OBSERVATIONES CIRCA INTEGRALIA FORMULARUM $\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1}$ POSITO POST INTEGRATIONEM $x=1$

Commentatio 321 indicis ENESTROEMIANI
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1. Formulam integralem

$$\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1}$$

seu hoc modo expressam

$$\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}$$

hic consideraturus assumo exponentes n, p et q esse numeros integros positivos, quandoquidem, si tales non essent, facile ad hanc formam perduci possent. Deinde huius formulae integrale non in genere hic perpendere constitui, sed eius tantum valorem, quem recipit, si post integrationem statuatur $x=1$, postquam scilicet integratio ita fuerit instituta, ut integrale evanescat posito $x=0$. Primum enim nullum est dubium, quin hoc casu $x=1$ integrale multo simplicius exprimatur; ac praeterea quoties in Analysi ad huiusmodi formulas pervenitur, plerumque non tam integrale indefinitum pro quocunque valore ipsius x quam definitum valori $x=1$ utpote praecipuo desiderari solet.

2. Constat autem casu, quo post integrationem ponitur $x=1$, integrale $\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}$ hoc modo per productum infinitorum factorum exprimi, ut sit

$$\frac{p+q}{pq}\cdot\frac{n(p+q+n)}{(p+n)(q+n)}\cdot\frac{2n(p+q+2n)}{(p+2n)(q+2n)}\cdot\frac{3n(p+q+3n)}{(p+3n)(q+3n)}\cdot\text{etc.},$$

cuius quidem primus factor $\frac{p+q}{pq}$ non legi sequentium adstringitur. Hoc tamen non obstante perspicuum est exponentes p et q inter se esse commutabiles, ita ut sit

$$\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}=\int \frac{x^{q-1}dx}{\sqrt[n]{(1-x^n)^{n-p}}},$$

quae quidem aequalitas etiam facile per se ostenditur. Verum productum istud infinitum nos ad alia multo maiora perducet, quibus haec integralia magis illustrabuntur.

3. Ut autem brevitati in scribendo consulam neque opus habeam scripturam huius formulae $\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}$ toties repetere, pro quovis exponente n eius loco scribam

$$\left(\frac{p}{q}\right),$$

ita ut $\left(\frac{p}{q}\right)$ denotet valorem formulae integralis $\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}$ casu, quo post integrationem ponitur $x=1$. Et quoniam vidimus esse hoc casu

$$\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}=\int \frac{x^{q-1}dx}{\sqrt[n]{(1-x^n)^{n-p}}},$$

manifestum est fore

$$\left(\frac{p}{q}\right)=\left(\frac{q}{p}\right),$$

ita ut pro quovis valore exponentis n hae expressiones $\left(\frac{p}{q}\right)$ et $\left(\frac{q}{p}\right)$ eandem significent quantitatem. Ita si fuerit exempli gratia $n=4$, erit

$$\left(\frac{3}{2}\right)=\left(\frac{2}{3}\right)=\int \frac{x^2dx}{\sqrt[4]{(1-x^4)^2}}=\int \frac{xdx}{\sqrt[4]{(1-x^4)}}.$$

Per productum autem infinitum habebitur

$$\left(\frac{3}{2}\right)=\left(\frac{2}{3}\right)=\frac{5}{2\cdot 3}\cdot\frac{4\cdot 9}{6\cdot 7}\cdot\frac{8\cdot 13}{10\cdot 11}\cdot\frac{12\cdot 17}{14\cdot 15}\cdot\text{etc.}$$

4. Iam primum observo, si exponentes p et q fuerint maiores exponente n, formulam integralem semper ad aliam reduci posse, in qua hi exponentes infra n deprimantur. Cum enim sit

$$\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}=\frac{p-n}{p+q-n}\int\frac{x^{p-n-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}},$$

erit recepto hic scribendi more

$$\left(\frac{p}{q}\right)=\frac{p-n}{p+q-n}\left(\frac{p-n}{q}\right),$$

quo, si fuerit $p>n$, formula ad aliam, in qua exponens p minor sit quam n, revocatur, quod etiam ob commutabilitatem de altero exponente q est tenendum. Quamobrem nobis has formulas examinaturis sufficiet pro quovis exponente n exponentes p et q ipso n minores accipere, quoniam his expeditis omnes casus, quibus maiores habituri essent valores, eo reduci possunt.

5. Statim autem patet casus, quibus est vel $p=n$ vel $q=n$, absolute seu algebraice esse integrabiles. Si enim fuerit $q=n$, ob

$$\left(\frac{p}{n}\right)=\int x^{p-1}dx=\frac{x^p}{p}$$

posito $x=1$ erit $\left(\frac{p}{n}\right)=\frac{1}{p}$ similique modo $\left(\frac{n}{q}\right)=\frac{1}{q}$. Atque hi soli sunt casus, quibus integrale nostrae formulae absolute exhiberi potest, si quidem p et q exponentem n non excedant. Reliquis casibus omnibus integratio vel quadraturam circuli vel adeo altiores quadraturas implicabit, quas hic accuratius perpendere animus est. Post eas igitur formulas $\left(\frac{p}{n}\right)$ seu $\left(\frac{n}{p}\right)$, quarum valor absolute est $=\frac{1}{p}$, veniunt eae, quarum valor per solam circuli qua-

draturam exprimitur; tum vero sequentur eae, quae altiorem quandam quadraturam postulant, atque has altiores quadraturas tam ad simplicissimam formam quam ad minimum numerum revocare conabor.

6. Cum numeri p et q exponente n minores ponantur, eae formulae $\left(\frac{p}{q}\right)$ per solam circuli quadraturam integrabiles existunt, in quibus est $p+q=n$. Sit enim $q=n-p$ et formula nostra

$$\left(\frac{p}{n-p}\right)=\left(\frac{n-p}{p}\right)=\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^p}}=\int\frac{x^{q-1}dx}{\sqrt[n]{(1-x^n)^q}}$$

hoc producto infinito exprimetur

$$\frac{n}{p(n-p)}\cdot\frac{n\cdot 2n}{(n+p)(2n-p)}\cdot\frac{2n\cdot 3n}{(2n+p)(3n-p)}\cdot\frac{3n\cdot 4n}{(3n+p)(4n-p)}\cdot\text{etc.},$$

quod hoc modo repraesentatum

$$\frac{1}{p}\cdot\frac{nn}{nn-pp}\cdot\frac{4nn}{4nn-pp}\cdot\frac{9nn}{9nn-pp}\cdot\text{etc.}$$

congruit cum eo producto, quo sinus angulorum expressi. Quare si π sumatur ad semicircumferentiam circuli, cuius radius sit $=1$, simulque mensuram duorum angulorum rectorum exhibeat, erit

$$\left(\frac{p}{n-p}\right)=\left(\frac{n-p}{p}\right)=\frac{\pi}{n\sin.\frac{p\pi}{n}}=\frac{\pi}{n\sin.\frac{q\pi}{n}}.$$

7. Ceteris casibus, quibus neque $p=n$ neque $q=n$ neque $p+q=n$, integrale etiam neque absolute neque per quadraturam circuli exhiberi potest, sed aliam quandam altiorem quadraturam complectitur. Neque vero singuli casus diversi peculiarem huiusmodi quadraturam exigunt, sed plures dantur reductiones, quibus diversas formulas inter se comparare licet. Hae autem reductiones derivantur ex producto infinito supra exhibito; cum enim sit

$$\left(\frac{p}{q}\right)=\frac{p+q}{pq}\cdot\frac{n(p+q+n)}{(p+n)(q+n)}\cdot\frac{2n(p+q+2n)}{(p+2n)(q+2n)}\cdot\text{etc.},$$

erit simili modo

$$\left(\frac{p+q}{r}\right)=\frac{p+q+r}{(p+q)r}\cdot\frac{n(p+q+r+n)}{(p+q+n)(r+n)}\cdot\frac{2n(p+q+r+2n)}{(p+q+2n)(r+2n)}\cdot\text{etc.},$$

quibus in se invicem ductis obtinetur

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\frac{p+q+r}{pqr}\cdot\frac{nn(p+q+r+n)}{(p+n)(q+n)(r+n)}\cdot\frac{4nn(p+q+r+2n)}{(p+2n)(q+2n)(r+2n)}\cdot\text{etc.},$$

ubi ternae quantitates p, q, r sunt inter se permutabiles.

8. Hinc ergo permutandis his quantitatibus p, q, r consequimur sequentes reductiones

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right)=\left(\frac{q}{r}\right)\left(\frac{q+r}{p}\right),$$

unde ex datis aliquot formulis plures aliae determinari possunt. Veluti si sit $q+r=n$ seu $r=n-q$, ob

$$\left(\frac{q+r}{p}\right)=\frac{1}{p} \quad \text{et} \quad \left(\frac{q}{r}\right)=\frac{\pi}{n\sin.\frac{q\pi}{n}}$$

erit

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{n-q}\right)=\frac{\pi}{np\sin.\frac{q\pi}{n}}$$

nec non

$$\left(\frac{p}{n-q}\right)\left(\frac{n+p-q}{q}\right)=\frac{\pi}{np\sin.\frac{q\pi}{n}}.$$

Deinde si sit $p+q+r=n$ seu $r=n-p-q$, erit

$$\frac{\pi}{n\sin.\frac{r\pi}{n}}\left(\frac{p}{q}\right)=\frac{\pi}{n\sin.\frac{q\pi}{n}}\left(\frac{p}{r}\right)=\frac{\pi}{n\sin.\frac{p\pi}{n}}\left(\frac{q}{r}\right),$$

unde insignes reductiones aliarum ad alias oriuntur, quibus multitudo quadraturarum ad nostrum scopum necessariarum vehementer diminuitur.

9. Praeterea vero pro p, q, r numeris determinatis assumendis sequentes adipiscimur productorum ex binis formulis aequalitates

$$\left(\frac{1}{1}\right)\left(\frac{2}{2}\right)=\left(\frac{2}{1}\right)\left(\frac{3}{1}\right),$$
$$\left(\frac{1}{1}\right)\left(\frac{3}{2}\right)=\left(\frac{3}{1}\right)\left(\frac{4}{1}\right),$$
$$\left(\frac{2}{1}\right)\left(\frac{3}{3}\right)=\left(\frac{3}{1}\right)\left(\frac{4}{2}\right)=\left(\frac{3}{2}\right)\left(\frac{5}{1}\right),$$
$$\left(\frac{2}{2}\right)\left(\frac{4}{3}\right)=\left(\frac{3}{2}\right)\left(\frac{5}{2}\right),$$
$$\left(\frac{3}{1}\right)\left(\frac{4}{3}\right)=\left(\frac{3}{3}\right)\left(\frac{6}{1}\right),$$
$$\left(\frac{3}{2}\right)\left(\frac{5}{3}\right)=\left(\frac{3}{3}\right)\left(\frac{6}{2}\right),$$
$$\left(\frac{2}{2}\right)\left(\frac{4}{4}\right)=\left(\frac{4}{2}\right)\left(\frac{6}{2}\right),$$
$$\left(\frac{3}{1}\right)\left(\frac{4}{4}\right)=\left(\frac{4}{1}\right)\left(\frac{5}{3}\right)=\left(\frac{4}{3}\right)\left(\frac{7}{1}\right),$$
$$\left(\frac{2}{1}\right)\left(\frac{5}{3}\right)=\left(\frac{5}{1}\right)\left(\frac{6}{2}\right)=\left(\frac{5}{2}\right)\left(\frac{7}{1}\right),$$
$$\left(\frac{1}{1}\right)\left(\frac{6}{2}\right)=\left(\frac{6}{1}\right)\left(\frac{7}{1}\right)$$

etc.,

ubi quidem plures occurrunt, quae iam in reliquis continentur.

10. His quasi principiis praemissis formulam generalem $\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}$, in qua numeros p et q exponentem n non superare assumo, in classes ex exponente n petitas distinguam, ita ut valores $n=1$, $n=2$, $n=3$, $n=4$ etc. classes primam, secundam, tertiam etc. sint praebituri.

Ac prima quidem classis, qua $n=1$, unicam formulam complectitur $\left(\frac{1}{1}\right)$, cuius valor est $=1$. Secunda classis, qua $n=2$, has formulas $\left(\frac{1}{1}\right)$, $\left(\frac{2}{1}\right)$ et $\left(\frac{2}{2}\right)$ continet, quarum evolutio per se est manifesta. Tertia classis, qua $n=3$, has habet

$$\left(\frac{1}{1}\right),\ \left(\frac{2}{1}\right),\ \left(\frac{3}{1}\right),\ \left(\frac{2}{2}\right),\ \left(\frac{3}{2}\right),\ \left(\frac{3}{3}\right).$$

Quarta vero classis, qua $n=4$, istas

$$\left(\frac{1}{1}\right),\ \left(\frac{2}{1}\right),\ \left(\frac{3}{1}\right),\ \left(\frac{4}{1}\right),\ \left(\frac{2}{2}\right),\ \left(\frac{3}{2}\right),\ \left(\frac{4}{2}\right),\ \left(\frac{3}{3}\right),\ \left(\frac{4}{3}\right),\ \left(\frac{4}{4}\right);$$

sicque in sequentibus classibus formularum numerus secundum numeros trigonales crescit. Has igitur classes ordine percurramus.

Classis 2^dae^ formae $\int \frac{x^{p-1}dx}{\sqrt[2]{(1-x^2)^{2-q}}}=\left(\frac{p}{q}\right)$

Perspicuum hic quidem est istas formulas vel absolute vel per quadraturam circuli exprimi; nam hae $\left(\frac{2}{1}\right)$ et $\left(\frac{2}{2}\right)$ absolute dantur et reliqua $\left(\frac{1}{1}\right)$ ob $1+1=2$ est $\frac{\pi}{2\sin.\frac{\pi}{2}}=\frac{\pi}{2}$; si ergo brevitatis causa ponamus $\frac{\pi}{2}=\alpha$, uti scilicet in sequentibus classibus faciemus, omnes formulae huius classis ita definiuntur:

$$\left(\frac{2}{1}\right)=1,\quad \left(\frac{2}{2}\right)=\frac{1}{2};$$
$$\left(\frac{1}{1}\right)=\alpha.$$

Classis 3^ae^ formae $\int \frac{x^{p-1}dx}{\sqrt[3]{(1-x^3)^{3-q}}}=\left(\frac{p}{q}\right)$

Cum hic sit $n=3$, formula quadraturam circuli involvens est

$$\left(\frac{2}{1}\right)=\frac{\pi}{3\sin.\frac{\pi}{3}};$$

ponamus ergo $\left(\frac{2}{1}\right)=\alpha$; reliquae autem formulae, quae non absolute dantur, altiorem quadraturam involvunt et quidem unicam $\left(\frac{1}{1}\right)$, quam littera A indicemus; qua concessa valores omnium formularum huius classis assignare poterimus:

$$\left(\frac{3}{1}\right)=1,\quad \left(\frac{3}{2}\right)=\frac{1}{2},\quad \left(\frac{3}{3}\right)=\frac{1}{3};$$
$$\left(\frac{2}{1}\right)=\alpha,\quad \left(\frac{2}{2}\right)=\frac{\alpha}{A};$$
$$\left(\frac{1}{1}\right)=A.$$

Classis 4^tae^ formae $\int \frac{x^{p-1}dx}{\sqrt[4]{(1-x^4)^{4-q}}}=\left(\frac{p}{q}\right)$

Cum hic sit $n=4$, duas habemus formulas a quadratura circuli pendentes, quarum valores, quia sunt cogniti, ita indicemus:

$$\left(\frac{3}{1}\right)=\frac{\pi}{4\sin.\frac{\pi}{4}}=\alpha \quad \text{et} \quad \left(\frac{2}{2}\right)=\frac{\pi}{4\sin.\frac{2\pi}{4}}=\beta.$$

Praeterea vero unica opus est formula altiorem quadraturam involvente, qua concessa reliquas omnes cognoscemus. Ponamus enim $\left(\frac{2}{1}\right)=A$ et omnes formulae huius classis ita determinabuntur:

$$\left(\frac{4}{1}\right)=1,\quad \left(\frac{4}{2}\right)=\frac{1}{2},\quad \left(\frac{4}{3}\right)=\frac{1}{3},\quad \left(\frac{4}{4}\right)=\frac{1}{4};$$
$$\left(\frac{3}{1}\right)=\alpha,\quad \left(\frac{3}{2}\right)=\frac{\beta}{A},\quad \left(\frac{3}{3}\right)=\frac{\alpha}{2A};$$
$$\left(\frac{2}{1}\right)=A,\quad \left(\frac{2}{2}\right)=\beta;$$
$$\left(\frac{1}{1}\right)=\frac{\alpha A}{\beta}.$$

Classis 5^tae^ formae $\int \frac{x^{p-1}dx}{\sqrt[5]{(1-x^5)^{5-q}}}=\left(\frac{p}{q}\right)$

Cum hic sit $n=5$, notemus statim formulas a quadratura circuli pendentes

$$\left(\frac{4}{1}\right)=\frac{\pi}{5\sin.\frac{\pi}{5}}=\alpha,\quad \left(\frac{3}{2}\right)=\frac{\pi}{5\sin.\frac{2\pi}{5}}=\beta.$$

Duabus autem insuper novis quadraturis opus est huic classi peculiaribus, quas ita designemus

$$\left(\frac{3}{1}\right)=A \quad \text{et} \quad \left(\frac{2}{2}\right)=B,$$

ex quibus reliquae omnes ita definientur:

$$\left(\frac{5}{1}\right)=1,\quad \left(\frac{5}{2}\right)=\frac{1}{2},\quad \left(\frac{5}{3}\right)=\frac{1}{3},\quad \left(\frac{5}{4}\right)=\frac{1}{4},\quad \left(\frac{5}{5}\right)=\frac{1}{5};$$

$$\left(\frac{4}{1}\right)=\alpha,\quad \left(\frac{4}{2}\right)=\frac{\beta}{A},\quad \left(\frac{4}{3}\right)=\frac{\beta}{2B},\quad \left(\frac{4}{4}\right)=\frac{\alpha}{3A};$$

$$\left(\frac{3}{1}\right)=A,\quad \left(\frac{3}{2}\right)=\beta,\quad \left(\frac{3}{3}\right)=\frac{\beta\beta}{\alpha B};$$

$$\left(\frac{2}{1}\right)=\frac{\alpha B}{\beta},\quad \left(\frac{2}{2}\right)=B;$$

$$\left(\frac{1}{1}\right)=\frac{\alpha A}{\beta}.$$

Classis 6^tae^ formae $\int\frac{x^{p-1}dx}{\sqrt[6]{(1-x^6)^{6-q}}}=\left(\frac{p}{q}\right)$

Hic est $n=6$ et formulae quadraturam circuli involventes sunt

$$\left(\frac{5}{1}\right)=\frac{\pi}{6\sin.\frac{\pi}{6}}=\alpha,\quad \left(\frac{4}{2}\right)=\frac{\pi}{6\sin.\frac{2\pi}{6}}=\beta,\quad \left(\frac{3}{3}\right)=\frac{\pi}{6\sin.\frac{3\pi}{6}}=\gamma.$$

Reliquarum vero omnium valores insuper a binis hisce quadraturis pendent

$$\left(\frac{4}{1}\right)=A \quad\text{et}\quad \left(\frac{3}{2}\right)=B$$

atque ita se habere deprehenduntur:

$$\left(\frac{6}{1}\right)=1,\quad \left(\frac{6}{2}\right)=\frac{1}{2},\quad \left(\frac{6}{3}\right)=\frac{1}{3},\quad \left(\frac{6}{4}\right)=\frac{1}{4},\quad \left(\frac{6}{5}\right)=\frac{1}{5},\quad \left(\frac{6}{6}\right)=\frac{1}{6};$$

$$\left(\frac{5}{1}\right)=\alpha,\quad \left(\frac{5}{2}\right)=\frac{\beta}{A},\quad \left(\frac{5}{3}\right)=\frac{\gamma}{2B},\quad \left(\frac{5}{4}\right)=\frac{\beta}{3B},\quad \left(\frac{5}{5}\right)=\frac{\alpha}{4A};$$

$$\left(\frac{4}{1}\right)=A,\quad \left(\frac{4}{2}\right)=\beta,\quad \left(\frac{4}{3}\right)=\frac{\beta\gamma}{\alpha B},\quad \left(\frac{4}{4}\right)=\frac{\beta\gamma A}{2\alpha BB};$$

$$\left(\frac{3}{1}\right)=\frac{\alpha B}{\beta},\quad \left(\frac{3}{2}\right)=B,\quad \left(\frac{3}{3}\right)=\gamma;$$

$$\left(\frac{2}{1}\right)=\frac{\alpha B}{\gamma},\quad \left(\frac{2}{2}\right)=\frac{\alpha BB}{\gamma A};$$

$$\left(\frac{1}{1}\right)=\frac{\alpha A}{\beta}.$$

Classis 7^mae^ formae $\int\frac{x^{p-1}dx}{\sqrt[7]{(1-x^7)^{7-q}}}=\left(\frac{p}{q}\right)$

Quia $n=7$, formulae a quadratura circuli pendentes ita designentur

$$\left(\frac{6}{1}\right)=\frac{\pi}{7\sin.\frac{\pi}{7}}=\alpha,\quad \left(\frac{5}{2}\right)=\frac{\pi}{7\sin.\frac{2\pi}{7}}=\beta,\quad \left(\frac{4}{3}\right)=\frac{\pi}{7\sin.\frac{3\pi}{7}}=\gamma,$$

praeterea vero hae quadraturae introducantur

$$\left(\frac{5}{1}\right)=A,\quad \left(\frac{4}{2}\right)=B,\quad \left(\frac{3}{3}\right)=C,$$

quibus datis omnes formulae ita determinabuntur:

$$\left(\frac{7}{1}\right)=1,\quad \left(\frac{7}{2}\right)=\frac{1}{2},\quad \left(\frac{7}{3}\right)=\frac{1}{3},\quad \left(\frac{7}{4}\right)=\frac{1}{4},\quad \left(\frac{7}{5}\right)=\frac{1}{5},\quad \left(\frac{7}{6}\right)=\frac{1}{6},\quad \left(\frac{7}{7}\right)=\frac{1}{7};$$

$$\left(\frac{6}{1}\right)=\alpha,\quad \left(\frac{6}{2}\right)=\frac{\beta}{A},\quad \left(\frac{6}{3}\right)=\frac{\gamma}{2B},\quad \left(\frac{6}{4}\right)=\frac{\gamma}{3C},\quad \left(\frac{6}{5}\right)=\frac{\beta}{4B},\quad \left(\frac{6}{6}\right)=\frac{\alpha}{5A};$$

$$\left(\frac{5}{1}\right)=A,\quad \left(\frac{5}{2}\right)=\beta,\quad \left(\frac{5}{3}\right)=\frac{\beta\gamma}{\alpha B},\quad \left(\frac{5}{4}\right)=\frac{\gamma\gamma A}{2\alpha BC},\quad \left(\frac{5}{5}\right)=\frac{\beta\gamma A}{3\alpha BC};$$

$$\left(\frac{4}{1}\right)=\frac{\alpha B}{\beta},\quad \left(\frac{4}{2}\right)=B,\quad \left(\frac{4}{3}\right)=\gamma,\quad \left(\frac{4}{4}\right)=\frac{\gamma\gamma}{\alpha C};$$

$$\left(\frac{3}{1}\right)=\frac{\alpha C}{\gamma},\quad \left(\frac{3}{2}\right)=\frac{\alpha BC}{\gamma A},\quad \left(\frac{3}{3}\right)=C;$$

$$\left(\frac{2}{1}\right)=\frac{\alpha B}{\gamma},\quad \left(\frac{2}{2}\right)=\frac{\alpha\beta BC}{\gamma\gamma A};$$

$$\left(\frac{1}{1}\right)=\frac{\alpha A}{\beta}.$$

Classis 8^vae^ formae $\int\frac{x^{p-1}dx}{\sqrt[8]{(1-x^8)^{8-q}}}=\left(\frac{p}{q}\right)$

Quia hic est $n=8$, formulae quadraturam circuli implicantes erunt

$$\left(\frac{7}{1}\right)=\frac{\pi}{8\sin.\frac{\pi}{8}}=\alpha,\quad \left(\frac{6}{2}\right)=\frac{\pi}{8\sin.\frac{2\pi}{8}}=\beta,$$

$$\left(\frac{5}{3}\right)=\frac{\pi}{8\sin.\frac{3\pi}{8}}=\gamma,\quad \left(\frac{4}{4}\right)=\frac{\pi}{8\sin.\frac{4\pi}{8}}=\delta.$$

Nunc vero tres frequentes formulae tanquam cognitae spectentur

$$\left(\frac{6}{1}\right)=A,\quad \left(\frac{5}{2}\right)=B \quad\text{et}\quad \left(\frac{4}{3}\right)=C$$

atque ex his omnes formulae huius classis ita determinabuntur:

$$\left(\frac{8}{1}\right)=1,\ \left(\frac{8}{2}\right)=\frac{1}{2},\ \left(\frac{8}{3}\right)=\frac{1}{3},\ \left(\frac{8}{4}\right)=\frac{1}{4},\ \left(\frac{8}{5}\right)=\frac{1}{5},\ \left(\frac{8}{6}\right)=\frac{1}{6},\ \left(\frac{8}{7}\right)=\frac{1}{7},\ \left(\frac{8}{8}\right)=\frac{1}{8};$$

$$\left(\frac{7}{1}\right)=\alpha,\ \left(\frac{7}{2}\right)=\frac{\beta}{A},\ \left(\frac{7}{3}\right)=\frac{\gamma}{2B},\ \left(\frac{7}{4}\right)=\frac{\delta}{3C},\ \left(\frac{7}{5}\right)=\frac{\gamma}{4C},\ \left(\frac{7}{6}\right)=\frac{\beta}{5B},\ \left(\frac{7}{7}\right)=\frac{\alpha}{6A};$$

$$\left(\frac{6}{1}\right)=A,\quad \left(\frac{6}{2}\right)=\beta,\quad \left(\frac{6}{3}\right)=\frac{\beta\gamma}{\alpha B},\ \left(\frac{6}{4}\right)=\frac{\gamma\delta A}{2\alpha BC},\ \left(\frac{6}{5}\right)=\frac{\gamma\delta A}{3\alpha CC},\quad \left(\frac{6}{6}\right)=\frac{\beta\gamma A}{4\alpha BC};$$

$$\left(\frac{5}{1}\right)=\frac{\alpha B}{\beta},\ \left(\frac{5}{2}\right)=B,\ \left(\frac{5}{3}\right)=\gamma,\ \left(\frac{5}{4}\right)=\frac{\gamma\delta}{\alpha C},\quad \left(\frac{5}{5}\right)=\frac{\gamma\gamma\delta A}{2\alpha\beta CC};$$

$$\left(\frac{4}{1}\right)=\frac{\alpha C}{\gamma},\quad \left(\frac{4}{2}\right)=\frac{\alpha BC}{\gamma A},\quad \left(\frac{4}{3}\right)=C,\quad \left(\frac{4}{4}\right)=\delta;$$

$$\left(\frac{3}{1}\right)=\frac{\alpha C}{\delta},\quad \left(\frac{3}{2}\right)=\frac{\alpha\beta CC}{\gamma\delta A},\quad \left(\frac{3}{3}\right)=\frac{\alpha CC}{\delta A};$$

$$\left(\frac{2}{1}\right)=\frac{\alpha B}{\gamma},\quad \left(\frac{2}{2}\right)=\frac{\alpha\beta BC}{\gamma\delta A};$$

$$\left(\frac{1}{1}\right)=\frac{\alpha A}{\beta}.$$

Hinc istas reductiones ad sequentes classes, quousque libuerit, continuare licet. Quemadmodum ergo hinc in genere singularum formularum integralia se sint habitura, exponamus.

Evolutio formae generalis $\displaystyle\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}=\left(\frac{p}{q}\right)$

Primo ergo absolute integrabiles sunt hae formulae

$$\left(\frac{n}{1}\right)=1,\quad \left(\frac{n}{2}\right)=\frac{1}{2},\quad \left(\frac{n}{3}\right)=\frac{1}{3},\quad \left(\frac{n}{4}\right)=\frac{1}{4}\quad\text{etc.},$$

deinde formulae a quadratura circuli pendentes sunt

$$\left(\frac{n-1}{1}\right)=\alpha,\quad \left(\frac{n-2}{2}\right)=\beta,\quad \left(\frac{n-3}{3}\right)=\gamma,\quad \left(\frac{n-4}{4}\right)=\delta\quad\text{etc.},$$

quarum quantitatum progressio tandem in se revertitur, cum sit etiam

$$\left(\frac{4}{n-4}\right)=\delta,\quad \left(\frac{3}{n-3}\right)=\gamma,\quad \left(\frac{2}{n-2}\right)=\beta,\quad \left(\frac{1}{n-1}\right)=\alpha.$$

Praeterea vero altiores quadraturae in subsidium vocari debent, quae ita repraesentantur

$$\left(\frac{n-2}{1}\right)=A,\quad \left(\frac{n-3}{2}\right)=B,\quad \left(\frac{n-4}{3}\right)=C,\quad \left(\frac{n-5}{4}\right)=D\quad\text{etc.},$$

quarum numerus quovis casu sponte determinatur, quia hae formulae tandem in se revertuntur.

