunt hae aequationes

$$\begin{array}{l|r|l}
(1, 3)(4, 2) = (1, 2)(3, 3) & \frac{AABRSS}{DDEP} = (3, 3)\frac{AQ}{C} & (3, 3) = \frac{ABCRSS}{DDEPQ}, \\
(1, 4)(5, 2) = (1, 2)(3, 4) & \frac{AABRSS}{CDEP} = (3, 4)\frac{AQ}{C} & (3, 4) = \frac{ABRSS}{DEPQ}, \\
(1, 5)(6, 2) = (1, 2)(3, 5) & \frac{AAQRS}{CDP} = (3, 5)\frac{AQ}{C} & (3, 5) = \frac{ARS}{DP}.
\end{array}$$

Unica ergo formula restat determinanda, scilicet (4, 4), quae ex hac aequatione

$$(1, 4)(5, 3) = (1, 3)(4, 4)$$

definietur; erit enim

$$\frac{AARSS}{DEP} = (4, 4)\frac{AR}{D} \quad \text{ideoque} \quad (4, 4) = \frac{ASS}{EPP}.$$

11. Ut nunc omnes has determinationes simul aspectui exponamus, quoniam in hoc ordine $n=10$ omnino 45 formulae integrales occurrunt, si ex iis ut cognitae spectentur novem sequentes

$$(1, 9) = A,\quad (2, 8) = B,\quad (3, 7) = C,\quad (4, 6) = D,\quad (5, 5) = E,$$
$$(1, 8) = P,\quad (2, 7) = Q,\quad (3, 6) = R,\quad (4, 5) = S,$$

reliquae triginta sex ex his sequenti modo determinabuntur:

$$\begin{array}{rl|rl}
1. & (9, 9) = \frac{A}{8P} & 8. & (9, 2) = \frac{B}{P}, \\
\hline
2. & (9, 8) = \frac{B}{7Q} & 9. & (1, 1) = \frac{AP}{B}, \\
3. & (9, 7) = \frac{C}{6R} & 10. & (1, 2) = \frac{AQ}{C}, \\
4. & (9, 6) = \frac{D}{5S} & 11. & (1, 3) = \frac{AR}{D}, \\
5. & (9, 5) = \frac{E}{4S} & 12. & (1, 4) = \frac{AS}{E}, \\
6. & (9, 4) = \frac{D}{3R} & 13. & (1, 5) = \frac{AS}{D}, \\
7. & (9, 3) = \frac{C}{2Q} & 14. & (1, 6) = \frac{AR}{C},
\end{array}$$

15. $(1, 7) = \frac{AQ}{B}$	26. $(8, 8) = \frac{BCP}{6AQR}$,
16. $(3, 8) = \frac{BC}{AQ}$	27. $(2, 2) = \frac{ABQR}{CDP}$,
17. $(4, 7) = \frac{CD}{AR}$	28. $(2, 3) = \frac{ABRS}{DEP}$,
18. $(5, 6) = \frac{DE}{AS}$	29. $(2, 4) = \frac{ABSS}{DEP}$,
19. $(2, 6) = \frac{AQR}{CP}$	30. $(2, 5) = \frac{ABRS}{CDP}$,
20. $(3, 5) = \frac{ARS}{DP}$	31. $(5, 7) = \frac{CDEP}{2ABRS}$,
21. $(4, 4) = \frac{ASS}{EP}$	32. $(6, 6) = \frac{DDEP}{2ABSS}$,
22. $(4, 8) = \frac{CDP}{2AQR}$	33. $(3, 4) = \frac{ABRSS}{DEPQ}$,
23. $(5, 8) = \frac{DEP}{3ARS}$	34. $(6, 7) = \frac{DDEPQ}{3ABRSS}$,
24. $(6, 8) = \frac{DEP}{4ASS}$	35. $(7, 7) = \frac{CDEPQ}{4ABRSS}$,
25. $(7, 8) = \frac{CDP}{5ARS}$	36. $(3, 3) = \frac{ABCRSS}{DDEPQ}$.

12. Eadem methodo, qua hic usi sumus pro casu $n = 10$, haud difficile erit ordines altiores evolvere; neque tamen hinc adhuc elucet, quanam lege omnes determinationes progrediantur, quandoquidem valores certarum formularum continuo magis evadunt complicati. Ceterum valores, quos hic invenimus, omnibus aequationibus in forma generali

$$(a, b)\,(a + b, c) = (a, c)\,(a + c, b)$$

contentis satisfacere deprehenduntur, ita ut perpetuo aequatio identica resultet neque idcirco inde ulla nova relatio inter litteras nostras maiusculas deduci queat. Tandem probe hic notasse iuvabit, quod in omnibus ordinibus praeter formulas a circulo pendentes commodissime eae formulae, quae in ordine proxime praecedente erant circulares, hic etiam tamquam cognitae accipi queant, quippe quibus determinationes omnes optimo successu perfici possunt.

METHODUS GENERALIS DETERMINANDI VALORES FORMULAE

$$(p, q) = \int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}} = \int \frac{x^{q-1}\partial x}{\sqrt[n]{(1-x^n)^{n-p}}}$$

A TERMINO $x = 0$ USQUE AD $x = 1$ EXTENSAE

UBI PRAETER FORMULAS CIRCULUM INVOLVENTES IN QUIBUS EST $p + q = n$ ETIAM ILLAE PRO COGNITIS ACCIPIUNTUR IN QUIBUS EST $p + q = n - 1$

I. Cum aequatio generalis, unde omnes hae determinationes sunt petendae, sit

$$(a, b)\,(a + b, c) = (a, c)\,(a + c, b),$$

sumatur primo $a = n - \alpha$, $b = \alpha$ et $c = \alpha - 1$ eritque aequatio

$$(n - \alpha, \alpha)\,(n, \alpha - 1) = (n - \alpha, \alpha - 1)\,(n - 1, \alpha),$$

ubi est [§ 7]

$$(n, \alpha - 1) = \frac{1}{\alpha - 1}.$$

In primo autem factore ob $p = n - \alpha$ et $q = \alpha$ est $p + q = n$ ideoque datur. In tertio autem factore, ubi $p = n - \alpha$ et $q = \alpha - 1$, est $p + q = n - 1$ ideoque pariter datur. Hinc ergo colligimus

$$(n - 1, \alpha) = \frac{1}{\alpha - 1} \cdot \frac{(n - \alpha, \alpha)}{(n - \alpha, \alpha - 1)},$$

ubi esse debet $\alpha > 1$, ita ut pro α accipi queant omnes numeri a 2 usque ad $n - 1$; at vero casu $\alpha = 1$ valor formulae per se est notus.

II. In aequatione generali iam sumatur $a = \beta$, $b = n - \beta - 1$ et $c = 1$ eritque nostra aequatio

$$(\beta, n - \beta - 1)\,(n - 1, 1) = (\beta, 1)\,(\beta + 1, n - \beta - 1),$$

ex qua aequatione colligitur

$$(\beta, 1) = \frac{(\beta, n - \beta - 1)\,(n - 1, 1)}{(\beta + 1, n - \beta - 1)},$$

ubi esse debet $\beta < n-1$, ita ut hinc omnes formulae $(\beta, 1)$ definiantur a valore $\beta = 1$ usque ad $\beta = n-1$, quo posteriore casu formula $(n-1, 1)$ per se cognoscitur.

III. Ut hinc etiam alias formas eliciamus, sumamus $a = 1$, $b = n-2$, $c = \gamma$, ut oriatur haec aequatio

$$(1, n-2)(n-1, \gamma) = (1, \gamma)(1+\gamma, n-2),$$

ubi primus factor ac tertius dantur per II, secundus vero per I; unde quartus derivatur, scilicet

$$(1+\gamma, n-2) = \frac{(1, n-2)(n-1, \gamma)}{(1, \gamma)},$$

ubi valores ipsius $1+\gamma$ a 2 usque ad $n-2$ augeri possunt. Cum igitur per I sit

$$(n-1, \gamma) = \frac{1}{\gamma - 1} \cdot \frac{(n-\gamma, \gamma)}{(n-\gamma, \gamma-1)},$$

tum vero per II sit

$$(\gamma, 1) = \frac{(\gamma, n-\gamma-1)(n-1, 1)}{(\gamma+1, n-\gamma-1)},$$

his valoribus substitutis fiet

$$(n-2, 1+\gamma) = \frac{1}{\gamma-1} \cdot \frac{(1, n-2)(n-\gamma, \gamma)(\gamma+1, n-\gamma-1)}{(n-\gamma, \gamma-1)(\gamma, n-\gamma-1)(n-1, 1)}.$$

IV. Sumamus nunc $a = 1$, $b = n-3$, $c = \delta$ prodibitque haec aequatio

$$(1, n-3)(n-2, \delta) = (1, \delta)(1+\delta, n-3),$$

unde colligitur

$$(n-3, 1+\delta) = \frac{(n-3, 1)(n-2, \delta)}{(\delta, 1)},$$

ubi ergo $1+\delta$ continet numeros 2, 3, 4, ... $n-3$, ita ut hinc excludatur [formula] $(n-3, 1)$, quae autem per II datur. At si valores ante reperti substituantur, fiet

$$(n-3, 1+\delta) = \frac{1}{\delta-2} \cdot \frac{(n-3, 2)(n-2, 1)(n-\delta+1, \delta-1)(\delta, n-\delta)(\delta+1, n-\delta-1)}{(n-2, 2)(n-\delta+1, \delta-2)(\delta-1, n-\delta)(n-1, 1)(\delta, n-\delta-1)},$$

unde patet esse debere $\delta > 2$ eodemque modo pro praecedente formula $\gamma > 1$, ita ut hic excludantur casus $(n-3, 1)$, $(n-3, 2)$, quorum quidem prior per II datur, alter vero per se.

V. Statuamus nunc $a = 1$, $b = n-4$ et $c = \varepsilon$ prodibitque haec aequatio

$$(1, n-4)(n-3, \varepsilon) = (1, \varepsilon)(1+\varepsilon, n-4),$$

unde concluditur

$$(n-4, 1+\varepsilon) = \frac{(n-4, 1)(n-3, \varepsilon)}{(1, \varepsilon)};$$

ubi si loco $(n-3, \varepsilon)$ valor ante inventus substitueretur, factor absolutus ingrederetur $\frac{1}{\varepsilon-3}$, ita ut esse debeat $\varepsilon > 3$ ideoque $1+\varepsilon > 4$, unde hic excluduntur casus $(n-4, 1)$, $(n-4, 2)$, $(n-4, 3)$, quorum quidem primus ex II, tertius autem per se datur, medius vero revera manet incognitus.

VI. Statuamus porro $a = 1$, $b = n-5$, $c = \zeta$ et aequatio erit

$$(1, n-5)(n-4, \zeta) = (1, \zeta)(1+\zeta, n-5),$$

unde fit

$$(n-5, 1+\zeta) = \frac{(n-5, 1)(n-4, \zeta)}{(1, \zeta)},$$

ubi ob formulam $(n-4, \zeta)$ debet esse $\zeta > 4$ ideoque $1+\zeta > 5$, unde hinc excluduntur casus $(n-5, 1)$, $(n-5, 2)$, $(n-5, 3)$, $(n-5, 4)$, quorum quidem primus ex II constat, quartus vero per se datur, ita ut hic occurrant duo casus etiamnunc incogniti $(n-5, 2)$ et $(n-5, 3)$.

VII. Simili modo si ulterius sumamus $a = 1$, $b = n-6$ et $c = \eta$, prodibit

$$(n-6, 1+\eta) = \frac{(n-6, 1)(n-5, \eta)}{(1, \eta)},$$

ubi revera occurrunt tres sequentes casus $(n-6, 2)$, $(n-6, 3)$, $(n-6, 4)$, qui adhuc manent incogniti, atque hoc modo progredi licebit, quousque necesse fuerit; unde patet numerum casuum incognitorum continuo augeri, ita ut terminorum p et q alter futurus sit vel 2 vel 3 vel 4 etc., qui igitur casus adhuc definiendi restant.

VIII. Sumamus nunc primo $a = 1$, $b = \theta$, $c = 1$, ut aequatio nostra fiat

$$(1, \theta)(1+\theta, 1) = (1, 1)(2, \theta),$$

unde concludimus

$$(2, \theta) = \frac{(1, \theta)(1+\theta, 1)}{(1, 1)},$$

quae formula iam omnes casus exclusos suppeditat, in quibus alter terminus erat 2.

IX. Deinde sumamus $a = 2$, $b = \varkappa$ et $c = 1$, ut aequatio prodeat

$$(2, \varkappa)(2 + \varkappa, 1) = (2, 1)(3, \varkappa),$$

unde fit

$$(3, \varkappa) = \frac{(2, \varkappa)(2 + \varkappa, 1)}{(2, 1)};$$

ubi cum $(2, \varkappa)$ per praecedentem numerum detur, nunc etiam ii casus innotescunt, ubi alter terminus erat 3.

X. Sumatur porro $a = 3$, $b = \varkappa$, $c = 1$ eritque

$$(3, \varkappa)(3 + \varkappa, 1) = (3, 1)(4, \varkappa),$$

unde fit

$$(4, \varkappa) = \frac{(3, \varkappa)(3 + \varkappa, 1)}{(3, 1)},$$

unde igitur ii casus eliciuntur, ubi alter terminus erat 4.

Eodem modo pro reliquis proceditur sicque omnes plane casus in formula proposita contenti plene sunt determinati.

