LEONHARDI EULERI OPERA OMNIA
SUB AUSPICIIS SOCIETATIS SCIENTIARUM NATURALIUM HELVETICAE
EDENDA CURAVERUNT
FERDINAND RUDIO · ADOLF KRAZER · PAUL STÄCKEL

SERIES I · OPERA MATHEMATICA · VOLUMEN XVIII

LEONHARDI EULERI

COMMENTATIONES ANALYTICAE

AD THEORIAM INTEGRALIUM PERTINENTES

VOLUMEN SECUNDUM

EDIDERUNT

AUGUST GUTZMER ET ALEXANDER LIAPOUNOFF

LIPSIAE ET BEROLINI
TYPIS ET IN AEDIBUS B. G. TEUBNERI
MCMXX

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VORWORT

Die Abhandlungen 475—594 des vorliegenden Bandes sind von A. GUTZMER, dem Herausgeber des Bandes I_{17}, bearbeitet worden, der auch, zusammen mit dem Redaktionskomitee, die Korrektur und die Revision der zugehörigen Druckbogen besorgt hat.

Die Bearbeitung des Restes, nämlich der Abhandlungen 606—653, sowie des ganzen folgenden Bandes I_{19} war von A. LIAPOUNOFF übernommen worden. Er hat diesen seinen Anteil an der Eulerausgabe dem Redaktionskomitee auch rechtzeitig eingeliefert; es war ihm aber leider nicht beschieden, sich an der Drucklegung zu beteiligen. Nachdem die Kriegsverhältnisse schon vor Jahren jede Verbindung mit ihm abgeschnitten und den Druck des fertig gesetzten, aber erst zur Hälfte korrigierten Bandes I_{18} aufs ungewisse hinausgeschoben hatten, ist er am 3. November 1918 gestorben. Die Schweizerische Naturforschende Gesellschaft wird seiner Mitarbeit und seines lebhaften Interesses für die Eulerausgabe stets dankbar gedenken.

Unter Mitwirkung von A. GUTZMER sind nun Korrektur und Revision der zweiten Hälfte des vorliegenden Bandes von A. KRAZER und dem Unterzeichneten durchgeführt worden, die hierfür auch die Verantwortung übernehmen.

Zürich, im Juli 1920.

FERDINAND RUDIO.

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Insunt in hoc volumine indicis Enestroemiani commentationes
475, 499, 500, 521, 539, 572, 587, 588, 589, 594, 606, 620, 621, 629, 630, 635, 640, 651, 653

SPECULATIONES ANALYTICAE

Commentatio 475 indicis ENESTROEMIANI
Novi commentarii academiae scientiarum Petropolitanae 20 (1775), 1776, p. 59—79
Summarium ibidem p. 15—18

SUMMARIUM

Speculationes analyticae ab Illustr. EULERO hic traditae super formula integrali $\int \frac{x^{\alpha} - x^{\beta}}{lx} dx$ versantur. Cum enim eius valorem, si a termino $x = 0$ usque ad terminum $x = 1$ extendatur, invenisset $- l \frac{\alpha + 1}{\beta + 1}$, haec integratio, quippe cuius veritatem per methodos consuetas ostendere hactenus non licuerat, haud parum attentionis mereri ipsi videbatur; quamobrem considerationes, quae super hac formula Viro Illustr. sese obtulerunt, hic exponuntur variaque inde elegantissima deducuntur theoremata, quorum praecipua hic ante oculos ponemus lectorem uberioris investigationis curiosum ad ipsam dissertationem ablegantes.

Si formula $\int \frac{dx \sin. nlx}{lx}$ a termino $x = 0$ usque ad terminum $x = 1$ extendatur, eius valor integralis aequetur arcui circuli, cuius tangens est n; cuius theorematis veritas ex consideratione tam exponentium imaginariorum quam sequentis seriei est petenda. Cum enim sit

$$\sin. nlx = \frac{nlx}{1} - \frac{n^3 (lx)^3}{1 \cdot 2 \cdot 3} + \frac{n^5 (lx)^5}{1 \cdots 5} - \frac{n^7 (lx)^7}{1 \cdots 7} + \text{etc.},$$

inde statim elicitur

$$\int \frac{dx \sin. nlx}{lx} = n - \frac{n^3}{3} + \frac{n^5}{5} - \frac{n^7}{7} + \frac{n^9}{9} - \text{etc.},$$

cuius seriei summa manifesto est A tang. n, ita ut posito $n = 1$ fiat $\int \frac{dx \sin. lx}{lx} = \frac{\pi}{4}$ denotante π semiperipheriam circuli, cuius radius $= 1$.

Si formula $\int \frac{dx \sin. plx \sin. qlx}{lx}$ a termino $x = 0$ ad terminum $x = 1$ extendatur, eius valor integralis deprehenditur esse $= \frac{1}{4} l \frac{1 + (p - q)^2}{1 + (p + q)^2}$, in quo igitur nullus arcus circularis occurrit, etiamsi in hoc theoremate praecedens contineri videatur. Manifestum autem est $\sin. qlx$ ad unitatem reduci non posse, nisi quantitas q variabilis accipiatur. Ex formula

autem generali quomodo huius theorematis integrale deducendum sit, investigandum indicavit Vir Illustr.; quem in finem hanc considerat formam $\int \frac{dx(x^\alpha - x^\beta)(x^\gamma - x^\delta)}{lx}$, quam in has duas resolvit $\int \frac{dx(x^{\alpha+\gamma} - x^{\beta+\gamma})}{lx} - \int \frac{dx(x^{\alpha+\delta} - x^{\beta+\delta})}{lx}$, quarum utraque cum formula generali initio memorata manifesto convenit; hinc autem labore haud operoso ad formulas in theoremate expressas pervenitur.

Si formula $\int \frac{x^m dx \sin. nlx}{lx}$ a termino $x = 0$ ad terminum $x = 1$ extendatur, ea semper huic valori A tang. $\frac{n}{m+1}$ aequetur. Hic observandum est hoc theorema ad primum reduci posito $m = 1$; tum vero quoties $\frac{n}{m+1}$ eundem induit valorem, toties etiam formae integrales inter se aequales evadent.

Formula $\int \frac{x^\alpha - x^\beta}{1 + x^n} \cdot \frac{dx}{xlx} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix}$ semper aequetur huic formulae

$$l\frac{\alpha}{\beta} \cdot \frac{\beta + n}{\alpha + n} \cdot \frac{\alpha + 2n}{\beta + 2n} \cdot \frac{\beta + 3n}{\alpha + 3n} \cdot \text{etc.},$$

cuius producti valor per ea, quae Vir Illustr. in Miscellaneorum Berolin. Tomo VII, p. 114,[1]) circa huiusmodi productum

$$\frac{a}{b} \cdot \frac{c+b}{c+a} \cdot \frac{a+k}{b+k} \cdot \frac{c+b+k}{c+a+k} \cdot \frac{a+2k}{b+2k} \cdot \frac{c+b+2k}{c+a+2k} \cdot \text{etc.}$$

docuerat, deprehenditur

$$= \frac{\int z^{n-1} dz (1 - z^{2n})^{\frac{\beta - 2n}{2n}}}{\int z^{n-1} dz (1 - z^{2n})^{\frac{\alpha - 2n}{2n}}}.$$

Denotante i numerum infinitum formula $\int \frac{z^{\alpha i} - z^{\beta i}}{z - 1} \cdot \frac{dz}{z} \begin{bmatrix} \text{ab } z = 0 \\ \text{ad } z = 1 \end{bmatrix}$ semper est $= l\frac{\alpha}{\beta}$.

Denotantibus litteris $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc. huiusmodi producta

$$\mathfrak{A} = (\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)(\alpha - \varepsilon) \text{ etc.},$$
$$\mathfrak{B} = (\beta - \alpha)(\beta - \gamma)(\beta - \delta)(\beta - \varepsilon) \text{ etc.},$$
$$\mathfrak{C} = (\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)(\gamma - \varepsilon) \text{ etc.}$$
etc.

(littera vero N sit $= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots (n-2)$) semper erit

$$\int \frac{dx}{x(lx)^{n-1}} \left(\frac{x^\alpha}{\mathfrak{A}} + \frac{x^\beta}{\mathfrak{B}} + \frac{x^\gamma}{\mathfrak{C}} + \frac{x^\delta}{\mathfrak{D}} + \text{etc.} \right) \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix}$$
$$= \frac{\alpha^{n-2} l\alpha}{N\mathfrak{A}} + \frac{\beta^{n-2} l\beta}{N\mathfrak{B}} + \frac{\gamma^{n-2} l\gamma}{N\mathfrak{C}} + \frac{\delta^{n-2} l\delta}{N\mathfrak{D}} + \text{etc.}$$

1) Vide notam p. 13. A. G.

Cum nuper[1]) invenissem integrale huius formulae differentialis $\frac{(x^\alpha - x^\beta)dx}{lx}$, si ita capiatur, ut evanescat posito $x = 0$, tum vero statuatur $x = 1$, aequari huic valori $l\frac{\alpha + 1}{\beta + 1}$, haec integratio eo magis attentione digna mihi videbatur, quod eius veritas per nullas methodos hactenus usitatas ostendi posset. Quamobrem nullum plane est dubium, quin ea plurimum in recessu habeat et ad multa alia praeclara inventa in Analysi perducere queat. Haud igitur ingratum Geometris fore arbitror, si nonnullas speculationes, quae super hac formula se mihi obtulerunt, exposuero.

1. Quoniam ista integratio se ad omnes plane exponentes pro litteris α et β assumtos extendit atque adeo valores imaginarii non excluduntur, ponamus

$$\alpha = n\sqrt{-1} \quad \text{et} \quad \beta = -n\sqrt{-1}$$

eritque

$$x^\alpha - x^\beta = x^{n\sqrt{-1}} - x^{-n\sqrt{-1}};$$

quae formula cum reducatur ad hanc $e^{nlx\sqrt{-1}} - e^{-nlx\sqrt{-1}}$, notum est valorem esse $= 2\sqrt{-1} \cdot \sin. nlx$, quo valore substituto prodit

$$2\sqrt{-1} \cdot \int \frac{dx \sin. nlx}{lx} = l\frac{1 + n\sqrt{-1}}{1 - n\sqrt{-1}}.$$

Constat autem huius formulae $l\frac{1 + n\sqrt{-1}}{1 - n\sqrt{-1}}$ valorem esse $2\sqrt{-1} \cdot \text{A tang.}\, n$, quandoquidem sumto n variabili eius differentiatio dat

$$d.\ l\frac{1 + n\sqrt{-1}}{1 - n\sqrt{-1}} = \frac{2dn\sqrt{-1}}{1 + nn},$$

cuius integrale manifesto est $2\sqrt{-1} \cdot \text{A tang.}\, n$; hinc igitur adipiscimur sequens theorema:

1) Vide L. Euleri Commentationem 464 (indicis Enestroemiani): *Nova methodus quantitates integrales determinandi*, Novi comment. acad. sc. Petrop. 19 (1774), 1775, p. 66; Leonhardi Euleri *Opera omnia*, series I, vol. 17, p. 421, imprimis 427. A. G.

THEOREMA 1

Ista formula integralis

$$\int \frac{dx \sin. nlx}{lx}$$

a termino $x = 0$ *usque ad terminum* $x = 1$ *extensa exprimit arcum circuli, cuius tangens* $= n$; *unde sumto* $n = 1$ *erit*

$$\int \frac{dx \sin. nlx}{lx} = \frac{\pi}{4}$$

denotante π *semiperipheriam circuli, cuius radius* $= 1$.

2. Quamvis autem haec integratio ex nostra forma generali, quae aliis methodis inaccessa videtur, sit deducta, tamen eius veritas per resolutiones consuetas sequenti modo ostendi potest sicque ex hoc casu integratio generalis eo maius firmamentum accipiet. Cum enim per seriem infinitam sit

$$\sin. nlx = \frac{nlx}{1} - \frac{n^3(lx)^3}{1\cdot 2\cdot 3} + \frac{n^5(lx)^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \text{etc.},$$

erit

$$\int \frac{dx \sin. nlx}{lx} = \int dx\left(n - \frac{n^3(lx)^2}{1\cdot 2\cdot 3} + \frac{n^5(lx)^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \text{etc.}\right);$$

constat autem esse

$$\int dx(lx)^2 = x(lx)^2 - 2\int dxlx = x(lx)^2 - 2xlx + 2\cdot 1x,$$

quae expressio posito $x = 1$ reducitur ad $2\cdot 1$; simili modo fiet

$$\int dx(lx)^4 = x(lx)^4 - 4\int dx(lx)^3 = x(lx)^4 - 4x(lx)^3 + 4\cdot 3\int dx(lx)^2,$$

quae posito $x = 1$ ob $l1 = 0$ praebet $4\cdot 3\cdot 2\cdot 1$; eodemque modo erit $\int dx(lx)^6 = 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1$. His igitur singulis valoribus integralibus introductis proveniet

$$\int \frac{dx \sin. nlx}{lx} = n - \frac{2\cdot 1n^3}{1\cdot 2\cdot 3} + \frac{4\cdot 3\cdot 2\cdot 1n^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{6\cdots 1n^7}{1\cdots 7} + \text{etc.}$$

$$= n - \frac{n^3}{3} + \frac{n^5}{5} - \frac{n^7}{7} + \text{etc.},$$

cuius seriei summa manifesto est A tang. n.

3. Hic casus nobis ansam praebet etiam hanc formulam integralem investigandi $\int \frac{dx \cos. nlx}{lx}$, quae quidem non immediate in nostra forma generali continetur; et quia est

$$\cos. nlx = 1 - \frac{nn(lx)^2}{1\cdot 2} + \frac{n^4(lx)^4}{1\cdot 2\cdot 3\cdot 4} - \frac{n^6(lx)^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \text{etc.},$$

ex primo termino oritur $\int \frac{dx}{lx}$, cuius quidem valorem ostendi esse infinitum. Pro sequentibus autem terminis erit

$$\int dxlx = xlx - x = -1 \quad \text{et} \quad \int dx(lx)^3 = -1\cdot 2\cdot 3 \quad \text{et} \quad \int dx(lx)^5 = -1\cdots 5 \quad \text{etc.},$$

quibus valoribus substitutis obtinebimus

$$\int \frac{dx \cos. nlx}{lx} = \int \frac{dx}{lx} + \frac{nn}{2} - \frac{n^4}{4} + \frac{n^6}{6} - \frac{n^8}{8} + \text{etc.},$$

quae expressio manifesto reducitur ad hanc

$$\int \frac{dx}{lx} + \frac{1}{2} l(1 + nn).$$

Quia autem primus terminus hanc summam reddit infinitam, hinc subtrahamus aliam similem

$$\int \frac{dx \cos. mlx}{lx} = \int \frac{dx}{lx} + \frac{1}{2} l(1 + mm)$$

et habebimus

$$\int \frac{dx(\cos. nlx - \cos. mlx)}{lx} = \frac{1}{2} l \frac{1 + nn}{1 + mm}$$

atque haec integratio non minus notatu digna videtur quam praecedens.

4. Cum autem in genere sit $\cos. a - \cos. b = 2 \sin. \frac{b + a}{2} \sin. \frac{b - a}{2}$, erit

$$\cos. nlx - \cos. mlx = 2 \sin. \frac{m + n}{2} lx \sin. \frac{m - n}{2} lx,$$

ita ut sit

$$\int dx \frac{\sin. \frac{m + n}{2} lx \sin. \frac{m - n}{2} lx}{lx} = \frac{1}{4} l \frac{1 + nn}{1 + mm};$$

quodsi ergo ponamus $m = p + q$ et $n = p - q$, sequens adipiscemur theorema maxime notatu dignum:

THEOREMA 2

Ista forma integralis

$$\int \frac{dx}{lx} \sin. plx \sin. qlx$$

est

$$= \frac{1}{4}\, l \frac{1+(p-q)^2}{1+(p+q)^2},$$

si scilicet integratio a termino $x=0$ *usque ad terminum* $x=1$ *extenditur.*

Quae integratio eo magis est notatu digna, quia in ea nullus arcus circularis occurrit, etiamsi priorem in se complecti videatur, quod autem secus se habet, quia $\sin. qlx$ ad unitatem reduci nequit, quin simul quantitas q reddatur variabilis.

5. Operae igitur pretium erit investigare, quomodo etiam integrale huius theorematis ex forma nostra generali derivari queat. Hunc in finem consideremus istam formam integralem

$$\int \frac{dx}{lx} (x^\alpha - x^\beta)(x^\gamma - x^\delta),$$

quae in has duas resolvitur

$$\int \frac{dx}{lx} (x^{\alpha+\gamma} - x^{\beta+\gamma}) - \int \frac{dx}{lx} (x^{\alpha+\delta} - x^{\beta+\delta}),$$

cuius prioris valor est $l\frac{\alpha+\gamma+1}{\beta+\gamma+1}$, posterioris vero $l\frac{\alpha+\delta+1}{\beta+\delta+1}$, ita ut habeamus

$$\int \frac{dx}{lx} (x^\alpha - x^\beta)(x^\gamma - x^\delta) = l\frac{(\alpha+\gamma+1)(\beta+\delta+1)}{(\beta+\gamma+1)(\alpha+\delta+1)}.$$

Nunc igitur statuamus

$$\alpha = p\sqrt{-1} \quad \text{et} \quad \beta = -p\sqrt{-1},$$

deinde

$$\gamma = q\sqrt{-1} \quad \text{et} \quad \delta = -q\sqrt{-1},$$

ut fiat

$$x^\alpha - x^\beta = 2\sqrt{-1}\cdot \sin. plx \quad \text{et} \quad x^\gamma - x^\delta = 2\sqrt{-1}\cdot \sin. qlx;$$

sic enim nostra forma integralis induet hanc formam

$$-4\int \frac{dx}{lx} \sin. plx \sin. qlx.$$

Pro eius autem valore reperimus

$$\alpha+\gamma+1 = 1+(p+q)\sqrt{-1} \quad \text{et} \quad \beta+\delta+1 = 1-(p+q)\sqrt{-1},$$

$$\beta+\gamma+1 = 1+(q-p)\sqrt{-1} \quad \text{et} \quad \alpha+\delta+1 = 1+(p-q)\sqrt{-1},$$

quibus valoribus substitutis valor integralis prodit

$$l\frac{1+(p+q)^2}{1+(p-q)^2} = -l\frac{1+(p-q)^2}{1+(p+q)^2},$$

unde manifesto sequitur integratio postremi theorematis

$$\int \frac{dx}{lx} \sin. plx \sin. qlx = \frac{1}{4}\, l\frac{1+(p-q)^2}{1+(p+q)^2}.$$

6. Hinc occasionem arripimus etiam hanc formam generalem evolvendi

$$\int \frac{dx}{lx} (x^\alpha - x^\beta)(x^\gamma + x^\delta),$$

cuius valor erit

$$= l\frac{\alpha+\gamma+1}{\beta+\gamma+1} + l\frac{\alpha+\delta+1}{\beta+\delta+1} = l\frac{(\alpha+\gamma+1)(\alpha+\delta+1)}{(\beta+\gamma+1)(\beta+\delta+1)}.$$

Nunc iterum faciamus

$$\alpha = p\sqrt{-1} \quad \text{et} \quad \beta = -p\sqrt{-1},$$

tum vero

$$\gamma = q\sqrt{-1} \quad \text{et} \quad \delta = -q\sqrt{-1}$$

fietque

$$x^\alpha - x^\beta = 2\sqrt{-1}\cdot \sin. plx \quad \text{et} \quad x^\gamma + x^\delta = 2\cos. qlx,$$

ita ut ipsa formula integralis oriatur

$$4\sqrt{-1}\cdot\int \frac{dx}{lx} \sin. plx \cos. qlx.$$

Pro valore autem integrali habebimus

$$\alpha+\gamma+1=1+(p+q)\sqrt{-1}, \quad \beta+\gamma+1=1+(q-p)\sqrt{-1},$$
$$\alpha+\delta+1=1+(p-q)\sqrt{-1}, \quad \beta+\delta+1=1-(p+q)\sqrt{-1},$$

unde valor integralis prodit

$$l\frac{1+(p+q)\sqrt{-1}}{1-(p+q)\sqrt{-1}}\cdot\frac{1+(p-q)\sqrt{-1}}{1-(p-q)\sqrt{-1}}=l\frac{1+(p+q)\sqrt{-1}}{1-(p+q)\sqrt{-1}}+l\frac{1+(p-q)\sqrt{-1}}{1-(p-q)\sqrt{-1}}.$$

Est vero

$$l\frac{1+(p+q)\sqrt{-1}}{1-(p+q)\sqrt{-1}}=2\sqrt{-1}\cdot \text{A tang.}(p+q)$$

eodemque modo

$$l\frac{1+(p-q)\sqrt{-1}}{1-(p-q)\sqrt{-1}}=2\sqrt{-1}\cdot \text{A tang.}(p-q),$$

quibus valoribus substitutis resultat ista integratio

$$\int\frac{dx}{lx}\sin.plx\cos.qlx=\frac{1}{2}\text{A tang.}(p+q)+\frac{1}{2}\text{A tang.}(p-q).$$

Cum igitur sit in genere $\text{A tang.}a+\text{A tang.}b=\text{A tang.}\frac{a+b}{1-ab}$, erit summa arcuum modo inventa $=\text{A tang.}\frac{2p}{1-pp+qq}$ et valor integralis

$$\frac{1}{2}\text{A tang.}\frac{2p}{1-pp+qq};$$

hinc sequens

THEOREMA 3

Ista formula integralis

$$\int\frac{dx}{lx}\sin.plx\cos.qlx$$

a termino $x=0$ *usque ad* $x=1$ *extensa aequalis est huic valori*

$$\frac{1}{2}\text{A tang.}\frac{2p}{1-pp+qq}.$$

7. Quodsi ergo sumamus $q=p$, ob $\sin.plx\cos.qlx=\frac{1}{2}\sin.2plx$ prodibit ista integratio

$$\frac{1}{2}\int\frac{dx}{lx}\sin.2plx=\frac{1}{2}\text{A tang.}2p,$$

id quod prorsus convenit cum theoremate primo; at vero etiam in genere ad theorema primum reduci potest. Cum enim sit

$$\sin.a\cos.b=\frac{1}{2}\sin.(a+b)+\frac{1}{2}\sin.(a-b),$$

formula nostra in has partes dividitur

$$\frac{1}{2}\int\frac{dx}{lx}\sin.(p+q)lx+\frac{1}{2}\int\frac{dx}{lx}\sin.(p-q)lx.$$

Prioris igitur partis valor erit ex theoremate

$$=\frac{1}{2}\text{A tang.}(p+q),$$

posterioris vero partis

$$=\frac{1}{2}\text{A tang.}(p-q),$$

quae forma utique reducitur ad eam, quam hic dedimus.

8. Nunc autem integrationem nostram generalem

$$\int\frac{dx}{lx}(x^\alpha-x^\beta)=l\frac{\alpha+1}{\beta+1}$$

aliquanto generalius ad angulos reducamus ponendo

$$\alpha=m+n\sqrt{-1}, \quad \beta=m-n\sqrt{-1},$$

ut fiat

$$x^\alpha-x^\beta=x^m(x^{n\sqrt{-1}}-x^{-n\sqrt{-1}})=2\sqrt{-1}\cdot x^m\sin.nlx;$$

tum vero erit

$$\frac{\alpha+1}{\beta+1}=\frac{1+m+n\sqrt{-1}}{1+m-n\sqrt{-1}},$$

quae fractio posito $n = k(m+1)$ reducitur ad hanc $\frac{1+k\sqrt{-1}}{1-k\sqrt{-1}}$. Est vero

$$l\frac{1+k\sqrt{-1}}{1-k\sqrt{-1}} = 2\sqrt{-1}\cdot \text{A tang.}\, k = 2\sqrt{-1}\cdot \text{A tang.}\, \frac{n}{m+1}$$

sicque impetramus sequens theorema:

THEOREMA 4

Ista formula integralis

$$\int \frac{dx}{lx} x^m \sin. nlx$$

a termino $x = 0$ *usque ad terminum* $x = 1$ *extensa semper aequalis erit huic valori*

$$\text{A tang.}\, \frac{n}{m+1}.$$

Quod theorema sumto $m = 0$ ad primum reducitur; ubi imprimis notatu dignum occurrit, quod, quoties $\frac{n}{m+1}$ eundem habet valorem, toties etiam formae integrales aequales inter se evadunt.

9. Verum etiam hoc theorema in genere ad primum reduci potest. Si enim ponatur $x^{m+1} = y$, erit

$$x^m dx = \frac{dy}{m+1} \quad \text{et} \quad lx = \frac{ly}{m+1};$$

his valoribus substitutis fiet

$$\int \frac{dy}{ly} \sin. \frac{n}{m+1} ly;$$

quae cum similis sit primo theoremati, eius valor manifesto est $= \text{A tang.}\, \frac{n}{m+1}$; quoniam autem hic posuimus $x^{m+1} = y$, ambo termini integrationis hic etiam erunt $y = 0$ et $y = 1$.

10. Per hoc ergo theorema, cum sit

$$\text{A tang.}\, \frac{1}{2} + \text{A tang.}\, \frac{1}{3} = \text{A tang.}\, 1 = \frac{\pi}{4},$$

pro priore erit $n = 1$ et $m = 1$, pro posteriore vero $n = 1$ et $m = 2$; hinc igitur habebitur ista integratio

$$\int \frac{dx}{lx}(x + xx) \sin. lx = \frac{\pi}{4}.$$

Deinde cum per seriem infinitam sit

$$\frac{\pi}{4} = \text{A tang.}\, \frac{1}{2} + \text{A tang.}\, \frac{1}{8} + \text{A tang.}\, \frac{1}{18} + \text{A tang.}\, \frac{1}{32} + \text{A tang.}\, \frac{1}{50} + \text{etc.},$$

cuius seriei terminus generalis est $\text{A tang.}\, \frac{1}{2nn}$, habebimus hanc integrationem satis memorabilem

$$\int \frac{dx}{xlx}(x^2 + x^8 + x^{18} + x^{32} + \text{etc.}) \sin. lx = \frac{\pi}{4},$$

quod eo magis est notatu dignum, quod series infinita $x^2 + x^8 + x^{18} + x^{32} + \text{etc.}$ nullo modo ad summam finitam reduci potest.

11. Sed revertamur ad nostram integrationem principalem, qua est

$$\int \frac{dx}{lx}(x^\alpha - x^\beta) = l\frac{\alpha+1}{\beta+1},$$

cuius veritatem etiam hoc modo ostendere licet. Cum sit $x^\alpha = e^{\alpha lx}$ denotante e numerum, cuius logarithmus hyperbolicus $= 1$, erit per seriem infinitam

$$x^\alpha = 1 + \frac{\alpha lx}{1} + \frac{\alpha\alpha(lx)^2}{1\cdot 2} + \frac{\alpha^3(lx)^3}{1\cdot 2\cdot 3} + \frac{\alpha^4(lx)^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}$$

hincque colligitur fore

$$x^\alpha - x^\beta = (\alpha - \beta)\frac{lx}{1} + (\alpha\alpha - \beta\beta)\frac{(lx)^2}{1\cdot 2} + (\alpha^3 - \beta^3)\frac{(lx)^3}{1\cdot 2\cdot 3} + \text{etc.},$$

quae series per $\frac{dx}{lx}$ multiplicata et integrata ob

$$\int dx(lx)^n = \pm 1\cdot 2\cdot 3\cdots n$$

(ubi signum superius valet, si n est numerus par, inferius, si impar) praebet posito $x=1$ sequentem seriem

$$\alpha-\beta-\frac{\alpha^2-\beta^2}{2}+\frac{\alpha^3-\beta^3}{3}-\frac{\alpha^4-\beta^4}{4}+\text{etc.},$$

quae series manifesto praebet

$$l(1+\alpha)-l(1+\beta)=l\frac{\alpha+1}{\beta+1}.$$

12. Quo valor huius formulae succinctius exprimatur, loco α et β scribamus $\alpha-1$ et $\beta-1$, ut habeamus

$$\int\frac{dx}{lx}(x^{\alpha-1}-x^{\beta-1})=\int\frac{dx}{xlx}(x^{\alpha}-x^{\beta})=l\frac{\alpha}{\beta}.$$

Quodsi ergo sumamus $\alpha=e^m$ et $\beta=e^n$, nanciscemur sequentem integrationem satis concinnam

$$\int\frac{dx}{xlx}(x^{e^m}-x^{e^n})=m-n.$$

13. Investigemus nunc integrale huius formulae differentialis

$$\frac{dx}{xlx}\cdot\frac{x^{\alpha}-x^{\beta}}{1+x^n},$$

et cum sit

$$\frac{1}{1+x^n}=1-x^n+x^{2n}-x^{3n}+x^{4n}-\text{etc.},$$

colligitur hinc integrale quaesitum

$$l\frac{\alpha}{\beta}-l\frac{\alpha+n}{\beta+n}+l\frac{\alpha+2n}{\beta+2n}-l\frac{\alpha+3n}{\beta+3n}+\text{etc.},$$

unde nanciscimur sequens theorema:

THEOREMA 5

Ista formula integralis

$$\int\frac{dx}{xlx}\cdot\frac{x^{\alpha}-x^{\beta}}{1+x^n}$$

a termino $x=0$ usque ad terminum $x=1$ extensa semper aequatur huic formulae logarithmicae

$$l\frac{\alpha}{\beta}\cdot\frac{\beta+n}{\alpha+n}\cdot\frac{\alpha+2n}{\beta+2n}\cdot\frac{\beta+3n}{\alpha+3n}\cdot\frac{\alpha+4n}{\beta+4n}\cdot\text{etc.}$$

14. Cum igitur alibi[1]) demonstraverim huius producti in infinitum continuati

$$\frac{a}{b}\cdot\frac{c+b}{c+a}\cdot\frac{a+k}{b+k}\cdot\frac{c+b+k}{c+a+k}\cdot\frac{a+2k}{b+2k}\cdot\frac{c+b+2k}{c+a+2k}\cdot\text{etc.}$$

valorem aequari huic expressioni

$$\frac{\int z^{c-1}dz(1-z^k)^{\frac{b-k}{k}}}{\int z^{c-1}dz(1-z^k)^{\frac{a-k}{k}}},$$

applicatione ad nostrum casum facta erit

$$a=\alpha,\quad b=\beta,\quad c=n,\quad k=2n$$

hincque valor nostri producti infiniti

$$=\frac{\int z^{n-1}dz(1-z^{2n})^{\frac{\beta-2n}{2n}}}{\int z^{n-1}dz(1-z^{2n})^{\frac{\alpha-2n}{2n}}},$$

quae ambae formulae integrales a termino $z=0$ usque ad terminum $z=1$ sunt extendendae; atque hinc colligimus sequens theorema:

THEOREMA 6

Ista formula integralis

$$\int\frac{dx}{xlx}\cdot\frac{x^{\alpha}-x^{\beta}}{1+x^n}$$

a termino $x=0$ usque ad terminum $x=1$ extensa aequalis est huic valori

$$l\frac{P}{Q}$$

existente

$$P=\int z^{n-1}dz(1-z^{2n})^{\frac{\beta-2n}{2n}}\quad et\quad Q=\int z^{n-1}dz(1-z^{2n})^{\frac{\alpha-2n}{2n}},$$

dum scilicet etiam hae formulae integrales posteriores a termino $z=0$ usque ad terminum $z=1$ extenduntur.

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1) Vide L. EULERI Commentationes 59 et 122 (indicis ENESTROEMIANI): *Theoremata circa reductionem formularum integralium ad quadraturam circuli*, Miscellanea Berolin. 7, 1743, p. 91, et *De productis ex infinitis factoribus ortis*, Comment. acad. sc. Petrop. 11 (1739), 1750, p. 3; *LEONHARDI EULERI Opera omnia*, series I, vol. 17 et 14. A. G.

15. Sumamus igitur $n=1$, ut formula nostra integralis fiat

$$\int\frac{dx}{xlx}\cdot\frac{x^\alpha-x^\beta}{1+x},$$

ac tum erit

$$P=\int dz(1-zz)^{\frac{\beta-2}{2}}\quad\text{et}\quad Q=\int dz(1-zz)^{\frac{\alpha-2}{2}},$$

unde pro α et β sequentes casus evolvamus. Sit primo $\alpha=2$ et $\beta=1$; erit $P=\text{A}\sin.z=\frac{\pi}{2}$ et $Q=z=1$ ideoque $\frac{P}{Q}=\frac{\pi}{2}$, unde colligimus fore

$$\int\frac{dx}{lx}\cdot\frac{x-1}{x+1}=l\frac{\pi}{2}.$$

16. Sumamus nunc $\alpha=3$ et $\beta=1$, ut fiat $\frac{x^\alpha-x^\beta}{1+x}=x(x-1)$, hincque formula nostra integralis erit $\int\frac{dx}{lx}(x-1)$, cuius valorem novimus esse $=l2$; at vero ex formula nostra generali erit

$$P=\frac{\pi}{2}\quad\text{et}\quad Q=\int dz\sqrt{(1-zz)}=\int\frac{dz}{\sqrt{(1-zz)}}-\int\frac{zzdz}{\sqrt{(1-zz)}}.$$

At vero per reductiones notas est

$$\int\frac{zzdz}{\sqrt{(1-zz)}}=\frac{1}{2}\int\frac{dz}{\sqrt{(1-zz)}}$$

sicque erit

$$Q=\frac{1}{2}\int\frac{dz}{\sqrt{(1-zz)}}=\frac{1}{2}\cdot\frac{\pi}{2},$$

unde fit $\frac{P}{Q}=2$, qui valor perfecte congruit cum ante assignato.

17. Quoniam in quantitate P non occurrit exponens α, in altero vero Q tantum α occurrit, superius theorema ita in duas partes distribuere licebit, ut sit

$$\int\frac{dx}{lx}\cdot\frac{x^{\alpha-1}}{1+x^n}=C-l\int z^{n-1}dz(1-z^{2n})^{\frac{\alpha-2n}{2n}}$$

et

$$\int\frac{dx}{lx}\cdot\frac{x^{\beta-1}}{1+x^n}=C-l\int z^{n-1}dz(1-z^{2n})^{\frac{\beta-2n}{2n}},$$

ubi C denotant certam constantem, quae autem in differentia duarum huiusmodi formularum integralium e calculo egreditur.

18. Possumus etiam nostram formulam integralem principalem

$$\int\frac{dx}{xlx}(x^\alpha-x^\beta)=l\frac{\alpha}{\beta}$$

ita transformare, ut in ea exponentes infiniti occurrant, quae ob hoc ipsum attentione non indigna videtur. Denotet igitur i numerum infinite magnum, et quia lx ita exprimere licet, ut sit $lx=i(x^{\frac{1}{i}}-1)$, formula nostra hanc induet formam

$$\int\frac{dx}{ix(x^{\frac{1}{i}}-1)}(x^\alpha-x^\beta)=l\frac{\alpha}{\beta}.$$

Nunc igitur ad exponentem fractum tollendum statuamus $x^{\frac{1}{i}}=z$, ut sit $x=z^i$ hincque $\frac{dx}{x}=\frac{idz}{z}$; tum vero $x^\alpha=z^{\alpha i}$ et $x^\beta=z^{\beta i}$, et quia adhuc iidem termini integrationis habentur $z=0$ et $z=1$, hinc sequens theorema resultat:

THEOREMA 7

Denotante i numerum infinite magnum ista formula integralis

$$\int\frac{dz(z^{\alpha i}-z^{\beta i})}{z(z-1)}$$

a termino $z=0$ usque ad terminum $z=1$ extensa semper aequalis est huic valori

$$l\frac{\alpha}{\beta}.$$

19. Cum sit

$$\frac{z^{\alpha i}}{z-1}=z^{\alpha i-1}+z^{\alpha i-2}+z^{\alpha i-3}+z^{\alpha i-4}+\text{etc.},$$

erit

$$\int\frac{z^{\alpha i}dz}{z(z-1)}=\frac{1}{\alpha i-1}+\frac{1}{\alpha i-2}+\frac{1}{\alpha i-3}+\frac{1}{\alpha i-4}+\text{etc.}+C$$

eodemque modo erit

$$\int\frac{z^{\beta i}dz}{z(z-1)}=\frac{1}{\beta i-1}+\frac{1}{\beta i-2}+\frac{1}{\beta i-3}+\frac{1}{\beta i-4}+\text{etc.}+C,$$

unde patet differentiam harum duarum serierum esse $l\frac{\alpha}{\beta}$; ita si fuerit $\alpha = 2$ et $\beta = 1$, prodibit ista integratio

$$\int \frac{dz(z^{2i}-z^{i})}{z(z-1)} = \frac{1}{2i-1} + \frac{1}{2i-2} + \frac{1}{2i-3} + \frac{1}{2i-4} + \cdots + \frac{1}{i},$$

quoniam sequentes termini per seriem posteriorem tolluntur. Constat autem huius seriei summam esse $l2$.

20. Plurima adhuc alia consectaria ex ista integratione memorabili deduci possent, quibus autem hic non immorabor, sed potius ipsam analysin, quae ad hanc integrationem perduxit, accuratius perpendam. Consideravi scilicet potestatem x^u, cuius exponens u pro lubitu sive ut constans sive ut variabilis spectari queat, et cum sit

$$\int \frac{x^u dx}{x} = \frac{x^u}{u},$$

ideoque si post integrationem sumatur $x = 1$, erit $\int \frac{dx}{x} x^u = \frac{1}{u}$; quae formula ergo fundamentum constituat, unde sequentia deducemus.

21. Hanc iam formulam per du multiplicatam integremus spectata x ut constante, et quia summa $\int x^u du = \frac{x^u}{lx}$, tum vero constat hanc integrationem ab altera, ubi x erat variabilis, non turbari, habebimus nunc istam integrationem

$$\int \frac{dx}{xlx} x^u = lu + A,$$

ubi A denotat constantem per integrationem ingressam, quae igitur e medio tolletur, si duas huiusmodi formas a se invicem subtrahamus; unde si primo sumamus $u = \alpha$, tum vero $u = \beta$ et posterius integrale a priori subtrahamus, prodibit nostra forma principalis initio commemorata

$$\int \frac{dx}{xlx}(x^\alpha - x^\beta) = l\frac{\alpha}{\beta}.$$

22. Simili autem modo a formula integrali $\int \frac{dx}{xlx} x^u = lu + A$ ulterius progrediamur, qua per du multiplicata et ex sola variabilitate ipsius u integrata

ob $\int x^u du = \frac{x^u}{lx}$ ut ante perveniemus ad istam integrationem

$$\int \frac{dx}{x(lx)^2} x^u = \int dulu + Au + B = ulu - u + Au + B,$$

ubi ergo ternas formulas particulares inter se combinari oportet, ut ambae constantes A et B ex calculo deturbentur; quia autem loco A scribere licet $A+1$, erit

$$\int \frac{dx}{x(lx)^2} x^u = ulu + Au + B.$$

23. Quodsi iam denuo per du multiplicemus et integremus, mutatis litteris constantibus, quo formula concinnior reddatur, reperiemus

$$\int \frac{dx}{x(lx)^3} x^u = \frac{1}{2} uulu + Auu + Bu + C$$

eodemque modo ulterius

$$\int \frac{dx}{x(lx)^4} x^u = \frac{1}{6} u^3 lu + Au^3 + Buu + Cu + D,$$

$$\int \frac{dx}{x(lx)^5} x^u = \frac{1}{24} u^4 lu + Au^4 + Bu^3 + Cuu + Du + E$$

etc.

Unde intelligitur continuo plures casus particulares invicem coniungi debere, ut omnes quantitates constantes A, B, C, D etc. ex calculo expellantur.

24. Hoc igitur modo evolvamus formulam § 22 inventam et exponenti u tribuamus hos tres valores α, β et γ, ut obtineamus istas tres formulas

I. $$\int \frac{dx}{x(lx)^2} x^\alpha = \alpha l\alpha + A\alpha + B,$$

II. $$\int \frac{dx}{x(lx)^2} x^\beta = \beta l\beta + A\beta + B,$$

III. $$\int \frac{dx}{x(lx)^2} x^\gamma = \gamma l\gamma + A\gamma + B,$$

unde eliminando B duas hasce nanciscimur aequationes

$$\mathrm{I}-\mathrm{II}=\alpha l\alpha-\beta l\beta+A(\alpha-\beta) \quad \text{et} \quad \mathrm{II}-\mathrm{III}=\beta l\beta-\gamma l\gamma+A(\beta-\gamma),$$

$$(\mathrm{I}-\mathrm{II})(\beta-\gamma)-(\mathrm{II}-\mathrm{III})(\alpha-\beta)$$
$$=(\beta-\gamma)\alpha l\alpha-(\beta-\gamma)\beta l\beta-(\alpha-\beta)\beta l\beta+(\alpha-\beta)\gamma l\gamma,$$

quae reducitur ad hanc

$$\mathrm{I}(\beta-\gamma)+\mathrm{II}(\gamma-\alpha)+\mathrm{III}(\alpha-\beta)=(\beta-\gamma)\alpha l\alpha+(\gamma-\alpha)\beta l\beta+(\alpha-\beta)\gamma l\gamma.$$

25. Hinc igitur pro formulis ad istud genus referendis constituere poterimus sequens theorema fundamentale:

THEOREMA 8

Ista formula integralis

$$\int\frac{dx}{x(lx)^2}\left((\beta-\gamma)x^\alpha+(\gamma-\alpha)x^\beta+(\alpha-\beta)x^\gamma\right)$$

a termino $x=0$ *usque ad terminum* $x=1$ *extensa semper aequalis est huic valori*

$$(\beta-\gamma)\alpha l\alpha+(\gamma-\alpha)\beta l\beta+(\alpha-\beta)\gamma l\gamma.$$

26. Circa hanc formam imprimis notasse iuvabit formulam

$$(\beta-\gamma)x^\alpha+(\gamma-\alpha)x^\beta+(\alpha-\beta)x^\gamma$$

non solum per $x-1$ esse divisibilem, sed etiam per $(x-1)^2$; prius inde patet, quod posito $x=1$ fit $\beta-\gamma+\gamma-\alpha+\alpha-\beta=0$, posterius vero, quod eius etiam differentiale posito $x=1$ fit $\alpha(\beta-\gamma)+\beta(\gamma-\alpha)+\gamma(\alpha-\beta)=0$, id quod natura rei postulat, quia in denominatore $(lx)^2$ posito $x=1$ continetur quadratum quantitatis evanescentis.

27. Quo vis huius integrationis generalis clarius perspiciatur, casum evolvisse operae pretium erit, quo ponitur $\alpha=n+2$, $\beta=n+1$ et $\gamma=n$, quandoquidem obtinebitur ista integratio

$$\int\frac{x^{n-1}dx(x-1)^2}{(lx)^2}=(n+2)l(n+2)-2(n+1)l(n+1)+nln=l\frac{(n+2)^{n+2}n^n}{(n+1)^{2(n+1)}}.$$

28. Tractemus eodem modo formulam integralem gradus tertii, in qua occurrit $(lx)^3$, tribuendo exponenti n quatuor valores $\alpha, \beta, \gamma, \delta$, unde oriuntur sequentes aequationes

$$\text{I.} \quad \int\frac{dx}{x(lx)^3}x^\alpha=\frac{1}{2}\alpha\alpha l\alpha+A\alpha\alpha+B\alpha+C,$$

$$\text{II.} \quad \int\frac{dx}{x(lx)^3}x^\beta=\frac{1}{2}\beta\beta l\beta+A\beta\beta+B\beta+C,$$

$$\text{III.} \quad \int\frac{dx}{x(lx)^3}x^\gamma=\frac{1}{2}\gamma\gamma l\gamma+A\gamma\gamma+B\gamma+C,$$

$$\text{IV.} \quad \int\frac{dx}{x(lx)^3}x^\delta=\frac{1}{2}\delta\delta l\delta+A\delta\delta+B\delta+C.$$

Atque hinc erit primo

$$\mathrm{I}-\mathrm{II}=\frac{1}{2}\alpha\alpha l\alpha-\frac{1}{2}\beta\beta l\beta+A(\alpha\alpha-\beta\beta)+B(\alpha-\beta),$$

unde fit

$$\frac{\mathrm{I}-\mathrm{II}}{\alpha-\beta}=\frac{\alpha\alpha l\alpha-\beta\beta l\beta}{2(\alpha-\beta)}+A(\alpha+\beta)+B.$$

Eodemque modo erit

$$\frac{\mathrm{II}-\mathrm{III}}{\beta-\gamma}=\frac{\beta\beta l\beta-\gamma\gamma l\gamma}{2(\beta-\gamma)}+A(\beta+\gamma)+B,$$

quarum formularum differentia dat

$$\frac{\mathrm{I}-\mathrm{II}}{\alpha-\beta}-\frac{\mathrm{II}-\mathrm{III}}{\beta-\gamma}=\frac{\alpha\alpha l\alpha-\beta\beta l\beta}{2(\alpha-\beta)}-\frac{\beta\beta l\beta-\gamma\gamma l\gamma}{2(\beta-\gamma)}+A(\alpha-\gamma),$$

qua per $\alpha-\gamma$ divisa prodit

$$\frac{\mathrm{I}-\mathrm{II}}{(\alpha-\beta)(\alpha-\gamma)}-\frac{\mathrm{II}-\mathrm{III}}{(\beta-\gamma)(\alpha-\gamma)}=\frac{\alpha\alpha l\alpha-\beta\beta l\beta}{2(\alpha-\beta)(\alpha-\gamma)}-\frac{\beta\beta l\beta-\gamma\gamma l\gamma}{2(\beta-\gamma)(\alpha-\gamma)}+A;$$

eodemque modo erit

$$\frac{\mathrm{II}-\mathrm{III}}{(\beta-\gamma)(\beta-\delta)}-\frac{\mathrm{III}-\mathrm{IV}}{(\gamma-\delta)(\beta-\delta)}=\frac{\beta\beta l\beta-\gamma\gamma l\gamma}{2(\beta-\gamma)(\beta-\delta)}-\frac{\gamma\gamma l\gamma-\delta\delta l\delta}{2(\gamma-\delta)(\beta-\delta)}+A,$$

quae postrema a superiori sublata relinquit

$$\frac{\mathrm{I}-\mathrm{II}}{(\alpha-\beta)(\alpha-\gamma)}-\frac{\mathrm{II}-\mathrm{III}}{(\beta-\gamma)(\alpha-\gamma)}-\frac{\mathrm{II}-\mathrm{III}}{(\beta-\gamma)(\beta-\delta)}+\frac{\mathrm{III}-\mathrm{IV}}{(\gamma-\delta)(\beta-\delta)}$$
$$=\frac{\alpha\alpha l\alpha-\beta\beta l\beta}{2(\alpha-\beta)(\alpha-\gamma)}-\frac{\beta\beta l\beta-\gamma\gamma l\gamma}{2(\beta-\gamma)(\alpha-\gamma)}-\frac{\beta\beta l\beta-\gamma\gamma l\gamma}{2(\beta-\gamma)(\beta-\delta)}+\frac{\gamma\gamma l\gamma-\delta\delta l\delta}{2(\gamma-\delta)(\beta-\delta)};$$

sicque iam omnes tres constantes A, B, C sunt elisae.

29. Quodsi iam singula huius aequationis membra evolvantur et tam secundum numeros I, II, III, IV quam secundum formulas $\alpha\alpha l\alpha$, $\beta\beta l\beta$, $\gamma\gamma l\gamma$, $\delta\delta l\delta$ in ordinem disponantur, obtinebitur sequens aequatio

$$\frac{\mathrm{I}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\mathrm{II}(\alpha-\delta)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}+\frac{\mathrm{III}(\alpha-\delta)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}+\frac{\mathrm{IV}(\alpha-\delta)}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}$$
$$=\frac{\alpha\alpha l\alpha}{2(\alpha-\beta)(\alpha-\gamma)}+\frac{(\alpha-\delta)\beta\beta l\beta}{2(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}+\frac{(\alpha-\delta)\gamma\gamma l\gamma}{2(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}+\frac{(\alpha-\delta)\delta\delta l\delta}{2(\delta-\alpha)(\delta-\beta)(\delta-\gamma)},$$

quae aequatio per $\alpha-\delta$ divisa ad pulcherrimam uniformitatem reducitur; quo facto sequens nanciscimur theorema ad hunc casum accommodatum:

THEOREMA 9

Ista formula integralis

$$\int\frac{dx}{x(lx)^3}\left\{\begin{array}{l}\dfrac{x^\alpha}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}+\dfrac{x^\beta}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}\\+\dfrac{x^\gamma}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}+\dfrac{x^\delta}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}\end{array}\right\}$$

a termino $x=0$ usque ad terminum $x=1$ extensa aequatur sequenti formulae

$$\frac{\alpha\alpha l\alpha}{2(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}+\frac{\beta\beta l\beta}{2(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}$$
$$+\frac{\gamma\gamma l\gamma}{2(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}+\frac{\delta\delta l\delta}{2(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}.$$

Ex qua forma perspicitur, quomodo ad casus magis compositos facile progredi liceat.

30. Ad hunc modum etiam praecedentes casus repraesentare operae pretium erit. Ita pro divisore lx habebimus sequentem formam integralem

$$\int\frac{dx}{xlx}\left(\frac{x^\alpha}{\alpha-\beta}+\frac{x^\beta}{\beta-\alpha}\right)=\frac{l\alpha}{\alpha-\beta}+\frac{l\beta}{\beta-\alpha}.$$

Deinde theorema § 24 allatum ita referetur

$$\int\frac{dx}{x(lx)^2}\left(\frac{x^\alpha}{(\alpha-\beta)(\alpha-\gamma)}+\frac{x^\beta}{(\beta-\alpha)(\beta-\gamma)}+\frac{x^\gamma}{(\gamma-\alpha)(\gamma-\beta)}\right)$$
$$=\frac{\alpha l\alpha}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta l\beta}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma l\gamma}{(\gamma-\alpha)(\gamma-\beta)}$$

atque istam formam sequitur illa, quam in theoremate ultimo retulimus.

31. Nunc igitur hoc negotium in genere expedire poterimus pro quacunque potestate ipsius lx, quae in denominatore formulae integralis occurrit, cuius exponens sit $=n-1$, ut numerus membrorum fiat $=n$; tum igitur accipiantur pro lubitu numeri α, β, γ, δ etc., quorum numerus sit $=n$, et quaerantur hinc sequentes valores

$$\mathfrak{A}=(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\varepsilon)\text{ etc.},$$
$$\mathfrak{B}=(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\varepsilon)\text{ etc.},$$
$$\mathfrak{C}=(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\varepsilon)\text{ etc.},$$
$$\mathfrak{D}=(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\varepsilon)\text{ etc.}$$
etc.,

tum vero ponatur etiam brevitatis gratia hoc productum

$$1\cdot2\cdot3\cdot4\cdot5\cdots(n-2)=N$$

atque obtinebitur sequens forma integralis generalissima

$$\int\frac{dx}{x(lx)^{n-1}}\left(\frac{x^\alpha}{\mathfrak{A}}+\frac{x^\beta}{\mathfrak{B}}+\frac{x^\gamma}{\mathfrak{C}}+\frac{x^\delta}{\mathfrak{D}}+\text{etc.}\right)=\frac{\alpha^{n-2}l\alpha}{N\mathfrak{A}}+\frac{\beta^{n-2}l\beta}{N\mathfrak{B}}+\frac{\gamma^{n-2}l\gamma}{N\mathfrak{C}}+\frac{\delta^{n-2}l\delta}{N\mathfrak{D}}+\text{etc.},$$

ubi notandum casu $n=2$ fore $N=1$.

32. Ad haec uberius illustranda meminisse iuvabit me iam pridem[1]) insigne theorema arithmeticum demonstrasse circa huiusmodi fractiones $\frac{1}{\mathfrak{A}} + \frac{1}{\mathfrak{B}} + \frac{1}{\mathfrak{C}} + \text{etc.}$, quorum numerus sit ut ante $= n$, ubi ostendi omnes sequentes formulas nihilo aequari:

$$\text{I.} \quad \frac{1}{\mathfrak{A}} + \frac{1}{\mathfrak{B}} + \frac{1}{\mathfrak{C}} + \frac{1}{\mathfrak{D}} + \text{etc.} = 0,$$

$$\text{II.} \quad \frac{\alpha}{\mathfrak{A}} + \frac{\beta}{\mathfrak{B}} + \frac{\gamma}{\mathfrak{C}} + \frac{\delta}{\mathfrak{D}} + \text{etc.} = 0,$$

$$\text{III.} \quad \frac{\alpha\alpha}{\mathfrak{A}} + \frac{\beta\beta}{\mathfrak{B}} + \frac{\gamma\gamma}{\mathfrak{C}} + \frac{\delta\delta}{\mathfrak{D}} + \text{etc.} = 0,$$

$$\text{IV.} \quad \frac{\alpha^3}{\mathfrak{A}} + \frac{\beta^3}{\mathfrak{B}} + \frac{\gamma^3}{\mathfrak{C}} + \frac{\delta^3}{\mathfrak{D}} + \text{etc.} = 0$$

etc.,

donec perveniatur ad potestatem exponentis $n - 2$; at vero sumto exponente $= n - 1$ semper fore demonstravimus

$$\frac{\alpha^{n-1}}{\mathfrak{A}} + \frac{\beta^{n-1}}{\mathfrak{B}} + \frac{\gamma^{n-1}}{\mathfrak{C}} + \frac{\delta^{n-1}}{\mathfrak{D}} + \text{etc.} = 1.$$

1) Vide epistolas d. 25. Sept. et 9. Nov. 1762 ab EULERO ad CHR. GOLDBACH scriptas, *Correspondance math. et phys. publiée par P. H. Fuss*, St.-Pétersbourg 1843, t. I, p. 659 et 663; *LEONHARDI EULERI Opera omnia*, series III. Vide porro L. EULERI *Institutionum calculi integralis* vol. II, § 1169, Petropoli 1769; *LEONHARDI EULERI Opera omnia*, series I, vol. 12, p. 341. Vide praeterea EULERI Commentationem 794 (indicis ENESTROEMIANI): *Theorema arithmeticum eiusque demonstratio*, *Comment. arithm.* 2, 1849, p. 588; *LEONHARDI EULERI Opera omnia*, series I, vol. 6.
A. G.

DE INTEGRATIONE FORMULAE $\int \frac{dx\,lx}{\sqrt{(1-xx)}}$ AB $x = 0$ AD $x = 1$ EXTENSA

Commentatio 499 indicis ENESTROEMIANI
Acta academiae scientiarum Petropolitanae 1777: II, 1780, p. 3—28

1. Methodus maxime naturalis huiusmodi formulas $\int p\,dx\,lx$ tractandi in hoc consistit, ut eae ad alias huiusmodi formas $\int q\,dx$ reducantur, in quibus littera q sit functio algebraica ipsius x, quandoquidem regulae integrandi potissimum ad tales formulas sunt accommodatae. Huiusmodi autem reductio nulla prorsus laborat difficultate, quando functio p ita est comparata, ut integrale $\int p\,dx$ algebraice exhiberi queat. Si enim fuerit $\int p\,dx = P$, ita ut formula proposita sit $\int dP\,lx$, ea sponte reducitur ad hanc expressionem

$$Plx - \int \frac{Pdx}{x}$$

sicque iam totum negotium ad integrationem huius formulae $\int \frac{Pdx}{x}$ est perductum. Quando vero formula $\int p\,dx$ integrationem algebraicam non admittit, quemadmodum evenit in nostra formula proposita $\int \frac{dx\,lx}{\sqrt{1-xx}}$, talis reductio successu penitus caret. Cum enim sit $\int \frac{dx}{\sqrt{1-xx}} = \text{A sin.}\,x$, ista reductio daret

$$\int \frac{dx\,lx}{\sqrt{1-xx}} = \text{A sin.}\,x\,lx - \int \frac{dx}{x}\,\text{A sin.}\,x$$

sicque post signum integrationis nova quantitas transcendens $\text{A sin.}\,x$ occurreret, cuius integratio aeque est abscondita ac ipsius propositae. Quare cum nuper singulari methodo invenissem esse

$$\int \frac{dx\,lx}{\sqrt{(1-xx)}} \begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = -\frac{\pi}{2}\,l2,$$

expressio integralis eo maiori attentione digna est censenda, quod eius investigatio neutiquam est obvia; unde operae pretium esse duxi eius veritatem etiam ex aliis fontibus ostendisse, antequam ipsam methodum, quae me eo perduxit, exponerem.

PRIMA DEMONSTRATIO INTEGRATIONIS PROPOSITAE

2. Quoniam hic potissimum ad series infinitas est recurrendum, formula autem lx talem resolutionem simplicem respuit, adhibeamus substitutionem $\sqrt{(1-xx)}=y$, unde fit $x=\sqrt{1-yy}$ hincque porro

$$lx = -\frac{yy}{2} - \frac{y^4}{4} - \frac{y^6}{6} - \frac{y^8}{8} - \text{etc.};$$

hoc igitur modo formula integralis proposita $\int \frac{dx\,lx}{\sqrt{1-xx}}$ transformatur in sequentem formam

$$\int \frac{dy}{\sqrt{1-yy}} \left(\frac{yy}{2} + \frac{y^4}{4} + \frac{y^6}{6} + \frac{y^8}{8} + \text{etc.}\right),$$

ubi, cum sit $y=\sqrt{(1-xx)}$, notetur integrationem extendi debere ab $y=1$ usque ad $y=0$; quare si hos terminos integrationis permutare velimus, signum totius formae mutari oportet.

3. Quo autem minus tali signorum mutatione confundamur, designemus valorem quaesitum littera S, ut sit

$$S = \int \frac{dx\,lx}{\sqrt{1-xx}} \begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix},$$

atque facta substitutione $y=\sqrt{1-xx}$ habebimus, uti modo monuimus,

$$S = -\int \frac{dy}{\sqrt{1-yy}} \left(\frac{yy}{2} + \frac{y^4}{4} + \frac{y^6}{6} + \text{etc.}\right) \begin{bmatrix}\text{ab } y=0\\ \text{ad } y=1\end{bmatrix}.$$

Sub his autem integrationis terminis, scilicet ab $y=0$ ad $y=1$, iam satis notum est singulas partes, quae hic occurrunt, ad sequentes valores reduci:

$$\int \frac{yy\,dy}{\sqrt{(1-yy)}} = \frac{1}{2}\cdot\frac{\pi}{2},$$
$$\int \frac{y^4 dy}{\sqrt{(1-yy)}} = \frac{1\cdot 3}{2\cdot 4}\cdot\frac{\pi}{2},$$
$$\int \frac{y^6 dy}{\sqrt{(1-yy)}} = \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{\pi}{2},$$
$$\int \frac{y^8 dy}{\sqrt{(1-yy)}} = \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{\pi}{2},$$
$$\int \frac{y^{10} dy}{\sqrt{(1-yy)}} = \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8\cdot 10}\cdot\frac{\pi}{2}$$

etc.,

ubi nimirum est $\frac{\pi}{2} = \int \frac{dy}{\sqrt{(1-yy)}}$, ita ut $1:\pi$ exprimat rationem diametri ad peripheriam circuli.

4. Quodsi ergo singulos istos valores introducamus, pro valore quaesito S impetrabimus sequentem seriem infinitam

$$S = -\frac{\pi}{2}\left(\frac{1}{2^2} + \frac{1\cdot 3}{2\cdot 4^2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6^2} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8^2} + \text{etc.}\right)$$

sicque nunc totum negotium eo est reductum, ut istius seriei infinitae summa investigetur; qui labor fortasse haud minus operosus videri potest quam id ipsum, quod nobis exsequi est propositum. Interim tamen ad cognitionem summae huius seriei haud difficulter sequenti modo nobis pertingere licebit.

5. Cum sit

$$\frac{1}{\sqrt{(1-zz)}} = 1 + \frac{1}{2}zz + \frac{1\cdot 3}{2\cdot 4}z^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}z^6 + \text{etc.},$$

si utrinque per $\frac{dz}{z}$ multiplicemus et integremus, obtinebimus

$$\int \frac{dz}{z\sqrt{1-zz}} = lz + \frac{1}{2^2}zz + \frac{1\cdot 3}{2\cdot 4^2}z^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6^2}z^6 + \text{etc.}$$

sicque ad ipsam seriem nostram sumus perducti, cuius ergo valor quaeri debet ex hac expressione $\int\frac{dz}{z\sqrt{1-zz}}-lz$, integrali scilicet ita sumto, ut evanescat posito $z=0$; quo facto statuatur $z=1$ ac prodibit ipsa series

$$\frac{1}{2^2}+\frac{1\cdot 3}{2\cdot 4^2}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6^2}+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8^2}+\text{etc.}$$

Hoc igitur modo totum negotium perductum est ad istam formulam integralem $\int\frac{dz}{z\sqrt{1-zz}}$, quae posito $\sqrt{1-zz}=v$ transit in hanc formam $\frac{-dv}{1-vv}$, cuius integrale constat esse

$$-\frac{1}{2}l\frac{1+v}{1-v}=-l\frac{1+v}{\sqrt{(1-vv)}}.$$

Quodsi loco v restituatur valor $\sqrt{1-zz}$, tota expressio, qua indigemus, ita se habebit:

$$\int\frac{dz}{z\sqrt{(1-zz)}}-lz=-l\frac{1+\sqrt{1-zz}}{z}-lz+C=C-l(1+\sqrt{1-zz}),$$

ubi constans ita accipi debet, ut valor evanescat posito $z=0$, ideoque erit $C=l2$. Quamobrem posito $z=1$ summa seriei quaesita erit $l2$ hincque valor ipsius formulae integralis propositae erit

$$\int\frac{dx\,lx}{\sqrt{(1-xx)}}=S=-\frac{\pi}{2}l2,$$

prorsus uti longe alia methodo inveneram, ex quo iam satis intelligitur istam veritatem utique altioris esse indaginis ideoque attentione Geometrarum maxime dignam.

ALIA DEMONSTRATIO INTEGRATIONIS PROPOSITAE

6. Cum sit $\frac{dx}{\sqrt{1-xx}}$ elementum arcus circuli, cuius sinus $=x$, ponamus istum angulum $=\varphi$, ita ut sit $x=\sin.\varphi$ et $\frac{dx}{\sqrt{1-xx}}=d\varphi$, atque facta hac substitutione valor quantitatis S, in quem inquirimus, ita repraesentabitur

$$S=\int d\varphi\, l\sin.\varphi \begin{bmatrix}\text{a } \varphi=0\\ \text{ad } \varphi=90^0\end{bmatrix}.$$

Cum enim ante termini fuissent $x=0$ et $x=1$, iis nunc respondent $\varphi=0$ et $\varphi=90^0$ sive $\varphi=\frac{\pi}{2}$. Hic igitur totum negotium eo redit, ut formula $l\sin.\varphi$ commode in seriem infinitam convertatur. Hunc in finem ponamus $l\sin.\varphi=s$ eritque $ds=\frac{d\varphi\cos.\varphi}{\sin.\varphi}$. Novimus autem esse

$$\frac{\cos.\varphi}{\sin.\varphi}=2\sin.2\varphi+2\sin.4\varphi+2\sin.6\varphi+2\sin.8\varphi+\text{etc.}$$

Si enim utrinque per $\sin.\varphi$ multiplicemus, ob

$$2\sin.n\varphi\sin.\varphi=\cos.(n-1)\varphi-\cos.(n+1)\varphi$$

utique prodit

$$\begin{aligned}\cos.\varphi=\cos.\varphi&+\cos.3\varphi+\cos.5\varphi+\cos.7\varphi+\cos.9\varphi+\text{etc.}\\ &-\cos.3\varphi-\cos.5\varphi-\cos.7\varphi-\cos.9\varphi-\text{etc.}\end{aligned}$$

Hac igitur serie pro $\frac{\cos.\varphi}{\sin.\varphi}$ in usum vocata erit

$$s=C-\cos.2\varphi-\frac{1}{2}\cos.4\varphi-\frac{1}{3}\cos.6\varphi-\frac{1}{4}\cos.8\varphi-\frac{1}{5}\cos.10\varphi-\text{etc.},$$

ubi, cum sit $s=l\sin.\varphi$ ideoque $s=0$, quando $\sin.\varphi=1$ ideoque $\varphi=\frac{\pi}{2}$, constantem C ita definire oportet, ut posito $\varphi=\frac{\pi}{2}=90^0$ evadat $s=0$, ex quo colligitur fore

$$C=-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\text{etc.}=-l2.$$

7. Cum igitur sit

$$l\sin.\varphi=-l2-\cos.2\varphi-\frac{1}{2}\cos.4\varphi-\frac{1}{3}\cos.6\varphi-\frac{1}{4}\cos.8\varphi-\text{etc.},$$

erit valor formulae propositae

$$\begin{aligned}\int d\varphi\, l\sin.\varphi=C-\varphi l2-\frac{1}{2}\sin.2\varphi-\frac{1}{8}\sin.4\varphi-\frac{1}{18}\sin.6\varphi\\ -\frac{1}{32}\sin.8\varphi-\frac{1}{50}\sin.10\varphi-\text{etc.};\end{aligned}$$

quae expressio cum evanescere debeat posito $\varphi=0$, constans hic ingressa erit $C=0$, ita ut iam in genere sit

$$\int d\varphi\, l\sin.\varphi = -\varphi l2 - \frac{2\sin.2\varphi}{2^2} - \frac{2\sin.4\varphi}{4^2} - \frac{2\sin.6\varphi}{6^2} - \frac{2\sin.8\varphi}{8^2}$$

$$- \frac{2\sin.10\varphi}{10^2} - \frac{2\sin.12\varphi}{12^2} - \text{etc.}$$

Quodsi iam hic capiatur $\varphi = 90^0 = \frac{\pi}{2}$, omnium angulorum 2φ, 4φ, 6φ, 8φ etc., qui hic occurrunt, sinus evanescunt ideoque valor quaesitus erit

$$S = \int d\varphi\, l\sin.\varphi \begin{bmatrix}\text{a } \varphi = 0\\ \text{ad } \varphi = 90^0\end{bmatrix} = -\frac{\pi}{2} l2,$$

quemadmodum etiam in priore demonstratione ostendimus.

8. Ista autem demonstratio praecedenti ideo longe antecellit, quod nobis non solum valorem formulae propositae exhibeat casu, quo $\varphi = 90^0$, sed etiam verum eius valorem ostendat, quicunque angulus pro φ accipiatur, id quod ad ipsam formulam propositam $\int \frac{dx\,lx}{\sqrt{1-xx}}$ transferri poterit, cuius adeo valorem pro quolibet valore ipsius x assignare poterimus. Quodsi enim istius formulae valorem desideremus ab $x=0$ usque ad $x=a$, quaeratur angulus α, cuius sinus sit aequalis ipsi a, atque semper habebitur

$$\int \frac{dx\,lx}{\sqrt{1-xx}} \begin{bmatrix}\text{ab } x=0\\ \text{ad } x=a\end{bmatrix} = -\alpha l2 - \frac{2\sin.2\alpha}{2^2} - \frac{2\sin.4\alpha}{4^2} - \frac{2\sin.6\alpha}{6^2} - \frac{2\sin.8\alpha}{8^2} - \text{etc.}$$

Unde patet, quoties fuerit $\alpha = \frac{i\pi}{2}$ denotante i numerum integrum quemcunque, quoniam omnes sinus evanescunt, valorem formulae his casibus finite exprimi per $-\frac{i\pi}{2} l2$; aliis vero casibus valor nostrae formulae per seriem infinitam satis concinnam exprimetur. Ita si capiatur $a = \frac{1}{\sqrt{2}}$, ut sit $\alpha = \frac{\pi}{4}$, valor nostrae formulae erit

$$-\frac{\pi}{4} l2 - \frac{2}{2^2} + \frac{2}{6^2} - \frac{2}{10^2} + \frac{2}{14^2} - \frac{2}{18^2} + \frac{2}{22^2} - \text{etc.},$$

quae series elegantius ita exprimitur

$$-\frac{\pi}{4} l2 - \frac{1}{2}\left(1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} + \text{etc.}\right),$$

sicque hic occurrit series satis memorabilis

$$1 - \frac{1}{9} + \frac{1}{25} - \frac{1}{49} + \frac{1}{81} - \frac{1}{121} + \text{etc.},$$

cuius summam nullo adhuc modo ad mensuras cognitas revocare licuit.