His autem formulis admissis omnes omnino ad eandem classem pertinentes definiri poterunt. Habebimus autem a formula $\left(\frac{n-1}{1}\right)=\alpha$, uti supra istas formulas ordinavimus, deorsum descendendo

$$\left(\frac{n-1}{1}\right)=\alpha,\quad \left(\frac{n-2}{1}\right)=A,\quad \left(\frac{n-3}{1}\right)=\frac{\alpha B}{\beta},\quad \left(\frac{n-4}{1}\right)=\frac{\alpha C}{\gamma},$$

$$\left(\frac{n-5}{1}\right)=\frac{\alpha D}{\delta},\quad \left(\frac{n-6}{1}\right)=\frac{\alpha E}{\varepsilon}\quad\text{etc.},$$

qui valores retro sumti ita se habent

$$\left(\frac{1}{1}\right)=\frac{\alpha A}{\beta},\quad \left(\frac{2}{1}\right)=\frac{\alpha B}{\gamma},\quad \left(\frac{3}{1}\right)=\frac{\alpha C}{\delta}\quad\text{etc.}$$

Tum vero ab eadem formula $\left(\frac{n-1}{1}\right)=\alpha$ horizontaliter progrediendo definiuntur istae formulae

$$\left(\frac{n-1}{1}\right)=\alpha,\quad \left(\frac{n-1}{2}\right)=\frac{\beta}{A},\quad \left(\frac{n-1}{3}\right)=\frac{\gamma}{2B},\quad \left(\frac{n-1}{4}\right)=\frac{\delta}{3C}\quad\text{etc.},$$

quarum ultima erit

$$\left(\frac{n-1}{n-1}\right)=\frac{\alpha}{(n-2)A},$$

penultima

$$\left(\frac{n-1}{n-2}\right)=\frac{\beta}{(n-3)B},$$

antepenultima

$$\left(\frac{n-1}{n-3}\right)=\frac{\gamma}{(n-4)C}$$

etc.

Simili modo a formula $\left(\frac{n-2}{2}\right)=\beta$ tam descendendo quam progrediendo horizontaliter valores aliarum impetrabimus ac descendendo quidem

$$\left(\frac{n-2}{2}\right)=\beta,\quad \left(\frac{n-3}{2}\right)=B,\quad \left(\frac{n-4}{2}\right)=\frac{\alpha BC}{\gamma A},\quad \left(\frac{n-5}{2}\right)=\frac{\alpha\beta CD}{\gamma\delta A},$$

$$\left(\frac{n-6}{2}\right)=\frac{\alpha\beta DE}{\delta\varepsilon A},\quad \left(\frac{n-7}{2}\right)=\frac{\alpha\beta EF}{\varepsilon\zeta A}\quad \text{etc.},$$

ubi erit ultima

$$\left(\frac{2}{2}\right)=\frac{\alpha\beta BC}{\gamma\delta A},$$

penultima

$$\left(\frac{3}{2}\right)=\frac{\alpha\beta CD}{\delta\varepsilon A}$$

etc.;

at horizontaliter progrediendo

$$\left(\frac{n-2}{2}\right)=\beta,\quad \left(\frac{n-2}{3}\right)=\frac{\beta\gamma}{\alpha B},\quad \left(\frac{n-2}{4}\right)=\frac{\gamma\delta A}{2\alpha BC},\quad \left(\frac{n-2}{5}\right)=\frac{\delta\varepsilon A}{3\alpha CD},$$

$$\left(\frac{n-2}{6}\right)=\frac{\varepsilon\zeta A}{4\alpha DE},\quad \left(\frac{n-2}{7}\right)=\frac{\zeta\eta A}{5\alpha EF}\quad \text{etc.},$$

quarum erit ultima

$$\left(\frac{n-2}{n-2}\right)=\frac{\beta\gamma A}{(n-4)\alpha BC},$$

penultima

$$\left(\frac{n-2}{n-3}\right)=\frac{\gamma\delta A}{(n-5)\alpha CD}$$

etc.

Porro a formula $\left(\frac{n-3}{n-3}\right)=\gamma$ descendendo pervenimus ad has formulas

$$\left(\frac{n-3}{3}\right)=\gamma,\quad \left(\frac{n-4}{3}\right)=C,\quad \left(\frac{n-5}{3}\right)=\frac{\alpha CD}{\delta A},\quad \left(\frac{n-6}{3}\right)=\frac{\alpha\beta CDE}{\delta\varepsilon AB},$$

$$\left(\frac{n-7}{3}\right)=\frac{\alpha\beta\gamma DEF}{\delta\varepsilon\zeta AB},\quad \left(\frac{n-8}{3}\right)=\frac{\alpha\beta\gamma EFG}{\varepsilon\zeta\eta AB}\quad \text{etc.}$$

et horizontaliter progrediendo

$$\left(\frac{n-3}{3}\right)=\gamma,\quad \left(\frac{n-3}{4}\right)=\frac{\gamma\delta}{\alpha C},\quad \left(\frac{n-3}{5}\right)=\frac{\gamma\delta\varepsilon A}{2\alpha\beta CD},\quad \left(\frac{n-3}{6}\right)=\frac{\delta\varepsilon\zeta AB}{3\alpha\beta CDE},$$

$$\left(\frac{n-3}{7}\right)=\frac{\varepsilon\zeta\eta AB}{4\alpha\beta DEF},\quad \left(\frac{n-3}{8}\right)=\frac{\zeta\eta\theta AB}{5\alpha\beta EFG}\quad \text{etc.}$$

Pari modo a formula $\left(\frac{n-4}{4}\right)=\delta$ descendendo nanciscimur

$$\left(\frac{n-4}{4}\right)=\delta,\quad \left(\frac{n-5}{4}\right)=D,\quad \left(\frac{n-6}{4}\right)=\frac{\alpha DE}{\varepsilon A},\quad \left(\frac{n-7}{4}\right)=\frac{\alpha\beta DEF}{\varepsilon\zeta AB},$$

$$\left(\frac{n-8}{4}\right)=\frac{\alpha\beta\gamma DEFG}{\varepsilon\zeta\eta ABC},\quad \left(\frac{n-9}{4}\right)=\frac{\alpha\beta\gamma\delta EFGH}{\varepsilon\zeta\eta\theta ABC}\quad \text{etc.}$$

et horizontaliter progrediendo

$$\left(\frac{n-4}{4}\right)=\delta,\quad \left(\frac{n-4}{5}\right)=\frac{\delta\varepsilon}{\alpha D},\quad \left(\frac{n-4}{6}\right)=\frac{\delta\varepsilon\zeta A}{2\alpha\beta DE},\quad \left(\frac{n-4}{7}\right)=\frac{\delta\varepsilon\zeta\eta AB}{3\alpha\beta\gamma DEF},$$

$$\left(\frac{n-4}{8}\right)=\frac{\varepsilon\zeta\eta\theta ABC}{4\alpha\beta\gamma DEFG},\quad \left(\frac{n-4}{9}\right)=\frac{\zeta\eta\theta\iota ABC}{5\alpha\beta\gamma EFGH}\quad \text{etc.}$$

Atque hac ratione tandem omnium formularum valores reperiuntur.

Accommodemus has generales reductiones ad

Classem 9^{nae} *formae* $\int\frac{x^{p-1}dx}{\sqrt[9]{(1-x^9)^{9-q}}}=\left(\frac{p}{q}\right)$

Ubi ob $n=9$ formulae quadraturam circuli involventes erunt

$$\left(\frac{8}{1}\right)=\alpha,\quad \left(\frac{7}{2}\right)=\beta,\quad \left(\frac{6}{3}\right)=\gamma,\quad \left(\frac{5}{4}\right)=\delta;$$

hinc $\varepsilon=\delta,\ \zeta=\gamma,\ \eta=\beta,\ \theta=\alpha.$

Deinde novae quadraturae huc requisitae ponantur

$$\left(\frac{7}{1}\right)=A,\quad \left(\frac{6}{2}\right)=B,\quad \left(\frac{5}{3}\right)=C,\quad \left(\frac{4}{4}\right)=D$$

sicque erit

$$E=C,\quad F=B\quad \text{et}\quad G=A;$$

atque his quatuor valoribus concessis omnium formularum nonae classis valores assignari poterunt, quos simili ordine, ut hactenus fecimus, repraesentemus:

$$\left(\frac{9}{1}\right)=1,\quad \left(\frac{9}{2}\right)=\frac{1}{2},\quad \left(\frac{9}{3}\right)=\frac{1}{3},\quad \left(\frac{9}{4}\right)=\frac{1}{4},\quad \left(\frac{9}{5}\right)=\frac{1}{5},$$
$$\left(\frac{9}{6}\right)=\frac{1}{6},\quad \left(\frac{9}{7}\right)=\frac{1}{7},\quad \left(\frac{9}{8}\right)=\frac{1}{8},\quad \left(\frac{9}{9}\right)=\frac{1}{9};$$

$$\left(\frac{8}{1}\right)=\alpha,\quad \left(\frac{8}{2}\right)=\frac{\beta}{A},\quad \left(\frac{8}{3}\right)=\frac{\gamma}{2B},\quad \left(\frac{8}{4}\right)=\frac{\delta}{3C},\quad \left(\frac{8}{5}\right)=\frac{\delta}{4D},$$
$$\left(\frac{8}{6}\right)=\frac{\gamma}{5C},\quad \left(\frac{8}{7}\right)=\frac{\beta}{6B},\quad \left(\frac{8}{8}\right)=\frac{\alpha}{7A};$$

$$\left(\frac{7}{1}\right)=A,\quad \left(\frac{7}{2}\right)=\beta,\quad \left(\frac{7}{3}\right)=\frac{\beta\gamma}{\alpha B},\quad \left(\frac{7}{4}\right)=\frac{\gamma\delta A}{2\alpha BC},\quad \left(\frac{7}{5}\right)=\frac{\delta\delta A}{3\alpha CD},$$
$$\left(\frac{7}{6}\right)=\frac{\gamma\delta A}{4\alpha CD},\quad \left(\frac{7}{7}\right)=\frac{\beta\gamma A}{5\alpha BC};$$

$$\left(\frac{6}{1}\right)=\frac{\alpha B}{\beta},\quad \left(\frac{6}{2}\right)=B,\quad \left(\frac{6}{3}\right)=\gamma,\quad \left(\frac{6}{4}\right)=\frac{\gamma\delta}{\alpha C},\quad \left(\frac{6}{5}\right)=\frac{\gamma\delta\delta A}{2\alpha\beta CD},$$
$$\left(\frac{6}{6}\right)=\frac{\gamma\delta\delta AB}{3\alpha\beta CCD};$$

$$\left(\frac{5}{1}\right)=\frac{\alpha C}{\gamma},\quad \left(\frac{5}{2}\right)=\frac{\alpha BC}{\gamma A},\quad \left(\frac{5}{3}\right)=C,\quad \left(\frac{5}{4}\right)=\delta,\quad \left(\frac{5}{5}\right)=\frac{\delta\delta}{\alpha D};$$

$$\left(\frac{4}{1}\right)=\frac{\alpha D}{\delta},\quad \left(\frac{4}{2}\right)=\frac{\alpha\beta CD}{\gamma\delta A},\quad \left(\frac{4}{3}\right)=\frac{\alpha CD}{\delta A},\quad \left(\frac{4}{4}\right)=D;$$

$$\left(\frac{3}{1}\right)=\frac{\alpha C}{\delta},\quad \left(\frac{3}{2}\right)=\frac{\alpha\beta CD}{\delta\delta A},\quad \left(\frac{3}{3}\right)=\frac{\alpha\beta CCD}{\delta\delta AB};$$

$$\left(\frac{2}{1}\right)=\frac{\alpha B}{\gamma},\quad \left(\frac{2}{2}\right)=\frac{\alpha\beta BC}{\gamma\delta A};$$

$$\left(\frac{1}{1}\right)=\frac{\alpha A}{\beta}.$$

Ordo harum formularum etiam in genere diagonaliter a sinistra ad dextram procedendo notari meretur, ubi quidem duo genera progressionum occurrunt, prout vel a prima serie verticali vel a suprema horizontali incipimus. Hoc modo primum a serie verticali incipiendo:

$$\left(\frac{n-1}{1}\right)=\alpha,\quad \left(\frac{n-2}{2}\right)=\frac{\beta}{\alpha}\times\left(\frac{n-1}{1}\right),\quad \left(\frac{n-3}{3}\right)=\frac{\gamma}{\beta}\times\left(\frac{n-2}{2}\right),\quad \left(\frac{n-4}{4}\right)=\frac{\delta}{\gamma}\times\left(\frac{n-3}{3}\right)$$
$$\left(\frac{n-2}{1}\right)=A,\quad \left(\frac{n-3}{2}\right)=\frac{B}{A}\times\left(\frac{n-2}{1}\right),\quad \left(\frac{n-4}{3}\right)=\frac{C}{B}\times\left(\frac{n-3}{2}\right),\quad \left(\frac{n-5}{4}\right)=\frac{D}{C}\times\left(\frac{n-4}{3}\right)$$
$$\left(\frac{n-3}{1}\right)=\frac{\alpha B}{\beta},\quad \left(\frac{n-4}{2}\right)=\frac{\beta C}{\gamma A}\times\left(\frac{n-3}{1}\right),\quad \left(\frac{n-5}{3}\right)=\frac{\gamma D}{\delta B}\times\left(\frac{n-4}{2}\right),\quad \left(\frac{n-6}{4}\right)=\frac{\delta E}{\varepsilon C}\times\left(\frac{n-5}{3}\right)$$
$$\left(\frac{n-4}{1}\right)=\frac{\alpha C}{\gamma},\quad \left(\frac{n-5}{2}\right)=\frac{\beta D}{\delta A}\times\left(\frac{n-4}{1}\right),\quad \left(\frac{n-6}{3}\right)=\frac{\gamma E}{\varepsilon B}\times\left(\frac{n-5}{2}\right),\quad \left(\frac{n-7}{4}\right)=\frac{\delta F}{\zeta C}\times\left(\frac{n-6}{3}\right)$$
$$\left(\frac{n-5}{1}\right)=\frac{\alpha D}{\delta},\quad \left(\frac{n-6}{2}\right)=\frac{\beta E}{\varepsilon A}\times\left(\frac{n-5}{1}\right),\quad \left(\frac{n-7}{3}\right)=\frac{\gamma F}{\zeta B}\times\left(\frac{n-6}{2}\right),\quad \left(\frac{n-8}{4}\right)=\frac{\delta G}{\eta C}\times\left(\frac{n-7}{3}\right)$$
$$\left(\frac{n-6}{1}\right)=\frac{\alpha E}{\varepsilon},\quad \left(\frac{n-7}{2}\right)=\frac{\beta F}{\zeta A}\times\left(\frac{n-6}{1}\right),\quad \left(\frac{n-8}{3}\right)=\frac{\gamma G}{\eta B}\times\left(\frac{n-7}{2}\right),\quad \left(\frac{n-9}{4}\right)=\frac{\delta H}{\theta C}\times\left(\frac{n-8}{3}\right)$$

etc.,

deinde a suprema horizontali incipiendo:

$$\left(\frac{n}{1}\right)=1,\quad \left(\frac{n-1}{2}\right)=\frac{\beta}{A}\times\left(\frac{n}{1}\right),\quad \left(\frac{n-2}{3}\right)=\frac{\gamma A}{\alpha B}\times\left(\frac{n-1}{2}\right),\quad \left(\frac{n-3}{4}\right)=\frac{\delta B}{\beta C}\times\left(\frac{n-2}{3}\right)$$
$$\left(\frac{n}{2}\right)=\frac{1}{2},\quad \left(\frac{n-1}{3}\right)=\frac{\gamma}{B}\times\left(\frac{n}{2}\right),\quad \left(\frac{n-2}{4}\right)=\frac{\delta A}{\alpha C}\times\left(\frac{n-1}{3}\right),\quad \left(\frac{n-3}{5}\right)=\frac{\varepsilon B}{\beta D}\times\left(\frac{n-2}{4}\right)$$
$$\left(\frac{n}{3}\right)=\frac{1}{3},\quad \left(\frac{n-1}{4}\right)=\frac{\delta}{C}\times\left(\frac{n}{3}\right),\quad \left(\frac{n-2}{5}\right)=\frac{\varepsilon A}{\alpha D}\times\left(\frac{n-1}{4}\right),\quad \left(\frac{n-3}{6}\right)=\frac{\zeta B}{\beta E}\times\left(\frac{n-2}{5}\right)$$
$$\left(\frac{n}{4}\right)=\frac{1}{4},\quad \left(\frac{n-1}{5}\right)=\frac{\varepsilon}{D}\times\left(\frac{n}{4}\right),\quad \left(\frac{n-2}{6}\right)=\frac{\zeta A}{\alpha E}\times\left(\frac{n-1}{5}\right),\quad \left(\frac{n-3}{7}\right)=\frac{\eta B}{\beta F}\times\left(\frac{n-2}{6}\right)$$
$$\left(\frac{n}{5}\right)=\frac{1}{5},\quad \left(\frac{n-1}{6}\right)=\frac{\zeta}{E}\times\left(\frac{n}{5}\right),\quad \left(\frac{n-2}{7}\right)=\frac{\eta A}{\alpha F}\times\left(\frac{n-1}{6}\right),\quad \left(\frac{n-3}{8}\right)=\frac{\theta B}{\beta G}\times\left(\frac{n-2}{7}\right)$$

etc.

Ubi lex, qua hae formulae a se invicem pendent, satis est perspicua, si modo notemus in utraque litterarum serie α, β, γ, δ etc. et A, B, C, D etc. terminos primum antecedentes inter se esse aequales.

CONCLUSIO

Cum igitur formulas secundae classis sola concessa circuli quadratura exhibere valeamus, formulae tertiae classis insuper requirunt quadraturam contentam vel hac formula

$$\int\frac{dx}{\sqrt[3]{(1-x^3)^2}}=A \quad\text{vel hac}\quad \int\frac{xdx}{\sqrt[3]{(1-x^3)}}=\frac{\alpha}{A},$$

quandoquidem, data una, simul altera datur. Quodsi istas formulas per productum infinitum exprimamus, earum valor reperitur

$$\int\frac{dx}{\sqrt[3]{(1-x^3)^2}}=\frac{2}{1}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{6\cdot 8}{7\cdot 7}\cdot\frac{9\cdot 11}{10\cdot 10}\cdot\frac{12\cdot 14}{13\cdot 13}\cdot\text{etc.},$$

unde eius quantitas vero proxime satis expedite colligi potest; simili modo est

$$\int\frac{xdx}{\sqrt[3]{(1-x^3)}}=1\cdot\frac{3\cdot 7}{5\cdot 5}\cdot\frac{6\cdot 10}{8\cdot 8}\cdot\frac{9\cdot 13}{11\cdot 11}\cdot\frac{12\cdot 16}{14\cdot 14}\cdot\text{etc.}$$

Deinde omnes formulas quartae classis integrare poterimus, si modo praeter circuli quadraturam una ex his quatuor formulis fuerit cognita $\left(\frac{2}{1}\right)$, $\left(\frac{1}{1}\right)$, $\left(\frac{3}{2}\right)$, $\left(\frac{3}{3}\right)$, quae praebent has formas

$$\int\frac{xdx}{\sqrt[4]{(1-x^4)^3}}=\frac{1}{2}\int\frac{dx}{\sqrt[4]{(1-xx)^3}}=\int\frac{dx}{\sqrt{(1-x^4)}}=A,$$

$$\int\frac{dx}{\sqrt[4]{(1-x^4)^3}}=\frac{\alpha A}{\beta},\quad \int\frac{xxdx}{\sqrt[4]{(1-x^4)}}=\frac{\alpha}{2A},$$

$$\int\frac{xxdx}{\sqrt{(1-x^4)}}=\int\frac{xdx}{\sqrt[4]{(1-x^4)}}=\frac{1}{2}\int\frac{dx}{\sqrt[4]{(1-xx)}}=\frac{\beta}{A};$$

at per productum infinitum erit

$$A=\frac{3}{1\cdot 2}\cdot\frac{4\cdot 7}{5\cdot 6}\cdot\frac{8\cdot 11}{9\cdot 10}\cdot\frac{12\cdot 15}{13\cdot 14}\cdot\frac{16\cdot 19}{17\cdot 18}\cdot\text{etc.}$$

Quinta classis postulat duas quadraturas altiores $\left(\frac{3}{1}\right)=A$ et $\left(\frac{2}{2}\right)=B$, quarum loco aliae binae ab his pendentes assumi possunt, quae quidem faciliores videantur, etsi ob 5 numerum primum aliae aliis vix simpliciores reputari queant.

Pro sexta classe etiam duae quadraturae requiruntur $\left(\frac{4}{1}\right)=A$ et $\left(\frac{3}{2}\right)=B$. Verum hic loco alterius ea, quae in tertia classe opus erat, assumi potest,

ut unica tantum nova sit adhibenda. Cum enim sit

$$\left(\frac{2}{2}\right)=\int\frac{xdx}{\sqrt[6]{(1-x^6)^4}}=\frac{1}{2}\int\frac{dx}{\sqrt[3]{(1-x^3)^2}}=\frac{\alpha BB}{\gamma A},$$

erit

$$\frac{2\alpha BB}{\gamma A}=\int\frac{dx}{\sqrt[3]{(1-x^3)^2}},$$

quae est formula ad classem tertiam requisita. Hac ergo data si insuper innotescat formula

$$\left(\frac{3}{2}\right)=\int\frac{xdx}{\sqrt{(1-x^6)}}=\frac{1}{2}\int\frac{dx}{\sqrt{(1-x^3)}}=B$$

vel etiam haec

$$\left(\frac{4}{3}\right)=\int\frac{xxdx}{\sqrt[3]{(1-x^6)}}=\frac{1}{3}\int\frac{dx}{\sqrt[3]{(1-xx)}}=\frac{\beta\gamma}{\alpha B},$$

quae sunt simplicissimae in hoc genere, reliquae omnes per has definiri poterunt. His autem combinatis patet fore

$$\int\frac{dx}{\sqrt{(1-x^3)}}\cdot\int\frac{dx}{\sqrt[3]{(1-xx)}}=\frac{6\beta\gamma}{\alpha}=\frac{\pi}{\sqrt{3}}.$$

Simili modo ex formulis quartae classis colligitur

$$\int\frac{dx}{\sqrt{(1-x^4)}}\cdot\int\frac{dx}{\sqrt[4]{(1-x^2)}}=\frac{\pi}{2},$$

cuiusmodi theorematum ingens multitudo hinc deduci potest, inter quae hoc imprimis est notabile

$$\int\frac{dx}{\sqrt[m]{(1-x^n)}}\cdot\int\frac{dx}{\sqrt[n]{(1-x^m)}}=\frac{\pi\sin.\frac{(m-n)\pi}{mn}}{(m-n)\sin.\frac{\pi}{m}\cdot\sin.\frac{\pi}{n}},$$

quod, si m et n sint numeri fracti, in hanc formam transmutatur

$$\int\frac{x^{q-1}dx}{\sqrt[r]{(1-x^p)^s}}\cdot\int\frac{x^{s-1}dx}{\sqrt[p]{(1-x^r)^q}}=\frac{\pi\sin.\left(\frac{s}{r}-\frac{q}{p}\right)\pi}{(ps-qr)\sin.\frac{q\pi}{p}\cdot\sin.\frac{s\pi}{r}}.$$

In genere vero est

$$\left(\frac{n-p}{q}\right)\left(\frac{n-q}{p}\right)=\frac{\left(\frac{n-p}{p}\right)\left(\frac{n-q}{q}\right)}{(q-p)\left(\frac{n-q+p}{q-p}\right)},$$

quod hanc formam praebet

$$\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^q}}\cdot\int\frac{x^{q-1}dx}{\sqrt[n]{(1-x^n)^p}}=\frac{\pi\sin.\frac{(q-p)\pi}{n}}{n(q-p)\sin.\frac{p\pi}{n}\cdot\sin.\frac{q\pi}{n}},$$

unde non solum praecedentia theoremata, sed alia plura facile derivantur. Posito enim $n=\frac{pq}{m}$ habebimus

$$\int\frac{x^{m-1}dx}{\sqrt[p]{(1-x^q)^m}}\cdot\int\frac{x^{m-1}dx}{\sqrt[q]{(1-x^p)^m}}=\frac{\pi\sin.\left(\frac{m}{p}-\frac{m}{q}\right)\pi}{m(q-p)\sin.\frac{m\pi}{q}\cdot\sin.\frac{m\pi}{p}},$$

quam ita latius extendere licet

$$\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^m)^q}}\cdot\int\frac{x^{q-1}dx}{\sqrt[m]{(1-x^n)^p}}=\frac{\pi\sin.\left(\frac{q}{n}-\frac{p}{m}\right)\pi}{(mq-np)\sin.\frac{p\pi}{m}\cdot\sin.\frac{q\pi}{n}};$$

in qua si ponatur $n=2q$, erit

$$\int\frac{x^{p-1}dx}{\sqrt{(1-x^m)}}\cdot\int\frac{x^{q-1}dx}{\sqrt[m]{(1-x^{2q})^p}}=\frac{\pi\cos.\frac{p\pi}{m}}{q(m-2p)\sin.\frac{p\pi}{m}}.$$

At in posteriori formula integrali si ponatur $x^{2q}=1-y^m$, erit

$$\int\frac{x^{q-1}dx}{\sqrt[m]{(1-x^{2q})^p}}=\frac{m}{2q}\int\frac{y^{m-p-1}dy}{\sqrt{(1-y^m)}},$$

unde scripto x pro y

$$\int\frac{x^{p-1}dx}{\sqrt{(1-x^m)}}\cdot\int\frac{x^{m-p-1}dx}{\sqrt{(1-x^m)}}=\frac{2\pi\cos.\frac{p\pi}{m}}{m(m-2p)\sin.\frac{p\pi}{m}}.$$

Simili modo si in genere ponatur pro altera formula integrali $1-x^n=y^m$, fiet

$$\int\frac{x^{q-1}dx}{\sqrt[m]{(1-x^n)^p}}=\frac{m}{n}\int\frac{y^{m-p-1}dy}{\sqrt[n]{(1-y^m)^{n-q}}},$$

unde scripto iterum x pro y obtinebitur

$$\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^m)^q}}\cdot\int\frac{x^{m-p-1}dx}{\sqrt[n]{(1-x^m)^{n-q}}}=\frac{n\pi\sin.\left(\frac{q}{n}-\frac{p}{m}\right)\pi}{m(mq-np)\sin.\frac{p\pi}{m}\cdot\sin.\frac{q\pi}{n}},$$

qui valor reducitur ad $\frac{n\pi}{m(mq-np)}\left(\cot.\frac{p\pi}{m}-\cot.\frac{q\pi}{n}\right)$. Atque hinc ista forma concinnior resultat

$$\int\frac{x^{\frac{m-r}{2}-1}dx}{\sqrt[n]{(1-x^m)^{\frac{n-s}{2}}}}\cdot\int\frac{x^{\frac{m+r}{2}-1}dx}{\sqrt[n]{(1-x^m)^{\frac{n+s}{2}}}}=\frac{2n\pi\left(\operatorname{tang}.\frac{r\pi}{2m}-\operatorname{tang}.\frac{s\pi}{2n}\right)}{m(nr-ms)}.$$

Cum fundamentum harum reductionum situm sit in hac aequalitate

$$\left(\frac{n-p}{q}\right)\left(\frac{n-q}{p}\right)=\frac{\left(\frac{n-p}{p}\right)\left(\frac{n-q}{q}\right)}{(q-p)\left(\frac{n-q+p}{q-p}\right)},$$

quae ad hanc formam reducitur

$$\left(\frac{n-p}{q}\right)\left(\frac{n-q}{p}\right)\left(\frac{n-q+p}{q-p}\right)=\left(\frac{n}{q-p}\right)\left(\frac{n-p}{p}\right)\left(\frac{n-q}{q}\right),$$

eius veritas hoc modo directe ostendi potest.