QUATUOR THEOREMATA MAXIME NOTATU DIGNA IN CALCULO INTEGRALI

Conventui exhibita die 1. Iulii 1776

Commentatio 651 indicis ENESTROEMIANI
Nova acta academiae scientiarum Petropolitanae 7 (1789), 1793, p. 22—41
Summarium ibidem p. 37—38

SUMMARIUM

Le sujet de ce Mémoire est encore, comme celui du Mémoire précédent[1]), une suite des recherches multipliées de l'Auteur sur les courbes algébriques, dont les arcs indéfinis s sont exprimés par une même formule intégrale, savoir

$$s = \int \partial\varphi \sin.\varphi^{n-1}.$$

Car comme les ordonnées d'une courbe quelconque, qui répondent à cet arc indéfini s, sont

$$x = \int \partial s \cos.\omega \quad \text{et} \quad y = \int \partial s \sin.\omega,$$

ω étant l'angle de courbure, tout revient à trouver pour cet angle ω une valeur telle que x et y puissent être exprimés algébriquement. Or M. EULER a trouvé que la valeur

$$\omega = (n + 2i + 1)\varphi$$

satisfait à cette condition, où n désigne un nombre quelconque, entier ou fractionnaire, positif ou négatif, et i un nombre entier positif quelconque; propriété qui est démontrée dans le premier et le second Théorème, où l'Auteur donne pour $\int \partial s \sin.\omega$ et pour $\int \partial s \cos.\omega$ les intégrales algébriques composées la première des sinus, l'autre des cosinus d'angles qui

1) Memoire 650 (suivant l'Index d'ENESTRÖM): *De formulis differentialibus, quae per duas pluresve quantitates datas multiplicatae fiant integrabiles*, Nova acta acad. sc. Petrop. 7 (1789), 1793, p. 3; *LEONHARDI EULERI Opera omnia*, series I, vol. 23. A. L.

forment une progression arithmétique décroissante dont le premier terme est $(n+2i)\varphi$, le dernier $n\varphi$ et la différence 2φ. La démonstration de ces deux premiers Théorèmes est fondée sur une réduction générale tirée de la différentielle des deux formules

$$\sin.\varphi^n \sin.\lambda\varphi \quad \text{et} \quad \sin.\varphi^n \cos.\lambda\varphi.$$

C'est par des expressions semblables et démontrées de la même manière que dans les deux derniers Théorèmes M. EULER présente les intégrales des mêmes formules $\int\partial s \cos.\omega$ et $\int\partial s \sin.\omega$, ∂s étant $=\partial\varphi \cos.\varphi^{n-1}$, de sorte que ces deux Théorèmes combinés peuvent servir à trouver une infinité de courbes algébriques dont les arcs indéfinis sont exprimés par la même formule intégrale $\int\partial\varphi \cos.\varphi^{n-1}$, les deux premiers Théorèmes ayant fourni une infinité de courbes algébriques, dont les arcs indéfinis sont $=\int\partial\varphi \sin.\varphi^{n-1}$.

Finalement tous les quatre Théorèmes combinés frayent le chemin à la solution du Problème de trouver une infinité de courbes algébriques dont les arcs indéfinis sont exprimés plus généralement par la forme

$$\int\partial\varphi \sqrt{(aa \sin.\varphi^{2n-2} + bb \cos.\varphi^{2n-2})},$$

Problème dont la solution termine ce Mémoire.

Nous ne pouvons pas passer sous silence une chose digne d'être relevée, c'est que M. EULER a résolu au § 20, pour ainsi dire en passant, un problème qui l'avoit beaucoup occupé autrefois et dont il avoit désespéré plus d'une fois de trouver la solution.[1])

THEOREMA 1

1. *Denotante* φ *angulum quemcunque variabilem si* n *significet numerum quemcunque sive integrum sive fractum sive positivum sive negativum, tum vero statuatur*

$$\partial s = \partial\varphi \sin.\varphi^{n-1},$$

sequentes formulae integrales omnes algebraice exhiberi possunt:

$$\text{I.} \quad \int\partial s \sin.(n+1)\varphi = \frac{\sin.\varphi^n}{n}\sin.n\varphi,$$

$$\text{II.} \quad \int\partial s \sin.(n+3)\varphi = \frac{\sin.\varphi^n}{n+1}\left(\sin.(n+2)\varphi + \frac{1}{n}\sin.n\varphi\right),$$

1) Voir le mémoire 639 (suivant l'Index d'ENESTRÖM): *De innumeris curvis algebraicis, quarum longitudinem per arcus ellipticos metiri licet*, Nova acta acad. sc. Petrop. 5 (1787), 1789, p. 71; *LEONHARDI EULERI Opera omnia*, series I, vol. 21, p. 163, surtout p. 178. A. L.

$$\text{III.} \quad \int\partial s \sin.(n+5)\varphi$$
$$= \frac{\sin.\varphi^n}{n+2}\left(\sin.(n+4)\varphi + \frac{2}{n+1}\sin.(n+2)\varphi + \frac{2}{n+1}\cdot\frac{1}{n}\sin.n\varphi\right),$$

$$\text{IV.} \quad \int\partial s \sin.(n+7)\varphi$$
$$= \frac{\sin.\varphi^n}{n+3}\left(\sin.(n+6)\varphi + \frac{3}{n+2}\sin.(n+4)\varphi + \frac{3}{n+2}\cdot\frac{2}{n+1}\sin.(n+2)\varphi + \frac{3}{n+2}\cdot\frac{2}{n+1}\cdot\frac{1}{n}\sin.n\varphi\right),$$

$$\text{V.} \quad \int\partial s \sin.(n+9)\varphi$$
$$= \frac{\sin.\varphi^n}{n+4}\left(\sin.(n+8)\varphi + \frac{4}{n+3}\sin.(n+6)\varphi + \frac{4}{n+3}\cdot\frac{3}{n+2}\sin.(n+4)\varphi\right.$$
$$\left. + \frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\sin.(n+2)\varphi + \frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\cdot\frac{1}{n}\sin.n\varphi\right),$$

$$\text{VI.} \quad \int\partial s \sin.(n+11)\varphi$$
$$= \frac{\sin.\varphi^n}{n+5}\left(\sin.(n+10)\varphi + \frac{5}{n+4}\sin.(n+8)\varphi + \frac{5}{n+4}\cdot\frac{4}{n+3}\sin.(n+6)\varphi\right.$$
$$+ \frac{5}{n+4}\cdot\frac{4}{n+3}\cdot\frac{3}{n+2}\sin.(n+4)\varphi + \frac{5}{n+4}\cdot\frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\sin.(n+2)\varphi$$
$$\left. + \frac{5}{n+4}\cdot\frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\cdot\frac{1}{n}\sin.n\varphi\right)$$

etc.

Unde si i *denotet numerum positivum quemcunque, generaliter habebimus*

$$\int\partial s \sin.(n+2i+1)\varphi$$
$$= \frac{\sin.\varphi^n}{n+i}\left(\sin.(n+2i)\varphi + \frac{i}{n+i-1}\sin.(n+2i-2)\varphi\right.$$
$$+ \frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\sin.(n+2i-4)\varphi$$
$$+ \frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cdot\frac{i-2}{n+i-3}\sin.(n+2i-6)\varphi$$
$$+ \frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cdot\frac{i-2}{n+i-3}\cdot\frac{i-3}{n+i-4}\sin.(n+2i-8)\varphi$$
$$\left. + \frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cdot\frac{i-2}{n+i-3}\cdot\frac{i-3}{n+i-4}\cdot\frac{i-4}{n+i-5}\sin.(n+2i-10)\varphi + \text{etc.}\right),$$

quae terminorum progressio quovis casu sponte abrumpitur.

DEMONSTRATIO

2. Ad veritatem huius theorematis demonstrandam consideretur ista formula

$$Z = \sin.\varphi^n \sin.\lambda\varphi,$$

quae differentiata dat

$$\partial Z = \partial\varphi \sin.\varphi^{n-1}(n\cos.\varphi\sin.\lambda\varphi + \lambda\sin.\varphi\cos.\lambda\varphi).$$

At per reductiones cognitas est

$$\cos.\varphi\sin.\lambda\varphi = +\frac{1}{2}\sin.(\lambda-1)\varphi + \frac{1}{2}\sin.(\lambda+1)\varphi$$

et

$$\sin.\varphi\cos.\lambda\varphi = -\frac{1}{2}\sin.(\lambda-1)\varphi + \frac{1}{2}\sin.(\lambda+1)\varphi,$$

quibus valoribus substitutis, quoniam posuimus $\partial\varphi\sin.\varphi^{n-1} = \partial s$, erit

$$2\partial Z = \partial s((n-\lambda)\sin.(\lambda-1)\varphi + (n+\lambda)\sin.(\lambda+1)\varphi),$$

unde denuo per partes integrando deducimus

$$\int\partial s\sin.(\lambda+1)\varphi = \frac{2Z}{\lambda+n} + \frac{\lambda-n}{\lambda+n}\int\partial s\sin.(\lambda-1)\varphi$$

sive

$$\int\partial s\sin.(\lambda+1)\varphi = \frac{2\sin.\varphi^n\sin.\lambda\varphi}{\lambda+n} + \frac{\lambda-n}{\lambda+n}\int\partial s\sin.(\lambda-1)\varphi.$$

3. Stabilita igitur hac postrema reductione generali capiamus $\lambda = n$, ut adipiscamur istam integrationem absolutam

$$\int\partial s\sin.(n+1)\varphi = \frac{\sin.\varphi^n}{n}\sin.n\varphi.$$

Nunc vero statuamus $\lambda = n+2$ et forma illa generalis dabit

$$\int\partial s\sin.(n+3)\varphi = \frac{\sin.\varphi^n}{n+1}\sin.(n+2)\varphi + \frac{1}{n+1}\int\partial s\sin.(n+1)\varphi$$

sicque haec integratio ad praecedentem est reducta. Iam ponamus $\lambda = n+4$ et forma generalis suppeditabit

$$\int\partial s\sin.(n+5)\varphi = \frac{\sin.\varphi^n}{n+2}\sin.(n+4)\varphi + \frac{2}{n+2}\int\partial s\sin.(n+3)\varphi,$$

quae ergo integratio iterum ad praecedentem est reducta. Sit porro $\lambda = n+6$ et ex forma generali prodibit

$$\int\partial s\sin.(n+7)\varphi = \frac{\sin.\varphi^n}{n+3}\sin.(n+6)\varphi + \frac{3}{n+3}\int\partial s\sin.(n+5)\varphi$$

sicque augendis continuo valoribus ipsius λ binario ulterius progredi licebit.

4. Quodsi iam singulos valores integrales antecedentes in sequentibus substituamus, sequentes orientur integrationes absolutae:

$$\text{I.}\quad \int\partial s\sin.(n+1)\varphi = \frac{\sin.\varphi^n}{n}\sin.n\varphi,$$

$$\text{II.}\quad \int\partial s\sin.(n+3)\varphi = \frac{\sin.\varphi^n}{n+1}\left(\sin.(n+2)\varphi + \frac{1}{n}\sin.n\varphi\right),$$

$$\text{III.}\quad \int\partial s\sin.(n+5)\varphi$$

$$= \frac{\sin.\varphi^n}{n+2}\left(\sin.(n+4)\varphi + \frac{2}{n+1}\sin.(n+2)\varphi + \frac{2}{n+1}\cdot\frac{1}{n}\sin.n\varphi\right).$$

$$\text{IV.}\quad \int\partial s\sin.(n+7)\varphi$$

$$= \frac{\sin.\varphi^n}{n+3}\left(\sin.(n+6)\varphi + \frac{3}{n+2}\sin.(n+4)\varphi + \frac{3}{n+2}\cdot\frac{2}{n+1}\sin.(n+2)\varphi + \frac{3}{n+2}\cdot\frac{2}{n+1}\cdot\frac{1}{n}\sin.n\varphi\right);$$

quae cum sint eae ipsae formulae, quas in theoremate annunciavimus, eius veritas sufficienter est evicta.

THEOREMA 2

5. *Denotante φ angulum quemcunque variabilem si n denotet numerum quemcunque ac brevitatis gratia ponatur ut ante*

$$\partial s = \partial\varphi \sin.\varphi^{n-1},$$

etiam omnes sequentes integrationes per algebraicos valores exhiberi possunt:

$$\text{I.}\quad \int \partial s \cos.(n+1)\varphi = \frac{\sin.\varphi^n}{n}\cos.n\varphi,$$

$$\text{II.}\quad \int \partial s \cos.(n+3)\varphi = \frac{\sin.\varphi^n}{n+1}\left(\cos.(n+2)\varphi + \frac{1}{n}\cos.n\varphi\right),$$

$$\text{III.}\quad \int \partial s \cos.(n+5)\varphi$$

$$= \frac{\sin.\varphi^n}{n+2}\left(\cos.(n+4)\varphi + \frac{2}{n+1}\cos.(n+2)\varphi + \frac{2}{n+1}\cdot\frac{1}{n}\cos.n\varphi\right),$$

$$\text{IV.}\quad \int \partial s \cos.(n+7)\varphi$$

$$= \frac{\sin.\varphi^n}{n+3}\left(\cos.(n+6)\varphi + \frac{3}{n+2}\cos.(n+4)\varphi + \frac{3}{n+2}\cdot\frac{2}{n+1}\cos.(n+2)\varphi + \frac{3}{n+2}\cdot\frac{2}{n+1}\cdot\frac{1}{n}\cos.n\varphi\right),$$

$$\text{V.}\quad \int \partial s \cos.(n+9)\varphi$$

$$= \frac{\sin.\varphi^n}{n+4}\left(\cos.(n+8)\varphi + \frac{4}{n+3}\cos.(n+6)\varphi + \frac{4}{n+3}\cdot\frac{3}{n+2}\cos.(n+4)\varphi\right.$$

$$\left. + \frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\cos.(n+2)\varphi + \frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\cdot\frac{1}{n}\cos.n\varphi\right),$$

$$\text{VI.}\quad \int \partial s \cos.(n+11)\varphi$$

$$= \frac{\sin.\varphi^n}{n+5}\left(\cos.(n+10)\varphi + \frac{5}{n+4}\cos.(n+8)\varphi + \frac{5}{n+4}\cdot\frac{4}{n+3}\cos.(n+6)\varphi\right.$$

$$+ \frac{5}{n+4}\cdot\frac{4}{n+3}\cdot\frac{3}{n+2}\cos.(n+4)\varphi + \frac{5}{n+4}\cdot\frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\cos.(n+2)\varphi$$

$$\left. + \frac{5}{n+4}\cdot\frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\cdot\frac{1}{n}\cos.n\varphi\right)$$

etc.

Unde patet, si i denotet numerum positivum quemcunque, fore in genere

$$\int \partial s \cos.(n+2i+1)\varphi$$

$$= \frac{\sin.\varphi^n}{n+i}\left(\cos.(n+2i)\varphi + \frac{i}{n+i-1}\cos.(n+2i-2)\varphi\right.$$

$$+ \frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cos.(n+2i-4)\varphi$$

$$\left. + \frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cdot\frac{i-2}{n+i-3}\cos.(n+2i-6)\varphi + \text{etc.}\right),$$

quos terminos quovis casu eousque continuari oportet, donec sponte evanescant.