9. Quoniam tam egregia series hic se quasi praeter exspectationem obtulit, etiam alios casus evolvamus notabiliores sumamusque $a = \frac{1}{2}$, ut sit $\alpha = 30^0 = \frac{\pi}{6}$, atque nostrae formulae hoc casu valor erit

$$-\frac{\pi}{6} l2 - \frac{\sqrt{3}}{2^2} - \frac{\sqrt{3}}{4^2} + \frac{\sqrt{3}}{8^2} + \frac{\sqrt{3}}{10^2} - \frac{\sqrt{3}}{14^2} - \frac{\sqrt{3}}{16^2} + \text{etc.},$$

quae expressio ita exhiberi potest

$$-\frac{\pi}{6} l2 - \frac{\sqrt{3}}{4}\left(1 + \frac{1}{2^2} - \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} - \frac{1}{10^2} - \frac{1}{11^2} + \text{etc.}\right),$$

in qua serie quadrata multiplorum ternarii deficiunt.

Sumamus nunc simili modo $a = \frac{\sqrt{3}}{2}$, ut sit $\alpha = 60^0 = \frac{\pi}{3}$, ac valor nostrae formulae hoc casu prodibit

$$-\frac{\pi}{3} l2 - \frac{\sqrt{3}}{2^2} + \frac{\sqrt{3}}{4^2} - \frac{\sqrt{3}}{8^2} + \frac{\sqrt{3}}{10^2} - \frac{\sqrt{3}}{14^2} + \frac{\sqrt{3}}{16^2} - \text{etc.}$$

sive hoc modo exprimetur

$$-\frac{\pi}{3} l2 - \frac{\sqrt{3}}{4}\left(1 - \frac{1}{2^2} + \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{8^2} + \frac{1}{10^2} - \frac{1}{11^2} + \text{etc.}\right).$$

ADHUC ALIA DEMONSTRATIO INTEGRATIONIS PROPOSITAE

10. Introducatur in formulam nostram angulus φ, cuius cosinus sit $=x$, sive sit $x = \cos.\varphi$ et formula nostra induet hanc formam $-\int d\varphi\, l\cos.\varphi$, quod integrale a $\varphi = 90^0$ usque ad $\varphi = 0$ erit extendendum. Quodsi autem hos terminos permutemus, valor S, quem quaerimus, ita exprimetur

$$S = \int d\varphi\, l\cos.\varphi \begin{bmatrix}\text{a } \varphi = 0\\ \text{ad } \varphi = 90^0\end{bmatrix}.$$

Ut hic $l\cos.\varphi$ in seriem idoneam convertamus, statuamus ut ante $s = l\cos.\varphi$ eritque $ds = -\frac{d\varphi \sin.\varphi}{\cos.\varphi}$. Constat autem per seriem esse

$$\frac{\sin.\varphi}{\cos.\varphi} = 2\sin.2\varphi - 2\sin.4\varphi + 2\sin.6\varphi - 2\sin.8\varphi + \text{etc.}$$

Cum enim in genere sit

$$2\sin.n\varphi\cos.\varphi = \sin.(n+1)\varphi + \sin.(n-1)\varphi,$$

si utrinque per $\cos.\varphi$ multiplicemus, orietur

$$\begin{aligned}\sin.\varphi = \quad & \sin.3\varphi - \sin.5\varphi + \sin.7\varphi - \sin.9\varphi + \text{etc.}\\ + \sin.\varphi - {} & \sin.3\varphi + \sin.5\varphi - \sin.7\varphi + \sin.9\varphi - \text{etc.};\end{aligned}$$

quare cum sit $ds = -\frac{d\varphi \sin.\varphi}{\cos.\varphi}$, erit nunc

$$s = C + \frac{\cos.2\varphi}{1} - \frac{\cos.4\varphi}{2} + \frac{\cos.6\varphi}{3} - \frac{\cos.8\varphi}{4} + \frac{\cos.10\varphi}{5} - \text{etc.}$$

Quia igitur est $s = l\cos.\varphi$, evidens est posito $\varphi = 0$ fieri debere $s = 0$, unde colligitur

$$C = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \text{etc.} = -l2,$$

sicque erit

$$l\cos.\varphi = -l2 + \frac{\cos.2\varphi}{1} - \frac{\cos.4\varphi}{2} + \frac{\cos.6\varphi}{3} - \frac{\cos.8\varphi}{4} + \text{etc.},$$

quae series ducta in $d\varphi$ et integrata praebet

$$S = \int d\varphi\, l\cos.\varphi = C - \varphi l2 + \frac{\sin.2\varphi}{2} - \frac{\sin.4\varphi}{8} + \frac{\sin.6\varphi}{18} - \frac{\sin.8\varphi}{32} + \frac{\sin.10\varphi}{50} - \text{etc.};$$

quae expressio quia sponte evanescit posito $\varphi = 0$, inde patet fore $C = 0$ sicque habebimus

$$\int d\varphi\, l\cos.\varphi = -\varphi l2 + \frac{1}{2}\left(\frac{\sin.2\varphi}{1} - \frac{\sin.4\varphi}{2^2} + \frac{\sin.6\varphi}{3^2} - \frac{\sin.8\varphi}{4^2} + \frac{\sin.10\varphi}{5^2} - \text{etc.}\right).$$

Sumto igitur $\varphi = \frac{\pi}{2} = 90^0$ oritur ut ante $S = -\frac{\pi}{2}l2$. Praeterea vero etiam hinc integrale ad quemvis terminum usque extendere licet.

11. Quodsi formulam posteriorem a praecedente subtrahamus, adipiscemur in genere hanc integrationem

$$\int d\varphi\, l\,\text{tang.}\varphi = -\sin.2\varphi - \frac{1}{3^2}\sin.6\varphi - \frac{1}{5^2}\sin.10\varphi - \text{etc.},$$

unde patet hoc integrale evanescere casibus $\varphi = 90^0$ et in genere $\varphi = i\frac{\pi}{2}$. Postquam igitur istam integrationem triplici modo demonstravimus, ipsam analysin, quae me primum huc perduxit, hic dilucide sum expositurus.

ANALYSIS AD INTEGRATIONEM FORMULAE $\int \frac{dx\,lx}{\sqrt{1-xx}}$ ALIARUMQUE SIMILIUM PERDUCENS

12. Tota haec analysis innititur sequenti lemmate a me iam olim[1]) demonstrato:

Posito brevitatis gratia

$$(1-x^n)^{\frac{m-n}{n}} = X$$

si hinc duae formulae integrales formentur

$$\int Xx^{p-1}dx \quad \text{et} \quad \int Xx^{q-1}dx,$$

quae a termino $x = 0$ usque ad terminum $x = 1$ extendantur, ratio horum valorum sequenti modo ad productum ex infinitis factoribus conflatum reduci potest

$$\frac{\int Xx^{p-1}dx}{\int Xx^{q-1}dx} = \frac{(m+p)q}{p(m+q)} \cdot \frac{(m+p+n)(q+n)}{(p+n)(m+q+n)} \cdot \frac{(m+p+2n)(q+2n)}{(p+2n)(m+q+2n)} \cdot \text{etc.},$$

ubi scilicet singuli factores tam numeratoris quam denominatoris continuo eadem quantitate n augentur. Hic autem probe tenendum est veritatem istius lemmatis subsistere non posse, nisi singulae m, n, p et q denotent numeros positivos, quos tamen semper tanquam integros spectare licet.

1) Vide L. EULERI Commentationem 122 (indicis ENESTROEMIANI): *De productis ex infinitis factoribus ortis*, Comment. acad. sc. Petrop. 11 (1739), 1750, p. 3; *LEONHARDI EULERI Opera omnia*, series I, vol. 14. A. G.

13. Circa has duas formulas integrales a termino $x=0$ usque ad $x=1$ extensas duo casus imprimis seorsim notari merentur, quibus integratio actu succedit verusque valor absolute assignari potest.

Prior casus locum habet, si fuerit $p=n$, ita ut formula sit $\int Xx^{n-1}dx$. Posito enim $x^n=y$ fiet

$$X=(1-y)^{\frac{m-n}{n}} \quad \text{et} \quad x^{n-1}dx=\frac{1}{n}dy$$

sicque ista formula evadet $\frac{1}{n}\int dy(1-y)^{\frac{m-n}{n}}$ pariter a termino $y=0$ usque ad $y=1$ extendenda, quae porro posito $1-y=z$ abit in hanc formulam $-\frac{1}{n}\int z^{\frac{m-n}{n}}dz$ a termino $z=1$ usque ad $z=0$ extendendam; eius ergo integrale manifesto est $-\frac{1}{m}z^{\frac{m}{n}}+\frac{1}{m}$, unde facto $z=0$ valor erit $=\frac{1}{m}$. Consequenter pro casu $p=n$ habebimus

$$\int Xx^{n-1}dx \begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=\frac{1}{m}$$

sicque, si fuerit vel $p=n$ vel $q=n$, integrale absolute innotescit.

14. Alter casus notatu dignus est, quo $p=n-m$, ita ut formula integranda sit $\int Xx^{n-m-1}dx$; tum enim si ponatur $x(1-x^n)^{\frac{-1}{n}}$ sive $\frac{x}{(1-x^n)^{\frac{1}{n}}}=y$, posito $x=0$ fiet $y=0$, at posito $x=1$ fiet $y=\infty$; tum autem erit

$$y^{n-m}=\frac{x^{n-m}}{(1-x^n)^{\frac{n-m}{n}}}=Xx^{n-m},$$

unde formula integranda erit $\int y^{n-m}\frac{dx}{x}$. Cum igitur sit $\frac{x}{(1-x^n)^{\frac{1}{n}}}=y$, erit $\frac{x^n}{1-x^n}=y^n$, unde colligitur $x^n=\frac{y^n}{1+y^n}$ ideoque $nlx=nly-l(1+y^n)$, cuius differentiatio praebet $\frac{dx}{x}=\frac{dy}{y(1+y^n)}$, quo valore substituto formula nostra integranda erit

$$\int \frac{y^{n-m-1}dy}{1+y^n}$$

a termino $y=0$ usque ad $y=\infty$ extendenda, quae formula ideo est notatu digna, quod ab omni irrationalitate est liberata.

15. Quoniam igitur hoc casu ad formulam rationalem sumus perducti, ex elementis calculi integralis constat eius integrationem semper per logarithmos et arcus circulares absolvi posse; tum vero pro hoc casu non ita pridem[1]) ostendi huius formulae $\int \frac{x^{m-1}dx}{1+x^n}$ integrale ab $x=0$ usque ad $x=\infty$ extensum reduci ad valorem $\frac{\pi}{n\sin.\frac{m\pi}{n}}$. Facta igitur applicatione pro nostro casu habebimus

$$\int \frac{y^{n-m-1}dy}{1+y^n}=\frac{\pi}{n\sin.\frac{(n-m)\pi}{n}}=\frac{\pi}{n\sin.\frac{m\pi}{n}};$$

quamobrem pro casu $p=n-m$ valor integralis sequenti modo absolute exprimi potest eritque

$$\int Xx^{n-m-1}dx \begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=\frac{\pi}{n\sin.\frac{m\pi}{n}},$$

quod idem manifesto tenendum est, si fuerit $q=n-m$.

16. His praemissis ponamus porro brevitatis gratia

$$\int Xx^{p-1}dx \begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=P \quad \text{et} \quad \int Xx^{q-1}dx \begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=Q$$

atque lemma allatum nobis praebet hanc aequationem

$$\frac{P}{Q}=\frac{(m+p)q}{p(m+q)}\cdot\frac{(m+p+n)(q+n)}{(p+n)(m+q+n)}\cdot\frac{(m+p+2n)(q+2n)}{(p+2n)(m+q+2n)}\cdot\text{etc.}$$

Hinc igitur sumendis logarithmis deducimus

$$\begin{aligned} lP-lQ=&\,l(m+p)-lp+l(m+p+n)-l(p+n)+l(m+p+2n)-l(p+2n)+\text{etc.}\\ &+lq-l(m+q)+l(q+n)-l(m+q+n)+l(q+2n)-l(m+q+2n)+\text{etc.}\end{aligned}$$

haecque aequalitas semper locum habebit, quicunque valores litteris m, n, p et q tribuantur, dummodo fuerint positivi.

1) Vide *Institutionum calculi integralis* vol. I, § 351, Petropoli 1768; *Leonhardi Euleri Opera omnia*, series I, vol. 11, p. 222. A. G.

17. Cum igitur haec aequalitas in genere subsistat, etiam veritati erit consentanea, quando quaepiam harum litterarum m, n, p et q infinite parum immutantur sive tanquam variabiles spectantur. Hanc ob rem consideremus solam quantitatem p tanquam variabilem, ita ut reliquae litterae m, n et q maneant constantes, ideoque etiam quantitas Q erit constans, dum altera P variabitur; ex quo differentiando nanciscemur hanc aequationem

$$\frac{dP}{P}=\frac{dp}{m+p}-\frac{dp}{p}+\frac{dp}{m+p+n}-\frac{dp}{p+n}+\frac{dp}{m+p+2n}-\frac{dp}{p+2n}$$
$$+\frac{dp}{m+p+3n}-\frac{dp}{p+3n}+\text{etc.},$$

ubi totum negotium eo redit, quemadmodum differentiale formulae P, quae est integralis, exprimi oporteat.

18. Cum igitur P sit formula integralis solam quantitatem x tanquam variabilem involvens, quandoquidem in eius integratione exponens p ut constans tractari debet, demum post integrationem ipsam quantitatem P tanquam functionem duarum variabilium x et p spectare licebit, unde quaestio huc redit, quomodo valorem hoc charactere $\left(\frac{dP}{dp}\right)$ exprimi solitum investigari oporteat; qui si indicetur littera Π, aequatio ante inventa hanc induet formam

$$\frac{\Pi}{P}=\frac{1}{m+p}-\frac{1}{p}+\frac{1}{m+p+n}-\frac{1}{p+n}+\frac{1}{m+p+2n}-\frac{1}{p+2n}+\text{etc.}$$

Hanc vero seriem infinitam haud difficulter ad expressionem finitam revocare licebit hoc modo. Ponatur

$$s=\frac{v^{m+p}}{m+p}-\frac{v^{p}}{p}+\frac{v^{m+p+n}}{m+p+n}-\frac{v^{p+n}}{p+n}+\frac{v^{m+p+2n}}{m+p+2n}-\frac{v^{p+2n}}{p+2n}+\text{etc.},$$

ita ut facto $v=1$ littera s nobis exhibeat valorem quaesitum $\frac{\Pi}{P}$; at vero differentiatio nobis dabit

$$\frac{ds}{dv}=v^{m+p-1}-v^{p-1}+v^{m+p+n-1}-v^{p+n-1}+v^{m+p+2n-1}-v^{p+2n-1}+\text{etc.},$$

cuius seriei infinitae summa manifesto est

$$\frac{v^{m+p-1}-v^{p-1}}{1-v^{n}}=\frac{v^{p-1}(v^{m}-1)}{1-v^{n}}.$$

Hinc igitur vicissim concludimus fore

$$s=\int\frac{v^{p-1}(v^{m}-1)dv}{1-v^{n}},$$

quae formula integralis a $v=0$ usque ad $v=1$ est extendenda; sicque habebimus

$$\frac{\Pi}{P}=\int\frac{v^{p-1}(v^{m}-1)dv}{1-v^{n}}\begin{bmatrix}\text{a } v=0\\ \text{ad } v=1\end{bmatrix}.$$

19. Ad valorem autem $\left(\frac{dP}{dp}\right)$, quem hic littera Π indicavimus, investigandum ex principiis calculi integralis ad functiones duarum variabilium applicati iam satis notum est differentiale formulae integralis $P=\int Xx^{p-1}dx$ ex sola variabilitate ipsius p oriundum obtineri, si formula post signum integrationis posita Xx^{p-1} ex sola variabilitate ipsius p differentietur atque elementum dp signo integrationis praefigatur; at vero quia X non continet p, hic ut constans tractari debet, potestatis vero x^{p-1} differentiale hinc natum erit $x^{p-1}dplx$; quamobrem ex hac differentiatione orietur

$$dP=dp\int Xx^{p-1}dx\,lx,$$

ita ut tantum post signum integrationis factor lx accesserit, ex quo manifestum est fore

$$\Pi=\int Xx^{p-1}dx\,lx\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix};$$

hinc igitur sequens theorema generale constituere licebit.

THEOREMA GENERALE

20. *Posito brevitatis gratia* $X=(1-x^{n})^{\frac{m-n}{n}}$ *si sequentes formulae integrales omnes a termino* $x=0$ *ad terminum* $x=1$ *extendantur, sequens aequalitas semper erit veritati consentanea*

$$\frac{\int Xx^{p-1}dx\,lx}{\int Xx^{p-1}dx}=\int\frac{x^{p-1}(x^{m}-1)dx}{1-x^{n}}.$$

Nihil enim obstabat, quominus loco v scriberemus x, quandoquidem isti valores tantum a terminis integrationis pendent.

21. Hoc igitur modo deducti sumus ad integrationem huiusmodi formularum $\int Xx^{p-1}dx\,lx$, in quibus quantitas logarithmica lx post signum integrationis tanquam factor inest, quarum valorem exprimere licuit per binas formulas integrales ordinarias, cum sit

$$\int Xx^{p-1}dx\,lx = \int Xx^{p-1}dx \cdot \int \frac{x^{p-1}(x^m-1)dx}{1-x^n},$$

integralibus scilicet ab $x=0$ ad $x=1$ extensis, ubi brevitatis gratia posuimus $(1-x^n)^{\frac{m-n}{n}} = X$. Hinc igitur pro binis casibus memorabilibus supra [§ 13—15] expositis bina theoremata particularia derivemus.

THEOREMA PARTICULARE 1 QUO $p=n$

22. Quoniam supra [§ 13] vidimus casu $p=n$ fieri $\int Xx^{n-1}dx = \frac{1}{m}$, hoc valore substituto habebimus istam aequationem satis elegantem

$$\int Xx^{n-1}dx\,lx = \frac{1}{m}\int \frac{x^{n-1}(x^m-1)dx}{1-x^n},$$

dum scilicet ambo integralia ab $x=0$ ad $x=1$ extenduntur.

THEOREMA PARTICULARE 2 QUO $p=n-m$

23. Quoniam pro hoc casu, quo $p=n-m$, supra [§ 15] ostendimus esse

$$\int Xx^{n-m-1}dx = \frac{\pi}{n\sin.\frac{m\pi}{n}},$$

nunc deducimur ad sequentem integrationem maxime notatu dignam

$$\int Xx^{n-m-1}dx\,lx = \frac{\pi}{n\sin.\frac{m\pi}{n}}\int \frac{x^{n-m-1}(x^m-1)dx}{1-x^n},$$

siquidem haec ambo integralia ab $x=0$ usque ad $x=1$ extendantur; ubi meminisse oportet esse $X=(1-x^n)^{\frac{m-n}{n}}$.

24. Hic probe notetur theorema generale latissime patere, propterea quod in eo insunt tres exponentes indefiniti, scilicet m, n et p, qui penitus arbitrio nostro relinquuntur, quos ergo infinitis modis pro lubitu definire licet,

dummodo singulis valores positivi tribuantur, ita ut semper valor huius formulae integralis $\int Xx^{p-1}dx\,lx$, quam ob factorem lx tanquam transcendentem spectari oportet, per formulas integrales ordinarias exprimi queat; quae cum sint generalissima, operae pretium erit nonnullos casus speciales evolvere.

I. EVOLUTIO CASUS QUO $m=1$ ET $n=2$

25. Hoc igitur casu erit $X = \frac{1}{\sqrt{1-xx}}$, unde pro hoc casu theorema generale ita se habebit:

$$\int \frac{x^{p-1}dx\,lx}{\sqrt{1-xx}} = -\int \frac{x^{p-1}dx}{\sqrt{1-xx}} \cdot \int \frac{x^{p-1}dx}{1+x},$$

siquidem singula haec integralia ab $x=0$ ad $x=1$ extendantur. Quoniam igitur hic tantum exponens p arbitrio nostro relinquitur, hinc sequentia exempla perlustremus.

EXEMPLUM 1 QUO $p=1$

26. Hoc igitur casu aequatio superior hanc induet formam

$$\int \frac{dx\,lx}{\sqrt{1-xx}} = -\int \frac{dx}{\sqrt{1-xx}} \cdot \int \frac{dx}{1+x},$$

ubi integralibus ab $x=0$ ad $x=1$ extensis notum est fieri

$$\int \frac{dx}{\sqrt{1-xx}} = \frac{\pi}{2} \quad \text{et} \quad \int \frac{dx}{1+x} = l2,$$

ita ut iam habeamus

$$\int \frac{dx\,lx}{\sqrt{1-xx}} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = -\frac{\pi}{2}\,l2,$$

quae est ea ipsa formula, quam initio huius dissertationis tractavimus et cuius veritatem iam triplici demonstratione corroboravimus.

27. Eundem valorem elicere licet ex theoremate particulari secundo, quo erat $p=n-m$, siquidem nunc ob $n=2$ et $m=1$ erit $p=1$; inde enim ob

$X=\frac{1}{\sqrt{1-xx}}$ istud theorema praebet

$$\int\frac{dxlx}{\sqrt{1-xx}}=\frac{\pi}{2\sin.\frac{\pi}{2}}\int-\frac{dx}{1+x}=-\frac{\pi}{2}l2.$$

EXEMPLUM 2 QUO $p=2$

28. Hoc igitur casu aequatio superior hanc induet formam

$$\int\frac{xdxlx}{\sqrt{1-xx}}=-\int\frac{xdx}{\sqrt{1-xx}}\cdot\int\frac{xdx}{1+x}.$$

Iam vero integralibus ab $x=0$ ad $x=1$ extensis notum est fore

$$\int\frac{xdx}{\sqrt{1-xx}}=1 \quad \text{et} \quad \int\frac{xdx}{1+x}=1-l2,$$

ita ut habeamus

$$\int\frac{xdxlx}{\sqrt{1-xx}}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=l2-1.$$

29. Quoniam in hac formula integrale $\int\frac{xdx}{\sqrt{1-xx}}$ algebraice exhiberi potest, cum sit $=1-\sqrt{1-xx}$, valor quaesitus etiam per reductiones consuetas erui potest, cum sit

$$\int\frac{xdxlx}{\sqrt{1-xx}}=\left(1-\sqrt{1-xx}\right)lx-\int\frac{dx}{x}\left(1-\sqrt{1-xx}\right);$$

positoque $x=1$ erit

$$\int\frac{xdxlx}{\sqrt{1-xx}}=-\int\frac{dx}{x}\left(1-\sqrt{1-xx}\right),$$

ad quam formam integrandam fiat

$$1-\sqrt{1-xx}=z,$$

unde colligitur $xx=2z-zz$, ergo $2lx=lz+l(2-z)$, sicque fiet

$$\frac{dx}{x}=\frac{dz(1-z)}{z(2-z)},$$

quibus valoribus substitutis erit

$$+\int\frac{dx}{x}\left(1-\sqrt{1-xx}\right)=+\int\frac{dz(1-z)}{2-z},$$

qui ergo valor erit $=C-z-l(2-z)$. Quia igitur posito $x=0$ fit $z=0$, constans erit $C=+l2$; facto igitur $x=1$, quia tum fit $z=1$, iste valor integralis erit $l2-1$, prorsus ut ante.

30. Eundem valorem suppeditat theorema prius supra allatum, quo erat $p=n=2$; inde enim statim fit $\int\frac{xdxlx}{\sqrt{1-xx}}=\int-\frac{xdx}{1+x}$. Ante autem vidimus esse $\int\frac{xdx}{1+x}=1-l2$, ita ut etiam hinc prodeat valor quaesitus $l2-1$.

EXEMPLUM 3 QUO $p=3$

31. Hoc igitur casu aequatio in theoremate generali allata hanc induet formam

$$\int\frac{xxdxlx}{\sqrt{1-xx}}=-\int\frac{xxdx}{\sqrt{1-xx}}\cdot\int\frac{xxdx}{1+x}.$$

Per reductiones autem notissimas constat esse

$$\int\frac{xxdx}{\sqrt{1-xx}}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=\frac{1}{2}\cdot\frac{\pi}{2};$$

at vero fractio spuria $\frac{xx}{1+x}$ resolvitur in has partes

$$x-1+\frac{1}{1+x},$$

unde erit

$$\int\frac{xxdx}{1+x}=\frac{1}{2}xx-x+l(1+x),$$

quod integrale iam evanescit posito $x=0$; facto ergo $x=1$ eius valor erit $=-\frac{1}{2}+l2$; quamobrem integrale, quod quaerimus, erit

$$\int\frac{xxdxlx}{\sqrt{1-xx}}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=-\frac{\pi}{4}\left(l2-\frac{1}{2}\right).$$

EXEMPLUM 4 QUO $p=4$

32. Hoc igitur casu aequatio superior hanc induet formam

$$\int \frac{x^3 dx\,lx}{\sqrt{1-xx}} = -\int \frac{x^3 dx}{\sqrt{1-xx}} \cdot \int \frac{x^3 dx}{1+x}.$$

Per reductiones autem notissimas constat esse

$$\int \frac{x^3 dx}{\sqrt{1-xx}} \begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = \frac{2}{3};$$

tum vero fractio spuria $\frac{x^3}{1+x}$ resolvitur in has partes

$$xx - x + 1 - \frac{1}{x+1},$$

unde integrando fit

$$\int \frac{x^3 dx}{1+x} = \frac{1}{3}x^3 - \frac{1}{2}xx + x - l(1+x),$$

ex quo valor formulae erit $= \frac{5}{6} - l2$. His ergo valoribus substitutis adipiscimur hanc integrationem

$$\int \frac{x^3 dx\,lx}{\sqrt{1-xx}} \begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = -\frac{2}{3}\left(\frac{5}{6} - l2\right).$$

EXEMPLUM 5 QUO $p=5$

33. Hoc igitur casu aequatio superior hanc induet formam

$$\int \frac{x^4 dx\,lx}{\sqrt{1-xx}} = -\int \frac{x^4 dx}{\sqrt{1-xx}} \cdot \int \frac{x^4 dx}{1+x}.$$

Constat autem esse

$$\int \frac{x^4 dx}{\sqrt{1-xx}} \begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = \frac{1\cdot 3}{2\cdot 4}\cdot\frac{\pi}{2};$$

tum vero fractio spuria $\frac{x^4}{1+x}$ manifesto resolvitur in has partes

$$x^3 - xx + x - 1 + \frac{1}{x+1},$$

unde integrando fit

$$\int \frac{x^4 dx}{1+x} = \frac{1}{4}x^4 - \frac{1}{3}x^3 + \frac{1}{2}xx - x + l(1+x),$$

ex quo valor formulae erit $= -\frac{7}{12} + l2$. His igitur valoribus substitutis prodibit ista integratio

$$\int \frac{x^4 dx\,lx}{\sqrt{1-xx}} \begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = -\frac{1\cdot 3}{2\cdot 4}\cdot\frac{\pi}{2}\left(l2 - \frac{7}{12}\right).$$

EXEMPLUM 6 QUO $p=6$

34. Hoc igitur casu aequatio superior induet hanc formam

$$\int \frac{x^5 dx\,lx}{\sqrt{1-xx}} = -\int \frac{x^5 dx}{\sqrt{1-xx}} \cdot \int \frac{x^5 dx}{1+x}.$$

Constat autem per reductiones notas esse

$$\int \frac{x^5 dx}{\sqrt{1-xx}} \begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = \frac{2\cdot 4}{3\cdot 5};$$

tum vero fractio spuria $\frac{x^5}{1+x}$ resolvitur in has partes

$$x^4 - x^3 + xx - x + 1 - \frac{1}{x+1},$$

unde integrando nanciscimur

$$\int \frac{x^5 dx}{1+x} = \frac{1}{5}x^5 - \frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}xx + x - l(1+x),$$

ex quo valor huius formulae erit $= \frac{47}{60} - l2$; quibus valoribus substitutis prodibit ista integratio

$$\int \frac{x^5 dx\,lx}{\sqrt{1-xx}} \begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = -\frac{2\cdot 4}{3\cdot 5}\left(\frac{47}{60} - l2\right).$$

II. EVOLUTIO CASUS QUO $m=3$ ET $n=2$

35. Hic ergo erit $X = \sqrt{(1-xx)}$, unde theorema nostrum generale nobis praebebit hanc aequationem

$$\int x^{p-1} dx\,lx \cdot \sqrt{1-xx} = \int x^{p-1} dx \sqrt{1-xx} \cdot \int \frac{x^{p-1}(x^3-1)dx}{1-xx};$$

ubi cum sit

$$\frac{x^3-1}{1-xx} = \frac{-xx-x-1}{x+1} = -x - \frac{1}{x+1},$$

erit postrema formula integralis

$$-\int x^p\,dx - \int \frac{x^{p-1}dx}{1+x},$$

quae integrata ab $x=0$ ad $x=1$ dat

$$-\frac{1}{p+1} - \int \frac{x^{p-1}dx}{1+x},$$

quamobrem habebimus

$$\int x^{p-1}dx\,lx\cdot\sqrt{1-xx} = -\int x^{p-1}dx\sqrt{1-xx}\cdot\left(\frac{1}{p+1}+\int\frac{x^{p-1}dx}{1+x}\right).$$

Hinc igitur sequentia exempla notasse iuvabit.

EXEMPLUM 1 QUO $p=1$

36. Pro hoc igitur casu postremus factor evadet $\frac{1}{2}+l2$, ita ut sit

$$\int dx\,lx\cdot\sqrt{1-xx} = -\left(\frac{1}{2}+l2\right)\int dx\sqrt{1-xx}.$$

Pro formula autem $\int dx\sqrt{1-xx}$ statuatur

$$\sqrt{1-xx} = 1-vx$$

fietque $x=\frac{2v}{1+vv}$ et $\sqrt{1-xx}=\frac{1-vv}{1+vv}$ atque $dx=\frac{2dv(1-vv)}{(1+vv)^2}$, unde fiet

$$dx\sqrt{1-xx} = \frac{2dv(1-vv)^2}{(1+vv)^3},$$

cuius integrale resolvitur in has partes

$$\frac{2v}{(1+vv)^2} - \frac{v}{1+vv} + \text{A tang.}\, v;$$

quae expressio cum extendi debeat ab $x=0$ usque ad $x=1$, prior terminus erit $v=0$, alter vero terminus est $v=1$, ita ut integrale illud a

$v=0$ usque ad $v=1$ extendi debeat. At vero illa expressio sponte evanescit posito $v=0$, facto autem $v=1$ valor integralis erit $=\frac{\pi}{4}$; quamobrem habebimus

$$\int dx\,lx\cdot\sqrt{1-xx}\left[\begin{matrix}\text{ab } x=0\\ \text{ad } x=1\end{matrix}\right] = -\frac{\pi}{4}\left(\frac{1}{2}+l2\right).$$

37. Hic quidem calculum per longas ambages evolvimus, prouti reductio ad rationalitatem formulae $\sqrt{1-xx}$ manuduxit; at vero solus aspectus formulae $\int dx\sqrt{1-xx}$ statim declarat eam exprimere aream quadrantis circuli, cuius radius $=1$, quem novimus esse $=\frac{\pi}{4}$. Caeterum adhiberi potuisset ista reductio

$$\int dx\sqrt{1-xx} = \frac{1}{2}x\sqrt{1-xx} + \frac{1}{2}\int\frac{dx}{\sqrt{1-xx}},$$

cuius valor ab $x=0$ ad $x=1$ extensus manifesto dat $\frac{\pi}{4}$.

EXEMPLUM 2 QUO $p=2$

38. Hoc ergo casu postremus factor fit

$$\frac{1}{3}+\int\frac{xdx}{1+x} = \frac{4}{3} - l2$$

sicque habebimus

$$\int xdx\,lx\cdot\sqrt{1-xx} = -\left(\frac{4}{3}-l2\right)\int xdx\sqrt{1-xx};$$

perspicuum autem est esse

$$\int xdx\sqrt{1-xx} = C - \frac{1}{3}(1-xx)^{\frac{3}{2}},$$

qui valor ab $x=0$ ad $x=1$ extensus praebet $\frac{1}{3}$, ita ut habeamus

$$\int xdx\,lx\cdot\sqrt{1-xx}\left[\begin{matrix}\text{ab } x=0\\ \text{ad } x=1\end{matrix}\right] = -\frac{1}{3}\left(\frac{4}{3}-l2\right).$$

III. EVOLUTIO CASUS QUO $m=1$ ET $n=3$

39. Hoc igitur casu erit $X=\frac{1}{\sqrt[3]{(1-x^3)^2}}$, unde theorema generale nobis praebet hanc aequationem

$$\int \frac{x^{p-1}dx\,lx}{\sqrt[3]{(1-x^3)^2}}=\int \frac{x^{p-1}dx}{\sqrt[3]{(1-x^3)^2}}\cdot\int \frac{x^{p-1}(x-1)dx}{1-x^3},$$

ubi postrema formula reducitur ad hanc $-\int \frac{x^{p-1}dx}{xx+x+1}$, ita ut habeamus

$$\int \frac{x^{p-1}dx\,lx}{\sqrt[3]{(1-x^3)^2}}=-\int \frac{x^{p-1}dx}{\sqrt[3]{(1-x^3)^2}}\cdot\int \frac{x^{p-1}dx}{xx+x+1}.$$

Sequentia igitur exempla adiungamus.

EXEMPLUM 1 QUO $p=1$

40. Hoc igitur casu postremus factor evadit $\int \frac{dx}{xx+x+1}$, cuius integrale indefinitum reperitur $\frac{2}{\sqrt{3}}$ A tang. $\frac{x\sqrt{3}}{2+x}$, qui valor posito $x=1$ abit in $\frac{\pi}{3\sqrt{3}}$; quocirca hoc casu habebimus

$$\int \frac{dx\,lx}{\sqrt[3]{(1-x^3)^2}}=-\frac{\pi}{3\sqrt{3}}\int \frac{dx}{\sqrt[3]{(1-x^3)^2}};$$

at vero formula integralis $\int \frac{dx}{\sqrt[3]{(1-x^3)^2}}$ peculiarem quantitatem transcendentem involvit, quam neque per logarithmos neque per arcus circulares explicare licet.

EXEMPLUM 2 QUO $p=2$

41. Hoc igitur casu postremus factor erit $\int \frac{xdx}{1+x+xx}$, qui in has partes resolvatur

$$\frac{1}{2}\int \frac{2xdx+dx}{1+x+xx}-\frac{1}{2}\int \frac{dx}{1+x+xx},$$

ubi partis prioris integrale est

$$\frac{1}{2}l(1+x+xx)=\frac{1}{2}l3 \quad \text{(posito scilicet } x=1),$$

alterius vero partis integrale est $-\frac{1}{2}\cdot\frac{\pi}{3\sqrt{3}}$; quo valore substituto habebimus

$$\int \frac{xdx\,lx}{\sqrt[3]{(1-x^3)^2}}=-\frac{1}{2}\left(l3-\frac{\pi}{3\sqrt{3}}\right)\int \frac{xdx}{\sqrt[3]{(1-x^3)^2}}.$$

Nunc vero istam formulam integralem commode assignare licet per reductionem supra initio indicatam; cum enim hic sit $m=1$ et $n=3$, tum vero sumserimus $p=2$, erit $p=n-m$. Supra autem (§ 15) invenimus hoc casu integrale fore $=\frac{\pi}{n \sin.\frac{m\pi}{n}}$, qui valor nostro casu abit in $\frac{\pi}{3\sin.\frac{\pi}{3}}=\frac{2\pi}{3\sqrt{3}}$. Hoc igitur valore substituto nostram formulam per meras quantitates cognitas exprimere poterimus hoc modo

$$\int \frac{xdx\,lx}{\sqrt[3]{(1-x^3)^2}}\left[\begin{matrix}\text{ab } x=0\\ \text{ad } x=1\end{matrix}\right]=-\frac{\pi}{3\sqrt{3}}\left(l3-\frac{\pi}{3\sqrt{3}}\right).$$

IV. EVOLUTIO CASUS QUO $m=2$ ET $n=3$

42. Hoc igitur casu erit $X=\frac{1}{\sqrt[3]{(1-x^3)}}$, unde theorema generale praebet istam aequationem

$$\int \frac{x^{p-1}dx\,lx}{\sqrt[3]{(1-x^3)}}=\int \frac{x^{p-1}dx}{\sqrt[3]{(1-x^3)}}\cdot\int \frac{x^{p-1}(xx-1)dx}{1-x^3},$$

ubi forma postrema transmutatur in hanc $-\int \frac{x^{p-1}dx(1+x)}{1+x+xx}$; unde fiet

$$\int \frac{x^{p-1}dx\,lx}{\sqrt[3]{(1-x^3)}}=-\int \frac{x^{p-1}dx}{\sqrt[3]{(1-x^3)}}\cdot\int \frac{x^{p-1}dx(1+x)}{1+x+xx},$$

unde sequentia exempla expediamus.

EXEMPLUM 1 QUO $p=1$

43. Hoc ergo casu membrum postremum erit $\int \frac{dx(1+x)}{1+x+xx}$, cuius integrale in has partes distribuatur

$$\frac{1}{2}\int \frac{2xdx+dx}{1+x+xx}+\frac{1}{2}\int \frac{dx}{1+x+xx},$$

unde manifesto pro casu $x=1$ prodit $\frac{1}{2}\left(l3+\frac{\pi}{3\sqrt{3}}\right)$; quamobrem nostra aequatio erit

$$\int\frac{dxlx}{\sqrt[3]{1-x^3}}=-\frac{1}{2}\left(l3+\frac{\pi}{3\sqrt{3}}\right)\int\frac{dx}{\sqrt[3]{(1-x^3)}}.$$

In hac autem formula integrali ob $m=2$ et $n=3$, quia sumsimus $p=1$, erit $p=n-m$; pro hoc ergo casu per § 15 valor istius formulae absolute exprimi poterit eritque $\int\frac{dx}{\sqrt[3]{(1-x^3)}}=\frac{2\pi}{3\sqrt{3}}$; consequenter etiam hoc casu per quantitates absolutas consequimur hanc formam

$$\int\frac{dxlx}{\sqrt[3]{(1-x^3)}}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=-\frac{\pi}{3\sqrt{3}}\left(l3+\frac{\pi}{3\sqrt{3}}\right).$$

44. Quodsi hanc formam cum postrema casus praecedentis, quae itidem absolute prodiit expressa, combinemus, earum summa primo dabit

$$\int\frac{xdxlx}{\sqrt[3]{(1-x^3)^2}}+\int\frac{dxlx}{\sqrt[3]{(1-x^3)}}=-\frac{2\pi l3}{3\sqrt{3}},$$

sin autem posterior a priore subtrahatur, orietur ista aequatio

$$\int\frac{xdxlx}{\sqrt[3]{(1-x^3)^2}}-\int\frac{dxlx}{\sqrt[3]{(1-x^3)}}=\frac{2\pi\pi}{27}.$$

Quoniam hoc modo ad expressiones satis simplices sumus perducti, operae pretium erit ambas aequationes sub alia forma repraesentare, qua binae partes integrales commode in unam coniungi queant; statuamus scilicet

$$\frac{x}{\sqrt[3]{(1-x^3)}}=z,$$

unde fit $\frac{xx}{\sqrt[3]{(1-x^3)^2}}=zz$, sicque prior formula induet hanc speciem $\int\frac{zzdxlx}{x}$, posterior vero istam $\int\frac{zdxlx}{x}$; tum vero habebimus $\frac{x^3}{1-x^3}=z^3$, unde fit $x^3=\frac{z^3}{1+z^3}$ ideoque

$$lx=lz-\frac{1}{3}l(1+z^3)=l\frac{z}{\sqrt[3]{(1+z^3)}}$$

hincque porro

$$\frac{dx}{x}=\frac{dz}{z}-\frac{zzdz}{1+z^3}=\frac{dz}{z(1+z^3)},$$

quare his valoribus adhibitis prior formula integralis evadit $\int\frac{zdz}{1+z^3}l\frac{z}{\sqrt[3]{(1+z^3)}}$, altera vero formula erit $\int\frac{dz}{1+z^3}l\frac{z}{\sqrt[3]{(1+z^3)}}$.

45. Quoniam autem integralia ab $x=0$ ad $x=1$ extendi debent, notandum est casu $x=0$ fieri $z=0$, at vero casu $x=1$ prodire $z=\infty$, ita ut novas istas formas a $z=0$ ad $z=\infty$ extendi oporteat. Quo observato prior harum formularum dabit

$$\int\frac{zdz}{1+z^3}l\frac{z}{\sqrt[3]{(1+z^3)}}\begin{bmatrix}\text{a } z=0\\ \text{ad } z=\infty\end{bmatrix}=-\frac{\pi l3}{3\sqrt{3}}+\frac{\pi\pi}{27},$$

posterior vero

$$\int\frac{dz}{1+z^3}l\frac{z}{\sqrt[3]{(1+z^3)}}\begin{bmatrix}\text{a } z=0\\ \text{ad } z=\infty\end{bmatrix}=-\frac{\pi l3}{3\sqrt{3}}-\frac{\pi\pi}{27}.$$

Hinc igitur summa harum formularum erit

$$\int\frac{dz(1+z)}{1+z^3}l\frac{z}{\sqrt[3]{(1+z^3)}}=-\frac{2\pi l3}{3\sqrt{3}},$$

at vero differentia

$$\int\frac{dz(z-1)}{1+z^3}l\frac{z}{\sqrt[3]{(1+z^3)}}=\frac{2\pi\pi}{27}.$$

46. Hic non inutile erit observasse istum logarithmum $l\frac{z}{\sqrt[3]{(1+z^3)}}$ commode in seriem infinitam satis simplicem converti posse; cum enim sit

$$l\frac{z}{\sqrt[3]{(1+z^3)}}=\frac{1}{3}l\frac{z^3}{1+z^3}=-\frac{1}{3}l\frac{1+z^3}{z^3},$$

erit per seriem

$$l\frac{z}{\sqrt[3]{(1+z^3)}}=-\frac{1}{3}\left(\frac{1}{z^3}-\frac{1}{2z^6}+\frac{1}{3z^9}-\frac{1}{4z^{12}}+\frac{1}{5z^{15}}-\text{etc.}\right);$$

verum ista resolutio nullum usum praestare potest ad integralia haec per series evolvenda, propterea quod potestates ipsius z in denominatoribus occurrunt ideoque singulae partes non ita integrari possunt, ut evanescant posito $z=0$.

EXEMPLUM 2 QUO $p=2$

47. Hoc igitur casu factor postremus evadit $\int\frac{xdx(1+x)}{1+x+xx}$, qui in has duas partes discerpitur

$$\int dx-\int\frac{dx}{1+x+xx},$$

cuius ergo integrale ab $x=0$ ad $x=1$ extensum est $=1-\frac{\pi}{3\sqrt{3}}$. Hinc igitur deducimur ad hanc aequationem

$$\int\frac{xdxlx}{\sqrt[3]{(1-x^3)}}=-\left(1-\frac{\pi}{3\sqrt{3}}\right)\int\frac{xdx}{\sqrt[3]{(1-x^3)}}.$$

Hic autem notandum est istam formulam integralem nullo modo absolute exhiberi posse, sed peculiarem quandam quantitatem transcendentem involvere.

V. EVOLUTIO CASUS QUO $m=2$ ET $n=4$

48. Hoc igitur casu erit $X=\frac{1}{\sqrt{(1-x^4)}}$, unde theorema nostrum generale nobis dabit hanc aequationem

$$\int\frac{x^{p-1}dxlx}{\sqrt{(1-x^4)}}=-\int\frac{x^{p-1}dx}{\sqrt{(1-x^4)}}\cdot\int\frac{x^{p-1}dx}{1+xx};$$

at vero problema particulare prius pro hoc casu praebet

$$\int\frac{x^3dxlx}{\sqrt{(1-x^4)}}=-\frac{1}{2}\int\frac{x^3dx}{1+xx}.$$

Cum autem sit

$$\int\frac{x^3dx}{1+xx}=\frac{1}{2}-\frac{1}{2}l2,$$

erit absolute

$$\int\frac{x^3dxlx}{\sqrt{(1-x^4)}}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=-\frac{1}{4}(1-l2),$$

at vero hic casus congruit cum supra (§ 28) tractato. Si enim hic ponamus $xx=y$, quo facto termini integrationis manent $y=0$ et $y=1$, erit $lx=\frac{1}{2}ly$ et $xdx=\frac{1}{2}dy$; quibus valoribus substitutis nostra aequatio abibit in hanc formam $\frac{1}{4}\int\frac{ydyly}{\sqrt{(1-yy)}}=-\frac{1}{4}(1-l2)$ sive $\int\frac{ydyly}{\sqrt{(1-yy)}}=l2-1$, prorsus ut supra.

49. Alterum vero theorema particulare ad praesentem casum accommodatum dabit

$$\int\frac{xdxlx}{\sqrt{(1-x^4)}}=-\frac{\pi}{4}\int\frac{xdx}{1+xx};$$

est vero

$$\int\frac{xdx}{1+xx}=l\sqrt{1+xx}=\frac{1}{2}l2,$$

ita ut habeamus

$$\int\frac{xdxlx}{\sqrt{(1-x^4)}}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=-\frac{\pi}{8}l2.$$

Quodsi vero hic ut ante statuamus $xx=y$, obtinebitur $\int\frac{dyly}{\sqrt{(1-yy)}}=-\frac{\pi}{2}l2$, qui est casus supra (§ 26) tractatus. His duobus casibus exponens p erat numerus par, unde casus impares evolvi conveniet.

EXEMPLUM 1 QUO $p=1$

50. Hoc igitur casu formula integralis postrema fiet

$$\int\frac{dx}{1+xx}=\text{A tang. } x,$$

ita ut posito $x=1$ prodeat $\frac{\pi}{4}$; tum vero aequatio nostra erit

$$\int\frac{dxlx}{\sqrt{(1-x^4)}}=-\frac{\pi}{4}\int\frac{dx}{\sqrt{(1-x^4)}},$$

integralibus scilicet ab $x=0$ ad $x=1$ extensis; ubi formula $\int\frac{dx}{\sqrt{(1-x^4)}}$ arcum curvae elasticae rectangulae exprimit ideoque absolute exhiberi nequit.

EXEMPLUM 2 QUO $p=3$

51. Hoc ergo casu formula integralis postrema erit

$$\int\frac{xxdx}{1+xx}=\int dx-\int\frac{dx}{1+xx},$$

cuius integrale posito $x=1$ fit $=1-\frac{\pi}{4}$, ita ut nunc aequatio nostra evadat

$$\int\frac{xxdxlx}{\sqrt{(1-x^4)}}=-\left(1-\frac{\pi}{4}\right)\int\frac{xxdx}{\sqrt{(1-x^4)}},$$

quae formula integralis pariter absolute exhiberi nequit; exprimit enim applicatam curvae elasticae rectangulae.

52. Quanquam autem haec duo exempla ad formulas inextricabiles perduxerunt, tamen iam pridem[1]) demonstravi productum horum duorum integralium $\int\frac{dx}{\sqrt{(1-x^4)}}\cdot\int\frac{xxdx}{\sqrt{(1-x^4)}}$ aequari areae circuli, cuius diameter $=1$, sive esse $=\frac{\pi}{4}$; quamobrem binis exemplis coniungendis hoc insigne theorema adipiscimur

$$\int\frac{dxlx}{\sqrt{(1-x^4)}}\cdot\int\frac{xxdxlx}{\sqrt{(1-x^4)}}=\frac{\pi^2}{16}\left(1-\frac{\pi}{4}\right).$$

Facile autem patet innumera alia huiusmodi theoremata ex hoc fonte hauriri posse, quae per se spectata profundissimae indaginis sunt censenda.

1) Vide L. EULERI Commentationes 59 et 122 (indicis ENESTROEMIANI): *Theoremata circa reductionem formularum integralium ad quadraturam circuli*, Miscellanea Berolin. 7, 1743, p. 91, et *De productis ex infinitis factoribus ortis*, Comment. acad. sc. Petrop. 11 (1739), 1750, p. 3; *LEONHARDI EULERI Opera omnia*, series I, vol. 17 et 14. Vide etiam epistolam ab EULERO d. 20. Dec. 1738 ad IOH. BERNOULLI datam, Biblioth. Mathem. 5_3, 1904, p. 285, imprimis p. 291, atque epistolam, quam EULERUS d. 9. Sept. 1741 ad CHR. GOLDBACH misit, *Correspondance math. et phys. publiée par P. H. Fuss*, St. Pétersbourg 1843, t. I, p. 105, imprimis p. 107; *LEONHARDI EULERI Opera omnia*, series III. A. G.

DE VALORE FORMULAE INTEGRALIS
$\int\frac{x^{a-1}dx}{lx}\cdot\frac{(1-x^b)(1-x^c)}{1-x^n}$
A TERMINO $x=0$ USQUE AD $x=1$ EXTENSAE

Commentatio 500 indicis ENESTROEMIANI
Acta academiae scientiarum Petropolitanae 1777: II (1780), p. 29—47

1. Quae non ita pridem de integratione eiusmodi formularum differentialium, in quarum denominatore occurrit lx, in medium attuli, ubi ostendi[1]) valorem huius formulae integralis $\int\frac{x^{a-1}-x^{b-1}}{lx}dx$ ab $x=0$ ad $x=1$ extensae esse $=l\frac{a}{b}$, non solum summa attentione digna, sed etiam quasi novum campum in methodo integrandi aperire sunt visa, propterea quod huiusmodi formularum integratio prorsus singularia artificia postulat, at ex principiis etiamnunc parum cognitis erat deducta. Tunc quidem temporis ista investigatio non admodum late patere videbatur, dum praeter formulam modo allegatam ad paucas alias eam mihi quidem extendere licuit; nunc autem, postquam hoc argumentum accuratius sum perscrutatus, deprehendi formulam multo generaliorem, eam scilicet, quae hic in titulo conspicitur, pari successu expediri posse. Quin etiam methodus, quam hic sum expositurus, etiam ad formulas adhuc generaliores facile extendi potest, unde haud contemnenda incrementa in universam Analysin redundare videntur.

1) Vide § 6 Commentationis 464 (indicis ENESTROEMIANI): *Nova methodus quantitates integrales determinandi*, Novi comment. acad. sc. Petrop. 19 (1774), 1775, p. 66; *LEONHARDI EULERI Opera omnia*, series I, vol. 17, p. 421, imprimis p. 426. A. G.

2. Designemus igitur littera S valorem formulae propositae, quem scilicet induit, si eius integratio a termino $x=0$ usque ad $x=1$ extendatur, ita ut sit

$$S=\int \frac{x^{a-1}dx}{lx}\cdot\frac{(1-x^b)(1-x^c)}{1-x^n}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix},$$

ad quem valorem investigandum ante omnia observari convenit fractionem $\frac{(1-x^b)(1-x^c)}{1-x^n}$ ita esse comparatam, ut posito $x=1$ penitus evanescat. Cum enim in numeratore tam $1-x^b$ quam $1-x^c$ factorem $1-x$ involvat ideoque totus numerator factorem habeat $(1-x)^2$, dum in denominatore tantum factor simplex $1-x$ inest, evidens est posito $x=1$ totam fractionem evanescere debere; id quod etiam inde intelligitur, quod casu $x=1$ tam numerator quam denominator evanescit, unde, si iuxta regulam notissimam tam loco numeratoris, qui evolutus est $1-x^b-x^c+x^{b+c}$, quam loco denominatoris utriusque differentialia scribantur, prodit ista fractio

$$\frac{-bx^{b-1}-cx^{c-1}+(b+c)x^{b+c-1}}{-nx^{n-1}}$$

illi aequalis casu $x=1$; posito autem $x=1$ ista fractio abit in hanc $\frac{-b-c+b+c}{-n}$, quae manifesto est $=0$.

3. Cum numerator fractionis modo consideratae sit $1-x^b-x^c+x^{b+c}$, si is per $1-x^n$ dividatur, ex quaternis terminis orientur quatuor sequentes series geometricae infinitae

I. $1+x^n+x^{2n}+x^{3n}+x^{4n}+x^{5n}+\text{etc.},$

II. $-x^b-x^{n+b}-x^{2n+b}-x^{3n+b}-x^{4n+b}-x^{5n+b}-\text{etc.},$

III. $-x^c-x^{n+c}-x^{2n+c}-x^{3n+c}-x^{4n+c}-x^{5n+c}-\text{etc.},$

IV. $x^{b+c}+x^{n+b+c}+x^{2n+b+c}+x^{3n+b+c}+x^{4n+b+c}+x^{5n+b+c}+\text{etc.}$

Harum igitur serierum singulos terminos duci oportet in formulam $\frac{x^{a-1}dx}{lx}$; tum enim omnium integralia ab $x=0$ ad $x=1$ extensa, si in unam summam colligantur, dabunt valorem quaesitum littera S designatum.

4. Hoc ergo modo totum negotium reducitur ad integrationem talis formulae $\frac{x^m dx}{lx}$ ab $x=0$ ad $x=1$ extendendam. Haec autem formula continet fundamentum principale, unde omnia, quae olim[1]) de hoc argumento sum commentatus, sunt deducta; tum autem ad eius integrale inveniendum usus sum doctrina circa functiones duarum variabilium versante, quam ad praesens institutum non satis commode applicare liceret; quamobrem hic aliam methodum in medium sum allaturus, cuius beneficio ista integratio, qua indigemus, multo facilius et clarius institui poterit et qua simul omnia, quae huc pertinent, haud mediocriter illustrabuntur.

5. Cum sit $lx^m=mlx$, si littera e denotet numerum, cuius logarithmus hyperbolicus unitati aequatur, posito brevitatis gratia $mlx=y$ erit $lx^m=y=yle$ hincque vicissim fiet $x^m=e^y=e^{mlx}$. Cum igitur per seriem notissimam sit

$$e^y=1+\frac{y}{1}+\frac{yy}{1\cdot 2}+\frac{y^3}{1\cdot 2\cdot 3}+\frac{y^4}{1\cdot 2\cdot 3\cdot 4}+\text{etc.},$$

erit pro nostro casu

$$x^m=1+\frac{mlx}{1}+\frac{mm}{1\cdot 2}(lx)^2+\frac{m^3}{1\cdot 2\cdot 3}(lx)^3+\frac{m^4}{1\cdot 2\cdot 3\cdot 4}(lx)^4+\text{etc.};$$

hac igitur serie in usum vocata erit

$$\frac{x^m}{lx}=\frac{1}{lx}+\frac{m}{1}+\frac{mm}{1\cdot 2}lx+\frac{m^3}{1\cdot 2\cdot 3}(lx)^2+\frac{m^4}{1\cdot 2\cdot 3\cdot 4}(lx)^3+\frac{m^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}(lx)^4+\text{etc.}$$

Huius igitur seriei singulos terminos in dx ductos integrari oportet, unde quidem ex termino primo orietur formula $\int\frac{dx}{lx}$, cuius valorem ab $x=0$ ad $x=1$ extensum esse infinitum ostendi[2]), cuius loco hic ubique scribamus characterem Δ; tum vero ex termino secundo oritur integrale $\frac{m}{1}x=m$.

1) Vide Commentationem 463 (indicis ENESTROEMIANI): *De valore formulae integralis* $\int\frac{z^{\lambda-\omega}\pm z^{\lambda+\omega}}{1\pm z^{2\lambda}}\cdot\frac{dz}{z}(lz)^\mu$ *casu, quo post integrationem ponitur* $z=1$, Novi comment. acad. sc. Petrop. 19 (1774), 1775, p. 30; *LEONHARDI EULERI Opera omnia*, series I, vol. 17, p. 384; vide etiam Commentationem 464 supra (nota p. 51) laudatam. A. G.

2) Vide *Institutionum calculi integralis* vol. I, § 228, Petropoli 1768; *LEONHARDI EULERI Opera omnia*, series I, vol. 11, p. 127. A. G.

6. Pro integralibus ex reliquis terminis oriundis ex elementis calculi integralis satis liquet, si integralia ab $x=0$ ad $x=1$ extendantur, fore ut sequitur:

$$\int dx lx = -1, \quad \int dx(lx)^2 = +1\cdot 2, \quad \int dx(lx)^3 = -1\cdot 2\cdot 3,$$

$$\int dx(lx)^4 = +1\cdot 2\cdot 3\cdot 4, \quad \int dx(lx)^5 = -1\cdot 2\cdot 3\cdot 4\cdot 5 \quad \text{etc.};$$

his igitur valoribus substitutis reperiemus fore

$$\int \frac{x^m dx}{lx} = \varDelta + m - \frac{mm}{2} + \frac{m^3}{3} - \frac{m^4}{4} + \frac{m^5}{5} - \frac{m^6}{6} + \frac{m^7}{7} - \text{etc.}$$

Ex doctrina autem logarithmorum constat esse

$$l(1+m) = m - \frac{mm}{2} + \frac{m^3}{3} - \frac{m^4}{4} + \text{etc.},$$

quo valore substituto habebimus

$$\int \frac{x^m dx}{lx} = \varDelta + l(1+m),$$

qui ergo est valor huius formulae integralis a termino $x=0$ ad $x=1$ extensae, quos terminos in sequentibus semper subintelligi oportet, unde eos non amplius commemorabimus.

7. Iste quidem valor integralis insigni incommodo laborare videtur, propterea quod characterem $\varDelta$ implicat, cuius valor non solum est incognitus, sed adeo infinitus; verum quia pro omnibus huiusmodi formulis perpetuo idem manet, ita ut sit

$$\int \frac{x^n dx}{lx} = \varDelta + l(1+n),$$

evidens est, si harum formularum altera ab altera subtrahatur, istum characterem penitus ex calculo egredi ac prodire

$$\int \frac{x^m - x^n}{lx} dx = l\frac{1+m}{1+n},$$

qui est ille ipse casus, ad quem primo initio sum perductus. Quo autem

clarius appareat, quibusnam casibus iste character $\varDelta$ penitus ex calculo sit excessurus, contemplemur hanc formam indefinitam

$$X = Ax^\alpha + Bx^\beta + Cx^\gamma + Dx^\delta + Ex^\varepsilon + \text{etc.}$$

ac per integrale illud inventum erit

$$\int \frac{X dx}{lx} = A\varDelta + B\varDelta + C\varDelta + D\varDelta + \text{etc.}$$
$$+ Al(1+\alpha) + Bl(1+\beta) + Cl(1+\gamma) + Dl(1+\delta) + \text{etc.},$$

Quocirca si coefficientes A, B, C, D etc. ita fuerint comparati, ut sit $A+B+C+D+\text{etc.}=0$, semper istud integrale ita exprimetur

$$\int \frac{X dx}{lx} = Al(1+\alpha) + Bl(1+\beta) + Cl(1+\gamma) + Dl(1+\delta) + \text{etc.},$$

perinde ac si formula canonica fuisset $\int \frac{x^m dx}{lx} = l(1+m)$ reiecto charactere $\varDelta$.

8. Quoties igitur fuerit

$$X = Ax^\alpha + Bx^\beta + Cx^\gamma + Dx^\delta + \text{etc.}$$

existente $A+B+C+D+\text{etc.}=0$, tum integrale $\int \frac{X dx}{lx}$ non amplius charactere $\varDelta$ inquinabitur atque singulas integrationes ita instituere licebit, quasi revera foret

$$\int \frac{x^m dx}{lx} = l(1+m).$$

Cum igitur series $A+B+C+D+\text{etc.}$ exhibeat valorem ipsius X, si ponatur $x=1$, manifestum est istam integrationem perpetuo succedere, si X eiusmodi exprimat functionem ipsius x, ut posito $x=1$ ea in nihilum abeat. Quare cum formula, quam hic tractare suscepimus,

$$X = \frac{x^{a-1}(1-x^b)(1-x^c)}{1-x^n},$$

uti iam observavimus, ad nihilum redigitur posito $x=1$, eius integrationem rite absolvere licebit ope formulae canonicae $\int \frac{x^m dx}{lx} = l(1+m)$, nullo scilicet respectu habito ad characterem $\varDelta$ initio introductum.

9. Quoniam igitur iam supra perducti sumus ad quatuor series infinitas, quas per formulam $\frac{x^{n-1}dx}{lx}$ multiplicari, tum vero integrari oportet, si hanc operationem in singulis terminis instituamus, valor quaesitus S per sequentes quatuor series infinitas expressus reperietur:

$$S = \begin{cases} \text{I.} & la + l(a+n) + l(a+2n) + l(a+3n) + l(a+4n) + \text{etc.} \\ \text{II.} & -l(a+b) - l(a+b+n) - l(a+b+2n) - l(a+b+3n) \\ & \quad - l(a+b+4n) - \text{etc.} \\ \text{III.} & -l(a+c) - l(a+c+n) - l(a+c+2n) - l(a+c+3n) \\ & \quad - l(a+c+4n) - \text{etc.} \\ \text{IV.} & l(a+b+c) + l(a+b+c+n) + l(a+b+c+2n) + l(a+b+c+3n) \\ & \quad + l(a+b+c+4n) + \text{etc.} \end{cases}$$

Hoc igitur modo tota quaestio huc est reducta, ut expressiones finitae investigentur, quae istis logarithmorum seriebus infinitis sint aequales.

10. Cum igitur valor quaesitus S infinitis logarithmis aequalis sit inventus, eum ipsum tanquam logarithmum spectari conveniet; quamobrem statuamus $S = lO$ atque a logarithmis ad numeros regrediendo valor ipsius O sequenti modo per factores exprimi deprehendetur

$$O = \frac{a(a+b+c)}{(a+b)(a+c)} \cdot \frac{(a+n)(a+b+c+n)}{(a+b+n)(a+c+n)} \cdot \frac{(a+2n)(a+b+c+2n)}{(a+b+2n)(a+c+2n)} \cdot \frac{(a+3n)(a+b+c+3n)}{(a+b+3n)(a+c+3n)} \cdot \text{etc.},$$

quam expressionem in membra puncto separata distinximus, quorum quodlibet continet binos factores in numeratore totidemque in denominatore, qui factores in singulis membris ita sunt comparati, ut summa factorum numeratoris semper aequalis sit summae factorum denominatoris. Praeterea vero notetur sumendo i pro numero infinito membrum infinitesimum esse

$$\frac{(a+in)(a+b+c+in)}{(a+b+in)(a+c+in)},$$

quod evolutum praebet

$$\frac{a(a+b+c) + in(2a+b+c) + iinn}{(a+b)(a+c) + in(2a+b+c) + iinn},$$

cuius valor ob partes primas finitas evanescentes manifesto unitati aequatur;

unde intelligitur hanc expressionem valorem finitum ac determinatum esse habituram, et quo plura membra actu in se invicem ducantur, eo propius continuo ad valorem ipsius O appropinquatum iri, quandoquidem membra satis remota continuo minus ab unitate discrepant.

11. Ut nunc in verum valorem litterae O inquiramus, in subsidium vocemus insigne lemma, cuius veritatem iam in *Calculo integrali*[1]) fusius demonstravi, quod ita se habet. Si ponatur

$$P = \int x^{p-1}dx(1-x^n)^{\frac{m-n}{n}} \quad \text{et} \quad Q = \int x^{q-1}dx(1-x^n)^{\frac{m-n}{n}},$$

tum erit

$$\frac{P}{Q} = \frac{(m+p)q}{p(m+q)} \cdot \frac{(m+p+n)(q+n)}{(p+n)(m+q+n)} \cdot \frac{(m+p+2n)(q+2n)}{(p+2n)(m+q+2n)} \cdot \frac{(m+p+3n)(q+3n)}{(p+3n)(m+q+3n)} \cdot \text{etc.},$$

quae expressio pariter ex infinitis membris constat, in quorum singulis tam numerator quam denominator etiam binis factoribus constat, prorsus uti nostra expressio pro O inventa, unde haud difficulter litterae p, q et m ita definiri poterunt, ut prodeat $O = \frac{P}{Q}$, siquidem littera n utrinque eundem significatum retinet; hocque modo valor litterae O saltem ad formulas integrales ordinarias P et Q reducetur. Hic autem probe est recordandum singulas litteras p, q, m et n numeros positivos designare debere, id quod etiam de nostris litteris a, b et c est tenendum, quandoquidem formula nostra canonica $\int \frac{x^m dx}{lx} = l(1+m)$ cum veritate consistere nequit, nisi $1+m$ fuerit numerus positivus, quia alioquin logarithmi numerorum negativorum hinc prodeuntes forent imaginarii.

12. Ad hanc conformitatem $\frac{P}{Q}$ et O constituendam sufficiet membra prima, quae sunt

$$\frac{a(a+b+c)}{(a+b)(a+c)} \quad \text{et} \quad \frac{(m+p)q}{p(m+q)},$$

ad identitatem perduxisse, propterea quod deinceps omnia sequentia membra

1) Vide *Institutionum calculi integralis* vol. I, § 360, Petropoli 1768; *Leonhardi Euleri Opera omnia*, series I, vol. 11, p. 231. Vide etiam lemma 4 Commentationis 59 supra (nota p. 50) laudatae. A. G.

sponte inter se convenient. Ista autem identitas duplici modo obtineri poterit; sumto enim $q = a$ vel statui poterit $m + q = a + b$ vel $m + q = a + c$, ita ut priori modo sit $m = b$, posteriori vero modo $m = c$; at vero tum pro priori modo erit $p = a + c$, unde sponte fiet $m + p = a + b + c$; pro posteriori vero modo, quo $m = c$, sumi debet $p = a + b$, unde denuo sponte fit $m + p = a + b + c$; quamobrem hinc geminos valores pro p et q nanciscemur, unde etiam geminae solutiones orientur, quae sunt:

$$\text{I. Solutio} \begin{cases} P = \int x^{a+c-1}\,dx(1-x^n)^{\frac{b-n}{n}} \\ Q = \int x^{a-1}\,dx(1-x^n)^{\frac{b-n}{n}} \end{cases}$$

$$\text{II. Solutio} \begin{cases} P = \int x^{a+b-1}\,dx(1-x^n)^{\frac{c-n}{n}} \\ Q = \int x^{a-1}\,dx(1-x^n)^{\frac{c-n}{n}} \end{cases}$$

utrinque enim erit $O = \frac{P}{Q}$, et cum sit $S = lO$, erit $S = lP - lQ$ sicque valorem ipsius S per formulas finitas expressum invenimus.

13. Circa valores autem litterarum p et q duos casus imprimis memorabiles notari convenit, quibus eos adeo absolute exhibere licet; alter enim praebet

$$\int x^{n-1}dx(1-x^n)^{\frac{m-n}{n}} = \frac{1}{m},$$

alter vero in hoc consistit, ut sit

$$\int x^{n-m-1}dx(1-x^n)^{\frac{m-n}{n}} = \frac{\pi}{n \sin. \frac{m\pi}{n}},$$

ubi π denotat 180° sive semiperipheriam circuli, cuius radius $= 1$. Quare cum pro nostra solutione priore sit $m = b$, videamus, utrum p et q ad istos valores absolutos reducere liceat. Hoc autem evenit, quando $b = c$ et insuper $a = n - b$, quo casu ambae solutiones inter se congruent, quem ergo casum seorsim evolvisse operae pretium erit.