Sumtis in reductione § 8 tradita pro numeris ternis p, q, r his $n-q$, $q-p$, q habebimus

$$\left(\frac{n-q}{q-p}\right)\left(\frac{n-p}{q}\right)=\left(\frac{n-q}{q}\right)\left(\frac{n}{q-p}\right);$$

tum vero sumtis eorum loco $n-q$, $q-p$, p erit

$$\left(\frac{n-q}{p}\right)\left(\frac{n-q+p}{q-p}\right)=\left(\frac{n-q}{q-p}\right)\left(\frac{n-p}{p}\right),$$

quibus aequationibus in se invicem ductis et formula $\left(\frac{n-q}{q-p}\right)$ utrinque communi per divisionem sublata erit

$$\left(\frac{n-p}{q}\right)\left(\frac{n-q}{p}\right)\left(\frac{n-q+p}{q-p}\right)=\left(\frac{n}{q-p}\right)\left(\frac{n-p}{p}\right)\left(\frac{n-q}{q}\right).$$

Quin etiam huiusmodi ternarum aequalitas ab exponente n non pendens exhiberi potest, scilicet

$$\left(\frac{s}{p}\right)\left(\frac{r+s}{q}\right)\left(\frac{p+s}{r}\right)=\left(\frac{r+s}{p}\right)\left(\frac{s}{q}\right)\left(\frac{q+s}{r}\right)=\left(\frac{r}{p}\right)\left(\frac{r+s}{q}\right)\left(\frac{p+r}{s}\right)=\left(\frac{r+s}{p}\right)\left(\frac{r}{q}\right)\left(\frac{q+r}{s}\right),$$

quae quatuor adeo litteras ab n non pendentes involvit ac similis est aequalitati inter binarum formularum producta

$$\left(\frac{r}{p}\right)\left(\frac{p+r}{q}\right)=\left(\frac{q+r}{p}\right)\left(\frac{r}{q}\right)=\left(\frac{q}{p}\right)\left(\frac{p+q}{r}\right).$$

Aequalitas autem inter ternarum formularum producta habetur etiam ista

$$\left(\frac{p}{q}\right)\left(\frac{r}{s}\right)\left(\frac{p+q}{r+s}\right)=\left(\frac{q}{r}\right)\left(\frac{s}{p}\right)\left(\frac{q+r}{p+s}\right)=\left(\frac{p}{r}\right)\left(\frac{q}{s}\right)\left(\frac{p+r}{q+s}\right)$$
$$=\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right)=\left(\frac{p}{q}\right)\left(\frac{p+q}{s}\right)\left(\frac{p+q+s}{r}\right) \quad \text{etc.}$$

In his enim litterae p, q, r, s utcumque inter se permutari possunt.

DE FORMULIS INTEGRALIBUS DUPLICATIS

Commentatio 391 indicis ENESTROEMIANI
Novi commentarii academiae scientiarum Petropolitanae 14 (1769): I, 1770, p. 72—103
Summarium ibidem p. 13—15

SUMMARIUM

Disquisitio de corporum soliditatibus et superficiebus quum ad eiusmodi formulas integrales deducatur, quae ex producto differentialium duarum variabilium x et y et functione quadam harum quantitatum componantur adeoque duplicem integrationem requirant, antequam valor ipsis competens determinari possit, res omnino fuit maximi momenti in naturam et proprietates harum formularum accuratius inquirere. De formulis autem eiusmodi integralibus, quas duplicatas appellare Illustr. Auctori visum est, tenendum est, quodsi binae variabiles x et y plane a se invicem non pendeant, duplicem earum integrationem ita instituendam esse, ut in una earum sola x variabilis, in altera vero sola y ponatur, tum vero loco constantium quantitatum duas quaslibet functiones singularum x et y adiici oportere, ut integrale completum inveniatur, et perinde omnino esse, quo ordine eiusmodi instituatur integratio, quum semper idem prodire debeat integrale. Hae autem formulae plane diversae sunt ab iis, quibus soliditas vel superficies corporum exprimitur; in his enim posterioribus omnino aliqua relatio inter x et y intercedit, unde earum integratio ita instituenda erit, ut, postquam in priori altera variabilium, ut x, pro constante assumta sit, hac integratione peracta, ea per omnes valores ipsius y extendi debeat et loco y extremus valor, quem recipit, substituendus erit, unde fit, ut in posteriori integratione y non amplius ab x sit independens, sed plerumque aliqua functione ipsius x exprimatur, adeo ut posteriorem integrationem unica variabilis x ingrediatur. Ad determinationem vero integrationum investigandam functionem, qua productum $dxdy$ multiplicatum est, unitati aequalem supponere licet; liquet enim aream basis hac formula $\iint dxdy$ exprimi, ex cuius formulae igitur integratione etiam istae conditiones, quae pro hac altera $\iint Zdxdy$ valent, praescribi

possunt. Insignes autem et plane singulares sunt affectiones harum formularum duplicatarum in earum transformatione conspicuae; scilicet quemadmodum variabiles x et y in alias t et v certa ratione ab ipsis dependentes transformari possunt, ita etiam pro x et y his earum valoribus inventis substitutis novae oriuntur formulae duplicatae alias variabiles involventes. Iam cum quam maxime probabile videri posset novas has formulas integrales non solum in se complectere tales, quas productum $dtdv$ ingreditur, sed praeter has quoque alias, quae ex dv^2 et dt^2 constant, facile tamen perspicitur hoc fieri non posse, quia posteriores hae formulae dv^2 et dt^2 in se complectentes ex numero formularum duplicatarum excludantur. Hoc autem dubium facile diluetur, si consideretur non plane necessarium esse, ut nova formula integralis duplicata priori prorsus sit aequalis, quoniam in hac posteriore aliae plane sunt conditiones, sub quibus integratio peragenda est, ac in priori. Potissimum igitur fundamentum, cui haec transformatio innititur, ex eo peti debet, quod prima integratio formae integralis per transformationem ortae ita institui debeat, ut vel v vel t pro constanti habeatur. Insignem autem hae transformationes saepius habent usum ad solutiones faciliores reddendas, quod imprimis exemplo famosi istius problematis Florentini illustratur, cuius plurimas solutiones elegantes Illustr. Auctor hac occasione adduxit, quarum, quae § 44 occurrit, generalissima est. Ceterum quoque notari meretur huic dissertationi occasionem dedisse elegans problema de invenienda figura corporis, quod inter omnia eiusdem soliditatis minima superficie contineretur; cuius tamen problematis solutio quomodo inveniri queat, nondum patet.

1. Si corporis cuiuscunque propositi vel soliditatem vel superficiem vel alias huiusmodi quantitates definire velimus, id per duplicem integrationem fieri solet; formula enim differentialis bis integranda tali forma $Zdxdy$ exprimitur binas variabiles x et y continente, quarum altera sola in priori integratione ut variabilis spectatur; posterior vero integratio ad alteram iam ut variabilem spectatam instituitur. Hinc quantitas per duplicem istam integrationem resultans duplex signum integrale praefigendo indicari solet hoc modo $\iint Zdxdy$, quippe qua duplicatione formula differentialis proposita $Zdxdy$ bis integrari debere est intelligenda. Huiusmodi igitur expressiones geminato signo summatorio affectas hic formulas integrales duplicatas appello; quarum usus cum latissime pateat, in earum indolem hic diligentius inquirere earumque proprietates et affectiones accuratius evolvere constitui.

2. Primum igitur cum x et y sint duae quantitates variabiles a se invicem non pendentes, Z vero denotet earum functionem quamcunque, formulae

integralis duplicatae $\iint Zdxdy$ vis ita exponi potest, ut quaerenda sit functio finita binarum istarum variabilium x et y, quae ita bis differentiata, ut in altera differentiatione sola x, in altera sola y pro variabili habeatur, ad formulam $Zdxdy$ deducat. Ita si fuerit $Z=a$, evidens fore $\iint adxdy = axy$; generalius vero erit $\iint adxdy = axy + X + Y$ denotante X functionem quamcunque ipsius x et Y ipsius y, quandoquidem hae duae quantitates per geminam illam differentiationem ex calculo tolluntur.

3. In genere autem si V fuerit eiusmodi functio ipsarum x et y, quae bis differentiata, ita ut modo est praeceptum, praebeat $Zdxdy$, erit quidem utique $V = \iint Zdxdy$, verum duplex integratio insuper functiones arbitrarias X et Y, illam ipsius x, hanc ipsius y, inducit, ut sit generalissime

$$\iint Zdxdy = V + X + Y.$$

Et statim perspicitur huiusmodi formulas differentiales necessario affectas esse producto $dxdy$ neque propterea secundum hanc significationem tales formulas $\iint Zdx^2$ vel $\iint Zdy^2$ quicquam significare, siquidem per ipsam rei naturam excluduntur, dum in altera integratione sola x, in altera vero sola y ut variabilis tractatur.

4. Constituta sic forma huiusmodi formularum integralium duplicatarum $\iint Zdxdy$, ita ut x et y sint duae quantitates variabiles a se invicem non pendentes et Z functio finita ex iis utcunque composita, haud difficile est duplicem integrationem, quam involvunt, instituere, quod quidem, prout primo vel x vel y sola variabilis consideratur, duplici modo fieri potest. Sumta scilicet primo y pro variabili altera x ut constans tractatur quaeriturque integrale $\int Zdy$, quod erit certa quaedam functio ipsarum x et y; qua inventa suscipiatur formula differentialis $dx\int Zdy$, in qua iam y ut constans solaque x ut variabilis tractetur, eiusque quaeratur integrale $\int dx\int Zdy$, qui erit valor quaesitus formulae integralis duplicatae propositae $\iint Zdxdy$. Si in hac duplici integratione ordo variabilium x et y invertatur, valor quaesitus ita exprimetur $\int dy\int Zdx$, qui ab illo non discrepabit.

5. Ob hunc consensum fit, ut talis forma $\iint Zdxdy$ promiscue sive hoc modo $\int dx\int Zdy$ sive hoc $\int dy\int Zdx$ exhiberi possit; utrovis autem utamur,

regulae vulgares integrationis sunt observandae, si modo notetur in ea integratione, in qua vel x vel y pro constante sumatur, constantem introductam eiusdem fore functionem quamcunque. Veluti si proponatur haec forma

$$\int\int\frac{dxdy}{xx+yy}=\int dx\int\frac{dy}{xx+yy},$$

ob

$$\int\frac{dy}{xx+yy}=\frac{1}{x}\,\text{A tang.}\frac{y}{x}+\frac{dX}{dx}$$

denotante $\frac{dX}{dx}$ functionem quamcunque ipsius x erit

$$\int\int\frac{dxdy}{xx+yy}=\int\frac{dx}{x}\,\text{A tang.}\frac{y}{x}+X,$$

ubi in integratione adhuc perficienda y pro constante habetur. Simili vero modo reperitur

$$\int\int\frac{dxdy}{xx+yy}=\int\frac{dy}{y}\,\text{A tang.}\frac{x}{y}+Y,$$

in qua integratione x constans assumitur; in quo quidem exemplo consensus binorum valorum inventorum non satis est perspicuus.

6. Interim tamen veritas consensus per series facile ostenditur; cum enim sit $\text{A tang.}\frac{x}{y}=\frac{\pi}{2}-\text{A tang.}\frac{y}{x}$ denotante $\frac{\pi}{2}$ angulum rectum et

$$\text{A tang.}\frac{y}{x}=\frac{y}{x}-\frac{y^3}{3x^3}+\frac{y^5}{5x^5}-\frac{y^7}{7x^7}+\frac{y^9}{9x^9}-\text{etc.},$$

erit

$$\int\frac{dx}{x}\,\text{A tang.}\frac{y}{x}=-\frac{y}{x}+\frac{y^3}{9x^3}-\frac{y^5}{25x^5}+\frac{y^7}{49x^7}-\text{etc.}+f:y$$

et

$$\int\frac{dy}{y}\,\text{A tang.}\frac{x}{y}=\frac{\pi}{2}\,ly-\frac{y}{x}+\frac{y^3}{9x^3}-\frac{y^5}{25x^5}+\frac{y^7}{49x^7}-\text{etc.}+f:x,$$

ex quarum utraque oritur

$$\int\int\frac{dxdy}{xx+yy}=X+Y-\frac{y}{x}+\frac{y^3}{9x^3}-\frac{y^5}{25x^5}+\frac{y^7}{49x^7}-\text{etc.}$$

Verum ubi ambae integrationes succedunt, convenientia sponte se offert; quod quidem pluribus exemplis ostendisse superfluum foret, cum eius ratio ex natura differentialium et integralium perfecte sit demonstrata.

7. Haec igitur tenenda sunt de istiusmodi formulis integralibus duplicatis, quando binae variabiles x et y nullo plane nexu inter se cohaerent, ita ut in altera integratione altera, in altera vero altera constans accipiatur. Verum tales formulae non confundendae sunt cum iis, quibus, ut initio dixi, soliditas et superficies corporum quorumcunque exprimi solet. Etsi enim hae formulae etiam duplicem integrationem requirunt et in priori altera binarum variabilium, puta y, sola ut variabilis tractatur altera x pro constante assumta, tamen priori integratione peracta, ea per omnes valores ipsius y extendi sicque tandem loco y extremus valor, quem recipere potest, statui debet, qui plerumque ab x pendet, ita ut hoc valore post primam integrationem loco y constituto in posteriori integratione y tanquam functio quaedam ipsius x ingrediatur neque propterea pro constanti haberi queat, qua conditione fit, ut altera integratio plurimum immutetur, etsi prior simili modo ut ante absolvatur.

8. Quod discrimen quo clarius perspiciatur, exemplum attulisse iuvabit. Quaeratur ergo soliditas sphaerae, cuius centrum sit C (Fig. 1) et radius $CA=a$, ac primo quidem portio eius quadranti ACB insistens, cuius elementum est columella $YZyz$ areolae $Yy=dxdy$ insistens positis $CP=x$ et $PY=y$, eritque eius altitudo $YZ=\sqrt{(aa-xx-yy)}$; hinc soliditas columellae elementaris $=dxdy\sqrt{(aa-xx-yy)}$, quam bis integrari oportet. Maneat primo intervallum $CP=x$ constans et integrale $\int dy\sqrt{(aa-xx-yy)}$ ita sumtum, ut evanescat posito $y=0$, dabit portiunculam areolae $PpYq$ insistentem, quae ergo erit

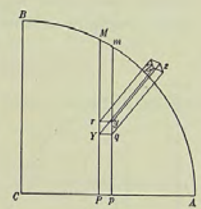

Fig. 1.

$$=\frac{1}{2}\,y\sqrt{(aa-xx-yy)}$$
$$+\frac{1}{2}(aa-xx)\,\text{A sin.}\frac{y}{\sqrt{(aa-xx)}}.$$

Iam hoc valore in altera integratione uti oportet, sed antequam is inducatur, per totam distantiam PM extendi debet, ut habeatur elementum soliditatis toti areolae $PpMm$ insistens; puncto autem Y ad M usque promoto fit $y=\sqrt{(aa-xx)}$, qui ergo valor loco y substitui debet, ita ut in sequente integratione quantitas y minime ut constans consideretur haecque tractandi methodus plurimum a praecedente discrepet.

9. Posito ergo $y = \sqrt{(aa - xx)}$ fit

$$\int dy \sqrt{(aa - xx - yy)} = \frac{\pi}{4}(aa - xx),$$

cum sit A sin. $1 = \frac{\pi}{2}$, sicque integratio adhuc absolvenda erit

$$\int dx \int dy \sqrt{(aa - xx - yy)} = \frac{\pi}{4} \int (aa - xx) dx,$$

ubi quidem unica variabilis x inest, sed non ideo, quod iam hic y pro constanti habeatur, sed quia pro y certa quaedam functio ipsius x est substituta. Haec altera vero integratio ita instituta, ut evanescat posito $x = 0$, dabit soliditatem portionis sphaerae, quae areae $CBMP$ insistit, quae idcirco erit $= \frac{\pi}{4}\left(aax - \frac{1}{3}x^3\right)$; unde sphaerae octans seu portio toti quadranti ACB insistens prodibit punctum P usque in A promovendo, ut fiat $x = a$. Tum ergo soliditas octantis sphaerae erit $= \frac{\pi}{6}a^3$ hincque totius sphaerae $= \frac{4\pi}{3}a^3$, uti constat. Ex quo exemplo intelligitur talem soliditatis investigationem plurimum differre ab integratione duplicata formularum primo exposita.

10. Quodsi non totum octantem sphaerae, sed eam tantum eius portionem, quae areae rectangulari $CEDF$ (Fig. 2) insistit, investigare velimus, prior integratio ut ante instituenda est, sed ea peracta ipsi y valor PM debet tribui, qui quidem est constans, et propterea haec investigatio ad primum genus videtur accedere, verum tamen eo discrepat, quod integrale determinatum prodeat, cum ibi functiones indefinitae X et Y inveherentur. Posito ergo ut ante sphaerae radio $CA = a$ sit rectanguli $CEFD$ latus $CD = e$ et $CE = f$ et solidum elementare areolae $PpYq$ insistens erit ut ante

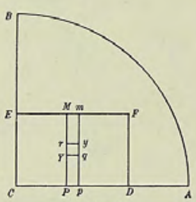

Fig. 2.

$$\frac{1}{2} y \sqrt{(aa - xx - yy)}$$
$$+ \frac{1}{2}(aa - xx) \text{ A sin.} \frac{y}{\sqrt{(aa - xx)}},$$

quod usque ad M extensum, ubi fit $y = f$, erit

$$\frac{1}{2} f \sqrt{(aa - ff - xx)} + \frac{1}{2}(aa - xx) \text{ A sin.} \frac{f}{\sqrt{(aa - xx)}},$$

unde solidum areae $CPEM$ insistens sequenti integrali exprimetur

$$\frac{1}{2} f \int dx \sqrt{(aa - ff - xx)} + \frac{1}{2} \int (aa - xx) dx \text{ A sin.} \frac{f}{\sqrt{(aa - xx)}},$$

si quidem ita definiatur, ut evanescat posito $x = 0$. Evolvamus ergo seorsim has binas formulas.

11. Ac prima quidem statim praebet

$$\int dx \sqrt{(aa - ff - xx)} = \frac{1}{2} x \sqrt{(aa - ff - xx)} + \frac{1}{2}(aa - ff) \text{ A sin.} \frac{x}{\sqrt{(aa - ff)}},$$

altera autem ob

$$d.\text{ A sin.} \frac{f}{\sqrt{(aa - xx)}} = \frac{fxdx}{(aa - xx)\sqrt{(aa - ff - xx)}}$$

ita transformatur

$$\int (aa - xx) dx \text{ A sin.} \frac{f}{\sqrt{(aa - xx)}}$$
$$= \left(aax - \frac{1}{3}x^3\right) \text{ A sin.} \frac{f}{\sqrt{(aa - xx)}} - f \int \frac{\left(aa - \frac{1}{3}xx\right) xxdx}{(aa - xx)\sqrt{(aa - ff - xx)}},$$

ad quam postremam partem integrandam notetur esse

$$\text{A sin.} \frac{fx}{\sqrt{(aa - ff)(aa - xx)}} = \int \frac{afdx}{(aa - xx)\sqrt{(aa - ff - xx)}};$$

huius ergo dabitur multiplum quoddam, quod illi formae adiectum praebeat talem formam

$$\int \frac{\left(aa - \frac{1}{3}xx\right) xxdx}{(aa - xx)\sqrt{(aa - ff - xx)}} + m \text{ A sin.} \frac{fx}{\sqrt{(aa - ff)(aa - xx)}}$$
$$= \int \frac{\left(aaxx - \frac{1}{3}x^4 + maf\right) dx}{(aa - xx)\sqrt{(aa - ff - xx)}},$$

ut $aaxx - \frac{1}{3}x^4 + maf$ fiat per $aa - xx$ divisibile, id quod fit sumendo $m = -\frac{2a^3}{3f}$; hincque erit

$$\int \frac{\left(aa - \frac{1}{3}xx\right) xxdx}{(aa - xx)\sqrt{(aa - ff - xx)}} = \frac{2a^3}{3f} \text{ A sin.} \frac{fx}{\sqrt{(aa - ff)(aa - xx)}} - \frac{1}{3} \int \frac{(2aa - xx)dx}{\sqrt{(aa - ff - xx)}}.$$

12. Cum igitur sit

$$\int\frac{(2aa-xx)dx}{\sqrt{(aa-ff-xx)}}=\frac{1}{2}(3aa+ff)\,\text{A}\sin.\frac{x}{\sqrt{(aa-ff)}}+\frac{1}{2}x\sqrt{(aa-ff-xx)},$$

erit

$$\int\frac{\left(aa-\frac{1}{3}xx\right)xxdx}{(aa-xx)\sqrt{(aa-ff-xx)}}$$

$$=\frac{2a^3}{3f}\,\text{A}\sin.\frac{fx}{\sqrt{(aa-ff)(aa-xx)}}-\frac{1}{6}(3aa+ff)\,\text{A}\sin.\frac{x}{\sqrt{(aa-ff)}}-\frac{1}{6}x\sqrt{(aa-ff-xx)}$$

hincque

$$\int(aa-xx)dx\,\text{A}\sin.\frac{f}{\sqrt{(aa-xx)}}$$

$$=\left(aax-\frac{1}{3}x^3\right)\text{A}\sin.\frac{f}{\sqrt{(aa-xx)}}-\frac{2}{3}a^3\,\text{A}\sin.\frac{fx}{\sqrt{(aa-ff)(aa-xx)}}$$

$$+\frac{1}{6}f(3aa+ff)\,\text{A}\sin.\frac{x}{\sqrt{(aa-ff)}}+\frac{1}{6}fx\sqrt{(aa-ff-xx)}.$$

Quare posito $x=CD=e$ erit soliditas portionis sphaerae rectangulo $CDEF$ insistentis

$$\frac{1}{4}ef\sqrt{(aa-ee-ff)}+\frac{1}{4}f(aa-ff)\,\text{A}\sin.\frac{e}{\sqrt{(aa-ff)}}$$

$$+\frac{1}{6}e(3aa-ee)\,\text{A}\sin.\frac{f}{\sqrt{(aa-ee)}}-\frac{1}{3}a^3\,\text{A}\sin.\frac{ef}{\sqrt{(aa-ee)(aa-ff)}}$$

$$+\frac{1}{12}f(3aa+ff)\,\text{A}\sin.\frac{e}{\sqrt{(aa-ff)}}+\frac{1}{12}ef\sqrt{(aa-ee-ff)},$$

quae expressio reducitur ad hanc

$$\frac{1}{3}ef\sqrt{(aa-ee-ff)}+\frac{1}{6}f(3aa-ff)\,\text{A}\sin.\frac{e}{\sqrt{(aa-ff)}}$$

$$+\frac{1}{6}e(3aa-ee)\,\text{A}\sin.\frac{f}{\sqrt{(aa-ee)}}-\frac{1}{3}a^3\,\text{A}\sin.\frac{ef}{\sqrt{(aa-ee)(aa-ff)}}.$$

13. Si ergo rectanguli terminus F usque ad peripheriam porrigatur, ut sit $ee+ff=aa$, primum membrum evanescit et arcus circulares tria reliqua afficientes abeunt in angulum rectum seu $\frac{\pi}{2}$ eritque soliditas

$$\frac{\pi}{2}\left(\frac{1}{2}aae+\frac{1}{2}aaf-\frac{1}{6}e^3-\frac{1}{6}f^3-\frac{1}{3}a^3\right)$$

seu ob $f=\sqrt{(aa-ee)}$

$$\frac{\pi}{12}\left((2aa+ee)\sqrt{(aa-ee)}-2a^3+3aae-e^3\right),$$

quod solidum fit maximum, si $f=e=\frac{a}{\sqrt{2}}$, fitque id tum $=\frac{\pi a^3(5-2\sqrt{2})}{12\sqrt{2}}$, dum soliditas octantis sphaerae est $=\frac{\pi}{6}a^3$, ita ut nostrum solidum sit ad octantem sphaerae ut $5-2\sqrt{2}$ ad $2\sqrt{2}$. Sin autem punctum F non ad peripheriam quadrantis pertingat fueritque $f=e$, erit soliditas quaesita

$$=\frac{1}{3}ee\sqrt{(aa-2ee)}+\frac{1}{3}e(3aa-ee)\,\text{A}\sin.\frac{e}{\sqrt{(aa-ee)}}-\frac{1}{3}a^3\,\text{A}\sin.\frac{ee}{aa-ee}.$$

Quare si fuerit

$$\text{A}\sin.\frac{e}{\sqrt{(aa-ee)}}:\text{A}\sin.\frac{ee}{aa-ee}=a^3:e(3aa-ee),$$

solidum algebraice exprimetur.

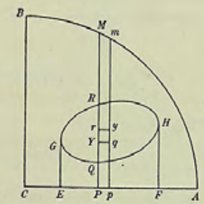

Fig. 3.

14. Quo autem rem generalius complectamur, quaeramus solidum areae cuicunque $GQHR$ (Fig. 3) insistens; cuius elementum cum areolae $Yy=dxdy$ insistat idque sit $=dxdy\sqrt{(aa-xx-yy)}$, prima integratio sumto x constante praebet

$$\frac{1}{2}dx\left(y\sqrt{(aa-xx-yy)}+(aa-xx)\,\text{A}\sin.\frac{y}{\sqrt{(aa-xx)}}\right).$$

Sint iam ex natura curvae $GQHR$ distantiae extremae $PQ=q$ et $PR=r$ atque solidum elementare areolae QR insistens erit

$$\frac{1}{2}dx\left\{\begin{aligned}&+r\sqrt{(aa-xx-rr)}+(aa-xx)\,\text{A}\sin.\frac{r}{\sqrt{(aa-xx)}}\\&-q\sqrt{(aa-xx-qq)}-(aa-xx)\,\text{A}\sin.\frac{q}{\sqrt{(aa-xx)}}\end{aligned}\right\}.$$

Quare cum q et r possint esse functiones quaecunque ipsius x, evidens est, quantum absit, quominus quantitas y in sequente integratione pro constanti habeatur. Sequens autem integratio a valore $x = CE$ usque ad valorem $x = CF$ est extendenda.

15. Si figura basis $GQHR$ a recta CA traiiciatur, ut quaeratur solidum basi CGH (Fig. 4) insistens, cuius natura exprimatur aequatione quacunque inter $CP = x$, $PR = r$, erit solidum

$$\frac{1}{2}\int dx\left(r\sqrt{(aa - xx - rr)} + (aa - xx)\,\text{A}\sin.\frac{r}{\sqrt{(aa - xx)}}\right),$$

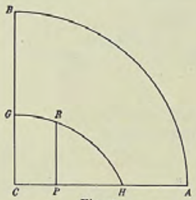

Fig. 4.

ubi problema non inelegans se offert, quo figura basis CGH quaeritur, ut solidum ei insistens algebraice exprimatur.

Statuatur in hunc finem $r = u\sqrt{(aa - xx)}$, ut solidum indefinitum areae $CPRG$ insistens sit

$$\frac{1}{2}\int (aa - xx)dx\left(u\sqrt{(1 - uu)} + \text{A}\sin.u\right),$$

quae expressio transformatur in hanc

$$\frac{1}{2}\left(aax - \frac{1}{3}x^3\right)\left(u\sqrt{(1 - uu)} + \text{A}\sin.u\right) - \int\left(aax - \frac{1}{3}x^3\right)du\sqrt{(1 - uu)}.$$

Fiat iam

$$\int\left(aax - \frac{1}{3}x^3\right)du\sqrt{(1 - uu)} = na^3\,\text{A}\sin.u + a^3U$$

existente U functione algebraica ipsius u, et cum sit soliditas

$$\frac{1}{2}\left(aax - \frac{1}{3}x^3\right)u\sqrt{(1 - uu)} - a^3U + \left(\frac{1}{2}aax - \frac{1}{6}x^3 - na^3\right)\text{A}\sin.u,$$

ea erit algebraica casu $-x^3 + 3aax = 6na^3$, dummodo u evanescat posito $x = 0$; tum enim soliditas erit $= na^3u\sqrt{(1 - uu)} - a^3U$.

16. Ponamus $dU = U'du$ ac prodibit haec inter x et u aequatio

$$aax - \frac{1}{3}x^3 = \frac{na^3}{1 - uu} + \frac{a^3U'}{\sqrt{(1 - uu)}}.$$

Fingatur $U = mu\sqrt{(1 - uu)}$; erit $U' = \frac{m - 2muu}{\sqrt{(1 - uu)}}$, et ut u evanescat posito $x = 0$, debet esse $m = -n$, ut fiat

$$aax - \frac{1}{3}x^3 = \frac{2na^3uu}{1 - uu} \quad \text{seu} \quad u = \sqrt{\frac{3aax - x^3}{6na^3 + 3aax - x^3}}$$

hincque

$$r = \sqrt{\frac{(aa - xx)(3aax - x^3)}{6na^3 + 3aax - x^3}}.$$

Iam ob

$$u\sqrt{(1 - uu)} = \frac{\sqrt{6na^3(3aax - x^3)}}{6na^3 + 3aax - x^3}$$

fit soliditas illa

$$= \frac{2na^3\sqrt{6na^3(3aax - x^3)}}{6na^3 + 3aax - x^3}.$$

Si haec soliditas locum habere debeat facto $x = a$, fit

$$n = \frac{1}{3}, \quad r = \sqrt{\frac{(aa - xx)(3aax - x^3)}{2a^3 + 3aax - x^3}} = \sqrt{\frac{x(a - x)(3aa - xx)}{(a + x)(2a - x)}}$$

ac posito $x = a$ erit soliditas $= \frac{1}{3}a^3$ et curva pro basi inventa est linea quarti ordinis.