DEMONSTRATIO

6. Ad veritatem horum integralium demonstrandam consideretur ista formula

$$Z = \sin.\varphi^n \cos.\lambda\varphi,$$

cuius differentiatio praebet

$$\partial Z = \partial\varphi \sin.\varphi^{n-1}(n\cos.\varphi\cos.\lambda\varphi - \lambda\sin.\varphi\sin.\lambda\varphi),$$

quae expressio ob

$$\cos.\varphi\cos.\lambda\varphi = \frac{1}{2}\cos.(\lambda-1)\varphi + \frac{1}{2}\cos.(\lambda+1)\varphi$$

et

$$\sin.\varphi\sin.\lambda\varphi = \frac{1}{2}\cos.(\lambda-1)\varphi - \frac{1}{2}\cos.(\lambda+1)\varphi,$$

si loco $\partial\varphi\sin.\varphi^{n-1}$ valorem assumtum ∂s scribamus, fiet

$$2\partial Z = \partial s((n-\lambda)\cos.(\lambda-1)\varphi + (n+\lambda)\cos.(\lambda+1)\varphi),$$

unde iterum per partes integrando erit

$$2\sin.\varphi^n\cos.\lambda\varphi = (n-\lambda)\int\partial s\cos.(\lambda-1)\varphi + (n+\lambda)\int\partial s\cos.(\lambda+1)\varphi,$$

atque hinc deducimus sequentem reductionem generalem

$$\int\partial s\cos.(\lambda+1)\varphi = \frac{2\sin.\varphi^n\cos.\lambda\varphi}{\lambda+n} + \frac{\lambda-n}{\lambda+n}\int\partial s\cos.(\lambda-1)\varphi.$$

7. Ponamus igitur primo $\lambda = n$, ut obtineamus hanc integrationem absolutam

$$\int \partial s \cos.(n+1)\varphi = \frac{\sin.\varphi^n}{n} \cos. n\varphi.$$

Fiat iam $\lambda = n + 2$ et forma generalis dabit

$$\int \partial s \cos.(n+3)\varphi = \frac{\sin.\varphi^n}{n+1} \cos.(n+2)\varphi + \frac{1}{n+1}\int \partial s \cos.(n+1)\varphi.$$

Statuatur porro $\lambda = n + 4$ et consequemur

$$\int \partial s \cos.(n+5)\varphi = \frac{\sin.\varphi^n}{n+2} \cos.(n+4)\varphi + \frac{2}{n+2}\int \partial s \cos.(n+3)\varphi.$$

Ponamus ulterius $\lambda = n + 6$ ac reperiemus

$$\int \partial s \cos.(n+7)\varphi = \frac{\sin.\varphi^n}{n+3} \cos.(n+6)\varphi + \frac{3}{n+3}\int \partial s \cos.(n+5)\varphi.$$

Faciamus simili modo ulterius $\lambda = n + 8$ ac nanciscemur

$$\int \partial s \cos.(n+9)\varphi = \frac{\sin.\varphi^n}{n+4} \cos.(n+8)\varphi + \frac{4}{n+4}\int \partial s \cos.(n+7)\varphi$$

etc.

8. Quodsi iam singulos valores integrales praecedentes in sequentes introducamus, perveniemus ad istas integrationes absolutas:

$$\text{I.}\quad \int \partial s \cos.(n+1)\varphi = \frac{\sin.\varphi^n}{n} \cos. n\varphi,$$

$$\text{II.}\quad \int \partial s \cos.(n+3)\varphi = \frac{\sin.\varphi^n}{n+1}\left(\cos.(n+2)\varphi + \frac{1}{n}\cos. n\varphi\right),$$

$$\text{III.}\quad \int \partial s \cos.(n+5)\varphi$$

$$= \frac{\sin.\varphi^n}{n+2}\left(\cos.(n+4)\varphi + \frac{2}{n+1}\cos.(n+2)\varphi + \frac{2}{n+1}\cdot\frac{1}{n}\cos. n\varphi\right),$$

$$\text{IV.}\quad \int \partial s \cos.(n+7)\varphi$$

$$= \frac{\sin.\varphi^n}{n+3}\left(\cos.(n+6)\varphi + \frac{3}{n+2}\cos.(n+4)\varphi + \frac{3}{n+2}\cdot\frac{2}{n+1}\cos.(n+2)\varphi + \frac{3}{n+2}\cdot\frac{2}{n+1}\cdot\frac{1}{n}\cos. n\varphi\right)$$

etc.,

quae manifesto sunt eae ipsae formulae, quas in theoremate produximus, quarum ergo veritas nunc solide est demonstrata.

COROLLARIUM

9. Haec duo theoremata combinata inservire possunt ad innumerabiles curvas algebraicas inveniendas, quarum arcus indefiniti s omnes per eandem formulam integralem $\int \partial\varphi \sin.\varphi^{n-1}$ exprimantur. Cum enim elementum curvae sit

$$\partial s = \partial\varphi \sin.\varphi^{n-1},$$

omnes plane curvae huic conditioni satisfacientes ita generaliter exhiberi possunt, ut earum coordinatae sint

$$x = \int \partial s \cos.\omega \quad \text{et} \quad y = \int \partial s \sin.\omega.$$

Nunc autem videmus ambas istas expressiones revera fore algebraicas, si angulus ω ita accipiatur, ut sit

$$\omega = (n + 2i + 1)\varphi,$$

ubi loco i numerum quemcunque integrum positivum accipere licet. Quamobrem numerum talium curvarum algebraicarum in infinitum augere licebit; curva autem simplicissima sine dubio prodibit ponendo $i = 0$. Hoc argumentum iam nuper fusius pertractavimus.[1])

THEOREMA 3

10. *Denotante φ angulum quemcunque variabilem si n significet numerum quemcunque sive integrum sive fractum sive positivum sive negativum, tum vero statuatur*

$$\partial s = \partial\varphi \cos.\varphi^{n-1},$$

sequentes formulae integrales omnes algebraice exhiberi possunt:

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1) Confer Commentationem 645 (indicis ENESTROEMIANI): *De curvis algebraicis, quarum longitudo exprimitur hac formula integrali* $\int \frac{v^{m-1}\partial v}{\sqrt{(1-v^{2n})}}$, Nova acta acad. sc. Petrop. 6 (1788), 1790, p. 36; *LEONHARDI EULERI Opera omnia*, series I, vol. 21, p. 180. A. L.

$$\text{I.}\quad \int \partial s \cos.(n+1)\varphi = \frac{\cos.\varphi^n}{n}\sin.n\varphi,$$

$$\text{II.}\quad \int \partial s \cos.(n+3)\varphi = \frac{\cos.\varphi^n}{n+1}\Big(\sin.(n+2)\varphi - \frac{1}{n}\sin.n\varphi\Big),$$

$$\text{III.}\quad \int \partial s \cos.(n+5)\varphi$$
$$= \frac{\cos.\varphi^n}{n+2}\Big(\sin\,(n+4)\varphi - \frac{2}{n+1}\sin.(n+2)\varphi + \frac{2}{n+1}\cdot\frac{1}{n}\sin.n\varphi\Big),$$

$$\text{IV.}\quad \int \partial s \cos.(n+7)\varphi$$
$$= \frac{\cos.\varphi^n}{n+3}\Big(\sin.(n+6)\varphi - \frac{3}{n+2}\sin.(n+4)\varphi + \frac{3}{n+2}\cdot\frac{2}{n+1}\sin.(n+2)\varphi - \frac{3}{n+2}\cdot\frac{2}{n+1}\cdot\frac{1}{n}\sin.n\varphi\Big),$$

$$\text{V.}\quad \int \partial s \cos.(n+9)\varphi$$
$$= \frac{\cos.\varphi^n}{n+4}\Big(\sin.(n+8)\varphi - \frac{4}{n+3}\sin.(n+6)\varphi + \frac{4}{n+3}\cdot\frac{3}{n+2}\sin.(n+4)\varphi$$
$$- \frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\sin.(n+2)\varphi + \frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\cdot\frac{1}{n}\sin.n\varphi\Big),$$

$$\text{VI.}\quad \int \partial s \cos.(n+11)\varphi$$
$$= \frac{\cos.\varphi^n}{n+5}\Big(\sin.(n+10)\varphi - \frac{5}{n+4}\sin.(n+8)\varphi + \frac{5}{n+4}\cdot\frac{4}{n+3}\sin.(n+6)\varphi$$
$$- \frac{5}{n+4}\cdot\frac{4}{n+3}\cdot\frac{3}{n+2}\sin.(n+4)\varphi + \frac{5}{n+4}\cdot\frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\sin.(n+2)\varphi$$
$$- \frac{5}{n+4}\cdot\frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\cdot\frac{1}{n}\sin.n\varphi\Big).$$

Ex quibus concluditur fore generaliter denotante i numerum integrum positivum quemcunque

$$\int \partial s \cos.(n+2i+1)\varphi$$
$$= \frac{\cos.\varphi^n}{n+i}\Big(\sin.(n+2i)\varphi - \frac{i}{n+i-1}\sin.(n+2i-2)\varphi$$
$$+ \frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\sin.(n+2i-4)\varphi$$
$$- \frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cdot\frac{i-2}{n+i-3}\sin.(n+2i-6)\varphi + \text{etc.}\Big).$$

DEMONSTRATIO

11. Ad veritatem huius theorematis demonstrandam consideretur ista formula

$$Z = \cos.\varphi^n \sin.\lambda\varphi,$$

quae differentiata dat

$$\partial Z = \delta\varphi \cos.\varphi^{n-1}(-n\sin.\varphi\sin.\lambda\varphi + \lambda\cos.\varphi\cos.\lambda\varphi),$$

quae per reductiones ante adhibitas transformatur in hanc formam

$$2\partial Z = \partial s((\lambda-n)\cos.(\lambda-1)\varphi + (\lambda+n)\cos.(\lambda+1)\varphi),$$

unde iterum per partes integrando nanciscimur

$$2Z = (\lambda-n)\int \partial s \cos.(\lambda-1)\varphi + (\lambda+n)\int \partial s \cos.(\lambda+1)\varphi,$$

hincque deducimus istam integrationem generalem

$$\int \partial s \cos.(\lambda+1)\varphi = \frac{2\cos.\varphi^n \sin.\lambda\varphi}{\lambda+n} - \frac{\lambda-n}{\lambda+n}\int \partial s \cos.(\lambda-1)\varphi.$$

12. Sumamus nunc primo $\lambda = n$, ut posterius integrale tollatur, ac prodibit

$$\int \partial s \cos.(n+1)\varphi = \frac{\cos.\varphi^n}{n}\sin.n\varphi.$$

Nunc autem porro ponamus $\lambda = n+2$ et forma nostra generalis nobis praebebit

$$\int \partial s \cos.(n+3)\varphi = \frac{\cos.\varphi^n}{n+1}\sin.(n+2)\varphi - \frac{1}{n+1}\int \partial s \cos.(n+1)\varphi,$$

ubi ergo posterius integrale iam est inventum. Fiat ulterius $\lambda = n+4$ et habebimus

$$\int \partial s \cos.(n+5)\varphi = \frac{\cos.\varphi^n}{n+2}\sin.(n+4)\varphi - \frac{2}{n+2}\int \partial s \cos.(n+3)\varphi,$$

quod postremum integrale itidem iam patet. Sumamus nunc $\lambda = n+6$ et forma generalis dabit

$$\int \partial s \cos.(n+7)\varphi = \frac{\cos.\varphi^n}{n+3}\sin.(n+6)\varphi - \frac{3}{n+3}\int \partial s \cos.(n+5)\varphi.$$

Simili modo si faciamus $\lambda = n + 8$, obtinebimus

$$\int \partial s \cos.(n+9)\varphi = \frac{\cos.\varphi^n}{n+4}\sin.(n+8)\varphi - \frac{4}{n+4}\int \partial s \cos.(n+7)\varphi.$$

Hocque modo ulterius progrediendo perpetuo sequentia integralia per praecedentia exprimere licebit.

13. Quodsi ergo valores integrales praecedentes in sequentibus substituamus, consequemur istas integrationes absolutas:

$$\text{I.}\quad \int \partial s \cos.(n+1)\varphi = \frac{\cos.\varphi^n}{n}\sin.n\varphi,$$

$$\text{II.}\quad \int \partial s \cos.(n+3)\varphi = \frac{\cos.\varphi^n}{n+1}\left(\sin.(n+2)\varphi - \frac{1}{n}\sin.n\varphi\right),$$

$$\text{III.}\quad \int \partial s \cos.(n+5)\varphi$$

$$= \frac{\cos.\varphi^n}{n+2}\left(\sin.(n+4)\varphi - \frac{2}{n+1}\sin.(n+2)\varphi + \frac{2}{n+1}\cdot\frac{1}{n}\sin.n\varphi\right),$$

$$\text{IV.}\quad \int \partial s \cos.(n+7)\varphi$$

$$= \frac{\cos.\varphi^n}{n+3}\left(\sin.(n+6)\varphi - \frac{3}{n+2}\sin.(n+4)\varphi + \frac{3}{n+2}\cdot\frac{2}{n+1}\sin.(n+2)\varphi - \frac{3}{n+2}\cdot\frac{2}{n+1}\cdot\frac{1}{n}\sin.n\varphi\right),$$

$$\text{V.}\quad \int \partial s \cos.(n+9)\varphi$$

$$= \frac{\cos.\varphi^n}{n+4}\left(\sin.(n+8)\varphi - \frac{4}{n+3}\sin.(n+6)\varphi + \frac{4}{n+3}\cdot\frac{3}{n+2}\sin.(n+4)\varphi\right.$$

$$\left. - \frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\sin.(n+2)\varphi + \frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\cdot\frac{1}{n}\sin.n\varphi\right)$$

etc.,

unde veritas nostri theorematis abunde elucet.