EVOLUTIO CASUS QUO $c = b$ ET $a = n - b$

14. Hoc igitur casu erit formula proposita

$$S = \int \frac{x^{n-b-1}dx}{lx} \cdot \frac{(1-x^b)^2}{1-x^n};$$

tum vero vidimus esse

$$P = \int x^{n-1}dx(1-x^n)^{\frac{b-n}{n}} = \frac{1}{b}$$

et

$$Q = \int x^{n-b-1}dx(1-x^n)^{\frac{b-n}{n}} = \frac{\pi}{n \sin. \frac{b\pi}{n}},$$

quamobrem, cum sit $S = lP - lQ = l\frac{P}{Q}$, erit his valoribus substitutis

$$S = l\frac{n \sin. \frac{b\pi}{n}}{b\pi},$$

ubi evidens est esse debere $b < n$, unde sequentia exempla considerasse iuvabit.

EXEMPLUM 1 QUO $b = 1$ ET $n = 2$

15. Hoc ergo casu erit $\sin. \frac{b\pi}{n} = 1$ hincque $S = l\frac{2}{\pi}$; quamobrem si formula proposita fuerit

$$S = \int \frac{dx}{lx} \cdot \frac{1-x}{1+x},$$

erit $S = l\frac{2}{\pi}$; at vero valorem ipsius S per logarithmos evolvendo, uti supra fecimus, ob $a = 1$, $b = c = 1$ et $n = 2$ prodibit

$$S = \begin{cases} l1 + l3 + l5 + l7 + l9 + l11 + \text{etc.} \\ -2l2 - 2l4 - 2l6 - 2l8 - 2l10 - \text{etc.} \\ +l3 + l5 + l7 + l9 + l11 + l13 + \text{etc.,} \end{cases}$$

quibus in ordinem redactis erit

$$S = l1 - 2l2 + 2l3 - 2l4 + 2l5 - 2l6 + 2l7 - 2l8 + 2l9 - \text{etc.}$$

16. Vicissim igitur si proponatur ista series logarithmorum

$$s = l1 - l2 + l3 - l4 + l5 - l6 + l7 - \text{etc.},$$

eius summa assignari poterit. Cum enim sit $S - 2s = -l1 = 0$, ob $S = l\frac{2}{\pi}$ erit $s = \frac{1}{2}l\frac{2}{\pi} = l\sqrt{\frac{2}{\pi}}$; sive cum sit $\pi > 2$, erit $s = -l\sqrt{\frac{\pi}{2}}$; ista scilicet summa s erit negativa.

EXEMPLUM 2 QUO $b = 1$ ET $n = 3$

17. Hoc igitur casu, quo $a = 2$, formula integranda proposita erit

$$S = \int \frac{x\,dx}{lx}\cdot\frac{(1-x)^2}{1-x^3} = \int \frac{x\,dx}{lx}\cdot\frac{1-x}{1+x+xx};$$

deinde cum sit $\sin.\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, valor quaesitus erit $S = l\frac{3\sqrt{3}}{2\pi}$; at vero idem valor S per seriem logarithmorum expressus ob $a = 2$, $b = c = 1$ et $n = 3$ erit

$$S = \begin{cases} l2 + l5 + l8 + l11 + l14 + l17 + \text{etc.} \\ -2l3 - 2l6 - 2l9 - 2l12 - 2l15 - \text{etc.} \\ +l4 + l7 + l10 + l13 + l16 + l19 + \text{etc.}; \end{cases}$$

sicque ergo erit

$$S = l2 - 2l3 + l4 + l5 - 2l6 + l7 + l8 - 2l9 + l10 + l11 - 2l12 + l13 + l14 - \text{etc.},$$

cuius ergo seriei satis regularis summa est $S = l\frac{3\sqrt{3}}{2\pi}$.

EXEMPLUM 3 QUO $b = 2$ ET $n = 3$

18. Hoc igitur casu erit $a = 1$ et formula nostra integralis fiet

$$S = \int \frac{dx}{lx}\cdot\frac{(1-xx)^2}{1-x^3} = \int \frac{dx}{lx}\cdot\frac{(1-x)(1+x)^2}{1+x+xx},$$

cuius ergo valor erit $S = l\frac{3\sqrt{3}}{4\pi}$; at vero idem valor S per seriem logarithmorum expressus ob $a = 1$, $b = c = 2$ et $n = 3$ erit

$$S = \begin{cases} l1 + l4 + l7 + l10 + l13 + \text{etc.} \\ -2l3 - 2l6 - 2l9 - 2l12 - \text{etc.} \\ +l5 + l8 + l11 + l14 + \text{etc.}; \end{cases}$$

sicque ergo erit

$$S = l1 - 2l3 + l4 + l5 - 2l6 + l7 + l8 - 2l9 + l10 + l11 - \text{etc.},$$

cuius seriei summa est $S = l\frac{3\sqrt{3}}{4\pi}$; unde, cum sit $S = l\frac{P}{Q}$, erit

$$\frac{P}{Q} = \frac{1\cdot 5}{3\cdot 3}\cdot\frac{4\cdot 8}{6\cdot 6}\cdot\frac{7\cdot 11}{9\cdot 9}\cdot\frac{10\cdot 14}{12\cdot 12}\cdot\text{etc.},$$

cuius ergo valor erit $= \frac{3\sqrt{3}}{4\pi}$.

EXEMPLUM 4 QUO $b = 1$ ET $n = 4$

19. Hinc ergo ob $a = 3$ formula nostra integralis erit

$$S = \int \frac{xx\,dx}{lx}\cdot\frac{(1-x)^2}{1-x^4} = \int \frac{xx\,dx}{lx}\cdot\frac{1-x}{(1+x)(1+xx)},$$

cuius ergo valor erit $= l\frac{2\sqrt{2}}{\pi}$; at vero idem valor per seriem logarithmorum ob $a = 3$, $b = c = 1$ et $n = 4$ hoc modo exprimetur

$$S = \begin{cases} l3 + l7 + l11 + l15 + \text{etc.} \\ -2l4 - 2l8 - 2l12 - \text{etc.} \\ +l5 + l9 + l13 + \text{etc.}; \end{cases}$$

hincque ergo erit

$$\frac{P}{Q} = \frac{3\cdot 5}{4\cdot 4}\cdot\frac{7\cdot 9}{8\cdot 8}\cdot\frac{11\cdot 13}{12\cdot 12}\cdot\text{etc.} = \frac{2\sqrt{2}}{\pi}.$$

EXEMPLUM 5 QUO $b = 3$ ET $n = 4$

20. Hoc ergo casu erit $a = 1$ et formula nostra integralis fiet

$$S = \int \frac{dx}{lx}\cdot\frac{(1-x^3)^2}{1-x^4},$$

cuius valor erit $S = l\frac{2\sqrt{2}}{3\pi}$, qui etiam hoc modo per seriem logarithmorum exprimetur

$$S = \begin{cases} l1 + l5 + l9 + l13 + l17 + \text{etc.} \\ -2l4 - 2l8 - 2l12 - 2l16 - \text{etc.} \\ +l7 + l11 + l15 + l19 + \text{etc.}; \end{cases}$$

hinc ergo fiet

$$\frac{P}{Q}=\frac{1\cdot 7}{4\cdot 4}\cdot\frac{5\cdot 11}{8\cdot 8}\cdot\frac{9\cdot 15}{12\cdot 12}\cdot\frac{13\cdot 19}{16\cdot 16}\cdot\text{etc.}=\frac{2\sqrt{2}}{3\pi}.$$

21. Praeter hos autem casus, quibus ambas formulas P et Q simul integrationem admittere observavimus, pro certo affirmare licet nullos alios insuper dari, quibus hoc eveniat. Interim tamen dantur innumerabiles alii casus, quibus valor nostrae formulae integralis S absolute sine formulis integralibus assignari potest, etiamsi neutra formularum P et Q seorsim integrari queat; qui casus cum per se sint notatu dignissimi, iis investigandis sequens problema destinemus.

PROBLEMA

22. *Investigare casus, quibus formulae integralis propositae valorem S absolute sine formulis integralibus exprimere licet.*

SOLUTIO

Totum ergo negotium huc redit, ut eiusmodi relationes inter exponentes a, b, c et n eruantur, quibus fractio supra adhibita $\frac{P}{Q}$ absolute exprimi queat, quamvis neutra harum formularum seorsim integrationem admittat; tum enim formulae propositae valor quaesitus erit $S=l\frac{P}{Q}$. Verum istam fractionem $\frac{P}{Q}$ vidimus designare istud productum in infinitum excurrens

$$\frac{P}{Q}=\frac{a(a+b+c)}{(a+b)(a+c)}\cdot\frac{(a+n)(a+b+c+n)}{(a+b+n)(a+c+n)}\cdot\frac{(a+2n)(a+b+c+2n)}{(a+b+2n)(a+c+2n)}\cdot\text{etc.}$$

23. Nunc vero meminisse iuvabit tam sinus quam cosinus angulorum per huiusmodi producta infinita exprimi solere; cum enim sit

$$\sin.\frac{p\pi}{2r}=\frac{p\pi}{2r}\cdot\frac{4rr-pp}{4rr}\cdot\frac{16rr-pp}{16rr}\cdot\frac{36rr-pp}{36rr}\cdot\text{etc.},$$

erit duabus huiusmodi expressionibus combinandis

$$\frac{\sin.\frac{p\pi}{2r}}{\sin.\frac{q\pi}{2r}}=\frac{p}{q}\cdot\frac{4rr-pp}{4rr-qq}\cdot\frac{16rr-pp}{16rr-qq}\cdot\frac{36rr-pp}{36rr-qq}\cdot\frac{64rr-pp}{64rr-qq}\cdot\text{etc.}$$

Quare si superior expressio pro $\frac{P}{Q}$ inventa ad hanc formam revocari queat, tum utique erit

$$S=l\sin.\frac{p\pi}{2r}-l\sin.\frac{q\pi}{2r}.$$

Quo autem ista reductio facilius succedat, posteriorem expressionem hac forma repraesentemus

$$\frac{\sin.\frac{p\pi}{2r}}{\sin.\frac{q\pi}{2r}}=\frac{p(2r-p)}{q(2r-q)}\cdot\frac{(2r+p)(4r-p)}{(2r+q)(4r-q)}\cdot\frac{(4r+p)(6r-p)}{(4r+q)(6r-q)}\cdot\text{etc.},$$

cuius expressionis membra manifesto ita progrediuntur, ut singuli factores tam numeratorum quam denominatorum continuo eodem incremento $2r$ augeantur. Quare cum in expressione $\frac{P}{Q}$ singuli factores capiant incrementum n, statui debebit $n=2r$, quo notato sufficiet prima membra ad conformitatem redigere, id quod eveniet sumendo

$$a=p,\quad a+b+c=2r-p,\quad a+b=q,\quad a+c=2r-q,$$

unde singulae litterae colliguntur

$$1^{\circ}.\ a=p,\qquad 2^{\circ}.\ b=q-p,\qquad 3^{\circ}.\ c=2r-p-q$$

existente $n=2r$. Hinc autem operae pretium erit notasse fore

$$2a+b+c=2r=n,$$

ita ut formula nostra generalis ad casum hunc semper accommodari queat, si modo fuerit $n=2a+b+c$; tum enim fit $p=a$, $q=a+b$ et $2r=2a+b+c$.

24. Quodsi vero formula nostra generalis evolvatur ac loco n scribatur iste valor $2a+b+c$, ea induet hanc formam

$$S=\int\frac{dx}{xlx}\cdot\frac{x^a-x^{a+b}-x^{a+c}+x^{a+b+c}}{1-x^{2a+b+c}},$$

cuius ergo valor, si loco p, q et r modo inventi valores scribantur, erit

$$S=l\frac{P}{Q}=l\sin.\frac{a\pi}{2a+b+c}-l\sin.\frac{(a+b)\pi}{2a+b+c},$$

quae formula utique ita est absoluta, ut nullam amplius formulam integralem involvat, prorsus uti desideratur. Patet igitur casum ante tractatum in hoc casu non contineri; cum enim in illo fuisset $a=n-b$ et $c=b$, hinc fiet $2a+b+c=2n$, cum praesenti casu sit $2a+b+c=n$.

25. Quodsi iam in hac expressione litteras p, q et r in calculum introducamus, formula nostra integralis ad hanc speciem reducetur

$$S=\int\frac{dx}{xlx}\cdot\frac{x^p-x^q-x^{2r-q}+x^{2r-p}}{1-x^{2r}},$$

cuius igitur valor ab $x=0$ ad $x=1$ extensus erit

$$S=l\sin.\frac{p\pi}{2r}-l\sin.\frac{q\pi}{2r},$$

ubi manifestum est hanc expressionem eandem manere, etiamsi loco p scribatur $2r-p$, loco q vero $2r-q$, propterea quod

$$\sin.\frac{(2r-p)\pi}{2r}=\sin.\frac{p\pi}{2r}\quad\text{et}\quad\sin.\frac{(2r-q)\pi}{2r}=\sin.\frac{q\pi}{2r};$$

at vero ipsa formula integralis facta sive alterutra substitutione sive utraque coniunctim prorsus non variatur.

26. Quodsi loco p et q scribamus $r-p$ et $r-q$, illi sinus transmutantur in cosinus; tum autem ipsa formula integralis erit

$$S=\int\frac{dx}{xlx}\cdot\frac{x^{r-p}-x^{r-q}-x^{r+q}+x^{r+p}}{1-x^{2r}},$$

cuius valor nunc erit

$$=l\cos.\frac{p\pi}{2r}-l\cos.\frac{q\pi}{2r},$$

ubi iterum manifestum est nullam mutationem oriri, sive litterae p et q valores habeant positivos sive negativos.

COROLLARIUM 1

27. Cum igitur his casibus neutra formularum integralium P et Q integrationem actu admittat, eo magis notatu dignum hic occurrit, quod nihilo-

minus valor fractionis $\frac{P}{Q}$ absolute exprimi possit, cum per sinus sit

$$\frac{P}{Q}=\frac{\sin.\frac{p\pi}{2r}}{\sin.\frac{q\pi}{2r}}.$$

Cum igitur hoc casu sit $a=p$, $b=q-p$, $c=2r-p-q$ et $n=2r$, valores integrales pro P et Q supra (§ 12) exhibiti in sequentes abibunt formas

$$P=\int\frac{x^{2r-q-1}dx}{(1-x^{2r})^{1+\frac{p-q}{2r}}}\quad\text{et}\quad Q=\int\frac{x^{p-1}dx}{(1-x^{2r})^{1+\frac{p-q}{2r}}}.$$

Quicunque ergo valores exponentibus tribuantur, semper erit $\frac{P}{Q}=\frac{\sin.\frac{p\pi}{2r}}{\sin.\frac{q\pi}{2r}}$.

COROLLARIUM 2

28. Quoniam hic loco p et q scribere licet $2r-p$ et $2r-q$, hinc quaternas formulas integrales exhibere possumus, ita ut pro singulis sit

$$\frac{P}{Q}=\frac{\sin.\frac{p\pi}{2r}}{\sin.\frac{q\pi}{2r}},$$

qui quaterni valores ita se habebunt.

$$\text{I.}\quad P=\int\frac{x^{2r-q-1}dx}{(1-x^{2r})^{1+\frac{p-q}{2r}}}\quad\text{et}\quad Q=\int\frac{x^{p-1}dx}{(1-x^{2r})^{1+\frac{p-q}{2r}}},$$

$$\text{II.}\quad P=\int\frac{x^{2r-q-1}dx}{(1-x^{2r})^{2-\frac{p+q}{2r}}}\quad\text{et}\quad Q=\int\frac{x^{2r-p-1}dx}{(1-x^{2r})^{2-\frac{p+q}{2r}}},$$

$$\text{III.}\quad P=\int\frac{x^{q-1}dx}{(1-x^{2r})^{\frac{p+q}{2r}}}\quad\text{et}\quad Q=\int\frac{x^{p-1}dx}{(1-x^{2r})^{\frac{p+q}{2r}}},$$

$$\text{IV.}\quad P=\int\frac{x^{q-1}dx}{(1-x^{2r})^{1+\frac{q-p}{2r}}}\quad\text{et}\quad Q=\int\frac{x^{2r-p-1}dx}{(1-x^{2r})^{1+\frac{q-p}{2r}}}.$$

COROLLARIUM 3

29. Quodsi hic loco p et q scribamus $r-p$ et $r-q$, quo pacto sinus in cosinus transmutantur, quaternas impetrabimus formulas integrales pro P et Q ita comparatas, ut pro omnibus sit

$$\frac{P}{Q} = \frac{\cos. \frac{p\pi}{2r}}{\cos. \frac{q\pi}{2r}},$$

qui quaterni valores erunt

$$\text{I.} \quad P = \int \frac{x^{r+q-1}\,dx}{(1-x^{2r})^{1+\frac{q-p}{2r}}} \quad \text{et} \quad Q = \int \frac{x^{r-p-1}\,dx}{(1-x^{2r})^{1+\frac{q-p}{2r}}},$$

$$\text{II.} \quad P = \int \frac{x^{r+q-1}\,dx}{(1-x^{2r})^{1+\frac{p+q}{2r}}} \quad \text{et} \quad Q = \int \frac{x^{r+p-1}\,dx}{(1-x^{2r})^{1+\frac{p+q}{2r}}},$$

$$\text{III.} \quad P = \int \frac{x^{r-q-1}\,dx}{(1-x^{2r})^{1-\frac{p+q}{2r}}} \quad \text{et} \quad Q = \int \frac{x^{r-p-1}\,dx}{(1-x^{2r})^{1-\frac{p+q}{2r}}},$$

$$\text{IV.} \quad P = \int \frac{x^{r-q-1}\,dx}{(1-x^{2r})^{1+\frac{p-q}{2r}}} \quad \text{et} \quad Q = \int \frac{x^{r+p-1}\,dx}{(1-x^{2r})^{1+\frac{p-q}{2r}}},$$

quae quaternae formulae tam pulchre inter se conspirant, ut aliter non discrepent nisi ratione signorum, quibus litterae p et q sunt affectae.

COROLLARIUM 4

30. Hae autem formulae prorsus sunt diversae ab illis, quas supra in evolutione § 14 habuimus, ubi erat $\frac{P}{Q} = \frac{n}{b\pi} \sin. \frac{b\pi}{n}$; quod discrimen quo clarius ob oculos ponatur, loco b et n scribamus p et $2r$, ut fiat $\frac{P}{Q} = \frac{2r}{p\pi} \sin. \frac{p\pi}{2r}$; tum autem fit

$$P = \int \frac{x^{2r-1}\,dx}{(1-x^{2r})^{1-\frac{p}{2r}}} \quad \text{et} \quad Q = \int \frac{x^{2r-p-1}\,dx}{(1-x^{2r})^{1-\frac{p}{2r}}},$$

quae formulae actu integrationem admittent, dum colligitur

$$P = \frac{1}{p} \quad \text{et} \quad Q = \frac{\pi}{2r \sin. \frac{p\pi}{2r}}.$$

COROLLARIUM 5

31. Quodsi in formulis penultimi corollarii capiamus $q=0$, ut fiat $\frac{P}{Q} = \cos. \frac{p\pi}{2r}$, binas tantum pro hoc casu diversas formulas pro P et Q nanciscemur, quae sunt

$$\text{I.} \quad P = \int \frac{x^{r-1}\,dx}{(1-x^{2r})^{1-\frac{p}{2r}}} \quad \text{et} \quad Q = \int \frac{x^{r-p-1}\,dx}{(1-x^{2r})^{1-\frac{p}{2r}}},$$

$$\text{II.} \quad P = \int \frac{x^{r-1}\,dx}{(1-x^{2r})^{1+\frac{p}{2r}}} \quad \text{et} \quad Q = \int \frac{x^{r+p-1}\,dx}{(1-x^{2r})^{1+\frac{p}{2r}}}.$$

Sin autem in formulis antepenultimi corollarii statuamus $q=r$, ut prodeat $\frac{P}{Q} = \sin. \frac{p\pi}{2r}$, iterum prodibunt binae formulae pro P et Q, quae sunt

$$\text{I.} \quad P = \int \frac{x^{r-1}\,dx}{(1-x^{2r})^{\frac{1}{2}+\frac{p}{2r}}} \quad \text{et} \quad Q = \int \frac{x^{p-1}\,dx}{(1-x^{2r})^{\frac{1}{2}+\frac{p}{2r}}},$$

$$\text{II.} \quad P = \int \frac{x^{r-1}\,dx}{(1-x^{2r})^{\frac{3}{2}-\frac{p}{2r}}} \quad \text{et} \quad Q = \int \frac{x^{2r-p-1}\,dx}{(1-x^{2r})^{\frac{3}{2}-\frac{p}{2r}}}.$$

COROLLARIUM 6

32. Quodsi in formulis Corollarii 2 statuamus $q = r - p$, ut fiat $\sin. \frac{q\pi}{2r} = \cos. \frac{p\pi}{2r}$, habebitur $\frac{P}{Q} = \text{tang.} \frac{p\pi}{2r}$ et quaterni valores pro formulis P et Q erunt

$$\text{I.} \quad P = \int \frac{x^{r+p-1}\,dx}{(1-x^{2r})^{\frac{1}{2}+\frac{p}{r}}} \quad \text{et} \quad Q = \int \frac{x^{p-1}\,dx}{(1-x^{2r})^{\frac{1}{2}+\frac{p}{r}}},$$

$$\text{II.} \quad P = \int \frac{x^{r+p-1}\,dx}{(1-x^{2r})^{\frac{3}{2}}} \quad \text{et} \quad Q = \int \frac{x^{2r-p-1}\,dx}{(1-x^{2r})^{\frac{3}{2}}},$$

$$\text{III.} \quad P = \int \frac{x^{r-p-1}\,dx}{(1-x^{2r})^{\frac{1}{2}}} \quad \text{et} \quad Q = \int \frac{x^{p-1}\,dx}{(1-x^{2r})^{\frac{1}{2}}},$$

$$\text{IV.} \quad P = \int \frac{x^{r-p-1}\,dx}{(1-x^{2r})^{\frac{3}{2}-\frac{p}{r}}} \quad \text{et} \quad Q = \int \frac{x^{2r-p-1}\,dx}{(1-x^{2r})^{\frac{3}{2}-\frac{p}{r}}}.$$

COROLLARIUM 7

33. Plurimum autem etiam intererit nosse ipsam formulam integralem S pro his casibus, quibus fit simpliciter vel $\frac{P}{Q} = \cos.\frac{p\pi}{2r}$ vel $\frac{P}{Q} = \sin.\frac{p\pi}{2r}$ vel $\frac{P}{Q} = \text{tang.}\frac{p\pi}{2r}$, fieri pro primo

$$S = \int \frac{dx}{xlx} \cdot \frac{x^{r-p} - 2x^r + x^{r+p}}{1 - x^{2r}} = l\cos.\frac{p\pi}{2r},$$

pro secundo casu

$$S = \int \frac{dx}{xlx} \cdot \frac{x^p - 2x^r + x^{2r-p}}{1 - x^{2r}} = l\sin.\frac{p\pi}{2r},$$

pro tertio autem casu

$$S = \int \frac{dx}{xlx} \cdot \frac{x^p - x^{r-p} - x^{r+p} + x^{2r-p}}{1 - x^{2r}} = l\,\text{tang.}\frac{p\pi}{2r},$$

quae postrema formula reducitur ad hanc

$$\int \frac{dx}{xlx} \cdot \frac{x^p - x^{r-p}}{1 + x^r} = l\,\text{tang.}\frac{p\pi}{2r},$$

quae est eadem integratio, quam non ita pridem ex diversissimis principiis elicueram.[1])

SCHOLION

34. Postremo autem circa omnes has varias formulas integrales probe notetur eas, in quibus exponens denominatoris reperitur unitate maior, utpote incongruas reiiciendas esse, propterea quod earum valores integrati posito $x = 1$ evadant infiniti, quod quidem, cum in utraque formula P et Q simul eveniat, non impedit, quominus fractio $\frac{P}{Q}$ assignatum obtineat valorem; sed quia eum hinc definire non licet, etiam istiusmodi formulae optatum usum non praestant. Commode autem evenit, ut plures formulae adsint, ex quibus valorem verum derivare liceat.

1) Vide § 42 Commentationis 463 supra (nota 1 p. 53) laudatae. A. G.

THEOREMES ANALYTIQUES

EXTRAITS DE DIFFERENTES LETTRES DE M. EULER A M. LE MARQUIS DE CONDORCET

Commentatio 521 indicis ENESTROEMIANI
Mémoires de l'académie des sciences de Paris 1778 (1781), p. 603—614
Résumé ibidem p. 42

RESUME

Ces deux théorèmes ont été proposés et démontrés par M. EULER; l'un donne pour une valeur déterminée l'expression de l'intégrale de plusieurs fonctions dont on ne peut connoître l'intégrale pour une valeur quelconque. Quoiqu'il ne soit ici question que de deux fonctions, il est aisé de voir que ce ne sont que des exemples d'une méthode plus générale, qui embrasse une classe de fonctions très-étendue. Le second théorème donne en un seul produit de facteurs, l'expression de la somme des carrés des coëfficiens de la formule du binome élevé à une puissance quelconque. La démonstration de M. EULER est beaucoup plus générale que l'énoncé du théorème, puisqu'elle donne une expression semblable pour la somme des coëfficiens d'une puissance du binome multipliée par les coëfficiens successifs d'une autre puissance quelconque.

On a joint à ce mémoire une autre démonstration des mêmes théorèmes, trouvée avant de connoître celle de M. EULER; l'auteur de cette démonstration a espéré qu'on ne le soupçonneroit pas de la présomption d'avoir voulu comparer son travail à celui d'un grand homme, dont il s'honore d'être l'admirateur et le disciple.

LETTRE DU $\frac{3}{14}$ NOVEMBRE 1775

L'intégrale de cette formule,

$$\frac{x^m - x^n}{lx} \cdot \frac{\partial x}{x},$$

prise depuis $x = 0$ *jusqu'à* $x = 1$, *est* $= l\frac{m}{n}$.

L'intégrale de cette formule,

$$\frac{x^{m-1}\partial x}{(1+x^n)lx},$$

prise depuis $x=0$ *jusqu'à* $x=\infty$, *est* $=l.\,\mathrm{tang.}\frac{m\pi}{n}$, *où* π *marque l'angle de* 180 *degrés.*

LETTRE DU 2 FEVRIER 1776

DEMONSTRATION DES DEUX THEOREMES PRECEDENS

Soit Q une fonction quelconque des deux variables x et y, et qu'on cherche la quantité Z, telle que

$$\left(\frac{\partial\partial z}{\partial x\partial y}\right)=Q,$$

où il s'agit d'une double intégration; l'une, où la seule x est prise pour variable, et l'autre, où la seule y varie; la première devra être étendue depuis $x=0$ jusqu'à $x=1$ et l'autre depuis $y=0$ jusqu'à $y=n$; par la nature de telles formules, on aura donc d'une double manière ou

$$Z=\int\partial x\int Q\partial y,$$

ou

$$Z=\int\partial y\int Q\partial x,$$

Maintenant, qu'on suppose

$$Q=x^y,$$

et on aura

$$\int Q\partial y=\frac{x^y}{lx}-\frac{1}{lx},$$

afin que cette intégrale évanouisse lorsque $y=0$. Soit donc à présent $y=n$, et nous aurons

$$\int Q\partial y=\frac{x^n-1}{lx}$$

et partant

$$Z=\int\frac{(x^n-1)\partial x}{lx};$$

ensuite, nous aurons

$$\int Q\partial x=\frac{x^{y+1}}{y+1},$$

qui évanouit lorsque $x=0$; posant donc $x=1$, il en résulte

$$\int Q\partial x=\frac{1}{y+1},$$

et de là,

$$Z=\int\frac{\partial y}{y+1}=l(y+1),$$

(expression qui disparoît lorsque $y=0$). Qu'on fasse donc $y=n$, et l'on aura $Z=l(n+1)$; par conséquent, il est certain que cette intégrale $\int\frac{\partial x(x^n-1)}{lx}$, prise depuis $x=0$ jusqu'à $x=1$, est $l(n+1)$.

Pour l'autre formule intégrale plus compliquée que je vous avois communiquée, j'avois supposé

$$Q=\frac{x^{m-y}+x^{m+y}}{(1+x^{2m})x};$$

de là, prenant d'abord x constante à cause de

$$\int x^{m-y}\partial y=-\frac{x^{m-y}}{lx}\quad\text{et de}\quad\int x^{m+y}\partial y=\frac{x^{m+y}}{lx},$$

on aura

$$\int Q\partial y=\frac{x^{m+y}-x^{m-y}}{(1+x^{2m})xlx},$$

ce qui devient $=0$ posant $y=0$. Faisant donc $y=n$, on aura

$$\int Q\partial y=\frac{x^{m+n}-x^{m-n}}{(1+x^{2m})xlx}$$

et partant

$$Z=\int\frac{(x^{m+n}-x^{m-n})\partial x}{(1+x^{2m})xlx}.$$

L'autre intégration donne d'abord

$$\int Q\partial x=\int\frac{(x^{m-y}+x^{m+y})\partial x}{(1+x^{2m})x},$$

dont l'intégrale doit être étendue depuis $x=0$ jusqu'à $x=1$; or, pour ce cas, j'ai démontré autrefois[1]) que cette intégrale se réduit à cette forme,

$$\frac{\pi}{2m\cos.\frac{\pi y}{2m}},$$

d'où nous tirons

$$Z = \int \frac{\pi\partial y}{2m\cos.\frac{\pi y}{2m}}.$$

Pour cette forme, posons $\frac{\pi y}{2m} = \varphi$, pour avoir

$$Z = \int \frac{\partial\varphi}{\cos.\varphi} = \int \frac{\partial\varphi}{\sin.(90^d + \varphi)},$$

dont l'intégrale est $l.\operatorname{tang.}\left(45^d + \frac{1}{2}\varphi\right)$ et partant, $Z = l.\operatorname{tang.}\left(45^d + \frac{\pi y}{4m}\right)$ qui en effet évanouit prenant $y = 0$. Faisons donc $y = n$, et nous aurons

$$Z = l.\operatorname{tang.}\left(45^d + \frac{\pi n}{4m}\right);$$

d'où il est clair que sous les conditions présentes, on aura

$$\int \frac{(x^{m+n-1} - x^{m-n-1})\partial x}{(1+x^{2m})lx} \begin{Bmatrix}\text{depuis } x = 0\\ \text{jusqu'à } x = 1\end{Bmatrix} = l.\operatorname{tang.}\left(45^d + \frac{\pi n}{4m}\right).$$

Par ces deux exemples, on verra aisément que cette spéculation mérite toute l'attention des géomètres. La première idée qui m'a conduit à cette recherche, étoit tirée d'un principe entièrement différent, que voici. J'avois considéré cette formule

$$\int \frac{(x-1)\partial x}{lx},$$

où, au lieu de lx, j'ai écrit cette valeur $\frac{x^\omega - 1}{\omega}$, en supposant ω infiniment petit, ou bien

$$lx = i(x^{\frac{1}{i}} - 1),$$

1) Voir § 3 du mémoire 463 (suivant *l'Index* d'Eneström): *De valore formulae integralis* $\int \frac{z^{\lambda-\omega} \pm z^{\lambda+\omega}}{1 \pm z^{2\lambda}} \cdot \frac{dz}{z}(lz)^\mu$ *casu quo post integrationem ponitur* $z = 1$, Novi comment. acad. sc. Petrop. 19 (1774), 1775, p. 30; *Leonhardi Euleri Opera omnia*, series I, vol. 17. A. G.

en prenant pour i un nombre infiniment grand. Qu'on pose à présent $x^{\frac{1}{i}} = z$, ou bien $x = z^i$, où il faut remarquer que les termes de l'intégration $x = 0$ et $x = 1$ se réduisent à $z = 0$ et à $z = 1$; cette valeur étant substituée, transforme notre formule en celle-ci, $\frac{(z^i - 1)z^{i-1}\partial z}{z-1}$; or, la fraction $\frac{z^i - 1}{z - 1}$ ou bien $\frac{1 - z^i}{1 - z}$, se réduit à la série

$$1 + z + z^2 + z^3 + \cdots + z^{i-1},$$

qui étant multipliée [par $z^{i-1}\partial z$] et intégrée, donne

$$\frac{z^i}{i} + \frac{z^{i+1}}{i+1} + \frac{z^{i+2}}{i+2} + \frac{z^{i+3}}{i+3} + \cdots + \frac{z^{2i-1}}{2i-1},$$

et, posant $z = 1$, la valeur cherchée sera

$$\frac{1}{i} + \frac{1}{i+1} + \frac{1}{i+2} + \frac{1}{i+3} + \frac{1}{i+4} + \cdots + \frac{1}{2i-1},$$

dont la valeur est $l2$, de sorte que

$$\int \frac{(x-1)\partial x}{lx} \begin{Bmatrix}\text{depuis } x = 0\\ \text{jusqu'à } x = 1\end{Bmatrix} \text{ est } = l2.$$

Pour démontrer la somme de la série trouvée, qu'on appellera A, on n'a qu'à remarquer que

$$\begin{aligned} A = 1 &+ \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{i-1} \\ &+ \frac{1}{i} + \frac{1}{i+1} + \frac{1}{i+2} + \frac{1}{i+3} + \cdots + \frac{1}{2i-1} \\ &- \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{1-i}\right), \end{aligned}$$

où, parce que la série supérieure contient deux fois plus de termes que l'inférieure, on n'a qu'à soustraire chaque terme de la dernière de la supérieure alternativement, et l'on aura

$$A = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \cdots$$
$$+ \frac{1}{i-1} + \frac{1}{i} + \frac{1}{i+1} + \cdots + \frac{1}{2i-1}$$
$$- 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \text{etc.}$$

ou bien

$$A = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \text{etc.} = l2.$$

AUTRE THEOREME

En prenant les lettres α, β, γ, δ, etc. pour marquer les coëfficiens d'un binome élevé à l'exposant n, de sorte que

$$(1+x)^n = 1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \text{etc.},$$

on aura toujours

$$1 + \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \text{etc.} = \frac{2}{1} \cdot \frac{6}{2} \cdot \frac{10}{3} \cdot \frac{14}{4} \cdots \frac{4n-2}{n}.$$

Par exemple, si $n = 6$, on aura $\alpha = 6$, $\beta = 15$, $\gamma = 20$, $\delta = 15$, $\varepsilon = 6$, $\zeta = 1$ et les suivans $= 0$; et partant, on aura

$$1 + 6^2 + 15^2 + 20^2 + 15^2 + 6^2 + 1 = \frac{2}{1} \cdot \frac{6}{2} \cdot \frac{10}{3} \cdot \frac{14}{4} \cdot \frac{18}{5} \cdot \frac{22}{6},$$

dont la démonstration directe me paroît extrêmement difficile.

LETTRE DU $\frac{12}{23}$ SEPTEMBRE 1776

DEMONSTRATION DE CE THEOREME

. . . en supposant

$$(1+z)^n = 1 + \left(\frac{n}{1}\right) z + \left(\frac{n}{2}\right) z^2 + \left(\frac{n}{3}\right) z^3 + \text{etc.};$$

d'où l'on voit que $\left(\frac{n}{0}\right) = 1$, aussi bien que $\left(\frac{n}{n}\right)$, et de là, il s'ensuit que $\left(\frac{n}{p}\right) = \left(\frac{n}{n-p}\right)$; outre cela, il est clair que la valeur de la formule $\left(\frac{n}{p}\right)$ est

toujours égale à zéro, tant dans les cas où p est un nombre négatif, que dans ceux où il est un nombre plus grand que n, ce qui s'entend des nombres entiers; ensuite, on sait que la valeur développée de ce caractère $\left(\frac{n}{p}\right)$ est

$$= \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdots \frac{n-p+1}{p}.$$

Cela posé, si nous passons aux coëfficiens de la puissance suivante $(1+z)^{n+1}$, on sait qu'on aura

$$\left(\frac{n+1}{p+1}\right) = \left(\frac{n}{p}\right) + \left(\frac{n}{p+1}\right);$$

de sorte que réciproquement

$$\left(\frac{n}{p+1}\right) + \left(\frac{n}{p+2}\right) = \left(\frac{n+1}{p+2}\right);$$

ajoutons ces deux équations ensemble, et nous aurons

$$\left(\frac{n}{p}\right) + 2\left(\frac{n}{p+1}\right) + \left(\frac{n}{p+2}\right) = \left(\frac{n+1}{p+1}\right) + \left(\frac{n+1}{p+2}\right) = \left(\frac{n+2}{p+2}\right);$$

de la même manière, nous aurons

$$\left(\frac{n}{p+1}\right) + 2\left(\frac{n}{p+2}\right) + \left(\frac{n}{p+3}\right) = \left(\frac{n+2}{p+3}\right);$$

cette équation, ajoutée à la précédente, donne

$$\left(\frac{n}{p}\right) + 3\left(\frac{n}{p+1}\right) + 3\left(\frac{n}{p+2}\right) + \left(\frac{n}{p+3}\right) = \left(\frac{n+2}{p+2}\right) + \left(\frac{n+2}{p+3}\right) = \left(\frac{n+3}{p+3}\right);$$

ensuite

$$\left(\frac{n}{p+1}\right) + 3\left(\frac{n}{p+2}\right) + 3\left(\frac{n}{p+3}\right) + \left(\frac{n}{p+4}\right) = \left(\frac{n+3}{p+4}\right),$$

qui, encore ajoutée à la précédente, donne

$$\left(\frac{n}{p}\right) + 4\left(\frac{n}{p+1}\right) + 6\left(\frac{n}{p+2}\right) + 4\left(\frac{n}{p+3}\right) + \left(\frac{n}{p+4}\right) = \left(\frac{n+3}{p+3}\right) + \left(\frac{n+3}{p+4}\right) = \left(\frac{n+4}{p+4}\right),$$

et de là, il est aisé à conclure qu'on aura en général

$$1\left(\frac{n}{p}\right) + \left(\frac{m}{1}\right)\left(\frac{n}{p+1}\right) + \left(\frac{m}{2}\right)\left(\frac{n}{p+2}\right) + \left(\frac{m}{3}\right)\left(\frac{n}{p+3}\right) + \text{etc.} = \left(\frac{n+m}{p+m}\right).$$

Voilà donc une progression bien générale, dont chaque terme est le produit de deux coëfficiens de puissances différentes du binome, dont le terme général peut être exprimé par la formule $\binom{m}{x}\binom{n}{p+x}$, où, mettant pour x successivement les nombres 0, 1, 2, 3, 4, etc., jusqu'à ce qu'on parvienne à des termes évanouissans, la somme de toute cette progression sera infailliblement $=\binom{n+m}{p+m}=\binom{n+m}{n-p}$. C'est de là que résulte le théorème que je vous ai communiqué, en faisant $m=n$ et $p=0$, de sorte qu'il est un cas infiniment plus particulier que la série que je viens de sommer ici. Dans ce cas, on aura cette sommation:

$$1^2+\binom{n}{1}^2+\binom{n}{2}^2+\binom{n}{3}^2+\text{etc.}=\binom{2n}{n};$$

or, cette formule développée donne

$$\frac{2n}{1}\cdot\frac{2n-1}{2}\cdot\frac{2n-2}{3}\cdot\frac{2n-3}{4}\cdots\frac{n+1}{n},$$

ce qui, comme il est aisé à démontrer, est égal à

$$\frac{2}{1}\cdot\frac{6}{2}\cdot\frac{10}{3}\cdot\frac{14}{4}\cdots\frac{4n-2}{n}.$$

Il est fort remarquable que cette sommation a aussi lieu, lors même que les exposans m et n sont des fractions quelconques, pourvu que, par la voie d'interpolation, on puisse assigner la juste valeur de $\binom{m+n}{m+p}$; et si le développement n'a pas lieu dans ce cas, il faut recourir à des formules intégrales; or, posant pour abréger $l\frac{1}{x}=u$, on aura toujours

$$\binom{m+n}{m+p}=\frac{\int u^{m+n}\partial x}{\int u^{m+p}\partial x\cdot\int u^{n-p}\partial x}\left\{\begin{matrix}\text{de } x=0\\ \text{à } x=1\end{matrix}\right\};$$

or, si λ marque un nombre entier positif quelconque, on sait qu'il y aura

$$\int u^{\lambda}\partial x=1\cdot2\cdot3\cdot4\cdots\lambda,$$

et de là, on tirera

$$\int u^{\lambda+1}\partial x=(\lambda+1)\int u^{\lambda}\partial x,$$
$$\int u^{\lambda+2}\partial x=(\lambda+1)(\lambda+2)\int u^{\lambda}\partial x,$$
etc.

et cette réduction aura toujours lieu, quelque nombre qu'on prenne pour λ. Prenant donc $\lambda=-\frac{1}{2}$, j'ai démontré autrefois[1]) qu'on aura

$$\int\frac{\partial x}{\sqrt{u}}=\sqrt{\pi}\quad\text{et}\quad\int\partial x\sqrt{u}=\tfrac{1}{2}\sqrt{\pi},$$

π désignant la circonférence d'un cercle dont le diamètre $=1$. Maintenant, si l'on met $m=n$ et $p=0$, puisque les coëfficiens de $(1+z)^{\frac{1}{2}}$ sont

$$1,\ +\frac{1}{2},\ -\frac{1\cdot1}{2\cdot4},\ +\frac{1\cdot1\cdot3}{2\cdot4\cdot6},\ -\frac{1\cdot1\cdot3\cdot5}{2\cdot4\cdot6\cdot8},\ \text{etc.},$$

nous en tirons cette série des carrés,

$$1^2+\left(\frac{1}{2}\right)^2+\left(\frac{1\cdot1}{2\cdot4}\right)^2+\left(\frac{1\cdot1\cdot3}{2\cdot4\cdot6}\right)^2+\text{etc.}$$

dont la somme sera

$$\frac{\int u\partial x}{\int\partial x\sqrt{u}\cdot\int\partial x\sqrt{u}}=\frac{4}{\pi},$$

à cause de

$$\int u\partial x=1\quad\text{et}\quad\int\partial x\sqrt{u}=\tfrac{1}{2}\sqrt{\pi},$$

ce qui s'accorde parfaitement avec la somme qu'on trouve par la voie de l'approximation.

J'ai cru pouvoir joindre ici une autre démonstration de deux des théorèmes précédens, quoique la méthode qui y est employée, soit fort inférieure à celle de M. Euler; mais il peut être quelquefois utile de voir comment différentes routes peuvent conduire aux mêmes vérités. D'ailleurs, M. Euler ayant daigné honorer ces recherches de son approbation, c'est lui donner une marque de mon respect que de les rendre publiques.

1) Voir § 16 et 28 du mémoire 421 (suivant *l'Index* d'Eneström): *Evolutio formulae integralis $\int x^{f-1}dx(lx)^{\frac{m}{n}}$ integratione a valore $x=0$ ad $x=1$ extensa*, Novi comment. acad. sc. Petrop. 16 (1771), 1772, p. 91; *Leonhardi Euleri Opera omnia*, series I, vol. 17, p. 316. Voir aussi § 11 du mémoire 19 (suivant *l'Index* d'Eneström): *De progressionibus transcendentibus, seu quarum termini generales algebraice dari nequeunt*, Comment. acad. sc. Petrop. 5 (1730/1), 1738, p. 36; *Leonhardi Euleri Opera omnia*, series I, vol. 14. A. G.

Soit la fonction $\int \frac{x^m}{lx}\cdot\frac{\partial x}{x}$ et qu'on l'intègre en série par la méthode des intégrations par parties, on aura

$$\int \frac{x^m}{lx}\cdot\frac{\partial x}{x} = x^m\left(\frac{1}{mlx}+\frac{1}{m^2lx^2}+\frac{2}{m^3lx^3}+\frac{2\cdot 3}{m^4lx^4}+\cdots\right).$$

Ainsi, la valeur de cette intégrale, prise depuis $x=B$ jusqu'à $x=A$, sera

$$(S)\qquad \left\{\begin{aligned} &B^m\left(\frac{1}{mlB}+\frac{1}{m^2lB^2}+\frac{2}{m^3lB^3}+\frac{2\cdot 3}{m^4lB^4}+\cdots\right)\\ &-A^m\left(\frac{1}{mlA}+\frac{1}{m^2lA^2}+\frac{2}{m^3lA^3}+\frac{2\cdot 3}{m^4lA^4}+\cdots\right).\end{aligned}\right.$$

Pour avoir maintenant la valeur de cette fonction en m, je la différencie par rapport à m et j'ai pour sa valeur:

$$\begin{aligned} B^m\Big(&\frac{\partial m}{m}+\frac{\partial m}{m^2lB}+\frac{2\partial m}{m^3lB^2}+\frac{2\cdot 3\partial m}{m^4lB^3}\cdots\\ &-\frac{\partial m}{m^2lB}-\frac{2\partial m}{m^3lB^2}-\frac{2\cdot 3\partial m}{m^4lB^3}\cdots\Big)\\ -A^m\Big(&\frac{\partial m}{m}+\frac{\partial m}{m^2lA}+\frac{2\partial m}{m^3lA^2}+\frac{2\cdot 3\partial m}{m^4lA^3}\cdots\\ &-\frac{\partial m}{m^2lA}-\frac{2\partial m}{m^3lA^2}-\frac{2\cdot 3\partial m}{m^4lA^3}\cdots\Big),\end{aligned}$$

valeur qui se réduit à

$$(B^m-A^m)\frac{\partial m}{m}.$$

La valeur de la série (S) sera donc

$$\int (B^m-A^m)\frac{\partial m}{m}+C;$$

et, si on suppose $A=0$ et $B=1$, [la valeur se réduit] à

$$\int \frac{\partial m}{m}+C=lm+C$$

C étant une constante indépendante de m; par la même raison, on aura pour valeur de $\int \frac{x^n}{lx}\cdot\frac{\partial x}{x}$, prise depuis $x=0$ jusqu'à $x=1$, la fonction $ln+C$; donc, la valeur $\int \frac{x^n-x^m}{lx}\cdot\frac{\partial x}{x}$, prise depuis $x=0$ jusqu'à $x=1$, sera $ln-lm$ ou $l\frac{n}{m}$.

On seroit parvenu à la même conclusion, sans employer les séries; en effet, le problème se réduit ici à trouver

$$\int\left(\frac{B^m}{lB}\cdot\frac{\partial B}{B}-\frac{A^m}{lA}\cdot\frac{\partial A}{A}\right);$$

or, différenciant cette fonction par rapport à m, elle devient

$$\int (B^{m-1}\partial B-A^{m-1}\partial A)\partial m=(B^m-A^m)\frac{\partial m}{m},$$

comme on l'a trouvé ci-dessus.

On auroit aussi trouvé immédiatement, en cherchant la valeur de

$$\int\left(\frac{B^n-B^m}{lB}\cdot\frac{\partial B}{B}-\frac{A^n-A^m}{lA}\cdot\frac{\partial A}{A}\right),$$

que cette fonction différenciée par rapport à n et à m, devient

$$(B^n-A^n)\frac{\partial n}{n}-(B^m-A^m)\frac{\partial m}{m},$$

dont l'intégrale est, lorsque $B=1$ et $A=0$, $l\frac{n}{m}+C$; mais, pour le cas de $m=n$, il est clair que cette intégrale doit être zéro; donc $C=0$; donc l'intégrale cherchée, est égale à $l\frac{n}{m}$.

Soit en général une fonction $\int X\partial x$, que X contienne des constantes indéterminées $m, n, \ldots$ et qu'on cherche des valeurs de $\int X\partial x$, prises depuis $x=A$ jusqu'à $x=B$, la valeur de cette fonction sera égale à l'intégrale de

$$\left(\int\frac{\partial X}{\partial m}\partial B-\int\frac{\partial X}{\partial m}\partial A\right)\partial m+\left(\int\frac{\partial X}{\partial n}\partial B-\int\frac{\partial X}{\partial n}\partial A\right)\partial n+\cdots$$

prise par rapport aux $m, n, \ldots$; ainsi, toutes les fois que les fonctions $\frac{\partial X}{\partial m}\partial x$, $\frac{\partial X}{\partial n}\partial x$, . . . seront intégrables, la formule ci-dessus sera débarrassée de signes d'intégration et l'on pourra chercher pour quelles valeurs de A et de B elle devient intégrable.

Sur quoi nous observerons:

1°. Que comme il faut ajouter une arbitraire à cette intégrale, dans le cas où l'on n'auroit pas des moyens de la déterminer, ce ne seroit pas la valeur de $\int X\partial x$, prise depuis $x=A$ jusqu'à $x=B$, qu'on pourroit trouver par cette méthode, mais celle de $\int (X'\partial x - X\partial x)$, X' étant ce que devient X, en y mettant au lieu de $m, n, \ldots m', n', \ldots$

Par exemple, soit repris l'exemple ci-dessus, où $X = \frac{x^m}{xlx}$, nous avons $\int \frac{x^m}{lx} \cdot \frac{\partial x}{x} = lm + C$; il est clair que lorsque $m = 0$, la valeur de l'intégrale est $ll.1 - ll.0$; or, elle est aussi $l0 + C$; donc, à cause de $l1 = 0$, on a $C = -ll.0$, et

$$\int \frac{x^m}{lx} \frac{\partial x}{x} = lm - ll.0;$$

mais, si on n'avoit pas connu la valeur de l'intégrale pour une particulière de m, C seroit resté inconnu et la méthode n'auroit donné aucun résultat, au lieu que même lorsqu'on ne peut connoître C, elle auroit toujours donné

$$\int \left(\frac{x^{m'}}{xlx} - \frac{x^m}{xlx}\right) \partial x = l\frac{m'}{m}\cdot$$

2°. Que pour trouver, par cette méthode, la valeur de $\int X\partial x$, il faut que, pour satisfaire à l'équation

$$\int \left(\int \frac{\partial X}{\partial m} \partial x\right) \partial m = \int X \partial x,$$

on ne soit pas obligé d'ajouter à la première intégrale, prise par rapport à m, une fonction de x; d'où il résulte qu'il y a encore une infinité de cas où la méthode ne pouvant être employée à trouver $\int X\partial x$, peut l'être à trouver $\int (X' - X)\partial x$.

3°. Que pour réduire l'intégration de $\int X\partial x$ à celle de $\int \left(\int \frac{\partial X}{\partial m} \partial x\right) \partial m$, il suffit de savoir intégrer $\int \frac{\partial X}{\partial m} \partial x$; mais, par la même raison, l'intégration de $\int \frac{\partial X}{\partial m} \partial x$ dépendra de l'intégration de $\int \frac{\partial^2 X}{\partial m^2} \partial x$; en sorte qu'en général, pourvu qu'on puisse trouver $\int \frac{\partial^p X}{\partial m^p} \partial x$, on pourra faire dépendre l'intégration de $\int X\partial x$, d'intégrales prises par rapport à m.

DEMONSTRATION DU SECOND THEOREME

Soit maintenant la fonction

$$(1+z)^n = 1 + A'z + A''z^2 + A'''z^3 + \cdots$$

en sorte que le coëfficient de z^m soit A^m, nous avons à prouver que

$$1 + A'^2 + A''^2 + A'''^2 + \cdots = \frac{2\cdot 6\cdot 10 \cdots (4n-2)}{1\cdot 2 \cdots n};$$

mettons $n+1$ à la place de n et faisons

$$1 + A'^2 + A''^2 + \cdots = Z,$$

nous aurons

$$\begin{aligned} Z + \Delta Z = 1 &+ A'^2 + A''^2 + \cdots \\ &+ 2A'\Delta A' + 2A''\Delta A'' + \cdots \\ &+ \Delta A'^2 + \Delta A''^2 + \cdots \\ &= \frac{2\cdot 6\cdot 10\cdots(4n+2)}{1\cdot 2\cdots(n+1)} = \frac{Z(4n+2)}{n+1}; \end{aligned}$$

mais

$$\Delta A' = 1, \quad \Delta A'' = A', \quad \Delta A''' = A'', \ldots$$

donc nous aurons

$$\left.\begin{array}{l} 1 + A'^2 + A''^2 + A'''^2 + \cdots \\ + 2A' + 2A'A'' + 2A''A''' + \cdots \\ + 1 + A'^2 + A''^2 + \cdots \end{array}\right\} = \frac{Z(4n+2)}{n+1}$$

ou

$$2Z + 2(A' + A'A'' + A''A''' + \cdots) = \frac{Z(4n+2)}{n+1},$$

d'où

$$(n+1)(A' + A'A'' + A''A''' + \cdots) = n(1 + A'^2 + A''^2 + A'''^2 + \cdots);$$

or, nous avons

$$A' = n, \quad A'A'' = A'^2\frac{n-1}{2}, \quad A''A''' = A''^2\frac{n-2}{3}, \ \cdots$$

et

$$A' = \frac{A'^2}{n}, \quad A'A'' = A''^2\frac{2}{n-1}, \quad A''A''' = A'''^2\frac{3}{n-2}, \ \cdots$$

Substituant ces valeurs dans l'équation ci-dessus, elle devient

$$\frac{n+1}{n}\left\{\begin{array}{l} an + \frac{1-a}{n}A'^2 + \frac{2}{n-1}(1-a')A''^2 + \frac{3}{n-2}(1-a'')A'''^2 + \cdots \\ \qquad + \frac{n-1}{2}a'A'^2 + \frac{n-2}{3}a''A''^2 + \frac{n-3}{4}a'''A'''^2 + \cdots \end{array}\right\}.$$

$$= 1 + A'^2 + A''^2 + A'''^2 + \cdots$$

a, a', a'', a'''... étant des coëfficiens indéterminés; d'où, comparant terme à terme et faisant

$$a=\frac{1}{n+1},\quad a'=\frac{2}{n+1},\quad a''=\frac{3}{n+1},\quad a'''=\frac{4}{n+1},\ \ldots$$

on concluera l'identité des deux formules.

Cette manière d'employer la méthode des coëfficiens indéterminés peut facilement s'étendre à différens théorèmes du même genre.

SUPPLEMENTUM CALCULI INTEGRALIS PRO INTEGRATIONE FORMULARUM IRRATIONALIUM

Commentatio 539 indicis ENESTROEMIANI
Acta academiae scientiarum Petropolitanae 1780: I (1783), p. 3—31

PROBLEMA 1

1. *Si functio X praeter ipsam variabilem x etiam formulam irrationalem*

$$s=\sqrt{(a+bx)}$$

involvat, ita tamen, ut X sit functio rationalis binarum quantitatum x et s, formulam differentialem Xdx ab irrationalitate liberare.

SOLUTIO

Cum irrationalitas tantum in formula $s=\sqrt{(a+bx)}$ insit, hanc tantum ita per idoneam substitutionem tolli oportet, ut inde valor ipsius x non fiat irrationalis. Hoc autem praestabitur ponendo $a+bx=zz$, ut fiat $s=z$ et

$$x=\frac{zz-a}{b}\quad\text{hincque}\quad dx=\frac{2}{b}zdz;$$

quibus valoribus substitutis tota formula differentialis Xdx ad rationalem, novam variabilem z complectentem, perducitur.

EXEMPLUM 1

2. Si fuerit

$$dy = \frac{dx}{\sqrt{(a+bx)}} \quad \text{seu} \quad dy = \frac{dx}{s},$$

posito $\sqrt{(a+bx)} = z$ fiet

$$dy = \frac{2}{b}dz$$

et integrando $y = \frac{2z}{b}$, unde facta substitutione colligitur

$$y = \frac{2}{b}\sqrt{(a+bx)} + C.$$

EXEMPLUM 2

3. Si fuerit

$$dy = dx\sqrt{(a+bx)} = sdx,$$

sumto $\sqrt{(a+bx)} = z$ erit

$$dy = zdx = \frac{2}{b}zzdz,$$

unde integrando fit $y = \frac{2}{3b}z^3$ et facta substitutione prodit

$$y = \frac{2}{3b}(a+bx)^{\frac{3}{2}} + C.$$

Quod integrale si debeat evanescere facto $x = 0$, fiet $C = -\frac{2a\sqrt{a}}{3b}$ ideoque

$$y = \frac{2(a+bx)^{\frac{3}{2}} - 2a\sqrt{a}}{3b}.$$

EXEMPLUM 3

4. Si fuerit

$$dy = \frac{xdx}{\sqrt{(a+bx)}},$$

facta substitutione $\sqrt{(a+bx)} = z$ erit

$$dy = \frac{2(zz-a)dz}{bb} = \frac{2zzdz - 2adz}{bb},$$

unde fit integrando $y = \frac{2}{3bb}z^3 - \frac{2a}{bb}z + C$ et facta restitutione

$$y = \frac{2}{3bb}(a+bx)^{\frac{3}{2}} - \frac{2a}{bb}\sqrt{(a+bx)} + C = \frac{2\sqrt{(a+bx)}}{bb}\left(\frac{1}{3}bx - \frac{2}{3}a\right) + C.$$

EXEMPLUM 4

5. Si fuerit

$$dy = \frac{dx}{(a+bx)^{\frac{3}{2}}},$$

facta substitutione $\sqrt{(a+bx)} = z$ erit $dy = \frac{dx}{z^3}$; quae formula porro ob $dx = \frac{2zdz}{b}$ abit in

$$dy = \frac{2dz}{bzz},$$

qua integrata fit $y = -\frac{2}{bz}$ seu facta restitutione

$$y = \frac{-2}{b\sqrt{(a+bx)}} + C.$$

Ubi notetur pro C sumi debere $\frac{2}{b\sqrt{a}}$ [si integrale debeat evanescere facto $x = 0$].

PROBLEMA 2

6. *Si fuerit X functio quaecunque rationalis binarum quantitatum x et s existente*

$$s = \sqrt[3]{(a+bx)},$$

formulam differentialem Xdx ab irrationalitate liberare.

SOLUTIO

Ponatur $\sqrt[3]{(a+bx)} = z$, ut sit $s = z$; erit $a + bx = z^3$ hincque

$$x = \frac{z^3 - a}{b} \quad \text{et} \quad dx = \frac{3zzdz}{b};$$

quibus valoribus substitutis tota formula fiet rationalis.

EXEMPLUM 1

7. Si fuerit

$$dy = \frac{dx}{\sqrt[3]{(a+bx)}} = \frac{dx}{s},$$

posito $\sqrt[3]{(a+bx)} = z$ et substituto valore hinc nato $dx = \frac{3zzdz}{b}$ erit

$$dy = \frac{3zdz}{b},$$

unde integrando fit

$$y = \frac{3}{2b}zz = \frac{3}{2b}\sqrt[3]{(a+bx)^2} + C.$$

EXEMPLUM 2

8. Si fuerit

$$dy = \frac{dx}{\sqrt[3]{(a+bx)^2}} = \frac{dx}{ss},$$

posito $\sqrt[3]{(a+bx)} = z$ fiet

$$dy = \frac{3dz}{b},$$

hinc integrando

$$y = \frac{3}{b}z = \frac{3}{b}\sqrt[3]{(a+bx)} + C.$$

EXEMPLUM 3

9. Si fuerit

$$dy = dx\sqrt[3]{(a+bx)} = sdx,$$

facta substitutione fit

$$dy = \frac{3z^3dz}{b},$$

hinc integrando

$$y = \frac{3}{4b}z^4 = \frac{3}{4b}(a+bx)\sqrt[3]{(a+bx)} + C.$$

PROBLEMA 3

10. *Si fuerit X functio rationalis binarum quantitatum x et s existente*

$$s = \sqrt[n]{(a+bx)},$$

formulam differentialem Xdx ab irrationalitate liberare.

SOLUTIO

Ponatur $\sqrt[n]{(a+bx)} = z$, ut sit $s = z$; erit $a + bx = z^n$, hinc

$$x = \frac{z^n - a}{b} \quad \text{et} \quad dx = \frac{nz^{n-1}dz}{b};$$

quibus valoribus substitutis formula proposita Xdx certe fiet rationalis, si modo numerus exponentialis n fuerit integer.

EXEMPLUM 1

11. Si fuerit

$$dy = \frac{dx}{\sqrt[n]{(a+bx)}} = \frac{dx}{s},$$

posito $\sqrt[n]{(a+bx)} = z$ ob valorem inde natum $dx = \frac{nz^{n-1}dz}{b}$ habebitur

$$dy = \frac{nz^{n-2}dz}{b},$$

unde integrando colligimus $y = \frac{n}{b(n-1)}z^{n-1} + C$ sive restitutis valoribus

$$y = \frac{n}{b(n-1)}(a+bx)^{\frac{n-1}{n}} + C = \frac{n}{b(n-1)} \cdot \frac{a+bx}{\sqrt[n]{(a+bx)}} + C.$$

EXEMPLUM 2

12. Si fuerit

$$dy = \frac{dx}{\sqrt[n]{(a+bx)^2}} = \frac{dx}{s^2},$$

posito $\sqrt[n]{(a+bx)} = z$ et substituto valore $dx = \frac{nz^{n-1}dz}{b}$ fiet

$$dy = \frac{nz^{n-1}dz}{bz^{\lambda}} = \frac{n}{b}z^{n-\lambda-1}dz,$$

cuius integrale dat

$$y = \frac{n}{b(n-\lambda)}(a+bx)^{\frac{n-\lambda}{n}} + C \quad \text{sive} \quad y = \frac{n}{b(n-\lambda)}\cdot\frac{a+bx}{\sqrt[n]{(a+bx)^{\lambda}}} + C.$$

Ex his autem exemplis iam apparet integrationem non impediri, etiamsi exponentes n et λ non fuerint numeri integri.

PROBLEMA 4

13. *Si fuerit X functio rationalis binarum quantitatum x et s existente*

$$s = \sqrt{(a + b\sqrt{(f+gx)})},$$

quae formula ergo duplicem irrationalitatem involvit, formulam differentialem Xdx ab hac duplici irrationalitate liberare.

SOLUTIO

Ponatur iterum $\sqrt{(a+b\sqrt{(f+gx)})} = z$, ut sit $s = z$; erit sumtis quadratis $a + b\sqrt{(f+gx)} = zz$, hinc $b\sqrt{(f+gx)} = zz - a$ ac sumtis denuo quadratis $bb(f+gx) = (zz-a)^2$, unde colligitur

$$x = \frac{(zz-a)^2}{bbg} - \frac{f}{g} \quad \text{hincque} \quad dx = \frac{4zdz(zz-a)}{bbg}.$$

Quibus valoribus substitutis tota formula reddetur rationalis.

COROLLARIUM

14. Perspicuum est eodem modo irrationalitatem tolli posse, si fuerit multo generalius

$$s = \sqrt[n]{(a + b\sqrt[m]{(f+gx)})}.$$

Posita enim hac formula $= z$ fiet

$$a + b\sqrt[m]{(f+gx)} = z^n \quad \text{et} \quad b\sqrt[m]{(f+gx)} = z^n - a.$$

Porro $b^m(f+gx) = (z^n-a)^m$ et hinc colligitur

$$x = \frac{(z^n-a)^m}{b^m g} - \frac{f}{g} \quad \text{ideoque} \quad dx = \frac{mnz^{n-1}dz(z^n-a)^{m-1}}{b^m g}.$$

Sicque etiam hoc modo tota formula rationalis evadet.

PROBLEMA 5

15. *Si fuerit X functio rationalis binarum quantitatum x et s existente*

$$s = \sqrt{\frac{a+bx}{f+gx}},$$

formulam differentialem Xdx ab irrationalitate liberare.

SOLUTIO

Ponatur $\sqrt{\frac{a+bx}{f+gx}} = z$ et sumtis quadratis erit $\frac{a+bx}{f+gx} = zz$ hincque

$$x = \frac{fzz-a}{b-gzz},$$

unde differentiando colligitur

$$dx = \frac{2bfzdz - 2agzdz}{(b-gzz)^2}.$$

Hisque valoribus substitutis formula proposita Xdx ad rationalitatem erit perducta.

EXEMPLUM 1

16. Si fuerit

$$dy = \frac{dx}{s} = \frac{dx\sqrt{(f+gx)}}{\sqrt{(a+bx)}},$$

posito $\sqrt{\frac{a+bx}{f+gx}} = z$ erit $dy = \frac{dx}{z}$ et substituto loco dx valore supra invento colligitur

$$dy = \frac{2(bf-ag)dz}{(b-gzz)^2};$$

quae formula, uti iam satis constat, reduci potest ad talem

$$\int \frac{dz}{b-gzz},$$

cuius autem integratio vel per logarithmos vel per arcus circulares expedietur.

EXEMPLUM 2

17. Sit specialius

$$dy = \frac{dx\sqrt{(1-x)}}{\sqrt{(1+x)}},$$

ubi $f=1$, $g=-1$, $a=1$ et $b=1$ ideoque

$$z = \frac{\sqrt{(1+x)}}{\sqrt{(1-x)}} \quad \text{et} \quad dx = \frac{4z\,dz}{(1+zz)^2};$$

quibus valoribus substitutis fiet

$$dy = \frac{4\,dz}{(1+zz)^2}.$$

Statuatur ergo

$$\int \frac{4\,dz}{(1+zz)^2} = \frac{Az}{1+zz} + B\int \frac{dz}{1+zz} = y,$$

unde sumtis differentialibus fiet

$$\frac{4}{(1+zz)^2} = \frac{A-Azz}{(1+zz)^2} + \frac{B}{1+zz} = \frac{A+B+(B-A)zz}{(1+zz)^2}.$$

Oportet igitur esse $A+B=4$ et $B-A=0$ ideoque $A=2$ et $B=2$; et quia $\int \frac{dz}{1+zz} = \text{A tang. } z$, adipiscimur $y = \frac{2z}{1+zz} + 2 \text{ A tang. } z$, quocirca facta restitutione ob $1+zz = \frac{2}{1-x}$ obtinebitur

$$y = \sqrt{(1-xx)} + 2 \text{ A tang.} \sqrt{\frac{1+x}{1-x}}.$$

Cum igitur huius arcus tangens sit $\sqrt{\frac{1+x}{1-x}}$, erit eius sinus $= \sqrt{\frac{1+x}{2}}$ et cosinus $= \sqrt{\frac{1-x}{2}}$; anguli vero dupli sinus erit $\sqrt{(1-xx)}$ et cosinus $= -x$, unde fiet

$$2 \text{ A tang.} \sqrt{\frac{1+x}{1-x}} = \text{A cos.} -x = \frac{\pi}{2} + \text{A sin. } x;$$

quocirca integrale quaesitum erit

$$y = \sqrt{(1-xx)} + \frac{\pi}{2} + \text{A sin. } x + C;$$

quod si ita capi debeat, ut evanescat posito $x=0$, erit $C = -1 - \frac{\pi}{2}$ ideoque

$$y = \sqrt{(1-xx)} - 1 + \text{A sin. } x.$$

Tum igitur si sumatur $x=1$, fiet $y = \frac{\pi}{2} - 1$, qui valor in fractionibus decimalibus dat 0,5707963.

PROBLEMA 6

18. *Si fuerit X functio rationalis binarum variabilium x et s existente*

$$s = \sqrt[n]{\frac{a+bx}{f+gx}},$$

formulam differentialem $X dx$ ad rationalitatem perducere.

SOLUTIO

Posito $s = \sqrt[n]{\frac{a+bx}{f+gx}} = z$ erit $\frac{a+bx}{f+gx} = z^n$ hincque

$$x = \frac{fz^n - a}{b - gz^n},$$

consequenter

$$dx = \frac{n(bf-ag)z^{n-1}dz}{(b-gz^n)^2};$$

hisque valoribus substitutis tota formula proposita $X dx$ ad rationalitatem erit perducta.