17. Quae hic de soliditate portionis sphaericae datae basi insistentis sunt tradita, simili calculo ad quaevis alia corpora accommodari possunt, cum tantum in formula $Zdxdy$ quantitas Z alio modo per x et y determinetur, dum hic erat $Z = \sqrt{(aa - xx - yy)}$. Quin etiam si superficies corporis cuiuscunque datae basi imminens definiri debeat, id integratione gemina similis formulae differentialis $Zdxdy$ eodem modo expedietur. Ita si corpus sit sphaera, elementum superficiei areolae elementari basis $dxdy$ imminentis est $\frac{adxdy}{\sqrt{(aa - xx - yy)}}$, ita ut sit $Z = \frac{a}{\sqrt{(aa - xx - yy)}}$, cuius gemina integratio pari modo pro ratione basis, cui imminens portio superficiei quaeritur, est instituenda. Atque in genere quantitates quaecunque aliae cuiusvis corporis, quae certae basi respondeant, ope similium operationum determinabuntur.

18. Quaecunque ergo Z fuerit functio ipsarum x et y, pro integrali duplicato $\iint Zdxdy$ primo quaeritur integrale $\int Zdy$ quantitate x ut constante spectata idque extendatur per totam quantitatem y sicque extremi valores

ipsius y in computum ingredientur, quae erunt functiones ipsius x ex basis figura cognitae; sicque pro $\int Zdy$ orietur functio ipsius x, quae in dx ducta denuo more solito debet integrari. Idem tenendum est, si ordine inverso primo formula $\int Zdx$ integretur spectato y ut constante; quod integrale dum per totum intervallum x extenditur, extremi valores ipsius x eidem y respondentes, qui erunt functiones ipsius y, invehentur sicque $\int Zdx$ abibit in functionem ipsius y tantum, quae per dy multiplicata denuo ita integrari debet, ut integrale per totum intervallum y extendatur. Utroque scilicet modo integratio per totam basin est extendenda eademque praecepta sunt observanda, qualiscunque Z fuerit functio ipsarum x et y.

19. Basi ergo data determinatio integrationum perinde se habet, ac si quantitas Z esset constans quaereretur que tantum integrale $\iint dxdy$, quo area basis exprimitur. Quare ad praecepta, quae in determinatione horum integralium observari oportet, stabilienda sufficiet posuisse $Z = 1$, ut integrale duplicatum $\iint dxdy$ definiendum sit; sive autem sumatur x sive y, extremi valores utriusque determinabuntur per aequationem basis figuram exprimentem. Scilicet priori integratione peracta, ubi punctum Y (Fig. 5) ubicunque intra terminos extremos erat assumtum, tum hoc punctum in peripheriam basis transferatur, quo pacto x et y fient coordinatae basis, inter quas aequatio datur, ex qua deinceps sive y per x sive x per y determinabitur.

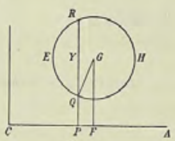

Fig. 5.

20. Quae quo clarius perspiciantur, sumamus basis figuram esse circulum centrum in G et radium GQ habentem ponamusque $CF = f$, $FG = g$ et $GQ = c$; erit puncto Y in peripheriam huius circuli translato

$$cc = (f-x)^2 + (g-y)^2.$$

Iam ad aream huius circuli investigandam sit primo x constans eritque $\int dy = y + C$, et quia y habet geminum valorem in nostra basi

$$y = g \pm \sqrt{(cc-(f-x)^2)},$$

haec integratio ita determinetur, ut integrale evanescat, dum ipsi y minor horum valorum $g - \sqrt{(cc-(f-x)^2)}$ tribuitur, ita ut sit

$$\int dy = y - g + \sqrt{(cc-(f-x)^2)}.$$

Nunc ergo y usque ad alterum terminum $y = g + \sqrt{(cc-(f-x)^2)}$ extenso erit

$$\int dy = 2\sqrt{(cc-(f-x)^2)},$$

quod iam per dx multiplicatum et integratum praebet

$$\int dx \int dy = C - (f-x)\sqrt{(cc-(f-x)^2)} - cc\,\mathrm{A\,sin.}\frac{f-x}{c};$$

quod ut evanescat posito $x = f - c$, fit $C = cc\,\mathrm{A\,sin.}\,1 = \frac{\pi}{2}cc$. Porro statuatur $x = f + c$ et ob $cc\,\mathrm{A\,sin.}\frac{f-x}{c} = -cc\,\mathrm{A\,sin.}\,1 = -\frac{\pi}{2}cc$ erit area quaesita tota $= \frac{\pi}{2}cc + \frac{\pi}{2}cc = \pi cc$, uti constat.

21. Si has determinationes accuratius perpendamus, videmus extremos valores ipsius x ita esse comparatos, ut alter sit maximus, [alter minimus,] siquidem basis tota quadam curva in se redeunte terminetur. Hi ergo ambo valores reperientur, si aequatio naturam basis exprimens differentietur et $dx = 0$ ponatur. Quando autem basis non una quadam linea curva terminatur, sed portione quapiam, veluti CGH (Fig. 4, p. 298), continetur, cuius basis CH sit maxima, tum minor terminus ipsius x manifesto est $= 0$, maior autem ipsi CH aequalis; eodemque casu termini applicatae PR abscissae $CP = x$ respondentis sunt alter $= 0$, alter vero $= CG$. Quacunque ergo basi proposita eius figura ante probe est examinanda ipsiusque termini quaquaversus explorandi, quam investigatio areae vel cuiusvis alius formulae integralis duplicatae suscipi queat; definitis autem terminis, quibus area continetur, inde determinationes integrationum sunt petendae.

22. His de integrationum determinatione expositis insignes maximeque notatu dignae affectiones huiusmodi formularum integralium duplicatarum perpendi merentur, quae in earum transformatione occurrunt. Scilicet quemadmodum coordinatae eiusdem curvae infinitis modis sumi possunt, ita hic loco binarum variabilium x et y binae quaecunque aliae variabiles in computum introduci possunt, sive eae pariter sint coordinatae sive aliae quantitates utcunque definitae. Ita talis transformatio in genere ita concipi potest,

ut loco x et y functiones quaecunque aliarum duarum variabilium t et v substituantur, hisque in aequationem pro basi datam introductis simili modo limites harum quantitatum t et v, quibus figura basis terminatur, definiri poterunt. Utcunque autem hae substitutiones sumantur, tandem post duplicem integrationem semper eadem quantitas resultet necesse est.

23. Si loco x et y aliae quaecunque binae coordinatae orthogonales introducantur, puta t et v, quod fit in genere ponendo

$$x = f + mt + v\sqrt{(1-mm)} \quad \text{et} \quad y = g + t\sqrt{(1-mm)} - mv,$$

manifestum est elementum areae basis, quod ante erat $dxdy$, nunc per $dtdv$ exprimi debere. Cum autem inde sit

$$dx = mdt + dv\sqrt{(1-mm)} \quad \text{et} \quad dy = dt\sqrt{(1-mm)} - mdv,$$

minime patet, quomodo loco $dxdy$ per has substitutiones oriri possit $dtdv$, dum potius prodiret

$$dxdy = mdt^2\sqrt{(1-mm)} + (1-2mm)dtdv - mdv^2\sqrt{(1-mm)},$$

quae autem formula, utcunque ad geminam integrationem adaptatur, semper in maximos errores inducet. Multo minus ergo hinc colligere licet, si loco x et y aliae functiones ipsarum t et v substituantur, cuiusmodi expressio loco $dxdy$ adhiberi debeat.

24. Ac primo quidem observo nullam hic esse rationem, cur expressio loco $dxdy$ in calculum introducenda ei aequalis esse debeat; quod tum demum necesse esset, si binae integrationes eodem modo ut ante secundum binas variabiles instituerentur. Cum autem nunc aliae variabiles t et v adsint atque altera integratio per variabilitatem ipsius t, altera ipsius v sit administranda, quae operationes a praecedentibus plurimum differunt, formula iam loco $dxdy$ inducenda non ex aequalitate aestimari, sed potius ad scopum, qui est propositus, accommodari debet. Et quoniam iam binas integrationes secundum binas variabiles t et v distingui oportet, manifestum est formulam loco $dxdy$ adhibendam necessario producto $dtdv$ affectam esse et huiusmodi formam $Zdtdv$ habere debere.

25. Quo haec certius expediantur, maneat primo x et loco y introducatur alia variabilis u, ita ut sit y functio quaecunque ipsarum x et u et $dy = Pdx + Qdu$. Si iam in priori integratione x constans sumatur, erit utique $dy = Qdu$, hinc $\iint dxdy = \int dx \int Qdu$, ita ut nunc loco formulae $dxdy$ habeatur $Qdxdu$, cuius integrale duplicatum proinde etiam hoc modo exprimi poterit $\int du \int Qdx$, ubi in priori integratione $\int Qdx$ quantitas u sumitur pro constante. Quodsi nunc simili modo u retineatur et loco x introducatur functio quaecunque ipsarum t et u, ut sit $dx = Rdt + Sdu$, in tractatione formulae $\int du \int Qdx$ prior integratio $\int Qdx$, in qua u constans statuitur, abibit in hanc $\int QRdt$, ita ut integrale duplicatum sit $\int du \int QRdt$ seu promiscue $\iint QRdtdu$, unde manifestum est ob has ambas substitutiones loco formulae $dxdy$ hanc $QRdtdu$ tractari debere.

26. Introducamus nunc statim loco x et y has duas novas variabiles t et u, per quas illae ita determinentur, ut sit

$$dx = Rdt + Sdu \quad \text{et} \quad dy = Tdt + Vdu,$$

unde valore ipsius dx in forma $dy = Pdx + Qdu$ substituto fit

$$dy = PRdt + (PS + Q)du,$$

ita ut sit $PR = T$ et $PS + Q = V$, unde fit $P = \frac{T}{R}$ et $\frac{ST}{R} + Q = V$ sicque $QR = VR - ST$. Quare vi harum substitutionum loco $dxdy$ uti debemus formula $(VR - ST)dtdu$, quae bis integrata iustis adhibitis determinationibus aeque aream totius basis praebere debet atque ipsa formula $dxdy$ bis integrata. Quod autem hic pro formula areae baseos $\iint dxdy$ est ostensum, locum habet pro quacunque alia formula $\iint Zdxdy$, quippe quae per easdem substitutiones transformatur in hanc $\iint Z(VR - ST)dtdu$, dummodo in Z loco x et y assumti valores substituantur. Pari enim modo binas integrationes ex figura basis determinari oportet.

27. Quodsi ergo ponatur

$$dx = Rdt + Sdu \quad \text{et} \quad dy = Tdt + Vdu,$$

loco $dxdy$ consequimur $(RV - ST)dtdu$, quae formula plurimum differt ab ea, cui productum $dxdy$ revera est aequale; etiamsi enim termini per dt^2 et du^2 affecti, utpote ad duplicem integrationem inepti, reiiciantur, tamen, quod restat, $(RV+ST)dtdu$ ratione signi a vera formula discrepat. Verum hic non leve dubium exoritur, quod, cum coordinatae x et y pari passu ambulent, nostra formula potius differentiam $RV-ST$ quam inversam $ST-RV$ complectatur; quod dubium eo magis augetur, quod, si superius ratiocinium respectu x et y invertissemus, eaedem substitutiones nos revera ad formulam $(ST-RV)dtdu$ perduxissent. Sed quia totum discrimen tantum in signo versatur alteraque formula alterius est negativa, hinc determinatio absoluta areae basis, quippe cuius quantitas absoluta quaeritur, nullam mutationem realem patitur.

28. Haec autem magis fient perspicua, si modum, quo supra (§ 20) ad aream $EQHR$ (Fig. 5, p. 300) inveniendam usi sumus, attentius consideremus. Primum scilicet ex integratione formulae $\iint dxdy$ deduximus hanc aream

$$=\int dx(PR-PQ),$$

ubi quidem PQ a PR subtraximus, quia manifesto erat $PR > PQ$; sed in ipso calculo nulla continetur ratio, quae praecipiat, ut potius PQ a PR quam vicissim PR a PQ subtrahamus, sicque non adversante calculo potuissemus aequo iure eandem aream per $\int dx(PQ-PR)$ exprimere, quo pacto ea negativa, sed priori aequalis proditura fuisset. Ex quo perspicuum est signum $+$ vel $-$ non quantitatem areae, quae quaeritur, afficere et calculum pari iure ad utrumque perducere posse. Quam ob causam superius dubium ita diluetur, ut dicamus aream quaesitam ita exprimi debere, ut sit $=\pm\iint dtdu(RV-ST)$, et ut area positive expressa prodeat, quovis casu eo signo utendum esse, quo $\pm(RV-ST)$ reddatur quantitas positiva.

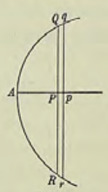

Fig. 6.

29. Hinc etiam dubia, quae forte oriri possent circa inventionem areae curvarum, quarum partes utrinque ad axem sunt dispositae et quibus tirones saepe non parum turbari solent, facile resolvuntur. Si enim curvae QAR (Fig. 6) ad axem AP relatae area tota QAR abscissae $AP=x$ respondens definiri debeat eiusque partes APQ et APR seorsim considerentur, certum est, si altera APQ affirmative spectetur, ut sit $=+Q$, alteram APR negative concipi debere, ut sit

$=-R$. Neque tamen hinc sequitur aream totam QAR fore $=Q-R$, quippe quae evanesceret, si ambae partes APQ et APR essent aequales; sed perinde ac si ambo puncta Q et R ad eandem axis partem sita essent, area perpetuo est $=\pm\int dx(PR-PQ)$, unde ob

$$\int PQ\cdot dx=Q \quad \text{et} \quad \int PR\cdot dx=-R$$

fit tota area $=\pm(Q+R)$, uti rei natura postulat.

30. Ope autem talium substitutionum, quibus loco binarum variabilium x et y binae quaecunque aliae introducantur t et u, saepenumero integrationes plurimum sublevari facilioresque reddi possunt et quovis casu haud difficile est substitutiones maxime idoneas reperire. Veluti si area circuli $EQHR$ (Fig. 5, p. 300) ad axem CP relati definiri debeat, ubi ob $CF=f$, $FG=g$, $GQ=c$ erat $cc=(f-x)^2+(g-y)^2$, poni conveniet

$$f-x=\frac{t}{\sqrt{(1+uu)}} \quad \text{et} \quad g-y=\frac{tu}{\sqrt{(1+uu)}},$$

ut fiat $tt=cc$ et $t=c$. Tum vero ob

$$dx=\frac{-dt}{\sqrt{(1+uu)}}+\frac{tudu}{(1+uu)^{\frac{3}{2}}} \quad \text{et} \quad dy=\frac{-udt}{\sqrt{(1+uu)}}-\frac{tdu}{(1+uu)^{\frac{3}{2}}}$$

loco $dxdy$ per § 27 adipiscimur $dtdu\left(\frac{t}{(1+uu)^2}+\frac{tuu}{(1+uu)^2}\right)=\frac{tdtdu}{1+uu}$, cuius duplex integrale ita exprimatur

$$\int\frac{du}{1+uu}\int tdt.$$

Iam vero est $\int tdt=\frac{1}{2}tt=\frac{1}{2}cc$ et area tota erit $\frac{1}{2}cc\int\frac{du}{1+uu}$, dum ipsi u omnes valores possibiles tribuuntur, quandoquidem u non amplius aequationem pro basi afficiebat.

31. Quo hunc usum clarius explicemus, consideremus iterum sphaeram centrum C et radium $CA=a$ habentem, cuius portio basi circulari perpendiculariter insistens quaeri debeat. Quia radium CA per centrum huius circuli G ducere licet, sit $FG=g=0$, ut fiat $cc=(f-x)^2+yy$ et solidum

quaesitum $= \iint dx dy \sqrt{(aa - xx - yy)}$; statuatur iam

$$x = \frac{t}{\sqrt{(1+uu)}} \quad \text{et} \quad y = \frac{tu}{\sqrt{(1+uu)}},$$

ut fiat $xx + yy = tt$ et $\sqrt{(aa - xx - yy)} = \sqrt{(aa - tt)}$ et pro $dxdy$ prodeat $\frac{tdtdu}{1+uu}$, ita ut soliditas quaesita ita exprimatur $\iint \frac{tdtdu\sqrt{(aa-tt)}}{1+uu}$, quae integrationes determinari debebunt ex aequatione hinc pro figura basis oriunda $cc = ff - \frac{2ft}{\sqrt{(1+uu)}} + tt$, unde fit

$$\text{vel} \quad t = \frac{f \pm \sqrt{(cc + ccuu - ffuu)}}{\sqrt{(1+uu)}} \quad \text{vel} \quad \sqrt{(1+uu)} = \frac{2ft}{ff - cc + tt}.$$

32. Consideretur primo t ut constans fietque integrale

$$= \int t dt \sqrt{(aa - tt)} \cdot \text{A tang.}\, u,$$

ubi constantem adiici non est necesse, quia evanescente u simul y evanescit; quaeramus enim primo solidum semicirculo insistens. At integrali hoc primo extenso ad terminum extremum ob A tang. u = A cos. $\frac{1}{\sqrt{(1+uu)}}$ fit id

$$\int t dt \sqrt{(aa - tt)} \cdot \text{A cos.}\, \frac{ff - cc + tt}{2ft},$$

cuius integrationis limites sunt $t = f - c$ et $t = f + c$. Si non soliditatem huius portionis sphaerae, sed eius superficiem basi quasi imminentem definire voluissemus, perventuri fuissemus ad hanc formulam

$$\int \frac{atdt}{\sqrt{(aa - tt)}} \text{A cos.}\, \frac{ff - cc + tt}{2ft};$$

at operae pretium non videtur eius integrationem fusius prosequi.

33. Methodus autem huiusmodi formulas integrales duplicatas tractandi haud parum illustrabitur, si eam ad problema illud quondam famosum Florentinum[1]) accommodemus, quo in superficie sphaerica portio geometrice assignabilis requirebatur, cuius superficies algebraice exprimi possit. Immineat talis

1) Quod problema primum anno 1692 a V. VIVIANI (1622—1703) ficto nomine *D. Pio Lisci Posillo geometra* propositum est, Acta erud. 1692, p. 274. Vide etiam notam p. 313 huius voluminis. A. G.

sphaerae portio curvae GRH (Fig. 4, p. 298), cuius propterea figura est determinanda; in qua si ponatur $CP = x$, $PR = y$, superficies sphaerae imminens hac formula integrali duplicata exprimitur $\iint \frac{adxdy}{\sqrt{(aa - xx - yy)}}$. Iam nulla substitutione adhibita si primo x pro constante habeatur, prodibit

$$\int adx \text{ A sin.}\, \frac{y}{\sqrt{(aa - xx)}},$$

qua portio sphaerae aream indefinitam $CPRG$ tegens exprimitur, et quaestio nunc huc redit, ut eiusmodi aequatio algebraica inter x et y assignetur, unde pro tota area $CHRG$ portio superficiei sphaericae ei respondentis fiat algebraice assignabilis.

34. Ponamus brevitatis gratia $\frac{y}{\sqrt{(aa - xx)}} = v$, ut sit $y = v\sqrt{(aa - xx)}$ ac posito $x = 0$ fiat $v = n$; quoniam superius integrale evanescere debet posito $x = 0$, erit ergo superficies sphaerica aream indefinitam $CPRG$ tegens

$$= ax \text{ A sin.}\, v - a \int \frac{xdv}{\sqrt{(1 - vv)}}$$

sumto hoc integrali ita, ut evanescat posito $x = 0$. Statuatur nunc

$$\int \frac{xdv}{\sqrt{(1 - vv)}} = f \text{ A sin.}\, v - aV$$

denotante V functionem quamcunque algebraicam ipsius v, quae abeat in N posito $x = 0$, eritque superficies nostra

$$= ax \text{ A sin.}\, v - af \text{ A sin.}\, v + aaV + af \text{ A sin.}\, n - aaN$$

atque x per v ita determinabitur, ut sit

$$x = f - \frac{adV\sqrt{(1 - vv)}}{dv};$$

sit iam $CH = h$ ac ponatur $x = h$, quo casu fiat $v = m$ et $V = M$, et cum superficies proposita sit

$$ah \text{ A sin.}\, m - af \text{ A sin.}\, m + aaM + af \text{ A sin.}\, n - aaN,$$

ea algebraica esse nequit, nisi sit

$$h \text{ A sin.}\, m - f \text{ A sin.}\, m + f \text{ A sin.}\, n = 0.$$

35. Hic igitur primo arcus, quorum sinus sunt m et n, inter se commensurabiles reddi debent, nisi forte sit $n=0$, quo casu sufficit fieri $h=f$. Quod etsi facile infinitis modis praestari potest, tamen hoc problema multo facilius adhibendis substitutionibus ante expositis resolvetur. Ponatur ergo

$$x=\frac{t}{\sqrt{(1+uu)}}\quad\text{et}\quad y=\frac{tu}{\sqrt{(1+uu)}},$$

ut fiat $xx+yy=tt$ et pro $dxdy$ prodeat $\frac{tdtdu}{1+uu}$, atque superficies portionis sphaericae hac formula integrali duplicata exprimetur $\iint\frac{atdtdu}{(1+uu)\sqrt{(aa-tt)}}$. Sumatur primo u constans; erit ea $=\int\frac{adu}{1+uu}(b-\sqrt{(aa-tt)})$, quae iam facile absolute integrabilis reddi potest; ponatur enim aequalis functioni algebraicae cuicunque ipsius u, quae sit $=V$, eritque $b-\sqrt{(aa-tt)}=\frac{dV(1+uu)}{adu}$ et portio superficiei sphaericae adeo indefinita erit $=V$, ubi pro V functionem algebraicam quamcunque ipsius u accipere licet.

36. Simplicissimae solutiones deducentur ex hac hypothesi

$$V=\frac{a(\alpha+\beta u)}{\sqrt{(1+uu)}},$$

unde fit $\frac{dV}{adu}=\frac{-\alpha u+\beta}{(1+uu)^{\frac{3}{2}}}$ hincque $b-\sqrt{(aa-tt)}=\frac{\beta-\alpha u}{\sqrt{1+uu}}$. Ponatur $b=0$, et cum per substitutiones sit $u=\frac{y}{x}$ et $t=\sqrt{(xx+yy)}$, erit pro curva quaesita

$$\sqrt{(xx+yy)(aa-xx-yy)}=\alpha y-\beta x$$

et pro superficie

$$V=\frac{a(\alpha x+\beta y)}{\sqrt{(xx+yy)}}.$$

Hinc casus simplicissimus oritur ponendo $\beta=0$ et $\alpha=a$, unde prodit $aaxx-(xx+yy)^2=0$ seu $yy=ax-xx$, ita ut curva GRH sit circulus diametro AC descriptus et $V=\frac{aax}{\sqrt{(xx+yy)}}$. Infiniti alii circuli diametrum $=a$ habentes ac per centrum sphaerae transeuntes reperiuntur, si sit $\beta=\sqrt{(aa-\alpha\alpha)}$,

unde fit

$$ax+y\sqrt{(aa-\alpha\alpha)}=xx+yy\quad\text{et}\quad V=\frac{a(\alpha x+y\sqrt{(aa-\alpha\alpha)})}{\sqrt{(xx+yy)}}=a\sqrt{(xx+yy)},$$

ubi notandum est quantitatem V pro natura rei constantem quandam assumere.

37. Concipiatur ergo octans sphaerae super quadrante ACB (Fig. 7) extractus, cuius radius $CA=a$, qui simul sit diameter semicirculi CRA; in quo si ducatur corda quaecunque CR et perpendiculum RP, ut sit $CP=x$ et $PR=y$, erit $CR=t$ et u erit tangens anguli ACR. Quoniam igitur posuimus $b=0$, prius integrale, quo u erat constans, est $\sqrt{(aa-tt)}$; quod cum evanescat, si $t=a$, evidens est id non per cordam $CR=t$, sed per eius complementum RS extendi. Hinc repetita integratio $\int\frac{adu}{1+uu}\sqrt{(aa-tt)}$ eam sphaericae superficiei portionem exprimit, quae trilineo $RVAS$ imminet, quae ergo ob $\sqrt{(aa-tt)}=\frac{au}{\sqrt{(1+uu)}}$ est

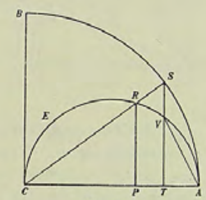

Fig. 7.

$$=\frac{-aa}{\sqrt{(1+uu)}}+aa,$$

integrali scilicet ita sumto, ut evanescat cum angulo ACR. Quare ob

$$\frac{1}{\sqrt{(1+uu)}}=\cos. ACR$$

ducto perpendiculo ST erit illa superficies $=a(a-CT)=CA\cdot AT=AV^2$ ducta corda AV. Consequenter portio superficiei sphaerae spatio $CERASB$ inter quadrantem et semicirculum intercepto imminens aequatur quadrato radii sphaerae.

38. Contemplemur autem adhuc eiusmodi casum, quo prima integratio evanescat posito $t=0$, seu sit $b=a$ ac ponatur $V=\frac{1}{2}aau$, quae expressio simul superficiem quaesitam praebet. Erit ergo

$$a-\sqrt{(aa-tt)}=\frac{1}{2}a(1+uu)\quad\text{et}\quad\sqrt{(aa-tt)}=\frac{1}{2}a(1-uu),$$

ita ut sit

$$t = \frac{1}{2} a\sqrt{(3 + 2uu - u^4)} \quad \text{seu} \quad t = \frac{1}{2} a\sqrt{(1 + uu)(3 - uu)},$$

ubi est $CR = t$ (Fig. 8) et u denotat tangentem anguli ACR. Ex hac aequatione patet, si sit $u = 0$, fore $t = \frac{a\sqrt{3}}{2}$; scilicet curva quaesita radio AC ita in E occurrit, ut sit $CE = CA \cdot \frac{\sqrt{3}}{2}$, eique perpendiculariter insistit. Tum si angulus ACR augeatur ad semirectum ACF, ut fiat $u = 1$, erit $t = a$ hocque casu curva per ipsum punctum F transit ibique quadrantem osculabitur; ac simul distantia t fit maxima. Dehinc curva introrsum reflectitur et t evanescit, si $u = \sqrt{3}$; hoc est, curva centro C ita immergitur, ut eius tangens in C cum radio CA faciat angulum 60°.

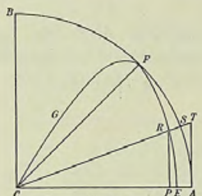

Fig. 8.

39. Tota ergo curva in quadrante descripta figuram habebit $ERFGC$ et ducta in ea ex C recta utcunque CR angulique ECR tangens sit $= u$; tum portio superficiei sphaericae sectori ECR imminens algebraice poterit assignari eritque ea $= \frac{1}{2} aau$. Quare si CR ad occursum cum tangente AT producatur, ob $AT = au$ ea portio praecise aequabitur triangulo CAT et portio imminens sectori ECF erit $= \frac{1}{2} aa$; si autem angulus ECR maior semirecto sumatur, ut sit $u > 1$, quia tum $\sqrt{(aa - tt)} = \sqrt{(aa - xx - yy)}$, quae est elevatio superficiei sphaericae supra quadrantem, fit negativa, superficies in inferiori octante capi debet. Quodsi huius curvae aequationem inter coordinatas $CP = x$ et $PR = y$ desideremus, ob $tt = xx + yy$ et $u = \frac{y}{x}$ habebimus

$$4xx + 4yy = aa\left(3 + \frac{2yy}{xx} - \frac{y^4}{x^4}\right) = \frac{aa(xx + yy)(3xx - yy)}{x^4},$$

quae divisa per $xx + yy$ praebet

$$4x^4 = 3aaxx - aayy \quad \text{seu} \quad yy = 3xx - \frac{4x^4}{aa}.$$

40. Hanc solutionem reddere possumus generaliorem ponendo $V = abu$ fietque $a - \sqrt{(aa - tt)} = b(1 + uu)$, hinc $\sqrt{(aa - tt)} = a - b - buu$, ergo

$$tt = 2ab - bb + 2(a - b)buu - bbu^4 = (1 + uu)(2ab - bb - bbuu).$$

Qua ad coordinatas orthogonales translata divisio per $xx + yy$ iterum succedet fietque

$$x^4 = (2ab - bb)xx - bbyy \quad \text{seu} \quad y = \frac{x}{b}\sqrt{(2ab - bb - xx)}$$

ac portio superficiei sphaericae sectori ECR huius curvae imminens erit $= \frac{aby}{x} = b \cdot AT$; quae expressio locum habet, quamdiu $uu < \frac{a-b}{b}$, hoc est, donec anguli ECR tangens fiat $= \sqrt{\frac{a-b}{b}}$, ubi fit $t = a$. Tum vero angulo ECR ultra aucto perpendiculares super curva erectae ad hemisphaerium inferius protendi debent, quo casu superficies eo magis augetur. Si ergo sit $b = a$, quia $\sqrt{(aa - tt)}$ ubique fit quantitas negativa, quantitas $b \cdot AT$ portionem sphaericae superficiei ad inferius hemisphaerium continuatae exprimit.