THEOREMA 4

14. *Denotante φ angulum quemcunque variabilem si n significet numerum quemcunque sive integrum sive fractum sive positivum sive negativum, tum vero statuatur*

$$\partial s = \partial\varphi \cos.\varphi^{n-1},$$

sequentes formulae integrales omnes algebraice exprimi poterunt:

$$\text{I.}\quad \int \partial s \sin.(n+1)\varphi = -\frac{\cos.\varphi^n}{n}\cos.n\varphi,$$

$$\text{II.}\quad \int \partial s \sin.(n+3)\varphi = -\frac{\cos.\varphi^n}{n+1}\left(\cos.(n+2)\varphi - \frac{1}{n}\cos.n\varphi\right),$$

$$\text{III.}\quad \int \partial s \sin.(n+5)\varphi$$

$$= -\frac{\cos.\varphi^n}{n+2}\left(\cos.(n+4)\varphi - \frac{2}{n+1}\cos.(n+2)\varphi + \frac{2}{n+1}\cdot\frac{1}{n}\cos.n\varphi\right),$$

$$\text{IV.}\quad \int \partial s \sin.(n+7)\varphi$$

$$= -\frac{\cos.\varphi^n}{n+3}\left(\cos.(n+6)\varphi - \frac{3}{n+2}\cos.(n+4)\varphi + \frac{3}{n+2}\cdot\frac{2}{n+1}\cos.(n+2)\varphi - \frac{3}{n+2}\cdot\frac{2}{n+1}\cdot\frac{1}{n}\cos.n\varphi\right),$$

$$\text{V.}\quad \int \partial s \sin.(n+9)\varphi$$

$$= -\frac{\cos.\varphi^n}{n+4}\left(\cos.(n+8)\varphi - \frac{4}{n+3}\cos.(n+6)\varphi + \frac{4}{n+3}\cdot\frac{3}{n+2}\cos.(n+4)\varphi\right.$$

$$\left. - \frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\cos.(n+2)\varphi + \frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\cdot\frac{1}{n}\cos.n\varphi\right),$$

$$\text{VI.}\quad \int \partial s \sin.(n+11)\varphi$$

$$= -\frac{\cos.\varphi^n}{n+5}\left(\cos.(n+10)\varphi - \frac{5}{n+4}\cos.(n+8)\varphi + \frac{5}{n+4}\cdot\frac{4}{n+3}\cos.(n+6)\varphi\right.$$

$$- \frac{5}{n+4}\cdot\frac{4}{n+3}\cdot\frac{3}{n+2}\cos.(n+4)\varphi + \frac{5}{n+4}\cdot\frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\cos.(n+2)\varphi$$

$$\left. - \frac{5}{n+4}\cdot\frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\cdot\frac{1}{n}\cos.n\varphi\right).$$

Unde manifesto patet, si i denotet numerum quemcunque integrum positivum, fore in genere

$$\int \partial s \sin.(n+2i+1)\varphi$$

$$= -\frac{\cos.\varphi^n}{n+i}\Big(\cos.(n+2i)\varphi - \frac{i}{n+i-1}\cos.(n+2i-2)\varphi$$

$$+\frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cos.(n+2i-4)\varphi$$

$$-\frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cdot\frac{i-2}{n+i-3}\cos.(n+2i-6)\varphi$$

$$+\frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cdot\frac{i-2}{n+i-3}\cdot\frac{i-3}{n+i-4}\cos.(n+2i-8)\varphi - \text{etc.}\Big).$$

DEMONSTRATIO

15. Ad hoc theorema demonstrandum consideretur formula

$$Z = \cos.\varphi^n \cos.\lambda\varphi,$$

quae differentiata praebet

$$\partial Z = -\partial\varphi \cos.\varphi^{n-1}(n \sin.\varphi \cos.\lambda\varphi + \lambda \cos.\varphi \sin.\lambda\varphi),$$

quae per notas reductiones reducitur ad hanc formam

$$2\partial Z = -\partial s((\lambda+n)\sin.(\lambda+1)\varphi + (\lambda-n)\sin.(\lambda-1)\varphi),$$

quae iterum per partes integrata dat

$$2Z = -(\lambda+n)\int \partial s \sin.(\lambda+1)\varphi - (\lambda-n)\int \partial s \sin.(\lambda-1)\varphi.$$

unde deducitur ista integratio generalis

$$\int \partial s \sin.(\lambda+1)\varphi = -\frac{2\cos.\varphi^n \cos.\lambda\varphi}{\lambda+n} - \frac{\lambda-n}{\lambda+n}\int \partial s \sin.(\lambda-1)\varphi.$$

16. Ut membrum integrale postremum e medio tollatur, capiamus $\lambda = n$ et forma generalis dabit

$$\int \partial s \sin.(n+1)\varphi = -\frac{\cos.\varphi^n}{n}\cos.n\varphi.$$

Statuamus nunc porro $\lambda = n+2$ ac proveniet

$$\int \partial s \sin.(n+3)\varphi = -\frac{\cos.\varphi^n}{n+1}\cos.(n+2)\varphi - \frac{1}{n+1}\int \partial s \sin.(n+1)\varphi.$$

Fiat porro $\lambda = n+4$, ut oriatur

$$\int \partial s \sin.(n+5)\varphi = -\frac{\cos.\varphi^n}{n+2}\cos.(n+4)\varphi - \frac{2}{n+2}\int \partial s \sin.(n+3)\varphi.$$

Sit iam $\lambda = n+6$; fiet

$$\int \partial s \sin.(n+7)\varphi = -\frac{\cos.\varphi^n}{n+3}\cos.(n+6)\varphi - \frac{3}{n+3}\int \partial s \sin.(n+5)\varphi.$$

Simili modo sit $\lambda = n+8$ ac resultabit

$$\int \partial s \sin.(n+9)\varphi = -\frac{\cos.\varphi^n}{n+4}\cos.(n+8)\varphi - \frac{4}{n+4}\int \partial s \sin.(n+7)\varphi$$

etc.,

ubi pariter sequentia integralia per praecedentia definiuntur.

17. Quamobrem si ubique valores integrales praecedentes substituantur, orientur sequentes integrationes absolutae:

$$\text{I.}\quad \int \partial s \sin.(n+1)\varphi = -\frac{\cos.\varphi^n}{n}\cos.n\varphi,$$

$$\text{II.}\quad \int \partial s \sin.(n+3)\varphi = -\frac{\cos.\varphi^n}{n+1}\Big(\cos.(n+2)\varphi - \frac{1}{n}\cos.n\varphi\Big),$$

$$\text{III.}\quad \int \partial s \sin.(n+5)\varphi$$

$$= -\frac{\cos.\varphi^n}{n+2}\Big(\cos.(n+4)\varphi - \frac{2}{n+1}\cos.(n+2)\varphi + \frac{2}{n+1}\cdot\frac{1}{n}\cos.n\varphi\Big),$$

$$\text{IV.}\quad \int \partial s \sin.(n+7)\varphi$$

$$= -\frac{\cos.\varphi^n}{n+3}\Big(\cos.(n+6)\varphi - \frac{3}{n+2}\cos.(n+4)\varphi + \frac{3}{n+2}\cdot\frac{2}{n+1}\cos.(n+2)\varphi - \frac{3}{n+2}\cdot\frac{2}{n+1}\cdot\frac{1}{n}\cos.n\varphi\Big).$$

$$\text{V.}\quad \int \partial s \sin.(n+9)\varphi$$

$$= -\frac{\cos.\varphi^n}{n+4}\Big(\cos.(n+8)\varphi - \frac{4}{n+3}\cos.(n+6)\varphi + \frac{4}{n+3}\cdot\frac{3}{n+2}\cos.(n+4)\varphi$$

$$- \frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\cos.(n+2)\varphi + \frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\cdot\frac{1}{n}\cos.n\varphi\Big)$$

etc.

sicque veritas theorematis propositi sufficienter est evicta.

COROLLARIUM 1

18. Si $\partial s = \partial\varphi \cos.\varphi^{n-1}$ denotet elementum cuiuspiam lineae curvae, cuius coordinatae orthogonales sint x et y, ita ut sit

$$\partial s^2 = \partial x^2 + \partial y^2,$$

huic conditioni generatim satisfiet sumendo

$$\partial x = \partial s \cos.\omega \quad \text{et} \quad \partial y = \partial s \sin.\omega.$$

Nunc igitur ex binis posterioribus theorematibus patet innumerabiles huiusmodi curvas algebraicas exhiberi posse, si scilicet capiatur

$$\omega = (n + 2i + 1)\varphi,$$

quandoquidem hinc valores ipsarum x et y algebraice exprimi possunt; ac simplicissima quidem curva prodibit ponendo $i = 0$; tum enim fiet

$$x = \int \partial s \cos.(n+1)\varphi = +\frac{\cos.\varphi^n}{n}\sin.n\varphi$$

et

$$y = \int \partial s \sin.(n+1)\varphi = -\frac{\cos.\varphi^n}{n}\cos.n\varphi.$$

COROLLARIUM 2

19. Quodsi sumatur $n = 1$, ut fieri debeat $\partial s = \partial\varphi$ ideoque $s = \varphi$, hoc est arcui circulari aequalis, tum facile ostendi potest, quicunque valor numero i tribuatur, curvas resultantes omnes fore circulos, ita ut hoc casu praeter circulum nulla alia curva algebraica satisfaciat, id quod pro casu $i = 3$ ostendisse sufficiat. Tum enim erit

$$x = \int \partial s \cos.8\varphi = \frac{1}{4}\cos.\varphi(\sin.7\varphi - \sin.5\varphi + \sin.3\varphi - \sin.\varphi),$$

quae forma per reductiones abit in hanc

$$x = \frac{1}{8}\sin.8\varphi.$$

Tum vero habebitur simili modo

$$y = \int \partial s \sin.8\varphi = -\frac{1}{4}\cos.\varphi(\cos.7\varphi - \cos.5\varphi + \cos.3\varphi - \cos.\varphi),$$

quae per similes reductiones praebet

$$y = \frac{1}{8}(1 - \cos.8\varphi) \quad \text{ideoque} \quad \frac{1}{8} - y = \frac{1}{8}\cos.8\varphi.$$

Ex his iam valoribus coniunctis manifestum est fore

$$xx + \left(\frac{1}{8} - y\right)^2 = \frac{1}{64},$$

quae utique est aequatio pro circulo. Eodem modo ostendi potest, quicunque valor numero i tribuatur, semper quoque circulum esse proditurum.

COROLLARIUM 3

20. Casus quoque, quo $n = -\frac{1}{2}$, omni attentione est dignus, pro quo curva simplicissima erit

$$x = \int \partial s \cos.\tfrac{1}{2}\varphi = \frac{2\sin.\frac{1}{2}\varphi}{\sqrt{\cos.\varphi}} \quad \text{et} \quad y = \int \partial s \sin.\tfrac{1}{2}\varphi = \frac{2\cos.\frac{1}{2}\varphi}{\sqrt{\cos.\varphi}},$$

ita ut elementum huius curvae futurum sit

$$\partial s = \frac{\partial\varphi}{\cos.\varphi\sqrt{\cos.\varphi}}.$$

Iam ad angulum φ eliminandum quoniam est

$$\left(\cos.\tfrac{1}{2}\varphi\right)^2 - \left(\sin.\tfrac{1}{2}\varphi\right)^2 = \cos.\varphi,$$

habebimus

$$yy - xx = 4 \quad \text{sive} \quad yy = 4 + xx,$$

quae est aequatio pro hyperbola aequilatera sive rectangula.[1])

SCHOLION 1

21. Quamquam autem in his quatuor theorematibus infinitae formulae integrabiles sunt exhibitae, tamen occurrere possunt certi casus, quibus integralia assignata evadunt incongrua atque adeo naturam quantitatum algebraicarum penitus amittunt. Tales casus oriuntur, quoties exponens n vel evanescit vel numero integro negativo fit aequalis. Hoc enim casu fieri potest, ut quispiam factor in denominatoribus in nihilum abeat, ideoque ipsi termini in infinitum excrescere videntur. Etiamsi enim hoc incommodum adiectione constantium pariter infinitarum evitari posset, tamen ipsi termini inde resultantes non amplius forent algebraici. Ita si esset $n = 0$, omnia prorsus integralia ibi exhibita penitus tollerentur. Si autem esset $n = -1$, tum tantum primae formulae relinquerentur, sequentes omnes autem evaderent inutiles. Si esset $n = -2$, tum binae priores formae tantum subsistere possent, solae autem ternae, si esset $n = -3$, etc. His autem casibus exceptis quicunque valores exponenti n tribuantur, singula theoremata innumerabiles suppeditant formulas integrabiles.

SCHOLION 2

22. Quemadmodum binis prioribus theorematibus iam sum usus ad innumerabiles curvas algebraicas inveniendas, quarum longitudo s hoc valore exprimatur

$$s = \int \partial\varphi \sin.\varphi^{n-1},$$

ita etiam bina posteriora theoremata innumerabilibus curvis algebraicis inveniendis inservire possunt, quarum longitudo sit

$$s = \int \partial\varphi \cos.\varphi^{n-1}.$$

1) Vide notam p. 436. A. L.