PROBLEMA 7

19. *Si fuerit X functio binarum quantitatum xx et s existente*

$$s=\sqrt{(a+bxx)},$$

formulam differentialem $\frac{X\,dx}{x}$ *ab irrationalitate liberare.*

SOLUTIO

Ponamus $s=\sqrt{(a+bxx)}=z$; erit $a+bxx=zz$, hinc

$$xx=\frac{zz-a}{b},$$

et quia in functione X tantum quadratum xx eiusque ergo potestates pares occurrunt, hac substitutione iam functio X evadet rationalis. Sumtis vero logarithmis $2lx=l(zz-a)-lb$ differentiando fit $\frac{2dx}{x}=\frac{2zdz}{zz-a}$ ideoque

$$\frac{dx}{x}=\frac{zdz}{zz-a}.$$

Hoc ergo modo formula proposita $X\frac{dx}{x}$ prorsus reddetur rationalis.

EXEMPLUM 1

20. Si fuerit

$$dy=\frac{xdx}{\sqrt{(a+bxx)}},$$

erit

$$dy=\frac{dx}{x}\cdot\frac{xx}{\sqrt{(a+bxx)}}=\frac{xx}{s}\cdot\frac{dx}{x}.$$

Posito ergo $\sqrt{(a+bxx)}=z$ erit

$$dy=\frac{dz}{b},$$

unde colligitur integrando

$$y=\frac{z}{b}=\frac{\sqrt{(a+bxx)}}{b}.$$

EXEMPLUM 2

21. Si fuerit

$$dy=\frac{x^3dx}{\sqrt{(a+bxx)}}=\frac{dx}{x}\cdot\frac{x^4}{s},$$

ponendo $\sqrt{(a+bxx)}=z$, ut sit

$$xx=\frac{zz-a}{b} \quad \text{et} \quad \frac{dx}{x}=\frac{zdz}{zz-a},$$

erit

$$dy=\frac{1}{bb}dz(zz-a)$$

hincque integrando adipiscimur $y=\frac{z}{3bb}(zz-3a)$; unde facta restitutione prodibit integrale quaesitum

$$y=\frac{bxx-2a}{3bb}\sqrt{(a+bxx)}+C.$$

EXEMPLUM 3

22. Si fuerit

$$dy=\frac{x^3dx}{\sqrt{(a+bxx)^3}},$$

erit $dy=\frac{dx}{x}\cdot\frac{x^4}{s^3}$; hinc posito $\sqrt{(a+bxx)}=s=z$ fiet

$$dy=\frac{dz}{bb}\cdot\frac{zz-a}{zz},$$

unde sumto integrali fiet $y=\frac{1}{bb}\cdot\frac{zz+a}{z}$, quocirca facta restitutione resultat

$$y=\frac{2a+bxx}{bb\sqrt{(a+bxx)}}+C.$$

PROBLEMA 8

23. *Si fuerit X functio rationalis binarum quantitatum* x^n *et s existente*

$$s=\sqrt[m]{(a+bx^n)},$$

formulam differentialem $X\frac{dx}{x}$ *ad rationalitatem perducere.*

SOLUTIO

Posito $s=\sqrt[m]{(a+bx^n)}=z$ fiet $a+bx^n=z^m$ et

$$x^n=\frac{z^m-a}{b}.$$

Quia igitur in functione X tantum potestas x^n occurrit, ea rationalis reddetur, si hi valores substituantur. Tum vero sumtis logarithmis habebitur

$$nlx=l(z^m-a)-lb$$

et differentiando

$$\frac{dx}{x}=\frac{mz^{m-1}dz}{n(z^m-a)};$$

sicque tota formula proposita fiet rationalis.

EXEMPLUM

24. Sit

$$dy=\frac{x^{n-1}dx}{\sqrt[m]{(a+bx^n)}}=\frac{dx}{x}\cdot\frac{x^n}{s}$$

factaque substitutione orietur haec aequatio

$$dy=\frac{mz^{m-2}dz}{bn},$$

qua integrata prodibit

$$y=\frac{mz^{m-1}}{nb(m-1)}=\frac{m}{nb(m-1)}\sqrt[m]{(a+bx^n)^{m-1}}+C$$

sive

$$y=\frac{m}{nb(m-1)}\cdot\frac{a+bx^n}{\sqrt[m]{(a+bx^n)}}+C.$$

PROBLEMA 9

25. *Si fuerit X functio rationalis quantitatum xx et s existente*

$$s=\sqrt{\frac{a+bxx}{f+gxx}},$$

formulam differentialem $X\frac{dx}{x}$ ab irrationalitate liberare.

SOLUTIO

Ponatur $s=\sqrt{\frac{a+bxx}{f+gxx}}=z$ eritque $\frac{a+bxx}{f+gxx}=zz$, hinc

$$xx=\frac{fzz-a}{b-gzz},$$

unde functio X penitus fit rationalis. Porro sumtis logarithmis

$$2lx=l(fzz-a)-l(b-gzz)$$

differentietur, ut prodeat

$$\frac{2dx}{x}=\frac{2fzdz}{fzz-a}+\frac{2gzdz}{b-gzz}=\frac{2(bf-ag)zdz}{(fzz-a)(b-gzz)},$$

unde fit

$$\frac{dx}{x}=\frac{(bf-ag)zdz}{(fzz-a)(b-gzz)};$$

sicque tota formula differentialis fiet rationalis.

EXEMPLUM

26. Si fuerit

$$dy=\frac{dx}{\sqrt{(f+gxx)}},$$

repraesentemus hanc formulam ita

$$dy=\frac{dx}{x}\cdot\frac{x}{\sqrt{(f+gxx)}}=\frac{dx}{x}\sqrt{\frac{xx}{f+gxx}}.$$

Hic ergo erit $a=0$, $b=1$ et $z=\frac{x}{\sqrt{(f+gxx)}}$, ita ut $dy=\frac{zdx}{x}$; erit autem $\frac{dx}{x}=\frac{dz}{z(1-gzz)}$, unde fit

$$dy=\frac{dz}{1-gzz},$$

cuius formulae integratio per logarithmos expeditur, si fuerit g numerus positivus; sin autem fuerit negativus, per arcus circulares absolvetur.

Sit igitur 1° $g=+bb$; erit

$$dy=\frac{dz}{1-bbzz}$$

ideoque

$$y=\frac{1}{2b}l\frac{1+bz}{1-bz}$$

et restitutis valoribus supra indicatis erit

$$y = \frac{1}{2b} l \frac{\sqrt{(f+bbxx)}+bx}{\sqrt{(f+bbxx)}-bx} = \frac{1}{b} l \frac{\sqrt{(f+bbxx)}+bx}{\sqrt{f}}.$$

Sit 2° g quantitas negativa, puta $g = -bb$; erit

$$dy = \frac{dz}{1+bbzz} = \frac{1}{b} \cdot \frac{bdz}{1+bbzz},$$

unde colligitur

$$y = \frac{1}{b} \text{A tang. } bz = \frac{1}{b} \text{A tang. } \frac{bx}{\sqrt{(f-bbxx)}}.$$

Ubi manifestum est f esse debere quantitatem positivam, quia alioquin formula differentialis esset imaginaria.

COROLLARIUM

27. Hinc ergo, si proponatur formula

$$dy = \frac{dx}{\sqrt{(1+xx)}},$$

ubi $f = 1$ et $g = 1$, ex casu priore ob $b = +1$ erit

$$\int \frac{dx}{\sqrt{(1+xx)}} = l\left(\sqrt{(1+xx)}+x\right).$$

At si fuerit

$$dy = \frac{dx}{\sqrt{(1-xx)}},$$

ubi $f = 1$ et $g = -1$, colligitur ex casu posteriore $y = \text{A tang. } \frac{x}{\sqrt{(1-xx)}}$, unde concluditur

$$\int \frac{dx}{\sqrt{(1-xx)}} = \text{A sin. } x = \text{A cos. } \sqrt{(1-xx)}.$$

PROBLEMA 10

28. *Si fuerit X functio rationalis quantitatum x^n et s existente*

$$s = \sqrt[n]{\frac{a+bx^n}{f+gx^n}},$$

formulam differentialem $X\frac{dx}{x}$ rationalem efficere.

SOLUTIO

Ponatur $s = \sqrt[n]{\frac{a+bx^n}{f+gx^n}} = z$ eritque $\frac{a+bx^n}{f+gx^n} = z^n$, hinc

$$x^n = \frac{fz^n-a}{b-gz^n};$$

tum autem sumtis logarithmis erit

$$nlx = l(fz^n-a) - l(b-gz^n)$$

et differentiando

$$\frac{dx}{x} = \frac{fz^{n-1}dz}{fz^n-a} + \frac{gz^{n-1}dz}{b-gz^n} = \frac{(bf-ag)z^{n-1}dz}{(fz^n-a)(b-gz^n)};$$

quibus valoribus substitutis formula proposita fit rationalis.

PROBLEMA 11

29. *Si fuerit X functio rationalis binarum quantitatum x^n et s existente*

$$s = \sqrt[m]{\frac{a+bx^n}{f+gx^n}},$$

formulam differentialem $X\frac{dx}{x}$ ab omni irrationalitate liberare.

SOLUTIO

Statuatur $s = \sqrt[m]{\frac{a+bx^n}{f+gx^n}} = z$ eritque $\frac{a+bx^n}{f+gx^n} = z^m$, unde fit

$$x^n = \frac{fz^m-a}{b-gz^m};$$

hinc sumtis logarithmis erit

$$nlx = l(fz^m-a) - l(b-gz^m),$$

hinc differentiando

$$\frac{ndx}{x} = \frac{m(bf-ag)z^{m-1}dz}{(fz^m-a)(b-gz^m)}$$

ideoque

$$\frac{dx}{x} = \frac{m(bf-ag)z^{m-1}dz}{n(fz^m-a)(b-gz^m)};$$

quibus valoribus substitutis irrationalitas formulae propositae penitus tollitur.

PROBLEMA 12

30. *Si fuerit X functio rationalis quaecunque binarum quantitatum x et s existente*

$$s = \sqrt{(\alpha + \beta x + \gamma xx)},$$

formulam differentialem Xdx ad rationalitatem perducere.

SOLUTIO

Hic duos casus a se invicem distingui convenit, prout γ fuerit vel quantitas positiva vel negativa.

I. Sit γ quantitas positiva ac ponatur $\gamma = cc$ et $\beta = 2bc$, ut habeatur

$$s = \sqrt{(\alpha + 2bcx + ccxx)} = \sqrt{(\alpha - bb + (b + cx)^2)},$$

ubi loco $\alpha - bb$ brevitatis ergo scribatur e, ut sit $s = \sqrt{(e + (b + cx)^2)}$. Iam statuatur $s = b + cx + z$ eritque

$$ss = e + (b + cx)^2 = (b + cx)^2 + 2(b + cx)z + zz,$$

unde sequitur

$$e - zz = 2z(b + cx) \quad \text{sive} \quad b + cx = \frac{e - zz}{2z};$$

hincque colligitur

$$x = \frac{e - zz}{2cz} - \frac{b}{c} \quad \text{seu} \quad x = \frac{e - 2bz - zz}{2cz}.$$

Aequatio autem $b + cx = \frac{e - zz}{2z}$ differentiata praebet

$$cdx = -\frac{edz}{2zz} - \frac{dz}{2} = -\frac{edz + zzdz}{2zz},$$

unde deducitur

$$dx = -\frac{dz(e + zz)}{2czz};$$

at ob $b + cx = \frac{e - zz}{2z}$ fiet

$$s = \frac{e + zz}{2z}.$$

His ergo valoribus substitutis formula nostra Xdx reddetur rationalis. Postquam igitur eius integrale fuerit inventum, loco z valor ante inventus $\sqrt{(e + (b + cx)^2)} - b - cx$ erit substituendus.

II. Sin autem γ fuerit quantitas negativa, ponatur $\gamma = -cc$ et $\beta = -2bc$, ut habeatur

$$s = \sqrt{(\alpha - 2bcx - ccxx)} = \sqrt{(\alpha + bb - (b + cx)^2)},$$

ubi evidens est quantitatem $\alpha + bb$ necessario esse debere positivam, quia alioquin s evaderet imaginarium. Quamobrem ponamus brevitatis gratia $\alpha + bb = aa$, ut fiat $s = \sqrt{(aa - (b + cx)^2)}$, ad quam formam rationalem efficiendam statuamus

$$\sqrt{(aa - (b + cx)^2)} = a - (b + cx)z,$$

unde sumtis quadratis erit

$$aa - (b + cx)^2 = aa - 2az(b + cx) + (b + cx)^2 zz,$$

quae aequatio reducitur ad hanc

$$-(b + cx) = -2az + (b + cx)zz,$$

unde reperitur

$$b + cx = \frac{2az}{1 + zz}$$

ideoque

$$x = \frac{2az - b - bzz}{c(1 + zz)}.$$

Illa autem aequatio differentiata dat

$$cdx = \frac{2adz(1 + zz) - 4azzdz}{(1 + zz)^2} = \frac{2adz(1 - zz)}{(1 + zz)^2},$$

unde fit

$$dx = \frac{2adz(1 - zz)}{c(1 + zz)^2}.$$

Porro autem cum sit $s = a - (b + cx)z$, ob $b + cx = \frac{2az}{1 + zz}$ erit

$$s = \frac{a(1 - zz)}{1 + zz},$$

quocirca, si loco x, s et dx inventi hi valores substituantur, formula proposita differentialis Xdx evadet rationalis et per variabilem z exprimetur; cuius integrale postquam fuerit inventum, loco z ubique eius restituatur valor assumtus

$$z = \frac{a - \sqrt{(aa - (b + cx)^2)}}{b + cx}$$

et integrale obtinebitur per solam variabilem x expressum.

EXEMPLUM 1

31. Si fuerit

$$dy = \frac{dx}{\sqrt{(e+(b+cx)^2)}},$$

quae formula ad casum priorem pertinet, erit

$$dy = \frac{dx}{s} = -\frac{dz}{cz}$$

ob

$$dx = -\frac{dz(e+zz)}{2czz} \quad \text{et} \quad s = \frac{e+zz}{2z},$$

cuius integrale est $y = -\frac{1}{c}lz$; restituto ergo valore

$$z = \sqrt{(e+(b+cx)^2)} - b - cx$$

erit

$$y = -\frac{1}{c}l\left(\sqrt{(e+(b+cx)^2)} - b - cx\right) + C;$$

quod integrale si evanescere debeat posito $x = 0$, fiet

$$C = \frac{1}{c}l\left(\sqrt{(e+bb)} - b\right).$$

COROLLARIUM

32. Si ponatur $b = 0$ et $c = 1$ sive

$$dy = \frac{dx}{\sqrt{(e+xx)}},$$

erit integrale

$$y = -l\left(\sqrt{(e+xx)} - x\right) + l\sqrt{e} = l\frac{\sqrt{e}}{\sqrt{(e+xx)}-x},$$

quae formula reducitur ad hanc

$$y = l\frac{\sqrt{(e+xx)}+x}{\sqrt{e}}.$$

Cum vero porro sit $d.\sqrt{(e+xx)} = \frac{xdx}{\sqrt{(e+xx)}}$, erit

$$\int\frac{xdx}{\sqrt{(e+xx)}} = \sqrt{(e+xx)}.$$

Si igitur hae duae formulae combinentur, habebitur ista integratio notatu digna

$$\int\frac{Adx+Bxdx}{\sqrt{(e+xx)}} = Al\frac{\sqrt{(e+xx)}+x}{\sqrt{e}} + B\sqrt{(e+xx)}.$$

EXEMPLUM 2

33. Sit

$$dy = \frac{dx}{\sqrt{(aa-(b+cx)^2)}},$$

quae formula ad casum secundum est referenda, ita ut sit $dy = \frac{dx}{s}$. Cum igitur sit

$$dx = \frac{2adz(1-zz)}{c(1+zz)^2} \quad \text{et} \quad s = \frac{a(1-zz)}{1+zz},$$

erit

$$dy = \frac{dx}{s} = \frac{2}{c}\cdot\frac{dz}{1+zz},$$

unde fit integrando $y = \frac{2}{c}$ A tang. z. Quia igitur est

$$z = \frac{a-\sqrt{(aa-(b+cx)^2)}}{b+cx},$$

erit

$$y = \frac{2}{c}\,\text{A tang.}\,\frac{a-\sqrt{(aa-(b+cx)^2)}}{b+cx} + C.$$

COROLLARIUM

34. Sit igitur iterum $b = 0$ et $c = 1$ seu formula differentialis proposita

$$dy = \frac{dx}{\sqrt{(aa-xx)}}$$

reperieturque

$$y = 2\,\text{A tang.}\,\frac{a-\sqrt{(aa-xx)}}{x} + C.$$

Quia igitur tangens huius arcus est $\frac{a-\sqrt{(aa-xx)}}{x}$, tangens dupli arcus erit $\frac{x}{\sqrt{(aa-xx)}}$, ita ut sit

$$y = \text{A tang.}\,\frac{x}{\sqrt{(aa-xx)}};$$

huius autem arcus sinus erit $\frac{x}{a}$ sicque integrale quaesitum

$$\int \frac{dx}{\sqrt{(aa-xx)}} = \text{A sin.} \frac{x}{a}.$$

Quia porro $d.\sqrt{(aa-xx)} = -\frac{xdx}{\sqrt{(aa-xx)}}$, erit

$$\int \frac{xdx}{\sqrt{(aa-xx)}} = -\sqrt{(aa-xx)},$$

quocirca ista generalior conficitur integratio

$$\int \frac{Adx + Bxdx}{\sqrt{(aa-xx)}} = A\text{A sin.} \frac{x}{a} - B\sqrt{(aa-xx)}.$$

PROBLEMA 13

35. *Si fuerit V functio rationalis binarum quantitatum v^n et s existente*

$$s = \sqrt{(\alpha + \beta v^n + \gamma v^{2n})},$$

formulam differentialem $Vv^{n-1}dv$ ab irrationalitate liberare.

SOLUTIO

Ponatur $v^n = x$; erit

$$s = \sqrt{(\alpha + \beta x + xx)} \quad \text{et} \quad v^{n-1}dv = \frac{dx}{n};$$

hic ergo iam erit V functio rationalis binarum quantitatum x et s existente $s = \sqrt{(\alpha + \beta x + \gamma xx)}$ et formula ab irrationalitate liberanda erit $\frac{Vdx}{n}$; qui casus prorsus convenit cum problemate praecedente ideoque eandem habebit solutionem.

SCHOLION

36. Praecepta hactenus tradita ad omnes fere formulas differentiales, quae quidem adhuc tractari potuerunt, extenduntur. Interim tamen eiusmodi casus occurrere possunt, quibus idonea substitutio ad irrationalitatem tollendam neccessaria non tam facile perspicitur, sed acri iudicio demum investigari licet; in quo negotio cum praecepta generalia tradere nondum liceat, exempla quaedam particularia speciminis loco in medium afferamus.

EXEMPLUM 1

37. *Si proposita fuerit haec formula irrationalis*

$$dP = \frac{dx(1+xx)}{(1-xx)\sqrt{(1+x^4)}},$$

eius integrale P investigare.

Si quis hic eiusmodi uti vellet substitutione, qua formula $\sqrt{(1+x^4)}$ ad rationalitatem perduceretur, oleum et operam esset perditurus; interim tamen singulari artificio sequens substitutio negotium conficere poterit. Statuatur

$$\frac{x\sqrt{2}}{1-xx} = p$$

eritque $1 + pp = \frac{1+x^4}{(1-xx)^2}$, hinc

$$\sqrt{(1+pp)} = \frac{\sqrt{(1+x^4)}}{1-xx};$$

tum vero erit differentiando

$$dp = \frac{dx\sqrt{2}(1+xx)}{(1-xx)^2},$$

ex quibus valoribus colligitur

$$\frac{dp}{\sqrt{(1+pp)}} = \frac{dx\sqrt{2}(1+xx)}{(1-xx)\sqrt{(1+x^4)}},$$

quae feliciter cum formula ipsa proposita convenit, ita ut sit

$$\frac{dp}{\sqrt{(1+pp)}} = dP\sqrt{2} \quad \text{sive} \quad dP = \frac{1}{\sqrt{2}} \cdot \frac{dp}{\sqrt{(1+pp)}},$$

unde colligitur integrando

$$P = \frac{1}{\sqrt{2}} l \left(\sqrt{(1+pp)} + p\right).$$

Quare si loco p et $\sqrt{(1+pp)}$ valores dati substituantur, haec obtinetur integratio satis memorabilis

$$P = \int \frac{dx(1+xx)}{(1-xx)\sqrt{(1+x^4)}} = \frac{1}{\sqrt{2}} l \frac{\sqrt{(1+x^4)} + x\sqrt{2}}{1-xx}.$$

EXEMPLUM 2

38. *Si proposita fuerit haec formula irrationalis*

$$\frac{dx(1-xx)}{(1+xx)\sqrt{(1+x^4)}},$$

eius integrale Q investigare.

Ad hoc praestandum fiat

$$\frac{x\sqrt{2}}{1+xx}=q$$

eritque

$$\sqrt{(1-qq)}=\frac{\sqrt{(1+x^4)}}{1+xx};$$

tum vero erit

$$dq=\frac{dx(1-xx)\sqrt{2}}{(1+xx)^2}$$

atque hinc colligitur

$$\frac{dq}{\sqrt{(1-qq)}}=\frac{dx(1-xx)\sqrt{2}}{(1+xx)\sqrt{(1+x^4)}}=dQ\sqrt{2},$$

unde fit

$$Q=\frac{1}{\sqrt{2}}\int\frac{dq}{\sqrt{(1-qq)}}=\frac{1}{\sqrt{2}}\,\text{A}\sin.q.$$

Restituto ergo pro q valore assumto ista obtinebitur integratio

$$Q=\int\frac{dx(1-xx)}{(1+xx)\sqrt{(1+x^4)}}=\frac{1}{\sqrt{2}}\,\text{A}\sin.\frac{x\sqrt{2}}{1+xx}.$$

SCHOLION

39. Cum istae duae formulae

$$\frac{dx(1+xx)\sqrt{2}}{(1-xx)\sqrt{(1+x^4)}}\quad\text{et}\quad\frac{dx(1-xx)\sqrt{2}}{(1+xx)\sqrt{(1+x^4)}}$$

perductae sint ad has simplices

$$\frac{dp}{\sqrt{(1+pp)}}\quad\text{et}\quad\frac{dq}{\sqrt{(1-qq)}},$$

quarum utraque facile ab irrationalitate liberatur, istae ipsae formulae propositae ope idoneae substitutionis ab irrationalitate liberari possunt; unde mirum non est earum integralia sive per logarithmum sive per arcum circularem exhiberi potuisse. Satis enim iam est ostensum omnium formularum differentialium rationalium integralia semper vel per logarithmos et arcus circulares vel adeo algebraice exhiberi posse; quod igitur etiam de illis formulis irrationalibus est tenendum, quas certae substitutionis ope ad rationalitatem perducere licet. Unde vicissim plures Geometrae concluserunt, si quae formula differentialis nullo plane modo ab irrationalitate liberari queat, tum eius integrale etiam neque per logarithmos nec arcus circulares, multo minus algebraice exprimi posse, sed ad aliud genus quantitatum transcendentium referri oportere. Ceterum combinatio duorum praecedentium exemplorum manuducit ad solutionem sequentium.

EXEMPLUM 3

40. *Si proposita fuerit haec formula differentialis*

$$dy=\frac{dx\sqrt{(1+x^4)}}{1-x^4},$$

eius integrale invenire.

Hanc formulam per neutram substitutionem ante usurpatam rationalem reddere licet, utraque tamen iuncta negotium confici poterit; namque eius integrale per logarithmos et arcus circulares sequenti artificio expedietur. Formula enim proposita in binas sequentes partes discerpi potest, quae sunt

$$dy=\frac{\frac{1}{2}dx(1+xx)}{(1-xx)\sqrt{(1+x^4)}}+\frac{\frac{1}{2}dx(1-xx)}{(1+xx)\sqrt{(1+x^4)}},$$

quippe quarum summa ipsam formulam nostram propositam producit; prodit enim

$$dy=\frac{\frac{1}{2}dx(1+xx)^2+\frac{1}{2}dx(1-xx)^2}{(1-x^4)\sqrt{(1+x^4)}}=\frac{dx(1+x^4)}{(1-x^4)\sqrt{(1+x^4)}}=\frac{dx\sqrt{(1+x^4)}}{1-x^4}.$$

Quodsi ergo duo praecedentia exempla in subsidium vocentur, manifesto fiet $dy=\frac{1}{2}dP+\frac{1}{2}dQ$, consequenter integrale quaesitum erit $y=\frac{1}{2}P+\frac{1}{2}Q$, quod sequenti modo exprimere licebit

$$\int \frac{dx\sqrt{(1+x^4)}}{1-x^4} = \frac{1}{2\sqrt{2}}\, l \frac{\sqrt{(1+x^4)}+x\sqrt{2}}{1-xx} + \frac{1}{2\sqrt{2}}\,\mathrm{A\,sin.} \frac{x\sqrt{2}}{1+xx}.$$

EXEMPLUM 4

41. *Si proposita fuerit haec formula differentialis*

$$dy = \frac{xxdx}{(1-x^4)\sqrt{(1+x^4)}},$$

eius integrale investigare.

Haec formula simili modo ac praecedens tractari potest; discerpatur enim in sequentes duas partes

$$\frac{\frac{1}{4}dx(1+xx)}{(1-xx)\sqrt{(1+x^4)}} - \frac{\frac{1}{4}dx(1-xx)}{(1+xx)\sqrt{(1+x^4)}},$$

quippe quae coniunctae producunt

$$dy = \frac{\frac{1}{4}dx(1+xx)^2 - \frac{1}{4}dx(1-xx)^2}{(1-x^4)\sqrt{(1+x^4)}} = \frac{\frac{1}{4}dx\cdot 4xx}{(1-x^4)\sqrt{(1+x^4)}} = \frac{xxdx}{(1-x^4)\sqrt{(1+x^4)}};$$

quae cum sit ipsa formula proposita, erit ex praecedentibus exemplis $dy = \frac{1}{4}dP - \frac{1}{4}dQ$, consequenter $y = \frac{1}{4}P - \frac{1}{4}Q$, hinc integrale quaesitum ita reperietur expressum

$$\int \frac{xxdx}{(1-x^4)\sqrt{(1+x^4)}} = \frac{1}{4\sqrt{2}}\, l \frac{\sqrt{(1+x^4)}+x\sqrt{2}}{1-xx} - \frac{1}{4\sqrt{2}}\,\mathrm{A\,sin.} \frac{x\sqrt{2}}{1+xx}.$$

SCHOLION

42. Haec duo postrema exempla si nullo plane modo ope cuiuspiam substitutionis ad rationalitatem perduci possent, insigne praeberent documentum, quod conclusio supra memorata quandoque fallere possit. Re autem attentius perpensa inveni omnia haec quatuor exempla ope unicae substitutionis immediate ad rationalitatem perduci ideoque integrari posse; id quod ostendisse utique operae erit pretium.

ALIA RESOLUTIO QUATUOR POSTREMORUM EXEMPLORUM

43. Statuatur pro primo exemplo

$$v = \frac{x\sqrt{2}}{\sqrt{(1+x^4)}}$$

eritque

$$\sqrt{(1+vv)} = \frac{1+xx}{\sqrt{(1+x^4)}};$$

tum vero

$$\sqrt{(1-vv)} = \frac{1-xx}{\sqrt{(1+x^4)}},$$

unde fit

$$\sqrt{\frac{1+vv}{1-vv}} = \frac{1+xx}{1-xx} \quad \text{et} \quad \sqrt{(1-v^4)} = \frac{1-x^4}{1+x^4}.$$

At differentiando adipiscimur

$$dv = \frac{dx(1-x^4)\sqrt{2}}{(1+x^4)\sqrt{(1+x^4)}}.$$

Cum nunc sit $\frac{1-x^4}{1+x^4} = \sqrt{(1-v^4)}$, erit $dv = \frac{dx\sqrt{2}\sqrt{(1-v^4)}}{\sqrt{(1+x^4)}}$ sive

$$\frac{dv}{\sqrt{(1-v^4)}} = \frac{dx\sqrt{2}}{\sqrt{(1+x^4)}},$$

quae aequalitas maxime est notatu digna. Quodsi iam haec aequatio multiplicetur per $\sqrt{\frac{1+vv}{1-vv}} = \frac{1+xx}{1-xx}$, nascetur haec aequatio

$$\frac{dv}{1-vv} = \frac{dx(1+xx)\sqrt{2}}{(1-xx)\sqrt{(1+x^4)}}$$

sicque erit

$$\int \frac{dx(1+xx)}{(1-xx)\sqrt{(1+x^4)}} = \frac{1}{\sqrt{2}} \int \frac{dv}{1-vv} = \frac{1}{2\sqrt{2}}\, l \frac{1+v}{1-v}.$$

Deinde aequatio

$$\frac{1}{\sqrt{2}} \cdot \frac{dv}{\sqrt{(1-v^4)}} = \frac{dx}{\sqrt{(1+x^4)}}$$

multiplicetur per $\sqrt{\frac{1-vv}{1+vv}} = \frac{1-xx}{1+xx}$ ac prodibit formula exempli secundi

$$\int \frac{dx(1-xx)}{(1+xx)\sqrt{(1+x^4)}} = \frac{1}{\sqrt{2}} \int \frac{dv}{1+vv} = \frac{1}{\sqrt{2}}\,\mathrm{A\,tang.}\, v.$$

Porro eadem aequatio

$$\frac{1}{\sqrt{2}}\cdot\frac{dv}{\sqrt{(1-v^4)}}=\frac{dx}{\sqrt{(1+x^4)}}$$

dividatur per $\sqrt{(1-v^4)}=\frac{1-x^4}{1+x^4}$ et prodibit

$$\frac{1}{\sqrt{2}}\cdot\frac{dv}{1-v^4}=\frac{dx\sqrt{(1+x^4)}}{1-x^4};$$

quae est ipsa formula exempli tertii, ita ut iam sit

$$\int\frac{dx\sqrt{(1+x^4)}}{1-x^4}=\frac{1}{\sqrt{2}}\int\frac{dv}{1-v^4}=\frac{1}{2\sqrt{2}}\int\frac{dv}{1+vv}+\frac{1}{2\sqrt{2}}\int\frac{dv}{1-vv},$$

quod integrale cum ante invento egregie convenit. Tandem postrema aequatio hic inventa

$$\frac{1}{\sqrt{2}}\cdot\frac{dv}{1-v^4}=\frac{dx\sqrt{(1+x^4)}}{1-x^4}$$

ducatur in $vv=\frac{2xx}{1+x^4}$, ut prodeat

$$\frac{1}{\sqrt{2}}\cdot\frac{vvdv}{1-v^4}=\frac{2xxdx\sqrt{(1+x^4)}}{(1-x^4)(1+x^4)}=\frac{2xxdx}{(1-x^4)\sqrt{(1+x^4)}},$$

unde pro exemplo quarto colligitur

$$\int\frac{xxdx}{(1-x^4)\sqrt{(1+x^4)}}=\frac{1}{2\sqrt{2}}\int\frac{vvdv}{1-v^4}=-\frac{1}{4\sqrt{2}}\int\frac{dv}{1+vv}+\frac{1}{4\sqrt{2}}\int\frac{dv}{1-vv},$$

unde, cum sit $v=\frac{x\sqrt{2}}{\sqrt{(1+x^4)}}$, erit

$$\int\frac{dv}{1-vv}=\frac{1}{2}\,l\,\frac{1+v}{1-v}=\frac{1}{2}\,l\,\frac{\sqrt{(1+x^4)}+x\sqrt{2}}{\sqrt{(1+x^4)}-x\sqrt{2}}=\frac{1}{2}\,l\,\frac{(\sqrt{(1+x^4)}+x\sqrt{2})^2}{(1-xx)^2}$$
$$=l\,\frac{\sqrt{(1+x^4)}+x\sqrt{2}}{1-xx}.$$

Deinde vero est

$$\int\frac{dv}{1+vv}=\text{A tang.}\,v=\text{A sin.}\,\frac{v}{\sqrt{(1+vv)}}=\text{A sin.}\,\frac{x\sqrt{2}}{1+xx}.$$

SCHOLION

44. Quanquam autem haec quatuor exempla ad rationalitatem reducere licuit, tamen conclusio supra memorata, quod omnes formulae integrales, quae nullo modo rationales effici queant, ad aliud pertineant transcendentium genus neque per solos logarithmos et arcus circulares expediri possint, non solum manet suspecta, sed etiam falsitas eius evidenter ob oculos poni potest. Sit enim functio

$$X=\frac{a}{\sqrt{(1+xx)}}+\frac{b}{\sqrt[3]{(1+x^3)}}+\frac{c}{\sqrt[4]{(1+x^4)}};$$

tum certe formula differentialis Xdx nullo modo ad rationalitatem perduci poterit; interim tamen singulae eius partes

$$\frac{adx}{\sqrt{(1+xx)}},\quad \frac{bdx}{\sqrt[3]{(1+x^3)}}\quad\text{et}\quad \frac{cdx}{\sqrt[4]{(1+x^4)}}$$

seorsim rationales effici et integralia per logarithmos et arcus circulares exhiberi possunt. Coronidis loco hic sequens problema notatu dignum adiungamus.

PROBLEMA 14

45. *Formularum integralium*

$$\int\frac{dx}{\sqrt{(1+x^4)}}\quad\textit{et}\quad\int\frac{dv}{\sqrt{(1-v^4)}}$$

valores per series investigare pro casibus, quibus ponitur tam $v=1$ *quam* $x=1$.

SOLUTIO

Cum posito $v=\frac{x\sqrt{2}}{\sqrt{(1+x^4)}}$, ut supra fecimus, evidens sit sumto $x=0$ fore etiam $v=0$ et sumto $x=1$ fore $v=1$, ita ut hae duae quantitates x et v simul evanescant et simul unitati aequentur, hinc deducimus istam aequationem differentialem attentione dignissimam

$$\frac{1}{\sqrt{2}}\cdot\frac{dv}{\sqrt{(1-v^4)}}=\frac{dx}{\sqrt{(1+x^4)}},$$

quas ergo ambas formulas in series converti oportet; erit autem

$$\frac{1}{\sqrt{(1-v^4)}}=(1-v^4)^{-\frac{1}{2}}=1+\frac{1}{2}v^4+\frac{1\cdot 3}{2\cdot 4}v^8+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}v^{12}+\text{etc.}$$

et

$$\frac{1}{\sqrt{(1+x^4)}}=(1+x^4)^{-\frac{1}{2}}=1-\frac{1}{2}x^4+\frac{1\cdot 3}{2\cdot 4}x^8-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^{12}+\text{etc.}$$

Illa iam per dv multiplicata et integrata praebet

$$\int\frac{dv}{\sqrt{(1-v^4)}}=v+\frac{1}{2\cdot 5}v^5+\frac{1\cdot 3}{2\cdot 4\cdot 9}v^9+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 13}v^{13}+\text{etc.},$$

unde posito $v=1$ valor huius integralis erit

$$1+\frac{1}{2\cdot 5}+\frac{1\cdot 3}{2\cdot 4\cdot 9}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 13}+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 17}+\text{etc.},$$

quam seriem littera A indicemus. Simili modo altera series in dx ducta et integrata producit

$$\int\frac{dx}{\sqrt{(1+x^4)}}=x-\frac{1}{2\cdot 5}x^5+\frac{1\cdot 3}{2\cdot 4\cdot 9}x^9-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 13}x^{13}+\text{etc.},$$

cuius valor facto $x=1$ erit

$$1-\frac{1}{2\cdot 5}+\frac{1\cdot 3}{2\cdot 4\cdot 9}-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 13}+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 17}-\text{etc.},$$

quem littera B designemus, ita ut sit $B=\frac{A}{\sqrt{2}}$ sive $A=B\sqrt{2}$, unde patet priorem seriem se habere ad posteriorem ut $\sqrt{2}:1$.

SCHOLION

46. Valor formulae integralis $\int\frac{dv}{\sqrt{(1-v^4)}}$ etiam hoc modo per seriem investigari potest. Cum sit

$$\frac{1}{\sqrt{(1-v^4)}}=\frac{(1+vv)^{-\frac{1}{2}}}{\sqrt{(1-vv)}}$$

et

$$(1+vv)^{-\frac{1}{2}}=1-\frac{1}{2}vv+\frac{1\cdot 3}{2\cdot 4}v^4-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}v^6+\text{etc.},$$

notetur esse $\int\frac{dv}{\sqrt{(1-vv)}}=\frac{\pi}{2}$. Deinde pro integratione reliquorum terminorum ponatur

$$\int\frac{v^{n+2}dv}{\sqrt{(1-vv)}}=Av^{n+1}\sqrt{(1-vv)}+B\int\frac{v^n dv}{\sqrt{(1-vv)}},$$

quae aequatio differentiata dat

$$\frac{v^{n+2}}{\sqrt{(1-vv)}}=(n+1)Av^n\sqrt{(1-vv)}-\frac{Av^{n+2}}{\sqrt{(1-vv)}}+\frac{Bv^n}{\sqrt{(1-vv)}},$$

unde per $\sqrt{(1-vv)}$ multiplicando prodit

$$v^{n+2}=(n+1)Av^n-(n+1)Av^{n+2}-Av^{n+2}+Bv^n.$$

Hinc termini, in quibus inest v^{n+2}, inter se aequati praebent $1=-(n+2)A$ ideoque $A=-\frac{1}{n+2}$, termini vero v^n continentes praebent $0=(n+1)A+B$, unde fit $B=\frac{n+1}{n+2}$, ita ut in genere sit

$$\int\frac{v^{n+2}dv}{\sqrt{(1-vv)}}=-\frac{1}{n+2}v^{n+1}\sqrt{(1-vv)}+\frac{n+1}{n+2}\int\frac{v^n dv}{\sqrt{(1-vv)}},$$

quod integrale, uti requiritur, evanescit posito $v=0$. Ponatur nunc $v=1$ eritque

$$\int\frac{v^{n+2}dv}{\sqrt{(1-vv)}}=\frac{n+1}{n+2}\int\frac{v^n dv}{\sqrt{(1-vv)}};$$

hinc ergo pro n scribendo successive valores 0, 2, 4, 6, 8 etc. erit

I. $$\int\frac{vv\,dv}{\sqrt{(1-vv)}}=\frac{1}{2}\cdot\frac{\pi}{2},$$

II. $$\int\frac{v^4 dv}{\sqrt{(1-vv)}}=\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2},$$

III. $$\int\frac{v^6 dv}{\sqrt{(1-vv)}}=\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

etc.,

quibus valoribus adhibitis erit casu $v = 1$

$$\int \frac{dv}{\sqrt{(1-v^4)}} = \frac{\pi}{2} - \frac{1^2}{2^2}\cdot\frac{\pi}{2} + \frac{1^2\cdot 3^2}{2^2\cdot 4^2}\cdot\frac{\pi}{2} - \frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}\cdot\frac{\pi}{2} + \text{etc.}$$

$$= \frac{\pi}{2}\left(1 - \frac{1^2}{2^2} + \frac{1^2\cdot 3^2}{2^2\cdot 4^2} - \frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2} + \frac{1^2\cdot 3^2\cdot 5^2\cdot 7^2}{2^2\cdot 4^2\cdot 6^2\cdot 8^2} - \text{etc.}\right),$$

ita ut sit ex problemate praecedente

$$1 - \frac{1}{2\cdot 5} + \frac{1\cdot 3}{2\cdot 4\cdot 9} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 9\cdot 13} + \text{etc.} = \frac{\pi}{2}\left(1 - \frac{1^2}{2^2} + \frac{1^2\cdot 3^2}{2^2\cdot 4^2} - \frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2} + \text{etc.}\right),$$

unde fit

$$\frac{\pi}{2} = \frac{1 - \frac{1}{2\cdot 5} + \frac{1\cdot 3}{2\cdot 4\cdot 9} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 9\cdot 13} + \text{etc.}}{1 - \frac{1^2}{2^2} + \frac{1^2\cdot 3^2}{2^2\cdot 4^2} - \frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2} + \text{etc.}}.$$

NOVA METHODUS INTEGRANDI FORMULAS DIFFERENTIALES RATIONALES SINE SUBSIDIO QUANTITATUM IMAGINARIARUM

Commentatio 572 indicis ENESTROEMIANI
Acta academiae scientiarum Petropolitanae 1781: I, 1784, p. 3—47

THEOREMA 1

1. *Si fuerit*

$$xx - 2x\cos.\omega + 1 = 0,$$

tum omnes potestates ipsius x reduci poterunt ad formam simplicem

$$\alpha x + \beta.$$

DEMONSTRATIO

Cum sit $xx - 2x\cos.\omega + 1 = 0$, erit $x^{\lambda+2} = 2x^{\lambda+1}\cos.\omega - x^{\lambda}$; unde si potestates x^{λ} et $x^{\lambda+1}$ ad formam praescriptam $\alpha x + \beta$ redigi queant, tum etiam potestas $x^{\lambda+2}$ per eandem formam exprimi poterit. Incipiamus igitur a potestatibus infimis, quas ita exhibeamus

$$x = \frac{x\sin.\omega}{\sin.\omega} \quad \text{et} \quad xx = \frac{2x\sin.\omega\cos.\omega - \sin.\omega}{\sin.\omega} = \frac{x\sin.2\omega - \sin.\omega}{\sin.\omega}.$$

His igitur constitutis, cum sit

$$2\cos.\omega\sin.\lambda\omega = \sin.(\lambda+1)\omega + \sin.(\lambda-1)\omega,$$

ex his duabus formulis facile eliciemus sequentes

$$x^3 = 2xx\cos.\omega - x = \frac{x\sin.3\omega - \sin.2\omega}{\sin.\omega},$$

$$x^4 = 2x^3\cos.\omega - xx = \frac{x\sin.4\omega - \sin.3\omega}{\sin.\omega},$$

hocque modo quousque libuerit progredi licet; atque hinc in genere concludimus fore

$$x^n = \frac{x\sin.n\omega - \sin.(n-1)\omega}{\sin.\omega},$$

quae ergo expressio formam habet $\alpha x + \beta$.

COROLLARIUM 1

2. Cum sit

$$\sin.(n-1)\omega = \sin.n\omega\cos.\omega - \cos.n\omega\sin.\omega,$$

hoc valore substituto fiet

$$x^n = \frac{x-\cos.\omega}{\sin.\omega}\sin.n\omega + \cos.n\omega;$$

quamobrem si fuerit $xx - 2x\cos.\omega + 1 = 0$, pro omnibus potestatibus ipsius x habebimus hanc reductionem

$$x^n = \frac{x-\cos.\omega}{\sin.\omega}\sin.n\omega + \cos.n\omega,$$

qua forma deinceps potissimum utemur.

COROLLARIUM 2

3. Hinc igitur erit

$$x^{k+n} = \frac{x-\cos.\omega}{\sin.\omega}\sin.(k+n)\omega + \cos.(k+n)\omega$$

et

$$x^{k-n} = \frac{x-\cos.\omega}{\sin.\omega}\sin.(k-n)\omega + \cos.(k-n)\omega,$$

quare his formulis addendis ob

$$\sin.(k+n)\omega + \sin.(k-n)\omega = 2\sin.k\omega\cos.n\omega$$

et

$$\cos.(k+n)\omega + \cos.(k-n)\omega = 2\cos.k\omega\cos.n\omega$$

fiet

$$x^{k+n} + x^{k-n} = \frac{2(x-\cos.\omega)}{\sin.\omega}\sin.k\omega\cos.n\omega + 2\cos.k\omega\cos.n\omega$$

sive

$$x^{k+n} + x^{k-n} = 2\cos.n\omega\left(\frac{x-\cos.\omega}{\sin.\omega}\sin.k\omega + \cos.k\omega\right).$$

COROLLARIUM 3

4. Sin autem potestatem posteriorem a priore subtrahamus, ob

$$\sin.(k+n)\omega - \sin.(k-n)\omega = 2\cos.k\omega\sin.n\omega$$

et

$$\cos.(k+n)\omega - \cos.(k-n)\omega = -2\sin.k\omega\sin.n\omega$$

habebimus

$$x^{k+n} - x^{k-n} = \frac{2(x-\cos.\omega)}{\sin.\omega}\cos.k\omega\sin.n\omega - 2\sin.k\omega\sin.n\omega;$$

hoc est

$$x^{k+n} - x^{k-n} = 2\sin.n\omega\left(\frac{x-\cos.\omega}{\sin.\omega}\cos.k\omega - \sin.k\omega\right).$$

COROLLARIUM 4

5. Etiamsi nostra demonstratio tantum ad potestates integras ipsius x perduxit, tamen ex indole harum formarum facile intelligitur eas etiam pro exponentibus fractis vel adeo irrationalibus locum habere, quandoquidem in ipsis his formulis nihil inest, quod tantum ad valores integros exponentis n restringatur; tum vero etiam nihil impedit, quominus exponenti n valores negativi tribuantur. Si enim verbi gratia sumamus $n = \frac{1}{2}$, per formulam Corollarii 1 esse debet

$$\sqrt{x} = \frac{x-\cos.\omega}{\sin.\omega}\sin.\frac{1}{2}\omega + \cos.\frac{1}{2}\omega,$$

unde sumtis quadratis ob

$$(x-\cos.\omega)^2 = xx - 2x\cos.\omega + \cos.\omega^2 = -\sin.\omega^2$$

habebimus

$$x = -\sin.\tfrac{1}{2}\omega^2 + \frac{2(x-\cos.\omega)}{\sin.\omega}\sin.\tfrac{1}{2}\omega\cos.\tfrac{1}{2}\omega + \cos.\tfrac{1}{2}\omega^2,$$

quae forma ob

$$2\sin.\tfrac{1}{2}\omega\cos.\tfrac{1}{2}\omega = \sin.\omega \quad \text{et} \quad \cos.\tfrac{1}{2}\omega^2 - \sin.\tfrac{1}{2}\omega^2 = \cos.\omega$$

abit in $x = x$, hoc est aequationem identicam.

SCHOLION

6. Formulae, quas hic sumus adepti, egregie conveniunt cum iis, quas calculus imaginariorum suppeditat. Cum enim aequatio $xx - 2x\cos.\omega + 1 = 0$ contineat has radices

$$x = \cos.\omega \pm \sin.\omega\sqrt{-1},$$

erit, uti in analysi est ostensum,

$$x^n = \cos.n\omega \pm \sin.n\omega\sqrt{-1};$$

quare cum sit

$$x^n - \cos.n\omega = \pm\sin.n\omega\sqrt{-1} \quad \text{et} \quad x - \cos.\omega = \pm\sin.\omega\sqrt{-1},$$

illa forma per hanc divisa dabit

$$\frac{x^n - \cos.n\omega}{x - \cos.\omega} = \frac{\sin.n\omega}{\sin.\omega},$$

unde sequitur fore

$$x^n - \cos.n\omega = \frac{x - \cos.\omega}{\sin.\omega}\sin.n\omega,$$

prorsus uti in Corollario 1 invenimus.

Ceterum nostrum theorema generalius proponi et ad aequationem

$$xx - 2ax\cos.\omega + aa = 0$$

extendi potuisset; tum enim prodiisset

$$x^n = \frac{a^{n-1}x\sin.n\omega - a^n\sin.(n-1)\omega}{\sin.\omega},$$

deinde etiam

$$x^n = \frac{a^{n-1}(x - a\cos.\omega)}{\sin.\omega}\sin.n\omega + a^n\cos.n\omega,$$

quae formulae a prioribus non discrepant, nisi quod hic littera a homogeneitatem dimensionum expleat. Hae scilicet formulae ex illis immediate sequentur, si ibi loco x scribatur $\frac{x}{a}$; sed brevitati et concinnitati consulentes eiusmodi tantum casus evolvemus, in quibus pro a commode unitatem scribere liceat.

THEOREMA 2

7. *Si fuerit*

$$xx - 2x\cos.\omega + 1 = 0,$$

omnes functiones rationales integrae, quaecunque potestates ipsius x in iis occurrant, semper reduci possunt ad hanc formam simplicem

$$\alpha x + \beta.$$

DEMONSTRATIO

Si functio proposita iam penitus fuerit evoluta, ita ut nullos factores complectatur, tum ea ope reductionis

$$x^n = \frac{x - \cos.\omega}{\sin.\omega}\sin.n\omega + \cos.n\omega$$

sponte redigitur ad talem formam $\frac{F(x - \cos.\omega)}{\sin.\omega} + G$. Verum si functio proposita duobus constet factoribus, veluti Pp, ac per istam reductionem prodierit

$$P = \frac{F(x - \cos.\omega)}{\sin.\omega} + G \quad \text{et} \quad p = \frac{f(x - \cos.\omega)}{\sin.\omega} + g,$$

tum facta multiplicatione ob $(x - \cos.\omega)^2 = -\sin.\omega^2$ colligitur fore

$$Pp = -Ff + Gg + \frac{(Fg + fG)(x - \cos.\omega)}{\sin.\omega},$$

quod ergo productum eiusdem est formae; unde simul patet, quotcunque eiusmodi dentur factores, eorum productum semper ad eandem formam reduci posse.

COROLLARIUM 1

8. Quodsi hoc modo prodierit

$$P=\frac{F(x-\cos.\omega)}{\sin.\omega}+G,$$

tum erit

$$P(x-\cos.\omega)=-F\sin.\omega+G(x-\cos.\omega),$$

quae expressio ideo est notatu digna, quod in sequentibus integrationibus ubique occurret.

COROLLARIUM 2

9. Si functio P factorem habuerit $xx-2x\cos.\omega+1$, tum posito, uti assumsimus,

$$xx-2x\cos.\omega+1=0,$$

valor ipsius P etiam evanescere debet. Hoc ergo casu formula $\frac{F(x-\cos.\omega)}{\sin.\omega}+G$ fiet $=0$, id quod ob x quantitatem indefinitam aliter evenire nequit, nisi fuerit et $F=0$ et $G=0$. Atque hinc vicissim, si facta reductione prodeat $P=0$, hoc certum erit signum ipsam functionem involvere factorem

$$xx-2x\cos.\omega+1.$$

THEOREMA 3

10. *Si fuerit*

$$xx-2x\cos.\omega+1=0,$$

tum etiam omnes functiones fractae rationales semper ad formam simplicem

$$\alpha x+\beta$$

reduci possunt.

DEMONSTRATIO

Sit enim proposita functio quaecunque fracta $\frac{P}{p}$ atque adhibita nostra reductione prodierit

$$P=\frac{F(x-\cos.\omega)}{\sin.\omega}+G \quad \text{et} \quad p=\frac{f(x-\cos.\omega)}{\sin.\omega}+g,$$

ita ut pervenerimus ad hanc fractionem

$$\frac{P}{p}=\frac{\frac{F(x-\cos.\omega)}{\sin.\omega}+G}{\frac{f(x-\cos.\omega)}{\sin.\omega}+g}.$$

Iam ut ipsam litteram x ex denominatore expellamus, multiplicemus tam numeratorem quam denominatorem per formulam $\frac{f(x-\cos.\omega)}{\sin.\omega}-g$; sic enim ob

$$(x-\cos.\omega)^2=-\sin.\omega^2$$

pro denominatore reperiemus $-ff-gg$, at vero pro numeratore

$$-Ff+\frac{(fG-Fg)(x-\cos.\omega)}{\sin.\omega}-Gg,$$

unde mutatis signis forma nostrae fractionis erit

$$\frac{P}{p}=\frac{\frac{(Fg-fG)(x-\cos.\omega)}{\sin.\omega}+Ff+Gg}{ff+gg}$$

sive concinnius

$$\frac{P}{p}=\frac{Fg-fG}{ff+gg}\cdot\frac{x-\cos.\omega}{\sin.\omega}+\frac{Ff+Gg}{ff+gg}.$$

PROBLEMA

11. *Proposita formula differentiali rationali quacunque, eam in suas fractiones partiales resolvere ac deinceps eius integrale investigare.*

SOLUTIO

Repraesentetur formula differentialis sub hac specie $\frac{P}{Q}\cdot\frac{dx}{x}=\frac{Pdx}{Qx}$, ita tamen, ut $\frac{P}{x}$ maneat functio integra, ne x sit factor denominatoris. Ante omnia quaerantur igitur ipsius Q omnes factores tam simplices quam duplices reales; et quia simplices nulla laborant difficultate, hic tantum duplices sum contemplaturus, quorum forma sit $xx-2x\cos.\omega+1$, ita ut posito $xx-2x\cos.\omega+1=0$ quantitas Q simul in nihilum abeat; ex qua conditione omnes valores anguli ω elici poterunt, ita ut hoc modo omnes factores denominatoris Q obtineantur. Nunc igitur fractionem $\frac{P}{Qx}$ in totidem fractiones partiales resolvi oportet, quot inventi fuerint factores formae

$$xx - 2x\cos.\omega + 1.$$

Sit igitur in genere fractio partialis ex isto factore nata

$$= \frac{\alpha x + \beta}{xx - 2x\cos.\omega + 1},$$

quandoquidem novimus eius numeratorem talem formam $\alpha x + \beta$ habere debere; pro reliquis autem fractionibus partialibus omnibus scribamus litteram R, ita ut esse debeat

$$\frac{P}{Qx} = \frac{\alpha x + \beta}{xx - 2x\cos.\omega + 1} + R,$$

et multiplicando per $xx - 2x\cos.\omega + 1$ habebimus

$$\frac{P(xx - 2x\cos.\omega + 1)}{Qx} = \alpha x + \beta + R(xx - 2x\cos.\omega + 1).$$

Quodsi ergo iam faciamus $xx - 2x\cos.\omega + 1 = 0$, erit

$$\alpha x + \beta = \frac{P(xx - 2x\cos.\omega + 1)}{Qx} = \frac{P}{x}\cdot\frac{xx - 2x\cos.\omega + 1}{Q},$$

ubi in priore factore $\frac{P}{x}$ ista substitutio nullam habet difficultatem; verum in altera fractione $\frac{xx - 2x\cos.\omega + 1}{Q}$, quia posito $xx - 2x\cos.\omega + 1 = 0$ non solum numerator, sed etiam denominator Q evanescit, secundum praecepta cognita utriusque loco eius differentiale scribamus, siquidem hoc casu fieri debet

$$\frac{xx - 2x\cos.\omega + 1}{Q} = \frac{2dx(x - \cos.\omega)}{dQ},$$

sicque obtinebitur numerator quaesitus

$$\alpha x + \beta = \frac{2Pdx(x - \cos.\omega)}{xdQ}.$$

Ponamus igitur per hanc substitutionem fieri

$$P = \frac{F(x - \cos.\omega)}{\sin.\omega} + G \quad \text{et} \quad \frac{xdQ}{dx} = \frac{f(x - \cos.\omega)}{\sin.\omega} + g,$$

ita ut sit

$$\frac{Pdx}{xdQ} = \frac{\frac{F(x - \cos.\omega)}{\sin.\omega} + G}{\frac{f(x - \cos.\omega)}{\sin.\omega} + g},$$

quae forma per theorema tertium reducitur ad hanc

$$\frac{Pdx}{xdQ} = \frac{Fg - fG}{ff + gg}\cdot\frac{x - \cos.\omega}{\sin.\omega} + \frac{Ff + Gg}{ff + gg},$$

quae ergo insuper per $2(x - \cos.\omega)$ multiplicata ob $(x - \cos.\omega)^2 = -\sin.\omega^2$ praebet numeratorem quaesitum

$$\alpha x + \beta = \frac{2(Ff + Gg)(x - \cos.\omega)}{ff + gg} + \frac{2(fG - Fg)}{ff + gg}\sin.\omega.$$

Multiplicetur igitur ista forma per $\frac{dx}{xx - 2x\cos.\omega + 1}$ atque obtinebitur pars integralis ex hac fractione partiali oriunda

$$\frac{2(Ff + Gg)}{ff + gg}\int\frac{dx(x - \cos.\omega)}{xx - 2x\cos.\omega + 1} + \frac{2(fG - Fg)}{ff + gg}\sin.\omega\int\frac{dx}{xx - 2x\cos.\omega + 1}.$$

Hic igitur pro priore parte manifesto est

$$\int\frac{dx(x - \cos.\omega)}{xx - 2x\cos.\omega + 1} = l\sqrt{(xx - 2x\cos.\omega + 1)},$$

quod integrale iam ita est sumtum, ut evanescat posito $x = 0$; pro altero autem membro facile reperitur

$$\int\frac{dx\sin.\omega}{1 - 2x\cos.\omega + xx} = \text{A tang.}\frac{x\sin.\omega}{1 - x\cos.\omega},$$

quod itidem evanescit posito $x = 0$, quocirca pars integralis ex denominatoris Q factore $xx - 2x\cos.\omega + 1$ orta erit

$$\frac{2(Ff + Gg)}{ff + gg}l\sqrt{(xx - 2x\cos.\omega + 1)} + \frac{2(fG - Fg)}{ff + gg}\text{A tang.}\frac{x\sin.\omega}{1 - x\cos.\omega}.$$

COROLLARIUM 1

12. Duo autem casus hic singularem evolutionem postulant, alter, quo $\omega = 0$, alter vero, quo $\omega = 180^\circ$; priore enim casu denominator $xx - 2x \cos. \omega + 1$ abit in $(x-1)^2$, posteriore vero in $(x+1)^2$. Cum autem hinc plus concludere non liceat quam vel $1-x$ vel $1+x$ esse factorem denominatoris, his casibus pars integralis in genere inventa tantum ad semissem redigi debet, quemadmodum in principiis calculi integralis fusius est ostensum.[1]) Ceterum his casibus posterior pars a circulo pendens semper evanescet.

COROLLARIUM 2

13. Praeter hos autem binos casus portio integralis ex formula $xx - 2x \cos. \omega + 1$ oriunda semper constabit duabus partibus, altera logarithmica, altera circulari, nisi forte fuerit vel $Ff + Gg = 0$ vel $fG - gF = 0$. Priore enim casu haec portio tantum arcum circularem involvet, posteriore vero tantum logarithmum.

SCHOLION

14. Quoniam assumsimus denominatoris Q factorem esse $xx - 2x \cos. \omega + 1$, alias denominatoris formas hic non contemplabimur, nisi quarum omnes factores tali formula exprimi queant. Tales autem formulae simpliciores occurrunt tres sequentes:

$$Q = 1 + x^{2k}, \quad Q = 1 - x^{2k}, \quad Q = 1 + 2x^k \cos. \eta + x^{2k},$$

ubi quidem in prioribus potestati ipsius x exponentem parem tribuimus, quoniam casus, quibus esset impar, facile ad hanc formam reduci possunt. Si enim denominator esset $1 \pm x^i$ denotante i numerum imparem, tantum loco x scribamus y^2 prodibitque talis forma $1 \pm y^{2i}$; at tali substitutione natura formulae differentialis neutiquam mutatur. Hos ergo tres casus in sequentibus tribus problematibus particularibus omni cura percurramus, quo magis praestantia istius novae methodi prae aliis, quae adhuc in usu fuerunt, eluceat.

1) Vide L. EULERI *Institutionum calculi integralis* vol. I, § 77, Petropoli 1768; *LEONHARDI EULERI Opera omnia*, series I, vol. 11, p. 41. Vide etiam EULERI Commentationem 462 (indicis ENESTROEMIANI): *De valore formulae integralis* $\int \frac{z^{m-1} \pm z^{n-m-1}}{1 \pm z^n} dz$ *casu, quo post integrationem ponitur* $z = 1$, Novi comment. acad. sc. Petrop. 19 (1774), 1775, p. 3; *LEONHARDI EULERI Opera omnia*, series I, vol. 17, p. 358, imprimis p. 369. Cf. porro eiusdem voluminis 17 Commentationes 60 et 162, imprimis p. 59, 126, 134. A. G.

PROBLEMA PARTICULARE 1

15. *Si fuerit*

$$Q = 1 + x^{2k},$$

investigare integrale huius formulae differentialis

$$\frac{P dx}{(1 + x^{2k}) x},$$

ubi quidem $\frac{P}{x}$ *sit functio integra, in qua nullae potestates altiores occurrant quam exponentis* $2k$, *ne scilicet ista fractio evadat spuria.*

SOLUTIO

Cum sit $Q = 1 + x^{2k}$, sit eius factor trinomialis quicunque

$$= xx - 2x \cos. \omega + 1,$$

ita ut numerus talium factorum sit $= k$; quare cum posito

$$xx - 2x \cos. \omega + 1 = 0$$

etiam ipsa formula $1 + x^{2k}$ evanescere debeat, facta substitutione debita secundum Theorema 2 fiet

$$Q = 1 + \frac{(x - \cos. \omega)}{\sin. \omega} \sin. 2k\omega + \cos. 2k\omega;$$

qui valor cum debeat evanescere, erit tam $\sin. 2k\omega = 0$ quam $1 + \cos. 2k\omega = 0$. Conditio ergo posterior praebet $\cos. 2k\omega = -1$; unde intelligitur angulum $2k\omega$ esse debere vel π vel 3π vel 5π vel in genere $(2i-1)\pi$ denotante $2i-1$ numerum imparem quemcunque. Valores igitur anguli ω erunt sequentes:

$$1. \ \omega = \frac{\pi}{2k}, \quad 2. \ \omega = \frac{3\pi}{2k}, \quad 3. \ \omega = \frac{5\pi}{2k}$$

et generatim

$$\omega = \frac{(2i-1)\pi}{2k};$$

quorum numerus cum esse debeat $= k$, ultimus valor erit $\omega = \frac{(2k-1)\pi}{2k}$; singulis autem istis valoribus simul prior conditio adimpletur, qua esse debet

$\sin. 2k\omega = 0$. Quodsi iam pro ω unusquisque horum valorum accipiatur atque ponatur

$$xx - 2x \cos. \omega + 1 = 0,$$

quicunque fuerit numerator P, sumamus facta hac substitutione fieri

$$P = \frac{F(x - \cos. \omega)}{\sin. \omega} + G;$$

tum vero erit $\frac{x\,dQ}{dx} = 2kx^{2k}$, unde, cum nostro casu fieri debeat $Q = 0$, erit utique $x^{2k} = -1$ sicque fiet $\frac{x\,dQ}{dx} = -2k$. Cum igitur haec formula in genere posita sit $\frac{f(x - \cos. \omega)}{\sin. \omega} + g$, erit nunc $f = 0$ et $g = -2k$, quo invento secundum praecepta ante tradita pars integralis ex hoc factore denominatoris

$$xx - 2x \cos. \omega + 1$$

oriunda erit

$$-\frac{G}{k} l \sqrt{(xx - 2x \cos. \omega + 1)} + \frac{F}{k} \text{A tang.} \frac{x \sin. \omega}{1 - x \cos. \omega};$$

consequenter si ex singulis valoribus anguli ω istae partes integralis formentur et in unam summam colligantur, impetrabitur totum integrale formulae differentialis propositae; et quia hoc casu nunquam fieri potest vel $\omega = 0$ vel $\omega = \pi$, cautione supra indicata non erit opus.

COROLLARIUM 1

16. Quodsi numerator P fuerit potestas simplex ipsius x, puta x^m existente $m > 0$, at $m < 2k$, ut formula integranda sit

$$\int \frac{x^{m-1} dx}{1 + x^{2k}},$$

posito $xx - 2x \cos. \omega + 1 = 0$ erit formula

$$P = x^m = \frac{(x - \cos. \omega)}{\sin. \omega} \sin. m\omega + \cos. m\omega$$

ideoque

$$F = \sin. m\omega \quad \text{et} \quad G = \cos. m\omega,$$

unde quaelibet portio integralis induet hanc formam

$$-\frac{\cos. m\omega}{k} l \sqrt{(xx - 2x \cos. \omega + 1)} + \frac{\sin. m\omega}{k} \text{A tang.} \frac{x \sin. \omega}{1 - x \cos. \omega},$$

et aggregatum omnium harum partium, siquidem loco ω successive singuli eius valores substituantur, dabit totum integrale formulae huius propositae ita sumtum, ut evanescat posito $x = 0$.

COROLLARIUM 2

17. Si numerator P ex pluribus huiusmodi terminis constet, ut sit $P = ax^\alpha + bx^\beta + cx^\gamma + \text{etc.}$, integratio maiore difficultate non laborat; erit enim

$$F = a \sin. \alpha\omega + b \sin. \beta\omega + c \sin. \gamma\omega + \text{etc.}$$

et

$$G = a \cos. \alpha\omega + b \cos. \beta\omega + c \cos. \gamma\omega + \text{etc.}$$

hincque totum integrale facile expedietur.

SCHOLION

18. Hic autem occurrit casus imprimis memorabilis, quo sumitur $P = x^{k-n} + x^{k+n}$, quem in sequente Problemate speciali seorsim evolvamus.

PROBLEMA SPECIALE

19. *Proposita formula differentiali*

$$\frac{x^{k-n} + x^{k+n}}{1 + x^{2k}} \cdot \frac{dx}{x}$$

eius totum integrale evolvere.

SOLUTIO

Cum hic sit $P = x^{k-n} + x^{k+n}$, si statuamus

$$xx - 2x \cos. \omega + 1 = 0,$$

fiet

$$P = \frac{x - \cos. \omega}{\sin. \omega} (\sin. (k - n)\omega + \sin. (k + n)\omega) + \cos. (k - n)\omega + \cos. (k + n)\omega,$$

unde sponte se produnt litterae F et G; cum autem in genere sit

$$\sin. p + \sin. q = 2 \sin. \frac{p+q}{2} \cos. \frac{p-q}{2}$$

et

$$\cos. p + \cos. q = 2 \cos. \frac{p+q}{2} \cos. \frac{p-q}{2},$$

facta hac reductione reperietur

$$F = 2\sin. k\omega \cos. n\omega \quad \text{et} \quad G = 2\cos. k\omega \cos. n\omega.$$

Cum autem in genere sit $\omega = \frac{(2i-1)\pi}{2k}$, erit $\sin. k\omega = \sin. \frac{(2i-1)\pi}{2}$, cuius valor est vel $+1$ vel -1; utrumvis autem locum habeat, semper erit $\cos. k\omega = 0$, ita ut sit

$$F = 2\sin. \frac{(2i-1)\pi}{2} \cos. n\frac{(2i-1)\pi}{2k} \quad \text{et} \quad G = 0;$$

quibus valoribus inventis pars integralis ex hoc factore generali oriunda erit

$$\frac{2}{k}\sin. \frac{(2i-1)\pi}{2} \cos. \frac{(2i-1)n\pi}{2k} \text{A tang.} \frac{x\sin. \frac{(2i-1)\pi}{2k}}{1 - x\cos. \frac{(2i-1)\pi}{2k}}.$$

Hinc ergo, si loco i successive scribamus valores 1, 2, 3, 4 etc. usque ad k, totum integrale quaesitum sequenti forma exprimetur:

$$\frac{2}{k}\cos. \frac{n\pi}{2k} \text{A tang.} \frac{x\sin. \frac{\pi}{2k}}{1 - x\cos. \frac{\pi}{2k}} - \frac{2}{k}\cos. \frac{3n\pi}{2k} \text{A tang.} \frac{x\sin. \frac{3\pi}{2k}}{1 - x\cos. \frac{3\pi}{2k}}$$

$$+ \frac{2}{k}\cos. \frac{5n\pi}{2k} \text{A tang.} \frac{x\sin. \frac{5\pi}{2k}}{1 - x\cos. \frac{5\pi}{2k}} - \frac{2}{k}\cos. \frac{7n\pi}{2k} \text{A tang.} \frac{x\sin. \frac{7\pi}{2k}}{1 - x\cos. \frac{7\pi}{2k}}$$

$$\vdots$$

$$+ \frac{2}{k}\sin. \frac{(2k-1)\pi}{2} \cos. \frac{(2k-1)n\pi}{2k} \text{A tang.} \frac{x\sin. \frac{(2k-1)\pi}{2k}}{1 - x\cos. \frac{(2k-1)\pi}{2k}},$$

ubi imprimis notatu dignum usu venit, ut omnes partes logarithmicae se mutuo destruxerint.