41. Sit adhuc $b = a$ ac ponatur $V = \frac{a^2(\alpha + \beta u)}{\sqrt{(1 + uu)}} - \alpha a^2$, ut superficies assignanda evanescat posito $u = 0$, eritque

$$a - \sqrt{(aa - tt)} = \frac{a(\beta - \alpha u)}{\sqrt{(1 + uu)}} \quad \text{et} \quad \sqrt{(aa - tt)} = a - \frac{a(\beta - \alpha u)}{\sqrt{(1 + uu)}},$$

ubi notandum est, si haec expressio fiat negativa, ibi in hemisphaerium inferius descendi. Ex his autem prodit

$$\frac{tt}{aa} = \frac{2(\beta - \alpha u)}{\sqrt{(1 + uu)}} - \frac{(\beta - \alpha u)^2}{1 + uu}.$$

Quare evanescente angulo ECR, cuius tangens $= u$, erit $\frac{tt}{aa} = 2\beta - \beta\beta$, at si $u = \frac{\beta}{\alpha}$, evanescit t. Pro altera parte axis CA fit u negativum ac posito $u = -v$ habetur superficies negative expressa $V = \frac{a^2(\alpha - \beta v)}{\sqrt{(1 + vv)}} - \alpha a^2$ et curva hac definietur aequatione

$$\frac{tt}{aa} = \frac{2(\beta + \alpha v)}{\sqrt{(1 + vv)}} - \frac{(\beta + \alpha v)^2}{1 + vv},$$

unde posito v infinito prodit $\frac{tt}{aa} = 2\alpha - \alpha\alpha$; ubi recta CR fit in curvam normalis, quod etiam evenit, ubi $v = \frac{\alpha}{\beta}$ et $\frac{tt}{aa} = 2\sqrt{(\alpha\alpha + \beta\beta)} - \alpha\alpha - \beta\beta$. Quare ne fiat t imaginarium, oportet sit $\sqrt{(\alpha\alpha + \beta\beta)} < 2$.

42. Consideremus casum, quo $\alpha = -\frac{1}{\sqrt{2}}$ et $\beta = \frac{1}{\sqrt{2}}$, ut sit superficies

$$V = aa\left(\frac{1}{\sqrt{2}} - \frac{1-u}{\sqrt{2(1+uu)}}\right) \quad \text{et} \quad \frac{tt}{aa} = \frac{2(1+u)}{\sqrt{2(1+uu)}} - \frac{(1+u)^2}{2(1+uu)},$$

ubi patet, si $u = -1$, fore $t = 0$; tum vero, ut sequitur,

$$\text{si } u = 0, \qquad \text{si } u = 1, \qquad \text{si } u = 7, \qquad \text{si } u = \infty,$$

erit

$$t = a\sqrt{\frac{2\sqrt{2}-1}{2}}, \qquad t = a, \qquad t = a\sqrt{\frac{24}{25}}, \qquad t = a\sqrt{\frac{2\sqrt{2}-1}{2}},$$

ubi notandum casibus $u = 1$ et $u = \infty$ rectam CR fore in curvam normalem. In hoc ergo quadrante curva nostra fere cum quadrante confunditur, cum ubique sit proxime $t = a$, cui portio superficiei sphaericae imminens erit $= aa\sqrt{2}$, quae deficit a superficie totius octantis, quae est $\frac{\pi}{2}aa$, parte satis parva $aa\left(\frac{\pi}{2} - \sqrt{2}\right) = 0{,}15658aa$. Ad alteram axis CA partem haec curva in centrum incidit, ubi tangens cum CA faciet angulum semirectum.

43. Verum solutio § 35 data multo magis amplificari potest; cum enim superficies sphaerae assignanda hac formula exprimatur $\int\frac{adu}{1+uu}\int\frac{tdt}{\sqrt{(aa-tt)}}$ et in integratione $\int\frac{tdt}{\sqrt{(aa-tt)}}$ quantitas u ut constans consideretur, integrale ita exhiberi poterit $U - \sqrt{(aa-tt)}$ denotante U functionem quamcunque ipsius u; quae formula quoniam evanescit, si $\sqrt{(aa-tt)} = U$ et $t = \sqrt{(aa-UU)}$, ab hoc termino quantitas t ulterius protendi est concipienda. Denotet iam V aliam quamcunque functionem ipsius u, quae abeat in C posito $u = 0$, ac ponatur superficies

$$\int\frac{adu}{1+uu}\left(U - \sqrt{(aa-tt)}\right) = aV - aC$$

eritque hinc

$$U - \sqrt{(aa-tt)} = \frac{dV(1+uu)}{du}$$

ideoque

$$\sqrt{(aa-tt)} = U - \frac{dV(1+uu)}{du},$$

unde alter terminus ipsius t definitur.

44. Hinc igitur solutio problematis Florentini[1]) ita generalissime adornabitur. Constituto quadrante circuli ACB (Fig. 9), cui octans sphaerae insistat, radio CA existente $= a$, ductoque radio quocunque CS vocetur anguli ACS tangens $= u$; tum primo curva EQG ita construatur, ut sit

$$CQ = \sqrt{(aa - UU)}$$

et perpendiculum ex Q ad sphaericam usque superficiem erectum

$$QM = U$$

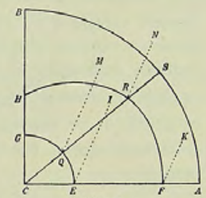

Fig. 9.

denotante U functionem quamcunque algebraicam ipsius u. Si $u = 0$, abeat CQ in CE et QM in EI. Deinde alia describatur curva FRH, ut sit

$$CR = \sqrt{\left(aa - \left(U - \frac{dV(1+uu)}{du}\right)^2\right)}$$

et perpendiculum ex R ad sphaeram usque pertingens

$$RN = U - \frac{dV(1+uu)}{du}$$

denotante V aliam quamcunque functionem algebraicam ipsius u, quae abeat in C, si $u = 0$; quo casu simul CR in CF et RN in FK abeat. Iam his duabus curvis constructis portio superficiei sphaericae areae $EQRF$ imminens et intra terminos I, K, M, N contenta algebraice exprimetur eritque $= a(V - C)$.

1) Primas solutiones problematis florentini (vide notam p. 306) dederunt G. G. L[EIBNIZ], *Constructio testudinis quadrabilis hemisphaericae*, Acta erud. 1692, p. 275, et I[AC.] B[ERNOULLI], *Aenigmatis florentini solutiones varie infinitae*, ib. p. 370; *Opera* p. 512. Vide porro I. WALLIS, *Algebra*, cap. 112: Problema Florentinum, de mira templi testudine quadrabili; *Opera* t. II, Oxoniae 1693, p. 478; G. GRANDI, *Geometrica demonstratio Vivianeorum problematum* etc., Florentiae 1699; E. OFFENBURG, *Annotationes in epistolam mensi Julio Act. erud. superioris anni insertam* etc., Acta erud. 1718, p. 164; I. HERMANN, *De epicycloidibus in superficie sphaerica descriptis*, Comment. acad. sc. Petrop. 1 (1726), 1728, p. 210; IOH. BERNOULLI, *Problème sur les epicycloïdes sphériques*, Mém. de l'acad. d. sc. de Paris 1732, p. 237; *Opera omnia* t. III, p. 216. A. G.

45. Haec de natura formularum integralium duplicatarum commentandi occasionem praebuit[1]) problema aeque elegans atque utile in Analysi, si quidem eius solutionem evolvere liceret. Quaerebatur scilicet inter omnia corpora eiusdem soliditatis id, quod minima superficie contineretur, quod quidem ad ternas coordinatas orthogonales x, y et z relatum posito $dz = pdx + qdy$ ita analytice exprimitur, ut inter omnes relationes harum trium variabilium, quae eandem quantitatem huius formulae integralis duplicatae $\iint zdxdy$ contineant, ea definiatur, cui minima quantitas huius $\iint dxdy\sqrt{(1+pp+qq)}$ respondeat. Quod problema si per theoriam variationum aggrediamur, effici oportebit, ut fiat

$$a\delta\iint dxdy\sqrt{(1+pp+qq)} = \delta\iint zdxdy,$$

ita ut totum negotium ad variationes huiusmodi formularum integralium duplicatarum indagandas reducatur.

46. Quoniam utraque formula duplicem integrationem exigit, si in priori x pro constante habeatur, nostra aequatio ita repraesentabitur

$$a\delta\int dx\int dy\sqrt{(1+pp+qq)} = \delta\int dx\int zdy.$$

Verum hic probe animadvertendum est, postquam integralia

$$\int dy\sqrt{(1+pp+qq)} \quad \text{et} \quad \int zdy$$

fuerint inventa, tum variabilem y non amplius indefinitam seu ab x non pendentem relinqui, quin potius pro y certam functionem ipsius x, quam figura corporis exigit, substitui oportere, ita ut in secunda integratione quantitas y non ut constans seu ab x non pendens spectari queat. Quia autem ob figuram corporis etiamnunc incognitam ista functio non constat, neutiquam apparet, quomodo variationes istiusmodi formularum duplicatarum determinari debeant.

1) Vide L. EULERI *Institutionum calculi integralis* vol. III, Appendix § 174; LEONHARDI EULERI *Opera omnia*, series I, vol. 13, p. 468. A. G.

47. Ipsa vero huius quaestionis natura alias praeterea determinationes requirere videtur, quarum ratio in solutione haberi debeat. Nam quemadmodum, si curva quaeritur, quae inter omnes alias eandem aream includentes brevissimo arcu contineatur, non solum basis AP (Fig. 10), sed etiam duo puncta B et M, per quae curva transeat, praescribi solent, ita etiam in nostro problemate non modo basis, cui corpus tanquam columna insistat, pro cognita assumi debere videtur, sed etiam ipsi extremi termini superficiei quaesitae. Quodsi enim hae res non praescribantur omnes, ne quaestioni quidem certae locus relinquitur; nam, etiamsi basis praescriberetur, termini vero supremi superficiei arbitrio nostro relinquerentur, manifestum est, quo altior fuerit columna, eo magis soliditatem auctum iri eadem manente superficie suprema, quandoquidem superficies laterum non in computum ducitur. Multo minus autem problema sine basis praescriptione ullam vim retineret, quoniam basi coarctanda quantumvis magna soliditas cum minima superficie posset esse coniuncta.

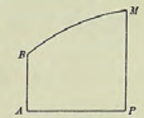

Fig. 10.

EVOLUTIO FORMULAE INTEGRALIS $\int x^{f-1}dx(lx)^{\frac{m}{n}}$ INTEGRATIONE A VALORE $x=0$ AD $x=1$ EXTENSA

Commentatio 421 indicis ENESTROEMIANI
Novi commentarii academiae scientiarum Petropolitanae **16** (1771), 1772, p. 91—139
Summarium ibidem p. 15

SUMMARIUM

Quum, qui in hac Dissertatione occurrunt, calculi analytici ita comparati sint, ut eorum summarium quoddam vix tradi queat, Mathematum peritos ad ipsam dissertationem ablegamus, id tantum monuisse contenti, quod Illustr. EULERUS heic fusius prosequutus sit, quae olim in Tomo V Comment. Petropol.[1]) de integrationibus formularum differentialium sub certis conditionibus peragendis iam exposuit.

THEOREMA 1

1. *Si n denotat numerum integrum positivum quemcunque et formulae*

$$\int x^{f-1}dx(1-x^g)^n$$

integratio a valore $x=0$ usque ad $x=1$ extendatur, erit eius valor

$$=\frac{g^n}{f}\cdot\frac{1\cdot 2\cdot 3\cdots n}{(f+g)(f+2g)(f+3g)\cdots(f+ng)}.$$

1) Vide p. 319. A. G.

DEMONSTRATIO

Notum est in genere integrationem formulae $\int x^{f-1}dx(1-x^g)^m$ reduci posse ad integrationem huius $\int x^{f-1}dx(1-x^g)^{m-1}$, quoniam quantitates constantes A et B ita definire licet, ut fiat

$$\int x^{f-1}dx(1-x^g)^m = A\int x^{f-1}dx(1-x^g)^{m-1} + Bx^f(1-x^g)^m;$$

sumtis enim differentialibus prodit haec aequatio

$$x^{f-1}dx(1-x^g)^m$$
$$= Ax^{f-1}dx(1-x^g)^{m-1} + Bfx^{f-1}dx(1-x^g)^m - Bmgx^{f+g-1}dx(1-x^g)^{m-1},$$

quae per $x^{f-1}dx(1-x^g)^{m-1}$ divisa dat

$$1-x^g = A + Bf(1-x^g) - Bmgx^g$$

seu

$$1-x^g = A - Bmg + B(f+mg)(1-x^g);$$

quae aequatio ut consistere possit, necesse est sit

$$1 = B(f+mg) \quad \text{et} \quad A = Bmg,$$

unde colligimus

$$B = \frac{1}{f+mg} \quad \text{et} \quad A = \frac{mg}{f+mg}.$$

Quocirca habebimus sequentem reductionem generalem

$$\int x^{f-1}dx(1-x^g)^m = \frac{mg}{f+mg}\int x^{f-1}dx(1-x^g)^{m-1} + \frac{1}{f+mg}x^f(1-x^g)^m;$$

quae cum evanescat posito $x=0$, siquidem sit $f>0$, constantis additione haud est opus. Quare extenso utroque integrali usque ad $x=1$ pars integralis postrema sponte evanescit eritque pro casu $x=1$

$$\int x^{f-1}dx(1-x^g)^m = \frac{mg}{f+mg}\int x^{f-1}dx(1-x^g)^{m-1}.$$

Cum igitur sumto $m=1$ sit

$$\int x^{f-1}dx(1-x^g)^0 = \frac{1}{f}x^f = \frac{1}{f}$$

posito $x=1$, nanciscimur pro eodem casu $x=1$ sequentes valores

$$\int x^{f-1}dx(1-x^g)^1=\frac{g}{f}\cdot\frac{1}{f+g},$$
$$\int x^{f-1}dx(1-x^g)^2=\frac{g^2}{f}\cdot\frac{1}{f+g}\cdot\frac{2}{f+2g},$$
$$\int x^{f-1}dx(1-x^g)^3=\frac{g^3}{f}\cdot\frac{1}{f+g}\cdot\frac{2}{f+2g}\cdot\frac{3}{f+3g}$$

hincque pro numero quocunque integro positivo n concludimus fore

$$\int x^{f-1}dx(1-x^g)^n=\frac{g^n}{f}\cdot\frac{1}{f+g}\cdot\frac{2}{f+2g}\cdot\frac{3}{f+3g}\cdots\frac{n}{f+ng},$$

si modo numeri f et g sint positivi.

COROLLARIUM 1

2. Hinc ergo vicissim valor huiusmodi producti ex quotcunque factoribus formati per formulam integralem exprimi potest, ita ut sit

$$\frac{1\cdot2\cdot3\cdots n}{(f+g)(f+2g)(f+3g)\cdots(f+ng)}=\frac{f}{g^n}\int x^{f-1}dx(1-x^g)^n$$

integrali hoc a valore $x=0$ usque ad $x=1$ extenso.

COROLLARIUM 2

3. Quodsi ergo huiusmodi habeatur progressio

$$\frac{1}{f+g},\quad\frac{1\cdot2}{(f+g)(f+2g)},\quad\frac{1\cdot2\cdot3}{(f+g)(f+2g)(f+3g)},\quad\frac{1\cdot2\cdot3\cdot4}{(f+g)(f+2g)(f+3g)(f+4g)}\quad\text{etc.},$$

eius terminus generalis, qui indici indefinito n convenit, commode hac forma integrali $\frac{f}{g^n}\int x^{f-1}dx(1-x^g)^n$ repraesentatur, cuius ope ea progressio interpolari eiusque termini indicibus fractis respondentes exhiberi poterunt.

COROLLARIUM 3

4. Si loco n scribamus $n-1$, habebimus

$$\frac{1\cdot2\cdot3\cdots(n-1)}{(f+g)(f+2g)(f+3g)\cdots(f+(n-1)g)}=\frac{f}{g^{n-1}}\int x^{f-1}dx(1-x^g)^{n-1},$$

quae per $\frac{n}{f+ng}$ multiplicata praebet

$$\frac{1\cdot2\cdot3\cdots n}{(f+g)(f+2g)(f+3g)\cdots(f+ng)}=\frac{f\cdot ng}{g^n(f+ng)}\int x^{f-1}dx(1-x^g)^{n-1}.$$

SCHOLION 1

5. Hanc posteriorem formam immediate ex praecedente derivare licuisset, cum modo demonstraverimus esse

$$\int x^{f-1}dx(1-x^g)^n=\frac{ng}{f+ng}\int x^{f-1}dx(1-x^g)^{n-1},$$

siquidem utrumque integrale a valore $x=0$ usque ad $x=1$ extendatur; quam integralium determinationem in sequentibus ubique subintelligi oportet. Deinde etiam perpetuo est tenendum quantitates f et g esse positivas, quippe quam conditionem demonstratio allata absolute postulat. Quod autem ad numerum n attinet, quatenus eo index cuiusque termini progressionis (§ 3) designatur, nihil impedit, quominus eo numeri quicunque sive positivi sive negativi denotentur, quandoquidem eius progressionis omnes termini etiam indicibus negativis respondentes per formulam integralem datam exhiberi censentur. Interim tamen probe tenendum est hanc reductionem

$$\int x^{f-1}dx(1-x^g)^m=\frac{mg}{f+mg}\int x^{f-1}dx(1-x^g)^{m-1}$$

non esse veritati consentaneam, nisi sit $m>0$, quia alioquin pars algebraica $\frac{1}{f+mg}x^f(1-x^g)^m$ non evanesceret posito $x=1$.

SCHOLION 2

6. Huiusmodi series, quas transcendentes appellare licet, quia termini indicibus fractis respondentes sunt quantitates transcendentes, iam olim in Comment. Petrop. Tomo V[1]) fusius sum prosecutus; unde hoc loco non tam istas progressiones quam eximias formularum integralium comparationes, quae inde derivantur, diligentius sum scrutaturus. Cum scilicet ostendissem huius pro-

1) L. Euleri Commentatio 19 (indicis Enestroemiani): *De progressionibus transcendentibus, seu quarum termini generales algebraice dari nequeunt,* Comment. acad. sc. Petrop. 5 (1730/1), 1738, p. 36; *Leonhardi Euleri Opera omnia*, series I, vol. 14. A. G.

ducti indefiniti $1\cdot 2\cdot 3\cdots n$ valorem hac formula integrali $\int dx\left(l\frac{1}{x}\right)^n$ ab $x=0$ ad $x=1$ extensa exprimi, quae res, quoties n est numerus integer positivus, per ipsam integrationem est manifesta, eos casus examini subieci, quibus pro n numeri fracti accipiuntur; ubi quidem ex ipsa formula integrali neutiquam patet, ad quodnam genus quantitatum transcendentium hi termini referri debeant. Singulari autem artificio eosdem terminos ad quadraturas magis cognitas reduxi, quod propterea maxime dignum videtur, ut maiori studio perpendatur.

PROBLEMA 1

7. *Cum demonstratum sit esse*

$$\frac{1\cdot 2\cdot 3\cdots n}{(f+g)(f+2g)(f+3g)\cdots(f+ng)}=\frac{f}{g^n}\int x^{f-1}dx(1-x^g)^n$$

integrali ab $x=0$ ad $x=1$ extenso, eiusdem producti casu, quo $g=0$, valorem per formulam integralem assignare.

SOLUTIO

Posito $g=0$ in formula integrali membrum $(1-x^g)^n$ evanescit, simul vero etiam denominator g^n, unde quaestio huc redit, ut fractionis $\frac{(1-x^g)^n}{g^n}$ valor definiatur casu $g=0$, quo tam numerator quam denominator evanescit. Hunc in finem spectetur g ut quantitas infinite parva, et cum sit $x^g=e^{glx}$, fiet $x^g=1+glx$ ideoque $(1-x^g)^n=g^n(-lx)^n=g^n\left(l\frac{1}{x}\right)^n$; ex quo pro hoc casu formula nostra integralis abit in $f\int x^{f-1}dx\left(l\frac{1}{x}\right)^n$, ita ut iam habeatur

$$\frac{1\cdot 2\cdot 3\cdots n}{f^n}=f\int x^{f-1}dx\left(l\frac{1}{x}\right)^n$$

seu

$$1\cdot 2\cdot 3\cdots n=f^{n+1}\int x^{f-1}dx\left(l\frac{1}{x}\right)^n.$$

COROLLARIUM 1

8. Quoties n est numerus integer positivus, integratio formulae $\int x^{f-1}dx\left(l\frac{1}{x}\right)^n$ succedit eaque ab $x=0$ ad $x=1$ extensa revera prodit id

productum, cui istam formulam aequalem invenimus. Sin autem pro n capiantur numeri fracti, eadem formula integralis inserviet huic progressioni hypergeometricae interpolandae

$$1,\quad 1\cdot 2,\quad 1\cdot 2\cdot 3,\quad 1\cdot 2\cdot 3\cdot 4,\quad 1\cdot 2\cdot 3\cdot 4\cdot 5\quad \text{etc.}$$

seu

$$1,\quad 2,\quad 6,\quad 24,\quad 120,\quad 720,\quad 5040\quad \text{etc.}$$

COROLLARIUM 2

9. Si expressio modo inventa per principalem dividatur, orietur productum, cuius factores in progressione arithmetica quacunque progrediuntur,

$$(f+g)(f+2g)(f+3g)\cdots(f+ng)=f^n g^n\frac{\int x^{f-1}dx\left(l\frac{1}{x}\right)^n}{\int x^{f-1}dx(1-x^g)^n},$$

cuius ergo etiam valores, si n sit numerus fractus, hinc assignare licebit.

COROLLARIUM 3

10. Cum sit

$$\int x^{f-1}dx(1-x^g)^n=\frac{ng}{f+ng}\int x^{f-1}dx(1-x^g)^{n-1},$$

erit etiam simili modo pro casu $g=0$

$$\int x^{f-1}dx\left(l\frac{1}{x}\right)^n=\frac{n}{f}\int x^{f-1}dx\left(l\frac{1}{x}\right)^{n-1}$$

hincque per istas alteras formulas integrales

$$1\cdot 2\cdot 3\cdots n=nf^n\int x^{f-1}dx\left(l\frac{1}{x}\right)^{n-1}$$

et

$$(f+g)(f+2g)\cdots(f+ng)=f^{n-1}g^{n-1}(f+ng)\frac{\int x^{f-1}dx\left(l\frac{1}{x}\right)^{n-1}}{\int x^{f-1}dx(1-x^g)^{n-1}}.$$

SCHOLION

11. Cum invenerimus esse

$$1\cdot 2\cdot 3\cdots n=f^{n+1}\int x^{f-1}dx\left(l\frac{1}{x}\right)^n,$$

patet hanc formulam integralem non a valore quantitatis f pendere, quod etiam facile perspicitur ponendo $x^f = y$, unde fit

$$fx^{f-1}dx = dy \quad \text{et} \quad l\frac{1}{x} = -lx = -\frac{1}{f}ly = \frac{1}{f}l\frac{1}{y}$$

ideoque

$$f^n\left(l\frac{1}{x}\right)^n = \left(l\frac{1}{y}\right)^n,$$

ita ut sit

$$1\cdot 2\cdot 3\cdots n = \int dy\left(l\frac{1}{y}\right)^n,$$

quae formula ex priori nascitur ponendo $f=1$. Pro interpolatione ergo huiusmodi formarum totum negotium huc reducitur, ut istius formulae integralis $\int dx\left(l\frac{1}{x}\right)^n$ valores definiantur, quando exponens n est numerus fractus. Veluti si n sit $=\frac{1}{2}$, assignari oportet valorem huius formulae $\int dx\sqrt{l\frac{1}{x}}$, quem olim iam ostendi[1]) esse $=\frac{1}{2}\sqrt{\pi}$ denotante π circuli peripheriam, cuius diameter $=1$; pro aliis autem numeris fractis eius valorem ad quadraturas curvarum algebraicarum altioris ordinis revocare docui. Quae reductio cum minime sit obvia atque tum solum locum habeat, quando formulae $\int dx\left(l\frac{1}{x}\right)^n$ integratio a valore $x=0$ ad $x=1$ extenditur, singulari attentione digna videtur. Etsi autem iam olim hoc argumentum tractavi, tamen, quia per plures ambages eo sum perductus, idem hic resumere et concinnius evolvere constitui.

THEOREMA 2

12. *Si formulae integrales a valore $x=0$ usque ad $x=1$ extendantur et n denotet numerum integrum positivum, erit*

$$\frac{1\cdot 2\cdot 3\cdots n}{(n+1)(n+2)(n+3)\cdots 2n} = \frac{1}{2}ng\int x^{f+ng-1}dx(1-x^g)^{n-1}\cdot\frac{\int x^{f-1}dx(1-x^g)^{n-1}}{\int x^{f-1}dx(1-x^g)^{2n-1}},$$

quicunque numeri positivi loco f et g accipiantur.