Etiamsi enim hi duo casus prorsus inter se conveniant, si quidem loco φ scribendo $90^\circ - \varphi$ altera formula in alteram transformatur, unde quis suspicari posset duo posteriora theoremata tuto omitti potuisse, tamen hos casus non tam plane ex prioribus deducere licet, quippe qui veritates per se notatu dignissimas involvere sunt censendi. Quin etiam omnia haec quatuor theoremata iunctim sumpta viam sternunt ad infinitas curvas algebraicas investigandas, quarum longitudo s formula multo magis complicata exprimatur; ad quod ostendendum ante oculos exponamus integrationes generales, ad quas singula theoremata nos duxerunt.

$$\text{I.} \quad \int \partial\varphi \sin.\varphi^{n-1} \sin.(n+2i+1)\varphi$$
$$= \frac{\sin.\varphi^n}{n+i}\Big(\sin.(n+2i)\varphi + \frac{i}{n+i-1}\sin.(n+2i-2)\varphi$$
$$+ \frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\sin.(n+2i-4)\varphi$$
$$+ \frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cdot\frac{i-2}{n+i-3}\sin.(n+2i-6)\varphi$$
$$+ \frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cdot\frac{i-2}{n+i-3}\cdot\frac{i-3}{n+i-4}\sin.(n+2i-8)\varphi + \text{etc.}\Big).$$

$$\text{II.} \quad \int \partial\varphi \sin.\varphi^{n-1} \cos.(n+2i+1)\varphi$$
$$= \frac{\sin.\varphi^n}{n+i}\Big(\cos.(n+2i)\varphi + \frac{i}{n+i-1}\cos.(n+2i-2)\varphi$$
$$+ \frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cos.(n+2i-4)\varphi$$
$$+ \frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cdot\frac{i-2}{n+i-3}\cos.(n+2i-6)\varphi$$
$$+ \frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cdot\frac{i-2}{n+i-3}\cdot\frac{i-3}{n+i-4}\cos.(n+2i-8)\varphi + \text{etc.}\Big).$$

$$\text{III.} \quad \int \partial\varphi \cos.\varphi^{n-1} \cos.(n+2i+1)\varphi$$
$$= \frac{\cos.\varphi^n}{n+i}\Big(\sin.(n+2i)\varphi - \frac{i}{n+i-1}\sin.(n+2i-2)\varphi$$
$$+ \frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\sin.(n+2i-4)\varphi$$
$$- \frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cdot\frac{i-2}{n+i-3}\sin.(n+2i-6)\varphi$$
$$+ \frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cdot\frac{i-2}{n+i-3}\cdot\frac{i-3}{n+i-4}\sin.(n+2i-8)\varphi - \text{etc.}\Big).$$

$$\text{IV.}\ \int \partial\varphi \cos.\varphi^{n-1} \sin.(n+2i+1)\varphi$$

$$= -\frac{\cos.\varphi^n}{n+i}\Big(\cos.(n+2i)\varphi - \frac{i}{n+i-1}\cos.(n+2i-2)\varphi$$

$$+\frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cos.(n+2i-4)\varphi$$

$$-\frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cdot\frac{i-2}{n+i-3}\cos.(n+2i-6)\varphi$$

$$+\frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cdot\frac{i-2}{n+i-3}\cdot\frac{i-3}{n+i-4}\cos.(n+2i-8)\varphi - \text{etc.}\Big).$$

PROBLEMA SINGULARE

23. *Invenire innumerabiles curvas algebraicas, quarum arcus indefiniti s ista formula integrali exprimantur*

$$s = \int \partial\varphi \sqrt{(aa \sin.\varphi^{2n-2} + bb \cos.\varphi^{2n-2})}.$$

SOLUTIO

Cum igitur elementum huius curvae sit

$$\partial s = \partial\varphi \sqrt{(aa \sin.\varphi^{2n-2} + bb \cos.\varphi^{2n-2})},$$

evidens est huic conditioni satisfieri, si elementa coordinatarum, quae primo sint X et Y, ita constituantur

$$\partial X = a\partial\varphi \sin.\varphi^{n-1} \quad \text{et} \quad \partial Y = b\partial\varphi \cos.\varphi^{n-1},$$

quandoquidem hinc manifesto fit

$$\partial X^2 + \partial Y^2 = \partial s^2.$$

Verum quia hae formulae paucissimis casibus exceptis non forent integrabiles, eae nostro instituto minus inserviunt; at vero ex iis alias coordinatas, quae sint x et y, formare licebit, ubi integratio certe succedet. Quodsi enim in genere statuamus

$$\partial x = \partial X \cos.\omega - \partial Y \sin.\omega \quad \text{et} \quad \partial y = \partial X \sin.\omega + \partial Y \cos.\omega,$$

hinc utique fiet

$$\partial x^2 + \partial y^2 = \partial X^2 + \partial Y^2 = \partial s^2.$$

Hae autem singulae partes revera integrationem admittent, si capiamus

$$\omega = (n+2i+1)\varphi;$$

quamobrem si loco ∂X et ∂Y valores assumtos restituamus, ambae coordinatae x et y ita algebraice exprimentur, ut sit

$$x = a\int \partial\varphi \sin.\varphi^{n-1}\cos.(n+2i+1)\varphi - b\int\partial\varphi \cos.\varphi^{n-1}\sin.(n+2i+1)\varphi$$

et

$$y = a\int \partial\varphi \sin.\varphi^{n-1}\sin.(n+2i+1)\varphi + b\int\partial\varphi \cos.\varphi^{n-1}\cos.(n+2i+1)\varphi,$$

ubi hae quatuor formulae integrales ope nostrorum theorematum algebraice exhiberi poterunt, ita ut, dum pro i omnes numeros integros positivos non excepta cyphra assumere licet, infinitae curvae algebraicae problemati satisfacientes assignari poterunt, quarum simplicissima sumendo $i = 0$ erit his formulis contenta

$$x = \frac{a}{n}\sin.\varphi^n \cos.n\varphi + \frac{b}{n}\cos.\varphi^n \cos.n\varphi$$

et

$$y = \frac{a}{n}\sin.\varphi^n \sin.n\varphi + \frac{b}{n}\cos.\varphi^n \sin.n\varphi,$$

quae ergo succincte ita referri possunt, ut sit

$$x = \frac{1}{n}\cos.n\varphi\,(a\sin.\varphi^n + b\cos.\varphi^n)$$

et

$$y = \frac{1}{n}\sin.n\varphi\,(a\sin.\varphi^n + b\cos.\varphi^n).$$

Hinc patet fore

$$\frac{y}{x} = \text{tang.}\,n\varphi$$

et

$$\sqrt{(xx+yy)} = \frac{1}{n}(a\sin.\varphi^n + b\cos.\varphi^n).$$

Unde haud difficile erit pro quovis casu aequationem inter ipsas coordinatas x et y elicere.

COROLLARIUM 1

24. Elementum curvae

$$\partial s = \partial\varphi\sqrt{(aa\sin.\varphi^{2n-2} + bb\cos.\varphi^{2n-2})}$$

in plures alias formas notatu dignas transfundere licet. Veluti si ponatur $\sin.\varphi = v$, ob $\partial\varphi = \frac{\partial v}{\sqrt{(1-vv)}}$ erit

$$\partial s = \frac{\partial v}{\sqrt{(1-vv)}}\left(aav^{2n-2} + bb(1-vv)^{n-1}\right),$$

ubi operae pretium est notasse casu $n = 2$ fieri

$$\partial s = \frac{\partial v}{\sqrt{(1-vv)}}\sqrt{((aa-bb)vv+bb)},$$

qua forma elementum ellipseos exprimitur, ita ut ope huius problematis infinitae curvae algebraicae reperiri queant, quarum longitudinem per arcus ellipticos metiri liceat.[1])

COROLLARIUM 2

25. Pro alia transformatione ponamus

$$\sin.\varphi = \sqrt{\frac{1-v}{2}} \quad \text{et} \quad \cos.\varphi = \sqrt{\frac{1+v}{2}}$$

eritque $\partial\varphi = -\frac{\partial v}{2\sqrt{(1-vv)}}$ hincque ergo fiet

$$\partial s = -\frac{\partial v}{2\sqrt{(1-vv)}}\sqrt{\frac{aa(1-v)^{n-1} + bb(1+v)^{n-1}}{2^{n-1}}},$$

quae formula casu $n = 2$ abit in hanc

$$\partial s = -\frac{\partial v}{2\sqrt{(1-vv)}}\sqrt{\frac{aa+bb+(bb-aa)v}{2}},$$

qua itidem elementum ellipticum exprimitur.

COROLLARIUM 3

26. Quodsi porro ponamus $\tang.\varphi = t$, erit

$$\sin.\varphi = \frac{t}{\sqrt{(1+tt)}} \quad \text{et} \quad \cos.\varphi = \frac{1}{\sqrt{(1+tt)}},$$

1) Confer Commentationem 639 p. 436 laudatam. A. L.

tum vero $\partial\varphi = \frac{\partial t}{1+tt}$, quibus substitutis elementum curvae nostrae erit

$$\partial s = \frac{\partial t}{1+tt}\sqrt{\frac{aat^{2n-2}+bb}{(1+tt)^{n-1}}} \quad \text{sive} \quad \partial s = \partial t\sqrt{\frac{aat^{2n-2}+bb}{(1+tt)^{n+1}}},$$

unde sumendo $n = 2$ iterum prodit elementum ellipticum

$$\partial s = \partial t\sqrt{\frac{aatt+bb}{(1+tt)^3}}.$$

SCHOLION

27. Ceterum quoniam in nostris theorematibus infiniti factores sunt indicati, per quos quaepiam formula differentialis multiplicata reddatur integrabilis, meminisse iuvabit in elementis calculi integralis methodum tradi solere, qua ex cognito uno tali factore innumerabiles alii reperiri possunt. Veluti si formula differentialis $v\partial x$ ducta in quantitatem p praebeat integrale $\int pv\partial x = q$, tum denotante Q functionem quamcunque ipsius q etiam multiplicator Qp formulam propositam $v\partial x$ reddet integrabilem. Cum enim sit $pv\partial x = \partial q$, erit $Qpv\partial x = Q\partial q$; unde quoties formula $\int Q\partial q$ est integrabilis, etiam factor ille Qp formulam propositam $v\partial x$ reddet integrabilem. Verum perspicuum est hunc casum toto coelo discrepare a formulis illis integralibus, quas in nostris theorematibus attulimus. Nam cum formula $\partial\varphi\sin.\varphi^{n-1}$ ducta in $\sin.(n+1)\varphi$ praebeat integrale $\frac{\sin.\varphi^n}{n}\sin.n\varphi$, hinc nemo certe secundum methodum memoratam reliquos multiplicatores idoneos, qui sunt

$$\sin.(n+3)\varphi, \quad \sin.(n+5)\varphi, \quad \sin.(n+7)\varphi \quad \text{etc.},$$

tum vero etiam

$$\cos.(n+1)\varphi, \quad \cos.(n+3)\varphi, \quad \cos.(n+5)\varphi \quad \text{etc.},$$

elicere valebit; quamobrem illa theoremata tanto magis omni attentione digna sunt censenda.

DE ITERATA INTEGRATIONE FORMULARUM INTEGRALIUM DUM ALIQUIS EXPONENS PRO VARIABILI ASSUMITUR

Conventui exhibita die 19. Augusti 1776

Commentatio 653 indicis ENESTROEMIANI
Nova acta academiae scientiarum Petropolitanae 7 (1789), 1793, p. 64—82
Summarium ibidem p. 40

SUMMARIUM

Pour mieux saisir l'idée de l'Auteur, le but de ce Mémoire, et la méthode d'intégrer qui y est développée, il suffira de suivre M. EULER dans la solution du premier Problème, où il traite la formule la plus simple $\int x^{\theta-1}\partial x$, dont la valeur prise depuis le terme $x=0$ jusqu'au terme $x=1$, est $\frac{1}{\theta}$, de sorte que $\int x^{\theta-1}\partial x=\frac{1}{\theta}$. Pour la nouvelle intégration de cette formule, on suppose l'exposant θ variable, et après avoir multiplié par $\partial\theta$, on prend de part et d'autre l'intégrale de manière qu'elle évanouisse en mettant $\theta=\alpha$. De cette manière on a

$$\int\frac{\partial x}{x}\int\partial\theta x^{\theta}=\int\frac{\partial\theta}{\theta}.$$

Mais à cause de $\frac{\partial x}{x}$ constant et

$$\int\partial\theta x^{\theta}=\frac{x^{\theta}-\alpha^{\theta}}{lx},$$

on aura

$$\int\frac{\partial x}{x}\cdot\frac{x^{\theta}-\alpha^{\theta}}{lx}=l\frac{\theta}{\alpha}.$$

D'une manière semblable, M. EULER traite plusieurs autres formules plus compliquées dont les intégrales sont connues pour certains termes d'intégration déterminés.

PROBLEMA 1

1. *Cum sit*

$$\int x^{\theta-1}\partial x\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=\frac{1}{\theta},$$

hanc formulam denuo integrare sumto exponente θ variabili.

SOLUTIO

Quoniam hic de integratione agitur, ut ea determinetur, integrale ita capi assumamus, ut evanescat certo casu, posito scilicet $\theta=\alpha$. Multiplicetur ergo utrimque per elementum $\partial\theta$ et integratione iuxta hanc legem instituta pro parte dextra habebimus

$$\int\frac{\partial\theta}{\theta}=l\theta-l\alpha=l\frac{\theta}{\alpha}.$$

At pro parte sinistra notum est hanc integrationem a signo summatorio $\int$ penitus non turbari, et quia iam sola littera θ pro variabili habetur, $\frac{\partial x}{x}$ vero ut constans spectatur, ob $x^{\theta-1}\partial x=\frac{\partial x}{x}x^{\theta}$ habebimus

$$\int x^{\theta}\partial\theta=\frac{x^{\theta}}{lx}-\frac{x^{\alpha}}{lx};$$

quo valore substituto membrum sinistrum erit

$$\int\frac{\partial x}{x}\cdot\frac{x^{\theta}-x^{\alpha}}{lx},$$

quamobrem ista integratio iterata nos perducit ad hanc aequationem [1])

$$\int\frac{(x^{\theta-1}-x^{\alpha-1})\partial x}{lx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=l\frac{\theta}{\alpha}.$$

COROLLARIUM 1

2. Si eodem modo formula integralis

$$\int x^{n+\theta-1}\partial x\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=\frac{1}{n+\theta}$$

1) Confer Commentationem 464 (indicis ENESTROEMIANI): *Nova methodus quantitates integrales determinandi*, Novi comment. acad. sc. Petrop. 19 (1774), 1775, p. 66; *LEONHARDI EULERI Opera omnia*, series I, vol. 17, p. 421, imprimis p. 427. A. L.

denuo integretur sumto θ variabili, reperietur haec aequatio integrata

$$\int (x^{n+\theta-1} - x^{n+\alpha-1}) \frac{\partial x}{lx} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = l\frac{n+\theta}{n+\alpha}.$$

At si θ negative capiatur, tum etiam α negative accipi debebit, unde aequatio denuo integrata haec prodibit

$$\int (x^{n-\theta-1} - x^{n-\alpha-1}) \frac{\partial x}{lx} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = l\frac{n-\theta}{n-\alpha}.$$

COROLLARIUM 2

3. Hic igitur notentur istae integrationes, quas in parte sinistra institui oportet et quibus pro aliis formulis in posterum erit utendum, ubi semper assumamus integralia ita capi debere, ut evanescant posito $\theta = \alpha$. Primo scilicet erit

$$\int x^{\theta} \partial\theta = \frac{x^{\theta} - x^{\alpha}}{lx},$$

praeterea vero simili modo

$$\int x^{n+\theta} \partial\theta = \frac{x^{n+\theta} - x^{n+\alpha}}{lx}$$

atque hinc porro intelligitur fore

$$\int x^{n+\lambda\theta} \partial\theta = \frac{x^{n+\lambda\theta} - x^{n+\lambda\alpha}}{\lambda lx},$$

unde patet, si λ capiatur negative, fore

$$\int x^{n-\lambda\theta} \partial\theta = \frac{x^{n-\lambda\theta} - x^{n-\lambda\alpha}}{-\lambda lx}.$$

PROBLEMA 2

4. *Cum sit, uti iam saepius*[1]) *est ostensum,*

$$\int \frac{x^{\theta-1}\partial x}{1+x^{\nu}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\pi}{\nu \sin. \frac{\theta\pi}{\nu}},$$

hanc aequationem denuo integrare sumto exponente θ pro variabili.