COROLLARIUM 1

20. Quodsi ergo sumamus $n = 0$, ita ut formula integranda sit

$$\int \frac{2x^k}{1 + x^{2k}} \cdot \frac{dx}{x},$$

eius integrale hoc modo exprimetur:

$$\frac{2}{k}\text{A tang.} \frac{x\sin. \frac{\pi}{2k}}{1 - x\cos. \frac{\pi}{2k}} - \frac{2}{k}\text{A tang.} \frac{x\sin. \frac{3\pi}{2k}}{1 - x\cos. \frac{3\pi}{2k}} + \frac{2}{k}\text{A tang.} \frac{x\sin. \frac{5\pi}{2k}}{1 - x\cos. \frac{5\pi}{2k}} - \cdots$$

$$+ \frac{2}{k}\sin. \frac{(2k-1)\pi}{2} \text{A tang.} \frac{x\sin. \frac{(2k-1)\pi}{2k}}{1 - x\cos. \frac{(2k-1)\pi}{2k}}.$$

At posito $x^k = z$ ob $\frac{dx}{x} = \frac{dz}{kz}$ formula integralis induet hanc formam $\int \frac{2dz}{k(1+zz)}$, cuius integrale manifesto est

$$\frac{2}{k}\text{A tang.} z = \frac{2}{k}\text{A tang.} x^k,$$

unde sequitur fore

$$\text{A tang.} x^k$$

$$= \text{A tang.} \frac{x\sin. \frac{\pi}{2k}}{1 - x\cos. \frac{\pi}{2k}} - \text{A tang.} \frac{x\sin. \frac{3\pi}{2k}}{1 - x\cos. \frac{3\pi}{2k}} + \text{A tang.} \frac{x\sin. \frac{5\pi}{2k}}{1 - x\cos. \frac{5\pi}{2k}} - \cdots$$

$$+ \sin \frac{(2k-1)\pi}{2} \text{A tang.} \frac{x\sin. \frac{(2k-1)\pi}{2k}}{1 - x\cos. \frac{(2k-1)\pi}{2k}},$$

quod sane est theorema maxima attentione dignum.

COROLLARIUM 2

21. Ad hoc theorema illustrandum sumamus $k = 1$ et ob $\sin. \frac{\pi}{2} = 1$ et $\cos. \frac{\pi}{2} = 0$ prodit manifesto $\text{A tang.} x = \text{A tang.} x$.

At sumto $k = 2$ ob

$$\sin. \frac{\pi}{4} = \frac{1}{\sqrt{2}}, \quad \cos. \frac{\pi}{4} = \frac{1}{\sqrt{2}}, \quad \sin. \frac{3\pi}{4} = \frac{1}{\sqrt{2}} \quad \text{et} \quad \cos. \frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$$

fiet

$$\text{A tang.} xx = \text{A tang.} \frac{x}{\sqrt{2} - x} - \text{A tang.} \frac{x}{\sqrt{2} + x}.$$

Cum autem in genere sit

$$\text{A tang.} p - \text{A tang.} q = \text{A tang.} \frac{p - q}{1 + pq},$$

hoc casu erit

$$p = \frac{x}{\sqrt{2} - x} \quad \text{et} \quad q = \frac{x}{\sqrt{2} + x}$$

ideoque

$$p - q = \frac{2xx}{2 - xx} \quad \text{et} \quad 1 + pq = \frac{2}{2 - xx},$$

unde manifesto prodit $\text{A tang.}\, xx = \text{A tang.}\, xx$.

Sumamus porro $k = 3$ et ob

$$\sin. \frac{\pi}{6} = \frac{1}{2}, \quad \cos. \frac{\pi}{6} = \frac{\sqrt{3}}{2}, \quad \sin. \frac{3\pi}{6} = 1, \quad \cos. \frac{3\pi}{6} = 0,$$
$$\sin. \frac{5\pi}{6} = \frac{1}{2} \quad \text{et} \quad \cos. \frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$$

reperietur

$$\text{A tang.}\, x^3 = \text{A tang.}\, \frac{x}{2 - x\sqrt{3}} - \text{A tang.}\, x + \text{A tang.}\, \frac{x}{2 + x\sqrt{3}},$$

ubi per reductionem superiorem arcus primus et tertius iunctim sumti ob

$$p = \frac{x}{2 - x\sqrt{3}} \quad \text{et} \quad q = \frac{-x}{2 + x\sqrt{3}}$$

praebent $\text{A tang.}\, \frac{x}{1 - xx}$; a quo si subtrahatur $\text{A tang.}\, x$, remanebit $\text{A tang.}\, x^3$.

SCHOLION

22. Ceterum veritas huius theorematis in genere commodissime sumendis differentialibus ostendi potest. Cum enim sit

$$d.\, \text{A tang.}\, x^k = \frac{kx^{k-1}dx}{1 + x^{2k}} \quad \text{et} \quad d.\, \text{A tang.}\, \frac{x \sin. \omega}{1 - x \cos. \omega} = \frac{dx \sin. \omega}{1 - 2x \cos. \omega + xx},$$

si loco ω valores debiti successive substituantur et per dx dividatur, resultabit sequens aequatio

$$\frac{kx^{k-1}}{1 + x^{2k}} = \frac{\sin. \frac{\pi}{2k}}{1 - 2x \cos. \frac{\pi}{2k} + xx} - \frac{\sin. \frac{3\pi}{2k}}{1 - 2x \cos. \frac{3\pi}{2k} + xx} + \cdots \pm \frac{\sin. \frac{(2k-1)\pi}{2k}}{1 - 2x \cos. \frac{(2k-1)\pi}{2k} + xx},$$

quae sunt eae ipsae fractiones partiales, in quas functio fracta $\frac{kx^{k-1}}{1 + x^{2k}}$

resolvitur. Ceterum cum in hac integratione omnes logarithmi excesserint, duplex quaestio circa integrale inventum institui potest, altera, qua quaeritur eius valor casu $x = \infty$, altera vero casu, quo sumitur $x = 1$.

QUAESTIO PRIOR

23. *Proposita formula differentiali*

$$\frac{x^{k-n} + x^{k+n}}{1 + x^{2k}} \cdot \frac{dx}{x}$$

eius integralis valorem investigare, qui oritur, si post integrationem ponitur $x = \infty$.

SOLUTIO

Cum quilibet arcus in expressione integralis inventi § 19 in genere sit huiusmodi $\text{A tang.}\, \frac{x \sin. \omega}{1 - x \cos. \omega}$, si statuatur $x = \infty$, is hanc induet formam $\text{A tang.}\, (-\text{tang.}\, \omega)$. Quia autem $-\text{tang.}\, \omega = +\text{tang.}\, (\pi - \omega)$, iste arcus fiet $= \pi - \omega$; quare si loco ω successive valores debitos substituamus, integrale quaesitum sequenti serie exprimetur

$$\frac{2}{k}\left(\pi - \frac{\pi}{2k}\right) \cos. \frac{n\pi}{2k} - \frac{2}{k}\left(\pi - \frac{3\pi}{2k}\right) \cos. \frac{3n\pi}{2k} + \frac{2}{k}\left(\pi - \frac{5\pi}{2k}\right) \cos. \frac{5n\pi}{2k}$$
$$- \frac{2}{k}\left(\pi - \frac{7\pi}{2k}\right) \cos. \frac{7n\pi}{2k} + \cdots \pm \frac{2}{k}\left(\pi - \frac{(2k-1)\pi}{2k}\right) \cos. \frac{(2k-1)n\pi}{2k},$$

cuius ultimum membrum habebit signum $+$, quoties fuerit $2k - 1$ numerus formae $4\alpha + 1$ sive $k = 2\alpha + 1$ ideoque k numerus impar; at vero signum $-$ valebit, si $2k - 1$ fuerit formae $4\alpha - 1$ sive $k = 2\alpha$ ideoque numerus par. Ad valorem huius seriei inveniendum ponamus

$$S = \left(1 - \frac{1}{2k}\right) \cos. \frac{n\pi}{2k} - \left(1 - \frac{3}{2k}\right) \cos. \frac{3n\pi}{2k} + \left(1 - \frac{5}{2k}\right) \cos. \frac{5n\pi}{2k}$$
$$- \left(1 - \frac{7}{2k}\right) \cos. \frac{7n\pi}{2k} + \left(1 - \frac{9}{2k}\right) \cos. \frac{9n\pi}{2k} - \cdots \pm \left(1 - \frac{2k-1}{2k}\right) \cos. \frac{(2k-1)n\pi}{2k},$$

ita ut valor noster quaesitus sit $\frac{2\pi S}{k}$. Quo nunc valorem ipsius S investigemus, multiplicemus utrinque per $2 \cos. \frac{n\pi}{2k}$, et cum in genere sit

$$2 \cos. \frac{n\pi}{2k} \cos. \frac{(2i-1)n\pi}{2k} = \cos. \frac{in\pi}{k} + \cos. \frac{(i-1)n\pi}{k},$$

adhibita ista reductione reperietur

$$\left\{\begin{array}{l} \qquad\qquad +\left(1-\frac{1}{2k}\right)\cos.\frac{n\pi}{k}-\left(1-\frac{3}{2k}\right)\cos.\frac{2n\pi}{k} \\ +\left(1-\frac{1}{2k}\right)-\left(1-\frac{3}{2k}\right)\cos.\frac{n\pi}{k}+\left(1-\frac{5}{2k}\right)\cos.\frac{2n\pi}{k} \end{array}\right.$$
$$\left.\begin{array}{l} +\left(1-\frac{5}{2k}\right)\cos.\frac{3n\pi}{k}-\left(1-\frac{7}{2k}\right)\cos.\frac{4n\pi}{k}+\text{etc.} \\ -\left(1-\frac{7}{2k}\right)\cos.\frac{3n\pi}{k}+\left(1-\frac{9}{2k}\right)\cos.\frac{4n\pi}{k}-\text{etc.} \end{array}\right\},$$

ubi patet quemlibet terminum superiorem cum sequente inferiori in unicum coalescere, ita ut tantum primus inferior, qui est $\left(1-\frac{1}{2k}\right)$, et ultimus superior, qui est $\pm\frac{1}{2k}\cos.n\pi$, solitarii relinquantur; facta ergo hac contractione reperietur

$$2S\cos.\frac{n\pi}{2k}=1-\frac{1}{2k}\pm\frac{1}{2k}\cos.n\pi$$
$$+\frac{1}{k}\cos.\frac{n\pi}{k}-\frac{1}{k}\cos.\frac{2n\pi}{k}+\frac{1}{k}\cos.\frac{3n\pi}{k}-\cdots\mp\frac{1}{k}\cos.\frac{(k-1)n\pi}{k},$$

ubi signum superius valet, si k fuerit numerus impar, inferius autem, si k fuerit numerus par. Ponamus porro ad hanc seriem summandam

$$T=\cos.\frac{n\pi}{k}-\cos.\frac{2n\pi}{k}+\cos.\frac{3n\pi}{k}-\cos.\frac{4n\pi}{k}+\cdots\mp\cos.\frac{(k-1)n\pi}{k},$$

ita ut hoc valore T invento futurum sit

$$2S\cos.\frac{n\pi}{2k}=1-\frac{1}{2k}\pm\frac{1}{2k}\cos.n\pi+\frac{T}{k}.$$

Multiplicemus simili modo utrinque per $2\cos.\frac{n\pi}{2k}$ et in subsidium vocata eadem reductione reperietur

$$2T\cos.\frac{n\pi}{2k}$$
$$=\left\{\begin{array}{l} \qquad\qquad +\cos.\frac{3n\pi}{2k}-\cos.\frac{5n\pi}{2k}+\cos.\frac{7n\pi}{2k}-\cdots\mp\cos.\frac{(2k-1)n\pi}{2k} \\ +\cos.\frac{n\pi}{2k}-\cos.\frac{3n\pi}{2k}+\cos.\frac{5n\pi}{2k}-\cos.\frac{7n\pi}{2k}+\cdots \end{array}\right\},$$

ubi omnes termini se mutuo destruunt praeter primum inferiorem et ultimum superiorem, ita ut obtineamus

$$2T\cos.\frac{n\pi}{2k}=\cos.\frac{n\pi}{2k}\mp\cos.\frac{(2k-1)n\pi}{2k}.$$

Quia autem $\frac{2k-1}{2k}n\pi=n\pi-\frac{n\pi}{2k}$, erit

$$\cos.\frac{(2k-1)n\pi}{2k}=\cos.n\pi\cos.\frac{n\pi}{2k}+\sin.n\pi\sin.\frac{n\pi}{2k};$$

quoniam vero n supponitur numerus integer, erit $\sin.n\pi=0$ ideoque

$$2T\cos.\frac{n\pi}{2k}=\cos.\frac{n\pi}{2k}\mp\cos.n\pi\cos.\frac{n\pi}{2k},$$

unde fit

$$T=\frac{1}{2}\mp\frac{1}{2}\cos.n\pi,$$

quo valore substituto fiet

$$2S\cos.\frac{n\pi}{2k}=1,$$

consequenter

$$S=\frac{1}{2\cos.\frac{n\pi}{2k}},$$

ideoque valor noster quaesitus erit

$$\frac{\pi}{k\cos.\frac{n\pi}{2k}},$$

unde nascitur sequens

THEOREMA 1

24. *Ista formula integralis*

$$\int\frac{x^{k-n}+x^{k+n}}{1+x^{2k}}\cdot\frac{dx}{x}$$

a termino $x=0$ *usque ad terminum* $x=\infty$ *extensa producit hunc valorem*

$$\frac{\pi}{k\cos.\frac{n\pi}{2k}}.$$

Cuius demonstratio ex praecedente paragrapho liquet. Huic adiungi potest sequens theorema, quod prorsus singulari demonstratione ex isto derivare licet.

THEOREMA 2

25. *Si tam ista formula integralis*

$$\int \frac{x^{k-n}}{1+x^{2k}} \cdot \frac{dx}{x}$$

quam haec

$$\int \frac{x^{k+n}}{1+x^{2k}} \cdot \frac{dx}{x}$$

a termino $x=0$ usque ad $x=\infty$ extendatur, utraque producet eandem summam, quae est

$$\frac{\pi}{2k\cos.\frac{n\pi}{2k}}.$$

DEMONSTRATIO

Ponatur $S=\int \frac{x^{k-n}}{1+x^{2k}} \cdot \frac{dx}{x}$, siquidem integratio a termino $x=0$ usque ad terminum $x=\infty$ extendatur, ac ponatur $x=\frac{1}{z}$, ita ut iam integratio absolvi debeat a termino ∞ usque ad 0, et ob $\frac{dx}{x}=-\frac{dz}{z}$ habebitur nunc

$$-\int \frac{z^{-k+n}}{1+z^{-2k}} \cdot \frac{dz}{z},$$

quae, si numerator ac denominator multiplicetur per z^{2k}, abit in hanc

$$S=-\int \frac{z^{k+n}}{1+z^{2k}} \cdot \frac{dz}{z}$$

integratione a termino $z=\infty$ usque ad $z=0$ extensa. Hinc permutatis terminis integrationis erit

$$S=\int \frac{z^{k+n}}{1+z^{2k}} \cdot \frac{dz}{z}$$

a termino $z=0$ usque ad $z=\infty$; unde si loco z scribatur x, manifestum est utramque formulam a termino $x=0$ usque ad $x=\infty$ extensam eandem habere summam S. Cum igitur ambae hae formulae iunctae praebeant summam $2S=\frac{\pi}{k}:\cos.\frac{n\pi}{2k}$, erit utique utriusque formulae valor seorsim

$$S=\frac{\pi}{2k\cos.\frac{n\pi}{2k}}.$$

QUAESTIO ALTERA

26. *Proposita formula differentiali*

$$\frac{x^{k-n}+x^{k+n}}{1+x^{2k}} \cdot \frac{dx}{x}$$

eius integralis valorem investigare, qui oritur, si post integrationem ponatur $x=1$.

SOLUTIO

Cum in forma integralis generali quilibet terminus inventus sit

$$\frac{2}{k}\cos.n\omega \text{ A tang.} \frac{x\sin.\omega}{1-x\cos.\omega},$$

fiat hic $x=1$ ac prodibit $\frac{2}{k}\cos.n\omega \text{ A tang.} \frac{\sin.\omega}{1-\cos.\omega}$, quae forma ob

$$\sin.\omega=2\sin.\tfrac{1}{2}\omega\cos.\tfrac{1}{2}\omega \quad \text{et} \quad 1-\cos.\omega=2\sin.\tfrac{1}{2}\omega^2$$

abit in hanc

$$\frac{2}{k}\cos.n\omega \text{ A tang.} \frac{\cos.\frac{1}{2}\omega}{\sin.\frac{1}{2}\omega},$$

quae, cum sit

$$\frac{\cos.\frac{1}{2}\omega}{\sin.\frac{1}{2}\omega}=\cot.\tfrac{1}{2}\omega=\text{tang.}\left(\frac{\pi}{2}-\frac{1}{2}\omega\right),$$

porro transformatur in hanc

$$\frac{2}{k}\cos.n\omega\left(\frac{\pi}{2}-\frac{1}{2}\omega\right)=\frac{1}{k}(\pi-\omega)\cos.n\omega.$$

Quodsi igitur hic loco ω successive scribamus eius valores, qui sunt $\frac{\pi}{2k}, \frac{3\pi}{2k}, \frac{5\pi}{2k}$ usque ad $\frac{(2k-1)\pi}{2k}$, valor integralis quaesitus exprimetur per hanc progressionem

$$\frac{1}{k}\left(\pi-\frac{\pi}{2k}\right)\cos.\frac{n\pi}{2k}-\frac{1}{k}\left(\pi-\frac{3\pi}{2k}\right)\cos.\frac{3n\pi}{2k}+\frac{1}{k}\left(\pi-\frac{5\pi}{2k}\right)\cos.\frac{5n\pi}{2k}$$

$$-\frac{1}{k}\left(\pi-\frac{7\pi}{2k}\right)\cos.\frac{7n\pi}{2k}+\cdots\pm\frac{1}{k}\left(\pi-\frac{(2k-1)\pi}{2k}\right)\cos.\frac{(2k-1)n\pi}{2k},$$

ubi signorum ambiguorum superius valet, si k fuerit numerus impar, inferius vero, si par. Comparemus hanc expressionem cum ea, ad quam in quaestione praecedente est perventum, ac reperiemus hanc illius praecise esse semissem, unde eius valor erit

$$\frac{\pi}{2k\cos.\frac{n\pi}{2k}},$$

sicque habebitur sequens

THEOREMA

27. *Ista formula integralis*

$$\int \frac{x^{k-n}+x^{k+n}}{1+x^{2k}}\cdot\frac{dx}{x}$$

a termino $x=0$ *usque ad terminum* $x=1$ *extensa producet hunc valorem*

$$\frac{\pi}{2k\cos.\frac{n\pi}{2k}}.$$

COROLLARIUM

28. Cum igitur huius formulae integrale a termino $x=0$ usque ad $x=1$ extensum sit dimidium eius, quod a termino $x=0$ usque ad $x=\infty$ extenditur, sequitur, si eadem formula integralis a termino $x=1$ usque ad $x=\infty$ extendatur, eius valorem quoque fore $\frac{\pi}{2k\cos.\frac{n\pi}{2k}}$; praeterea vero utriusque valor aequabitur huic formulae integrali $\int\frac{x^{k\pm n}}{1+x^{2k}}\cdot\frac{dx}{x}$, siquidem ab $x=0$ usque ad $x=\infty$ extendatur.

PROBLEMA PARTICULARE 2

29. *Si sumatur*

$$Q=1-x^{2k},$$

investigare integrale huius formulae differentialis

$$\frac{Pdx}{x(1-x^{2k})},$$

ubi quidem $\frac{P}{x}$ *sit functio integra, in qua nullae potestates altiores occurrant quam exponentis* $2k$, *ne scilicet ista fractio evadat spuria.*

SOLUTIO

Cum sit $Q=1-x^{2k}$, statim duo eius habentur factores simplices reales, qui sunt $1-x$ et $1+x$, quare partes integrales ex iis oriundas primum investigemus. Ponamus igitur pro factore $1-x$ fractionem

$$\frac{P}{x(1-x^{2k})}=\frac{\alpha}{1-x}+R,$$

ubi R complectitur omnes reliquas partes, unde per $1-x$ multiplicando habebimus

$$\frac{P(1-x)}{x(1-x^{2k})}=\alpha+R(1-x);$$

quare si faciamus $x=1$, nanciscemur

$$\alpha=\frac{P}{x}\cdot\frac{1-x}{1-x^{2k}},$$

cuius posterioris fractionis posito $x=1$ tam numerator quam denominator evanescit; hinc eorum loco scribamus eorum differentialia fietque

$$\alpha=\frac{P}{x}\cdot\frac{1}{2kx^{2k-1}}.$$

Fiat igitur nunc $x=1$, quo casu abeat P in B, ac prodibit $\alpha=\frac{B}{2k}$; ex fractione autem partiali $\frac{\alpha}{1-x}$ porro reperitur pars integralis inde nata

$$-\alpha l(1-x)=-\frac{B}{2k}l(1-x).$$

Pro altero factore $1+x$ faciamus simili modo

$$\frac{P}{x(1-x^{2k})}=\frac{\beta}{1+x}+R,$$

unde per $1+x$ multiplicando fit

$$\frac{P(1+x)}{x(1-x^{2k})}=\beta+R(1+x);$$

quare si faciamus $x=-1$, erit

$$\beta=\frac{P}{x}\cdot\frac{1+x}{1-x^{2k}},$$

ubi in posteriore fractione differentialia tam supra quam infra scribantur, ut prodeat

$$\beta = \frac{P}{x} \cdot \frac{1}{-2kx^{2k-1}} = -\frac{P}{2kx^{2k}} = -\frac{P}{2k},$$

posito scilicet $x=-1$. Ponamus ergo facto $x=-1$ functionem P abire in C fietque $\beta = -\frac{C}{2k}$ et ex fractione partiali $\frac{\beta}{1+x}$ obtinebitur pars inde nata integralis

$$\beta l(1+x) = -\frac{C}{2k} l(1+x)$$

sicque ex ambobus factoribus $1+x$ et $1-x$ nascuntur hae duae partes integrales

$$-\frac{B}{2k} l(1-x) - \frac{C}{2k} l(1+x).$$

His expeditis sit formulae $1-x^{2k}$ factor trinominalis quicunque

$$1 - 2x\cos.\omega + xx,$$

quo facto $=0$ ista formula $1-x^{2k}$ induet hanc formam

$$1 - \frac{x-\cos.\omega}{\sin.\omega} \sin.2k\omega - \cos.2k\omega;$$

quae formula cum debeat evanescere, has suppeditat conditiones

$$1.\ \sin.2k\omega = 0 \quad \text{et} \quad 2.\ \cos.2k\omega = 1;$$

ex posteriore statim intelligitur angulum ω sequentes valores accipere posse

$$1.\ \omega = 0, \quad 2.\ \omega = \frac{2\pi}{2k} = \frac{\pi}{k}, \quad 3.\ \omega = \frac{4\pi}{2k} = \frac{2\pi}{k}$$

et in genere

$$\omega = \frac{i\pi}{k}.$$

Quia igitur numerus horum valorum debet esse $=k$, primus autem $\omega=0$ tantum factori simplici respondet, numerus valorum debet sumi $k+1$, ita ut iam ultimus futurus sit $\frac{k\pi}{k} = \pi$, unde alter factor simplex $1+x$ nascitur; hoc autem modo simul primae conditioni satisfit, qua esse debet $\sin.2k\omega = 0$.

Nunc consideremus factorem generalem $xx - 2x\cos.\omega + 1$, quo posito $=0$ fiat

$$P = \frac{F(x-\cos.\omega)}{\sin.\omega} + G,$$

eritque $\frac{xdQ}{dx} = -2kx^{2k}$; at vero iam vidimus tum fieri $x^{2k}=1$ sicque $\frac{xdQ}{dx} = -2k$, pro qua forma in genere posuimus $\frac{f(x-\cos.\omega)}{\sin.\omega} + g$, quocirca pro nostro casu erit $f=0$ et $g=-2k$. Cum igitur pro hoc factore in genere inventa sit ista pars integralis

$$\frac{2(Ff+Gg)}{ff+gg} l\sqrt{(xx-2x\cos.\omega+1)} + \frac{2(fG-Fg)}{ff+gg} \text{A tang.} \frac{x\sin.\omega}{1-x\cos.\omega},$$

erit ista pars nostro casu

$$-\frac{G}{k} l\sqrt{(xx-2x\cos.\omega+1)} + \frac{F}{k} \text{A tang.} \frac{x\sin.\omega}{1-x\cos.\omega};$$

consequenter si loco ω successive scribantur valores indicati, scilicet $\omega=0$, $\omega=\frac{\pi}{k}$, $\omega=\frac{2\pi}{k}$ usque ad $\omega=\frac{k\pi}{k}$, et omnes istae partes in unam summam colligantur, obtinebitur totum integrale formulae propositae. Hic autem probe observandum est ex parte prima et ultima eas ipsas partes oriri, quas iam pro valoribus $1-x$ et $1+x$ assignavimus, quare eas penitus omitti conveniet.

COROLLARIUM 1

30. Quodsi numerator P fuerit potestas simplex ipsius x, puta x^m existente $m>1$ et $m<2k$, ut formula integranda sit

$$\int \frac{x^{m-1}dx}{1-x^{2k}},$$

pro factoribus simplicibus $1-x$ et $1+x$ erit $B=+1$ et $C=(-1)^m$, unde partes integralis hinc natae erunt

$$\frac{-1}{2k} l(1-x) - \frac{(-1)^m}{2k} l(1+x).$$

Pro reliquis vero partibus erit $F=\sin.m\omega$ et $G=\cos.m\omega$, unde quaelibet portio integralis induet hanc formam

$$-\frac{\cos. m\omega}{k}\, l\sqrt{(xx-2x\cos.\omega+1)}+\frac{\sin. m\omega}{k}\,\text{A tang.}\,\frac{x\sin.\omega}{1-x\cos.\omega},$$

ubi valores pro ω substituendi sunt $\frac{\pi}{k}$, $\frac{2\pi}{k}$, $\frac{3\pi}{k}$, $\ldots$ $\frac{(k-1)\pi}{k}$.

COROLLARIUM 2

31. Si numerator P pluribus huiusmodi terminis constet, ut sit

$$P=ax^{\alpha}+bx^{\beta}+cx^{\gamma}+\text{etc.},$$

integratio maiore difficultate non laborat; erit enim

$$F=a\sin.\alpha\omega+b\sin.\beta\omega+c\sin.\gamma\omega+\text{etc.}$$

et

$$G=a\cos.\alpha\omega+b\cos.\beta\omega+c\cos.\gamma\omega+\text{etc.}$$

hincque totum integrale facile expedietur.

SCHOLION

32. Casus prae ceteris memoratu dignus, qui hic occurrit, est, quo statuitur $P=x^{k-n}-x^{k+n}$, quippe quo omnes partes logarithmicae se mutuo tollere reperiuntur, unde eum in sequente problemate evolvamus.

PROBLEMA SPECIALE

33. *Proposita formula differentiali*

$$\frac{x^{k-n}-x^{k+n}}{1-x^{2k}}\cdot\frac{dx}{x}$$

eius totum integrale investigare.

SOLUTIO

Quia hic est $P=x^{k-n}-x^{k+n}=x^{k-n}(1-x^{2n})$, ob n numerum integrum posito tam $x=1$ quam $x=-1$ iste valor evanescat, unde fiet tam $B=0$

quam $C=0$, sicque partes integrales ex factoribus simplicibus natae sponte evanescunt. Pro factore autem duplici quocunque

$$1-2x\cos.\omega+xx$$

eo facto $=0$ reperietur

$$P=\frac{x-\cos.\omega}{\sin.\omega}(\sin.(k-n)\omega-\sin.(k+n)\omega)+\cos.(k-n)\omega-\cos.(k+n)\omega$$

hincque colligitur

$$F=\sin.(k-n)\omega-\sin.(k+n)\omega$$

et

$$G=\cos.(k-n)\omega-\cos.(k+n)\omega.$$

Cum autem in genere sit

$$\sin.p-\sin.q=2\sin.\frac{p-q}{2}\cos.\frac{p+q}{2}$$

et

$$\cos.p-\cos.q=2\sin.\frac{q-p}{2}\sin.\frac{p+q}{2},$$

ob

$$p=(k-n)\omega \quad \text{et} \quad q=(k+n)\omega$$

erit

$$F=-2\sin.n\omega\cos.k\omega \quad \text{et} \quad G=2\sin.n\omega\sin.k\omega.$$

Est vero, uti vidimus, in genere $\omega=\frac{i\pi}{k}$, unde fit

$$\sin.k\omega=\sin.i\pi=0 \quad \text{et} \quad \cos.k\omega=\pm 1,$$

scilicet valebit $+1$, si i est numerus par, et -1, si i impar. Ad hanc autem ambiguitatem evitandam retineamus $\cos.k\omega$ atque habebimus

$$F=-2\sin.n\omega\cos.k\omega \quad \text{et} \quad G=0.$$

Ex his igitur pars integralis quaecunque in genere erit

$$-\frac{2\sin.n\omega\cos.k\omega}{k}\,\text{A tang.}\,\frac{x\sin.\omega}{1-x\cos.\omega},$$

ubi tantum opus est loco ω valores indicatos successive substitui; et quia pro primo $\omega=0$ et ultimo $\omega=\pi$ partes integrales sponte evanescunt, perinde est, sive valores primus et ultimus reiiciantur sive retineantur, quamobrem totum integrale quaesitum sequenti modo exprimetur:

$$\frac{2}{k}\sin.\frac{n\pi}{k}\,\text{A tang.}\frac{x\sin.\frac{\pi}{k}}{1-x\cos.\frac{\pi}{k}}-\frac{2}{k}\sin.\frac{2n\pi}{k}\,\text{A tang.}\frac{x\sin.\frac{2\pi}{k}}{1-x\cos.\frac{2\pi}{k}}$$

$$+\frac{2}{k}\sin.\frac{3n\pi}{k}\,\text{A tang.}\frac{x\sin.\frac{3\pi}{k}}{1-x\cos.\frac{3\pi}{k}}-\frac{2}{k}\sin.\frac{4n\pi}{k}\,\text{A tang.}\frac{x\sin.\frac{4\pi}{k}}{1-x\cos.\frac{4\pi}{k}}$$

$$\cdots\cdots\cdots$$

$$\pm\frac{2}{k}\sin.\frac{(k-1)n\pi}{k}\,\text{A tang.}\frac{x\sin.\frac{(k-1)\pi}{k}}{1-x\cos.\frac{(k-1)\pi}{k}},$$

ubi signum superius valet, si n fuerit numerus par, inferius vero, si impar.

COROLLARIUM

34. Si hic sumere velimus $n=0$, formula integranda sponte evanescit, ita ut hic nihil memoratu dignum resultet, unde in valores huius integralis pro casibus $x=\infty$ et $x=1$ inquiramus.

QUAESTIO PRIOR

35. *Proposita formula differentiali*

$$\frac{x^{k-n}-x^{k+n}}{1-x^{2k}}\cdot\frac{dx}{x}$$

eius valorem integralem investigare, qui oritur, si post integrationem ponitur $x=\infty$.

SOLUTIO

Cum arcuum, qui hic occurrunt, forma generalis sit A tang. $\frac{x\sin.\omega}{1-x\cos.\omega}$, ea posito $x=\infty$, uti ante vidimus, abit in $\pi-\omega$, unde litterae ω valores suos ordine tribuendo valor quaesitus nostrae formulae integralis sequenti progressione exprimetur:

$$\frac{2}{k}\left(\pi-\frac{\pi}{k}\right)\sin.\frac{n\pi}{k}-\frac{2}{k}\left(\pi-\frac{2\pi}{k}\right)\sin.\frac{2n\pi}{k}+\frac{2}{k}\left(\pi-\frac{3\pi}{k}\right)\sin.\frac{3n\pi}{k}$$

$$-\frac{2}{k}\left(\pi-\frac{4\pi}{k}\right)\sin.\frac{4n\pi}{k}+\cdots\pm\frac{2}{k}\left(\pi-\frac{(k-1)\pi}{k}\right)\sin.\frac{(k-1)n\pi}{k}.$$

Ad huius valorem investigandum ponamus

$$S=\left(1-\frac{1}{k}\right)\sin.\frac{n\pi}{k}-\left(1-\frac{2}{k}\right)\sin.\frac{2n\pi}{k}+\left(1-\frac{3}{k}\right)\sin.\frac{3n\pi}{k}$$

$$-\left(1-\frac{4}{k}\right)\sin.\frac{4n\pi}{k}+\cdots\pm\left(1-\frac{k-1}{k}\right)\sin.\frac{(k-1)n\pi}{k},$$

ita ut valor, quem quaerimus, sit $\frac{2\pi S}{k}$. Multiplicemus igitur ut hactenus utrinque per $2\cos.\frac{n\pi}{2k}$, et cum in genere sit

$$2\sin.\frac{in\pi}{k}\cos.\frac{n\pi}{2k}=\sin.\frac{(2i+1)n\pi}{2k}+\sin.\frac{(2i-1)n\pi}{2k},$$

facta hac reductione perveniemus ad sequentem expressionem

$$2S\cos.\frac{n\pi}{2k}=\left\{\begin{array}{l} \qquad\qquad+\left(1-\frac{1}{k}\right)\sin.\frac{3n\pi}{2k}-\left(1-\frac{2}{k}\right)\sin.\frac{5n\pi}{2k}\\ \left(1-\frac{1}{k}\right)\sin.\frac{n\pi}{2k}-\left(1-\frac{2}{k}\right)\sin.\frac{3n\pi}{2k}+\left(1-\frac{3}{k}\right)\sin.\frac{5n\pi}{2k}\end{array}\right.$$

$$\left.\begin{array}{l}+\left(1-\frac{3}{k}\right)\sin.\frac{7n\pi}{2k}-\cdots\mp\left(1-\frac{k-2}{k}\right)\sin.\frac{(2k-3)n\pi}{2k}\pm\left(1-\frac{k-1}{k}\right)\sin.\frac{(2k-1)n\pi}{2k}\\ -\left(1-\frac{4}{k}\right)\sin.\frac{7n\pi}{2k}+\cdots\pm\left(1-\frac{k-1}{k}\right)\sin.\frac{(2k-3)n\pi}{2k}\end{array}\right\},$$

ubi quilibet terminus superior cum sequente inferiori in unum contrahi potest, unde primum inferiorem cum ultimo superiori seorsim exhibeamus hoc modo

$$2S\cos.\frac{n\pi}{2k}=\left(1-\frac{1}{k}\right)\sin.\frac{n\pi}{2k}\pm\frac{1}{k}\sin.\frac{(2k-1)n\pi}{2k}$$

$$+\frac{1}{k}\sin.\frac{3n\pi}{2k}-\frac{1}{k}\sin.\frac{5n\pi}{2k}+\frac{1}{k}\sin.\frac{7n\pi}{2k}-\cdots\mp\frac{1}{k}\sin.\frac{(2k-3)n\pi}{2k}.$$

Hoc igitur modo ultimus superior cum reliquis eandem legem sequitur, ita ut ponere liceat

$$2S\cos.\frac{n\pi}{2k}=\left(1-\frac{1}{k}\right)\sin.\frac{n\pi}{2k}$$

$$+\frac{1}{k}\sin.\frac{3n\pi}{2k}-\frac{1}{k}\sin.\frac{5n\pi}{2k}+\frac{1}{k}\sin.\frac{7n\pi}{2k}-\cdots\pm\frac{1}{k}\sin.\frac{(2k-1)n\pi}{2k}.$$

Statuamus porro

$$T = \sin.\frac{3n\pi}{2k} - \sin.\frac{5n\pi}{2k} + \sin.\frac{7n\pi}{2k} - \cdots \pm \sin.\frac{(2k-1)n\pi}{2k},$$

ut sit

$$2S\cos.\frac{n\pi}{2k} = \left(1 - \frac{1}{k}\right)\sin.\frac{n\pi}{2k} + \frac{T}{k}.$$

Iam iterum multiplicemus per $2\cos.\frac{n\pi}{2k}$ et adhibita eadem reductione reperiemus

$$2T\cos.\frac{n\pi}{2k} = \sin.\frac{2n\pi}{k} - \sin.\frac{3n\pi}{k} + \sin.\frac{4n\pi}{k} - \cdots \mp \sin.\frac{(k-1)n\pi}{k} \pm \sin.n\pi$$

$$+ \sin.\frac{n\pi}{k} - \sin.\frac{2n\pi}{k} + \sin.\frac{3n\pi}{k} - \sin.\frac{4n\pi}{k} + \cdots \pm \sin.\frac{(k-1)n\pi}{k},$$

ubi destructis terminis, qui se mutuo tollunt, obtinebitur

$$2T\cos.\frac{n\pi}{2k} = \sin.\frac{n\pi}{k} + \sin.n\pi = \sin.\frac{n\pi}{k}$$

ob $\sin.n\pi = 0$. Quia igitur est

$$\sin.\frac{n\pi}{k} = 2\sin.\frac{n\pi}{2k}\cos.\frac{n\pi}{2k},$$

erit

$$T = \sin.\frac{n\pi}{2k},$$

quo valore substituto fiet

$$2S\cos.\frac{n\pi}{2k} = \sin.\frac{n\pi}{2k}$$

ideoque

$$S = \frac{1}{2}\operatorname{tang}.\frac{n\pi}{2k};$$

consequenter nostrae formulae integralis casu $x = \infty$ valor erit

$$\frac{\pi}{k}\operatorname{tang}.\frac{n\pi}{2k},$$

unde nascitur sequens

THEOREMA

36. *Ista formula integralis*

$$\int \frac{x^{k-n} - x^{k+n}}{1 - x^{2k}} \cdot \frac{dx}{x}$$

a termino $x = 0$ *usque ad terminum* $x = \infty$ *extensa producit hunc valorem*

$$\frac{\pi}{k}\operatorname{tang}.\frac{n\pi}{2k}.$$

SCHOLION

37. Cum haec formula duabus constet partibus, si simili modo, ut supra factum est, in valorem utriusque seorsim inquirere velimus, utriusque valor adeo imaginarius esse deprehendetur, id quod facile inde percipitur, quod posito $x = 1$ ipsa fractio iam in infinitum excrescat. Tractemus autem ut supra partem priorem ponendo $S = \int \frac{x^{k-n}}{1 - x^{2k}} \cdot \frac{dx}{x}$ a termino $x = 0$ usque ad $x = \infty$ ac faciendo $x = \frac{1}{z}$ fiet

$$S = -\int \frac{z^{-k+n}}{1 - z^{-2k}} \cdot \frac{dz}{z} = -\int \frac{z^{k+n}}{z^{2k} - 1} \cdot \frac{dz}{z}$$

a termino $z = \infty$ usque ad $z = 0$, ergo mutatis terminis integrationis erit

$$S = -\int \frac{z^{k+n}}{1 - z^{2k}} \cdot \frac{dz}{z}$$

a termino $z = 0$ usque ad $z = \infty$; unde si loco z scribamus x et has formulas iungamus, erit

$$2S = \int \frac{x^{k-n} - x^{k+n}}{1 - x^{2k}} \cdot \frac{dx}{x} = \frac{\pi}{k}\operatorname{tang}.\frac{n\pi}{2k},$$

unde prodiret $S = \frac{\pi}{2k}\operatorname{tang}.\frac{n\pi}{2k}$. Haec autem conclusio admitti nequit, quoniam nostra formula integralis eatenus tantum ad arcum circularem reduci poterit, quatenus numerator $x^{k-n} - x^{k+n}$ cum denominatore $1 - x^{2k}$ factorem communem habet $1 - xx$, qui ergo semper per divisionem tolli posset. Verum sumta tantum alterutra parte iste factor $1-xx$ ex denominatore non tollitur ex eoque igitur necessario nasceretur pars integralis vel huius formae $\alpha l \frac{1+x}{1-x}$ vel huius $\alpha l(1-xx)$, quae utraque forma sumto $x=\infty$ fit imaginaria.

QUAESTIO ALTERA

38. *Proposita formula differentiali*

$$\frac{x^{k-n}-x^{k+n}}{1-x^{2k}}\cdot\frac{dx}{x}$$

eius integrale investigare, quod oritur, si post integrationem ponitur $x=1$.

SOLUTIO

Si in forma generali arcuum, quibus integrale exprimitur, quae est A tang. $\frac{x \sin.\omega}{1-x\cos.\omega}$, ponatur $x=1$, prodit, ut ante vidimus, $\frac{\pi}{2}-\frac{\omega}{2}$; qui valor cum sit dimidius eius, quem casu praecedente habuimus, statim patet valorem nostrum fore $\frac{\pi}{2k}\operatorname{tang.}\frac{n\pi}{2k}$, unde nascitur istud

THEOREMA

39. *Ista formula integralis*

$$\int\frac{x^{k-n}-x^{k+n}}{1-x^{2k}}\cdot\frac{dx}{x}$$

a termino $x=0$ *usque ad terminum* $x=1$ *extensa producet hunc valorem*

$$\frac{\pi}{2k}\operatorname{tang.}\frac{n\pi}{2k}.$$

COROLLARIUM

40. Hinc si eiusdem formulae integrale a termino $x=1$ usque ad $x=\infty$ extendatur, eius valor quoque erit $\frac{\pi}{2k}\operatorname{tang.}\frac{n\pi}{2k}$, quandoquidem hi duo valores iunctim sumti valorem casus praecedentis producere debent.

PROBLEMA PARTICULARE 3

41. *Si sumatur* $Q=1+2x^k\cos.\eta+x^{2k}$, *investigare integrale huius formulae differentialis*

$$\frac{P\,dx}{x(1+2x^k\cos.\eta+x^{2k})},$$

ubi quidem $\frac{P}{x}$ *sit functio integra, in qua nullae potestates altiores occurrant quam exponentis* $2k$.

SOLUTIO

Quia denominator Q alios factores simplices praeter imaginarios non admittit nisi casu, quo $\eta=180^0$, sit eius factor trinomialis in genere $1-2x\cos.\omega+xx$, quo posito $=0$ fiet

$$Q=\frac{x-\cos.\omega}{\sin.\omega}(\sin.2k\omega+2\cos.\eta\sin.k\omega)+\cos.2k\omega+2\cos.\eta\cos.k\omega+1;$$

quae forma quia debet esse nihilo aequalis, postulat has duas conditiones

$$\text{I.}\quad \sin.2k\omega+2\cos.\eta\sin.k\omega=0$$

et

$$\text{II.}\quad \cos.2k\omega+2\cos.\eta\cos.k\omega+1=0.$$

Cum igitur sit

$$\sin.2k\omega=2\sin.k\omega\cos.k\omega \quad \text{et} \quad \cos.2k\omega+1=2\cos.k\omega^2,$$

prior conditio dat

$$2\sin.k\omega(\cos.k\omega+\cos.\eta)=0$$

et secunda conditio

$$2\cos.k\omega(\cos.k\omega+\cos.\eta)=0;$$

utrique igitur conditioni satisfit simul, si fuerit

$$\cos.k\omega+\cos.\eta=0;$$

quod quo facilius fieri possit, sumamus $\eta=\pi-\theta$, ut habeatur $\cos.k\omega=\cos.\theta$. Omnes autem anguli cum θ communem cosinum habentes sunt θ, $2\pi\pm\theta$, $4\pi\pm\theta$, $6\pi\pm\theta$ et in genere $2i\pi\pm\theta$, quamobrem statuamus pro ω sequentes valores

$$\omega=\frac{\theta}{k},\quad \omega=\frac{2\pi+\theta}{k},\quad \omega=\frac{4\pi+\theta}{k}\quad \text{etc.}$$

et in genere $\omega = \frac{2i\pi+\theta}{k}$; quorum valorum numerus cum debeat esse $= k$, ultimus erit

$$\omega = \frac{2(k-1)\pi+\theta}{k} \quad \text{sive} \quad \omega = \frac{-2\pi+\theta}{k}.$$

His constitutis consideremus formulam $\frac{x\,dQ}{dx}$, quae erit

$$= 2k(x^k \cos.\eta + x^{2k}),$$

quae per conditionem $xx - 2x\cos.\omega + 1 = 0$ ob $\cos.\eta = -\cos.\theta$ reducitur ad hanc formam

$$2k\,\frac{x-\cos.\omega}{\sin.\omega}(\sin.2k\omega - \cos.\theta\sin.k\omega) + 2k(\cos.2k\omega - \cos.\theta\cos.k\omega),$$

pro qua in genere sumsimus

$$= \frac{f(x-\cos.\omega)}{\sin.\omega} + g,$$

sicque erit

$$f = 2k(\sin.2k\omega - \cos.\theta\sin.k\omega)$$

et

$$g = 2k(\cos.2k\omega - \cos.\theta\cos.k\omega).$$

Loco $\sin.2k\omega$ et $\cos.2k\omega$ scribamus valores ante indicatos prodibitque

$$f = 2k\sin.k\omega(2\cos.k\omega - \cos.\theta)$$

et

$$g = 2k(2\cos.k\omega^2 - 1 - \cos.\theta\cos.k\omega);$$

cum autem esse debeat $\cos.k\omega = \cos.\theta$, fiet

$$f = 2k\sin.k\omega\cos.\theta = k\sin.2k\omega$$

et

$$g = 2k(\cos.\theta^2 - 1) = -2k\sin.\theta^2.$$

Quia igitur in genere est $\omega = \frac{2i\pi+\theta}{k}$, erit $2k\omega = 4i\pi + 2\theta$, ita ut iam habeamus $f = k\sin.2\theta = 2k\sin.\theta\cos.\theta$, ita ut sit $ff + gg = 4kk\sin.\theta^2$; quocirca si posito $1 - 2x\cos.\omega + xx = 0$ functio P transformetur in hanc formam $\frac{F(x-\cos.\omega)}{\sin.\omega} + G$, ex denominatoris Q factore $1 - 2x\cos.\omega + xx$ orietur ista pars integralis

$$\frac{F\cos.\theta - G\sin.\theta}{k\sin.\theta}\, l\sqrt{(xx - 2x\cos.\omega + 1)} + \frac{G\cos.\theta + F\sin.\theta}{k\sin.\theta}\,\text{A tang.}\frac{x\sin.\omega}{1 - x\cos.\omega}.$$

Tantum igitur superest, ut loco ω ordine substituantur omnes eius valores, qui sunt $\frac{\theta}{k}, \frac{2\pi+\theta}{k}, \frac{4\pi+\theta}{k}, \ldots \frac{2(k-1)\pi+\theta}{k}$, et summa omnium harum formularum praebebit totum integrale quaesitum.

COROLLARIUM

42. Si fuerit numerator P simplex potestas ipsius x, scilicet $P = x^m$, tum fiet $F = \sin.m\omega$ et $G = \cos.m\omega$, unde pars integralis ex denominatoris factore indefinito $1 - 2x\cos.\omega + xx$ oriunda erit

$$\frac{\cos.\theta\sin.m\omega - \sin.\theta\cos.m\omega}{k\sin.\theta}\, l\sqrt{(xx - 2x\cos.\omega + 1)}$$
$$+ \frac{\cos.\theta\cos.m\omega + \sin.\theta\sin.m\omega}{k\sin.\theta}\,\text{A tang.}\frac{x\sin.\omega}{1 - x\cos.\omega},$$

unde simul patet, si functio P ex pluribus huiusmodi potestatibus fuerit composita, quemadmodum integrationem absolvi oporteat.

PROBLEMA SPECIALE

43. *Proposita formula differentiali*

$$\frac{x^{k-n} + x^{k+n}}{1 + 2x^k\cos.\eta + x^{2k}} \cdot \frac{dx}{x}$$

eius totum integrale investigare.

SOLUTIO

Cum hic sit $P = x^{k-n} + x^{k+n}$, erit

$$F = 2\sin.k\omega\cos.n\omega \quad \text{et} \quad G = 2\cos.k\omega\cos.n\omega,$$

ubi $\cos.k\omega = \cos.\theta$; quibus valoribus substitutis pro parte integralis logarithmica erit

$$\frac{F\cos.\theta - G\sin.\theta}{k\sin.\theta} = \frac{2\cos.\theta\cos.n\omega(\sin.k\omega - \sin.\theta)}{k\sin.\theta}.$$

Cum autem in genere sit $\omega = \frac{2i\pi+\theta}{k}$, erit $\sin.k\omega = \sin.\theta$, unde patet hanc formulam evanescere, ita ut omnes partes logarithmicae ex integrali excedant.

Pro partibus autem circularibus evadet coefficiens

$$\frac{G\cos.\theta + F\sin.\theta}{k\sin.\theta} = \frac{2\cos.n\omega}{k\sin.\theta}$$

sicque ex factore denominatoris indefinito $1 - 2x\cos.\omega + xx$ oritur ista pars integralis

$$\frac{2\cos.n\omega}{k\sin.\theta}\,\mathrm{A\,tang.}\frac{x\sin.\omega}{1 - x\cos.\omega}.$$

In hac ergo formula pro ω ordine scribamus eius valores, qui sunt $\frac{\theta}{k}$, $\frac{2\pi+\theta}{k}$, $\frac{4\pi+\theta}{k}$ etc. usque ad $\frac{2(k-1)\pi+\theta}{k}$, ubi meminisse oportet esse $\theta = \pi - \eta$, et quo formulae non nimis fiant perplexae, utamur sequentibus valoribus

$$\frac{2\pi}{k} = \alpha, \quad \frac{\theta}{k} = \beta, \quad \frac{2n\pi}{k} = \gamma \quad \text{et} \quad \frac{n\theta}{k} = \delta,$$

ut valores ipsius ω fiant

$$\beta, \ \alpha+\beta, \ 2\alpha+\beta, \ 3\alpha+\beta, \ \ldots \ (k-1)\alpha+\beta.$$

At vero omnes valores anguli $n\omega$ erunt ordine

$$\delta, \ \gamma+\delta, \ 2\gamma+\delta, \ 3\gamma+\delta, \ \ldots \ (k-1)\gamma+\delta.$$

His igitur valoribus adhibitis totum integrale, quod quaerimus, erit

$$\frac{2\cos.\delta}{k\sin.\theta}\,\mathrm{A\,tang.}\frac{x\sin.\beta}{1-x\cos.\beta} + \frac{2\cos.(\gamma+\delta)}{k\sin.\theta}\,\mathrm{A\,tang.}\frac{x\sin.(\alpha+\beta)}{1-x\cos.(\alpha+\beta)}$$
$$+ \frac{2\cos.(2\gamma+\delta)}{k\sin.\theta}\,\mathrm{A\,tang.}\frac{x\sin.(2\alpha+\beta)}{1-x\cos.(2\alpha+\beta)} + \frac{2\cos.(3\gamma+\delta)}{k\sin.\theta}\,\mathrm{A\,tang.}\frac{x\sin.(3\alpha+\beta)}{1-x\cos.(3\alpha+\beta)}$$
$$+ \cdots + \frac{2\cos.((k-1)\gamma+\delta)}{k\sin.\theta}\,\mathrm{A\,tang.}\frac{x\sin.((k-1)\alpha+\beta)}{1-x\cos.((k-1)\alpha+\beta)}.$$

COROLLARIUM 1

44. Evolvamus casum, quo $n = 0$ ideoque etiam $\gamma = 0$ et $\delta = 0$, et quia hoc casu formula integralis evadet

$$2\int \frac{x^{k-1}dx}{1 + 2x^k\cos.\eta + x^{2k}},$$

pro ea statuamus $x^k = z$ atque ob $x^{k-1}dx = \frac{dz}{k}$ habebitur

$$\frac{2}{k}\int \frac{dz}{1 + 2z\cos.\eta + zz},$$

cuius integrale erit

$$= \frac{2}{k\sin.\eta}\,\mathrm{A\,tang.}\frac{z\sin.\eta}{1 + z\cos.\eta} = \frac{2}{k\sin.\theta}\,\mathrm{A\,tang.}\frac{z\sin.\theta}{1 - z\cos.\theta};$$

unde cum in serie inventa omnes coefficientes arcuum fiant $\frac{2}{k\sin.\theta}$, per hunc coefficientem dividendo habebimus sequentem aequationem

$$\mathrm{A\,tang.}\frac{x^k\sin.\theta}{1 + x^k\cos.\theta} = \mathrm{A\,tang.}\frac{x\sin.\beta}{1 - x\cos.\beta} + \mathrm{A\,tang.}\frac{x\sin.(\alpha+\beta)}{1 - x\cos.(\alpha+\beta)}$$
$$+ \mathrm{A\,tang.}\frac{x\sin.(2\alpha+\beta)}{1 - x\cos.(2\alpha+\beta)} + \cdots + \mathrm{A\,tang.}\frac{x\sin.((k-1)\alpha+\beta)}{1 - x\cos.((k-1)\alpha+\beta)},$$

ubi recordandum est esse $\alpha = \frac{2\pi}{k}$, $\beta = \frac{\theta}{k}$.

COROLLARIUM 2

45. Ponamus esse $\theta = 90^0 = \frac{\pi}{2}$ et aequatio modo inventa hanc induet formam

$$\mathrm{A\,tang.}x^k = \mathrm{A\,tang.}\frac{x\sin.\frac{\pi}{2k}}{1 - x\cos.\frac{\pi}{2k}} + \mathrm{A\,tang.}\frac{x\sin.\frac{5\pi}{2k}}{1 - x\cos.\frac{5\pi}{2k}} + \mathrm{A\,tang.}\frac{x\sin.\frac{9\pi}{2k}}{1 - x\cos.\frac{9\pi}{2k}}$$
$$+ \mathrm{A\,tang.}\frac{x\sin.\frac{13\pi}{2k}}{1 - x\cos.\frac{13\pi}{2k}} + \cdots + \mathrm{A\,tang.}\frac{x\sin.\frac{(4k-3)\pi}{2k}}{1 - x\cos.\frac{(4k-3)\pi}{2k}}.$$

Sit $k = 1$ eritque $\mathrm{A\,tang.}x = \mathrm{A\,tang.}x$.

Sit $k = 2$ eritque

$$\mathrm{A\,tang.}xx = \mathrm{A\,tang.}\frac{x}{\sqrt{2} - x} + \mathrm{A\,tang.}\frac{-x}{\sqrt{2} + x} = \mathrm{A\,tang.}\frac{x}{\sqrt{2} - x} - \mathrm{A\,tang.}\frac{x}{\sqrt{2} + x}.$$

Sit $k = 3$ eritque

$$\mathrm{A\,tang.}x^3 = \mathrm{A\,tang.}\frac{x}{2 - x\sqrt{3}} + \mathrm{A\,tang.}\frac{x}{2 + x\sqrt{3}} - \mathrm{A\,tang.}x$$

etc.

Haec igitur series ab ea, quam supra (§ 20) invenimus, prorsus discrepat, etiamsi utriusque valor sit idem, scilicet $\mathrm{A\,tang.}x^k$.

QUAESTIO PRIOR

46. *Proposita formula differentiali*

$$\frac{x^{k-n}+x^{k+n}}{1+2x^k\cos.\eta+x^{2k}}\cdot\frac{dx}{x}$$

eius integralis valorem investigare, qui oritur, si post integrationem ponitur $x=1$.

SOLUTIO

Cum posito $x=\infty$ in genere sit

$$\text{A tang.}\,\frac{x\sin.\omega}{1-x\cos.\omega}=\pi-\omega,$$

valor integralis, quem quaerimus, hac serie exprimetur

$$\frac{2\cos.\delta}{k\sin.\theta}(\pi-\beta)+\frac{2\cos.(\gamma+\delta)}{k\sin.\theta}(\pi-\alpha-\beta)+\frac{2\cos.(2\gamma+\delta)}{k\sin.\theta}(\pi-2\alpha-\beta)$$
$$+\frac{2\cos.(3\gamma+\delta)}{k\sin.\theta}(\pi-3\alpha-\beta)+\cdots+\frac{2\cos.((k-1)\gamma+\delta)}{k\sin.\theta}(\pi-(k-1)\alpha-\beta).$$

Statuamus igitur

$$S=(\pi-\beta)\cos.\delta+(\pi-\alpha-\beta)\cos.(\gamma+\delta)+(\pi-2\alpha-\beta)\cos.(2\gamma+\delta)$$
$$+\cdots+(\pi-(k-1)\alpha-\beta)\cos.((k-1)\gamma+\delta),$$

ut sit valor quaesitus $\frac{2S}{k\sin.\theta}$. Multiplicemus utrinque per $2\sin.\frac{1}{2}\gamma$, et cum sit

$$2\sin.\tfrac{1}{2}\gamma\cos.q=\sin.\left(\tfrac{1}{2}\gamma+q\right)-\sin.\left(q-\tfrac{1}{2}\gamma\right),$$

hac reductione adhibita fiet

$$2S\sin.\tfrac{1}{2}\gamma=\left\{\begin{aligned}&-(\pi-\beta)\sin.\left(\delta-\tfrac{1}{2}\gamma\right)\\&+(\pi-\beta)\sin.\left(\tfrac{1}{2}\gamma+\delta\right)-(\pi-\alpha-\beta)\sin.\left(\tfrac{1}{2}\gamma+\delta\right)\\&+(\pi-\alpha-\beta)\sin.\left(\tfrac{3}{2}\gamma+\delta\right)-(\pi-2\alpha-\beta)\sin.\left(\tfrac{3}{2}\gamma+\delta\right)\\&+(\pi-2\alpha-\beta)\sin.\left(\tfrac{5}{2}\gamma+\delta\right)-(\pi-3\alpha-\beta)\sin.\left(\tfrac{5}{2}\gamma+\delta\right)\\&+\text{etc.,}\end{aligned}\right.$$

quae series contractis terminis similibus transit in hanc

$$2S\sin.\tfrac{1}{2}\gamma=-(\pi-\beta)\sin.\left(\delta-\tfrac{1}{2}\gamma\right)+\alpha\sin.\left(\tfrac{1}{2}\gamma+\delta\right)$$
$$+\alpha\sin.\left(\tfrac{3}{2}\gamma+\delta\right)+\alpha\sin.\left(\tfrac{5}{2}\gamma+\delta\right)+\cdots+\alpha\sin.\left(\frac{2k-3}{2}\gamma+\delta\right)$$
$$+(\pi-(k-1)\alpha-\beta)\sin.\left(\frac{2k-1}{2}\gamma+\delta\right);$$

ubi cum sit $\alpha=\frac{2\pi}{k}$ et $\beta=\frac{\theta}{k}$, erit

$$\pi-(k-1)\alpha-\beta=\alpha-\pi-\beta.$$

Ponatur

$$T=\sin.\left(\tfrac{1}{2}\gamma+\delta\right)+\sin.\left(\tfrac{3}{2}\gamma+\delta\right)+\sin.\left(\tfrac{5}{2}\gamma+\delta\right)$$
$$+\sin.\left(\tfrac{7}{2}\gamma+\delta\right)+\cdots+\sin.\left(\left(k-\tfrac{1}{2}\right)\gamma+\delta\right),$$

ut nanciscamur

$$2S\sin.\tfrac{1}{2}\gamma=-(\pi-\beta)\sin.\left(\delta-\tfrac{1}{2}\gamma\right)-(\pi+\beta)\sin.\left(\left(k-\tfrac{1}{2}\right)\gamma+\delta\right)+\alpha T,$$

quae expressio ob $k\gamma=2n\pi$ reducitur ad

$$-2\pi\sin.\left(\delta-\tfrac{1}{2}\gamma\right)+\alpha T.$$

Nunc igitur ad quantitatem T inveniendam multiplicemus utrinque per $2\sin.\frac{1}{2}\gamma$, et cum in genere sit

$$2\sin.\tfrac{1}{2}\gamma\sin.q=\cos.\left(q-\tfrac{1}{2}\gamma\right)-\cos.\left(q+\tfrac{1}{2}\gamma\right),$$

obtinebimus

$$2T\sin.\tfrac{1}{2}\gamma=\cos.\delta$$
$$-\cos.(\gamma+\delta)-\cos.(2\gamma+\delta)-\cos.(3\gamma+\delta)-\cos.(4\gamma+\delta)-\cdots$$
$$+\cos.(\gamma+\delta)+\cos.(2\gamma+\delta)+\cos.(3\gamma+\delta)+\cos.(4\gamma+\delta)+\cdots$$
$$-\cos.(k\gamma+\delta),$$

quae forma contrahitur in istam

$$2T\sin.\tfrac{1}{2}\gamma=\cos.\delta-\cos.(k\gamma+\delta).$$

Cum autem sit $\gamma = \frac{2n\pi}{k}$, erit $k\gamma = 2n\pi$ ideoque $\cos.(k\gamma + \delta) = \cos.\delta$, unde fit

$$2T\sin.\frac{1}{2}\gamma = 0,$$

ita ut nunc sit

$$2S\sin.\frac{1}{2}\gamma = 2\pi\sin.\left(\frac{1}{2}\gamma - \delta\right)$$

ideoque

$$S = \frac{\pi\sin.\left(\frac{1}{2}\gamma - \delta\right)}{\sin.\frac{1}{2}\gamma}.$$

Est vero $\frac{1}{2}\gamma = \frac{n\pi}{k}$ et $\delta = \frac{n\theta}{k}$ ideoque $\frac{1}{2}\gamma - \delta = \frac{n(\pi - \theta)}{k} = \frac{n\eta}{k}$ ob $\theta = \pi - \eta$ hocque modo habebimus

$$S = \frac{\pi\sin.\frac{n\eta}{k}}{\sin.\frac{n\pi}{k}};$$

consequenter valor integralis quaesiti concluditur fore

$$\frac{2\pi\sin.\frac{n\eta}{k}}{k\sin.\theta\sin.\frac{n\pi}{k}} = \frac{2\pi\sin.\frac{n\eta}{k}}{k\sin.\eta\sin.\frac{n\pi}{k}},$$

unde formetur sequens theorema.

THEOREMA 1

47. *Haec formula integralis*

$$\int \frac{x^{k-n} + x^{k+n}}{1 + 2x^k\cos.\eta + x^{2k}} \cdot \frac{dx}{x}$$

a termino $x = 0$ *usque ad* $x = \infty$ *extensa producit hunc valorem*

$$\frac{2\pi\sin.\frac{n\eta}{k}}{k\sin.\eta\sin.\frac{n\pi}{k}}.$$

Cui adiungatur adhuc sequens

THEOREMA 2

48. *Ista vero formula integralis*

$$\int \frac{x^{k \pm n - 1}dx}{1 + 2x^k\cos.\eta + x^{2k}}$$

pariter a termino $x = 0$ *usque ad terminum* $x = \infty$ *extensa valorem habet dimidium praecedentis, qui ergo erit*

$$\frac{\pi\sin.\frac{n\eta}{k}}{k\sin.\eta\sin.\frac{n\pi}{k}}.$$

Cuius demonstratio perinde succedet ac supra (§ 25).

QUAESTIO ALTERA

49. *Proposita formula differentiali*

$$\frac{x^{k-n} + x^{k+n}}{1 + 2x^k\cos.\eta + x^{2k}} \cdot \frac{dx}{x}$$

eius integralis valorem investigare, qui oritur, si post integrationem ponitur $x = 1$.

SOLUTIO

Posito $x = 1$ formula generalis $\text{A tang.}\frac{x\sin.\omega}{1 - x\cos.\omega}$, ut supra vidimus, reducitur ad $\frac{\pi - \omega}{2}$; unde patet singulas partes integralis duplo minores esse quam casu praecedente, unde valor quaesitus etiam erit duplo minor

$$= \frac{\pi\sin.\frac{n\eta}{k}}{k\sin.\eta\sin.\frac{n\pi}{k}},$$

unde nascitur sequens

THEOREMA

50. *Ista formula integralis*

$$\int \frac{x^{k-n} + x^{k+n}}{1 + 2x^k\cos.\eta + x^{2k}} \cdot \frac{dx}{x}$$

a termino $x = 0$ *usque ad* $x = 1$ *extensa producet hunc valorem*

$$\frac{\pi \sin. \frac{n\eta}{k}}{k \sin. \eta \sin. \frac{n\pi}{k}}.$$

SCHOLION

51. In his valoribus integralibus ii casus praecipue sunt notatu digni, quibus post integrationem statuitur $x = 1$, quandoquidem tum ista integralia commode per seriem infinitam exprimere licet. Ita pro casu § 26, quoniam est

$$\frac{1}{1+x^{2k}} = 1 - x^{2k} + x^{4k} - x^{6k} + \text{etc.},$$

si hanc seriem multiplicemus per $(x^{k-n} + x^{k+n})\frac{dx}{x}$ et integremus, tum vero ponamus $x = 1$, prodibit ista series infinita

$$\frac{1}{k-n} - \frac{1}{3k-n} + \frac{1}{5k-n} - \frac{1}{7k-n} + \text{etc.}$$
$$+ \frac{1}{k+n} - \frac{1}{3k+n} + \frac{1}{5k+n} - \frac{1}{7k+n} + \text{etc.},$$

cuius ergo seriei in infinitum continuatae summa est

$$\frac{\pi}{2k \cos. \frac{n\pi}{2k}}.$$

At pro casu § 38 ob

$$\frac{1}{1-x^{2k}} = 1 + x^{2k} + x^{4k} + x^{6k} + \text{etc.}$$

eodem modo operando pervenitur ad hanc seriem

$$\frac{1}{k-n} + \frac{1}{3k-n} + \frac{1}{5k-n} + \frac{1}{7k-n} + \text{etc.}$$
$$- \frac{1}{k+n} - \frac{1}{3k+n} - \frac{1}{5k+n} - \frac{1}{7k+n} - \text{etc.},$$

cuius ergo summa erit

$$\frac{\pi}{2k} \operatorname{tang.} \frac{n\pi}{2k}.$$

Denique pro casu, quem extremo loco tractavimus, cum sit, ut alibi[1]) ostendimus,

$$\frac{\sin. \eta}{1 + 2x^k \cos. \eta + x^{2k}} = \sin. \eta - x^k \sin. 2\eta + x^{2k} \sin. 3\eta - \text{etc.},$$

haec series ducta in $(x^{k-n} + x^{k+n})\frac{dx}{x}$ et integrata sumendo $x = 1$ producet hanc seriem

$$\frac{\sin. \eta}{k-n} - \frac{\sin. 2\eta}{2k-n} + \frac{\sin. 3\eta}{3k-n} - \frac{\sin. 4\eta}{4k-n} + \text{etc.}$$
$$+ \frac{\sin. \eta}{k+n} - \frac{\sin. 2\eta}{2k+n} + \frac{\sin. 3\eta}{3k+n} - \frac{\sin. 4\eta}{4k+n} + \text{etc.},$$

cuius ergo valor aequabitur illi, quem invenimus, valori ducto in $\sin. \eta$, ita ut summa huius seriei sit

$$= \frac{\pi \sin. \frac{n\eta}{k}}{k \sin. \frac{n\pi}{k}},$$

quae series eo magis sunt memorabiles, quod alio modo earum summa vix elici potest.

1) L. EULERI Commentatio 464 (indicis ENESTROEMIANI): *Nova methodus quantitates integrales determinandi*, Novi comment. acad. sc. Petrop. 19 (1774), 1775, p. 66; *LEONHARDI EULERI Opera omnia*, series I, vol. 17, p. 421, imprimis p. 441. A. G.

OBSERVATIONES IN ALIQUOT THEOREMATA ILLUSTRISSIMI DE LA GRANGE

Commentatio 587 indicis ENESTROEMIANI
Opuscula analytica 2, 1785, p. 16—41

Postquam aliquod theorema ex iis, quae non ita pridem demonstravi, quo ostendi[1]) formulae integralis $\int \frac{(x-1)dx}{lx}$, si post integrationem ponatur $x=1$, valorem esse $=l2$, cum illustri Domino DE LA GRANGE communicassem[2]), is novitate huius argumenti permotus non solum felicissimo successu eius demonstrationem penetravit, sed etiam plurima alia praeclara inventa inde deduxit, quorum uberior enucleatio scientiae analyticae maxima incrementa polliceri videtur, ex quo genere aliquot praeclarissima specimina mecum benevole communicavit, quae statim summo studio sum perscrutatus; et quoniam haec materia attentionem mereri videtur, meas meditationes, quae se mihi hac occasione obtulerunt, fusius sum expositurus. Cum autem hoc quasi novum Analyseos genus potissimum in eiusmodi formulis integralibus versetur, in quibus variabili post integrationem certus valor determinatus tribuitur, ad taediosas verborum ambages evitandas, quas perpetua talium conditionum commemoratio postularet, peculiarem signandi modum adhibebo, quem ante omnia accuratius explicare necesse erit.