DEMONSTRATIO

Cum supra (§ 4) ostenderimus esse

$$\frac{1\cdot 2\cdot 3\cdots n}{(f+g)(f+2g)\cdots(f+ng)} = \frac{f\cdot ng}{g^n(f+ng)}\int x^{f-1}dx(1-x^g)^{n-1},$$

1) Vide § 11 Commentationis 19 supra (p. 319) commemoratae. A. G.

habebimus, si loco n scribamus $2n$,

$$\frac{1\cdot 2\cdot 3\cdots 2n}{(f+g)(f+2g)\cdots(f+2ng)} = \frac{f\cdot 2ng}{g^{2n}(f+2ng)}\int x^{f-1}dx(1-x^g)^{2n-1}.$$

Dividatur nunc prima aequatio per secundam ac prodibit ista tertia

$$\frac{(f+(n+1)g)(f+(n+2)g)\cdots(f+2ng)}{(n+1)(n+2)\cdots 2n} = \frac{g^n(f+2ng)}{2(f+ng)}\cdot\frac{\int x^{f-1}dx(1-x^g)^{n-1}}{\int x^{f-1}dx(1-x^g)^{2n-1}}.$$

At si in prima aequatione loco f scribatur $f+ng$, orietur haec aequatio quarta

$$\frac{1\cdot 2\cdot 3\cdots n}{(f+(n+1)g)(f+(n+2)g)\cdots(f+2ng)} = \frac{(f+ng)ng}{g^n(f+2ng)}\int x^{f+ng-1}dx(1-x^g)^{n-1}.$$

Multiplicetur haec quarta aequatio per illam tertiam ac reperietur ipsa aequatio demonstranda

$$\frac{1\cdot 2\cdot 3\cdots n}{(n+1)(n+2)(n+3)\cdots 2n} = \frac{1}{2}ng\int x^{f+ng-1}dx(1-x^g)^{n-1}\cdot\frac{\int x^{f-1}dx(1-x^g)^{n-1}}{\int x^{f-1}dx(1-x^g)^{2n-1}}.$$

COROLLARIUM 1

13. Si in prima aequatione statuatur $f=n$ et $g=1$, orietur idem productum

$$\frac{1\cdot 2\cdot 3\cdots n}{(n+1)(n+2)\cdots 2n} = \frac{1}{2}n\int x^{n-1}dx(1-x)^{n-1},$$

qua aequatione cum illa collata adipiscimur

$$\frac{\int x^{n-1}dx(1-x)^{n-1}}{g\int x^{f+ng-1}dx(1-x^g)^{n-1}} = \frac{\int x^{f-1}dx(1-x^g)^{n-1}}{\int x^{f-1}dx(1-x^g)^{2n-1}}.$$

COROLLARIUM 2

14. Si in illa aequatione loco x scribamus x^g, fiet

$$\frac{1\cdot 2\cdot 3\cdots n}{(n+1)(n+2)\cdots 2n} = \frac{1}{2}ng\int x^{ng-1}dx(1-x^g)^{n-1},$$

ita ut iam consequamur istam comparationem inter sequentes formulas integrales

$$\int x^{ng-1}dx(1-x^g)^{n-1} = \int x^{f+ng-1}dx(1-x^g)^{n-1} \cdot \frac{\int x^{f-1}dx(1-x^g)^{n-1}}{\int x^{f-1}dx(1-x^g)^{2n-1}}.$$

COROLLARIUM 3

15. Si in aequatione theorematis ponamus $g=0$, ob $(1-x^g)^m = g^m\left(l\frac{1}{x}\right)^m$ potestates ipsius g se destruent orieturque haec aequatio

$$\frac{1\cdot 2\cdot 3\cdots n}{(n+1)(n+2)\cdots 2n} = \frac{1}{2}n\int x^{f-1}dx\left(l\frac{1}{x}\right)^{n-1} \cdot \frac{\int x^{f-1}dx\left(l\frac{1}{x}\right)^{n-1}}{\int x^{f-1}dx\left(l\frac{1}{x}\right)^{2n-1}},$$

unde colligimus

$$\frac{\left(\int x^{f-1}dx\left(l\frac{1}{x}\right)^{n-1}\right)^2}{\int x^{f-1}dx\left(l\frac{1}{x}\right)^{2n-1}} = g\int x^{ng-1}dx(1-x^g)^{n-1}$$

seu ob

$$\int x^{f-1}dx\left(l\frac{1}{x}\right)^{n-1} = \frac{f}{n}\int x^{f-1}dx\left(l\frac{1}{x}\right)^n$$

hanc

$$\frac{2f}{n}\cdot\frac{\left(\int x^{f-1}dx\left(l\frac{1}{x}\right)^n\right)^2}{\int x^{f-1}dx\left(l\frac{1}{x}\right)^{2n}} = g\int x^{ng-1}dx(1-x^g)^{n-1}.$$

COROLLARIUM 4

16. Ponamus hic $f=1$, $g=2$ et $n=\frac{m}{2}$, ut m sit numerus integer positivus, et ob

$$\int dx\left(l\frac{1}{x}\right)^m = 1\cdot 2\cdot 3\cdots m$$

erit

$$\frac{4}{m}\cdot\frac{\left(\int dx\left(l\frac{1}{x}\right)^{\frac{m}{2}}\right)^2}{1\cdot 2\cdot 3\cdots m} = 2\int x^{m-1}dx(1-x^2)^{\frac{m}{2}-1}$$

hincque

$$\int dx\left(l\frac{1}{x}\right)^{\frac{m}{2}} = \sqrt{1\cdot 2\cdot 3\cdots m\cdot\frac{m}{2}\int x^{m-1}dx(1-x^2)^{\frac{m}{2}-1}}$$

et sumendo $m=1$ ob

$$\int\frac{dx}{\sqrt{(1-xx)}} = \frac{\pi}{2}$$

habebitur

$$\int dx\sqrt{l\frac{1}{x}} = \sqrt{\frac{1}{2}\int\frac{dx}{\sqrt{(1-xx)}}} = \frac{1}{2}\sqrt{\pi}.$$

SCHOLION

17. En ergo succinctam demonstrationem theorematis olim a me prolati, quod sit $\int dx\sqrt{l\frac{1}{x}} = \frac{1}{2}\sqrt{\pi}$, eamque ab interpolationis ratione, qua tum usus fueram, liberam. Deducta scilicet hic ea ex hoc theoremate, quo inveni esse

$$\frac{\left(\int x^{f-1}dx\left(l\frac{1}{x}\right)^{n-1}\right)^2}{\int x^{f-1}dx\left(l\frac{1}{x}\right)^{2n-1}} = g\int x^{ng-1}dx(1-x^g)^{n-1}.$$

Principale autem theorema, unde hoc est deductum, ita se habet

$$g\frac{\int x^{f-1}dx(1-x^g)^{n-1}\cdot\int x^{f+ng-1}dx(1-x^g)^{n-1}}{\int x^{f-1}dx(1-x^g)^{2n-1}} = \int x^{n-1}dx(1-x)^{n-1};$$

utrumque enim membrum per integrationem ab $x=0$ ad $x=1$ extensam evolvitur in hoc productum numericum

$$\frac{1\cdot 2\cdot 3\cdots(n-1)}{(n+1)(n+2)\cdots(2n-1)}.$$

Ac si alteri membro speciem latius patentem tribuere velimus, theorema ita proponi poterit, ut sit

$$g\frac{\int x^{f-1}dx(1-x^g)^{n-1}\cdot\int x^{f+ng-1}dx(1-x^g)^{n-1}}{\int x^{f-1}dx(1-x^g)^{2n-1}} = k\int x^{nk-1}dx(1-x^k)^{n-1},$$

hicque si capiatur $g=0$, fit

$$\frac{\left(\int x^{f-1}dx\left(l\frac{1}{x}\right)^{n-1}\right)^2}{\int x^{f-1}dx\left(l\frac{1}{x}\right)^{2n-1}} = k\int x^{nk-1}dx(1-x^k)^{n-1}.$$

Imprimis igitur notandum est, quod illa aequalitas subsistat, quicunque numeri loco f et g accipiantur; casu quidem $f=g$ ea est manifesta, cum sit

$$\int x^{g-1}dx(1-x^g)^{n-1}=\frac{1-(1-x^g)^n}{ng}=\frac{1}{ng};$$

fiet enim

$$2g\int x^{ng+g-1}dx(1-x^g)^{n-1}=k\int x^{nk-1}dx(1-x^k)^{n-1},$$

et quia

$$\int x^{ng+g-1}dx(1-x^g)^{n-1}=\frac{1}{2}\int x^{ng-1}dx(1-x^g)^{n-1},$$

aequalitas est perspicua, quia k pro lubitu accipere licet. Eodem autem modo, quo ad hoc theorema perveni, ad alia similia pertingere licet.

THEOREMA 3

18. *Si sequentes formulae integrales a valore $x=0$ ad $x=1$ extendantur et n denotet numerum integrum positivum quemcunque, erit*

$$\frac{1\cdot 2\cdot 3\cdots n}{(2n+1)(2n+2)\cdots 3n}=\frac{2}{3}ng\int x^{f+2ng-1}dx(1-x^g)^{n-1}\cdot\frac{\int x^{f-1}dx(1-x^g)^{2n-1}}{\int x^{f-1}dx(1-x^g)^{3n-1}},$$

quicunque numeri positivi pro f et g accipiantur.

DEMONSTRATIO

In praecedente theoremate iam vidimus esse

$$\frac{1\cdot 2\cdot 3\cdots 2n}{(f+g)(f+2g)\cdots(f+2ng)}=\frac{f\cdot 2ng}{g^{2n}(f+2ng)}\int x^{f-1}dx(1-x^g)^{2n-1};$$

simili autem modo si in forma principali loco n scribamus $3n$, habebimus

$$\frac{1\cdot 2\cdot 3\cdots 3n}{(f+g)(f+2g)\cdots(f+3ng)}=\frac{f\cdot 3ng}{g^{3n}(f+3ng)}\int x^{f-1}dx(1-x^g)^{3n-1},$$

ex quo illa aequatio per hanc divisa producit

$$\frac{(f+(2n+1)g)(f+(2n+2)g)\cdots(f+3ng)}{(2n+1)(2n+2)\cdots 3n}=\frac{2g^n(f+3ng)}{3(f+2ng)}\cdot\frac{\int x^{f-1}dx(1-x^g)^{2n-1}}{\int x^{f-1}dx(1-x^g)^{3n-1}}.$$

Verum si in aequatione principali (§ 4) loco f scribamus $f+2gn$, adipiscimur hanc aequationem

$$\frac{1\cdot 2\cdot 3\cdots n}{(f+(2n+1)g)(f+(2n+2)g)\cdots(f+3ng)}=\frac{(f+2ng)ng}{g^n(f+3ng)}\int x^{f+2ng-1}dx(1-x^g)^{n-1}.$$

Multiplicetur nunc haec aequatio per praecedentem et orietur ipsa aequatio, quam demonstrari oportet,

$$\frac{1\cdot 2\cdot 3\cdots n}{(2n+1)(2n+2)\cdots 3n}=\frac{2}{3}ng\int x^{f+2ng-1}dx(1-x^g)^{n-1}\cdot\frac{\int x^{f-1}dx(1-x^g)^{2n-1}}{\int x^{f-1}dx(1-x^g)^{3n-1}}.$$

COROLLARIUM 1

19. Eundem valorem ex aequatione principali nanciscimur ponendo $f=2n$ et $g=1$, ita ut sit

$$\frac{1\cdot 2\cdot 3\cdots n}{(2n+1)(2n+2)\cdots 3n}=\frac{2}{3}n\int x^{2n-1}dx(1-x)^{n-1},$$

quae formula integralis loco x scribendo x^k transformatur in hanc

$$\frac{2}{3}nk\int x^{2nk-1}dx(1-x^k)^{n-1},$$

ita ut sit

$$g\int x^{f+2ng-1}dx(1-x^g)^{n-1}\cdot\frac{\int x^{f-1}dx(1-x^g)^{2n-1}}{\int x^{f-1}dx(1-x^g)^{3n-1}}=k\int x^{2nk-1}dx(1-x^k)^{n-1}.$$

COROLLARIUM 2

20. Si hic statuamus $g=0$, ob $1-x^g=gl\frac{1}{x}$ habebimus hanc aequationem

$$\int x^{f-1}dx\left(l\frac{1}{x}\right)^{n-1}\cdot\frac{\int x^{f-1}dx\left(l\frac{1}{x}\right)^{2n-1}}{\int x^{f-1}dx\left(l\frac{1}{x}\right)^{3n-1}}=k\int x^{2nk-1}dx(1-x^k)^{n-1};$$

cum igitur ante invenissemus

$$\frac{\left(\int x^{f-1}dx\left(l\frac{1}{x}\right)^{n-1}\right)^2}{\int x^{f-1}dx\left(l\frac{1}{x}\right)^{2n-1}}=k\int x^{nk-1}dx(1-x^k)^{n-1},$$

habebimus has aequationes in se multiplicando

$$\frac{\left(\int x^{f-1}dx\left(l\frac{1}{x}\right)^{n-1}\right)^3}{\int x^{f-1}dx\left(l\frac{1}{x}\right)^{3n-1}} = k^2\int x^{nk-1}dx(1-x^k)^{n-1}\cdot\int x^{2nk-1}dx(1-x^k)^{n-1}.$$

COROLLARIUM 3

21. Sine ulla restrictione hic ponere licet $f=1$; tum ergo sumto $n=\frac{1}{3}$ et $k=3$ erit

$$\frac{\left(\int dx\left(l\frac{1}{x}\right)^{-\frac{2}{3}}\right)^3}{\int dx\left(l\frac{1}{x}\right)^0} = 9\int dx(1-x^3)^{-\frac{2}{3}}\cdot\int x\,dx(1-x^3)^{-\frac{2}{3}}$$

et ob

$$\int dx\left(l\frac{1}{x}\right)^{-\frac{2}{3}} = 3\int dx\left(l\frac{1}{x}\right)^{\frac{1}{3}} \quad \text{et} \quad \int dx\left(l\frac{1}{x}\right)^0 = 1$$

$$\left(\int dx\left(l\frac{1}{x}\right)^{\frac{1}{3}}\right)^3 = \frac{1}{3}\int dx(1-x^3)^{-\frac{2}{3}}\cdot\int x\,dx(1-x^3)^{-\frac{2}{3}};$$

tum vero sumto $n=\frac{2}{3}$ et $k=3$ erit

$$\frac{\left(\int dx\left(l\frac{1}{x}\right)^{-\frac{1}{3}}\right)^3}{\int dx\left(l\frac{1}{x}\right)} = 9\int x\,dx(1-x^3)^{-\frac{1}{3}}\cdot\int x^3dx(1-x^3)^{-\frac{1}{3}}$$

seu

$$\left(\int dx\left(l\frac{1}{x}\right)^{\frac{2}{3}}\right)^3 = \frac{4}{3}\int x\,dx(1-x^3)^{-\frac{1}{3}}\cdot\int x^3dx(1-x^3)^{-\frac{1}{3}}.$$

THEOREMA GENERALE

22. *Si sequentes formulae integrales a valore $x=0$ usque ad $x=1$ extendantur et n denotet numerum integrum positivum quemcunque, erit*

$$\frac{1\cdot 2\cdot 3\cdots n}{(\lambda n+1)(\lambda n+2)\cdots(\lambda+1)n} = \frac{\lambda}{\lambda+1}ng\int x^{f+\lambda ng-1}dx(1-x^g)^{n-1}\cdot\frac{\int x^{f-1}dx(1-x^g)^{\lambda n-1}}{\int x^{f-1}dx(1-x^g)^{(\lambda+1)n-1}},$$

quicunque numeri positivi pro litteris f et g accipiantur.

DEMONSTRATIO

Cum sit, uti supra ostendimus,

$$\frac{1\cdot 2\cdots n}{(f+g)(f+2g)\cdots(f+ng)} = \frac{f\cdot ng}{g^n(f+ng)}\int x^{f-1}dx(1-x^g)^{n-1},$$

si hic loco n scribamus primo λn, tum vero $(\lambda+1)n$, nanciscemur has duas aequationes

$$\frac{1\cdot 2\cdots \lambda n}{(f+g)(f+2g)\cdots(f+\lambda ng)} = \frac{f\cdot \lambda ng}{g^{\lambda n}(f+\lambda ng)}\int x^{f-1}dx(1-x^g)^{\lambda n-1},$$

$$\frac{1\cdot 2\cdots (\lambda+1)n}{(f+g)(f+2g)\cdots(f+(\lambda+1)ng)} = \frac{f\cdot (\lambda+1)ng}{g^{(\lambda+1)n}(f+(\lambda+1)ng)}\int x^{f-1}dx(1-x^g)^{(\lambda+1)n-1},$$

quarum illa per hanc divisa praebet

$$\frac{(f+\lambda ng+g)(f+\lambda ng+2g)\cdots(f+\lambda ng+ng)}{(\lambda n+1)(\lambda n+2)\cdots(\lambda n+n)} = g^n\frac{\lambda(f+\lambda ng+ng)}{(\lambda+1)(f+\lambda ng)}\cdot\frac{\int x^{f-1}dx(1-x^g)^{\lambda n-1}}{\int x^{f-1}dx(1-x^g)^{(\lambda+1)n-1}}.$$

At si in aequatione prima loco f scribamus $f+\lambda ng$, obtinebimus

$$\frac{1\cdot 2\cdots n}{(f+\lambda ng+g)(f+\lambda ng+2g)\cdots(f+\lambda ng+ng)} = \frac{(f+\lambda ng)ng}{g^n(f+\lambda ng+ng)}\int x^{f+\lambda ng-1}dx(1-x^g)^{n-1},$$

quae duae aequationes in se ductae producunt ipsam aequalitatem demonstrandam

$$\frac{1\cdot 2\cdots n}{(\lambda n+1)(\lambda n+2)\cdots(\lambda n+n)} = \frac{\lambda ng}{\lambda+1}\int x^{f+\lambda ng-1}dx(1-x^g)^{n-1}\cdot\frac{\int x^{f-1}dx(1-x^g)^{\lambda n-1}}{\int x^{f-1}dx(1-x^g)^{(\lambda+1)n-1}}.$$

COROLLARIUM 1

23. Si in aequatione principali statuamus $f=\lambda n$ et $g=1$, reperiemus etiam

$$\frac{1\cdot 2\cdots n}{(\lambda n+1)(\lambda n+2)\cdots(\lambda n+n)} = \frac{\lambda n}{\lambda+1}\int x^{\lambda n-1}dx(1-x)^{n-1},$$

quae forma loco x scribendo x^k abit in hanc

$$\frac{\lambda nk}{\lambda+1}\int x^{\lambda nk-1}dx(1-x^k)^{n-1},$$

ita ut habeamus hoc theorema latissime patens

$$g\int x^{f+\lambda ng-1}dx(1-x^g)^{n-1}\cdot\frac{\int x^{f-1}dx(1-x^g)^{\lambda n-1}}{\int x^{f-1}dx(1-x^g)^{\lambda n+n-1}}=k\int x^{\lambda nk-1}dx(1-x^k)^{n-1}.$$

COROLLARIUM 2

24. Hoc iam theorema locum habet, etiamsi n non sit numerus integer; quin etiam, cum numerum λ pro lubitu accipere liceat, loco λn scribamus m et perveniemus ad hoc theorema

$$\frac{\int x^{f-1}dx(1-x^g)^{m-1}}{\int x^{f-1}dx(1-x^g)^{m+n-1}}=\frac{k\int x^{mk-1}dx(1-x^k)^{n-1}}{g\int x^{f+mg-1}dx(1-x^g)^{n-1}}.$$

COROLLARIUM 3

25. Si ponamus $g=0$, ob $1-x^g=gl\frac{1}{x}$ hoc theorema istam induet formam

$$\frac{\int x^{f-1}dx\left(l\frac{1}{x}\right)^{m-1}}{\int x^{f-1}dx\left(l\frac{1}{x}\right)^{m+n-1}}=\frac{k\int x^{mk-1}dx(1-x^k)^{n-1}}{\int x^{f-1}dx\left(l\frac{1}{x}\right)^{n-1}},$$

quae commodius ita repraesentatur

$$\frac{\int x^{f-1}dx\left(l\frac{1}{x}\right)^{n-1}\cdot\int x^{f-1}dx\left(l\frac{1}{x}\right)^{m-1}}{\int x^{f-1}dx\left(l\frac{1}{x}\right)^{m+n-1}}=k\int x^{mk-1}dx(1-x^k)^{n-1},$$

ubi evidens est numeros m et n inter se permutari posse.

SCHOLION

26. Duplicem ergo deteximus fontem, unde innumerabiles formularum integralium comparationes haurire licet; alter fons § 24 patefactus complectitur huiusmodi formulas integrales

$$\int x^{p-1}dx(1-x^g)^{q-1},$$

quas iam ante aliquod tempus pertractavi[1]) in observationibus circa integralia

1) Vide Commentationem 321 huius voluminis. A. G.

formularum

$$\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1}$$

a valore $x=0$ usque ad $x=1$ extensa, ubi ostendi primo litteras p et q inter se permutari posse, ut sit

$$\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1}=\int x^{q-1}dx(1-x^n)^{\frac{p}{n}-1},$$

tum vero etiam esse

$$\int\frac{x^{p-1}dx}{(1-x^n)^{\frac{p}{n}}}=\frac{\pi}{n\sin.\frac{p\pi}{n}};$$

imprimis autem demonstravi esse

$$\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}\cdot\int\frac{x^{p+q-1}dx}{\sqrt[n]{(1-x^n)^{n-r}}}=\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-r}}}\cdot\int\frac{x^{p+r-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}},$$

in qua aequatione comparatio in § 24 inventa iam continetur, ita ut hinc nihil novi, quod non iam evolvi, deduci queat. Alterum igitur fontem § 25 indicatum hic potissimum investigandum suscipio; ubi cum sine ulla restrictione sumi queat $f=1$, aequatio nostra primaria erit

$$\frac{\int dx\left(l\frac{1}{x}\right)^{n-1}\cdot\int dx\left(l\frac{1}{x}\right)^{m-1}}{\int dx\left(l\frac{1}{x}\right)^{m+n-1}}=k\int x^{mk-1}dx(1-x^k)^{n-1},$$

cuius beneficio valores formulae integralis $\int dx\left(l\frac{1}{x}\right)^{\lambda}$, quando λ non est numerus integer, ad quadraturas curvarum algebraicarum revocare licebit; quandoquidem, quoties λ est numerus integer, integratio habetur absoluta, quoniam est

$$\int dx\left(l\frac{1}{x}\right)^{\lambda}=1\cdot2\cdot3\cdots\lambda.$$

Maximi autem momenti quaestio versatur circa eos casus, quibus λ est numerus fractus; quos ergo pro ratione denominationis hic successive sum definiturus.

PROBLEMA 2

27. *Denotante i numerum integrum positivum definire valorem formulae integralis $\int dx\left(l\frac{1}{x}\right)^{\frac{i}{2}}$ integratione ab $x=0$ usque ad $x=1$ extensa.*

SOLUTIO

In aequatione nostra generali faciamus $m=n$ eritque

$$\frac{\left(\int dx\left(l\frac{1}{x}\right)^{n-1}\right)^2}{\int dx\left(l\frac{1}{x}\right)^{2n-1}}=k\int x^{nk-1}dx(1-x^k)^{n-1}.$$

Sit iam $n-1=\frac{i}{2}$ et ob $2n-1=i+1$ erit

$$\int dx\left(l\frac{1}{x}\right)^{2n-1}=1\cdot2\cdot3\cdots(i+1);$$

sumatur porro $k=2$, ut sit $nk-1=i+1$, fietque

$$\frac{\left(\int dx\sqrt{\left(l\frac{1}{x}\right)^i}\right)^2}{1\cdot2\cdot3\cdots(i+1)}=2\int x^{i+1}dx(1-x^2)^{\frac{i}{2}}$$

ideoque

$$\frac{\int dx\sqrt{\left(l\frac{1}{x}\right)^i}}{\sqrt{1\cdot2\cdot3\cdots(i+1)}}=\sqrt{2\int x^{i+1}dx\sqrt{(1-x^2)^i}},$$

ubi evidens est pro i numeros tantum impares sumi convenire, quoniam pro paribus evolutio per se est manifesta.

COROLLARIUM 1

28. Omnes autem casus facile reducuntur ad $i=1$ vel adeo ad $i=-1$; dummodo enim $i+1$ non sit numerus negativus, reductio inventa locum habet. Pro hoc ergo casu erit

$$\int\frac{dx}{\sqrt{l\frac{1}{x}}}=\sqrt{2\int\frac{dx}{\sqrt{(1-xx)}}}=\sqrt{\pi}$$

ob $\int\frac{dx}{\sqrt{(1-xx)}}=\frac{\pi}{2}$.

COROLLARIUM 2

29. Hoc autem casu principali expedito ob

$$\int dx\left(l\frac{1}{x}\right)^n=n\int dx\left(l\frac{1}{x}\right)^{n-1}$$

habebimus

$$\int dx\sqrt{l\frac{1}{x}}=\frac{1}{2}\sqrt{\pi},\quad \int dx\left(l\frac{1}{x}\right)^{\frac{3}{2}}=\frac{1\cdot3}{2\cdot2}\sqrt{\pi}$$

atque in genere

$$\int dx\left(l\frac{1}{x}\right)^{\frac{2n+1}{2}}=\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdots\frac{2n+1}{2}\sqrt{\pi}.$$

PROBLEMA 3

30. *Denotante i numerum integrum positivum definire valorem formulae integralis $\int dx\left(l\frac{1}{x}\right)^{\frac{i}{3}-1}$ integratione ab $x=0$ ad $x=1$ extensa.*

SOLUTIO

Inchoemus ab aequatione praecedentis problematis

$$\frac{\left(\int dx\left(l\frac{1}{x}\right)^{n-1}\right)^2}{\int dx\left(l\frac{1}{x}\right)^{2n-1}}=k\int x^{nk-1}dx(1-x^k)^{n-1}$$

atque in forma generali statuamus $m=2n$, ut habeatur

$$\frac{\int dx\left(l\frac{1}{x}\right)^{n-1}\cdot\int dx\left(l\frac{1}{x}\right)^{2n-1}}{\int dx\left(l\frac{1}{x}\right)^{3n-1}}=k\int x^{2nk-1}dx(1-x^k)^{n-1},$$

ac multiplicando has duas aequalitates adipiscimur

$$\frac{\left(\int dx\left(l\frac{1}{x}\right)^{n-1}\right)^3}{\int dx\left(l\frac{1}{x}\right)^{3n-1}}=kk\int x^{nk-1}dx(1-x^k)^{n-1}\cdot\int x^{2nk-1}dx(1-x^k)^{n-1}.$$

Hic iam ponatur $n=\frac{i}{3}$, ut sit

$$\int dx\left(l\frac{1}{x}\right)^{i-1}=1\cdot2\cdot3\cdots(i-1),$$

sumaturque $k=3$ ac prodibit

$$\frac{\left(\int dx\sqrt[3]{\left(l\frac{1}{x}\right)^{i-3}}\right)^3}{1\cdot 2\cdot 3\cdots(i-1)}=9\int x^{i-1}dx\sqrt[3]{(1-x^3)^{i-3}}\cdot\int x^{2i-1}dx\sqrt[3]{(1-x^3)^{i-3}},$$

unde concludimus

$$\frac{\int dx\sqrt[3]{\left(l\frac{1}{x}\right)^{i-3}}}{\sqrt[3]{1\cdot 2\cdot 3\cdots(i-1)}}=\sqrt[3]{9}\int\frac{x^{i-1}dx}{\sqrt[3]{(1-x^3)^{3-i}}}\cdot\int\frac{x^{2i-1}dx}{\sqrt[3]{(1-x^3)^{3-i}}}.$$

COROLLARIUM 1

31. Bini hic occurunt casus principales, a quibus reliqui omnes pendent, ponendo scilicet vel $i=1$ vel $i=2$, qui sunt

$$\text{I. }\int\frac{dx}{\sqrt[3]{\left(l\frac{1}{x}\right)^2}}=\sqrt[3]{9}\int\frac{dx}{\sqrt[3]{(1-x^3)^2}}\cdot\int\frac{xdx}{\sqrt[3]{(1-x^3)^2}},$$

$$\text{II. }\int\frac{dx}{\sqrt[3]{l\frac{1}{x}}}=\sqrt[3]{9}\int\frac{xdx}{\sqrt[3]{(1-x^3)}}\cdot\int\frac{x^3dx}{\sqrt[3]{(1-x^3)}},$$

quae posterior forma ob

$$\int\frac{x^3dx}{\sqrt[3]{(1-x^3)}}=\frac{1}{3}\int\frac{dx}{\sqrt[3]{(1-x^3)}}.$$

abit in

$$\int\frac{dx}{\sqrt[3]{l\frac{1}{x}}}=\sqrt[3]{3}\int\frac{dx}{\sqrt[3]{(1-x^3)}}\cdot\int\frac{xdx}{\sqrt[3]{(1-x^3)}}.$$

COROLLARIUM 2

32. Si uti in observationibus meis ante allegatis brevitatis gratia ponamus[1])

$$\int\frac{x^{p-1}dx}{\sqrt[3]{(1-x^3)^{3-q}}}=\left(\frac{p}{q}\right)$$

atque ut ibi pro hac classe

$$\left(\frac{2}{1}\right)=\frac{\pi}{3\sin.\frac{\pi}{3}}=\alpha,$$

1) Vide p. 269. A. G.

tum vero

$$\left(\frac{1}{1}\right)=\int\frac{dx}{\sqrt[3]{(1-x^3)^2}}=A,$$

erit

$$\text{I. }\int\frac{dx}{\sqrt[3]{\left(l\frac{1}{x}\right)^2}}=\sqrt[3]{9}\left(\frac{1}{1}\right)\left(\frac{2}{1}\right)=\sqrt[3]{9\alpha A},$$

$$\text{II. }\int\frac{dx}{\sqrt[3]{\left(l\frac{1}{x}\right)^1}}=\sqrt[3]{3}\left(\frac{1}{2}\right)\left(\frac{2}{2}\right)=\sqrt[3]{\frac{3\alpha\alpha}{A}}.$$

COROLLARIUM 3

33. Pro casu ergo priori habebimus

$$\int dx\sqrt[3]{\left(l\frac{1}{x}\right)^{-2}}=\sqrt[3]{9\alpha A},\quad \int dx\sqrt[3]{l\frac{1}{x}}=\frac{1}{3}\sqrt[3]{9\alpha A}$$

et

$$\int dx\sqrt[3]{\left(l\frac{1}{x}\right)^{3n+1}}=\frac{1}{3}\cdot\frac{4}{3}\cdot\frac{7}{3}\cdots\frac{3n+1}{3}\sqrt[3]{9\alpha A},$$

pro altero vero casu

$$\int dx\sqrt[3]{\left(l\frac{1}{x}\right)^{-1}}=\sqrt[3]{\frac{3\alpha\alpha}{A}},\quad \int dx\sqrt[3]{\left(l\frac{1}{x}\right)^2}=\frac{2}{3}\sqrt[3]{\frac{3\alpha\alpha}{A}}$$

et

$$\int dx\sqrt[3]{\left(l\frac{1}{x}\right)^{3n-1}}=\frac{2}{3}\cdot\frac{5}{3}\cdot\frac{8}{3}\cdots\frac{3n-1}{3}\sqrt[3]{\frac{3\alpha\alpha}{A}}.$$

PROBLEMA 4

34. *Denotante i numerum integrum positivum definire valorem formulae integralis $\int dx\left(l\frac{1}{x}\right)^{\frac{i}{k}-1}$ integratione ab $x=0$ ad $x=1$ extensa.*