1) Confer Commentationes 60 et 254 (indicis ENESTROEMIANI): *De inventione integralium, si post integrationem variabili quantitati determinatus valor tribuatur,* Miscellanea Berolin. 7, 1743, p. 129, et *De expressione integralium per factores*, Novi comment. acad. sc. Petrop. 6 (1756/7), 1761, p. 115; *LEONHARDI EULERI Opera omnia,* series I, vol. 17, p. 35 et 233, imprimis p. 54 et 260. A. L.

SOLUTIO

Perpetuo hic ut hactenus integralia ita accipi statuamus, ut evanescant posito $\theta = \alpha$; quo observato pro parte dextra habebimus

$$\int \frac{\pi\partial\theta}{\nu \sin. \frac{\theta\pi}{\nu}},$$

quae formula posito $\frac{\theta\pi}{\nu} = \varphi$ abit in hanc $\int \frac{\partial\varphi}{\sin. \varphi}$, cuius integrale novimus esse $l \operatorname{tang.} \frac{1}{2}\varphi$; quamobrem adiecta debita constante pro hac parte habebimus

$$\int \frac{\pi\partial\theta}{\nu \sin. \frac{\theta\pi}{\nu}} = l \operatorname{tang.} \frac{\theta\pi}{2\nu} - l \operatorname{tang.} \frac{\alpha\pi}{2\nu} = l \frac{\operatorname{tang.} \frac{\theta\pi}{2\nu}}{\operatorname{tang.} \frac{\alpha\pi}{2\nu}}.$$

Pro parte autem sinistra, ubi solus factor $x^{\theta-1}$ est variabilis, erit

$$\int x^{\theta-1}\partial\theta = \frac{x^{\theta-1} - x^{\alpha-1}}{lx}.$$

Hoc igitur valore introducto formula nostra integralis denuo integrata erit

$$\int \frac{\partial x (x^{\theta-1} - x^{\alpha-1})}{(1+x^{\nu}) lx} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = l \frac{\operatorname{tang.} \frac{\theta\pi}{2\nu}}{\operatorname{tang.} \frac{\alpha\pi}{2\nu}}.$$

COROLLARIUM

5. Quodsi ergo sumamus $\alpha = \frac{1}{2}\nu$, quoniam $\operatorname{tang.} \frac{\pi}{4} = 1$, hoc casu ponendo potius $\nu = 2\alpha$ habebimus hanc aequationem integralem satis memorabilem[1])

$$\int \frac{\partial x (x^{\theta-1} - x^{\alpha-1})}{(1+x^{2\alpha}) lx} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = l \operatorname{tang.} \frac{\theta\pi}{4\alpha}.$$

1) Confer Commentationem 463 (indicis ENESTROEMIANI): *De valore formulae integralis* $\int \frac{z^{\lambda-\omega} \pm z^{\lambda+\omega}}{1 \pm z^{2\lambda}} \cdot \frac{dz}{z} (lz)^{\mu}$ *casu, quo post integrationem ponitur* $z = 1$, Novi comment. acad. sc. Petrop. 19 (1774), 1775, p. 30; *LEONHARDI EULERI Opera omnia*, series I, vol. 17, p. 384, imprimis p. 415. Confer ibidem p. 421 Commentationem 464 supra p. 459 laudatam, imprimis p. 440. A. L.

PROBLEMA 3

6. *Cum sit, uti iam satis constat*[1]),

$$\int \frac{(x^{\theta-1}+x^{\nu-\theta-1})\partial x}{1+x^{\nu}}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=\frac{\pi}{\nu \sin.\frac{\theta\pi}{\nu}},$$

hanc aequationem denuo integrare per exponentem variabilem θ, ita ut integralia evanescant posito $\theta=\alpha$.

SOLUTIO

Multiplicando igitur per $\partial\theta$ et integrando pro parte dextra prorsus ut in praecedente problemate habebimus

$$l\frac{\text{tang.}\frac{\theta\pi}{2\nu}}{\text{tang.}\frac{\alpha\pi}{2\nu}}.$$

Pro parte autem sinistra, quia formula $\frac{\partial x}{1+x^{\nu}}$ est constans et exponens θ in duobus terminis occurrit, pro priore termino habebimus

$$\int x^{\theta-1}\partial\theta=\frac{x^{\theta-1}-x^{\alpha-1}}{lx},$$

pro altero vero termino ex § 3 habebimus

$$\int x^{\nu-\theta-1}\partial\theta=\frac{x^{\nu-\alpha-1}-x^{\nu-\theta-1}}{lx};$$

quibus valoribus substitutis orietur ista nova integratio

$$\int\frac{\partial x}{lx}\cdot\frac{x^{\theta-1}-x^{\alpha-1}+x^{\nu-\alpha-1}-x^{\nu-\theta-1}}{1+x^{\nu}}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=l\frac{\text{tang.}\frac{\theta\pi}{2\nu}}{\text{tang.}\frac{\alpha\pi}{2\nu}}.$$

1) Confer praeter Commentationes 60 et 463 modo laudatas etiam Commentationes 59 et 462 (indicis ENESTROEMIANI): *Theoremata circa reductionem formularum integralium ad quadraturam circuli*, Miscellanea Berolin. 7, 1743, p. 91, et *De valore formulae integralis* $\int\frac{z^{m-1}\pm z^{n-m-1}}{1\pm z^{n}}dz$ *casu, quo post integrationem ponitur* $z=1$, Novi comment. acad. sc. Petrop. 19 (1774), 1775, p. 3; *LEONHARDI EULERI Opera omnia*, series I, vol. 17, p. 1 et 358, imprimis p. 9 et 375. A. L.

COROLLARIUM 1

7. Ista aequatio aliquanto succinctius ita repraesentari potest

$$\int\frac{\partial x}{xlx}\cdot\frac{x^{\theta}-x^{\alpha}+x^{\nu-\alpha}-x^{\nu-\theta}}{1+x^{\nu}}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=l\frac{\text{tang.}\frac{\theta\pi}{2\nu}}{\text{tang.}\frac{\alpha\pi}{2\nu}};$$

ubi cum sit

$$x^{\nu-\alpha}-x^{\nu-\theta}=x^{\nu-\alpha-\theta}(x^{\theta}-x^{\alpha}),$$

ista aequatio ita commodius per factores repraesentari poterit

$$\int\frac{\partial x}{xlx}\cdot\frac{(x^{\theta}-x^{\alpha})(1+x^{\nu-\alpha-\theta})}{1+x^{\nu}}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=l\frac{\text{tang.}\frac{\theta\pi}{2\nu}}{\text{tang.}\frac{\alpha\pi}{2\nu}}.$$

COROLLARIUM 2

8. Quodsi hic capiamus $\theta=\nu-\alpha$, ut fiat $x^{\nu-\alpha-\theta}=1$, pro parte dextra erit $\text{tang.}\frac{(\nu-\alpha)\pi}{2\nu}=\text{cotang.}\frac{\alpha\pi}{2\nu}$, unde totum hoc membrum erit $2l\cot.\frac{\alpha\pi}{2\nu}$; quare cum pro parte sinistra factor $1+x^{\nu-\alpha-\theta}$ evadat $=2$, utrimque per 2 dividendo habebimus

$$\int\frac{\partial x}{xlx}\cdot\frac{x^{\nu-\alpha}-x^{\alpha}}{1+x^{\nu}}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=l\cot.\frac{\alpha\pi}{2\nu}.$$

COROLLARIUM 3

9. Quodsi sumamus $\nu=2\alpha$, ut fiat $\text{tang.}\frac{\alpha\pi}{2\nu}=1$, pro parte sinistra factor $1+x^{\nu-\alpha-\theta}$ abit in $1+x^{\alpha-\theta}$, dum prior factor $x^{\theta}-x^{\alpha}$ ita repraesentari potest $x^{\theta}(1-x^{\alpha-\theta})$, unde amborum productum erit $x^{\theta}(1-x^{2\alpha-2\theta})$; quamobrem integratio nostra ita se habebit

$$\int\frac{x^{\theta-1}\partial x}{lx}\cdot\frac{1-x^{2\alpha-2\theta}}{1+x^{2\alpha}}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=l\,\text{tang.}\frac{\theta\pi}{4\alpha}.$$

SCHOLION

10. Istae integrationes eo maiorem attentionem merentur, quod in iis tres exponentes α, θ, ν indefiniti occurrunt, quos singulos pro lubitu utcunque determinare licet, ita ut istae formulae multo latius pateant, quam eae, quas non ita pridem[1]) ex iisdem fundamentis derivavi.

PROBLEMA 4

11. *Cum sit, uti iam abunde est demonstratum*[2]),

$$\int \frac{x^{\theta-1}-x^{\nu-\theta-1}}{1-x^{\nu}}\,\partial x \begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = \frac{\pi}{\nu \operatorname{tang.} \frac{\theta\pi}{\nu}},$$

hanc formulam denuo integrare sumto exponente θ variabili, ita ut integralia evanescant posito $\theta=\alpha$.

SOLUTIO

Quodsi ergo hic per $\partial\theta$ multiplicemus, pro parte dextra habebimus

$$\frac{\pi\partial\theta}{\nu \operatorname{tang.} \frac{\theta\pi}{\nu}},$$

quae formula posito $\frac{\pi\theta}{\nu}=\varphi$ abit in

$$\frac{\partial\varphi}{\operatorname{tang.}\varphi} = \frac{\partial\varphi \cos.\varphi}{\sin.\varphi},$$

cuius integrale manifesto est $l\sin.\varphi$; quamobrem constanti debita adiecta pro parte dextra habebimus

$$l\sin.\frac{\theta\pi}{\nu} - l\sin.\frac{\alpha\pi}{\nu} = l\frac{\sin.\frac{\theta\pi}{\nu}}{\sin.\frac{\alpha\pi}{\nu}}.$$

Pro parte autem sinistra, quae ita repraesentetur

$$\int \frac{\partial x}{x}\cdot\frac{x^{\theta}-x^{\nu-\theta}}{1-x^{\nu}},$$

1) Confer Commentationes 463 et 464 modo laudatas. A. L.

2) Confer Commentationes 59, 60, 462 et 463 supra laudatas. A. L.

habebimus

$$\int x^{\theta}\partial\theta = \frac{x^{\theta}-x^{\alpha}}{lx} \quad \text{et} \quad \int x^{\nu-\theta}\partial\theta = \frac{x^{\nu-\alpha}-x^{\nu-\theta}}{lx},$$

quibus valoribus substitutis orietur sequens aequatio integrata

$$\int \frac{\partial x}{xlx}\cdot\frac{x^{\theta}-x^{\alpha}-x^{\nu-\alpha}+x^{\nu-\theta}}{1-x^{\nu}} \begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = l\frac{\sin.\frac{\theta\pi}{\nu}}{\sin.\frac{\alpha\pi}{\nu}},$$

ubi iterum tres exponentes indefiniti occurrunt α, θ, ν.

COROLLARIUM 1

12. Cum sit, uti iam ante observavimus,

$$x^{\nu-\alpha}-x^{\nu-\theta} = x^{\nu-\alpha-\theta}(x^{\theta}-x^{\alpha}),$$

formula nostra commodius ita per factores exprimi poterit

$$\int \frac{\partial x}{xlx}\cdot\frac{(x^{\theta}-x^{\alpha})(1-x^{\nu-\alpha-\theta})}{1-x^{\nu}} \begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = l\frac{\sin.\frac{\theta\pi}{\nu}}{\sin.\frac{\alpha\pi}{\nu}};$$

ubi si sumeremus $\nu=\alpha+\theta$, membrum sinistrum evanesceret, dextrum autem manifesto quoque evanesceret.

COROLLARIUM 2

13. Quodsi autem hic sumamus $\nu=2\alpha$, pro dextra foret $\sin.\frac{\alpha\pi}{\nu}=1$, unde hoc casu formula nostra integralis erit

$$\int \frac{\partial x}{xlx}\cdot\frac{(x^{\theta}-x^{\alpha})(1-x^{\alpha-\theta})}{1-x^{2\alpha}} \begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = l\sin.\frac{\theta\pi}{2\alpha},$$

quae forma evidenter in hanc contrahitur

$$\int \frac{x^{\theta-1}\partial x}{lx}\cdot\frac{(1-x^{\alpha-\theta})^2}{1-x^{2\alpha}} \begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = l\sin.\frac{\theta\pi}{2\alpha}.$$

SCHOLION

14. Has igitur egregias integrationes deduximus ex formulis integralibus iam pridem erutis, quatenus in iis exponentes indefiniti occurrunt; quodsi ergo aliae huiusmodi formulae integrales insuper innotescerent, eas simili

modo tractare liceret; verum hactenus nullae tales formulae sunt inventae, quae ad hunc scopum accommodari possunt, quam ob causam integrationes hic exhibitae summa attentione Geometrarum dignae sunt existimandae.