1) Vide § 5 Commentationis 464 supra, p. 155, laudatae. A. G.

2) Vide epistolas ab EULERO d. 26. Jan. et 23. Mart. 1775 ad I. L. LAGRANGE scriptas, *LEONHARDI EULERI Opera postuma* 1, p. 585 et 586; *Oeuvres de LAGRANGE* 14, p. 240 et 241; *LEONHARDI EULERI Opera omnia*, series III. A. G.

HYPOTHESIS

1. Hac signandi ratione

$$\int Pdx \begin{bmatrix} \text{ab } x=a \\ \text{ad } x=b \end{bmatrix}$$

declaratur integrale $\int Pdx$ ita esse assumtum, ut evanescat posito $x=a$, tum vero statui $x=b$; quo pacto manifestum est eius valorem penitus fore determinatum.

SCHOLION

2. Quo indoles huius determinationis clarius perspiciatur, quoniam P denotat functionem aliquam ipsius x, eius naturam repraesentemus linea quadam curva *ixabco* (Fig. 1) super axe *IO* exstructa, cuius quaecunque appli-

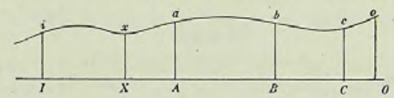

Fig. 1.

cata Xx abscissae $IX=x$ respondens exhibeat ipsam functionem P, ita ut formula integralis $\int Pdx$ indefinite exprimat aream huius curvae. Quodsi iam capiantur abscissae $IA=a$, $IB=b$, quibus respondeant applicatae Aa et Bb, formula proposita exprimet aream $AaBb$ inter applicatas Aa et Bb interceptam. Eodem modo, si alia quaepiam abscissa statuatur $IC=c$, area $AaCc$ exprimetur hac formula

$$\int Pdx \begin{bmatrix} \text{ab } x=a \\ \text{ad } x=c \end{bmatrix},$$

area autem $BbCc$ ista formula

$$\int Pdx \begin{bmatrix} \text{ab } x=b \\ \text{ad } x=c \end{bmatrix};$$

tum vero ab initio I incipiendo area $IiAa$ indicabitur per hanc formulam

$$\int Pdx \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=a \end{bmatrix};$$

unde sponte fluunt sequentia lemmata ita succincte expressa.

LEMMA 1

3. $$\int Pdx\left[\begin{smallmatrix}\text{ab } x=a\\ \text{ad } x=b\end{smallmatrix}\right]=-\int Pdx\left[\begin{smallmatrix}\text{ab } x=b\\ \text{ad } x=a\end{smallmatrix}\right].$$

Quoniam enim, si b ut maius spectetur quam a, formula posterior

$$\int Pdx\left[\begin{smallmatrix}\text{ab } x=b\\ \text{ad } x=a\end{smallmatrix}\right]$$

eandem aream $AaBb$ refert quam prior, sed ordine retrogrado, ista expressio pro negativa erit habenda sicque erit quoque

$$\int Pdx\left[\begin{smallmatrix}\text{ab } x=a\\ \text{ad } x=b\end{smallmatrix}\right]+\int Pdx\left[\begin{smallmatrix}\text{ab } x=b\\ \text{ad } x=a\end{smallmatrix}\right]=0.$$

LEMMA 2

4. $$\int Pdx\left[\begin{smallmatrix}\text{ab } x=a\\ \text{ad } x=b\end{smallmatrix}\right]+\int Pdx\left[\begin{smallmatrix}\text{ab } x=b\\ \text{ad } x=c\end{smallmatrix}\right]=\int Pdx\left[\begin{smallmatrix}\text{ab } x=a\\ \text{ad } x=c\end{smallmatrix}\right],$$

quemadmodum inspectio figurae manifesto declarat.

LEMMA 3

5. $$\int Pdx\left[\begin{smallmatrix}\text{ab } x=a\\ \text{ad } x=c\end{smallmatrix}\right]-\int Pdx\left[\begin{smallmatrix}\text{ab } x=a\\ \text{ad } x=b\end{smallmatrix}\right]=\int Pdx\left[\begin{smallmatrix}\text{ab } x=b\\ \text{ad } x=c\end{smallmatrix}\right],$$

ubi in binis prioribus formulis idem occurrit terminus *a quo*, scilicet $x=a$; terminorum vero *ad quem*, scilicet $x=c$ et $x=b$, posterior $x=b$ dat pro tertia formula terminum *a quo*, prior vero terminum *ad quem*.

LEMMA 4

6. $$\int Pdx\left[\begin{smallmatrix}\text{ab } x=a\\ \text{ad } x=c\end{smallmatrix}\right]-\int Pdx\left[\begin{smallmatrix}\text{ab } x=b\\ \text{ad } x=c\end{smallmatrix}\right]=\int Pdx\left[\begin{smallmatrix}\text{ab } x=a\\ \text{ad } x=b\end{smallmatrix}\right],$$

ubi notetur binas formulas priores eundem habere terminum *ad quem*, scilicet $x=c$, terminorum autem *a quo* priorem $x=a$ dare in tertia formula terminum *a quo*, posteriorem vero terminum *ad quem*.

LEMMA 5

7. $$\int Pdx\left[\begin{smallmatrix}\text{ab } x=a\\ \text{ad } x=b\end{smallmatrix}\right]+\int Pdx\left[\begin{smallmatrix}\text{ab } x=b\\ \text{ad } x=c\end{smallmatrix}\right]+\int Pdx\left[\begin{smallmatrix}\text{ab } x=c\\ \text{ad } x=a\end{smallmatrix}\right]=0.$$

SCHOLION

8. His igitur, quae per se sunt maxime perspicua, praemissis argumenta praecipua, quae Celeb. DE LA GRANGE mihi perscripsit, ordine percurram. Primo autem mentionem insignis paradoxi facit, cuius indolem ipse non satis perspicere fatetur, a quo igitur meas meditationes inchoabo.

RESOLUTIO INSIGNIS PARADOXI

9. Cum Vir celeb. etiam invenisset hoc theorema generale

$$\int\frac{x^n-x^m}{lx}\cdot\frac{dx}{x}\left[\begin{smallmatrix}\text{ab } x=0\\ \text{ad } x=1\end{smallmatrix}\right]=l\frac{n}{m},$$

cuius veritatem non ita pridem pluribus demonstrationibus adstruxi[1]), posuit $x^n=z$ et $x^m=y$; quo facto pars prior $\int\frac{x^{n-1}dx}{lx}$ transformatur in hanc $\int\frac{dz}{lz}$, simili vero modo altera $\int\frac{x^{m-1}dx}{lx}$ in hanc $\int\frac{dy}{ly}$; unde his partibus seorsim positis sequitur fore

$$\int\frac{dz}{lz}\left[\begin{smallmatrix}\text{a } z=0\\ \text{ad } z=1\end{smallmatrix}\right]-\int\frac{dy}{ly}\left[\begin{smallmatrix}\text{ab } y=0\\ \text{ad } y=1\end{smallmatrix}\right]=l\frac{n}{m}.$$

Quare cum hae duae formulae omnino sint similes atque iisdem terminis integrationis contentae, quis non crederet eos etiam inter se perfecte fore aequales sive esse

$$\int\frac{dz}{lz}\left[\begin{smallmatrix}\text{a } z=0\\ \text{ad } z=1\end{smallmatrix}\right]=\int\frac{dy}{ly}\left[\begin{smallmatrix}\text{ab } y=0\\ \text{ad } y=1\end{smallmatrix}\right]?$$

Interim tamen vidimus differentiam inter has formulas esse $l\frac{n}{m}$. Hic igitur se offert quaestio maximi momenti, quemadmodum istam manifestam contradictionem dirimere oporteat.

1) Vide § 6 Commentationis 464 supra, p. 155, laudatae. A. G.

10. Primo autem hic observari convenit ambas quantitates y et z certo quodam modo a se invicem pendere. Cum enim sit $y = x^m$ et $z = x^n$, erit $y^n = z^m$, quo tamen nexu non impeditur, quominus posito sive $y = 0$ sive $y = 1$ etiam fiat $z = 0$ sive $z = 1$. Interim tamen hinc neutiquam patet, cur ob hanc rationem istae binae formulae

$$\int \frac{dy}{ly} \begin{bmatrix} \text{ab} & y = 0 \\ \text{ad} & y = 1 \end{bmatrix} \text{ et } \int \frac{dz}{lz} \begin{bmatrix} \text{a} & z = 0 \\ \text{ad} & z = 1 \end{bmatrix}$$

dispares prodire queant; unde haec observatio ad dubium solvendum nihil plane conferre videtur.

11. Quin etiam nullo prorsus dubio obnoxia videtur haec aequatio multo generalior

$$\int \frac{dy}{ly} \begin{bmatrix} \text{ab} & y = a \\ \text{ad} & y = b \end{bmatrix} = \int \frac{dz}{lz} \begin{bmatrix} \text{a} & z = a \\ \text{ad} & z = b \end{bmatrix},$$

quandoquidem nihil plane impedit, quominus loco z scribamus y vel vicissim; verum plurima phaenomena in analysi observata satis luculenter docent huiusmodi aequalitates interdum exceptionem pati, quando valores evadunt infiniti. Haec autem circumstantia nostro casu utique locum habet, cum formula integralis $\int \frac{dy}{ly}$, si ab $y = 0$ ad $y = 1$ extendatur, utique in infinitum excrescat, quod etiam de altera $\int \frac{dz}{lz}$ est tenendum. Si enim [abscissa] fiat $= 1$, applicata nostrae curvae, quae est $\frac{1}{lz}$, manifesto fit infinite magna, unde superior aequalitas generalis

$$\int \frac{dy}{ly} \begin{bmatrix} \text{ab} & y = a \\ \text{ad} & y = b \end{bmatrix} - \int \frac{dz}{lz} \begin{bmatrix} \text{a} & z = a \\ \text{ad} & z = b \end{bmatrix} = 0$$

hanc restrictionem postulat, nisi vel a sit $= 1$ vel $b = 1$, quippe quibus casibus utraque formula fit infinita.

12. His perpensis nullum plane dubium mihi quidem superesse videtur, quin in hac circumstantia vera solutio propositi paradoxi sit quaerenda, quae scilicet in eo versatur, quod sit

$$\text{tam} \quad \int \frac{dy}{ly} \begin{bmatrix} \text{ab} & y = 0 \\ \text{ad} & y = 1 \end{bmatrix} = \infty \quad \text{quam} \quad \int \frac{dz}{lz} \begin{bmatrix} \text{a} & z = 0 \\ \text{ad} & z = 1 \end{bmatrix} = \infty,$$

ita ut horum infinitorum differentia possit aequari quantitati finitae cuicunque ideoque in se spectata prorsus non determinetur; quod autem ista differentia nostro casu sit $l\frac{n}{m}$ ideoque determinata, inde venit, quod sit $y^n = z^m$.

13. Simile aliquid evenire potest in formulis simplicioribus, quales sunt $\int \frac{dy}{y}$ et $\int \frac{dz}{z}$, quippe quarum valores a termino $y = 0$ et $z = 0$ sumti sunt infiniti, unde, etiamsi post integrationem idem terminus *ad quem* statuatur, scilicet $y = 1$ et $z = 1$, tamen hinc nullo modo sequitur differentiam absolute nihilo aequari, quin potius tanquam indeterminata spectari debebit, cum quidem pro aliis terminis integrationis certo sit

$$\int \frac{dy}{y} \begin{bmatrix} \text{ab} & y = a \\ \text{ad} & y = b \end{bmatrix} = \int \frac{dz}{z} \begin{bmatrix} \text{a} & z = a \\ \text{ad} & z = b \end{bmatrix},$$

dummodo neque a neque b fuerit $= 0$ vel $= \infty$.

14. Atque hinc etiam paradoxon proposito penitus simile proferri potest, quod ita se habet

$$\int \frac{dz}{z} \begin{bmatrix} \text{a} & z = 0 \\ \text{ad} & z = \infty \end{bmatrix} - \int \frac{dy}{y} \begin{bmatrix} \text{ab} & y = 0 \\ \text{ad} & y = \infty \end{bmatrix} = la;$$

cuius veritas cum in aprico sit posita, siquidem accipiatur $z = ay$, etiam paradoxon propositum rite dilutum erit censendum.

OBSERVATIONES IN HOC THEOREMA D. DE LA GRANGE

$$\int \frac{x^n - x^m}{lx} \cdot \frac{dx}{x} \begin{bmatrix} \text{ab} & x = a \\ \text{ad} & x = b \end{bmatrix} = \int (b^y - a^y) \frac{dy}{y} \begin{bmatrix} \text{ab} & y = m \\ \text{ad} & y = n \end{bmatrix}$$

15. Cum equidem ante aliquod tempus reductiones huiusmodi formularum tractassem, alios terminos integrationis praeterquam ab $x = 0$ ad $x = 1$ non sum contemplatus, unde hoc theorema mihi statim altioris indaginis est visum atque omnino dignum, quod summa cura expendatur. Primum igitur in eius veritatem per series inquirere constitui, quod negotium sequenti modo peregi.

16. Cum sit

$$x^\alpha = e^{\alpha lx} = 1 + \alpha lx + \frac{(\alpha lx)^2}{1 \cdot 2} + \frac{(\alpha lx)^3}{1 \cdot 2 \cdot 3} + \text{etc.},$$

erit

$$x^n - x^m = (n - m)\frac{lx}{1} + (n^2 - m^2)\frac{(lx)^2}{1 \cdot 2} + \frac{(n^3 - m^3)(lx)^3}{1 \cdot 2 \cdot 3} + \text{etc.}$$

Hanc ergo seriem ducamus in $\frac{dx}{xlx}$, et quia in genere

$$\int (lx)^{\lambda}\frac{dx}{xlx}\begin{bmatrix}\text{ab } x=a\\ \text{ad } x=b\end{bmatrix}=\frac{(lb)^{\lambda}-(la)^{\lambda}}{\lambda},$$

formulae ad sinistram partem scriptae valor per hanc seriem infinitam exprimetur

$$\frac{n-m}{1}\cdot\frac{lb-la}{1}+\frac{n^2-m^2}{1\cdot 2}\cdot\frac{(lb)^2-(la)^2}{2}+\frac{n^3-m^3}{1\cdot 2\cdot 3}\cdot\frac{(lb)^3-(la)^3}{3}+\text{etc.}$$

17. Simili modo pro formula ad dextram posita per seriem infinitam erit

$$b^y-a^y=y\,\frac{lb-la}{1}+y^2\frac{(lb)^2-(la)^2}{1\cdot 2}+y^3\frac{(lb)^3-(la)^3}{1\cdot 2\cdot 3}+\text{etc.},$$

quae ergo ducatur in $\frac{dy}{y}$, et quia in genere est

$$\int y^{\lambda}\frac{dy}{y}\begin{bmatrix}\text{ab } y=m\\ \text{ad } y=n\end{bmatrix}=\frac{n^{\lambda}-m^{\lambda}}{\lambda},$$

valor istius formulae per seriem hanc infinitam exprimetur

$$\frac{n-m}{1}\cdot\frac{lb-la}{1}+\frac{n^2-m^2}{2}\cdot\frac{(lb)^2-(la)^2}{1\cdot 2}+\frac{n^3-m^3}{3}\cdot\frac{(lb)^3-(la)^3}{1\cdot 2\cdot 3}+\text{etc.}$$

Quia igitur haec series cum praecedente perfecte congruit, veritas theorematis firmiter est evicta.

18. Verum hinc neutiquam perspicitur, quomodo sagacissimus Auctor ad hoc theorema sit perductus, quamobrem rebus probe perpensis viam inveni ex iisdem principiis, quibus antehac[1]) sum usus, ad easdem formulas perveniendi. Inchoandum autem est ab hac forma simplicissima

$$\int x^{\lambda}\frac{dx}{x}\begin{bmatrix}\text{ab } x=a\\ \text{ad } x=b\end{bmatrix}=\frac{b^{\lambda}-a^{\lambda}}{\lambda},$$

ubi utrinque per $d\lambda$ multiplicans denuo integrationem instituo, et cum, uti iam passim[2]) demonstratum reperitur, sit

$$\int d\lambda\int x^{\lambda}\frac{dx}{x}=\int\frac{dx}{x}\int x^{\lambda}d\lambda,$$

1) Vide Commentationem 464 supra, p. 155, laudatam. A. G.
2) Vide exempli gratia ibidem p. 432. A. G.

quaeri tantum debet hoc integrale $\int x^{\lambda}d\lambda$ spectata quantitate x ut constante, ita ut sola λ sit variabilis. Est vero

$$\int x^{\lambda}d\lambda=\frac{x^{\lambda}}{lx}+C,$$

quemadmodum ex elementis calculi exponentialis liquet. Hic vero cardo rei in hoc versatur, ut istud integrale certa lege definiatur, quam deinceps etiam in altera parte observari oportet. Statuamus ergo talia integralia ita capi, ut evanescant posito $\lambda=0$, eritque

$$\int x^{\lambda}d\lambda=\frac{x^{\lambda}-1}{lx},$$

quo pacto pro sinistra parte habebimus

$$\int d\lambda\int x^{\lambda}\frac{dx}{x}=\int\frac{dx}{x}\cdot\frac{x^{\lambda}-1}{lx}.$$

19. Pro parte autem dextra habebimus

$$\int\frac{d\lambda}{\lambda}(b^{\lambda}-a^{\lambda}),$$

qua formula eadem lege integrata, ut facto $\lambda=0$ prodeat nihilum, hunc valorem more hic recepto repraesentare licebit

$$\int\frac{dy}{y}(b^y-a^y)\begin{bmatrix}\text{ab } y=0\\ \text{ad } y=\lambda\end{bmatrix}.$$

Hic enim nil aliud fecimus, nisi quod pro λ scripsimus y et facta integratione loco y eius valorem λ restitui assumsimus, sicque assecuti sumus sequentem formulam

$$\int (x^{\lambda}-1)\frac{dx}{xlx}\begin{bmatrix}\text{ab } x=a\\ \text{ad } x=b\end{bmatrix}=\int\frac{dy}{y}(b^y-a^y)\begin{bmatrix}\text{ab } y=0\\ \text{ad } y=\lambda\end{bmatrix},$$

quam tanquam theorema utilissimum spectare licet.

20. Vi ergo huius theorematis nanciscimur sequentes reductiones

$$\int (x^{n}-1)\frac{dx}{xlx}\begin{bmatrix}\text{ab } x=a\\ \text{ad } x=b\end{bmatrix}=\int\frac{dy}{y}(b^y-a^y)\begin{bmatrix}\text{ab } y=0\\ \text{ad } y=n\end{bmatrix}$$

et

$$\int (x^{m}-1)\frac{dx}{xlx}\begin{bmatrix}\text{ab } x=a\\ \text{ad } x=b\end{bmatrix}=\int\frac{dy}{y}(b^y-a^y)\begin{bmatrix}\text{ab } y=0\\ \text{ad } y=m\end{bmatrix};$$

quare si formula posterior a priore subtrahatur, erit

$$\int (x^n - x^m)\frac{dx}{xlx}\begin{bmatrix}\text{ab} & x=a\\ \text{ad} & x=b\end{bmatrix} = \int \frac{dy}{y}(b^y - a^y)\begin{bmatrix}\text{ab} & y=0\\ \text{ad} & y=n\end{bmatrix} - \int \frac{dy}{y}(b^y - a^y)\begin{bmatrix}\text{ab} & y=0\\ \text{ad} & y=m\end{bmatrix};$$

verum ista formula ad dextram posita per reductionem in Lemmate 3 ostensam revocatur ad hanc formam simpliciorem

$$\int \frac{dy}{y}(b^y - a^y)\begin{bmatrix}\text{ab} & y=m\\ \text{ad} & y=n\end{bmatrix};$$

unde patet hoc modo ipsum hoc insigne theorema etiam ex nostris principiis investigari potuisse.

21. Hoc autem theoremate generalissimo Vir ingeniosissimus est usus ad theorema meum demonstrandum, quo ostendi esse

$$\int (x^n - x^m)\frac{dx}{xlx}\begin{bmatrix}\text{ab} & x=0\\ \text{ad} & x=1\end{bmatrix} = l\frac{n}{m};$$

tantum enim opus erat, ut caperetur $a=0$ et $b=1$, quo pacto formula ad dextram posita integralis abit in

$$\int \frac{dy}{y}\begin{bmatrix}\text{ab} & y=m\\ \text{ad} & y=n\end{bmatrix},$$

cuius valor manifesto fit $ln - lm = l\frac{n}{m}$, quae est nova demonstratio mei theorematis, cuiusmodi quidem dudum[1]) plures alias dederam.

OBSERVATIONES IN THEOREMA D. DE LA GRANGE

$$\int \frac{(x^n - x^m)dx}{(1+x^r)lx}\begin{bmatrix}\text{ab} & x=0\\ \text{ad} & x=\infty\end{bmatrix} = l\frac{\text{tang.}\frac{(n+1)\pi}{2r}}{\text{tang.}\frac{(m+1)\pi}{2r}}$$

22. Quia hic ambo exponentes m et n neque a se invicem neque ab exponente r pendent, manifestum est pro utraque potestate x^m et x^n seorsim integrale talem formam habere debere

$$\int \frac{x^n dx}{(1+x^r)lx} = l\,\text{tang.}\frac{(n+1)\pi}{2r} + C \quad\text{et}\quad \int \frac{x^m dx}{(1+x^r)lx} = l\,\text{tang.}\frac{(m+1)\pi}{2r} + C.$$

1) Vide Commentationem 464 supra, p. 155, laudatam. A. G.

Si enim posterior forma a priore subtrahatur, constans C ex calculo egreditur et ipsum integrale propositum resultat. Hic igitur plurimum intererit valorem istius constantis C determinasse.

23. Inter formulas integrales, quarum valores pro casu, quo post integrationem variabilis infinita statuitur, ex primis principiis calculi integralis assignavi[1]), reperitur ista

$$\int \frac{x^{k+n}}{1+x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{ab} & x=0\\ \text{ad} & x=\infty\end{bmatrix} = \frac{\pi}{2k\cos.\frac{n\pi}{2k}} = \frac{\pi}{2k\sin.\frac{(k+n)\pi}{2k}},$$

ubi autem assumitur exponentem n non maiorem capi quam k. Quodsi iam hic exponens n ut variabilis tractetur spectata ipsa x ut constante et utrinque per dn multiplicetur denuoque integretur, formula sinistra erit

$$\int dn \int \frac{x^{k+n}}{1+x^{2k}}\cdot\frac{dx}{x} = \int \frac{dx}{x(1+x^{2k})}\int x^{k+n}dn,$$

ubi postremum integrale fit

$$\int x^{k+n}dn = \frac{x^{k+n}}{lx} + C.$$

Ut autem hoc integrale determinetur, constantem ita definiamus, ut id evanescat posito $n=0$, unde obtinetur

$$\int x^{k+n}dn = \frac{x^{k+n} - x^k}{lx},$$

ita ut formula integralis ad sinistram posita futura sit

$$\int \frac{x^{k+n} - x^k}{1+x^{2k}}\cdot\frac{dx}{xlx}\begin{bmatrix}\text{ab} & x=0\\ \text{ad} & x=\infty\end{bmatrix}.$$

24. Pro parte dextra autem habebimus hoc integrale

$$\int \frac{\pi dn}{2k\sin.\frac{(k+n)\pi}{2k}}$$

etiam ita sumendum, ut evanescat posito $n=0$. Hunc in finem statuamus

1) L. Euleri Commentatio 59 (indicis Enestroemiani): *Theoremata circa reductionem formularum integralium ad quadraturam circuli*, Miscellanea Berolin. 7, 1743, p. 91; *Leonhardi Euleri Opera omnia*, series I, vol. 17, p. 1, imprimis p. 29; vide porro *Institutionum calculi integralis* vol. I, § 351; *Leonhardi Euleri Opera omnia*, series I, vol. 11, p. 225. A. G.

angulum $\frac{(k+n)\pi}{2k}=\varphi$, et quia hinc erit $d\varphi=\frac{\pi dn}{2k}$, formula nostra integranda erit $\int\frac{d\varphi}{\sin.\varphi}$, cuius integrale per regulas notas in genere est

$$l\,\text{tang.}\,\frac{1}{2}\varphi+C=l\,\text{tang.}\,\frac{(k+n)\pi}{4k}+C,$$

quod facto $n=0$ abit in $l\,\text{tang.}\,\frac{\pi}{4}+C$. Quare cum $\text{tang.}\,\frac{\pi}{4}=1$ et $l\,1=0$, evidens est constantem C fore $=0$, ita ut integrale hoc quaesitum sit $l\,\text{tang.}\,\frac{(k+n)\pi}{4k}$. Hinc ergo assecuti sumus istam reductionem generalem

$$\int\frac{x^{k+n}-x^k}{1+x^{2k}}\cdot\frac{dx}{xlx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=l\,\text{tang.}\,\frac{(k+n)\pi}{4k},$$

ubi autem probe notari oportet exponentes m et n maiores capi non licere quam k.

25. Cum igitur loco n alium numerum m sumendo simili modo sit

$$\int\frac{x^{k+m}-x^k}{1+x^{2k}}\cdot\frac{dx}{xlx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=l\,\text{tang.}\,\frac{(k+m)\pi}{4k},$$

subtrahatur ista formula a praecedente et obtinebitur ista

$$\int\frac{x^{k+n}-x^{k+m}}{1+x^{2k}}\cdot\frac{dx}{xlx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=l\frac{\text{tang.}\,\frac{(k+n)\pi}{4k}}{\text{tang.}\,\frac{(k+m)\pi}{4k}},$$

quae manifesto cum forma proposita congruit, si modo loco $k+n-1$ scribatur n et m loco $k+m-1$, at loco exponentis $2k$ scribatur r; tum enim manifesto fiet

$$\int\frac{x^n-x^m}{1+x^r}\cdot\frac{dx}{lx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=l\frac{\text{tang.}\,\frac{(n+1)\pi}{2r}}{\text{tang.}\,\frac{(m+1)\pi}{2r}}.$$

26. Quoniam ista analysis nos perduxit ad hanc formam

$$\int\frac{x^{k+n}-x^k}{1+x^{2k}}\cdot\frac{dx}{xlx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=l\,\text{tang.}\,\frac{(k+n)\pi}{4k},$$

hic maximi momenti erit observasse semper fore

$$\int\frac{x^k}{1+x^{2k}}\cdot\frac{dx}{xlx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=0,$$

id quod ita ostendere possum. Ponatur $x^k=z$; erit

$$x^{k-1}dx=\frac{dz}{k}\quad\text{et}\quad lx=\frac{lz}{k}$$

sicque ista formula induet hanc formam $\int\frac{dz}{(1+zz)lz}$, ubi termini integrationis etiamnunc sunt $z=0$ et $z=\infty$. Fiat porro $z=\text{tang.}\,\varphi$, unde termini integrationis erunt $\varphi=0$ et $\varphi=\frac{\pi}{2}$; hinc autem ob $d\varphi=\frac{dz}{1+zz}$ nascetur ista formula

$$\int\frac{d\varphi}{l\,\text{tang}\,\varphi}\begin{bmatrix}\text{a } \varphi=0\\ \text{ad } \varphi=\frac{\pi}{2}\end{bmatrix},$$

cuius valorem in nihilum abire ostendi debet.

27. Ad hoc demonstrandum statuatur axis $IH=\frac{\pi}{2}$ (Fig. 2), super quo ab initio I sumta abscissa indefinita $Ip=\varphi$ applicata sit $=\frac{1}{l\,\text{tang.}\,\varphi}$. Quodsi ergo hic axis IH in O bisecetur, ut sit $IO=\frac{\pi}{4}$, in hoc puncto applicata erit

$$=\frac{1}{l\,\text{tang.}\,\frac{\pi}{4}}=\infty.$$

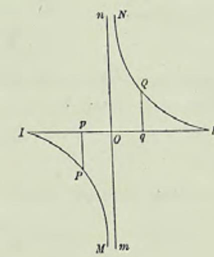

Fig. 2.

Iam ab hoc puncto O utrinque capiantur intervalla aequalia $Op=Oq=\omega$ et pro puncto p erit $\varphi=\frac{\pi}{4}-\omega$ sicque in hoc puncto p applicata erit

$$=\frac{1}{l\,\text{tang.}\left(\frac{\pi}{4}-\omega\right)};$$

est vero $\text{tang.}\left(\frac{\pi}{4}-\omega\right)=\text{cot.}\left(\frac{\pi}{4}+\omega\right)$, quare, cum sit $l\,\text{cot.}=-l\,\text{tang.}$, applicata in hoc puncto p erit

$$=\frac{-1}{l\,\text{tang.}\left(\frac{\pi}{4}+\omega\right)};$$

at quia est $Iq=\frac{\pi}{4}+\omega$, erit applicata in puncto q

$$=\frac{+1}{l\,\text{tang.}\left(\frac{\pi}{4}+\omega\right)}$$

sicque aequalis est applicatae in p, sed in contrarium vergens. Ita si applicata sursum directa fuerit qQ, in puncto p eadem applicata deorsum erit directa $pP = qQ$.

28. Quodsi ergo talis curva super axe $IH = \frac{\pi}{2}$ exstruatur, ita ut abscissae φ respondeat applicata $\frac{1}{l\,\text{tang.}\,\varphi}$, haec curva ex duabus portionibus inter se perfecte aequalibus constabit circa punctum medium O ita dispositis, ut curva sinistra sit IPM in infinitum descendens ad asymptotam Om, pars autem dextra simili modo a H sinistrorsum sursum ascendet ad asymptotam On. Quare cum formula integralis $\int \frac{d\varphi}{l\,\text{tang.}\,\varphi}$ a $\varphi = 0$ ad $\varphi = \frac{\pi}{2}$ extensa exprimat totius huius curvae ab I usque ad H protensae aream, evidens est totam hanc aream ad nihilum redigi, quia portio eius negative sumenda perfecte similis est portioni positive sumendae.

29. Sic igitur per demonstrationem omnino singularem evictum est semper esse

$$\int \frac{x^k}{1+x^{2k}} \cdot \frac{dx}{x\,lx} \begin{bmatrix} \text{ab} & x=0 \\ \text{ad} & x=\infty \end{bmatrix} = 0,$$

quod certe est theorema in hoc genere maxime notatu dignum. Quodsi ergo cum illustri D. DE LA GRANGE statuamus $2k = r$, erit

$$\int \frac{x^{\frac{1}{2}r-1}\,dx}{(1+x^r)\,lx} = 0;$$

praeterea vero pro nostra formula § 24 exhibita ob

$$\int \frac{x^k\,dx}{(1+x^{2k})\,x\,lx} \begin{bmatrix} \text{ab} & x=0 \\ \text{ad} & x=\infty \end{bmatrix} = 0$$

deducitur istud theorema omnino notabile

$$\int \frac{x^{k+n}}{1+x^{2k}} \cdot \frac{dx}{x\,lx} \begin{bmatrix} \text{ab} & x=0 \\ \text{ad} & x=\infty \end{bmatrix} = l\,\text{tang.}\,\frac{(k+n)\pi}{4k},$$

quod more D. DE LA GRANGE ita proponi potest

$$\int \frac{x^n\,dx}{(1+x^r)\,lx} \begin{bmatrix} \text{ab} & x=0 \\ \text{ad} & x=\infty \end{bmatrix} = l\,\text{tang.}\,\frac{(n+1)\pi}{2r};$$

sicque patet constantem illam supra (§ 22) a nobis inductam revera nihilo aequari.

30. Quoniam demonstratio huius theorematis methodo satis insueta innititur, eius veritatem per series ostendisse iuvabit. Ad hoc autem valorem formulae

$$\int \frac{x^{\lambda-1}\,dx}{(1+x^r)\,lx} \begin{bmatrix} \text{ab} & x=0 \\ \text{ad} & x=\infty \end{bmatrix}$$

in duas partes divelli necesse est (scilicet loco n scribendo $\lambda - 1$), quae sint

$$P = \int \frac{x^{\lambda-1}\,dx}{(1+x^r)\,lx} \begin{bmatrix} \text{ab} & x=0 \\ \text{ad} & x=1 \end{bmatrix} \quad \text{et} \quad Q = \int \frac{x^{\lambda-1}\,dx}{(1\mp x^r)\,lx} \begin{bmatrix} \text{ab} & x=1 \\ \text{ad} & x=\infty \end{bmatrix},$$

ita ut $P + Q$ exprimat valorem, quem quaerimus. Nunc in posteriore parte loco x scribamus $\frac{1}{z}$ fietque

$$Q = \int \frac{z^{-\lambda}}{1+z^{-r}} \cdot \frac{dz}{z\,lz} \begin{bmatrix} \text{a} & z=1 \\ \text{ad} & z=0 \end{bmatrix} = \int \frac{z^{r-\lambda}}{1+z^r} \cdot \frac{dz}{z\,lz} \begin{bmatrix} \text{a} & z=1 \\ \text{ad} & z=0 \end{bmatrix}$$

et commutatis terminis integrationis

$$Q = -\int \frac{z^{r-\lambda}}{1+z^r} \cdot \frac{dz}{z\,lz} \begin{bmatrix} \text{a} & z=0 \\ \text{ad} & z=1 \end{bmatrix}.$$

Nunc autem loco z scribamus x; quia termini integrationis utrinque sunt iidem, erit

$$P + Q = \int \frac{x^\lambda - x^{r-\lambda}}{1+x^r} \cdot \frac{dx}{x\,lx} \begin{bmatrix} \text{ab} & x=0 \\ \text{ad} & x=1 \end{bmatrix},$$

cuius ergo valor formulae propositae est aequalis.

31. Iam fractionem $\frac{1}{1+x^r}$ in seriem infinitam convertamus

$$1 - x^r + x^{2r} - x^{3r} + x^{4r} - \text{etc.},$$

cuius singuli termini in $\frac{dx}{x\,lx}(x^\lambda - x^{r-\lambda})$ ducti producunt

$$\frac{dx}{x\,lx}(x^\lambda - x^{r-\lambda}) - \frac{dx}{x\,lx}(x^{r+\lambda} - x^{2r-\lambda}) + \frac{dx}{x\,lx}(x^{2r+\lambda} - x^{3r-\lambda}) - \frac{dx}{x\,lx}(x^{3r+\lambda} - x^{4r-\lambda}) + \text{etc.}$$

Cum autem per theorema principale in hoc genere sit

$$\int \frac{dx}{x\,lx}(x^\alpha - x^\gamma) \begin{bmatrix} \text{ab} & x=0 \\ \text{ad} & x=1 \end{bmatrix} = l\,\frac{\alpha}{\beta},$$

singulis membris hoc modo integratis prodibit

$$P+Q=l\frac{\lambda}{r-\lambda}-l\frac{r+\lambda}{2r-\lambda}+l\frac{2r+\lambda}{3r-\lambda}-l\frac{3r+\lambda}{4r-\lambda}+\text{etc.}$$

32. Omnes hos logarithmos in unicum compingere licebit ratione habita signi cuiusque hocque modo reperietur fore

$$P+Q=l\frac{\lambda}{r-\lambda}\cdot\frac{2r-\lambda}{r+\lambda}\cdot\frac{2r+\lambda}{3r-\lambda}\cdot\frac{4r-\lambda}{3r+\lambda}\cdot\frac{4r+\lambda}{5r-\lambda}\cdot\frac{6r-\lambda}{5r+\lambda}\cdot\text{etc.}$$

At vero in *Introductione in Analysin Infinitorum* [1]) p. 147 ostendi esse

$$\text{tang.}\frac{m\pi}{2n}=\frac{m}{n-m}\cdot\frac{2n-m}{n+m}\cdot\frac{2n+m}{3n-m}\cdot\frac{4n-m}{3n+m}\cdot\frac{4n+m}{5n-m}\cdot\text{etc.},$$

quae series manifesto in inventam transformatur statuendo $m=\lambda$ et $n=r$, ita ut nunc sit $P+Q=l\,\text{tang.}\frac{\lambda\pi}{2r}$, prorsus uti supra est inventum.

ADDITAMENTUM

33. In dissertatione Actorum Tomo V, parte I, inserta [2]), unde desumsi hoc theorema

$$\int\frac{x^{k+n}}{1+x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{ab} & x=0\\ \text{ad} & x=\infty\end{bmatrix}=\frac{\pi}{2k\cos.\frac{n\pi}{2k}},$$

simul occurrunt sequentia

$$\int\frac{x^{k-n}+x^{k+n}}{1+x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{ab} & x=0\\ \text{ad} & x=\infty\end{bmatrix}=\frac{\pi}{k\cos.\frac{n\pi}{2k}},$$

$$\int\frac{x^{k-n}+x^{k+n}}{1+x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{ab} & x=0\\ \text{ad} & x=1\end{bmatrix}=\frac{\pi}{2k\cos.\frac{n\pi}{2k}},$$

$$\int\frac{x^{k-n}-x^{k+n}}{1-x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{ab} & x=0\\ \text{ad} & x=\infty\end{bmatrix}=\frac{\pi}{k}\,\text{tang.}\frac{n\pi}{2k},$$

$$\int\frac{x^{k-n}-x^{k+n}}{1-x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{ab} & x=0\\ \text{ad} & x=1\end{bmatrix}=\frac{\pi}{2k}\,\text{tang.}\frac{n\pi}{2k},$$

1) Vide *Introductionem in analysin infinitorum*, t. I cap. XI; *Leonhardi Euleri Opera omnia*, series I, vol. 8. A. G.

2) Vide Commentationem 572 huius voluminis, imprimis § 23 et seq. A. G.

$$\int\frac{x^{k-n}+x^{k+n}}{1+2x^k\cos.\eta+x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{ab} & x=0\\ \text{ad} & x=\infty\end{bmatrix}=\frac{2\pi\sin.\frac{n\eta}{k}}{k\sin.\eta\sin.\frac{n\pi}{k}},$$

$$\int\frac{x^{k-n}+x^{k+n}}{1+2x^k\cos.\eta+x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{ab} & x=0\\ \text{ad} & x=1\end{bmatrix}=\frac{\pi\sin.\frac{n\eta}{k}}{k\sin.\eta\sin.\frac{n\pi}{k}},$$

$$\int\frac{x^{k\pm n}}{1+2x^k\cos.\eta+x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{ab} & x=0\\ \text{ad} & x=\infty\end{bmatrix}=\frac{\pi\sin.\frac{n\eta}{k}}{k\sin.\eta\sin.\frac{n\pi}{k}},$$

quas formulas ergo simili modo tractare operae pretium erit.

34. Incipiamus igitur a formula

$$\int\frac{x^{k-n}+x^{k+n}}{1+x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{ab} & x=0\\ \text{ad} & x=1\end{bmatrix}=\frac{\pi}{2k\cos.\frac{n\pi}{2k}},$$

quia praecedens cum formula iam tractata prorsus conveniret; quae si ducatur in dn et ita integretur, ut integrale evanescat posito $n=0$, quoniam est

$$\int x^{k-n}dn=-\frac{x^{k-n}-x^k}{lx}\quad\text{et}\quad\int x^{k+n}dn=\frac{x^{k+n}-x^k}{lx},$$

tum vero, ut ante vidimus,

$$\int\frac{\pi\,dn}{2k\cos.\frac{n\pi}{2k}}=l\,\text{tang.}\frac{(k+n)\pi}{4k},$$

prodibit haec integratio

$$\int\frac{x^{k+n}-x^{k-n}}{1+x^{2k}}\cdot\frac{dx}{xlx}\begin{bmatrix}\text{ab} & x=0\\ \text{ad} & x=1\end{bmatrix}=l\,\text{tang.}\frac{(k+n)\pi}{4k},$$

qui ergo valor prorsus convenit cum eo, quem pro formula

$$\int\frac{x^{k+n}}{1+x^{2k}}\cdot\frac{dx}{xlx}\begin{bmatrix}\text{ab} & x=0\\ \text{ad} & x=\infty\end{bmatrix}$$

invenimus.

35. Simili modo tractemus sequentem formulam

$$\int\frac{x^{k-n}-x^{k+n}}{1-x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{ab} & x=0\\ \text{ad} & x=\infty\end{bmatrix}=\frac{\pi}{k}\,\text{tang.}\frac{n\pi}{2k},$$

quae ducta in dn et ut supra integrata praebet a parte sinistra

$$\int \frac{2x^k - x^{k-n} - x^{k+n}}{1 - x^{2k}} \cdot \frac{dx}{xlx} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix},$$

a parte autem dextra

$$\int \frac{\pi dn}{k} \operatorname{tang.} \frac{n\pi}{2k} = \int \frac{\pi dn \sin. \frac{n\pi}{2k}}{k \cos. \frac{n\pi}{2k}}.$$

Ad hoc integrandum fiat $\frac{n\pi}{2k} = \varphi$ eritque $\frac{\pi dn}{k} = 2d\varphi$ sicque formula integranda erit

$$2\int \frac{d\varphi \sin. \varphi}{\cos. \varphi} = -2l \cos. \varphi + C = -2l \cos. \frac{n\pi}{2k} + C.$$

Fiat igitur $n = 0$ esseque debebit $-2l1 + C = 0$ ideoque constans $C = 0$, quocirca haec integratio nobis suppeditat sequentem formulam

$$\int \frac{2x^k - x^{k-n} - x^{k+n}}{1 - x^{2k}} \cdot \frac{dx}{xlx} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = -2l \cos. \frac{n\pi}{2k};$$

sequens autem formula $\begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix}$ singulari evolutione non indiget, cum eius valor sit huius semissis.

36. Evolvamus casum, quo $k = 2$ et $n = 1$, et ex parte sinistra habemus

$$-\int \frac{(1-x)^2}{1-x^4} \cdot \frac{dx}{lx} = -\int \frac{1-x}{(1+x)(1+xx)} \cdot \frac{dx}{lx} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix};$$

at vero ex dextra parte $-2l \cos. \frac{\pi}{4} = 2l\sqrt{2} = l2$. Verum fractio $\frac{1-x}{(1+x)(1+xx)}$ resolvitur in has duas $\frac{1}{1+x} - \frac{x}{1+xx}$, unde formula nostra resolvitur in has duas

$$-\int \frac{dx}{(1+x)lx} + \int \frac{xdx}{(1+xx)lx} = l2.$$

Sed ex forma generali

$$\int \frac{x^{\lambda-1}dx}{(1+x^r)lx} = l \operatorname{tang.} \frac{\lambda\pi}{2r}$$

utriusque formulae valor in infinitum excrescit sicque nihil impedit, quominus differentia $= l2$.

37. Quodsi hic in posteriore formula statuamus $xx = z$, ea abibit in hanc $\int \frac{dz}{(1+z)lz}$, quae priori omnino est similis atque sub iisdem terminis integrationis continetur. Hic igitur iterum occurrit paradoxon prorsus simile illi, quod ab Illustr. DE LA GRANGE fuit memoratum; duae scilicet hic habentur formae prorsus pares $\int \frac{dx}{(1+x)lx}$ et $\int \frac{dz}{(1+z)lz}$, quarum utramque a termino 0 ad ∞ integrari oportet; nihilo tamen minus earum differentia non est nulla, sed, uti vidimus, $= l2$. Atque hinc solutio huius paradoxi in eo manifesto est sita, quod utriusque integralis valor in infinitum excrescit.

38. Quodsi binas postremas formulas eodem modo tractare et per dn multiplicatas integrare velimus, a parte sinistra resultat ista formula integralis

$$\int \frac{x^{k+n} - x^{k-n}}{1 + 2x^k \cos. \eta + x^{2k}} \cdot \frac{dx}{xlx} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix};$$

pro dextra autem parte nanciscimur hanc formulam integralem

$$\int \frac{2\pi dn \sin. \frac{n\eta}{k}}{k \sin. \eta \sin. \frac{n\pi}{k}}$$

a termino $n = 0$ extendendam. Verum haec integratio nullo modo succedit; si enim ponamus $\frac{n\pi}{k} = \varphi$, fiet $\frac{n\eta}{k} = \frac{\eta\varphi}{\pi} = \alpha\varphi$ ponendo $\frac{\eta}{\pi} = \alpha$, unde formula integranda erit $\frac{2}{\sin. \eta} \int \frac{d\varphi \sin. \alpha\varphi}{\sin. \varphi}$, cuius valor aliter nisi per signum summatorium exprimi non potest, sicque nulla concinna theoremata hinc derivare licet.

39. Quemadmodum autem hic exponentem n ut variabilem spectando transformationes per integrationem instituimus, ita etiam differentiatio egregias transformationes suppeditabit, quod argumentum unica formula principali illustrasse sufficiet. Consideremus scilicet hanc formulam

$$\int \frac{x^{k+n}}{1 + x^{2k}} \cdot \frac{dx}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\pi}{2k \cos. \frac{n\pi}{2k}},$$

quae sumto exponente n ut solo variabili continuo differentietur, ubi notandum est esse $d.x^{k+n} = x^{k+n} dnlx$. At vero pro formula $\frac{\pi}{2k \cos. \frac{n\pi}{2k}}$ scribamus litteram ν, quae ergo spectanda erit tanquam functio ipsius n, cuius ergo differentialia cuiusque ordinis sunt in nostra potestate. Hinc igitur sequentes reductiones consequemur

$$\int \frac{x^{k+n}}{1+x^{2k}} \cdot \frac{dx}{x} lx = \frac{d\nu}{dn}$$

sive

$$\int \frac{x^{k+n-1} dx lx}{1+x^{2k}} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=\infty \end{bmatrix} = \frac{d\nu}{dn},$$

$$\int \frac{x^{k+n-1} dx (lx)^2}{1+x^{2k}} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=\infty \end{bmatrix} = \frac{dd\nu}{dn^2},$$

$$\int \frac{x^{k+n-1} dx (lx)^3}{1+x^{2k}} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=\infty \end{bmatrix} = \frac{d^3\nu}{dn^3},$$

$$\int \frac{x^{k+n-1} dx (lx)^4}{1+x^{2k}} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=\infty \end{bmatrix} = \frac{d^4\nu}{dn^4},$$

$$\int \frac{x^{k+n-1} dx (lx)^5}{1+x^{2k}} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=\infty \end{bmatrix} = \frac{d^5\nu}{dn^5}$$

etc.

40. Cum igitur hinc totum negotium ad differentialia continua ipsius ν reducatur, ea sequenti modo commodissime reperire licebit. Cum enim sit

$$\nu = \frac{\pi}{2k \cos. \frac{n\pi}{2k}},$$

erit $\nu \cos. \frac{n\pi}{2k} = \frac{\pi}{2k}$ hincque continuo differentiando obtinebimus sequentes formulas

$$\frac{d\nu}{dn} \cos. \frac{n\pi}{2k} - \frac{\pi}{2k}\nu \sin. \frac{n\pi}{2k} = 0,$$

$$\frac{dd\nu}{dn^2} \cos. \frac{n\pi}{2k} - \frac{2\pi}{2k}\frac{d\nu}{dn} \sin. \frac{n\pi}{2k} - \frac{\pi\pi}{4kk}\nu \cos. \frac{n\pi}{2k} = 0,$$

$$\frac{d^3\nu}{dn^3} \cos. \frac{n\pi}{2k} - \frac{3\pi}{2k}\frac{dd\nu}{dn^2} \sin. \frac{n\pi}{2k} - \frac{3\pi\pi}{4kk}\frac{d\nu}{dn} \cos. \frac{n\pi}{2k} + \frac{\pi^3}{8k^3}\nu \sin. \frac{n\pi}{2k} = 0,$$

$$\frac{d^4\nu}{dn^4} \cos. \frac{n\pi}{2k} - \frac{4\pi}{2k}\frac{d^3\nu}{dn^3} \sin. \frac{n\pi}{2k} - \frac{6\pi\pi}{4kk}\frac{dd\nu}{dn^2} \cos. \frac{n\pi}{2k} + \frac{4\pi^3}{8k^3}\frac{d\nu}{dn} \sin. \frac{n\pi}{2k} + \frac{\pi^4}{16k^4}\nu \cos. \frac{n\pi}{2k} = 0$$

etc.,

unde singula differentialia altiora ex inferioribus formari possunt.

41. Quo autem hae operationes magis subleventur, statuamus brevitatis gratia $\frac{\pi}{2k} = \alpha$, ut sit $\nu = \frac{\alpha}{\cos. \alpha n}$, atque singula differentialia ex superioribus aequationibus sequenti modo determinabuntur

$$\frac{d\nu}{dn} = \alpha\nu \text{ tang. } \alpha n,$$

$$\frac{dd\nu}{dn^2} = 2\alpha\frac{d\nu}{dn} \text{ tang. } \alpha n + \alpha\alpha\nu,$$

$$\frac{d^3\nu}{dn^3} = 3\alpha\frac{dd\nu}{dn^2} \text{ tang. } \alpha n + 3\alpha\alpha\frac{d\nu}{dn} - \alpha^3\nu \text{ tang. } \alpha n,$$

$$\frac{d^4\nu}{dn^4} = 4\alpha\frac{d^3\nu}{dn^3} \text{ tang. } \alpha n + 6\alpha\alpha\frac{dd\nu}{dn^2} - 4\alpha^3\frac{d\nu}{dn} \text{ tang. } \alpha n - \alpha^4\nu,$$

$$\frac{d^5\nu}{dn^5} = 5\alpha\frac{d^4\nu}{dn^4} \text{ tang. } \alpha n + 10\alpha\alpha\frac{d^3\nu}{dn^3} - 10\alpha^3\frac{dd\nu}{dn^2} \text{ tang. } \alpha n - 5\alpha^4\frac{d\nu}{dn} + \alpha^5\nu \text{ tang. } \alpha n$$

etc.

Quodsi brevitatis gratia insuper statuamus tang. $\alpha n = t$ et praecedentes valores in sequentibus substituamus, reperiemus

$$\frac{d\nu}{dn} = \alpha\nu t,$$

$$\frac{dd\nu}{dn^2} = \alpha\alpha\nu(2tt+1),$$

$$\frac{d^3\nu}{dn^3} = \alpha^3\nu(6t^3+5t),$$

$$\frac{d^4\nu}{dn^4} = \alpha^4\nu(24t^4+28tt+5),$$

$$\frac{d^5\nu}{dn^5} = \alpha^5\nu(120t^5+180t^3+61t),$$

$$\frac{d^6\nu}{dn^6} = \alpha^6\nu(720t^6+1320t^4+662tt+61)$$

etc.

42. Ex consideratione harum expressionum facilis erui potest operatio, cuius ope ex qualibet earum expressionum sequens colligi potest. Sit enim pro differentiali ordinis indefiniti

$$\frac{d^\lambda\nu}{dn^\lambda} = \alpha^\lambda\nu P,$$

at pro ordine sequente

$$\frac{d^{\lambda+1}\nu}{dn^{\lambda+1}} = \alpha^{\lambda+1}\nu Q,$$

et quoniam vidimus valorem ipsius P talem habere formam

$$P = At^{\lambda} + Bt^{\lambda-2} + Ct^{\lambda-4} + Dt^{\lambda-6} + \text{etc.},$$

tum valor ipsius Q ex sequentibus binis seriebus erit compositus

$$\begin{aligned} Q = (\lambda+1)At^{\lambda+1} + (\lambda-1)Bt^{\lambda-1} + (\lambda-3)Ct^{\lambda-3} + (\lambda-5)Dt^{\lambda-5} + \text{etc.} \\ + \lambda At^{\lambda-1} + (\lambda-2)Bt^{\lambda-3} + (\lambda-4)Ct^{\lambda-5} + \text{etc.}, \end{aligned}$$

unde patet hanc determinationem ita repraesentari posse, ut sit

$$Q = \frac{td.Pt}{dt} + \frac{dP}{dt}.$$

43. Haec vero formula, qua ex cognito valore P sequens Q derivatur, etiam ex ipsa natura rei sequenti modo ostendi potest. Cum per hypothesin sit

$$\frac{d^{\lambda}\nu}{dn^{\lambda}} = \alpha^{\lambda}\nu P,$$

erit differentiando

$$\frac{d^{\lambda+1}\nu}{dn^{\lambda}} = \alpha^{\lambda} P d\nu + \alpha^{\lambda}\nu dP;$$

initio autem vidimus esse $\frac{d\nu}{dn} = \alpha\nu t$ sive $d\nu = \alpha\nu t dn$, quo valore substituto fit

$$\frac{d^{\lambda+1}\nu}{dn^{\lambda+1}} = \alpha^{\lambda+1}\nu Pt + \alpha^{\lambda}\nu\frac{dP}{dn};$$

tum vero assumsimus $t = \text{tang.}\,\alpha n$, unde differentiando fit $\alpha dn = \frac{dt}{1+tt}$, quo valore in postremo termino substituto obtinebitur

$$\frac{d^{\lambda+1}\nu}{dn^{\lambda+1}} = \alpha^{\lambda+1}\nu Pt + \alpha^{\lambda+1}\nu\frac{dP(1+tt)}{dt} = \alpha^{\lambda+1}\nu\left(Pt + \frac{dP(1+tt)}{dt}\right),$$

quae forma manifesto reducitur ad hanc

$$\frac{d^{\lambda+1}\nu}{dn^{\lambda+1}} = \alpha^{\lambda+1}\nu\frac{td.Pt + dP}{dt},$$

ita ut sit

$$Q = \frac{td.Pt + dP}{dt} = Pt + \frac{dP(1+tt)}{dt};$$

unde intelligitur, si sumatur $tt + 1 = 0$, quo facto in nostris formulis signa terminorum alternabuntur, et omissa littera t fieri $Q = P$; unde patet hoc casu omnes formulas superiores eundem valorem esse adepturas, id quod etiam ex formulis supra exhibitis manifestum est, ex quibus erit $2 - 1 = 1$, $6 - 5 = 1$, $24 - 28 + 5 = 1$, $120 - 180 + 61 = 1$, $720 - 1320 + 662 - 61 = 1$ etc., unde insigne criterium obtinetur, utrum formulae istae recte sint per calculum definitae.

INVESTIGATIO FORMULAE INTEGRALIS $\int \frac{x^{m-1}dx}{(1+x^k)^n}$ CASU QUO POST INTEGRATIONEM STATUITUR $x=\infty$

Commentatio 588 indicis ENESTROEMIANI
Opuscula analytica 2, 1785, p. 42—54

1. Iam satis notum est huius formulae integrale [casu, quo $n=1$] partim logarithmos, partim arcus circulares complecti et partes logarithmicas hanc progressionem constituere

$$\begin{aligned}
&-\frac{2}{k}\cos.\frac{m\pi}{k}\, l\sqrt{\left(1-2x\cos.\frac{\pi}{k}+xx\right)}\\
&-\frac{2}{k}\cos.\frac{3m\pi}{k}\, l\sqrt{\left(1-2x\cos.\frac{3\pi}{k}+xx\right)}\\
&-\frac{2}{k}\cos.\frac{5m\pi}{k}\, l\sqrt{\left(1-2x\cos.\frac{5\pi}{k}+xx\right)}\\
&-\frac{2}{k}\cos.\frac{7m\pi}{k}\, l\sqrt{\left(1-2x\cos.\frac{7\pi}{k}+xx\right)}\\
&\quad\vdots\\
&-\frac{2}{k}\cos.\frac{im\pi}{k}\, l\sqrt{\left(1-2x\cos.\frac{i\pi}{k}+xx\right)},
\end{aligned}$$

ubi i denotat numerum imparem non maiorem quam k. Hinc si k fuerit numerus par, erit $i=k-1$; ac si k fuerit numerus impar, hanc progressionem continuari oportet usque ad $i=k$, eius vero coefficiens duplo minor capi debet seu loco $-\frac{2}{k}$ tantum scribi debet $-\frac{1}{k}$, cuius irregularitatis ratio in *Calculo Integrali*[1]) est exposita.

1) Vide notam p. 122. A. G.

2. Cum hae partes sponte iam evanescant posito $x=0$, statuamus statim $x=\infty$, et cum in genere sit

$$\sqrt{(1-2x\cos.\omega+xx)}=x-\cos.\omega,$$

erit

$$l\sqrt{(1-2x\cos.\omega+xx)}=l(x-\cos.\omega)=lx-\frac{\cos.\omega}{x}=lx$$

ob $\frac{\cos.\omega}{x}=0$; omnes ergo illi logarithmi reducuntur ad eandem formam lx, quae multiplicanda est per hanc seriem

$$-\frac{2}{k}\cos.\frac{m\pi}{k}-\frac{2}{k}\cos.\frac{3m\pi}{k}-\frac{2}{k}\cos.\frac{5m\pi}{k}-\cdots-\frac{2}{k}\cos.\frac{im\pi}{k},$$

ubi, ut diximus, i denotat maximum numerum imparem ipso k non maiorem, hac tamen restrictione, ut, si k fuerit impar ideoque $i=k$, ultimum membrum ad dimidium reduci debeat. Quamobrem si huius progressionis summam investigare velimus, duo casus erunt constituendi, alter, quo k est numerus par et $i=k-1$, alter vero, quo k est impar et $i=k$.

EVOLUTIO CASUS PRIORIS QUO k EST NUMERUS PAR ET $i=k-1$

3. Hoc ergo casu posito $x=\infty$ formula $-\frac{2}{k}lx$ multiplicatur per hanc cosinuum seriem

$$\cos.\frac{m\pi}{k}+\cos.\frac{3m\pi}{k}+\cos.\frac{5m\pi}{k}+\cos.\frac{7m\pi}{k}+\cdots+\cos.\frac{(k-1)m\pi}{k},$$

cuius summam statuamus $=S$. Ducamus hanc seriem in $\sin.\frac{m\pi}{k}$, et cum in genere sit

$$\sin.\frac{m\pi}{k}\cos.\frac{im\pi}{k}=\frac{1}{2}\sin.\frac{(i+1)m\pi}{k}-\frac{1}{2}\sin.\frac{(i-1)m\pi}{k},$$

facta hac reductione habebimus

$$\begin{aligned}
&S\sin.\frac{m\pi}{k}\\
&=\frac{1}{2}\sin.\frac{2m\pi}{k}+\frac{1}{2}\sin.\frac{4m\pi}{k}+\frac{1}{2}\sin.\frac{6m\pi}{k}+\cdots+\frac{1}{2}\sin.\frac{(k-2)m\pi}{k}+\frac{1}{2}\sin.\frac{km\pi}{k}\\
&-\frac{1}{2}\sin.\frac{2m\pi}{k}-\frac{1}{2}\sin.\frac{4m\pi}{k}-\frac{1}{2}\sin.\frac{6m\pi}{k}-\cdots-\frac{1}{2}\sin.\frac{(k-2)m\pi}{k},
\end{aligned}$$

ubi omnes termini praeter ultimum manifesto se destruunt, ita ut sit

$$S \sin.\frac{m\pi}{k} = \frac{1}{2}\sin. m\pi.$$

Iam vero, quia nostri coefficientes m et k supponuntur integri, utique erit $\sin. m\pi = 0$ ideoque etiam $S = 0$, nisi forte etiam fuerit $\sin.\frac{m\pi}{k} = 0$, qui autem casus locum habere nequit, quoniam in integratione formulae propositae $\frac{x^{m-1}dx}{(1+x^k)^n}$ semper assumi solet esse $m < k$. Hoc igitur modo evictum est casu, quo post integrationem statuitur $x = \infty$, omnes partes logarithmicas integralis se destruere.

EVOLUTIO CASUS ALTERIUS QUO EST k NUMERUS IMPAR ET $i = k$

4. Hoc ergo casu sumto $x = \infty$ formula lx multiplicatur per hanc seriem

$$-\frac{2}{k}\cos.\frac{m\pi}{k} - \frac{2}{k}\cos.\frac{3m\pi}{k} - \frac{2}{k}\cos.\frac{5m\pi}{k} - \cdots - \frac{1}{k}\cos.\frac{km\pi}{k},$$

ubi terminus penultimus est $-\frac{2}{k}\cos.\frac{(k-2)m\pi}{k}$, pro ultimo vero termino erit $\cos. m\pi = \pm 1$ signo superiore valente, si n sit numerus par, inferiore, si impar; quare remoto termino ultimo pro reliquis ponamus

$$\cos.\frac{m\pi}{k} + \cos.\frac{3m\pi}{k} + \cos.\frac{5m\pi}{k} + \cdots + \cos.\frac{(k-2)m\pi}{k} = S,$$

ita ut multiplicator ipsius logarithmi x sit

$$-\frac{2S}{k} - \frac{1}{k}\cos. m\pi.$$

Hinc procedendo ut ante fiet

$$S \sin.\frac{m\pi}{k}$$

$$= \frac{1}{2}\sin.\frac{2m\pi}{k} + \frac{1}{2}\sin.\frac{4m\pi}{k} + \frac{1}{2}\sin.\frac{6m\pi}{k} + \cdots + \frac{1}{2}\sin.\frac{(k-3)m\pi}{k} + \frac{1}{2}\sin.\frac{(k-1)m\pi}{k}$$

$$-\frac{1}{2}\sin.\frac{2m\pi}{k} - \frac{1}{2}\sin.\frac{4m\pi}{k} - \frac{1}{2}\sin.\frac{6m\pi}{k} - \cdots - \frac{1}{2}\sin.\frac{(k-3)m\pi}{k},$$

ubi iterum omnes termini praeter ultimum se mutuo tollunt, ita ut hinc prodeat

$$S \sin.\frac{m\pi}{k} = \frac{1}{2}\sin.\frac{(k-1)m\pi}{k} = \frac{1}{2}\sin.\left(m\pi - \frac{m\pi}{k}\right);$$

at vero est

$$\sin.\left(m\pi - \frac{m\pi}{k}\right) = \sin. m\pi \cos.\frac{m\pi}{k} - \cos. m\pi \sin.\frac{m\pi}{k},$$

ubi notetur esse $\sin. m\pi = 0$ ob m numerum integrum; habebimus ergo

$$S \sin.\frac{m\pi}{k} = -\frac{1}{2}\cos. m\pi \sin.\frac{m\pi}{k} \quad \text{sive} \quad S = -\frac{1}{2}\cos. m\pi,$$

consequenter multiplicator ipsius lx erit

$$= \frac{1}{k}\cos. m\pi - \frac{1}{k}\cos. m\pi = 0$$

sicque manifestum est, sive k sit numerus par sive impar, omnia membra logarithmica in nostro integrali se mutuo destruere, siquidem post integrationem statuamus $x = \infty$, quemadmodum hic semper supponimus.

5. Consideremus nunc etiam partes a circulo pendentes, ex quibus integrale nostrae formulae componitur. Hae autem partes sequentem progressionem constituere sunt compertae:

$$\frac{2}{k}\sin.\frac{m\pi}{k}\,\text{A tang.}\frac{x\sin.\frac{\pi}{k}}{1 - x\cos.\frac{\pi}{k}} + \frac{2}{k}\sin.\frac{3m\pi}{k}\,\text{A tang.}\frac{x\sin.\frac{3\pi}{k}}{1 - x\cos.\frac{3\pi}{k}}$$

$$+ \frac{2}{k}\sin.\frac{5m\pi}{k}\,\text{A tang.}\frac{x\sin.\frac{5\pi}{k}}{1 - x\cos.\frac{5\pi}{k}} + \frac{2}{k}\sin.\frac{7m\pi}{k}\,\text{A tang.}\frac{x\sin.\frac{7\pi}{k}}{1 - x\cos.\frac{7\pi}{k}}$$

$$+ \cdots + \frac{2}{k}\sin.\frac{im\pi}{k}\,\text{A tang.}\frac{x\sin.\frac{i\pi}{k}}{1 - x\cos.\frac{i\pi}{k}},$$

ubi in ultimo membro est vel $i = k - 1$ vel $i = k$; prius scilicet valet, si i est numerus par, posterius, si impar.

6. Cum etiam omnia haec membra evanescant posito $x = 0$, faciamus pro instituto nostro $x = \infty$. In genere igitur fiet

$$\text{A tang.}\frac{x\sin.\frac{i\pi}{k}}{1 - x\cos.\frac{i\pi}{k}} = \text{A tang.}\left(-\text{tang.}\frac{i\pi}{k}\right).$$

Est vero

$$-\text{tang.}\,\frac{i\pi}{k} = +\,\text{tang.}\,\frac{(k-i)\pi}{k},$$

ex quo hic arcus fit $= \frac{(k-i)\pi}{k}$. Hinc ergo loco i scribendo successive numeros 1, 3, 5, 7 etc. istae partes nostri integralis quaesiti erunt

$$\frac{2(k-1)\pi}{kk}\sin.\frac{m\pi}{k} + \frac{2(k-3)\pi}{kk}\sin.\frac{3m\pi}{k} + \frac{2(k-5)\pi}{kk}\sin.\frac{5m\pi}{k} + \frac{2(k-7)\pi}{kk}\sin.\frac{7m\pi}{k}$$
$$+ \frac{2(k-9)\pi}{kk}\sin.\frac{9m\pi}{k} + \cdots + \frac{2(k-i)\pi}{kk}\sin.\frac{im\pi}{k},$$

ubi casu, quo k est numerus par, progredi oportet usque ad $i = k-1$, ac si k sit numerus impar, usque ad $i = k$.

7. Statuamus brevitatis gratia

$$(k-1)\sin.\frac{m\pi}{k} + (k-3)\sin.\frac{3m\pi}{k} + (k-5)\sin.\frac{5m\pi}{k} + \cdots + (k-i)\sin.\frac{im\pi}{k} = S,$$

ita ut integrale quaesitum sit $\frac{2\pi S}{kk}$, quandoquidem partes logarithmicae se mutuo destruxerunt. Multiplicemus nunc utrinque per $2\sin.\frac{m\pi}{k}$, et cum in genere sit

$$2\sin.\frac{m\pi}{k}\sin.\frac{im\pi}{k} = \cos.\frac{(i-1)m\pi}{k} - \cos.\frac{(i+1)m\pi}{k},$$

facta substitutione erit

$$2S\sin.\frac{m\pi}{k} = (k-1)\cos.\frac{0m\pi}{k}$$
$$+(k-3)\cos.\frac{2m\pi}{k} + (k-5)\cos.\frac{4m\pi}{k} + \cdots + \quad (k-i)\cos.\frac{(i-1)m\pi}{k}$$
$$-(k-1)\cos.\frac{2m\pi}{k} - (k-3)\cos.\frac{4m\pi}{k} - \cdots - (k-i+2)\cos.\frac{(i-1)m\pi}{k} - (k-i)\cos.\frac{(i+1)m\pi}{k},$$

quae series manifesto contrahitur in sequentem

$$2S\sin.\frac{m\pi}{k} = k-1-2\cos.\frac{2m\pi}{k} - 2\cos.\frac{4m\pi}{k} - 2\cos.\frac{6m\pi}{k} - \cdots - 2\cos.\frac{(i-1)m\pi}{k}$$
$$-(k-i)\cos.\frac{(i+1)m\pi}{k},$$

ubi primo et ultimo membro sublatis regularem termini intermedii constituunt seriem, pro cuius valore investigando ponamus

$$T = \cos.\frac{2m\pi}{k} + \cos.\frac{4m\pi}{k} + \cos.\frac{6m\pi}{k} + \cdots + \cos.\frac{(i-1)m\pi}{k},$$

ita ut sit

$$2S\sin.\frac{m\pi}{k} = k-1-2T-(k-i)\cos.\frac{(i+1)m\pi}{k}.$$

Hic autem iterum convenit duos casus perpendere, prout k fuerit par vel impar.