SOLUTIO

In solutione problematis praecedentis perducti sumus ad hanc aequationem

$$\frac{\left(\int dx\left(l\frac{1}{x}\right)^{n-1}\right)^3}{\int dx\left(l\frac{1}{x}\right)^{3n-1}}=kk\int\frac{x^{nk-1}dx}{(1-x^k)^{1-n}}\cdot\int\frac{x^{2nk-1}dx}{(1-x^k)^{1-n}};$$

forma generalis autem sumendo $m=3n$ praebet

$$\frac{\int dx\left(l\frac{1}{x}\right)^{n-1}\cdot\int dx\left(l\frac{1}{x}\right)^{3n-1}}{\int dx\left(l\frac{1}{x}\right)^{4n-1}}=k\int\frac{x^{3nk-1}dx}{(1-x^k)^{1-n}},$$

quibus coniungendis adipiscimur

$$\frac{\left(\int dx\left(l\frac{1}{x}\right)^{n-1}\right)^4}{\int dx\left(l\frac{1}{x}\right)^{4n-1}}=k^3\int\frac{x^{nk-1}dx}{(1-x^k)^{1-n}}\cdot\int\frac{x^{2nk-1}dx}{(1-x^k)^{1-n}}\cdot\int\frac{x^{3nk-1}dx}{(1-x^k)^{1-n}}.$$

Sit nunc $n=\frac{i}{4}$ et sumatur $k=4$ fietque

$$\frac{\int dx\left(l\frac{1}{x}\right)^{\frac{i}{4}-1}}{\sqrt[4]{1\cdot2\cdot3\cdots(i-1)}}=\sqrt[4]{4^3}\int\frac{x^{i-1}dx}{\sqrt[4]{(1-x^4)^{4-i}}}\cdot\int\frac{x^{2i-1}dx}{\sqrt[4]{(1-x^4)^{4-i}}}\cdot\int\frac{x^{3i-1}dx}{\sqrt[4]{(1-x^4)^{4-i}}}.$$

COROLLARIUM 1

35. Si igitur sit $i=1$, habebimus

$$\int dx\sqrt[4]{\left(l\frac{1}{x}\right)^{-3}}=\sqrt[4]{4^3}\int\frac{dx}{\sqrt[4]{(1-x^4)^3}}\cdot\int\frac{xdx}{\sqrt[4]{(1-x^4)^3}}\cdot\int\frac{xxdx}{\sqrt[4]{(1-x^4)^3}};$$

quae expressio si littera P designetur, erit in genere

$$\int dx\sqrt[4]{\left(l\frac{1}{x}\right)^{4n-3}}=\frac{1}{4}\cdot\frac{5}{4}\cdot\frac{9}{4}\cdots\frac{4n-3}{4}P.$$

COROLLARIUM 2

36. Pro altero casu principali sumamus $i=3$ eritque

$$\int dx\sqrt[4]{\left(l\frac{1}{x}\right)^{-1}}=\sqrt[4]{2\cdot4^3}\int\frac{x^2dx}{\sqrt[4]{(1-x^4)}}\cdot\int\frac{x^5dx}{\sqrt[4]{(1-x^4)}}\cdot\int\frac{x^8dx}{\sqrt[4]{(1-x^4)}}$$

seu facta reductione ad simpliciores formas

$$\int dx\sqrt[4]{\left(l\frac{1}{x}\right)^{-1}}=\sqrt[4]{8}\int\frac{xxdx}{\sqrt[4]{(1-x^4)}}\cdot\int\frac{xdx}{\sqrt[4]{(1-x^4)}}\cdot\int\frac{dx}{\sqrt[4]{(1-x^4)}};$$

quae expressio si littera Q designetur, erit generatim

$$\int dx\sqrt[4]{\left(l\frac{1}{x}\right)^{4n-1}}=\frac{3}{4}\cdot\frac{7}{4}\cdot\frac{11}{4}\cdots\frac{4n-1}{4}Q.$$

SCHOLION

37. Si formulam integralem $\int\frac{x^{p-1}dx}{\sqrt[4]{(1-x^4)^{4-q}}}$ hoc signo $\left(\frac{p}{q}\right)$ indicemus, solutio problematis ita se habebit

$$\int dx\sqrt[4]{\left(l\frac{1}{x}\right)^{i-4}}=\sqrt[4]{1\cdot2\cdot3\cdots(i-1)4^3}\left(\frac{i}{i}\right)\left(\frac{2i}{i}\right)\left(\frac{3i}{i}\right)$$

et pro binis casibus evolutis fit

$$P=\sqrt[4]{4^3}\left(\frac{1}{1}\right)\left(\frac{2}{1}\right)\left(\frac{3}{1}\right)\quad\text{et}\quad Q=\sqrt[4]{8}\left(\frac{3}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right).$$

Statuamus nunc pro iis formulis, quae a circulo pendent,

$$\left(\frac{3}{1}\right)=\frac{\pi}{4\sin.\frac{\pi}{4}}=\alpha\quad\text{et}\quad\left(\frac{2}{2}\right)=\frac{\pi}{4\sin.\frac{2\pi}{4}}=\beta,$$

pro transcendentibus autem altioris ordinis

$$\left(\frac{2}{1}\right)=\int\frac{xdx}{\sqrt[4]{(1-x^4)^3}}=\int\frac{dx}{\sqrt{(1-x^4)}}=A,$$

quippe a qua omnes reliquae pendent, ac reperimus

$$P=\sqrt[4]{4^3\frac{\alpha\alpha}{\beta}AA}\quad\text{et}\quad Q=\sqrt[4]{4\alpha\alpha\beta\frac{1}{AA}},$$

unde patet esse

$$PQ=4\alpha=\frac{\pi}{\sin.\frac{\pi}{4}}.$$

Cum autem sit $\alpha=\frac{\pi}{2\sqrt{2}}$ et $\beta=\frac{\pi}{4}$, erit

$$P=\sqrt[4]{32\pi AA}\quad\text{et}\quad Q=\sqrt[4]{\frac{\pi^3}{8AA}}\quad\text{et}\quad\frac{P}{Q}=\frac{4A}{\sqrt{\pi}}.$$

PROBLEMA 5

38. *Denotante i numerum integrum positivum definire valorem formulae integralis* $\int dx \sqrt[5]{\left(l\frac{1}{x}\right)^{i-5}}$ *integratione ab* $x=0$ *ad* $x=1$ *extensa.*

SOLUTIO

Ex praecedentibus solutionibus iam satis est perspicuum pro hoc casu tandem perventum iri ad hanc formam

$$\frac{\int dx \sqrt[5]{\left(l\frac{1}{x}\right)^{i-5}}}{\sqrt[5]{1\cdot 2\cdot 3\cdots(i-1)}} = \sqrt[5]{5^4}\int\frac{x^{i-1}dx}{\sqrt[5]{(1-x^5)^{5-i}}}\cdot\int\frac{x^{2i-1}dx}{\sqrt[5]{(1-x^5)^{5-i}}}\cdot\int\frac{x^{3i-1}dx}{\sqrt[5]{(1-x^5)^{5-i}}}\cdot\int\frac{x^{4i-1}dx}{\sqrt[5]{(1-x^5)^{5-i}}},$$

quae formulae integrales ad classem quintam dissertationis meae supra allegatae[1]) sunt referendae. Quare si modo ibi recepto signum $\left(\frac{p}{q}\right)$ denotet hanc formulam $\int\frac{x^{p-1}dx}{\sqrt[5]{(1-x^5)^{5-q}}}$, valorem quaesitum ita commodius exprimere licebit, ut sit

$$\int dx \sqrt[5]{\left(l\frac{1}{x}\right)^{i-5}} = \sqrt[5]{1\cdot 2\cdot 3\cdots(i-1)5^4\left(\frac{i}{i}\right)\left(\frac{2i}{i}\right)\left(\frac{3i}{i}\right)\left(\frac{4i}{i}\right)},$$

ubi quidem sufficit ipsi i valores quinario minores tribuisse; quando autem numeratores quinarium superant, tenendum est esse

$$\left(\frac{5+m}{i}\right) = \frac{m}{m+i}\left(\frac{m}{i}\right),$$

tum vero porro

$$\left(\frac{10+m}{i}\right) = \frac{m}{m+i}\cdot\frac{m+5}{m+i+5}\left(\frac{m}{i}\right),$$

$$\left(\frac{15+m}{i}\right) = \frac{m}{m+i}\cdot\frac{m+5}{m+i+5}\cdot\frac{m+10}{m+i+10}\left(\frac{m}{i}\right).$$

Deinde vero pro hac classe binae formulae quadraturam circuli involvunt, quae sint

$$\left(\frac{4}{1}\right) = \frac{\pi}{5\sin.\frac{\pi}{5}} = \alpha \quad \text{et} \quad \left(\frac{3}{2}\right) = \frac{\pi}{5\sin.\frac{2\pi}{5}} = \beta,$$

1) Vide Commentationem 321, p. 275 huius voluminis. A. G.

duae autem quadraturas altiores continent, quae ponantur

$$\left(\frac{3}{1}\right) = \int\frac{xxdx}{\sqrt[5]{(1-x^5)^4}} = \int\frac{dx}{\sqrt[5]{(1-x^5)^2}} = A \quad \text{et} \quad \left(\frac{2}{2}\right) = \int\frac{xdx}{\sqrt[5]{(1-x^5)^3}} = B,$$

atque ex his valores omnium reliquarum formularum huius classis assignavi, scilicet

$$\left(\frac{5}{1}\right) = 1, \quad \left(\frac{5}{2}\right) = \frac{1}{2}, \quad \left(\frac{5}{3}\right) = \frac{1}{3}, \quad \left(\frac{5}{4}\right) = \frac{1}{4}, \quad \left(\frac{5}{5}\right) = \frac{1}{5};$$

$$\left(\frac{4}{1}\right) = \alpha, \quad \left(\frac{4}{2}\right) = \frac{\beta}{A}, \quad \left(\frac{4}{3}\right) = \frac{\beta}{2B}, \quad \left(\frac{4}{4}\right) = \frac{\alpha}{3A};$$

$$\left(\frac{3}{1}\right) = A, \quad \left(\frac{3}{2}\right) = \beta, \quad \left(\frac{3}{3}\right) = \frac{\beta\beta}{\alpha B};$$

$$\left(\frac{2}{1}\right) = \frac{\alpha B}{\beta}, \quad \left(\frac{2}{2}\right) = B;$$

$$\left(\frac{1}{1}\right) = \frac{\alpha A}{\beta}.$$

COROLLARIUM 1

39. Sumto exponente $i=1$ erit

$$\int dx \sqrt[5]{\left(l\frac{1}{x}\right)^{-4}} = \sqrt[5]{5^4\left(\frac{1}{1}\right)\left(\frac{2}{1}\right)\left(\frac{3}{1}\right)\left(\frac{4}{1}\right)} = \sqrt[5]{5^4\frac{\alpha^3}{\beta^2}A^2B},$$

unde in genere concludimus fore denotante n numerum integrum quemcunque

$$\int dx \sqrt[5]{\left(l\frac{1}{x}\right)^{5n-4}} = \frac{1}{5}\cdot\frac{6}{5}\cdot\frac{11}{5}\cdots\frac{5n-4}{5}\sqrt[5]{5^4\frac{\alpha^3}{\beta^2}A^2B}.$$

COROLLARIUM 2

40. Sit nunc $i=2$, et cum prodeat

$$\int dx \sqrt[5]{\left(l\frac{1}{x}\right)^{-3}} = \sqrt[5]{1\cdot 5^4\left(\frac{2}{2}\right)\left(\frac{4}{2}\right)\left(\frac{6}{2}\right)\left(\frac{8}{2}\right)},$$

ob

$$\left(\frac{6}{2}\right) = \frac{1}{3}\left(\frac{1}{2}\right) = \frac{1}{3}\left(\frac{2}{1}\right) \quad \text{et} \quad \left(\frac{8}{2}\right) = \frac{3}{3}\left(\frac{3}{2}\right)$$

erit haec expressio

$$\sqrt[5]{5^3}\left(\frac{2}{2}\right)\left(\frac{4}{2}\right)\left(\frac{2}{1}\right)\left(\frac{3}{2}\right)=\sqrt[5]{5^3\alpha\beta}\,\frac{BB}{A}$$

et in genere

$$\int dx\sqrt[5]{\left(l\frac{1}{x}\right)^{5n-3}}=\frac{2}{5}\cdot\frac{7}{5}\cdot\frac{12}{5}\cdots\frac{5n-3}{5}\sqrt[5]{5^3\alpha\beta}\,\frac{BB}{A}.$$

COROLLARIUM 3

41. Sit $i=3$ et forma inventa

$$\int dx\sqrt[5]{\left(l\frac{1}{x}\right)^{-2}}=\sqrt[5]{2\cdot 5^4}\left(\frac{3}{3}\right)\left(\frac{6}{3}\right)\left(\frac{9}{3}\right)\left(\frac{12}{3}\right)$$

ob

$$\left(\frac{6}{3}\right)=\frac{1}{4}\left(\frac{3}{1}\right),\quad \left(\frac{9}{3}\right)=\frac{4}{7}\left(\frac{4}{3}\right),\quad \left(\frac{12}{3}\right)=\frac{2}{5}\cdot\frac{7}{10}\left(\frac{3}{2}\right)$$

abit in

$$\sqrt[5]{2\cdot 5^2}\left(\frac{3}{3}\right)\left(\frac{3}{1}\right)\left(\frac{4}{3}\right)\left(\frac{3}{2}\right)=\sqrt[5]{5^2\frac{\beta^4}{\alpha}}\cdot\frac{A}{BB},$$

unde in genere colligitur

$$\int dx\sqrt[5]{\left(l\frac{1}{x}\right)^{5n-2}}=\frac{3}{5}\cdot\frac{8}{5}\cdot\frac{13}{5}\cdots\frac{5n-2}{5}\sqrt[5]{5^2\frac{\beta^4}{\alpha}}\cdot\frac{A}{BB}.$$

COROLLARIUM 4

42. Posito denique $i=4$ forma nostra

$$\int dx\sqrt[5]{\left(l\frac{1}{x}\right)^{-1}}=\sqrt[5]{6\cdot 5^4}\left(\frac{4}{4}\right)\left(\frac{8}{4}\right)\left(\frac{12}{4}\right)\left(\frac{16}{4}\right)$$

ob

$$\left(\frac{8}{4}\right)=\frac{3}{7}\left(\frac{4}{3}\right),\quad \left(\frac{12}{4}\right)=\frac{2}{6}\cdot\frac{7}{11}\left(\frac{4}{2}\right),\quad \left(\frac{16}{4}\right)=\frac{1}{5}\cdot\frac{6}{10}\cdot\frac{11}{15}\left(\frac{4}{1}\right)$$

transformabitur in hanc

$$\sqrt[5]{6\cdot 5}\left(\frac{4}{4}\right)\left(\frac{4}{3}\right)\left(\frac{4}{2}\right)\left(\frac{4}{1}\right)=\sqrt[5]{5\frac{\alpha\alpha\beta\beta}{AAB}},$$

ita ut sit in genere

$$\int dx\sqrt[5]{\left(l\frac{1}{x}\right)^{5n-1}}=\frac{4}{5}\cdot\frac{9}{5}\cdot\frac{14}{5}\cdots\frac{5n-1}{5}\sqrt[5]{5\alpha\alpha\beta\beta\,\frac{1}{AAB}}.$$

SCHOLION

43. Si valorem formulae integralis $\int dx\left(l\frac{1}{x}\right)^{\lambda}$ hoc signo $[\lambda]$ repraesentemus, casus hactenus evoluti praebent

$$\left[-\frac{4}{5}\right]=\sqrt[5]{5^4\frac{\alpha^3}{\beta^2}\cdot A^2B},\qquad \left[+\frac{1}{5}\right]=\frac{1}{5}\sqrt[5]{5^4\frac{\alpha^3}{\beta^2}\cdot A^2B},$$

$$\left[-\frac{3}{5}\right]=\sqrt[5]{5^3\alpha\beta\cdot\frac{BB}{A}},\qquad \left[+\frac{2}{5}\right]=\frac{2}{5}\sqrt[5]{5^3\alpha\beta\cdot\frac{BB}{A}},$$

$$\left[-\frac{2}{5}\right]=\sqrt[5]{5^2\frac{\beta^4}{\alpha}\cdot\frac{A}{BB}},\qquad \left[+\frac{3}{5}\right]=\frac{3}{5}\sqrt[5]{5^2\frac{\beta^4}{\alpha}\cdot\frac{A}{BB}},$$

$$\left[-\frac{1}{5}\right]=\sqrt[5]{5\alpha^2\beta^2\cdot\frac{1}{AAB}},\qquad \left[+\frac{4}{5}\right]=\frac{4}{5}\sqrt[5]{5\alpha^2\beta^2\cdot\frac{1}{AAB}},$$

unde binis, quarum indices simul sumti fiunt $=0$, coniungendis colligimus

$$\left[+\frac{1}{5}\right]\cdot\left[-\frac{1}{5}\right]=\alpha=\frac{\pi}{5\sin.\frac{\pi}{5}},$$

$$\left[+\frac{2}{5}\right]\cdot\left[-\frac{2}{5}\right]=2\beta=\frac{2\pi}{5\sin.\frac{2\pi}{5}},$$

$$\left[+\frac{3}{5}\right]\cdot\left[-\frac{3}{5}\right]=3\beta=\frac{3\pi}{5\sin.\frac{3\pi}{5}},$$

$$\left[+\frac{4}{5}\right]\cdot\left[-\frac{4}{5}\right]=4\alpha=\frac{4\pi}{5\sin.\frac{4\pi}{5}}.$$

Ex antecedente autem problemate simili modo deducimus

$$\left[-\frac{3}{4}\right]=P=\sqrt[4]{4^3\frac{\alpha\alpha}{\beta}\cdot AA},\qquad \left[+\frac{1}{4}\right]=\frac{1}{4}\sqrt[4]{4^3\frac{\alpha\alpha}{\beta}\cdot AA},$$

$$\left[-\frac{1}{4}\right]=Q=\sqrt[4]{4\alpha\alpha\beta\cdot\frac{1}{AA}},\qquad \left[+\frac{3}{4}\right]=\frac{3}{4}\sqrt[4]{4\alpha\alpha\beta\cdot\frac{1}{AA}}$$

hincque

$$\left[+\frac{1}{4}\right]\cdot\left[-\frac{1}{4}\right]=\alpha=\frac{\pi}{4\sin.\frac{\pi}{4}},$$

$$\left[+\frac{3}{4}\right]\cdot\left[-\frac{3}{4}\right]=3\alpha=\frac{3\pi}{4\sin.\frac{3\pi}{4}},$$

unde in genere hoc theorema adipiscimur, quod sit

$$[\lambda]\cdot[-\lambda]=\frac{\lambda\pi}{\sin.\lambda\pi},$$

cuius ratio ex methodo interpolandi olim[1]) exposita ita reddi potest. Cum sit

$$[\lambda]=\frac{1^{1-\lambda}\cdot 2^{\lambda}}{1+\lambda}\cdot\frac{2^{1-\lambda}\cdot 3^{\lambda}}{2+\lambda}\cdot\frac{3^{1-\lambda}\cdot 4^{\lambda}}{3+\lambda}\cdot\text{etc.},$$

erit

$$[-\lambda]=\frac{1^{1+\lambda}\cdot 2^{-\lambda}}{1-\lambda}\cdot\frac{2^{1+\lambda}\cdot 3^{-\lambda}}{2-\lambda}\cdot\frac{3^{1+\lambda}\cdot 4^{-\lambda}}{3-\lambda}\cdot\text{etc.}$$

hincque

$$[\lambda]\cdot[-\lambda]=\frac{1\cdot 1}{1-\lambda\lambda}\cdot\frac{2\cdot 2}{4-\lambda\lambda}\cdot\frac{3\cdot 3}{9-\lambda\lambda}\cdot\text{etc.}=\frac{\lambda\pi}{\sin.\lambda\pi},$$

uti alibi[2]) demonstravi.

PROBLEMA 6 GENERALE

44. *Si litterae i et n denotent numeros integros positivos, definire valorem formulae integralis*

$$\int dx\left(l\frac{1}{x}\right)^{\frac{i-n}{n}}\quad\textit{seu}\quad\int dx\sqrt[n]{\left(l\frac{1}{x}\right)^{i-n}}$$

integratione ab $x=0$ ad $x=1$ extensa.

SOLUTIO

Methodus hactenus usitata quaesitum valorem sequenti modo per quadraturas curvarum algebraicarum expressum exhibebit

$$\frac{\int dx\sqrt[n]{\left(l\frac{1}{x}\right)^{i-n}}}{\sqrt[n]{1\cdot 2\cdot 3\cdot(i-1)}}=\sqrt[n]{n^{n-1}}\int\frac{x^{i-1}dx}{\sqrt[n]{(1-x^n)^{n-i}}}\cdot\int\frac{x^{2i-1}dx}{\sqrt[n]{(1-x^n)^{n-i}}}\cdots\int\frac{x^{(n-1)i-1}dx}{\sqrt[n]{(1-x^n)^{n-i}}}.$$

1) Vide L. EULERI Commentationem 19 (indicis ENESTROEMIANI): *De progressionibus transcendentibus, seu quarum termini generales algebraice dari nequeunt,* Comment. acad. sc. Petrop. 5 (1730/1), 1738, p. 36; *LEONHARDI EULERI Opera omnia* series I, vol. 14. A. G.

2) Vide L. EULERI Commentationem 128 (indicis ENESTROEMIANI): *Methodus facilis computandi angulorum sinus ac tangentes tam naturales quam artificiales,* Comment. acad. sc. Petrop. 11 (1739), 1750, p. 194; *LEONHARDI EULERI Opera omnia* series I, vol. 14. A. G.

Quodsi iam brevitatis gratia formulam integralem $\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}$ hoc charactere $\left(\frac{p}{q}\right)$, formulam vero $\int dx\sqrt[n]{\left(l\frac{1}{x}\right)^m}$ isthoc $\left[\frac{m}{n}\right]$ designemus, ita ut $\left[\frac{m}{n}\right]$ valorem huius producti indefiniti $1\cdot 2\cdot 3\cdots z$ denotet existente $z=\frac{m}{n}$, succinctius valor quaesitus hoc modo expressus prodibit

$$\left[\frac{i-n}{n}\right]=\sqrt[n]{1\cdot 2\cdot 3\cdots(i-1)\,n^{n-1}\left(\frac{i}{i}\right)\left(\frac{2i}{i}\right)\left(\frac{3i}{i}\right)\cdots\left(\frac{ni-i}{i}\right)},$$

unde etiam colligitur

$$\left[\frac{i}{n}\right]=\frac{i}{n}\sqrt[n]{1\cdot 2\cdot 3\cdots(i-1)\,n^{n-1}\left(\frac{i}{i}\right)\left(\frac{2i}{i}\right)\left(\frac{3i}{i}\right)\cdots\left(\frac{ni-i}{i}\right)}.$$

Hic semper numerum i ipso n minorem accepisse sufficiet, quoniam pro maioribus notum est esse

$$\left[\frac{i+n}{n}\right]=\frac{i+n}{n}\left[\frac{i}{n}\right],\quad\text{item}\quad\left[\frac{i+2n}{n}\right]=\frac{i+n}{n}\cdot\frac{i+2n}{n}\left[\frac{i}{n}\right]\quad\text{etc.},$$

hocque modo tota investigatio ad eos tantum casus reducitur, quibus fractionis $\frac{i}{n}$ numerator i denominatore n est minor. Praeterea vero de formulis integralibus

$$\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}=\left(\frac{p}{q}\right)$$

sequentia notasse iuvabit:

I. Litteras p et q inter se esse permutabiles, ut sit

$$\left(\frac{p}{q}\right)=\left(\frac{q}{p}\right).$$

II. Si alteruter numerorum p vel q ipsi exponenti n aequetur, valorem formulae integralis fore algebraicum, scilicet

$$\left(\frac{n}{p}\right)=\left(\frac{p}{n}\right)=\frac{1}{p}\quad\text{seu}\quad\left(\frac{n}{q}\right)=\left(\frac{q}{n}\right)=\frac{1}{q}.$$

III. Si summa numerorum $p+q$ ipsi exponenti n aequetur, formulae integralis $\left(\frac{p}{q}\right)$ valorem per circulum exhiberi posse, cum sit

$$\left(\frac{p}{n-p}\right)=\left(\frac{n-p}{p}\right)=\frac{\pi}{n\sin.\frac{p\pi}{n}}\quad\text{et}\quad\left(\frac{q}{n-q}\right)=\left(\frac{n-q}{q}\right)=\frac{\pi}{n\sin.\frac{q\pi}{n}}.$$

IV. Si alteruter numerorum p vel q maior sit exponente n, formulam integralem $\left(\frac{p}{q}\right)$ ad aliam revocari posse, cuius termini sint ipso n minores, quod fit ope huius reductionis

$$\left(\frac{p+n}{q}\right)=\frac{p}{p+q}\left(\frac{p}{q}\right).$$

V. Inter plures huiusmodi formulas integrales talem relationem intercedere, ut sit

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right)=\left(\frac{q}{r}\right)\left(\frac{q+r}{p}\right),$$

cuius ope omnes reductiones reperiuntur, quas in observationibus circa has formulas exposui.[1])

COROLLARIUM 1

45. Si hoc modo ope reductionis n^{o} IV indicatae formam inventam ad singulos casus accommodemus, eos sequenti ratione simplicissime exhibere poterimus. Ac primo quidem pro casu $n=2$, quo nulla opus est reductione, habebimus

$$\left[\frac{1}{2}\right]=\frac{1}{2}\sqrt[2]{2\left(\frac{1}{1}\right)}=\frac{1}{2}\sqrt[2]{\frac{\pi}{\sin.\frac{\pi}{2}}}=\frac{1}{2}\sqrt{\pi}.$$

COROLLARIUM 2

46. Pro casu $n=3$ habebimus has reductiones

$$\left[\frac{1}{3}\right]=\frac{1}{3}\sqrt[3]{3^2\left(\frac{1}{1}\right)\left(\frac{2}{1}\right)}$$

$$\left[\frac{2}{3}\right]=\frac{2}{3}\sqrt[3]{3\cdot1\left(\frac{2}{2}\right)\left(\frac{1}{2}\right)}.$$

COROLLARIUM 3

47. Pro casu $n=4$ hae tres reductiones obtinentur

1) Vide p. 269—272 huius voluminis. A. G.

$$\left[\frac{1}{4}\right]=\frac{1}{4}\sqrt[4]{4^3\left(\frac{1}{1}\right)\left(\frac{2}{1}\right)\left(\frac{3}{1}\right)},$$

ob $\left(\frac{4}{2}\right)=\frac{1}{2}$,

$$\left[\frac{2}{4}\right]=\frac{2}{4}\sqrt[4]{4^2\cdot2\left(\frac{2}{2}\right)^2\left(\frac{4}{2}\right)}=\frac{1}{2}\sqrt[2]{4\left(\frac{2}{2}\right)}$$

$$\left[\frac{3}{4}\right]=\frac{3}{4}\sqrt{4\cdot1\cdot2\left(\frac{3}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)};$$

cum in media sit $\left(\frac{2}{2}\right)=\left(\frac{2}{4-2}\right)=\frac{\pi}{4}$, erit utique ut ante

$$\left[\frac{2}{4}\right]=\left[\frac{1}{2}\right]=\frac{1}{2}\sqrt{\pi}.$$

COROLLARIUM 4

48. Sit nunc $n=5$ et prodeunt hae quatuor reductiones

$$\left[\frac{1}{5}\right]=\frac{1}{5}\sqrt[5]{5^4\left(\frac{1}{1}\right)\left(\frac{2}{1}\right)\left(\frac{3}{1}\right)\left(\frac{4}{1}\right)},$$

$$\left[\frac{2}{5}\right]=\frac{2}{5}\sqrt[5]{5^3.1\left(\frac{2}{2}\right)\left(\frac{4}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)},$$

$$\left[\frac{3}{5}\right]=\frac{3}{5}\sqrt[5]{5^2\cdot1\cdot2\left(\frac{3}{3}\right)\left(\frac{1}{3}\right)\left(\frac{4}{3}\right)\left(\frac{2}{3}\right)},$$

$$\left[\frac{4}{5}\right]=\frac{4}{5}\sqrt[5]{5\cdot1\cdot2\cdot3\left(\frac{4}{4}\right)\left(\frac{3}{4}\right)\left(\frac{2}{4}\right)\left(\frac{1}{4}\right)}.$$