ADDITAMENTUM

15. Cum nuper[1]) ostendissem huius formulae integralis

$$\int \frac{x^{a-1}\partial x}{lx} \cdot \frac{(1-x^b)(1-x^c)}{1-x^n}$$

a termino $x=0$ ad terminum $x=1$ extensae valorem ita exprimi, ut sit $l\frac{P}{Q}$ existente

$$P=\int \frac{x^{a+b-1}\partial x}{(1-x^n)^{1-\frac{c}{n}}} \quad \text{et} \quad Q=\int \frac{x^{a-1}\partial x}{(1-x^n)^{1-\frac{c}{n}}},$$

quae integralia denuo ab $x=0$ ad $x=1$ sunt extendenda, manifestum est in hac forma generali plerasque integrationes supra inventas contineri; quamobrem cum illis casibus valores integralium absolute exprimantur, operae pretium erit istam formam generalem ad illos casus applicare, ut relatio inter binas formulas integrales P et Q inde innotescat. Problema quidem primum et secundum huc plane non pertinent. Ex problemate igitur tertio et quarto eos perscrutemur casus, quos ad formam nostram generalem revocare licet.

EVOLUTIO FORMULAE INTEGRALIS SUPRA § 8 INVENTAE

$$\int \frac{\partial x}{xlx} \cdot \frac{x^{\nu-\alpha}-x^{\alpha}}{1+x^{\nu}} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = l \cot. \frac{\alpha\pi}{2\nu}$$

16. Quoniam hic denominator est $1+x^{\nu}$, ut is ad formam generalem reducatur, multiplicetur fractio supra et infra per $1-x^{\nu}$ et formula ista integralis hanc induet formam

$$\int \frac{\partial x}{xlx} \cdot \frac{(x^{\nu-\alpha}-x^{\alpha})(1-x^{\nu})}{1-x^{2\nu}}.$$

Hic ante omnia dispiciendum est, uter exponentium $\nu-\alpha$ et α sit maior, unde duos casus evolvi conveniet, prouti fuerit vel $\nu-\alpha<\alpha$, hoc est $\nu<2\alpha$, vel $\nu-\alpha>\alpha$, hoc est $\nu>2\alpha$.

1) Scilicet in Commentatione 500 huius voluminis p. 51. A. L.

17. Sit igitur primo $\nu<2\alpha$ seu $\alpha>\frac{1}{2}\nu$ atque formula integralis ita repraesentari poterit

$$\int \frac{x^{\nu-\alpha-1}\partial x}{lx} \cdot \frac{(1-x^{2\alpha-\nu})(1-x^{\nu})}{1-x^{2\nu}}.$$

Hinc iam comparatione cum formula generali instituta manifesto habebimus $a=\nu-\alpha$, $b=2\alpha-\nu$ et $c=\nu$, denique $n=2\nu$, ex quibus valoribus formabuntur sequentes formulae

$$P=\int \frac{x^{\alpha-1}\partial x}{\sqrt{(1-x^{2\nu})}} \quad \text{et} \quad Q=\int \frac{x^{\nu-\alpha-1}\partial x}{\sqrt{(1-x^{2\nu})}}.$$

Ponere etiam potuissemus $b=\nu$ et $c=2\alpha-\nu$ manentibus $a=\nu-\alpha$ et $n=2\nu$ hincque prodiissent valores

$$P=\int \frac{x^{2\nu-\alpha-1}\partial x}{(1-x^{2\nu})^{\frac{3\nu-2\alpha}{2\nu}}} \quad \text{et} \quad Q=\int \frac{x^{\nu-\alpha-1}\partial x}{(1-x^{2\nu})^{\frac{3\nu-2\alpha}{2\nu}}};$$

utrimque autem erit $l\frac{P}{Q}=l\cot.\frac{\alpha\pi}{2\nu}$.

18. Hinc igitur duas nanciscimur integrationes notatu dignissimas. Cum enim sit $\frac{P}{Q}=\cot.\frac{\alpha\pi}{2\nu}$, hae duae integrationes ita se habebunt

$$\text{I.} \quad \int \frac{x^{\alpha-1}\partial x}{\sqrt{(1-x^{2\nu})}} : \int \frac{x^{\nu-\alpha-1}\partial x}{\sqrt{(1-x^{2\nu})}} = \cot.\frac{\alpha\pi}{2\nu},$$

$$\text{II.} \quad \int \frac{x^{2\nu-\alpha-1}\partial x}{(1-x^{2\nu})^{\frac{3\nu-2\alpha}{2\nu}}} : \int \frac{x^{\nu-\alpha-1}\partial x}{(1-x^{2\nu})^{\frac{3\nu-2\alpha}{2\nu}}} = \cot.\frac{\alpha\pi}{2\nu}.$$

19. Sin autem fuerit $\nu>2\alpha$, ipsa formula generalis mutatis signis ita debet repraesentari

$$\int \frac{\partial x}{xlx} \cdot \frac{(x^{\alpha}-x^{\nu-\alpha})(1-x^{\nu})}{1-x^{2\nu}} = l\,\text{tang.}\frac{\alpha\pi}{2\nu},$$

cui aequationi nunc induamus hanc formam

$$\int \frac{x^{\alpha-1}\partial x}{lx} \cdot \frac{(1-x^{\nu-2\alpha})(1-x^{\nu})}{1-x^{2\nu}},$$

unde iam manifesto habemus $a=\alpha$, $b=\nu-2\alpha$, $c=\nu$ atque $n=2\nu$, unde deducuntur isti valores

$$P=\int\frac{x^{\nu-\alpha-1}\partial x}{\sqrt{(1-x^{2\nu})}} \quad \text{et} \quad Q=\int\frac{x^{\alpha-1}\partial x}{\sqrt{(1-x^{2\nu})}}.$$

Sin autem sumamus $c=\nu-2\alpha$ et $b=\nu$ manente $a=\alpha$ et $n=2\nu$, reperietur

$$P=\int\frac{x^{\alpha+\nu-1}\partial x}{(1-x^{2\nu})^{\frac{\nu+2\alpha}{2\nu}}} \quad \text{et} \quad Q=\int\frac{x^{\alpha-1}\partial x}{(1-x^{2\nu})^{\frac{\nu+2\alpha}{2\nu}}}.$$

20. Cum nunc utrimque sit $l\frac{P}{Q}=l\,\text{tang.}\frac{\alpha\pi}{2\nu}$ ideoque $\frac{P}{Q}=\text{tang.}\frac{\alpha\pi}{2\nu}$, hinc adipiscimur iterum has duas integrationes

$$\text{III.}\quad \int\frac{x^{\nu-\alpha-1}\partial x}{\sqrt{(1-x^{2\nu})}}:\int\frac{x^{\alpha-1}\partial x}{\sqrt{(1-x^{2\nu})}}=\text{tang.}\frac{\alpha\pi}{2\nu},$$

quae quidem convenit cum priore antecedentium, siquidem formulae P et Q tantum inter se permutantur; altera vero integratio est nova, scilicet

$$\text{IV.}\quad \int\frac{x^{\alpha+\nu-1}\partial x}{(1-x^{2\nu})^{\frac{\nu+2\alpha}{2\nu}}}:\int\frac{x^{\alpha-1}\partial x}{(1-x^{2\nu})^{\frac{\nu+2\alpha}{2\nu}}}=\text{tang.}\frac{\alpha\pi}{2\nu}.$$

EVOLUTIO FORMULAE INTEGRALIS § 9 ALLATAE

$$\int\frac{x^{\theta-1}\partial x}{lx}\cdot\frac{1-x^{2\alpha-2\theta}}{1+x^{2\alpha}}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=l\,\text{tang.}\frac{\theta\pi}{4\alpha}$$

21. Quo haec expressio ad formam praescriptam reducatur, multiplicetur supra et infra per $1-x^{2\alpha}$, ut habeamus hanc formam

$$\int\frac{x^{\theta-1}\partial x}{lx}\cdot\frac{(1-x^{2\alpha-2\theta})(1-x^{2\alpha})}{1-x^{4\alpha}}=l\,\text{tang.}\frac{\theta\pi}{4\alpha},$$

quae sponte ad formam generalem revocatur sumendo $a=\theta$, $b=2\alpha-2\theta$, $c=2\alpha$ et $n=4\alpha$, si modo fuerit $\alpha>\theta$. Si enim fuerit $\theta>\alpha$, alio modo comparatio institui debet, uti deinceps videbimus. Ex his autem valoribus conficietur

$$P=\int\frac{x^{2\alpha-\theta-1}\partial x}{\sqrt{(1-x^{4\alpha})}} \quad \text{et} \quad Q=\int\frac{x^{\theta-1}\partial x}{\sqrt{(1-x^{4\alpha})}},$$

unde ergo deducitur

$$\text{V.}\quad \int\frac{x^{2\alpha-\theta-1}\partial x}{\sqrt{(1-x^{4\alpha})}}:\int\frac{x^{\theta-1}\partial x}{\sqrt{(1-x^{4\alpha})}}=\text{tang.}\frac{\theta\pi}{4\alpha}.$$

22. Possumus etiam valores litterarum b et c inter se permutare, ut sit $b=2\alpha$ et $c=2\alpha-2\theta$ manentibus $a=\theta$ et $n=4\alpha$; tum autem fiet

$$P=\int\frac{x^{2\alpha+\theta-1}\partial x}{(1-x^{4\alpha})^{\frac{\alpha+\theta}{2\alpha}}} \quad \text{et} \quad Q=\int\frac{x^{\theta-1}\partial x}{(1-x^{4\alpha})^{\frac{\alpha+\theta}{2\alpha}}}$$

hincque deducitur reductio

$$\text{VI.}\quad \int\frac{x^{2\alpha+\theta-1}\partial x}{(1-x^{4\alpha})^{\frac{\alpha+\theta}{2\alpha}}}:\int\frac{x^{\theta-1}\partial x}{(1-x^{4\alpha})^{\frac{\alpha+\theta}{2\alpha}}}=\text{tang.}\frac{\theta\pi}{4\alpha},$$

quae autem aeque ac praecedens locum non habet, nisi sit $\alpha>\theta$.

23. Quodsi autem θ superet α, aequationem nostram in aliam formam transfundi oportet signa utrimque mutando, unde prodibit

$$\int\frac{x^{2\alpha-\theta-1}\partial x}{lx}\cdot\frac{(1-x^{2\theta-2\alpha})(1-x^{2\alpha})}{1-x^{4\alpha}}=l\,\text{cot.}\frac{\theta\pi}{4\alpha}.$$

Hic iam iterum duplex comparatio institui potest; primo scilicet sumamus $a=2\alpha-\theta$, $b=2\theta-2\alpha$, $c=2\alpha$ et $n=4\alpha$, unde formamus

$$P=\int\frac{x^{\theta-1}\partial x}{\sqrt{(1-x^{4\alpha})}} \quad \text{et} \quad Q=\int\frac{x^{2\alpha-\theta-1}\partial x}{\sqrt{(1-x^{4\alpha})}},$$

hincque oritur septima relatio haec

$$\text{VII.}\quad \int\frac{x^{\theta-1}\partial x}{\sqrt{(1-x^{4\alpha})}}:\int\frac{x^{2\alpha-\theta-1}\partial x}{\sqrt{(1-x^{4\alpha})}}=\text{cot.}\frac{\theta\pi}{4\alpha},$$

quae manifesto cum quinta congruit.

24. Nova autem reductio obtinebitur, si statuamus $b=2\alpha$ et $c=2\theta-2\alpha$ manentibus $a=2\alpha-\theta$ et $n=4\alpha$; tum igitur erit

$$P=\int\frac{x^{4\alpha-\theta-1}\partial x}{(1-x^{4\alpha})^{\frac{3\alpha-\theta}{2\alpha}}} \quad \text{et} \quad Q=\int\frac{x^{2\alpha-\theta-1}\partial x}{(1-x^{4\alpha})^{\frac{3\alpha-\theta}{2\alpha}}}.$$

Hinc vero colligitur reductio octava

$$\text{VIII.}\quad \int \frac{x^{4\alpha-\theta-1}\partial x}{(1-x^{4\alpha})^{\frac{3\alpha-\theta}{2\alpha}}} : \int \frac{x^{2\alpha-\theta-1}\partial x}{(1-x^{4\alpha})^{\frac{3\alpha-\theta}{2\alpha}}} = \cot. \frac{\theta\pi}{4\alpha}.$$

25. Hic autem probe notandum est quaternas posteriores reductiones ex quatuor prioribus oriri, si in istis loco α scribatur θ, at ν loco 2α, ita ut quatuor posteriores reductiones iam in prioribus contineantur; quamobrem sive quatuor priores sive posteriores penitus omittere licebit, ita ut nobis tantum quatuor relinquantur, inter quas porro, quoniam tertia non discrepat a prima, tantum tres supererunt huiusmodi reductiones, quae quidem ex problemate tertio sunt natae.