EVOLUTIO CASUS PRIORIS QUO k EST NUMERUS PAR ET $i = k-1$

8. Hoc ergo casu habebimus

$$T = \cos.\frac{2m\pi}{k} + \cos.\frac{4m\pi}{k} + \cos.\frac{6m\pi}{k} + \cdots + \cos.\frac{(k-2)m\pi}{k}.$$

Multiplicemus denuo per $2\sin.\frac{m\pi}{k}$ et per reductiones supra indicatas habebimus

$$2T\sin.\frac{m\pi}{k} = \sin.\frac{3m\pi}{k} + \sin.\frac{5m\pi}{k} + \cdots + \sin.\frac{(k-3)m\pi}{k} + \sin.\frac{(k-1)m\pi}{k}$$
$$-\sin.\frac{m\pi}{k} - \sin.\frac{3m\pi}{k} - \sin.\frac{5m\pi}{k} - \cdots - \sin.\frac{(k-3)m\pi}{k};$$

deletis igitur terminis se mutuo tollentibus erit

$$2T\sin.\frac{m\pi}{k} = -\sin.\frac{m\pi}{k} + \sin.\frac{(k-1)m\pi}{k}.$$

Est vero

$$\sin.\frac{(k-1)m\pi}{k} = \sin.\left(m\pi - \frac{m\pi}{k}\right) = \sin.m\pi\cos.\frac{m\pi}{k} - \cos.m\pi\sin.\frac{m\pi}{k},$$

ubi $\sin.m\pi = 0$, quamobrem fiet

$$2T = -1 - \cos.m\pi.$$

9. Invento valore pro T colligitur fore

$$2S\sin.\frac{m\pi}{k} = k \quad \text{ideoque} \quad S = \frac{k}{2\sin.\frac{m\pi}{k}}.$$

Denique vero ipse valor formulae nostrae integralis, quem quaerimus, erit $\frac{2\pi S}{kk}$ et nunc manifestum est integrale nostrae formulae casu, quo S est numerus par, fore $\frac{\pi}{k\sin.\frac{m\pi}{k}}$, siquidem post integrationem statuatur $x=\infty$.

EVOLUTIO ALTERIUS CASUS QUO k EST NUMERUS IMPAR ET $i=k$

10. Hoc ergo casu est

$$T=\cos.\frac{2m\pi}{k}+\cos.\frac{4m\pi}{k}+\cos.\frac{6m\pi}{k}+\cdots+\cos.\frac{(k-1)m\pi}{k},$$

quae series multiplicata per $2\sin.\frac{m\pi}{k}$ producet ut ante

$$2T\sin.\frac{m\pi}{k}=\sin.\frac{3m\pi}{k}+\sin.\frac{5m\pi}{k}+\cdots+\sin.\frac{(k-2)m\pi}{k}+\sin.\frac{km\pi}{k}$$
$$-\sin.\frac{m\pi}{k}-\sin.\frac{3m\pi}{k}-\sin.\frac{5m\pi}{k}-\cdots-\sin.\frac{(k-2)m\pi}{k},$$

unde deletis terminis se mutuo tollentibus reperietur

$$2T\sin.\frac{m\pi}{k}=-\sin.\frac{m\pi}{k}+\sin.m\pi$$

ideoque

$$2T=-1+\frac{\sin.m\pi}{\sin.\frac{m\pi}{k}}=-1$$

ob $\sin.m\pi=0$, hincque porro fiet

$$2S\sin.\frac{m\pi}{k}=k;$$

quare cum valor integralis quaesitus sit $\frac{2\pi S}{kk}$, erit etiam hoc casu integrale nostrum $=\frac{\pi}{k\sin.\frac{m\pi}{k}}$, prorsus uti praecedente casu. Hinc ergo deducimus sequens

THEOREMA

11. *Si haec formula differentialis*

$$\frac{x^{m-1}dx}{1+x^k}$$

ita integretur, ut posito $x=0$ integrale evanescat, tum vero statuatur $x=\infty$,

valor inde resultans semper erit

$$\frac{\pi}{k\sin.\frac{m\pi}{k}},$$

sive k sit numerus par sive impar.

Huius theorematis demonstratio ex praecedentibus est manifesta.

12. In evolutione huius formulae assumsimus esse $m<k$, quia alioquin membra logarithmica se non destruissent; at vero ne hac quidem limitatione nunc amplius est opus. Casu enim, quo foret $m=k$, integrale formulae $\frac{x^{m-1}dx}{1+x^k}$ esset $\frac{1}{k}l(1+x^k)$, quod facto $x=\infty$ fieret etiam ∞; verum hoc idem indicat nostrum integrale esse $\frac{\pi}{k\sin.\pi}=\infty$. Dummodo ergo m non fuerit maius quam k, nostra formula veritati semper est consentanea.

13. Quin etiam ne quidem necesse est, ut exponentes m et k sint numeri integri, dummodo non fuerit $m>k$; si enim fuerit $m=\frac{\mu}{\lambda}$ et $k=\frac{\varkappa}{\lambda}$, erit valor per nostram formulam $\frac{\lambda\pi}{\varkappa\sin.\frac{\mu\pi}{\varkappa}}$, cuius veritas ita ostenditur. Quia hoc casu formula integranda est

$$\int\frac{x^{\frac{\mu}{\lambda}}}{1+x^{\frac{\varkappa}{\lambda}}}\cdot\frac{dx}{x},$$

statuatur $x=y^\lambda$; erit $\frac{dx}{x}=\frac{\lambda dy}{y}$ et formula fiet

$$\int\frac{y^\mu}{1+y^\varkappa}\cdot\frac{\lambda dy}{y}=\lambda\int\frac{y^{\mu-1}dy}{1+y^\varkappa},$$

cuius valor utique erit $\frac{\lambda\pi}{\varkappa\sin.\frac{\mu\pi}{\varkappa}}$.

ALIA DEMONSTRATIO THEOREMATIS

14. Denotet P valorem integralis $\int\frac{x^m}{1+x^k}\cdot\frac{dx}{x}$ a termino $x=0$ usque ad $x=1$, at Q valorem eiusdem integralis a termino $x=1$ usque ad $x=\infty$, ita ut $P+Q$ praebeat eum ipsum valorem, qui in theoremate continetur. Nunc pro valore Q inveniendo statuatur $x=\frac{1}{y}$, unde fit $\frac{dx}{x}=-\frac{dy}{y}$, fietque

$$Q = \int \frac{y^{-m}}{1+y^{-k}} \cdot \frac{-dy}{y} = -\int \frac{y^{k-m}}{1+y^k} \cdot \frac{dy}{y}$$

a termino $y = 1$ usque ad $y = 0$. Hinc igitur commutatis terminis erit

$$Q = +\int \frac{y^{k-m}}{1+y^k} \cdot \frac{dy}{y}$$

a termino $y = 0$ usque ad $y = 1$. Iam quia hoc integrali expedito littera y ex calculo egreditur, loco y scribere licebit x, ita ut sit

$$Q = \int \frac{x^{k-m}}{1+x^k} \cdot \frac{dx}{x},$$

quo facto habebimus

$$P + Q = \int \frac{x^m + x^{k-m}}{1+x^k} \cdot \frac{dx}{x}$$

a termino $x = 0$ usque ad terminum $x = 1$. Verum non ita pridem demonstravi[1]) valorem huius formulae integralis intra terminos $x = 0$ et $x = 1$ contentum esse $= \frac{\pi}{k \sin. \frac{m\pi}{k}}$. Hinc igitur nascitur sequens theorema non minus notatu dignum.

THEOREMA

15. *Valor huius formulae integralis*

$$\int \frac{x^m + x^{k-m}}{1+x^k} \cdot \frac{dx}{x}$$

intra terminos $x = 0$ et $x = 1$ contentus aequalis est valori istius integralis

$$\int \frac{x^m}{1+x^k} \cdot \frac{dx}{x}$$

intra terminos $x = 0$ et $x = \infty$ contento.

16. His expensis formulam integralem in titulo propositam aggrediamur, et quo eam ad formam hactenus tractatam reducamus, in subsidium vocemus

1) Vide praeter § 27 Commentationis praecedentis etiam Commentationem 463 (indicis ENESTROEMIANI): *De valore formulae integralis* $\int \frac{z^{k-\omega} \pm z^{k+\omega}}{1 \pm z^{2k}} \cdot \frac{dz}{z} (lz)^{\mu}$ *casu, quo post integrationem ponitur* $z = 1$, Novi comment. acad. sc. Petrop. 19 (1774), 1775, p. 30; *LEONHARDI EULERI Opera omnia*, series I, vol. 17, p. 384, imprimis p. 388. A. G.

sequentem reductionem

$$\int \frac{x^{m-1}dx}{(1+x^k)^{\lambda+1}} = \frac{Ax^m}{(1+x^k)^{\lambda}} + B\int \frac{x^{m-1}dx}{(1+x^k)^{\lambda}},$$

unde facta differentiatione prodit sequens aequatio

$$\frac{x^{m-1}dx}{(1+x^k)^{\lambda+1}} = \frac{mAx^{m-1}dx}{(1+x^k)^{\lambda}} - \frac{\lambda kAx^{m+k-1}dx}{(1+x^k)^{\lambda+1}} + \frac{Bx^{m-1}dx}{(1+x^k)^{\lambda}},$$

quae aequatio per $x^{m-1}dx$ divisa ac per $(1+x^k)^{\lambda}$ multiplicata terminum negativum a dextra ad sinistram transponendo erit

$$\frac{1+\lambda kAx^k}{1+x^k} = mA + B,$$

quae aequatio manifesto subsistere nequit, nisi sit $\lambda kA = 1$ sive $A = \frac{1}{\lambda k}$, unde erit $1 = mA + B = \frac{m}{\lambda k} + B$, sicque erit $B = 1 - \frac{m}{\lambda k}$.

17. Inventis his valoribus pro litteris A et B primum assumimus integralia ita capi, ut evanescant posito $x = 0$; tum vero posito $x = \infty$, quia exponens n minor supponitur quam k, membrum absolutum littera A affectum sponte evanescit, ita ut hoc casu $x = \infty$ fiat

$$\int \frac{x^{m-1}dx}{(1+x^k)^{\lambda+1}} = \left(1 - \frac{m}{\lambda k}\right) \int \frac{x^{m-1}dx}{(1+x^k)^{\lambda}}.$$

Quodsi iam primo capiamus $\lambda = 1$, quia ante invenimus pro eodem casu $x = \infty$ esse

$$\int \frac{x^{m-1}dx}{1+x^k} = \frac{\pi}{k \sin. \frac{m\pi}{k}},$$

habebimus valorem istius integralis

$$\int \frac{x^{m-1}dx}{(1+x^k)^2} = \left(1 - \frac{m}{k}\right) \frac{\pi}{k \sin. \frac{m\pi}{k}},$$

siquidem integrale etiam a termino $x = 0$ usque ad terminum $x = \infty$ extendatur.

18. Quodsi iam simili modo ponamus $\lambda = 2$, reperietur pro iisdem terminis integrationis

$$\int \frac{x^{m-1}dx}{(1+x^k)^3} = \left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\frac{\pi}{k \sin. \frac{m\pi}{k}};$$

eodem modo si litterae λ continuo maiores valores tribuantur, reperientur sequentes integralium formae omni attentione dignae

$$\int \frac{x^{m-1}dx}{(1+x^k)^4} = \left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\left(1-\frac{m}{3k}\right)\frac{\pi}{k \sin. \frac{m\pi}{k}},$$

$$\int \frac{x^{m-1}dx}{(1+x^k)^5} = \left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\left(1-\frac{m}{3k}\right)\left(1-\frac{m}{4k}\right)\frac{\pi}{k \sin. \frac{m\pi}{k}},$$

$$\int \frac{x^{m-1}dx}{(1+x^k)^6} = \left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\left(1-\frac{m}{3k}\right)\left(1-\frac{m}{4k}\right)\left(1-\frac{m}{5k}\right)\frac{\pi}{k \sin. \frac{m\pi}{k}}$$

etc.

19. Quare si littera n denotet numerum quemcunque integrum pro formula in titulo expressa, si eius integrale a termino $x = 0$ usque ad $x = \infty$ extendatur, eius valor sequenti modo se habebit:

$$\left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\left(1-\frac{m}{3k}\right)\left(1-\frac{m}{4k}\right)\cdots\left(1-\frac{m}{(n-1)k}\right)\frac{\pi}{k \sin. \frac{m\pi}{k}},$$

qui ergo conveniet huic formulae integrali

$$\int \frac{x^{m-1}dx}{(1+x^k)^n}.$$

20. Hic quidem necessario pro n alii numeri praeter integros accipi non licet; at vero per methodum interpolationum, quae fusius iam passim est explicata[1]), hanc integrationem etiam ad casus, quibus exponens n est numerus fractus, extendere licet. Quodsi enim sequentes formulae integrales a ter-

1) Vide Commentationem 254 (indicis ENESTROEMIANI): *De expressione integralium per factores*, Novi comment. acad. sc. Petrop. 6 (1756/7), 1761, p. 115; *LEONHARDI EULERI Opera omnia*, series I, vol. 17, p. 233. A. G.

mino $y = 0$ usque ad $y = 1$ extendantur, in genere valor nostrae formulae propositae ita repraesentari poterit

$$\int \frac{x^{m-1}dx}{(1+x^k)^n} = \frac{\pi}{k \sin. \frac{m\pi}{k}} \cdot \frac{\int y^{nk-m-1}dy(1-y^k)^{\frac{m}{k}-1}}{\int y^{k-m-1}dy(1-y^k)^{\frac{m}{k}-1}}.$$

Unde, si fuerit $m = 1$ et $k = 2$, sequitur fore

$$\int \frac{dx}{(1+xx)^n} = \frac{\pi}{2}\int \frac{y^{2(n-1)}dy}{\sqrt{(1-yy)}} : \int \frac{dy}{\sqrt{(1-yy)}} = \int \frac{y^{2(n-1)}dy}{\sqrt{(1-yy)}}.$$

Ita, si $n = \frac{3}{2}$, erit

$$\int \frac{dx}{(1+xx)^{\frac{3}{2}}} = \int \frac{ydy}{\sqrt{(1-yy)}},$$

cuius veritas sponte elucet, quia integrale prius generatim est $\frac{x}{\sqrt{(1+xx)}}$, posterius vero $= 1 - \sqrt{(1-yy)}$, quae facto $x = \infty$ et $y = 1$ utique fiunt aequalia. Caeterum pro hac integratione generali notasse iuvabit exponentem unitate minorem accipi non posse, quia alioquin valores amborum integralium in infinitum excrescerent.

INVESTIGATIO VALORIS INTEGRALIS

$$\int \frac{x^{m-1}dx}{1-2x^k \cos.\theta + x^{2k}}$$

A TERMINO $x=0$ USQUE AD $x=\infty$ EXTENSI

Commentatio 589 indicis Enestroemiani
Opuscula analytica 2, 1785, p. 55—75

1. Quaeramus primo integrale formulae propositae indefinitum atque adeo omnes operationes ex primis Analyseos principiis repetamus. Ac primo quidem, quoniam denominator in factores reales simplices resolvi nequit, sit in genere eius factor duplicatus quicunque $1-2x\cos.\omega + xx$; evidens enim est denominatorem fore productum ex k huiusmodi factoribus duplicatis. Cum igitur posito hoc factore $=0$ fiat $x=\cos.\omega \pm \sqrt{-1}\cdot\sin.\omega$, etiam ipse denominator duplici modo evanescere debebit, sive si ponatur

$$x = \cos.\omega + \sqrt{-1}\cdot\sin.\omega \quad \text{sive} \quad x=\cos.\omega - \sqrt{-1}\cdot\sin.\omega.$$

Constat autem omnes potestates harum formularum ita commode exprimi posse, ut sit

$$(\cos.\omega \pm \sqrt{-1}\cdot\sin.\omega)^{\lambda} = \cos.\lambda\omega \pm \sqrt{-1}\cdot\sin.\lambda\omega;$$

hinc igitur erit

$$x^k = \cos.k\omega \pm \sqrt{-1}\cdot\sin.k\omega \quad \text{et} \quad x^{2k} = \cos.2k\omega \pm \sqrt{-1}\cdot\sin.2k\omega.$$

Substituamus ergo hos valores et denominator noster evadet

$$1-2\cos.\theta\cos.k\omega + \cos.2k\omega \mp 2\sqrt{-1}\cdot\cos.\theta\sin.k\omega \pm \sqrt{-1}\cdot\sin.2k\omega.$$

2. Perspicuum igitur est huius aequationis tam terminos reales quam imaginarios seorsim se mutuo tollere debere, unde nascuntur hae duae aequationes

$$\text{I.} \quad 1-2\cos.\theta\cos.k\omega + \cos.2k\omega = 0,$$

$$\text{II.} \quad -2\cos.\theta\sin.k\omega + \sin.2k\omega = 0.$$

Cum igitur sit

$$\sin.2k\omega = 2\sin.k\omega\cos.k\omega,$$

posterior aequatio induet hanc formam

$$-2\cos.\theta\sin.k\omega + 2\sin.k\omega\cos.k\omega = 0,$$

quae per $2\sin.k\omega$ divisa dat

$$\cos.k\omega = \cos.\theta$$

ideoque

$$\cos.2k\omega = \cos.2\theta = \cos.\theta^2 - \sin.\theta^2 = 2\cos.\theta^2 - 1,$$

qui valores in aequatione priore substituti praebent aequationem identicam, ita ut utrique aequationi satisfiat sumendo $\cos.k\omega = \cos.\theta$.

3. Pro ω igitur eiusmodi angulum assumi oportet, ut fiat $\cos.k\omega = \cos.\theta$, unde quidem statim deducitur $k\omega = \theta$ ideoque $\omega = \frac{\theta}{k}$. Verum quia infiniti dantur anguli eundem cosinum habentes, qui praeter ipsum angulum θ sunt $2\pi \pm \theta$, $4\pi \pm \theta$, $6\pi \pm \theta$ etc. atque adeo in genere $2i\pi \pm \theta$ denotante i omnes numeros integros, quaesito nostro satisfiet faciendo $k\omega = 2i\pi \pm \theta$, unde colligitur angulus $\omega = \frac{2i\pi \pm \theta}{k}$, sicque pro ω nancisceremur innumerabiles angulos satisfacientes, quorum autem sufficiet tot assumsisse, quot exponens k continet unitates; successive igitur angulo ω sequentes tribuamus valores

$$\frac{\theta}{k},\ \frac{2\pi+\theta}{k},\ \frac{4\pi+\theta}{k},\ \frac{6\pi+\theta}{k},\ \frac{8\pi+\theta}{k},\ \ldots\ \frac{2(k-1)\pi+\theta}{k}.$$

Quodsi ergo angulo ω successive singulos istos valores, quorum numerus est $=k$, tribuamus, formula $1-2x\cos.\omega+xx$ omnes suppeditabit factores duplicatos nostri denominatoris $1-2x^k\cos.\theta + x^{2k}$, quorum numerus erit $=k$.

4. Inventis iam omnibus factoribus duplicatis nostri denominatoris fractio $\frac{x^{m-1}}{1-2x^k\cos.\theta+x^{2k}}$ resolvi debet in tot fractiones partiales, quarum deno-

minatores sint ipsi isti factores duplicati, quorum numerus est k, ita ut in genere talis fractio partialis habitura sit talem formam

$$\frac{A+Bx}{1-2x\cos.\omega+xx},$$

quam insuper resolvamus in binas simplices etsi imaginarias, et cum sit

$$xx-2x\cos.\omega+1=(x-\cos.\omega+\sqrt{-1}\cdot\sin.\omega)(x-\cos.\omega-\sqrt{-1}\cdot\sin.\omega),$$

statuantur ambae istae fractiones partiales

$$\frac{f}{x-\cos.\omega-\sqrt{-1}\cdot\sin.\omega}+\frac{g}{x-\cos.\omega+\sqrt{-1}\cdot\sin.\omega},$$

ita ut totum resolutionis negotium huc redeat, ut ambo numeratores f et g determinentur; iis enim inventis habebitur summa ambarum fractionum

$$=\frac{fx+gx-(f+g)\cos.\omega+\sqrt{-1}\cdot(f-g)\sin.\omega}{xx-2x\cos.\omega+1},$$

ubi igitur erit

$$B=f+g \quad \text{et} \quad A=(f-g)\sqrt{-1}\cdot\sin.\omega-(f+g)\cos.\omega.$$

5. Per methodum igitur fractiones quascunque in fractiones simplices resolvendi statuamus

$$\frac{x^{m-1}}{1-2x^k\cos.\theta+x^{2k}}=\frac{f}{x-\cos.\omega-\sqrt{-1}\cdot\sin.\omega}+R,$$

ubi R complectatur omnes reliquas fractiones partiales. Hinc per

$$x-\cos.\omega-\sqrt{-1}\cdot\sin.\omega$$

multiplicando habebitur

$$\frac{x^m-x^{m-1}(\cos.\omega+\sqrt{-1}\cdot\sin.\omega)}{1-2x^k\cos.\theta+x^{2k}}=f+R(x-\cos.\omega-\sqrt{-1}\cdot\sin.\omega);$$

quae aequatio cum vera esse debeat, quicunque valor ipsi x tribuatur, statuamus $x=\cos.\omega+\sqrt{-1}\cdot\sin.\omega$, ut membrum postremum prorsus e calculo tollatur; tum vero in parte sinistra, quia formula $x-\cos.\omega-\sqrt{-1}\cdot\sin.\omega$ simul est factor denominatoris, facta hac substitutione tam numerator quam denominator in nihilum abibunt, ita ut hinc nihil concludi posse videatur.

6. Hic igitur utamur regula notissima et loco tam numeratoris quam denominatoris eorum differentialia scribamus, unde nostra aequatio accipiet sequentem formam

$$\frac{mx^{m-1}-(m-1)x^{m-2}(\cos.\omega+\sqrt{-1}\cdot\sin.\omega)}{-2kx^{k-1}\cos.\theta+2kx^{2k-1}}$$
$$=\frac{mx^m-(m-1)x^{m-1}(\cos.\omega+\sqrt{-1}\cdot\sin.\omega)}{-2kx^k\cos.\theta+2kx^{2k}}=f,$$

posito scilicet $x=\cos.\omega+\sqrt{-1}\cdot\sin.\omega$. Tum autem erit

$$x^m=\cos.m\omega+\sqrt{-1}\cdot\sin.m\omega$$

et

$$x^{m-1}(\cos.\omega+\sqrt{-1}\cdot\sin.\omega)=x^m=\cos.m\omega+\sqrt{-1}\cdot\sin.m\omega$$

et pro denominatore

$$x^k=\cos.k\omega+\sqrt{-1}\cdot\sin.k\omega \quad \text{et} \quad x^{2k}=\cos.2k\omega+\sqrt{-1}\cdot\sin.2k\omega;$$

unde fit numerator

$$x^m=\cos.m\omega+\sqrt{-1}\cdot\sin.m\omega$$

et denominator

$$-2k\cos.\theta\cos.k\omega+2k\cos.2k\omega-2k\sqrt{-1}\cdot\cos.\theta\sin.k\omega+2k\sqrt{-1}\cdot\sin.2k\omega.$$

7. Pro denominatore reducendo recordemur iam supra inventum esse $\cos.k\omega=\cos.\theta$, unde fit $\sin.k\omega=\sin.\theta$, tum vero

$$\cos.2k\omega=\cos.2\theta=2\cos.\theta^2-1 \quad \text{et} \quad \sin.2k\omega=2\sin.\theta\cos.\theta,$$

quibus valoribus adhibitis denominator noster erit

$$2k\cos.\theta^2-2k+2k\sqrt{-1}\cdot\sin.\theta\cos.\theta=-2k\sin.\theta^2+2k\sqrt{-1}\cdot\sin.\theta\cos.\theta$$
$$=-2k\sin.\theta(\sin.\theta-\sqrt{-1}\cdot\cos.\theta),$$

quamobrem hoc valore adhibito habebimus

$$f=\frac{\cos.m\omega+\sqrt{-1}\cdot\sin.m\omega}{2k\sin.\theta(\sqrt{-1}\cdot\cos.\theta-\sin.\theta)}.$$

Simul vero hinc sine novo calculo deducemus valorem g, quippe qui ab f ratione signi $\sqrt{-1}$ tantum discrepat, sicque erit

$$g=\frac{\cos.m\omega-\sqrt{-1}\cdot\sin.m\omega}{-2k\sin.\theta(\sin.\theta+\sqrt{-1}\cdot\cos.\theta)}.$$

8. Inventis autem his litteris f et g pro litteris A et B colligemus primo

$$f+g=\frac{\cos.\theta\sin.m\omega-\sin.\theta\cos.m\omega}{k\sin.\theta}=\frac{\sin.(m\omega-\theta)}{k\sin.\theta},$$

tum vero erit

$$f-g=-\frac{\sqrt{-1}\cdot\cos.(m\omega-\theta)}{k\sin.\theta}.$$

Ex his igitur reperiemus

$$B=\frac{\sin.(m\omega-\theta)}{k\sin.\theta}$$

et

$$A=\frac{\sin.\omega\cos.(m\omega-\theta)-\cos.\omega\sin.(m\omega-\theta)}{k\sin.\theta}=-\frac{\sin.((m\omega-\theta)-\omega)}{k\sin.\theta},$$

ubi ergo imaginaria sponte se mutuo destruxerunt.

9. Inventis his valoribus A et B investigari oportet integrale partiale

$$\int\frac{(A+Bx)dx}{1-2x\cos.\omega+xx},$$

ubi, cum denominatoris differentiale sit

$$2xdx-2dx\cos.\omega=2dx(x-\cos.\omega),$$

statuamus

$$A+Bx=B(x-\cos.\omega)+C$$

eritque $C=A+B\cos.\omega$; hinc igitur erit

$$C=\frac{\cos.\omega\sin.(m\omega-\theta)-\sin.((m\omega-\theta)-\omega)}{k\sin.\theta}.$$

Quia vero $-\sin.((m\omega-\theta)-\omega)=-\sin.(m\omega-\theta)\cos.\omega+\cos.(m\omega-\theta)\sin.\omega$, erit

$$C=\frac{\sin.\omega\cos.(m\omega-\theta)}{k\sin.\theta}.$$

Hac ergo forma adhibita formula integranda $\frac{(A+Bx)dx}{1-2x\cos.\omega+xx}$ discerpatur in has duas partes

$$\frac{B(x-\cos.\omega)dx}{1-2x\cos.\omega+xx}+\frac{Cdx}{1-2x\cos.\omega+xx}.$$

Hic igitur prioris partis integrale manifesto est

$$Bl\sqrt{(1-2x\cos.\omega+xx)},$$

alterius vero partis facile patet integrale per arcum circuli expressum iri, cuius tangens sit $\frac{x\sin.\omega}{1-x\cos.\omega}$. Ad hoc integrale inveniendum ponamus

$$\int\frac{Cdx}{1-2x\cos.\omega+xx}=D\,\text{A tang.}\frac{x\sin.\omega}{1-x\cos.\omega}$$

et sumtis differentialibus, quia $d.\text{A tang.}t$ aequale est $\frac{dt}{1+tt}$, habebimus

$$\frac{Cdx}{1-2x\cos.\omega+xx}=D\frac{dx\sin.\omega}{1-2x\cos.\omega+xx},$$

unde manifesto fit

$$D=\frac{C}{\sin.\omega}=\frac{\cos.(m\omega-\theta)}{k\sin.\theta}.$$

10. Substituamus igitur loco B et D valores modo inventos et ex singulis factoribus denominatoris $1-2x^k\cos.\theta+x^{2k}$, quorum forma est $1-2x\cos.\omega+xx$, oritur pars integralis constans ex membro logarithmico et arcu circulari, quae erit

$$\frac{\sin.(m\omega-\theta)}{k\sin.\theta}l\sqrt{(1-2x\cos.\omega+xx)}+\frac{\cos.(m\omega-\theta)}{k\sin.\theta}\text{A tang.}\frac{x\sin.\omega}{1-x\cos.\omega},$$

quae evanescit sumto $x=0$. In hac igitur forma tantum opus est, ut loco ω successive scribamus valores supra indicatos, scilicet

$$\omega=\frac{\theta}{k},\quad\frac{2\pi+\theta}{k},\quad\frac{4\pi+\theta}{k},\quad\frac{6\pi+\theta}{k}\quad\text{etc.},$$

donec perveniatur ad $\frac{2(k-1)\pi+\theta}{k}$; tum enim summa omnium harum formarum praebebit totum integrale indefinitum formulae propositae.

11. Postquam igitur integrale indefinitum elicuimus, nihil aliud superest, nisi ut in eo faciamus $x=\infty$, quo facto pars logarithmica ob

$$\sqrt{(1-2x\cos.\omega+xx)}=x-\cos.\omega$$

erit $Bl(x-\cos.\omega)$. Est vero

$$l(x-\cos.\omega)=lx-\frac{\cos.\omega}{x}=lx$$

ob $\frac{\cos.\omega}{x}=0$; quamobrem facto $x=\infty$ quaelibet pars logarithmica habebit hanc formam $\frac{\sin.(m\omega-\theta)}{k\sin.\theta}lx$. Deinde pro partibus a circulo pendentibus facto $x=\infty$ fit

$$\frac{x\sin.\omega}{1-x\cos.\omega}=-\text{tang.}\,\omega=\text{tang.}(\pi-\omega)$$

sicque arcus, cuius haec est tangens, erit $=\pi-\omega$ hincque pars circularis quaecunque fiet $\frac{\cos.(m\omega-\theta)}{k\sin.\theta}(\pi-\omega)$.

12. Cum quilibet valor anguli ω in genere hanc habeat formam $\frac{2i\pi+\theta}{k}$, erit angulus

$$m\omega-\theta=\frac{2im\pi-\theta(k-m)}{k}\quad\text{et}\quad\pi-\omega=\frac{\pi(k-2i)-\theta}{k}.$$

Ponamus brevitatis gratia

$$\frac{\theta(k-m)}{k}=\zeta\quad\text{et}\quad\frac{m\pi}{k}=\alpha,$$

ut sit

$$m\omega-\theta=2i\alpha-\zeta,$$

ubi loco i scribi debent successive numeri 0, 1, 2, 3 etc. usque ad $k-1$. Hinc igitur, si omnes partes logarithmicas in unam summam colligamus, ea ita repraesentari poterit

$$\frac{lx}{k\sin.\theta}\left\{\begin{array}{l}-\sin.\zeta+\sin.(2\alpha-\zeta)+\sin.(4\alpha-\zeta)+\sin.(6\alpha-\zeta)\\ \qquad+\sin.(8\alpha-\zeta)+\cdots+\sin.(2(k-1)\alpha-\zeta)\end{array}\right\};$$

ubi quidem ex iis, quae hactenus sunt tradita, facile suspicari licet totam hanc progressionem ad nihilum redigi. Verum hoc ipsum firma demonstratione muniri necesse est.

13. Ad hoc ostendendum ponamus

$$S=-\sin.\zeta+\sin.(2\alpha-\zeta)+\sin.(4\alpha-\zeta)+\cdots+\sin.(2(k-1)\alpha-\zeta);$$

multiplicemus utrinque per $2\sin.\alpha$, et cum sit

$$2\sin.\alpha\sin.\varphi=\cos.(\alpha-\varphi)-\cos.(\alpha+\varphi),$$

huius reductionis ope obtinebimus sequentem expressionem

$$\begin{aligned}2S\sin.\alpha&=\cos.(\alpha+\zeta)\\&-\cos.(\alpha-\zeta)-\cos.(3\alpha-\zeta)-\cos.(5\alpha-\zeta)-\cdots\\&+\cos.(\alpha-\zeta)+\cos.(3\alpha-\zeta)+\cos.(5\alpha-\zeta)+\cdots\\&-\cos.((2k-1)\alpha-\zeta),\end{aligned}$$

unde deletis terminis se mutuo destruentibus habebitur

$$2S\sin.\alpha=\cos.(\alpha+\zeta)-\cos.((2k-1)\alpha-\zeta).$$

14. Ponamus hos duos angulos, qui sunt relicti,

$$\alpha+\zeta=p\quad\text{et}\quad(2k-1)\alpha-\zeta=q$$

eritque eorum summa $p+q=2\alpha k$. Quia porro est $\alpha=\frac{m\pi}{k}$, erit $p+q=2m\pi$, hoc est multiplo totius circuli peripheriae ob m numerum integrum. Quare cum sit $q=2m\pi-p$, erit $\cos.q=\cos.p$; unde patet summam inventam nihilo esse aequalem sicque manifestum est omnes partes logarithmicas, quae in integrale formulae nostrae ingrediuntur, casu $x=\infty$ se mutuo destruere.

15. Progrediamur igitur ad partes circulares, quarum forma generalis, ut vidimus, est $\frac{\cos.(m\omega-\theta)}{k\sin.\theta}(\pi-\omega)$, quae posito $\alpha=\frac{m\pi}{k}$ et $\zeta=\frac{\theta(k-m)}{k}$ fit

$$\frac{\cos.(2i\alpha-\zeta)}{k\sin.\theta}\left(\pi-\frac{2i\pi+\theta}{k}\right)=\frac{\cos.(2i\alpha-\zeta)}{k\sin.\theta}\left(\pi-\frac{2i\pi}{k}-\frac{\theta}{k}\right).$$

Hic ponatur porro $\frac{\pi}{k}=\beta$ et $\pi-\frac{\theta}{k}=\gamma$, ut forma generalis sit

$$\frac{\cos.(2i\alpha-\zeta)}{k\sin.\theta}(\gamma-2i\beta).$$

Quare si loco i scribamus ordine valores 0, 1, 2, 3, 4 usque ad $k-1$, omnes partes circulares hanc progressionem constituent

$$\frac{1}{k\sin.\theta}(\gamma\cos.\zeta+(\gamma-2\beta)\cos.(2\alpha-\zeta)+(\gamma-4\beta)\cos.(4\alpha-\zeta)+\cdots$$
$$+(\gamma-2(k-1)\beta)\cos.(2(k-1)\alpha-\zeta)).$$

Ponamus igitur

$$S=\gamma\cos.\zeta+(\gamma-2\beta)\cos.(2\alpha-\zeta)+(\gamma-4\beta)\cos.(4\alpha-\zeta)+\cdots$$
$$+(\gamma-2(k-1)\beta)\cos.(2(k-1)\alpha-\zeta),$$

ut summa omnium partium circularium sit $\frac{S}{k\sin.\theta}$, quae ergo praebebit valorem quaesitum formulae integralis propositae casu, quo post integrationem statuitur $x=\infty$, ita ut totum negotium in investigando valore ipsius S versetur.

16. Hunc in finem multiplicemus utrinque per $2\sin.\alpha$, et cum in genere sit

$$2\sin.\alpha\cos.\varphi=\sin.(\alpha+\varphi)-\sin.(\varphi-\alpha),$$

hac reductione in singulis terminis facta perveniemus ad hanc aequationem

$$2S\sin.\alpha=\gamma\sin.(\alpha+\zeta)$$
$$+\gamma\sin.(\alpha-\zeta)+(\gamma-2\beta)\sin.(3\alpha-\zeta)+(\gamma-4\beta)\sin.(5\alpha-\zeta)+\cdots$$
$$-(\gamma-2\beta)\sin.(\alpha-\zeta)-(\gamma-4\beta)\sin.(3\alpha-\zeta)-(\gamma-6\beta)\sin.(5\alpha-\zeta)-\cdots$$
$$+(\gamma-2(k-1)\beta)\sin.((2k-1)\alpha-\zeta),$$

ubi praeter primum et ultimum terminum omnes reliqui contrahi possunt, ita ut prodeat

$$2S\sin.\alpha=\gamma\sin.(\alpha+\zeta)+2\beta\sin.(\alpha-\zeta)+2\beta\sin.(3\alpha-\zeta)+2\beta\sin.(5\alpha-\zeta)$$
$$+\cdots+2\beta\sin.((2k-3)\alpha-\zeta)+(\gamma-2(k-1)\beta)\sin.((2k-1)\alpha-\zeta).$$

17. Iam pro hac serie summanda ponamus porro

$$T=2\sin.(\alpha-\zeta)+2\sin.(3\alpha-\zeta)+2\sin.(5\alpha-\zeta)+\cdots+2\sin.((2k-3)\alpha-\zeta),$$

ut habeamus

$$2S\sin.\alpha=\gamma\sin.(\alpha+\zeta)+(\gamma-2(k-1)\beta)\sin.((2k-1)\alpha-\zeta)+\beta T.$$

Iam multiplicemus ut hactenus per $\sin.\alpha$, et cum sit

$$2\sin.\alpha\sin.\varphi=\cos.(\varphi-\alpha)-\cos.(\varphi+\alpha),$$

facta hac reductione nanciscimur

$$T\sin.\alpha=+\cos.\zeta$$
$$+\cos.(2\alpha-\zeta)+\cos.(4\alpha-\zeta)+\cdots+\cos.(2(k-2)\alpha-\zeta)$$
$$-\cos.(2\alpha-\zeta)-\cos.(4\alpha-\zeta)-\cdots-\cos.(2(k-2)\alpha-\zeta)$$
$$-\cos.(2(k-1)\alpha-\zeta),$$

unde deletis terminis, quae se mutuo destruunt, remanebit tantum ista expressio

$$T\sin.\alpha=\cos.\zeta-\cos.(2(k-1)\alpha-\zeta).$$

Cum igitur sit $\alpha=\frac{m\pi}{k}$, erit $2(k-1)\alpha=2m\pi-\frac{2m\pi}{k}$, cuius loco scribere licet $-\frac{2m\pi}{k}$, unde ob $\zeta=\frac{\theta(k-m)}{k}$ erit

$$T\sin.\alpha=\cos.\frac{\theta(k-m)}{k}-\cos.\frac{2m\pi+\theta(k-m)}{k}.$$

18. Nunc vero notetur in genere esse

$$\cos.p-\cos.q=2\sin.\frac{q+p}{2}\sin.\frac{q-p}{2};$$

quare cum sit

$$p=\frac{\theta(k-m)}{k}\quad\text{et}\quad q=\frac{2m\pi+\theta(k-m)}{k},$$

erit

$$\frac{q+p}{2}=\frac{m\pi+\theta(k-m)}{k}\quad\text{et}\quad\frac{q-p}{2}=\frac{m\pi}{k},$$

unde sequitur fore

$$T\sin.\alpha=2\sin.\frac{m\pi+\theta(k-m)}{k}\sin.\frac{m\pi}{k}$$

ideoque

$$T=2\sin.\frac{m\pi+\theta(k-m)}{k}$$

ob $\alpha=\frac{m\pi}{k}$.

19. Hoc igitur valore T invento reperiemus porro

$$2S\sin.\alpha=\gamma\sin.(\alpha+\zeta)+(\gamma-2(k-1)\beta)\sin.((2k-1)\alpha-\zeta)$$
$$+2\beta\sin.\frac{m\pi+\theta(k-m)}{k},$$

quae ob $\frac{m\pi + \theta(k-m)}{k} = \alpha + \zeta$ reducitur ad hanc formam

$$2S \sin.\alpha = (\gamma + 2\beta)\sin.(\alpha+\zeta) + (\gamma - 2(k-1)\beta)\sin.((2k-1)\alpha - \zeta),$$

quae ita repraesentari potest

$$2S \sin.\alpha = (\gamma + 2\beta)\big(\sin.(\alpha+\zeta) + \sin.((2k-1)\alpha - \zeta)\big) - 2k\beta \sin.((2k-1)\alpha - \zeta),$$

ubi pro parte priore ob

$$\sin.p + \sin.q = 2\sin.\frac{p+q}{2}\cos.\frac{p-q}{2}$$

erit

$$\frac{p+q}{2} = \alpha k \quad \text{et} \quad \frac{p-q}{2} = (k-1)\alpha - \zeta,$$

unde pars ipsa prior fit

$$2(\gamma + 2\beta)\sin.\alpha k \cos.((k-1)\alpha - \zeta);$$

ubi cum sit $\alpha k = m\pi$, erit $\sin.\alpha k = 0$, ita ut tantum supersit

$$2S\sin.\alpha = -2\beta k \sin.((2k-1)\alpha - \zeta)$$

hincque

$$S = -\frac{\beta k \sin.((2k-1)\alpha - \zeta)}{\sin.\alpha}.$$

Est vero

$$(2k-1)\alpha - \zeta = 2m\pi - \frac{m\pi}{k} - \frac{\theta(k-m)}{k};$$

omisso termino $2m\pi$ erit igitur

$$S = +\frac{\pi \sin.\frac{m\pi + \theta(k-m)}{k}}{\sin.\frac{m\pi}{k}}$$

ideoque valor quaesitus erit

$$\frac{S}{k\sin.\theta} = +\frac{\pi\sin.\frac{m\pi + \theta(k-m)}{k}}{k\sin.\theta \sin.\frac{m\pi}{k}},$$

quae forma reducitur ad hanc

$$\frac{\pi \sin.\frac{m(\pi-\theta) + k\theta}{k}}{k\sin.\theta\sin.\frac{m\pi}{k}}.$$

20. Contemplemur hic ante omnia casum, quo $\theta = \frac{\pi}{2}$, et formula integralis proposita abit in hanc

$$\int \frac{x^{m-1}\,dx}{1+x^{2k}},$$

cuius ergo valor, si post integrationem ponatur $x = \infty$, evadet

$$= \frac{\pi\sin.\left(\frac{\pi}{2} + \frac{m\pi}{2k}\right)}{k\sin.\frac{m\pi}{k}} = \frac{\pi\cos.\frac{m\pi}{2k}}{k\sin.\frac{m\pi}{k}}.$$

Quia igitur est $\sin.\frac{m\pi}{k} = 2\sin.\frac{m\pi}{2k}\cos.\frac{m\pi}{2k}$, prodibit iste valor

$$= \frac{\pi}{2k\sin.\frac{m\pi}{2k}},$$

qui valor egregie convenit cum eo, quem non ita pridem pro formula $\int \frac{x^{m-1}\,dx}{1+x^k}$ assignavimus[1]), siquidem loco k scribatur $2k$.

21. Evolvamus etiam casum, quo $\theta = \pi$, et formula nostra integralis [abit in hanc]

$$\int \frac{x^{m-1}\,dx}{(1+x^k)^2},$$

cuius ergo facto $x = \infty$ valor erit

$$\frac{\pi\sin.\left(\frac{m(\pi-\theta)}{k} + \theta\right)}{k\sin.\theta\sin.\frac{m\pi}{k}} = \frac{\pi}{k\sin.\frac{m\pi}{k}}\cdot\frac{\sin.\left(\frac{m(\pi-\theta)}{k}+\theta\right)}{\sin.\theta}.$$

Huius autem posterioris fractionis casu $\theta = \pi$ tam numerator quam denominator evanescit; quare ut eius verus valor eruatur, loco utriusque eius differentiale scribamus, quo facto ista fractio abibit in hanc

$$\frac{d\theta\left(1-\frac{m}{k}\right)\cos.\left(\frac{m(\pi-\theta)}{k}+\theta\right)}{d\theta\cos.\theta},$$

cuius valor facto $\theta = \pi$ nunc manifesto est $1 - \frac{m}{k}$; sicque valor integralis quaesitus erit $\left(1-\frac{m}{k}\right)\frac{\pi}{k\sin.\frac{m\pi}{k}}$, prorsus uti in superiore dissertatione[2]) invenimus.

1) Vide § 11 Commentationis praecedentis. A. G.
2) Vide § 17 Commentationis praecedentis. A. G.

22. Quo autem valorem generalem inventum commodiorem reddamus, ponamus $\pi - \theta = \eta$ fietque $\sin.\theta = \sin.\eta$ et $\cos.\theta = -\cos.\eta$; tum vero erit angulus

$$\frac{m(\pi-\theta)}{k} + \theta = \frac{m\eta}{k} + \pi - \eta,$$

cuius sinus est $\sin.\left(1-\frac{m}{k}\right)\eta$, unde valor quaesitus nostrae formulae erit

$$\frac{\pi \sin.\left(1-\frac{m}{k}\right)\eta}{k \sin.\eta \sin.\frac{m\pi}{k}},$$

atque hinc tandem sequens adepti sumus

THEOREMA

23. *Si haec formula integralis*

$$\int \frac{x^{m-1}dx}{1+2x^k \cos.\eta + x^{2k}}$$

a termino $x=0$ usque ad terminum $x=\infty$ extendatur, eius valor erit

$$=\frac{\pi \sin.\left(1-\frac{m}{k}\right)\eta}{k \sin.\eta \sin.\frac{m\pi}{k}},$$

sive cum sit

$$\sin.\left(1-\frac{m}{k}\right)\eta = \sin.\eta \cos.\frac{m\eta}{k} - \cos.\eta \sin.\frac{m\eta}{k},$$

iste valor etiam hoc modo exprimi potest

$$\frac{\pi \cos.\frac{m\eta}{k}}{k \sin.\frac{m\pi}{k}} - \frac{\pi \sin.\frac{m\eta}{k}}{k \operatorname{tang}.\eta \sin.\frac{m\pi}{k}}.$$

24. Consideremus nunc alio modo hanc formulam integralem

$$\int \frac{x^{m-1}dx}{1+2x^k \cos.\eta + x^{2k}},$$

cuius valor a termino $x=0$ usque ad $x=1$ ponatur $=P$, eiusdem vero

valor ab $x=1$ usque ad $x=\infty$ ponatur $=Q$, ita ut $P+Q$ exhibere debeat ipsum valorem ante inventum. Nunc vero pro valore Q inveniendo ponamus $x=\frac{1}{y}$ et formula nostra ita repraesentata

$$\frac{x^m}{1+2x^k \cos.\eta + x^{2k}} \cdot \frac{dx}{x}$$

ob $\frac{dx}{x} = -\frac{dy}{y}$ induet hanc formam

$$-\int \frac{y^{-m}}{1+2y^{-k}\cos.\eta + y^{-2k}} \cdot \frac{dy}{y} = -\int \frac{y^{2k-m-1}dy}{y^{2k}+2y^k \cos.\eta + 1},$$

cuius valor a termino $y=1$ usque ad $y=0$ extendi debet. Commutatis igitur his terminis habebimus

$$Q = +\int \frac{y^{2k-m-1}dy}{y^{2k}+2y^k \cos.\eta + 1}$$

a termino $y=0$ usque ad $y=1$.

25. Quia in utraque forma pro P et Q eadem conditio integrationis praescribitur, a termino 0 usque ad 1, nihil impedit, quominus in posteriore loco y scribamus x, unde pro $P+Q$ habebimus hanc formam integralem

$$\int \frac{x^{m-1}+x^{2k-m-1}}{1+2x^k \cos.\eta + x^{2k}} dx,$$

cuius valor a termino $x=0$ usque ad $x=1$ extensus aequabitur huic expressioni $\frac{\pi \sin.\left(1-\frac{m}{k}\right)\eta}{k \sin.\eta \sin.\frac{m\pi}{k}}$. Comparatis igitur his binis formulis integralibus nanciscemur sequens theorema notatu maxime dignum.

THEOREMA

26. *Haec formula integralis*

$$\int \frac{x^{m-1}+x^{2k-m-1}}{1+2x^k \cos.\eta + x^{2k}} dx$$

a termino $x=0$ usque ad terminum $x=1$ extensa aequalis est huic formulae integrali

$$\int \frac{x^{m-1}dx}{1+2x^k \cos.\eta + x^{2k}}$$

a termino $x=0$ *usque ad terminum* $x=\infty$ *extensae; utriusque enim valor erit*

$$\frac{\pi \sin.\left(1-\frac{m}{k}\right)\eta}{k\sin.\eta\sin.\frac{m\pi}{k}}.$$

27. Quodsi hanc fractionem $\frac{\sin.\eta}{1+2x^k\cos.\eta+x^{2k}}$ in seriem infinitam evolvamus, quae sit

$$\sin.\eta+Ax^k+Bx^{2k}+Cx^{3k}+Dx^{4k}+Ex^{5k}+\text{etc.},$$

per denominatorem multiplicando perveniemus ad hanc expressionem infinitam

$$\begin{array}{lllllll}
\sin.\eta=\sin.\eta+ & Ax^k+ & Bx^{2k}+ & Cx^{3k}+ & Dx^{4k}+ & Ex^{5k}+\text{etc.}, \\
\quad +2\sin.\eta\cos.\eta & +2A\cos.\eta & +2B\cos.\eta & +2C\cos.\eta & +2D\cos.\eta & +\text{etc.} \\
 & +\ \sin.\eta & +\ A & +\ B & +\ C & +\text{etc.}
\end{array}$$

unde singulis terminis ad nihilum reductis reperiemus

$$\begin{array}{llll}
1. & A+2\sin.\eta\cos.\eta=0 & \text{hincque} & A=-\sin.2\eta, \\
2. & B+2A\cos.\eta+\sin.\eta=0, & \text{unde fit} & B=\sin.3\eta, \\
3. & C+2B\cos.\eta+A=0, & \text{unde fit} & C=-\sin.4\eta, \\
4. & D+2C\cos.\eta+B=0, & \text{unde fit} & D=\sin.5\eta \\
 & \text{etc.} & & \text{etc.},
\end{array}$$

ita ut nostra fractio $\frac{\sin.\eta}{1+2x^k\cos.\eta+x^{2k}}$ resolvatur in hanc seriem

$$\sin.\eta-x^k\sin.2\eta+x^{2k}\sin.3\eta-x^{3k}\sin.4\eta+x^{4k}\sin.5\eta-\text{etc.}$$

28. Multiplicemus nunc hanc seriem per

$$x^{m-1}dx+x^{2k-m-1}dx$$

et post integrationem faciamus $x=1$, ut obtineamus valorem huius formulae

$$\sin.\eta\int\frac{x^{m-1}+x^{2k-m-1}}{1+2x^k\cos.\eta+x^{2k}}dx$$

pro casu $x=1$, hocque modo perveniemus ad geminas sequentes series

$$\frac{\sin.\eta}{m}-\frac{\sin.2\eta}{m+k}+\frac{\sin.3\eta}{m+2k}-\frac{\sin.4\eta}{m+3k}+\frac{\sin.5\eta}{m+4k}-\text{etc.},$$

$$\frac{\sin.\eta}{2k-m}-\frac{\sin.2\eta}{3k-m}+\frac{\sin.3\eta}{4k-m}-\frac{\sin.4\eta}{5k-m}+\frac{\sin.5\eta}{6k-m}-\text{etc.}$$

Aggregatum igitur harum duarum serierum iunctim sumtarum aequabitur huic valori

$$\frac{\pi\sin.\left(1-\frac{m}{k}\right)\eta}{k\sin.\frac{m\pi}{k}},$$

unde subiungamus adhuc istud theorema.

THEOREMA

29. *Si* η *denotet angulum quemcunque, litterae vero* m *et* k *pro lubitu accipiantur ex iisque binae sequentes series formentur*

$$P=\frac{\sin.\eta}{m}-\frac{\sin.2\eta}{m+k}+\frac{\sin.3\eta}{m+2k}-\frac{\sin.4\eta}{m+3k}+\frac{\sin.5\eta}{m+4k}-\text{etc.},$$

$$Q=\frac{\sin.\eta}{2k-m}-\frac{\sin.2\eta}{3k-m}+\frac{\sin.3\eta}{4k-m}-\frac{\sin.4\eta}{5k-m}+\frac{\sin.5\eta}{6k-m}-\text{etc.},$$

neutrius quidem summa exhiberi potest, utriusque autem iunctim sumtae summa erit

$$P+Q=\frac{\pi\sin.\left(1-\frac{m}{k}\right)\eta}{k\sin.\frac{m\pi}{k}}.$$

COROLLARIUM

30. Quodsi ergo angulum η infinite parvum capiamus, ut fiat

$$\sin.\eta=\eta,\quad \sin.2\eta=2\eta,\quad \sin.3\eta=3\eta\quad \text{etc.},$$

quia in formula summae fiet

$$\sin.\left(1-\frac{m}{k}\right)\eta=\left(1-\frac{m}{k}\right)\eta,$$

si utrinque per η dividamus, obtinebimus sequentem seriem geminatam

$$\frac{1}{m} - \frac{2}{m+k} + \frac{3}{m+2k} - \frac{4}{m+3k} + \frac{5}{m+4k} - \text{etc.}$$
$$+ \frac{1}{2k-m} - \frac{2}{3k-m} + \frac{3}{4k-m} - \frac{4}{5k-m} + \frac{5}{6k-m} - \text{etc.},$$

cuius ergo summa erit $\left(1-\frac{m}{k}\right)\frac{\pi}{k \sin.\frac{m\pi}{k}}$; ubi notetur ambas istas series non incongrue in hanc simplicem contrahi posse

$$\frac{2k}{m(2k-m)} - \frac{8k}{(m+k)(3k-m)} + \frac{18k}{(m+2k)(4k-m)} - \frac{32k}{(m+3k)(5k-m)} + \text{etc.},$$

ubi numeratores sunt numeri quadrati duplicati.

31. Formulae autem, quarum valores hactenus invenimus, multo concinnius et elegantius exprimi possunt, si loco exponentis m scribamus $k-n$; tum enim in valore integrali invento fiet $\left(1-\frac{m}{k}\right)\eta = \frac{n\eta}{k}$, at vero pro denominatore fiet $\frac{m\pi}{k} = \pi - \frac{n\pi}{k}$, cuius sinus erit $\sin.\frac{n\pi}{k}$; sicque nostra formula inventa hanc induet formam $\frac{\pi \sin.\frac{n\eta}{k}}{k \sin.\eta \sin.\frac{n\pi}{k}}$, quae ergo exprimet valorem huius formulae integralis

$$\int \frac{x^{k-n-1}dx}{1+2x^k \cos.\eta + x^{2k}}$$

ab $x=0$ usque ad $x=\infty$, ut et huius formulae

$$\int \frac{x^{k-n-1}+x^{k+n-1}}{1+2x^k \cos.\eta + x^{2k}}dx$$

a termino $x=0$ usque ad terminum $x=1$; et quia utriusque valor est $\frac{\pi \sin.\frac{n\eta}{k}}{k \sin.\eta \sin.\frac{n\pi}{k}}$, perspicuum est eum manere eundem, etsi loco n scribatur $-n$, ex quo prior formula ita repraesentari poterit

$$\int \frac{x^{k\pm n-1}dx}{1+2x^k \cos.\eta + x^{2k}};$$

at posterior formula ob hanc ambiguitatem nullam plane mutationem patitur.

32. Ponendo $m=k-n$ etiam series nostra geminata pulchriorem accipiet faciem; habebitur enim

$$\frac{\sin.\eta}{k-n} - \frac{\sin.2\eta}{2k-n} + \frac{\sin.3\eta}{3k-n} - \frac{\sin.4\eta}{4k-n} + \text{etc.}$$
$$+ \frac{\sin.\eta}{k+n} - \frac{\sin.2\eta}{2k+n} + \frac{\sin.3\eta}{3k+n} - \frac{\sin.4\eta}{4k+n} + \text{etc.},$$

cuius ergo summa erit $\frac{\pi \sin.\frac{n\eta}{k}}{k \sin.\frac{n\pi}{k}}$. Tum vero si hae geminae series in unam contrahantur et utrinque per $2k$ dividatur, obtinebitur sequens summatio memoratu digna

$$\frac{\pi \sin.\frac{n\eta}{k}}{2kk \sin.\frac{n\pi}{k}} = \frac{\sin.\eta}{kk-nn} - \frac{2\sin.2\eta}{4kk-nn} + \frac{3\sin.3\eta}{9kk-nn} - \frac{4\sin.4\eta}{16kk-nn} + \text{etc.}$$

33. Quodsi haec postrema series differentietur sumendo solum angulum η variabilem, ob $d \sin.\frac{n\eta}{k} = \frac{n\,d\eta}{k}\cos.\frac{n\eta}{k}$ habebimus

$$\frac{\pi n \cos.\frac{n\eta}{k}}{2k^3 \sin.\frac{n\pi}{k}} = \frac{\cos.\eta}{kk-nn} - \frac{4\cos.2\eta}{4kk-nn} + \frac{9\cos.3\eta}{9kk-nn} - \frac{16\cos.4\eta}{16kk-nn} + \text{etc.}$$

Unde si sumatur $\eta=0$, orietur ista summatio

$$\frac{\pi n}{2k^3 \sin.\frac{n\pi}{k}} = \frac{1}{kk-nn} - \frac{4}{4kk-nn} + \frac{9}{9kk-nn} - \frac{16}{16kk-nn} + \text{etc.};$$

sin autem sumatur $\eta = 90^0 = \frac{\pi}{2}$, erit

$$\cos.\eta = 0, \quad \cos.2\eta = -1, \quad \cos.3\eta = 0, \quad \cos.4\eta = +1 \text{ etc.},$$

unde nascitur sequens series

$$\frac{n\pi \cos.\frac{n\pi}{2k}}{2k^3 \sin.\frac{n\pi}{k}} = \frac{4}{4kk-nn} - \frac{16}{16kk-nn} + \frac{36}{36kk-nn} - \frac{64}{64kk-nn} + \text{etc.}$$

Quia autem $\sin.\frac{n\pi}{k} = 2\sin.\frac{n\pi}{2k}\cos.\frac{n\pi}{2k}$, erit eiusdem seriei summa $\frac{n\pi}{4k^3 \sin.\frac{n\pi}{2k}}$.

34. At si series illa § 32 exhibita in $d\eta$ ducatur et integretur, ob $\int d\eta \sin.\frac{n\eta}{k} = -\frac{k}{n}\cos.\frac{n\eta}{k}$ erit

$$C - \frac{\pi \cos.\frac{n\eta}{k}}{2nk \sin.\frac{n\pi}{k}} = -\frac{\cos.\eta}{kk-nn} + \frac{\cos.2\eta}{4kk-nn} - \frac{\cos.3\eta}{9kk-nn} + \frac{\cos.4\eta}{16kk-nn} - \text{etc.}$$

Ut autem hic constantem addendam C definiamus, sumamus $\eta = 0$ fietque

$$C - \frac{\pi}{2nk \sin.\frac{n\pi}{k}} = -\frac{1}{kk-nn} + \frac{1}{4kk-nn} - \frac{1}{9kk-nn} + \text{etc.};$$

quare si huius seriei summa aliunde pateat, constans C definiri poterit. Series autem haec in sequentem geminatam resolvi potest

$$2nC - \frac{\pi}{k \sin.\frac{n\pi}{k}} = \frac{1}{k+n} - \frac{1}{2k+n} + \frac{1}{3k+n} - \frac{1}{4k+n} + \text{etc.}$$
$$- \frac{1}{k-n} + \frac{1}{2k-n} - \frac{1}{3k-n} + \frac{1}{4k-n} - \text{etc.}$$

35. Cum igitur in *Introductione in Analysin Infinitorum*[1]) pag. 142 ad hanc pervenissem seriem

$$\frac{1}{kk-nn} - \frac{1}{4kk-nn} + \frac{1}{9kk-nn} - \frac{1}{16kk-nn} + \text{etc.} = \frac{\pi}{2kn \sin.\frac{n\pi}{k}} - \frac{1}{2nn}$$

(hic scilicet loco litterarum ibi adhibitarum m et n scripsi n et k), hoc valore adhibito nostra aequatio erit

$$C - \frac{\pi}{2nk \sin.\frac{n\pi}{k}} = \frac{1}{2nn} - \frac{\pi}{2nk \sin.\frac{n\pi}{k}},$$

unde fit $C = \frac{1}{2nn}$. Hinc ergo habebimus istam summationem

$$\frac{\pi \cos.\frac{n\eta}{k}}{2nk \sin.\frac{n\pi}{k}} - \frac{1}{2nn} = \frac{\cos.\eta}{kk-nn} - \frac{\cos.2\eta}{4kk-nn} + \frac{\cos.3\eta}{9kk-nn} - \frac{\cos.4\eta}{16kk-nn} + \text{etc.},$$

quae series utique omni attentione digna videtur.

1) Vide *Introductionem in analysin infinitorum*, t. I cap. X; *Leonhardi Euleri Opera omnia*, series I, vol. 8. A. G.

METHODUS INVENIENDI FORMULAS INTEGRALES QUAE CERTIS CASIBUS DATAM INTER SE TENEANT RATIONEM UBI SIMUL METHODUS TRADITUR FRACTIONES CONTINUAS SUMMANDI

Commentatio 594 indicis Enestroemiani
Opuscula analytica 2, 1785, p. 178—216

1. Quemadmodum in seriebus recurrentibus quilibet terminus ex uno pluribusve praecedentibus secundum legem quandam constantem determinatur, ita hic eiusmodi series sum consideraturus, in quibus quilibet terminus ex uno pluribusve praecedentibus secundum quampiam legem variabilem determinatur. Quoniam autem in talibus seriebus formula generalis singulos terminos exprimens plerumque non est algebraica sed transcendens, singulos terminos per formulas integrales exhiberi conveniet; quae ut valores determinatos praebeant, post integrationem quantitati variabili valorem determinatum tribui assumo, ita ut singuli termini prodeant quantitates determinatae; atque nunc quaestio principalis huc redit, quemadmodum istae formulae integrales debeant esse comparatae, ut quilibet terminus secundum datam legem ex uno pluribusve praecedentibus determinetur.

2. Quod quo clarius perspiciatur, contemplemur seriem notissimam harum formularum integralium

$$\int \frac{dx}{\sqrt{(1-xx)}}, \int \frac{xxdx}{\sqrt{(1-xx)}}, \int \frac{x^4dx}{\sqrt{(1-xx)}}, \int \frac{x^6dx}{\sqrt{(1-xx)}} \text{ etc.};$$

quae si singulae ita integrentur, ut evanescant posito $x=0$, tum vero variabili x tribuatur valor $=1$, quilibet terminus a praecedente ita pendet, ut sit

$$\int\frac{xxdx}{\sqrt{(1-xx)}}=\frac{1}{2}\int\frac{dx}{\sqrt{(1-xx)}},$$

$$\int\frac{x^4dx}{\sqrt{(1-xx)}}=\frac{3}{4}\int\frac{xxdx}{\sqrt{(1-xx)}},$$

$$\int\frac{x^6dx}{\sqrt{(1-xx)}}=\frac{5}{6}\int\frac{x^4dx}{\sqrt{(1-xx)}}$$

atque in genere

$$\int\frac{x^ndx}{\sqrt{(1-xx)}}=\frac{n-1}{n}\int\frac{x^{n-2}dx}{\sqrt{(1-xx)}}.$$

Unde patet hanc formulam generalem spectari posse tanquam terminum generalem illius seriei atque quemlibet terminum ex praecedente oriri, si iste multiplicetur per $\frac{n-1}{n}$.

3. Ad similitudinem igitur huius casus seriem formularum integralium ita in genere constituamus

$$\int dv,\quad \int xdv,\quad \int xxdv,\quad \int x^3dv,\quad \int x^4dv \text{ etc.},$$

ita ut terminus indici n respondens sit

$$\int x^{n-1}dv,$$

quae singula integralia ita accipi sumamus, ut evanescant posito $x=0$; post integrationem autem quantitati variabili x tribuamus quempiam valorem constantem, veluti $x=1$ vel alii cuipiam numero. Quibus positis quaestio huc redit, qualis pro v assumi debeat functio ipsius x, ut quilibet terminus per unum vel duos pluresve praecedentes secundum legem quandam datam, utcunque variabilem sive ab indice n pendentem, determinetur; ubi quidem imprimis eo erit respiciendum, ad quot dimensiones index n in scala relationis proposita ascendat; plerumque autem non ultra primam dimensionem assurgere erit opus. Hinc igitur sequentia problemata pertractemus.

PROBLEMA 1

4. *Invenire functionem v, ut ista relatio inter binos terminos sibi succedentes locum habeat*

$$\int x^ndv=\frac{\alpha n+a}{\beta n+b}\int x^{n-1}dv.$$

SOLUTIO

Requiritur igitur hic, ut sit

$$(\alpha n+a)\int x^{n-1}dv=(\beta n+b)\int x^ndv,$$

si scilicet post integrationem variabili x certus valor tribuatur. Quoniam igitur ista conditio tum demum locum habere debet, postquam variabili x iste valor constans fuerit datus, ponamus in genere, dum x est variabilis, hanc aequationem locum habere

$$(\alpha n+a)\int x^{n-1}dv=(\beta n+b)\int x^ndv+V,$$

quantitatem autem V ita esse comparatam, ut evanescat, postquam variabili ille valor determinatus fuerit assignatus. Praeterea vero, quia ambo integralia ita capi assumimus, ut evanescant posito $x=0$, necesse est, ut etiam ista quantitas V eodem quoque casu evanescat.

5. Quoniam haec aequalitas subsistere debet pro omnibus indicibus n, quos quidem semper ut positivos spectamus, facile intelligitur quantitatem istam V factorem habere debere x^n; quo pacto iam isti conditioni satisfit, ut posito $x=0$ etiam fiat $V=0$. Quamobrem statuamus

$$V=x^nQ,$$

ubi Q denotet functionem ipsius x proposito accommodatam, et quam simul ita comparatam esse desideramus, ut evanescat, si ipsi x certus quidam valor tribuatur.

6. Cum igitur esse debeat

$$(\alpha n+a)\int x^{n-1}dv=(\beta n+b)\int x^ndv+x^nQ,$$

differentietur ista aequatio ac differentiali per x^{n-1} diviso pervenietur ad hanc aequationem differentialem

$$(\alpha n + a)dv = (\beta n + b)x dv + nQdx + xdQ;$$

quae cum subsistere debeat pro omnibus valoribus ipsius n, termini ista littera affecti seorsim se tollere debent, unde nanciscimur has duas aequalitates

$$\text{I.}\quad (\alpha - \beta x)dv = Qdx \quad \text{et} \quad \text{II.}\quad (a - bx)dv = xdQ.$$

Ex priore fit $dv = \frac{Qdx}{\alpha - \beta x}$, ex altera vero $dv = \frac{xdQ}{a - bx}$, qui duo valores inter se aequati suppeditant hanc aequationem $\frac{dQ}{Q} = \frac{dx}{x} \cdot \frac{a - bx}{\alpha - \beta x}$, quae aequatio resolvitur in has partes

$$\frac{dQ}{Q} = \frac{a}{\alpha} \cdot \frac{dx}{x} + \frac{a\beta - b\alpha}{\alpha} \cdot \frac{dx}{\alpha - \beta x},$$

cuius ergo integrale erit

$$lQ = \frac{a}{\alpha} lx - \frac{a\beta - b\alpha}{\alpha\beta} l(\alpha - \beta x),$$

unde deducitur

$$Q = Cx^{\frac{a}{\alpha}} (\alpha - \beta x)^{\frac{b\alpha - a\beta}{\alpha\beta}}.$$

7. Ex hoc valore pro Q invento statim patet eum evanescere casu $x = \frac{\alpha}{\beta}$, si modo fuerit $\frac{b\alpha - a\beta}{\alpha\beta} > 0$; sin autem secus eveniat, non patet, quomodo haec quantitas ullo casu evanescere queat. Invento autem hoc valore Q inde reperietur

$$dv = Cx^{\frac{a}{\alpha}} dx(\alpha - \beta x)^{\frac{b\alpha - a\beta}{\alpha\beta} - 1}$$

hincque nostrae seriei terminus indici n respondens erit

$$\int x^{n-1} dv = C\int x^{n + \frac{a}{\alpha} - 1} dx(\alpha - \beta x)^{\frac{b\alpha - a\beta}{\alpha\beta} - 1},$$

tum vero erit

$$V = Cx^{n + \frac{a}{\alpha}} (\alpha - \beta x)^{\frac{b\alpha - a\beta}{\alpha\beta}}.$$

Ubi res imprimis eo redit, ut ista quantitas praeter casum $x = 0$ insuper alio casu evanescat.