EVOLUTIO FORMULAE INTEGRALIS § 12 ALLATAE

$$\int \frac{\partial x}{x\,lx} \cdot \frac{(x^{\theta}-x^{\alpha})(1-x^{\nu-\alpha-\theta})}{1-x^{\nu}} \begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = l\,\frac{\sin.\frac{\theta\pi}{\nu}}{\sin.\frac{\alpha\pi}{\nu}}$$

26. Ista expressio iam congruit cum forma nostra generali neque idcirco ulteriori transformatione indiget. Hic quidem duo casus essent distinguendi, prouti fuerit vel $\theta > \alpha$ vel $\theta < \alpha$; verum hac etiam distinctione carere possumus, propterea quod binae litterae α et θ inter se sunt permutabiles; iis enim permutatis signa utrimque invertuntur. Hanc ob causam, quoscunque valores habuerint ambae litterae α et θ, minorem semper littera θ, maiorem vero littera α designare licebit, unde aequatio nostra ita repraesentabitur

$$\int \frac{x^{\theta-1}\partial x}{lx} \cdot \frac{(1-x^{\alpha-\theta})(1-x^{\nu-\alpha-\theta})}{1-x^{\nu}} = l\,\frac{\sin.\frac{\theta\pi}{\nu}}{\sin.\frac{\alpha\pi}{\nu}}.$$

27. Nihilo vero minus duo casus distinguendi etiam hic occurrunt, prouti fuerit vel $\nu > \alpha+\theta$ vel $\nu < \alpha+\theta$. Sit igitur primo $\nu > \alpha+\theta$ et forma exposita manebit invariata, quae denuo duplicem comparationem cum generali admittit. Primo igitur statuamus $a=\theta$, $b=\alpha-\theta$, $c=\nu-\alpha-\theta$ et $n=\nu$, qui valores nobis suppeditant

$$P = \int \frac{x^{\alpha-1}\partial x}{(1-x^{\nu})^{\frac{\alpha+\theta}{\nu}}} \quad \text{et} \quad Q = \int \frac{x^{\theta-1}\partial x}{(1-x^{\nu})^{\frac{\alpha+\theta}{\nu}}},$$

sicque ex hac evolutione habebimus sequentem reductionem

$$\text{I.}\quad \int \frac{x^{\alpha-1}\partial x}{(1-x^{\nu})^{\frac{\alpha+\theta}{\nu}}} : \int \frac{x^{\theta-1}\partial x}{(1-x^{\nu})^{\frac{\alpha+\theta}{\nu}}} = \frac{\sin.\frac{\theta\pi}{\nu}}{\sin.\frac{\alpha\pi}{\nu}}.$$

28. Secunda nascetur reductio permutandis litteris b et c, ita ut sit $a=\theta$, $b=\nu-\alpha-\theta$, $c=\alpha-\theta$ et $n=\nu$, unde formantur hae formulae

$$P = \int \frac{x^{\nu-\alpha-1}\partial x}{(1-x^{\nu})^{\frac{\nu-\alpha+\theta}{\nu}}} \quad \text{et} \quad Q = \int \frac{x^{\theta-1}\partial x}{(1-x^{\nu})^{\frac{\nu-\alpha+\theta}{\nu}}};$$

quare secunda reductio hinc orta erit

$$\text{II.}\quad \int \frac{x^{\nu-\alpha-1}\partial x}{(1-x^{\nu})^{\frac{\nu-\alpha+\theta}{\nu}}} : \int \frac{x^{\theta-1}\partial x}{(1-x^{\nu})^{\frac{\nu-\alpha+\theta}{\nu}}} = \frac{\sin.\frac{\theta\pi}{\nu}}{\sin.\frac{\alpha\pi}{\nu}},$$

quae duae reductiones postulant, ut sit $\nu > \alpha+\theta$.

29. Sin autem fuerit $\nu < \alpha+\theta$, ipsa aequationis forma hoc modo immutari debebit

$$\int \frac{x^{\nu-\alpha-1}\partial x}{lx} \cdot \frac{(1-x^{\alpha-\theta})(1-x^{\alpha+\theta-\nu})}{1-x^{\nu}} = l\,\frac{\sin.\frac{\alpha\pi}{\nu}}{\sin.\frac{\theta\pi}{\nu}},$$

ubi iterum gemina comparatio institui potest. Sit igitur primo $a=\nu-\alpha$, $b=\alpha-\theta$, $c=\alpha+\theta-\nu$ et $n=\nu$, unde oriuntur hae formulae

$$P = \int \frac{x^{\nu-\theta-1}\partial x}{(1-x^{\nu})^{\frac{2\nu-\alpha-\theta}{\nu}}} \quad \text{et} \quad Q = \int \frac{x^{\nu-\alpha-1}\partial x}{(1-x^{\nu})^{\frac{2\nu-\alpha-\theta}{\nu}}}.$$

Hinc igitur concluditur tertia reductio

$$\text{III.}\quad \int \frac{x^{\nu-\theta-1}\partial x}{(1-x^{\nu})^{\frac{2\nu-\alpha-\theta}{\nu}}} : \int \frac{x^{\nu-\alpha-1}\partial x}{(1-x^{\nu})^{\frac{2\nu-\alpha-\theta}{\nu}}} = \frac{\sin.\frac{\alpha\pi}{\nu}}{\sin.\frac{\theta\pi}{\nu}}.$$

30. Denique statuamus $a=\nu-\alpha$, $b=\alpha+\theta-\nu$, $c=\alpha-\theta$ et $n=\nu$ et formulae hinc sequentes nascentur

$$P = \int \frac{x^{\theta-1}\partial x}{(1-x^{\nu})^{\frac{\nu-\alpha+\theta}{\nu}}} \quad \text{et} \quad Q = \int \frac{x^{\nu-\alpha-1}\partial x}{(1-x^{\nu})^{\frac{\nu-\alpha+\theta}{\nu}}},$$

ita ut quarta hinc oriatur reductio

$$\text{IV.} \quad \int \frac{x^{\theta-1}\partial x}{(1-x^{\nu})^{\frac{\nu-\alpha+\theta}{\nu}}} : \int \frac{x^{\nu-\alpha-1}\partial x}{(1-x^{\nu})^{\frac{\nu-\alpha+\theta}{\nu}}} = \frac{\sin. \frac{\alpha\pi}{\nu}}{\sin. \frac{\theta\pi}{\nu}}.$$

31. Quatuor igitur hic nacti sumus formularum integralium paria, quae eandem inter se tenent rationem ac sinus duorum angulorum, dum evolutiones praecedentes tantum tria huiusmodi paria praebuerant, quarum ratio $P:Q$ tangenti cuiuspiam anguli aequatur, ubi quidem evidens est secundam et quartam inter se convenire. Cum igitur huiusmodi reductiones altioris sint indaginis ac sine dubio insignem usum habere queant, operae pretium erit eas clarius ob oculos exponere.

PROBLEMA

32. *Invenire binas formulas integrales P et Q ab $x=0$ ad $x=1$ extensas, ut fiat*

$$\frac{P}{Q} = \text{tang.} \frac{m\pi}{2n}.$$

SOLUTIO

Triplici igitur modo hoc fieri potest secundum evolutionem primam supra institutam.

I. Ex prima enim reductione, cum sit $\cot. \frac{\alpha\pi}{2\nu} = \text{tang.} \frac{(\nu-\alpha)\pi}{2\nu}$, fiet $\nu - \alpha = m$ et $\nu = n$, ita ut sit $\alpha = n - m$. Hinc igitur erit

$$P = \int \frac{x^{n-m-1}\partial x}{\sqrt{(1-x^{2n})}} \quad \text{et} \quad Q = \int \frac{x^{m-1}\partial x}{\sqrt{(1-x^{2n})}},$$

quae ergo est solutio prima.

II. Secunda reductio supra allata erat $\frac{P}{Q} = \cot. \frac{\alpha\pi}{2\nu} = \text{tang.} \frac{(\nu-\alpha)\pi}{2\nu}$, ubi ergo iterum est $\alpha = n - m$ et $\nu = n$, sicque secunda solutio huius problematis constabit his formulis

$$P = \int \frac{x^{m+n-1}\partial x}{(1-x^{2n})^{\frac{2m+n}{2n}}} \quad \text{et} \quad Q = \int \frac{x^{m-1}\partial x}{(1-x^{2n})^{\frac{2m+n}{2n}}}.$$

Hae autem formulae tantum valent, quando fuerit $m < \frac{1}{2}n$ ideoque ipse angulus $\frac{m\pi}{2n}$ minor semirecto.

III. Quoniam tertia reductio ibi allata cum prima convenit, ex quarta, ubi erat $\frac{P}{Q} = \text{tang.} \frac{\alpha\pi}{2\nu}$ ideoque pro nostro casu $\alpha = m$ et $\nu = n$, tertia solutio ita se habebit

$$P = \int \frac{x^{m+n-1}\partial x}{(1-x^{2n})^{\frac{2m+n}{2n}}} \quad \text{et} \quad Q = \int \frac{x^{m-1}\partial x}{(1-x^{2n})^{\frac{2m+n}{2n}}};$$

qui valores quoniam a praecedentibus non sint diversi, duas tantum adipisci mur solutiones nostri problematis, quarum secunda limitatione quadam indiget, scilicet $m < \frac{1}{2}n$, prior vero ad omnes angulos recto non maiores patet. Hae ergo duae solutiones ita repraesententur

$$\text{I.} \quad P = \int \frac{x^{n-m-1}\partial x}{\sqrt{(1-x^{2n})}}, \quad Q = \int \frac{x^{m-1}\partial x}{\sqrt{(1-x^{2n})}},$$

$$\text{II.} \quad P = \int \frac{x^{m+n-1}\partial x}{(1-x^{2n})^{\frac{2m+n}{2n}}}, \quad Q = \int \frac{x^{m-1}\partial x}{(1-x^{2n})^{\frac{2m+n}{2n}}};$$

ex utraque igitur erit $\frac{P}{Q} = \text{tang.} \frac{m\pi}{2n}$.

PROBLEMA

33. *Invenire binas formulas integrales P et Q, ut fiat*

$$\frac{P}{Q} = \frac{\sin. \frac{p\pi}{2n}}{\sin. \frac{q\pi}{2n}},$$

siquidem ambo illa integralia ab $x=0$ ad $x=1$ extendantur.

SOLUTIO

Ad hanc igitur formam transferamus quatuor illas reductiones in evolutione tertia traditas, et cum pro prima et secunda esset

$$\frac{P}{Q} = \frac{\sin. \frac{\theta\pi}{\nu}}{\sin. \frac{\alpha\pi}{\nu}},$$

pro forma hic praescripta erit $\theta = p$, $\alpha = q$ et $\nu = 2n$; quamobrem hinc nanciscimur duas sequentes solutiones

$$\text{I.}\quad P = \int \frac{x^{q-1}\partial x}{(1-x^{2n})^{\frac{p+q}{2n}}} \quad \text{et} \quad Q = \int \frac{x^{p-1}\partial x}{(1-x^{2n})^{\frac{p+q}{2n}}},$$

$$\text{II.}\quad P = \int \frac{x^{2n-q-1}\partial x}{(1-x^{2n})^{\frac{2n-q+p}{2n}}} \quad \text{et} \quad Q = \int \frac{x^{p-1}\partial x}{(1-x^{2n})^{\frac{2n-q+p}{2n}}}.$$

Tertia vero et quarta reductio habebant

$$\frac{P}{Q} = \frac{\sin. \frac{\alpha\pi}{\nu}}{\sin. \frac{\theta\pi}{\nu}},$$

pro qua igitur erit $\alpha = p$, $\theta = q$, $\nu = 2n$, unde ambae solutiones sequentes deducuntur

$$\text{III.}\quad P = \int \frac{x^{2n-q-1}\partial x}{(1-x^{2n})^{\frac{4n-p-q}{2n}}} \quad \text{et} \quad Q = \int \frac{x^{2n-p-1}\partial x}{(1-x^{2n})^{\frac{4n-p-q}{2n}}},$$

$$\text{IV.}\quad P = \int \frac{x^{q-1}\partial x}{(1-x^{2n})^{\frac{2n-p+q}{2n}}} \quad \text{et} \quad Q = \int \frac{x^{2n-p-1}\partial x}{(1-x^{2n})^{\frac{2n-p+q}{2n}}}.$$

Hinc igitur patet quadruplici modo fieri posse

$$\frac{P}{Q} = \frac{\sin. \frac{p\pi}{2n}}{\sin. \frac{q\pi}{2n}}.$$

COROLLARIUM 1

34. Si assumamus $q = n$, ut fiat $\sin. \frac{q\pi}{2n} = 1$ ideoque prodire debeat $\frac{P}{Q} = \sin. \frac{p\pi}{2n}$, pro hoc casu quatuor inventae solutiones dabunt

$$\text{I.}\quad P = \int \frac{x^{n-1}\partial x}{(1-x^{2n})^{\frac{p+n}{2n}}} \quad \text{et} \quad Q = \int \frac{x^{p-1}\partial x}{(1-x^{2n})^{\frac{p+n}{2n}}},$$

$$\text{II.}\quad P = \int \frac{x^{n-1}\partial x}{(1-x^{2n})^{\frac{n+p}{2n}}} \quad \text{et} \quad Q = \int \frac{x^{p-1}\partial x}{(1-x^{2n})^{\frac{n+p}{2n}}},$$

$$\text{III.}\quad P = \int \frac{x^{n-1}\partial x}{(1-x^{2n})^{\frac{3n-p}{2n}}} \quad \text{et} \quad Q = \int \frac{x^{2n-p-1}\partial x}{(1-x^{2n})^{\frac{3n-p}{2n}}},$$

$$\text{IV.}\quad P = \int \frac{x^{n-1}\partial x}{(1-x^{2n})^{\frac{3n-p}{2n}}} \quad \text{et} \quad Q = \int \frac{x^{2n-p-1}\partial x}{(1-x^{2n})^{\frac{3n-p}{2n}}},$$

ubi ergo solutio prima cum secunda et tertia cum quarta convenit.

COROLLARIUM 2

35. Sumamus nunc esse $q = n - p$, ut fiat $\sin. \frac{q\pi}{2n} = \cos. \frac{p\pi}{2n}$ ideoque prodire debeat $\frac{P}{Q} = \text{tang.} \frac{p\pi}{2n}$. Pro hoc ergo casu quatuor solutiones inventae evadent

$$\text{I.}\quad P = \int \frac{x^{n-p-1}\partial x}{\sqrt{(1-x^{2n})}} \quad \text{et} \quad Q = \int \frac{x^{p-1}\partial x}{\sqrt{(1-x^{2n})}},$$

$$\text{II.}\quad P = \int \frac{x^{n+p-1}\partial x}{(1-x^{2n})^{\frac{n+2p}{2n}}} \quad \text{et} \quad Q = \int \frac{x^{p-1}\partial x}{(1-x^{2n})^{\frac{n+2p}{2n}}},$$

$$\text{III.}\quad P = \int \frac{x^{n+p-1}\partial x}{(1-x^{2n})^{\frac{3}{2}}} \quad \text{et} \quad Q = \int \frac{x^{2n-p-1}\partial x}{(1-x^{2n})^{\frac{3}{2}}},$$

$$\text{IV.}\quad P = \int \frac{x^{n-p-1}\partial x}{(1-x^{2n})^{\frac{3n-2p}{2n}}} \quad \text{et} \quad Q = \int \frac{x^{2n-p-1}\partial x}{(1-x^{2n})^{\frac{3n-2p}{2n}}}$$

hincque erit

$$\frac{P}{Q} = \text{tang.} \frac{p\pi}{2n},$$

ubi prima et secunda forma cum iis, quas in praecedente problemate invenimus, prorsus conveniunt; tertia autem forma ob $(1-x^{2n})^{\frac{3}{2}}$ fit incongrua, quia inde P et Q in infinitum excrescerent; quarta autem novam formam dare videtur.