COROLLARIUM 1

8. Hic duo casus occurrunt, qui peculiarem evolutionem postulant; prior est, quo $\alpha = 0$; tum autem inchoandum erit ab aequatione $\frac{dQ}{Q} = -\frac{(a - bx)dx}{\beta xx}$, unde integrando elicitur $lQ = \frac{a}{\beta x} + \frac{b}{\beta} lx$, hincque sumendo e pro numero, cuius logarithmus hyperbolicus $= 1$, colligitur

$$Q = e^{\frac{a}{\beta x}} x^{\frac{b}{\beta}},$$

quae formula in nihilum abire nequit, nisi fiat $\frac{a}{\beta x} = -\infty$ ideoque $x = 0$, sicque non duo haberentur casus, quibus fieret $V = 0$, cum tamen duo desiderentur. Interim autem hinc fiet

$$dv = \frac{e^{\frac{a}{\beta x}} x^{\frac{b}{\beta}} dx}{-\beta x}.$$

COROLLARIUM 2

9. Alter casus peculiarem integrationem postulans erit $\beta = 0$; tum autem erit $\frac{dQ}{Q} = \frac{dx(a - bx)}{\alpha x}$, unde fit $lQ = \frac{a}{\alpha} lx - \frac{bx}{\alpha}$ ideoque $Q = x^{\frac{a}{\alpha}} e^{\frac{-bx}{\alpha}}$, quae formula casu $x = \infty$ evanescit, si modo fuerit $\frac{b}{\alpha}$ numerus positivus; sin autem $\frac{b}{\alpha}$ fuerit numerus negativus, tum Q evanescit casu $x = -\infty$. Porro vero hoc casu fiet

$$dv = \frac{x^{\frac{a}{\alpha}} e^{\frac{-bx}{\alpha}} dx}{\alpha}.$$

SCHOLION

9[a][1]). His in genere observatis aliquot casus speciales evolvamus, quibus litteris α, β et a, b certos valores tribuemus, qui ad casus iam satis cognitos perducant.

EXEMPLUM 1

9[b][1]). *Quaerantur formulae integrales, ut fiat*

$$\int x^n dv = \frac{2n - 1}{2n} \int x^{n-1} dv.$$

1) In editione principe falso numeri 8 et 9 iterantur. A. G.

Cum igitur hic esse debeat $(2n-1)\int x^{n-1}dv = 2n\int x^n dv$, erit hoc casu $\alpha = 2$ et $a = -1$, tum vero $\beta = 2$ et $b = 0$; hinc fit

$$\frac{dQ}{Q} = -\frac{dx}{2x(1-x)} = -\frac{dx}{2x} - \frac{dx}{2(1-x)},$$

inde integrando

$$lQ = -\frac{1}{2}lx + \frac{1}{2}l(1-x)$$

ideoque

$$Q = C\sqrt{\frac{1-x}{x}}, \quad \text{ergo} \quad V = Cx^n\sqrt{\frac{1-x}{x}}.\text{[1]}$$

Porro cum hic sit $dv = \frac{Qdx}{2(1-x)}$, erit

$$dv = \frac{Cdx\sqrt{\frac{1-x}{x}}}{2(1-x)} = \frac{Cdx}{2\sqrt{(x-xx)}};$$

sumto ergo $C = 2$ erit $dv = \frac{dx}{\sqrt{(x-xx)}}$ et formula nostra generalis

$$\int x^{n-1}dv = \int\frac{x^{n-1}dx}{\sqrt{(x-xx)}};$$

unde cum sit $V = x^n\sqrt{\frac{1-x}{x}}$, haec quantitas manifesto evanescit sumto $x = 1$, ita ut nostra formula, si post integrationem statuatur $x = 1$, quaesito satisfaciat. Quodsi iam ponamus $x = yy$, ista formula induet hanc formam

$$2\int\frac{y^{2n-2}dy}{\sqrt{(1-yy)}},$$

quae posito post integrationem $y = 1$ praebet hanc relationem

$$\int\frac{y^{2n}dy}{\sqrt{(1-yy)}} = \frac{2n-1}{2n}\int\frac{y^{2n-2}dy}{\sqrt{(1-yy)}},$$

1) In editione principe factor C huic formulae deest. Factores autem constantes EULERUS etiam in formulis sequentibus saepenumero neglexit. A. G.

quae continet relationes supra (§ 2) commemoratas; hinc enim fiet

$$\int\frac{yydy}{\sqrt{(1-yy)}} = \frac{1}{2}\int\frac{dy}{\sqrt{(1-yy)}},$$

$$\int\frac{y^4dy}{\sqrt{(1-yy)}} = \frac{3}{4}\int\frac{yydy}{\sqrt{(1-yy)}}$$

et

$$\int\frac{y^6dy}{\sqrt{(1-yy)}} = \frac{5}{6}\int\frac{y^4dy}{\sqrt{(1-yy)}}.$$

EXEMPLUM 2

10. *Quaerantur formulae integrales, ut fiat*

$$\int x^n dv = \frac{\alpha n-1}{\alpha n}\int x^{n-1}dv.$$

Cum igitur hic esse debeat $(\alpha n-1)\int x^{n-1}dv = \alpha n\int x^n dv$, erit hoc casu $a = -1$, $\beta = \alpha$ et $b = 0$, unde per formulas supra datas colligitur

$$Q = Cx^{\frac{-1}{\alpha}}(\alpha-\alpha x)^{\frac{-\alpha}{\alpha}} = Cx^{\frac{-1}{\alpha}}(1-x)^{\frac{+1}{\alpha}},$$

quae quantitas manifesto evanescit posito $x = 1$. Tum autem erit

$$dv = \frac{x^{\frac{-1}{\alpha}}(1-x)^{\frac{+1}{\alpha}}dx}{1-x},$$

unde formula nostra generalis erit

$$\int x^{n-1}dv = \int x^{n-\frac{1}{\alpha}-1}(1-x)^{+\frac{1}{\alpha}-1}dx = \int\frac{x^{n-\frac{1}{\alpha}-1}dx}{(1-x)^{1-\frac{1}{\alpha}}},$$

quae concinnior redditur faciendo $x = y^\alpha$; tum enim ea induet hanc formam

$$\int\frac{y^{\alpha n-2}dy}{(1-y^\alpha)^{\frac{\alpha-1}{\alpha}}},$$

ubi iterum post integrationem statui debet $y = 1$. Erit hinc

$$\int\frac{y^{\alpha n+\alpha-2}dy}{(1-y^\alpha)^{\frac{\alpha-1}{\alpha}}} = \frac{\alpha n-1}{\alpha n}\int\frac{y^{\alpha n-2}dy}{(1-y^\alpha)^{\frac{\alpha-1}{\alpha}}}$$

atque hinc orientur sequentes casus speciales

$$\int \frac{y^{2\alpha-2}dy}{(1-y^{\alpha})^{\frac{\alpha-1}{\alpha}}} = \frac{\alpha-1}{\alpha}\int \frac{y^{\alpha-2}dy}{(1-y^{\alpha})^{\frac{\alpha-1}{\alpha}}}$$

et

$$\int \frac{y^{3\alpha-2}dy}{(1-y^{\alpha})^{\frac{\alpha-1}{\alpha}}} = \frac{2\alpha-1}{2\alpha}\int \frac{y^{2\alpha-2}dy}{(1-y^{\alpha})^{\frac{\alpha-1}{\alpha}}}.$$

11. Hinc igitur si sumatur $\alpha = 1$, ut fieri debeat

$$\int x^n dv = \frac{n-1}{n}\int x^{n-1}dv,$$

formula nostra generalis iam in y expressa erit $\int y^{n-2}dy$, cuius ergo valor est $\frac{1}{n-1}y^{n-1} = \frac{1}{n-1}$, unde tota series nostrarum formularum integralium abibit in hanc

$$\frac{1}{0},\quad \frac{1}{1},\quad \frac{1}{2},\quad \frac{1}{3},\quad \frac{1}{4},\quad \frac{1}{5},\quad \frac{1}{6},\quad \frac{1}{7}\quad \text{etc.}$$

12. Sumamus etiam $\alpha = \frac{1}{2}$ et iam non amplius opus erit ad y procedere. Hoc igitur casu erit

$$Q = \frac{(1-x)^2}{xx} \quad \text{et} \quad dv = \frac{(1-x)dx}{xx},$$

unde formula nostra generalis fit

$$\int x^{n-1}dv = \int x^{n-3}(1-x)dx,$$

cuius ergo valor algebraice expressus erit

$$\frac{1}{n-2}x^{n-2} - \frac{1}{n-1}x^{n-1} = \frac{1}{(n-1)(n-2)},$$

unde series nostrarum formularum evadet

$$\frac{1}{0\cdot -1},\quad \frac{1}{0\cdot 1},\quad \frac{1}{1\cdot 2},\quad \frac{1}{2\cdot 3},\quad \frac{1}{3\cdot 4},\quad \frac{1}{4\cdot 5}\quad \text{etc.}$$

EXEMPLUM 3

13. *Quaerantur formulae integrales, ut sit*

$$\int x^n dv = n\int x^{n-1}dv.$$

Cum igitur esse debeat $n\int x^{n-1}dv = 1\int x^n dv$, erit $\alpha = 1$, $a = 0$, $b = 1$, $\beta = 0$. Cum igitur sit $\beta = 0$, casus Corollarii 2 hic locum habet indeque erit $Q = e^{-x}$ ideoque $V = e^{-x}x^n$, quae quantitas his duobus casibus evanescit $x = 0$ et $x = \infty$. Porro vero erit $dv = e^{-x}dx$ hincque formula nostra generalis fiet $\int x^{n-1}dx e^{-x}$, unde ipsi seriei termini ab initio sequenti modo se habebunt:

$$\int e^{-x}dx,\quad \int e^{-x}xdx,\quad \int e^{-x}xxdx,\quad \int e^{-x}x^3dx \quad \text{etc.},$$

quibus integratis ita, ut evanescant posito $x = 0$, tum vero posito $x = \infty$ orietur sequens series satis simplex

$$1,\quad 1,\quad 1\cdot 2,\quad 1\cdot 2\cdot 3,\quad 1\cdot 2\cdot 3\cdot 4,\quad 1\cdot 2\cdot 3\cdot 4\cdot 5 \quad \text{etc.},$$

quae est series hypergeometrica WALLISII[1]), cuius ergo terminus generalis est

$$\int x^{n-1}e^{-x}dx = 1\cdot 2\cdot 3\cdot 4\cdots(n-1).$$

14. Ope ergo huius termini generalis hanc seriem interpolare licebit. Ita si quaeratur terminus medius inter duos primos, poni debet $n = \frac{3}{2}$ ac valor huius termini erit $\int e^{-x}dx\sqrt{x}$, cuius autem valor nullo modo algebraice exprimi potest. Inveni autem singulari modo hunc ipsum terminum aequari $\frac{1}{2}\sqrt{\pi}$ denotante π peripheriam circuli, cuius diameter $= 1$, unde hic vicissim cognoscimus esse $\int e^{-x}dx\sqrt{x} = \frac{\sqrt{\pi}}{2}$, posito scilicet post integrationem $x = \infty$. Terminus autem hunc praecedens indici $\frac{1}{2}$ respondens erit $= \sqrt{\pi}$, cui ergo aequatur formula $\int \frac{e^{-x}dx}{\sqrt{x}}$. Quodsi hic ponamus $e^x = y$, ita ut posito $x = 0$

1) Vide I. WALLISII *Arithmeticam infinitorum*, Oxonii 1656, scholium adiectum ad propositionem 190; *Opera*, t. I, Oxoniae 1695, p. 466. Quo in scholio *progressio hypergeometrica* vocatur „progressio facta ex termini primi continua multiplicatione in quotlibet succedentes numeros inaequales, sive crescentes sive decrescentes; (puta 1, 2, 6, 24, etc. ex continue multiplicatis $1 \times 2 \times 3 \times 4$ etc. vel 1, $\frac{3}{2}$, $\frac{15}{8}$, $\frac{105}{48}$, etc. ex continue multiplicatis $1 \times \frac{3}{2} \times \frac{5}{4} \times \frac{7}{6}$ etc.)" A. G.

sit $y=1$, at posito $x=\infty$ fiat $y=\infty$, tum ergo ista formula $\int\frac{e^{-x}dx}{\sqrt{x}}$ abit in hanc $\int\frac{dy}{yy\sqrt{ly}}$, quae formula, si ita integretur, ut evanescat posito $y=1$, tum vero fiat $y=\infty$, praebet valorem ipsius $\sqrt{\pi}$. Si porro fiat $y=\frac{1}{z}$, erunt termini integrationis $z=1$ et $z=0$ et formula integralis erit

$$-\int\frac{dz}{\sqrt{-lz}}\begin{bmatrix}\text{a} & z=1\\ \text{ad} & z=0\end{bmatrix}=\sqrt{\pi}$$

sive permutatis terminis integrationis erit

$$\int\frac{dz}{\sqrt{-lz}}\begin{bmatrix}\text{a} & z=0\\ \text{ad} & z=1\end{bmatrix}=\sqrt{\pi},$$

quemadmodum iam olim observavi.[1])

EXEMPLUM 4

15. *Quaerantur formulae integrales, ut sit*

$$\int x^n dv=\frac{1}{n}\int x^{n-1}dv \quad \textit{sive} \quad \int x^{n-1}dv=n\int x^n dv.$$

Hic est $\alpha=0$ et $a=1$, $\beta=1$ et $b=0$; qui ergo est casus in Corollario 1 tractatus, unde colligitur fore $Q=e^{\frac{1}{x}}$ ideoque $V=x^n e^{\frac{1}{x}}$, quae formula nequidem evanescit sumto $x=0$, quandoquidem formula $e^{\frac{1}{0}}$ aequivalet infinito infinitesimae potestatis. Hic autem miro modo evenit, ut casus $x=-0$ reddat formulam $e^{\frac{-1}{0}}$ subito evanescentem. Scilicet si ω denotet quantitatem infinite parvam, erit $e^{\frac{1}{\omega}}=\infty^\infty$, tum vero repente fiet $e^{\frac{-1}{\omega}}=\frac{1}{\infty^\infty}=0$, quam ob causam formulam hinc exhibere non licet scopo nostro respondentem. Reperietur quidem $dv=-e^{\frac{1}{x}}\frac{dx}{x}$, ita ut formula nostra generalis futura sit $-\int x^{n-2}dxe^{\frac{1}{x}}$, quae autem nobis nullum usum praestare potest.

1) Vide Commentationem 19 (indicis ENESTROEMIANI): *De progressionibus transcendentibus, seu quarum termini generales algebraice dari nequeunt*, Comment. acad. sc. Petrop. **5** (1730/1), 1738, p. 36; *LEONHARDI EULERI Opera omnia*, series I, vol. 14; vide porro Commentationem 421: *Evolutio formulae integralis* $\int x^{f-1}dx(lx)^{\frac{m}{n}}$ *integratione a valore* $x=0$ *ad* $x=1$ *extensa*, Novi comment. acad. sc. Petrop. **16** (1771), 1772, p. 91; *LEONHARDI EULERI Opera omnia*, series I, vol. **17**. A. G.

16. Quodsi hic ponamus $\frac{1}{x}=y$, formula ista generalis transit in hanc $+\int\frac{e^y dy}{y^n}$. At vero nunc erit $V=\frac{e^y}{y^n}$, quae formula evanescit posito $y=-\infty$. Quomodocunque autem hanc expressionem transformemus, semper idem incommodum occurret. Interim tamen etiam hunc casum sequenti modo resolvere licebit. Sit enim seriei, quam quaerimus, primus terminus $=\omega$, ex quo per regulam praescriptam sequentes ordine ita procedent

$$\begin{matrix}1 & 2 & 3 & 4 & 5 & & n\\ \omega, & \frac{\omega}{1}, & \frac{\omega}{1\cdot 2}, & \frac{\omega}{1\cdot 2\cdot 3}, & \frac{\omega}{1\cdot 2\cdot 3\cdot 4}, & \cdots & \frac{\omega}{1\cdot 2\cdot 3\cdots(n-1)}.\end{matrix}$$

Supra autem vidimus huius formulae $1\cdot 2\cdot 3\cdot 4\cdot 5\cdots(n-1)$ valorem exprimi per hoc integrale $\int x^{n-1}e^{-x}dx$ integratione ab $x=0$ ad $x=\infty$ extensa; tantum igitur opus est, ut hanc formulam integralem in denominatorem transferamus, et seriei, quam quaerimus, terminus generalis erit

$$\frac{1}{\int x^{n-1}e^{-x}dx},$$

unde satis intelligitur negotium non per simplicem formulam integralem expediri posse, quod idem quoque tenendum est de aliis casibus, quibus quantitas V non duobus casibus evanescere potest; tum enim tantum opus est fractionem $\frac{\alpha n+a}{\beta n+b}$ invertere atque formulam integralem in denominatorem transferre.

SCHOLION

17. Nisi sit vel $\alpha=0$ vel $\beta=0$, quos casus iam expedivimus, resolutio nostri problematis semper reduci potest ad casum, quo ambae litterae α et β sunt aequales unitati. Cum enim esse debeat

$$\int x^n dv=\frac{\alpha n+a}{\beta n+b}\int x^{n-1}dv,$$

ponatur $x=\frac{\alpha y}{\beta}$ fietque

$$\frac{\alpha}{\beta}\int y^n dv=\frac{\alpha n+a}{\beta n+b}\int y^{n-1}dv,$$

quae aequatio reducitur ad hanc formam

$$\int y^n dv=\frac{n+a:\alpha}{n+b:\beta}\int y^{n-1}dv.$$

Quodsi iam nunc loco $\frac{a}{\alpha}$ scribamus a et b loco $\frac{b}{\beta}$, resolvenda erit haec formula

$$\int y^n dv = \frac{n+a}{n+b} \int y^{n-1} dv,$$

cuius resolutio, si loco x scribamus y et loco litterarum α et β unitatem, ex superiori solutione praebet primo

$$Q = Cy^a (1-y)^{b-a},$$

quod ergo evanescit posito $y = 1$, si modo fuerit $b > a$; tum autem erit ipsa formula

$$\int y^{n-1} dv = C \int y^{n+a-1} dy (1-y)^{b-a-1};$$

sin autem fuerit $b < a$, haec solutio, uti vidimus, locum habere nequit; verum hoc casu pro termino nostrae seriei assumi debet haec forma $\frac{1}{\int y^{n-1} dv}$, ita ut tum esse debeat

$$\frac{1}{\int y^n dv} = \frac{n+a}{n+b} \cdot \frac{1}{\int y^{n-1} dv}$$

sive

$$\int y^n dv = \frac{n+b}{n+a} \int y^{n-1} dv,$$

cuius resolutio permutatis litteris a et b praebet

$$Q = Cy^b (1-y)^{a-b},$$

quae iam casu $y = 1$ evanescit, si fuerit $a > b$; atque tum erit formula generalis

$$\int y^{n-1} dv = C \int y^{n+b-1} dy (1-y)^{a-b-1}.$$

Sive igitur sit $b > a$ sive $a > b$, solutio nulla amplius laborat difficultate.

18. Sin autem fuerit vel $\alpha = 0$ vel $\beta = 0$, loco alterius etiam scribi poterit unitas; unde si esse debeat

$$\int x^n dv = \frac{n+a}{b} \int x^{n-1} dv,$$

ob $\alpha = 1$ et $\beta = 0$ solutio nostra generalis dat

$$\frac{dQ}{Q} = \frac{dx}{x}(a - bx),$$

unde colligitur $Q = Cx^a e^{-bx}$, quae formula evanescit posito $x = \infty$, si modo b fuerit numerus positivus; tum autem fit terminus generalis

$$\int x^{n-1} dv = C \int x^{n+a-1} dx e^{-bx}.$$

At vero numerus b negativus esse nequit, quia alioquin conditio praescripta esset incongrua.

19. Consideremus etiam alterum casum, quo $\alpha = 0$ et $\beta = 1$ ideoque conditio praescripta

$$\int x^n dv = \frac{a}{n+b} \int x^{n-1} dv,$$

unde fit

$$\frac{dQ}{Q} = -\frac{dx}{xx}(a - bx).$$

Hinc autem pro Q orietur valor, qui praeter casum $x = 0$ evanescere non posset; quam ob causam formula generalis statui debet $\frac{1}{\int x^{n-1} dv}$, ita ut esse debeat

$$\int x^n dv = \frac{n+b}{a} \int x^{n-1} dv,$$

unde prodit

$$\frac{dQ}{Q} = \frac{dx}{x}(b - ax) \quad \text{ideoque} \quad Q = Ce^{-ax} x^b,$$

quae expressio evanescit posito $x = \infty$, quoniam a necessario debet esse numerus positivus; tum autem erit

$$dv = Ce^{-ax} x^b dx,$$

unde formula generalis seriei erit

$$\frac{1}{C \int x^{n+b-1} dx e^{-ax}}.$$

PROBLEMA 2

20. *Denotet T terminum indici n respondentem in serie, quam considerandam suscepimus, at vero T' terminum sequentem atque proponatur haec conditio adimplenda*

$$T' = \frac{(\alpha n + a)(\alpha' n + a')}{(\beta n + b)(\beta' n + b')} T.$$

SOLUTIO

Quoniam hic valores geminati occurrunt, huic conditioni commodissime satisfiet, si terminus generalis T tanquam productum ex duobus factoribus spectetur. Statuatur igitur $T = RS$ sitque terminus sequens $= R'S'$ et quaerantur formulae R et S, ut fiat

$$R' = \frac{\alpha n + a}{\beta n + b} R \quad \text{et} \quad S' = \frac{\alpha' n + a'}{\beta' n + b'} S;$$

tum enim sumendo $T = RS$ conditioni praescriptae manifesto satisfiet. Hoc igitur modo pro R et S vel huiusmodi formulae $\int x^{n-1} dv$ vel inversae $\frac{1}{\int x^{n-1} dv}$ reperientur, id quod pro solutione generali sufficit, unde rem exemplo illustremus.

EXEMPLUM

21. *Quaeratur formula generalis T, ut fiat*

$$T' = \frac{nn - cc}{nn} T.$$

Resolvamus igitur T in duos factores R et S ac statuamus

$$R' = \frac{n - c}{n} R \quad \text{et} \quad S' = \frac{n + c}{n} S.$$

Pro priore forma si statuamus $R = \int x^{n-1} dv$, ex solutione generali, ubi erit $\alpha = 1$, $a = -c$, $\beta = 1$ et $b = 0$, fiet

$$Q = Cx^{-c}(1 - x)^c,$$

quae forma manifesto evanescit posito $x = 1$; hincque quia fit

$$V = Cx^{n-c}(1 - x)^c,$$

haec forma etiam casu $x = 0$ evanescit, si modo n fuerit $> c$, id quod tuto assumi potest, quia exponentem n successive in infinitum crescere assumimus ac plerumque pro c fractiones tantum accipi solent. Hinc ergo erit

$$R = C\int x^{n-c-1}(1 - x)^{c-1} dx.$$

22. Hinc iam alter valor litterae S deduci posset scribendo tantum $-c$ loco c, tum autem non amplius fieret $Q = 0$ posito $x = 1$, quamobrem pro S formulam inversam $\frac{1}{\int x^{n-1} dv}$ assumi oportet, ut esse debeat

$$\int x^n dv = \frac{n}{n + c} \int x^{n-1} dv;$$

ubi cum sit $\alpha = 1$, $a = 0$, $\beta = 1$ et $b = c$, reperitur

$$Q = C(1 - x)^c,$$

quae forma manifesto fit $= 0$ posito $x = 1$; hinc autem prodit

$$dv = C(1 - x)^{c-1} dx,$$

ergo habebimus

$$S = \frac{1}{C\int x^{n-1}(1 - x)^{c-1} dx};$$

consequenter formula nostra generalis quaesita erit

$$T = \frac{\int x^{n-c-1}(1 - x)^{c-1} dx}{\int x^{n-1}(1 - x)^{c-1} dx}.$$

23. Quodsi ergo nostrae seriei per factores procedentis primum terminum ponamus $= A$, ipsa series erit

$$\overset{1}{A}, \quad \overset{2}{\frac{1 - cc}{1} A}, \quad \overset{3}{\frac{1 - cc}{1} \cdot \frac{4 - cc}{4} A}, \quad \overset{4}{\frac{1 - cc}{1} \cdot \frac{4 - cc}{4} \cdot \frac{9 - cc}{9} A} \quad \text{etc.};$$

unde si sumamus $c = \frac{1}{2}$, erit haec series

$$A, \quad \frac{1 \cdot 3}{2 \cdot 2} A, \quad \frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{3 \cdot 5}{4 \cdot 4} A, \quad \frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{3 \cdot 5}{4 \cdot 4} \cdot \frac{5 \cdot 7}{6 \cdot 6} A \quad \text{etc.},$$

cuius ergo terminus indici n respondens est

$$\frac{\int x^{n-\frac{3}{2}}(1 - x)^{-\frac{1}{2}} dx}{\int x^{n-1}(1 - x)^{-\frac{1}{2}} dx},$$

qui posito $x = yy$ transit in hanc formam

$$\frac{\int y^{2n-2}(1 - yy)^{-\frac{1}{2}} dy}{\int y^{2n-1}(1 - yy)^{-\frac{1}{2}} dy},$$

unde patet terminum primum fore

$$A = \int \frac{dy}{\sqrt{(1-yy)}} : \int \frac{y\,dy}{\sqrt{(1-yy)}} = \frac{\pi}{2},$$

posito scilicet post integrationem $y = 1$.

PROBLEMA 3

24. *Denotet T terminum seriei indici n respondentem sintque T' et T'' termini sequentes pro indicibus $n+1$ et $n+2$; si proponatur inter ternos terminos se insequentes talis relatio, ut sit*

$$(\alpha n + a)T = (\beta n + b)T' + (\gamma n + c)T'',$$

investigare formulam pro T, qua terminus generalis huius seriei exprimatur.

SOLUTIO

Assumatur pro T formula integralis $\int x^{n-1}dv$ huiusque integrale ita capiatur, ut evanescat posito $x = 0$, eruntque termini sequentes $T' = \int x^n dv$ et $T'' = \int x^{n+1}dv$, siquidem post integrationem variabili x certus valor determinatus tribuatur. Quamdiu autem haec quantitas x ut variabilis spectatur, ponamus esse

$$(\alpha n + a)T = (\beta n + b)T' + (\gamma n + c)T'' + x^n Q$$

ac perspicuum est Q eiusmodi functionem esse debere ipsius x, quae evanescat, si loco x valor ille determinatus substituatur, quem autem a cyphra diversum esse oportet, quoniam iam assumsimus omnes istas formulas in nihilum abire posito $x = 0$. Quodsi vero absoluto calculo huic conditioni nullo modo satisfieri poterit, id erit indicio problema nostrum hac ratione resolvi non posse, ut scilicet eius terminus generalis T per talem formulam differentialem simplicem $\int x^{n-1}dv$ exhibeatur.

25. Differentiemus nunc aequationem modo stabilitam ac divisione facta per x^{n-1} sequens prodibit aequatio

$$(\alpha n + a)dv = (\beta n + b)x\,dv + (\gamma n + c)xx\,dv + nQ\,dx + x\,dQ,$$

quae, quia termini littera n affecti seorsim se destruere debent, discerpetur in binas sequentes aequationes

$$1.\quad \alpha\,dv = \beta x\,dv + \gamma xx\,dv + Q\,dx,$$

$$2.\quad a\,dv = bx\,dv + cxx\,dv + x\,dQ,$$

ex quarum priore fit

$$dv = \frac{Q\,dx}{\alpha - \beta x - \gamma xx},$$

ex altera vero fit

$$dv = \frac{x\,dQ}{a - bx - cxx},$$

quorum valorum posterior per priorem divisus praebet

$$\frac{dQ}{Q} = \frac{dx(a - bx - cxx)}{x(\alpha - \beta x - \gamma xx)},$$

ex cuius ergo integratione valor ipsius Q elici debet, quo facto facile patebit, utrum is certo quodam casu praeter $x = 0$ evanescere possit. Imprimis autem hic notari convenit, si hoc integrale involvat huiusmodi factorem $e^{\frac{1}{x}}$, tum solutionem quoque successu esse carituram, quandoquidem posito $x = 0$ iste factor tantam involvet infiniti potestatem, ut, etiamsi per x^n multiplicetur, productum etiamnum infinitum maneat.

26. Quodsi igitur his conditionibus praescriptis satisfacere licuerit, tum invento valore litterae Q, quem ponamus fieri $= 0$ posito $x = f$, habebitur

$$dv = \frac{Q\,dx}{\alpha - \beta x - \gamma xx}$$

et formula generalis naturam seriei complectens erit

$$T = \int x^{n-1}dv = \int \frac{x^{n-1}Q\,dx}{\alpha - \beta x - \gamma xx},$$

quippe cuius integrale a termino $x = 0$ usque ad terminum $x = f$ extensum praebebit valorem termini T indici cuicunque n respondentis.

SCHOLION

27. Inventa autem tali relatione inter ternos terminos cuiuspiam seriei sibi invicem succedentes inde more solito formari poterit fractio continua, cuius valorem assignare licebit. Si enim characteres

$$T',\ T'',\ T''',\ T'''' \text{ etc.}$$

denotent ordine omnes terminos post T sequentes in infinitum, ex relationibus, quas inter se tenent, sequentes formulae deducentur. Ex relatione

$$(\alpha n+a)T=(\beta n+b)T'+(\gamma n+c)T''$$

deducitur

$$(\alpha n+a)\frac{T}{T'}=\beta n+b+\frac{(\gamma n+c)(\alpha n+\alpha+a)}{(\alpha n+\alpha+a)T':T''}.$$

Ex relatione sequente

$$(\alpha n+\alpha+a)T'=(\beta n+\beta+b)T''+(\gamma n+\gamma+c)T'''$$

deducitur

$$(\alpha n+\alpha+a)\frac{T'}{T''}=\beta n+\beta+b+\frac{(\gamma n+\gamma+c)(\alpha n+2\alpha+a)}{(\alpha n+2\alpha+a)T'':T'''}.$$

Simili modo sequentes relationes suppeditabunt

$$(\alpha n+2\alpha+a)\frac{T''}{T'''}=\beta n+2\beta+b+\frac{(\gamma n+2\gamma+c)(\alpha n+3\alpha+a)}{(\alpha n+3\alpha+a)T''':T''''},$$

$$(\alpha n+3\alpha+a)\frac{T'''}{T''''}=\beta n+3\beta+b+\frac{(\gamma n+3\gamma+c)(\alpha n+4\alpha+a)}{(\alpha n+4\alpha+a)T'''':T'''''};$$

unde manifestum est, si in prima formula continuo sequentes valores ordine substituantur, prodituram esse fractionem continuam, cuius valor aequalis erit formulae $(\alpha n+a)\frac{T}{T'}$.

28. Quodsi ergo loco n successive scribamus numeros 1, 2, 3, 4 etc., sequens problema circa fractiones continuas resolvere poterimus.

PROBLEMA 4

Proposita fractione continua huius formae

$$\beta+b+\cfrac{(\gamma+c)(2\alpha+a)}{2\beta+b+\cfrac{(2\gamma+c)(3\alpha+a)}{3\beta+b+\cfrac{(3\gamma+c)(4\alpha+a)}{4\beta+b+\cfrac{(4\gamma+c)(5\alpha+a)}{5\beta+b+\cfrac{(5\gamma+c)(6\alpha+a)}{6\beta+b+\text{ etc.}}}}}}$$

eius valorem investigare.

SOLUTIO

Consideretur in genere ista relatio inter ternas quantitates sibi succedentes T, T', T'', quae sit

$$(\alpha n+a)T=(\beta n+b)T'+(\gamma n+c)T'',$$

atque ex praecedente problemate quaeratur valor ipsius T, siquidem fieri potest, hoc modo expressus

$$T=\int x^{n-1}dv=\int\frac{x^{n-1}Qdx}{\alpha-\beta x-\gamma xx},$$

cuius integrale ab $x=0$ usque ad $x=f$ extendatur, qua formula inventa ponatur

$$\int\frac{Qdx}{\alpha-\beta x-\gamma xx}=A \quad\text{et}\quad \int\frac{xQdx}{\alpha-\beta x-\gamma xx}=B,$$

ita ut A et B sint valores ipsius T pro casibus $n=1$ et $n=2$; quibus definitis fractionis continuae propositae valor per praecedentia erit $=\frac{(\alpha+a)A}{B}$. Hanc igitur investigationem ad sequentia exempla accommodemus.

EXEMPLUM 1

29. *Investigare valorem fractionis continuae notissimae, quam olim* BROUNCKERUS[1]) *pro quadratura circuli protulit, quae est*

$$2+\cfrac{1\cdot1}{2+\cfrac{3\cdot3}{2+\cfrac{5\cdot5}{2+\text{ etc.}}}}$$

1) Hanc celebrem fractionem continuam BROUNCKERUS epistola cum WALLISIO communicaverat. Vide I. WALLISII *Arithmeticam infinitorum*, Oxonii 1656, p. 182; *Opera*, t. I, Oxoniae

Quia omnes partes integrae laevam respicientes sunt constantes $=2$, pro nostra forma generali fiet

$$\beta+b=2, \quad 2\beta+b=2, \quad 3\beta+b=2 \quad \text{etc.};$$

erit ergo $\beta=0$ et $b=2$; at pro numeratoribus sequentium fractionum, quandoquidem constant binis factoribus, erit pro factoribus prioribus

$$\gamma+c=1, \quad 2\gamma+c=3, \quad 3\gamma+c=5, \quad 4\gamma+c=7 \quad \text{etc.},$$

unde concluditur $\gamma=2$ et $c=-1$, pro alteris vero erit

$$2\alpha+a=1, \quad 3\alpha+a=3, \quad 4\alpha+a=5 \quad \text{etc.},$$

unde $\alpha=2$ et $a=-3$. Ex his autem valoribus colligimus hanc aequationem

$$\frac{dQ}{Q}=-\frac{dx(3+2x-xx)}{2x(1-xx)},$$

quae per $1+x$ depressa praebet

$$\frac{dQ}{Q}=-\frac{dx(3-x)}{2x(1-x)},$$

unde integrando fit

$$lQ=-\frac{3}{2}lx+l(1-x) \quad \text{et hinc} \quad Q=\frac{1-x}{x^{\frac{3}{2}}},$$

ex quo valore porro sequitur

$$A=\int\frac{(1-x)dx}{2x^{\frac{3}{2}}(1-xx)}=\int\frac{dx}{2x(1+x)\sqrt{x}},$$

$$B=\int\frac{(1-x)dx}{2x^{\frac{1}{2}}(1-xx)}=\int\frac{dx}{2(1+x)\sqrt{x}}.$$

1695, p. 469. Notandum est fractionem BROUNCKERIANAM apud WALLISIUM hoc modo scriptam

$$1\,\frac{1}{2}\,\frac{9}{2}\,\frac{25}{2}\,\frac{49}{2}\,\frac{81}{2} \text{ etc.}$$

valorem fractionis $\frac{4}{\pi}$ repraesentare ideoque a fractione huius exempli unitate differre.

Vide etiam L. EULERI *Introductionem in analysin infinitorum*, Lausannae 1748, t. I, p. 305; *LEONHARDI EULERI Opera omnia*, series I, vol. 8. A. G.

30. In his autem valoribus istud incommodum deprehenditur, quod prius integrale evanescens reddi nequit posito $x=0$. Hoc autem incommodum facile removeri potest, si fractionem continuam supremo membro truncemus et quaeramus valorem istius fractionis

$$2+\cfrac{3\cdot 3}{2+\cfrac{5\cdot 5}{2+\text{etc.};}}$$

qui si repertus fuerit $=s$, erit ipsius propositae valor $=b+\frac{1}{s}$. Nunc vero comparatione instituta fit quidem ut ante $\beta=0$ et $b=2$, tum vero $\gamma=2$ et $c=+1$, $\alpha=2$ et $a=-1$, unde sequitur

$$\frac{dQ}{Q}=-\frac{dx(1+2x+xx)}{2x(1-xx)}=-\frac{dx(1+x)}{2x(1-x)},$$

unde integrando fit

$$lQ=-\frac{1}{2}lx+l(1-x) \quad \text{ideoque} \quad Q=\frac{1-x}{\sqrt{x}},$$

ex quo valore iam habebimus

$$A=\int\frac{(1-x)dx}{2(1-xx)\sqrt{x}}=\frac{1}{2}\int\frac{dx}{(1+x)\sqrt{x}}$$

et

$$B=\frac{1}{2}\int\frac{dx\sqrt{x}}{1+x};$$

ubi cum sit $Q=\frac{1-x}{\sqrt{x}}$, eius valor manifesto evanescit posito $x=1$, quamobrem illa integralia a termino $x=0$ usque ad $x=1$ sunt extendenda.

31. Quo nunc haec integralia facilius eruamus, statuamus $x=zz$, ita ut termini integrationis etiamnunc sint $z=0$ et $z=1$, eritque

$$A=\int\frac{dz}{1+zz}=\text{A tang.}\,z=\frac{\pi}{4}$$

et

$$B=\int\frac{zzdz}{1+zz}=1-\frac{\pi}{4}$$

sicque habebimus $s=\frac{\pi}{4-\pi}$, quocirca ipsius fractionis BROUNCKERIANAE valor est $1+\frac{4}{\pi}$, omnino uti olim BROUNCKERUS iam invenerat.[1])

1) Sed vide notam p. 227. A. G.

EXEMPLUM 2

31[a].[1]). *Investigare valorem huius fractionis continuae* BROUNCKERIANAE[2]) *latius patentis*

$$b+\cfrac{1\cdot 1}{b+\cfrac{3\cdot 3}{b+\cfrac{5\cdot 5}{b+\text{etc.}}}}$$

Ut hic incommodum superius evitemus, omittamus membrum supremum et quaeramus

$$s=b+\cfrac{3\cdot 3}{b+\cfrac{5\cdot 5}{b+\text{etc.}}},$$

quandoquidem tum erit valor quaesitus $=b+\frac{1}{s}$. Nunc igitur erit $\beta=0$ et $b=b$, $\gamma=2$, $c=1$, $\alpha=2$ et $a=-1$, unde fit

$$\frac{dQ}{Q}=-\frac{dx(1+bx+xx)}{2x(1-xx)}$$

ac proinde

$$lQ=-\frac{1}{2}lx-\frac{b-2}{4}l(1+x)+\frac{b+2}{4}l(1-x)$$

hincque

$$Q=\frac{(1-x)^{\frac{b+2}{4}}}{(1+x)^{\frac{b-2}{4}}\sqrt{x}},$$

1) In editione principe falso numerus 31 iteratur. A. G.

2) Num vero BROUNCKERUS ipse huiusmodi fractiones continuas casu $b\gtrless 2$ tractaverit, non liquet, cum investigationes eius nonnisi ex relatione a WALLISIO in propositione 191 *Arithmeticae infinitorum* (vide notam p. 227) data noverimus. Extant autem in fine huius propositionis haec WALLISII verba: „Atque hactenus Nobilissimi Viri mentem, quanta potui brevitate simul atque perspicuitate exposui; quaeque de ipsius methodo dicenda habui breviter indicavi". Vide I. WALLIS, *Opera*, t. I, p. 476. Vide etiam, id quod EULERUS ipse ad hanc quaestionem scripsit in Commentatione 123 (indicis ENESTROEMIANI): *De fractionibus continuis observationes*, Comment. acad. sc. Petrop. 11 (1739) 1750, p. 32, imprimis p. 39—42; *LEONHARDI EULERI Opera omnia*, series I, vol. 14.

Ceterum ne WALLISIUS quidem alios casus nisi $b=4n+2$ et $b=4n$ tractavit. Vide *Arithmeticam infinitorum*, p. 182—183; *Opera*, t. I, p. 470. A. G.

quae formula manifesto fit $=0$ ponendo $x=1$, siquidem $b+2$ fuerit numerus positivus, unde fit

$$dv=\frac{(1-x)^{\frac{b-2}{4}}dx}{2(1+x)^{\frac{b+2}{4}}\sqrt{x}}.$$

Hinc autem colligetur

$$A=\frac{1}{2}\int\frac{(1-x)^{\frac{b-2}{4}}dx}{(1+x)^{\frac{b+2}{4}}\sqrt{x}}\quad\text{et}\quad B=\frac{1}{2}\int\frac{(1-x)^{\frac{b-2}{4}}dx\sqrt{x}}{(1+x)^{\frac{b+2}{4}}}$$

sive ponendo $x=zz$ habebimus

$$A=\int\frac{(1-zz)^{\frac{b-2}{4}}dz}{(1+zz)^{\frac{b+2}{4}}}\quad\text{et}\quad B=\int\frac{(1-zz)^{\frac{b-2}{4}}zzdz}{(1+zz)^{\frac{b+2}{4}}},$$

quae ambo integralia a $z=0$ usque ad $z=1$ sunt extendenda. Ex his autem valoribus A et B erit $s=\frac{A}{B}$; ipsius igitur fractionis propositae valor erit $=b+\frac{1}{s}=b+\frac{B}{A}$.

32. Quodsi hic ponatur $b=2$, prodit casus ante expositus a quadratura circuli pendens, quippe quo casu formula fit rationalis. Quando autem exponentes $\frac{b-2}{4}$ et $\frac{b+2}{4}$ non sunt numeri integri, tum litteras A et B neque per arcus circulares neque per logarithmos exprimere licet. Veluti si fuerit $b=4$, erit

$$A=\int\frac{dz\sqrt{(1-zz)}}{(1+zz)^{\frac{3}{2}}},$$

cuius valor per arcus ellipticos exhiberi posset. At si b fuerit numerus impar, hi valores multo magis evadunt transcendentes, ita ut his ipsis litteris A et B debeamus esse contenti. Contra autem si exponentes illi fiant numeri integri, totum negotium per arcus circulares expedire licebit.

33. Exponentes autem illi $\frac{b-2}{4}$ et $\frac{b+2}{4}$ erunt numeri integri, quoties fuerit b numerus huius formae

$$b=4i+2;$$

tum enim erit

$$A=\int\frac{(1-zz)^i dz}{(1+zz)^{i+1}} \quad \text{et} \quad B=\int\frac{(1-zz)^i zzdz}{(1+zz)^{i+1}};$$

quos ergo casus quomodo evolvi oporteat, operae pretium erit docere, quoniam WALLISIUS eos iam est contemplatus.[1])

34. Quoniam hoc negotium totum redit ad reductionem huiusmodi formularum integralium ad formas simpliciores, consideremus in genere formam $P=\frac{z^m}{(1+zz)^n}$, cuius differentiale sub sequentibus formis exhiberi potest:

$$1. \quad dP=\frac{mz^{m-1}dz}{(1+zz)^n}-\frac{2nz^{m+1}dz}{(1+zz)^{n+1}},$$

$$2. \quad dP=\frac{mz^{m-1}dz}{(1+zz)^{n+1}}-\frac{(2n-m)z^{m+1}dz}{(1+zz)^{n+1}},$$

$$3. \quad dP=-\frac{(2n-m)z^{m-1}dz}{(1+zz)^n}+\frac{2nz^{m-1}dz}{(1+zz)^{n+1}},$$

unde hanc triplicem reductionem integralium deducimus

$$\text{I.} \quad \int\frac{z^{m+1}dz}{(1+zz)^{n+1}}=\frac{m}{2n}\int\frac{z^{m-1}dz}{(1+zz)^n}-\frac{1}{2n}\cdot\frac{z^m}{(1+zz)^n},$$

$$\text{II.} \quad \int\frac{z^{m+1}dz}{(1+zz)^{n+1}}=\frac{m}{2n-m}\int\frac{z^{m-1}dz}{(1+zz)^{n+1}}-\frac{1}{2n-m}\cdot\frac{z^m}{(1+zz)^n},$$

$$\text{III.} \quad \int\frac{z^{m-1}dz}{(1+zz)^{n+1}}=\frac{2n-m}{2n}\int\frac{z^{m-1}dz}{(1+zz)^n}+\frac{1}{2n}\cdot\frac{z^m}{(1+zz)^n},$$

quarum reductionum ope casibus $b=4i+2$ totum negotium absolvi et ad formulam $\frac{\pi}{4}$ reduci poterit, siquidem post integrationem sumatur $z=1$.

35. Sit $i=1$ ideoque $b=6$ eritque

$$A=\int\frac{(1-zz)dz}{(1+zz)^2} \quad \text{et} \quad B=\int\frac{(1-zz)zzdz}{(1+zz)^2}.$$

Nunc igitur reperiemus per reductionem tertiam

$$\int\frac{dz}{(1+zz)^2}=\frac{1}{2}\int\frac{dz}{1+zz}+\frac{1}{2}\cdot\frac{z}{1+zz}=\frac{\pi}{8}+\frac{1}{4}$$

1) Vide notam 2 p. 230. A. G.

et per reductionem primam

$$\int\frac{zzdz}{(1+zz)^2}=\frac{1}{2}\int\frac{dz}{1+zz}-\frac{1}{2}\cdot\frac{z}{1+zz}=\frac{\pi}{8}-\frac{1}{4},$$

porro

$$\int\frac{z^4dz}{(1+zz)^2}=\frac{3}{2}\int\frac{zzdz}{1+zz}-\frac{1}{2}\cdot\frac{z^3}{1+zz}=\frac{5}{4}-\frac{3\pi}{8}.$$

Ex his iam valoribus colligitur $A=\frac{1}{2}$ et $B=\frac{\pi}{2}-\frac{3}{2}$ ideoque $\frac{B}{A}=\pi-3$, quocirca orietur ista summatio

$$3+\pi=6+\cfrac{1\cdot 1}{6+\cfrac{3\cdot 3}{6+\cfrac{5\cdot 5}{6+\cfrac{7\cdot 7}{6+\text{etc.}}}}}$$

36. Sit nunc $i=2$ et $b=10$ eritque

$$A=\int\frac{(1-zz)^2dz}{(1+zz)^3} \quad \text{et} \quad B=\int\frac{zz(1-zz)^2dz}{(1+zz)^3}.$$

Quo harum integralium valores investigemus, sequentes evolvamus formulas

$$\int\frac{dz}{(1+zz)^3}=\frac{3}{4}\int\frac{dz}{(1+zz)^2}+\frac{1}{4}\cdot\frac{z}{(1+zz)^2}=\frac{3\pi}{32}+\frac{1}{4},$$

$$\int\frac{zzdz}{(1+zz)^3}=\frac{1}{4}\int\frac{dz}{(1+zz)^2}-\frac{1}{4}\cdot\frac{z}{(1+zz)^2}=\frac{\pi}{32},$$

$$\int\frac{z^4dz}{(1+zz)^3}=\frac{3}{4}\int\frac{zzdz}{(1+zz)^2}-\frac{1}{4}\cdot\frac{z^3}{(1+zz)^2}=\frac{3\pi}{32}-\frac{1}{4},$$

$$\int\frac{z^6dz}{(1+zz)^3}=\frac{5}{4}\int\frac{z^4dz}{(1+zz)^2}-\frac{1}{4}\cdot\frac{z^5}{(1+zz)^2}=\frac{3}{2}-\frac{15\pi}{32}.$$

Ex quibus iam valoribus deducitur $A=\frac{\pi}{8}$ et $B=2-\frac{5\pi}{8}$ ideoque $\frac{B}{A}=\frac{16-5\pi}{\pi}$, unde emergit sequens summatio

$$\frac{5\pi+16}{\pi}=10+\cfrac{1\cdot 1}{10+\cfrac{3\cdot 3}{10+\cfrac{5\cdot 5}{10+\text{etc.}}}}$$

37. Si b esset numerus negativus, investigatio nulla prorsus laboraret difficultate. Si enim in genere fuerit

$$s = -a + \cfrac{\alpha}{-b + \cfrac{\beta}{-c + \cfrac{\gamma}{-d + \cfrac{\delta}{-e + \text{etc.},}}}}$$

semper erit

$$-s = a + \cfrac{\alpha}{b + \cfrac{\beta}{c + \cfrac{\gamma}{d + \cfrac{\delta}{e + \text{etc.},}}}}$$

unde, si habeatur valor istius expressionis, idem negative sumtus dabit valorem illius.

EXEMPLUM 3

38. *Proposita sit fractio continua, cuius valorem investigari oporteat, ista*

$$1 + \cfrac{1 \cdot 1}{3 + \cfrac{3 \cdot 3}{5 + \cfrac{5 \cdot 5}{7 + \cfrac{7 \cdot 7}{9 + \text{etc.}}}}}$$

Quo fractiones supra [§ 28] allegatas [adhibeamus,] omisso membro supremo sit

$$s = 3 + \cfrac{3 \cdot 3}{5 + \cfrac{5 \cdot 5}{7 + \cfrac{7 \cdot 7}{9 + \text{etc.}}}}$$

eritque $\beta + b = 3$, $2\beta + b = 5$ ideoque $\beta = 2$ et $b = 1$, tum vero ut ante $\alpha = 2$, $a = -1$, $\gamma = 2$ et $c = +1$; invento autem s erit valor quaesitus $= 1 + \frac{1}{s}$. Nunc igitur habebimus

$$\frac{dQ}{Q} = -\frac{dx(1 + x + xx)}{2x(1 - x - xx)}.$$

Est vero

$$\frac{1 + x + xx}{x(1 - x - xx)} = \frac{1}{x} + \frac{2 + 2x}{1 - x - xx},$$

unde fit

$$lQ = -\frac{1}{2}lx - \int \frac{dx(1 + x)}{1 - x - xx}.$$

Porro vero pro formula $\int \frac{dx(1+x)}{1-x-xx}$ invenienda statuamus denominatorem

$$1 - x - xx = (1 - fx)(1 - gx)$$

eritque $f + g = 1$ et $fg = -1$, unde fit

$$f = \frac{1 + \sqrt{5}}{2} \quad \text{et} \quad g = \frac{1 - \sqrt{5}}{2}.$$

Nunc statuatur

$$\frac{1 + x}{1 - x - xx} = \frac{\mathfrak{A}}{1 - fx} + \frac{\mathfrak{B}}{1 - gx},$$

unde reperietur

$$\mathfrak{A} = \frac{1 + f}{f - g} \quad \text{et} \quad \mathfrak{B} = -\frac{1 + g}{f - g},$$

sive substitutis pro f et g valoribus supra datis erit

$$\mathfrak{A} = \frac{\sqrt{5} + 3}{2\sqrt{5}} \quad \text{et} \quad \mathfrak{B} = \frac{\sqrt{5} - 3}{2\sqrt{5}},$$

quibus inventis erit

$$\int \frac{dx(1 + x)}{1 - x - xx} = -\frac{\mathfrak{A}}{f} l(1 - fx) - \frac{\mathfrak{B}}{g} l(1 - gx)$$

$$= -\frac{1 + \sqrt{5}}{2\sqrt{5}} l(1 - fx) - \frac{\sqrt{5} - 1}{2\sqrt{5}} l(1 - gx),$$

quocirca fiet

$$lQ = -\frac{1}{2}lx + \frac{\sqrt{5} + 1}{2\sqrt{5}} l(1 - fx) + \frac{\sqrt{5} - 1}{2\sqrt{5}} l(1 - gx),$$

consequenter

$$Q = \frac{(1 - fx)^{\frac{\sqrt{5}+1}{2\sqrt{5}}} (1 - gx)^{\frac{\sqrt{5}-1}{2\sqrt{5}}}}{\sqrt{x}},$$

qui valor duobus casibus evanescit, altero, quo

$$x = \frac{1}{f} = \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2},$$

altero vero, quo

$$x = \frac{1}{g} = -\frac{1+\sqrt{5}}{2};$$

utrovis autem utamur, res eodem redibit.

39. Ex hoc autem valore habebimus

$$A = \int \frac{Q\,dx}{1-x-xx} \quad \text{et} \quad B = \int \frac{Qx\,dx}{1-x-xx},$$

unde porro deducitur

$$s = (\alpha + a)\frac{A}{B} = \frac{A}{B};$$

hinc propositae fractionis summa erit $1 + \frac{B}{A}$. Hinc autem nihil ulterius concludere licet ob formulas differentiales non solum irrationales, sed etiam vere transcendentes ob exponentes surdos.

EXEMPLUM 4

40. *Proposita sit haec fractio continua*

$$b + \cfrac{1\cdot 1}{b + \cfrac{2\cdot 2}{b + \cfrac{3\cdot 3}{b + \cfrac{4\cdot 4}{b + \text{etc.}}}}},$$

ubi est $\beta = 0$, $b = b$.

Nunc consideremus hanc formam

$$s = b + \cfrac{2\cdot 2}{b + \cfrac{3\cdot 3}{b + \text{etc.}}},$$

quippe quo valore invento quaesitus erit $= b + \frac{1}{s}$. Habebimus igitur

$\gamma + c = 2$, $2\gamma + c = 3$ ideoque $\gamma = 1$ et $c = 1$, deinde erit $\alpha = \gamma = 1$, $a = 0$ et $c = 1$. Hinc igitur colligimus

$$\frac{dQ}{Q} = -\frac{dx(bx+xx)}{x(1-xx)} = -\frac{dx(b+x)}{1-xx}$$

ideoque

$$lQ = -\frac{b}{2}\,l\frac{1+x}{1-x} + \frac{1}{2}\,l(1-xx)$$

hincque

$$Q = \frac{(1-x)^{\frac{b}{2}}\sqrt{(1-xx)}}{(1+x)^{\frac{b}{2}}} = \frac{(1-x)^{\frac{b+1}{2}}}{(1+x)^{\frac{b-1}{2}}},$$

quae quantitas manifesto evanescit posito $x = 1$. Hinc igitur fiet

$$A = \int \frac{Q\,dx}{1-xx} = \int \frac{(1-x)^{\frac{b+1}{2}}\,dx}{(1+x)^{\frac{b-1}{2}}(1-xx)} = \int \frac{(1-x)^{\frac{b-1}{2}}\,dx}{(1+x)^{\frac{b+1}{2}}}$$

et

$$B = \int \frac{x(1-x)^{\frac{b-1}{2}}\,dx}{(1+x)^{\frac{b+1}{2}}},$$

tum autem erit $s = (\alpha + a)\frac{A}{B} = \frac{A}{B}$ ideoque summa quaesita $b + \frac{B}{A}$.

41. Percurramus nunc casus praecipuos ac primo sit $b = 1$ eritque

$$A = \int \frac{dx}{1+x} = l(1+x) = l2 \quad \text{et} \quad B = \int \frac{x\,dx}{1+x} = x - \int \frac{dx}{1+x} = 1 - l2$$

ideoque $b + \frac{B}{A} = \frac{1}{l2}$; ergo hinc prodibit ista summatio

$$\frac{1}{l2} = 1 + \cfrac{1\cdot 1}{1 + \cfrac{2\cdot 2}{1 + \cfrac{3\cdot 3}{1 + \text{etc.}}}}$$

42. Sit nunc $b = 2$ eritque

$$A = \int \frac{dx\sqrt{(1-x)}}{(1+x)^{\frac{3}{2}}} \quad \text{et} \quad B = \int \frac{x\,dx\sqrt{(1-x)}}{(1+x)^{\frac{3}{2}}}.$$

Ad has formulas rationales reddendas statuamus

$$\frac{\sqrt{(1-x)}}{\sqrt{(1+x)}} = z$$

eritque $x = \frac{1-zz}{1+zz}$, unde terminis integrationis $x = 0$ et $x = 1$ respondebunt $z = 1$ et $z = 0$; tum vero erit

$$1 + x = \frac{2}{1+zz} \quad \text{et} \quad dx = -\frac{4z\,dz}{(1+zz)^2}$$

hincque colligitur

$$A = -2\int\frac{zz\,dz}{1+zz} = -2z + 2\,\text{A tang.}\, z = 2 - \frac{\pi}{2},$$

porro fit

$$B = -2\int\frac{zz\,dz}{(1+zz)^2} + 2\int\frac{z^4\,dz}{(1+zz)^2}.$$

Per reductiones igitur supra (§ 35) monstratas, si hic scilicet terminos integrationis $z = 1$ et $z = 0$ permutemus, ut habeamus

$$B = +2\int\frac{zz\,dz}{(1+zz)^2} - 2\int\frac{z^4\,dz}{(1+zz)^2},$$

erit

$$B = 2\left(\frac{\pi}{8} - \frac{1}{4}\right) - 2\left(\frac{5}{4} - \frac{3\pi}{8}\right) = \pi - 3,$$

unde sequitur ista summatio

$$\frac{2}{4-\pi} = 2 + \cfrac{1\cdot 1}{2 + \cfrac{2\cdot 2}{2 + \cfrac{3\cdot 3}{2 + \cfrac{4\cdot 4}{2 + \text{etc.},}}}}$$

quae Brounckerianae simplicitate nihil cedit.

43. Si ponamus $b = 0$, fractio continua abit in sequens continuum productum

$$\frac{1\cdot 1}{2\cdot 2}\cdot\frac{3\cdot 3}{4\cdot 4}\cdot\frac{5\cdot 5}{6\cdot 6}\cdot\frac{7\cdot 7}{8\cdot 8}\cdot\text{etc.};$$

hoc autem casu fit

$$A = \int\frac{dx}{\sqrt{(1-xx)}} = \frac{\pi}{2} \quad \text{et} \quad B = \int\frac{x\,dx}{\sqrt{(1-xx)}} = 1,$$

unde istius producti valor colligitur $\frac{2}{\pi}$, id quod egregie convenit cum iam dudum cognitis, quandoquidem hoc productum est ipsa progressio Wallisiana.[1])

EXEMPLUM 5

44. *Proposita sit haec fractio continua, ubi $\beta = 0$, $b = b$ et numeratores numeri trigonales,*

$$b + \cfrac{1}{b + \cfrac{3}{b + \cfrac{6}{b + \cfrac{10}{b + \text{etc.}}}}}$$

Omisso supremo membro statuamus

$$s = b + \cfrac{3}{b + \cfrac{6}{b + \text{etc.}}}$$

et primo numeratores per producta repraesentemus hoc modo

$$3 = 2\cdot\frac{3}{2}, \quad 6 = 3\cdot\frac{4}{2}, \quad 10 = 4\cdot\frac{5}{2},$$

quorum priores comparentur cum formulis $\gamma + c$, $2\gamma + c$, $3\gamma + c$, posteriores vero cum formulis $2\alpha + a$, $3\alpha + a$, $4\alpha + a$, eritque $\gamma = 1$, $c = 1$, $\alpha = \frac{1}{2}$, $a = \frac{1}{2}$, unde erit

$$\frac{dQ}{Q} = \frac{dx(\frac{1}{2} - bx - xx)}{x(\frac{1}{2} - xx)} = \frac{dx(1 - 2bx - 2xx)}{x(1 - 2xx)}$$

sive

$$\frac{dQ}{Q} = \frac{dx}{x} - \frac{2b\,dx}{1 - 2xx},$$

1) Notandum autem est progressionem Wallisianam hanc formam habere

$$\frac{3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdot\ 9\cdot\text{etc.}}{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdot 10\cdot\text{etc.}} \quad \text{seu} \quad \frac{9\cdot 25\cdot 49\cdot 81\cdot\text{etc.}}{8\cdot 24\cdot 48\cdot 80\cdot\text{etc.}}$$

neque valorem $\frac{2}{\pi}$ sed $\frac{4}{\pi}$ repraesentare. Vide Wallisii *Arithmeticam infinitorum*, p. 179; *Opera*, t. I, p. 467—469. A. G.

cuius integrale est

$$lQ = lx - \frac{b}{\sqrt{2}}\, l\frac{1+x\sqrt{2}}{1-x\sqrt{2}},$$

ergo

$$Q = \frac{x(1-x\sqrt{2})^{\frac{b}{\sqrt{2}}}}{(1+x\sqrt{2})^{\frac{b}{\sqrt{2}}}},$$

quae formula evanescit casu $x = \frac{1}{\sqrt{2}}$. Hinc igitur erit

$$dv = \frac{2x(1-x\sqrt{2})^{\frac{b}{\sqrt{2}}}dx}{(1-2xx)(1+x\sqrt{2})^{\frac{b}{\sqrt{2}}}}.$$

Sit $\frac{b}{\sqrt{2}} = \lambda$ eritque

$$A = 2\int\frac{x(1-x\sqrt{2})^{\lambda}dx}{(1-2xx)(1+x\sqrt{2})^{\lambda}} = 2\int\frac{x(1-x\sqrt{2})^{\lambda-1}dx}{(1+x\sqrt{2})^{\lambda+1}}$$

et

$$B = 2\int\frac{xx(1-x\sqrt{2})^{\lambda-1}dx}{(1+x\sqrt{2})^{\lambda+1}},$$

ubi post integrationem statuitur $x = \frac{1}{\sqrt{2}}$; tum autem fit $s = \frac{A}{B}$ hincque valor fractionis propositae $= b + \frac{B}{A}$.

45. Nisi igitur fuerit $\lambda = \frac{b}{\sqrt{2}}$ numerus rationalis, hos valores commode assignare non licet. Sit igitur $b = \sqrt{2}$ sive $\lambda = 1$ eritque

$$A = 2\int\frac{xdx}{(1+x\sqrt{2})^2} \quad \text{et} \quad B = 2\int\frac{xxdx}{(1+x\sqrt{2})^2}.$$

Hinc integrando colligitur

$$A = l(1+x\sqrt{2}) - \frac{x\sqrt{2}}{1+x\sqrt{2}}$$

ideoque posito $x\sqrt{2} = 1$ fiet $A = l2 - \frac{1}{2}$; tum vero reperitur

$$B = \frac{3}{2\sqrt{2}} - \sqrt{2}\cdot l2,$$

quare ob $b = \sqrt{2}$ erit $b + \frac{B}{A} = \frac{1}{\sqrt{2}(2l2-1)}$, unde sequitur haec summatio

$$\frac{1}{\sqrt{2}(2l2-1)} = \sqrt{2} + \cfrac{1}{\sqrt{2} + \cfrac{3}{\sqrt{2} + \cfrac{6}{\sqrt{2} + \text{etc.}}}}$$

SCHOLION

46. Fractiones autem continuae, ad quas plerumque calculo numerico deducimur, huiusmodi formam habere solent

$$a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e + \text{etc.}}}}},$$

ubi omnes numeratores sunt unitates, denominatores vero a, b, c, d, e etc. numeri integri. Verum ope nostrae methodi difficulter talium formarum valores eruere licet, etiamsi numeri a, b, c, d, e progressionem arithmeticam constituant, id quod sequenti exemplo ostendamus.

EXEMPLUM

47. *Proposita sit ista fractio continua*

$$\beta + b + \cfrac{1}{2\beta + b + \cfrac{1}{3\beta + b + \cfrac{1}{4\beta + b + \cfrac{1}{5\beta + b + \text{etc.},}}}}$$

ubi $\alpha = 0$, $\gamma = 0$, $a = 1$, $c = 1$.

Hinc fit

$$\frac{dQ}{Q} = -\frac{dx(1-bx-xx)}{\beta xx},$$

unde

$$lQ = \frac{1}{\beta x} + \frac{b}{\beta}lx + \frac{x}{\beta} \quad \text{et} \quad Q = e^{\frac{1+xx}{\beta x}}x^{\frac{b}{\beta}},$$

quae autem expressio nullo casu evanescere potest, etiamsi per x^n multiplicetur, siquidem β fuerit numerus positivus. Verum si pro β sumamus numeros negativos, puta $\beta = -m$, tum valor $Q = x^{\frac{-b}{m}} e^{\frac{-(1+xx)}{mx}}$ manifesto evanescit, tam si $x = 0$ quam si $x = \infty$. Hinc autem erit

$$dv = \frac{x^{\frac{-b}{m}} e^{\frac{-1-xx}{mx}} dx}{mxx},$$

quamobrem habebimus

$$A = \frac{1}{m}\int \frac{dx}{x^{2+\frac{b}{m}} e^{\frac{1+xx}{mx}}} \quad \text{et} \quad B = \frac{1}{m}\int \frac{dx}{x^{1+\frac{b}{m}} e^{\frac{1+xx}{mx}}}.$$

His valoribus inventis formula $\frac{A}{B}$ exprimet summam huius fractionis continuae

$$-m+b+\cfrac{1}{-2m+b+\cfrac{1}{-3m+b+\cfrac{1}{-4m+b+\cfrac{1}{-5m+b+\text{etc.}}}}}$$

quamobrem formula illa negative sumta $-\frac{A}{B}$ exprimet valorem huius fractionis continuae

$$m-b+\cfrac{1}{2m-b+\cfrac{1}{3m-b+\cfrac{1}{4m-b+\text{etc.}}}}$$

quem igitur assignare liceret, si modo formulae integrales A et B expediri et a termino $x = 0$ ad $x = \infty$ extendi possent. Verum istae formulae ita sunt comparatae, ut earum integratio nullo plane casu per quantitates cognitas exprimi queat, quod tamen non impedit, quominus fractio $\frac{A}{B}$ valores satis cognitos involvere queat, etiamsi eos nullo adhuc modo assignare valeamus.

48[1]). Talium autem fractionum continuarum mihi quidem binae sequentes innotuere, quarum valores commode exhibere licet:

1) In editione principe huic paragrapho falso numerus 49 inscribitur. A. G.

$$n+\cfrac{1}{3n+\cfrac{1}{5n+\cfrac{1}{7n+\cfrac{1}{9n+\text{etc.}}}}} = \frac{e^{\frac{2}{n}}+1}{e^{\frac{2}{n}}-1} \text{ [1])}$$

et

$$n-\cfrac{1}{3n-\cfrac{1}{5n-\cfrac{1}{7n-\cfrac{1}{9n-\text{etc.}}}}} = \cot.\frac{1}{n}.$$

Harum fractionum prior cum formulis postremi exempli collata praebet $m - b = n$, $2m - b = 3n$ ideoque $m = 2n$ et $b = n$, unde fit

$$A = \frac{1}{2n}\int \frac{dx}{x^{\frac{5}{2}} e^{\frac{1+xx}{2nx}}} \quad \text{et} \quad B = \frac{1}{2n}\int \frac{dx}{x^{\frac{3}{2}} e^{\frac{1+xx}{2nx}}},$$

unde iam discimus, si hae duae formulae integrentur a termino $x = 0$ usque ad terminum $x = \infty$, tum fore

$$\frac{A}{B} = \frac{1+e^{\frac{2}{n}}}{1-e^{\frac{2}{n}}},$$

quanquam nulla adhuc via analytica patet hanc convenientiam demonstrandi.

1) Editio princeps (atque etiam editiones 594a, 594b, 594 A indicis ENESTROEMIANI): $\frac{e^{\frac{2}{n}}}{e^{\frac{2}{n}}-1}$. Correctam autem formulam EULERUS ipse iam dederat in Commentatione 71 (indicis ENESTROEMIANI): *De fractionibus continuis*, Comment. acad. sc. Petrop. 9 (1737), 1744, p. 98 (vide ibi secundam formulam § 30, ubi sufficit loco s scribere $\frac{n}{2}$); *LEONHARDI EULERI Opera omnia*, series I, vol. 14. Vide etiam EULERI Commentationem 595 (indicis ENESTROEMIANI): *Summatio fractionis continuae, cuius indices progressionem arithmeticam constituunt, dum numeratores omnes sunt unitates; ubi simul resolutio aequationis RICCATIANAE per huiusmodi fractiones docetur, Opuscula analytica* 2, 1785, p. 217; *LEONHARDI EULERI Opera omnia*, series I, vol. 23. Statim in § 1 huius Commentationis 595 invenitur correcta formula $\frac{e^{\frac{2}{n}}+1}{e^{\frac{2}{n}}-1}$ atque praeterea etiam formula pro $\cot.\frac{1}{n}$, „quarum quidem altera ex altera facile deducitur, si loco n scribatur $n\sqrt{-1}$". A. G.