COROLLARIUM 5

49. Sit $n=6$ et habebimus has reductiones

$$\left[\frac{1}{6}\right]=\frac{1}{6}\sqrt{6^5\left(\frac{1}{1}\right)\left(\frac{2}{1}\right)\left(\frac{3}{1}\right)\left(\frac{4}{1}\right)\left(\frac{5}{1}\right)},$$

$$\left[\frac{2}{6}\right]=\frac{2}{6}\sqrt[6]{6^4\cdot2\left(\frac{2}{2}\right)^2\left(\frac{4}{2}\right)^2\left(\frac{6}{2}\right)}=\frac{1}{3}\sqrt[3]{6^2\left(\frac{3}{2}\right)\left(\frac{4}{2}\right)},$$

$$\left[\frac{3}{6}\right]=\frac{3}{6}\sqrt[6]{6^3\cdot3\cdot3\left(\frac{3}{3}\right)^3\left(\frac{6}{3}\right)^2}=\frac{1}{2}\sqrt[2]{6\left(\frac{3}{3}\right)},$$

$$\left[\frac{4}{6}\right]=\frac{4}{6}\sqrt[6]{6^2\cdot2\cdot4\cdot2\left(\frac{4}{4}\right)^2\left(\frac{2}{4}\right)^2\left(\frac{6}{4}\right)}=\frac{2}{3}\sqrt[3]{6\cdot2\left(\frac{4}{4}\right)\left(\frac{2}{4}\right)},$$

$$\left[\frac{5}{6}\right]=\frac{5}{6}\sqrt[6]{6\cdot1\cdot2\cdot3\cdot4\left(\frac{5}{5}\right)\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)\left(\frac{1}{5}\right)}.$$

COROLLARIUM 6

50. Posito $n=7$ sequentes sex prodeunt aequationes

$$\left[\frac{1}{7}\right]=\frac{1}{7}\sqrt[7]{7^6}\left(\frac{1}{1}\right)\left(\frac{2}{1}\right)\left(\frac{3}{1}\right)\left(\frac{4}{1}\right)\left(\frac{5}{1}\right)\left(\frac{6}{1}\right),$$

$$\left[\frac{2}{7}\right]=\frac{2}{7}\sqrt[7]{7^5\cdot 1}\left(\frac{2}{2}\right)\left(\frac{4}{2}\right)\left(\frac{6}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right),$$

$$\left[\frac{3}{7}\right]=\frac{3}{7}\sqrt[7]{7^4\cdot 1\cdot 2}\left(\frac{3}{3}\right)\left(\frac{6}{3}\right)\left(\frac{2}{3}\right)\left(\frac{5}{3}\right)\left(\frac{1}{3}\right)\left(\frac{4}{3}\right),$$

$$\left[\frac{4}{7}\right]=\frac{4}{7}\sqrt[7]{7^3\cdot 1\cdot 2\cdot 3}\left(\frac{4}{4}\right)\left(\frac{1}{4}\right)\left(\frac{5}{4}\right)\left(\frac{2}{4}\right)\left(\frac{6}{4}\right)\left(\frac{3}{4}\right),$$

$$\left[\frac{5}{7}\right]=\frac{5}{7}\sqrt[7]{7^2\cdot 1\cdot 2\cdot 3\cdot 4}\left(\frac{5}{5}\right)\left(\frac{3}{5}\right)\left(\frac{1}{5}\right)\left(\frac{6}{5}\right)\left(\frac{4}{5}\right)\left(\frac{2}{5}\right),$$

$$\left[\frac{6}{7}\right]=\frac{6}{7}\sqrt[7]{7\cdot 1\cdot 2\cdot 3\cdot 4\cdot 5}\left(\frac{6}{6}\right)\left(\frac{5}{6}\right)\left(\frac{4}{6}\right)\left(\frac{3}{6}\right)\left(\frac{2}{6}\right)\left(\frac{1}{6}\right).$$

COROLLARIUM 7

51. Sit $n=8$ et septem hae reductiones impetrabuntur

$$\left[\frac{1}{8}\right]=\frac{1}{8}\sqrt[8]{8^7}\left(\frac{1}{1}\right)\left(\frac{2}{1}\right)\left(\frac{3}{1}\right)\left(\frac{4}{1}\right)\left(\frac{5}{1}\right)\left(\frac{6}{1}\right)\left(\frac{7}{1}\right),$$

$$\left[\frac{2}{8}\right]=\frac{2}{8}\sqrt[8]{8^6\cdot 2}\left(\frac{2}{2}\right)^2\left(\frac{4}{2}\right)^2\left(\frac{6}{2}\right)^2\left(\frac{8}{2}\right)=\frac{1}{4}\sqrt[4]{8^3}\left(\frac{2}{2}\right)\left(\frac{4}{2}\right)\left(\frac{6}{2}\right),$$

$$\left[\frac{3}{8}\right]=\frac{3}{8}\sqrt[8]{8^5\cdot 1\cdot 2}\left(\frac{3}{3}\right)\left(\frac{6}{3}\right)\left(\frac{1}{3}\right)\left(\frac{4}{3}\right)\left(\frac{7}{3}\right)\left(\frac{2}{3}\right)\left(\frac{5}{3}\right),$$

$$\left[\frac{4}{8}\right]=\frac{4}{8}\sqrt[8]{8^4\cdot 4\cdot 4\cdot 4}\left(\frac{4}{4}\right)^4\left(\frac{8}{4}\right)^3=\frac{1}{2}\sqrt[2]{8}\left(\frac{4}{4}\right),$$

$$\left[\frac{5}{8}\right]=\frac{5}{8}\sqrt[8]{8^3\cdot 1\cdot 2\cdot 3\cdot 4}\left(\frac{5}{5}\right)\left(\frac{2}{5}\right)\left(\frac{7}{5}\right)\left(\frac{4}{5}\right)\left(\frac{1}{5}\right)\left(\frac{6}{5}\right)\left(\frac{3}{5}\right),$$

$$\left[\frac{6}{8}\right]=\frac{6}{8}\sqrt[8]{8^2\cdot 4\cdot 2\cdot 6\cdot 4\cdot 2}\left(\frac{6}{6}\right)^2\left(\frac{4}{6}\right)^2\left(\frac{2}{6}\right)^2\left(\frac{8}{6}\right)=\frac{3}{4}\sqrt[4]{8\cdot 2\cdot 4}\left(\frac{6}{6}\right)\left(\frac{4}{6}\right)\left(\frac{2}{6}\right),$$

$$\left[\frac{7}{8}\right]=\frac{7}{8}\sqrt[8]{8\cdot 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\left(\frac{7}{7}\right)\left(\frac{6}{7}\right)\left(\frac{5}{7}\right)\left(\frac{4}{7}\right)\left(\frac{3}{7}\right)\left(\frac{2}{7}\right)\left(\frac{1}{7}\right).$$

SCHOLION

52. Superfluum foret hos casus ulterius evolvere, cum ex allatis ordo istarum formularum satis perspiciatur. Si enim in formula proposita $\left[\frac{m}{n}\right]$

numeri m et n sint inter se primi, lex est manifesta, cum fiat

$$\left[\frac{m}{n}\right]=\frac{m}{n}\sqrt[n]{n^{n-m}\cdot 1\cdot 2\cdots(m-1)}\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n-1}{m}\right);$$

sin autem hi numeri m et n communem habeant divisorem, expediet quidem fractionem $\frac{m}{n}$ ad minimam formam reduci et ex casibus praecedentibus quaesitum valorem peti; interim tamen etiam operatio hoc modo institui poterit. Cum expressio quaesita certe hanc habeat formam

$$\left[\frac{m}{n}\right]=\frac{m}{n}\sqrt[n]{n^{n-m}PQ},$$

ubi Q est productum ex $n-1$ formulis integralibus, P vero productum ex aliquot numeris absolutis, primum pro illo producto Q inveniendo continuetur haec formularum series $\left(\frac{m}{m}\right)\left(\frac{2m}{m}\right)\left(\frac{3m}{m}\right)$ etc., donec numerator superet exponentem n, eiusque loco excessus supra n scribatur; qui si ponatur $=\alpha$, ut iam formula nostra sit $\left(\frac{\alpha}{m}\right)$, hic ipse numerator α dabit factorem producti P; tum hinc formularum series porro statuatur $\left(\frac{\alpha}{m}\right)\left(\frac{\alpha+m}{m}\right)\left(\frac{\alpha+2m}{m}\right)$ etc., donec iterum ad numeratorem exponente n maiorem perveniatur formulaque prodeat $\left(\frac{n+\beta}{m}\right)$, cuius loco scribi oportet $\left(\frac{\beta}{m}\right)$, simulque hinc factor β in productum P inferatur sicque progredi conveniet, donec pro Q prodierint $n-1$ formulae.

Quae operationes quo facilius intelligantur, casum formulae

$$\left[\frac{9}{12}\right]=\frac{9}{12}\sqrt[12]{12^3PQ}$$

hoc modo evolvamus, ubi investigatio litterarum Q et P ita instituetur:

$$\text{Pro } Q\ldots\left(\frac{9}{9}\right)\left(\frac{6}{9}\right)\left(\frac{3}{9}\right)\left(\frac{12}{9}\right)\left(\frac{9}{9}\right)\left(\frac{6}{9}\right)\left(\frac{3}{9}\right)\left(\frac{12}{9}\right)\left(\frac{9}{9}\right)\left(\frac{6}{9}\right)\left(\frac{3}{9}\right),$$

$$\text{pro } P\ldots\quad 6\cdot 3\qquad 9\cdot 6\cdot 3\qquad 9\cdot 6\cdot 3$$

sicque reperitur

$$Q=\left(\frac{9}{9}\right)^3\left(\frac{6}{9}\right)^3\left(\frac{3}{9}\right)^3\left(\frac{12}{9}\right)^2\quad\text{et}\quad P=6^3\cdot 3^3\cdot 9^2.$$

Cum igitur sit $\left(\frac{12}{9}\right)=\frac{1}{9}$, fit $PQ=6^3\cdot 3^3\left(\frac{9}{9}\right)^3\left(\frac{6}{9}\right)^3\left(\frac{3}{9}\right)^3$ ideoque

$$\left[\frac{9}{12}\right]=\frac{3}{4}\sqrt[4]{12\cdot 6\cdot 3}\left(\frac{9}{9}\right)\left(\frac{6}{9}\right)\left(\frac{3}{9}\right).$$

THEOREMA

53. *Quicunque numeri integri positivi litteris m et n indicentur, erit semper signandi modo ante exposito*

$$\left[\frac{m}{n}\right] = \frac{m}{n}\sqrt[n]{n^{n-m}\cdot 1\cdot 2\cdot 3\cdots(m-1)\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n-1}{m}\right)}.$$

DEMONSTRATIO

Pro casu, quo m et n sunt numeri inter se primi, veritas theorematis in antecedentibus est evicta; quod autem etiam locum habeat, si illi numeri m et n commune divisore gaudeant, inde quidem non liquet; verum ex hoc ipso, quod pro casibus, quibus m et n sunt numeri primi, veritas constat, tuto concludere licet theorema in genere esse verum. Minime quidem difficteor hoc concludendi genus prorsus esse singulare ac plerisque suspectum videri debere. Quare quo nullum dubium relinquatur, quoniam pro casibus, quibus numeri m et n inter se sunt compositi, geminam expressionem sumus nacti, utriusque consensum pro casibus ante evolutis ostendisse iuvabit. Insigne autem iam suppeditat firmamentum casus $m=n$, quo forma nostra manifesto unitatem producit.[1])

COROLLARIUM 1

54. Primus casus consensus demonstrationem postulans est, quo $m=2$ et $n=4$, pro quo supra (§ 47) invenimus

$$\left[\frac{2}{4}\right] = \frac{2}{4}\sqrt{4^2\left(\frac{2}{2}\right)^2};$$

nunc autem vi theorematis est

$$\left[\frac{2}{4}\right] = \frac{2}{4}\sqrt[4]{4^2\cdot 1\left(\frac{1}{2}\right)\left(\frac{2}{2}\right)\left(\frac{3}{2}\right)},$$

unde comparatione instituta fit

$$\left(\frac{2}{2}\right) = \left(\frac{1}{2}\right)\left(\frac{3}{2}\right),$$

cuius veritas in observationibus supra allegatis[2]) est confirmata.

1) Vide Supplementum p. 354. 2) Vide Commentationem 321 huius voluminis. A. G.

COROLLARIUM 2

55. Si $m=2$ et $n=6$, ex superioribus (§ 49) est

$$\left[\frac{2}{6}\right] = \frac{2}{6}\sqrt[6]{6^4\left(\frac{2}{2}\right)^2\left(\frac{4}{2}\right)^2};$$

nunc vero per theorema

$$\left[\frac{2}{6}\right] = \frac{2}{6}\sqrt[6]{6^4\cdot 1\left(\frac{1}{2}\right)\left(\frac{2}{2}\right)\left(\frac{3}{2}\right)\left(\frac{4}{2}\right)\left(\frac{5}{2}\right)}$$

ideoque necesse est sit

$$\left(\frac{2}{2}\right)\left(\frac{4}{2}\right) = \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right),$$

cuius veritas indidem patet.

COROLLARIUM 3

56. Si $m=3$ et $n=6$, pervenitur ad hanc aequationem

$$\left(\frac{3}{3}\right)^2 = 1\cdot 2\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\left(\frac{4}{3}\right)\left(\frac{5}{3}\right);$$

at si $m=4$ et $n=6$, fit simili modo

$$2^2\left(\frac{4}{4}\right)\left(\frac{2}{4}\right) = 1\cdot 2\cdot 3\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)\left(\frac{5}{4}\right)$$

seu

$$\left(\frac{4}{4}\right)\left(\frac{2}{4}\right) = \frac{3}{2}\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)\left(\frac{5}{4}\right),$$

quod etiam verum deprehenditur.

COROLLARIUM 4

57. Casus $m=2$ et $n=8$ praebet hanc aequalitatem

$$\left(\frac{2}{2}\right)\left(\frac{4}{2}\right)\left(\frac{6}{2}\right) = \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\left(\frac{7}{2}\right),$$

at casus $m=4$ et $n=8$ hanc

$$\left(\frac{4}{4}\right)^3 = 1\cdot 2\cdot 3\left(\frac{1}{4}\right)\left(\frac{2}{4}\right)\left(\frac{3}{4}\right)\left(\frac{5}{4}\right)\left(\frac{6}{4}\right)\left(\frac{7}{4}\right)$$

casus denique $m = 6$ et $n = 8$ istam

$$2\cdot 4\left(\frac{6}{6}\right)\left(\frac{4}{6}\right)\left(\frac{2}{6}\right)=1\cdot 3\cdot 5\left(\frac{1}{6}\right)\left(\frac{3}{6}\right)\left(\frac{5}{6}\right)\left(\frac{7}{6}\right),$$

quae etiam veritati sunt consentaneae.

SCHOLION

58. In genere autem si numeri m et n communem habeant factorem 2 et formula proposita sit $\left[\frac{2m}{2n}\right]=\left[\frac{m}{n}\right]$, quia est

$$\left[\frac{m}{n}\right]=\frac{m}{n}\sqrt[n]{n^{n-m}}\cdot 1\cdot 2\cdot 3\cdots(m-1)\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n-1}{m}\right),$$

erit eadem ad exponentem $2n$ reducta

$$\frac{m}{n}\sqrt[2n]{2n^{2n-2m}\cdot 2^2\cdot 4^2\cdot 6^2\cdots(2m-2)^2\left(\frac{2}{2m}\right)^2\left(\frac{4}{2m}\right)^2\left(\frac{6}{2m}\right)^2\cdots\left(\frac{2n-2}{2m}\right)^2}.$$

Per theorema vero eadem expressio fit

$$\frac{m}{n}\sqrt[2n]{2n^{2n-2m}}\cdot 1\cdot 2\cdot 3\cdots(2m-1)\left(\frac{1}{2m}\right)\left(\frac{2}{2m}\right)\left(\frac{3}{2m}\right)\cdots\left(\frac{2n-1}{2m}\right),$$

unde pro exponente $2n$ erit

$$2\cdot 4\cdot 6\cdots(2m-2)\left(\frac{2}{2m}\right)\left(\frac{4}{2m}\right)\left(\frac{6}{2m}\right)\cdots\left(\frac{2n-2}{2m}\right)$$
$$=1\cdot 3\cdot 5\cdots(2m-1)\left(\frac{1}{2m}\right)\left(\frac{3}{2m}\right)\left(\frac{5}{2m}\right)\cdots\left(\frac{2n-1}{2m}\right).$$

Simili modo si communis divisor sit 3, pro exponente $3n$ reperietur

$$3^2\cdot 6^2\cdot 9^2\cdots(3m-3)^2\left(\frac{3}{3m}\right)^2\left(\frac{6}{3m}\right)^2\left(\frac{9}{3m}\right)^2\cdots\left(\frac{3n-3}{3m}\right)^2$$
$$=1\cdot 2\cdot 4\cdot 5\cdots(3m-2)(3m-1)\left(\frac{1}{3m}\right)\left(\frac{2}{3m}\right)\left(\frac{4}{3m}\right)\left(\frac{5}{3m}\right)\cdots\left(\frac{3n-1}{3m}\right),$$

quae aequatio concinnius ita exhiberi potest

$$\frac{1\cdot 2\cdot 4\cdot 5\cdot 7\cdot 8\cdot 10\cdots(3m-2)(3m-1)}{3^2\cdot 6^2\cdot 9^2\cdots(3m-3)^2}=\frac{\left(\frac{3}{3m}\right)^2\left(\frac{6}{3m}\right)^2\cdots\left(\frac{3n-3}{3m}\right)^2}{\left(\frac{1}{3m}\right)\left(\frac{2}{3m}\right)\left(\frac{4}{3m}\right)\left(\frac{5}{3m}\right)\left(\frac{7}{3m}\right)\cdots\left(\frac{3n-2}{3m}\right)\left(\frac{3n-1}{3m}\right)}.$$

In genere autem si communis divisor sit d et exponens dn, habebitur

$$\left(d\cdot 2d\cdot 3d\cdots(dm-d)\left(\frac{d}{dm}\right)\left(\frac{2d}{dm}\right)\left(\frac{3d}{dm}\right)\cdots\left(\frac{dn-d}{dm}\right)\right)^d$$
$$=1\cdot 2\cdot 3\cdot 4\cdots(dm-1)\left(\frac{1}{dm}\right)\left(\frac{2}{dm}\right)\left(\frac{3}{dm}\right)\cdots\left(\frac{dn-1}{dm}\right),$$

quae aequatio facile ad quosvis casus accommodari potest, unde sequens theorema notari meretur.

THEOREMA

59. *Si α fuerit divisor communis numerorum m et n haecque formula $\left(\frac{p}{q}\right)$ denotet valorem integralis $\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}$ ab $x=0$ usque ad $x=1$ extensi, erit*

$$\left(\alpha\cdot 2\alpha\cdot 3\alpha\cdots(m-\alpha)\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\left(\frac{n-\alpha}{m}\right)\right)^\alpha$$
$$=1\cdot 2\cdot 3\cdots(m-1)\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n-1}{m}\right).$$

DEMONSTRATIO

Ex praecedente scholio veritas huius theorematis perspicitur; cum enim ibi divisor communis esset $=d$ binique numeri propositi dm et dn, horum loco hic scripsi m et n, loco divisoris eorum autem d litteram α, quam divisoris rationem aequalitas enunciata ita complectitur, ut in progressione arithmetica α, 2α, 3α etc. continuata occurrere assumantur ipsi numeri m et n ideoque etiam $m-\alpha$ et $n-\alpha$. Ceterum fateri cogor hanc demonstrationem utpote inductioni potissimum innixam neutiquam pro rigorosa haberi posse; cum autem nihilominus de eius veritate simus convicti, hoc theorema eo maiori attentione dignum videtur; interim tamen nullum est dubium, quin uberior huiusmodi formularum integralium evolutio tandem perfectam demonstrationem sit largitura; quod autem iam ante nobis hanc veritatem perspicere licuerit, insigne hinc specimen analyticae investigationis elucet.[1]

1) Vide p. 356. A. G.

COROLLARIUM 1

60. Si loco signorum adhibitorum ipsas formulas integrales substituamus, theorema nostrum ita se habebit, ut sit

$$\alpha\cdot 2\alpha\cdot 3\alpha\cdots(m-\alpha)\int\frac{x^{\alpha-1}dx}{\sqrt[n]{(1-x^n)^{n-m}}}\cdot\int\frac{x^{2\alpha-1}dx}{\sqrt[n]{(1-x^n)^{n-m}}}\cdots\int\frac{x^{n-\alpha-1}dx}{\sqrt[n]{(1-x^n)^{n-m}}}$$
$$=\sqrt[\alpha]{}1\cdot 2\cdot 3\cdots(m-1)\int\frac{dx}{\sqrt[n]{(1-x^n)^{n-m}}}\cdot\int\frac{x\,dx}{\sqrt[n]{(1-x^n)^{n-m}}}\cdots\int\frac{x^{n-2}dx}{\sqrt[n]{(1-x^n)^{n-m}}}.$$

COROLLARIUM 2

61. Vel si ad abbreviandum statuamus $\sqrt[n]{(1-x^n)^{n-m}}=X$, erit

$$\alpha\cdot 2\alpha\cdot 3\alpha\cdots(m-\alpha)\int\frac{x^{\alpha-1}dx}{X}\cdot\int\frac{x^{2\alpha-1}dx}{X}\cdots\int\frac{x^{n-\alpha-1}dx}{X}$$
$$=\sqrt[\alpha]{}1\cdot 2\cdot 3\cdots(m-1)\int\frac{dx}{X}\cdot\int\frac{x\,dx}{X}\cdot\int\frac{x^2dx}{X}\cdots\int\frac{x^{n-2}dx}{X}.$$

THEOREMA GENERALE

62. *Si binorum numerorum m et n divisores communes sint α, β, γ etc. formulaque $\left(\frac{p}{q}\right)$ denotet valorem integralis $\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}$ ab $x=0$ ad $x=1$ extensi, sequentes expressiones ex huiusmodi formulis integralibus formatae inter se erunt aequales*

$$\left(\alpha\cdot 2\alpha\cdot 3\alpha\cdots(m-\alpha)\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\left(\frac{n-\alpha}{m}\right)\right)^{\alpha}$$
$$=\left(\beta\cdot 2\beta\cdot 3\beta\cdots(m-\beta)\left(\frac{\beta}{m}\right)\left(\frac{2\beta}{m}\right)\left(\frac{3\beta}{m}\right)\cdots\left(\frac{n-\beta}{m}\right)\right)^{\beta}$$
$$=\left(\gamma\cdot 2\gamma\cdot 3\gamma\cdots(m-\gamma)\left(\frac{\gamma}{m}\right)\left(\frac{2\gamma}{m}\right)\left(\frac{3\gamma}{m}\right)\cdots\left(\frac{n-\gamma}{m}\right)\right)^{\gamma}$$

etc.

DEMONSTRATIO

Ex praecedente theoremate huius veritas manifesto sequitur, cum quaelibet harum expressionum seorsim aequetur huic

$$1\cdot 2\cdot 3\cdots(m-1)\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n-1}{m}\right),$$

quae unitati utpote minimo communi divisori numerorum m et n convenit. Tot igitur huiusmodi expressiones inter se aequales exhiberi possunt, quot fuerint divisores communes binorum numerorum m et n.

COROLLARIUM 1

63. Cum sit haec formula $\left(\frac{n}{m}\right)=\frac{1}{m}$ ideoque $m\left(\frac{n}{m}\right)=1$, expressiones nostrae aequales succinctius hoc modo repraesentari possunt

$$\left(\alpha\cdot 2\alpha\cdot 3\alpha\cdots m\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\left(\frac{n}{m}\right)\right)^{\alpha}$$
$$=\left(\beta\cdot 2\beta\cdot 3\beta\cdots m\left(\frac{\beta}{m}\right)\left(\frac{2\beta}{m}\right)\left(\frac{3\beta}{m}\right)\cdots\left(\frac{n}{m}\right)\right)^{\beta}$$
$$=\left(\gamma\cdot 2\gamma\cdot 3\gamma\cdots m\left(\frac{\gamma}{m}\right)\left(\frac{2\gamma}{m}\right)\left(\frac{3\gamma}{m}\right)\cdots\left(\frac{n}{m}\right)\right)^{\gamma}.$$

Etsi enim hic factorum numerus est auctus, tamen ratio compositionis facilius in oculos incurrit.

COROLLARIUM 2

64. Si ergo sit $m=6$ et $n=12$, ob horum numerorum divisores communes 6, 3, 2, 1 quatuor sequentes formae inter se aequales habebuntur

$$\left(6\left(\frac{6}{6}\right)\left(\frac{12}{6}\right)\right)^{6}=\left(3\cdot 6\left(\frac{3}{6}\right)\left(\frac{6}{6}\right)\left(\frac{9}{6}\right)\left(\frac{12}{6}\right)\right)^{3}$$
$$=\left(2\cdot 4\cdot 6\left(\frac{2}{6}\right)\left(\frac{4}{6}\right)\left(\frac{6}{6}\right)\left(\frac{8}{6}\right)\left(\frac{10}{6}\right)\left(\frac{12}{6}\right)\right)^{2}$$
$$=1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\left(\frac{1}{6}\right)\left(\frac{2}{6}\right)\left(\frac{3}{6}\right)\cdots\left(\frac{12}{6}\right).$$

COROLLARIUM 3

65. Si ultima cum penultima combinetur, nascetur haec aequatio

$$\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}=\frac{\left(\frac{2}{6}\right)\left(\frac{4}{6}\right)\left(\frac{6}{6}\right)\left(\frac{8}{6}\right)\left(\frac{10}{6}\right)\left(\frac{12}{6}\right)}{\left(\frac{1}{6}\right)\left(\frac{3}{6}\right)\left(\frac{5}{6}\right)\left(\frac{7}{6}\right)\left(\frac{9}{6}\right)\left(\frac{11}{6}\right)},$$

ultima autem cum antepenultima comparata praebet

$$\frac{1\cdot 2\cdot 4\cdot 5}{3\cdot 3\cdot 6\cdot 6}=\frac{\left(\frac{3}{6}\right)\left(\frac{3}{6}\right)\left(\frac{6}{6}\right)\left(\frac{6}{6}\right)\left(\frac{9}{6}\right)\left(\frac{9}{6}\right)\left(\frac{12}{6}\right)\left(\frac{12}{6}\right)}{\left(\frac{1}{6}\right)\left(\frac{2}{6}\right)\left(\frac{4}{6}\right)\left(\frac{5}{6}\right)\left(\frac{7}{6}\right)\left(\frac{8}{6}\right)\left(\frac{10}{6}\right)\left(\frac{11}{6}\right)}.$$

SCHOLION

66. Infinitae igitur hinc consequuntur relationes inter formulas integrales formae

$$\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}=\left(\frac{p}{q}\right),$$

quae eo magis sunt notatu dignae, quod singulari prorsus methodo ad eas hic sumus perducti. Ac si quis de earum veritate adhuc dubitet, observationes meas[1]) circa has formulas integrales consultet indeque pro quovis casu oblato de veritate facile convincetur. Etsi autem illa tractatio huic confirmandae inservit, tamen relationes hic erutae eo maioris sunt momenti, quod in iis certus ordo cernitur eaeque per omnes classes, quantumvis exponentem n accipere lubeat, facili negotio continuentur, in priori vero tractatione calculus pro classibus altioribus continuo fiat operosior et intricatior.

SUPPLEMENTUM CONTINENS DEMONSTRATIONEM THEOREMATIS § 53 PROPOSITI

Demonstrationem hanc altius peti convenit; sumatur scilicet aequatio § 25 data, quae posito $f=1$ et mutatis litteris est

$$\frac{\int dx\left(l\frac{1}{x}\right)^{\nu-1}\cdot\int dx\left(l\frac{1}{x}\right)^{\mu-1}}{\int dx\left(l\frac{1}{x}\right)^{\mu+\nu-1}}=\varkappa\int\frac{x^{\varkappa\mu-1}dx}{(1-x^{\varkappa})^{1-\nu}},$$

1) Vide Commentationem 321 huius voluminis. A. G.

eaque per reductiones notas hac forma repraesentetur

$$\frac{\int dx\left(l\frac{1}{x}\right)^{\nu}\cdot\int dx\left(l\frac{1}{x}\right)^{\mu}}{\int dx\left(l\frac{1}{x}\right)^{\mu+\nu}}=\frac{\varkappa\mu\nu}{\mu+\nu}\int\frac{x^{\varkappa\mu-1}dx}{(1-x^{\varkappa})^{1-\nu}}.$$

Statuatur nunc $\nu=\frac{m}{n}$ et $\mu=\frac{\lambda}{n}$, tum vero $\varkappa=n$, ut habeamus

$$\frac{\int dx\left(l\frac{1}{x}\right)^{\frac{m}{n}}\cdot\int dx\left(l\frac{1}{x}\right)^{\frac{\lambda}{n}}}{\int dx\left(l\frac{1}{x}\right)^{\frac{\lambda+m}{n}}}=\frac{\lambda m}{\lambda+m}\int\frac{x^{\lambda-1}dx}{\sqrt[n]{(1-x^n)^{n-m}}},$$

quae brevitatis gratia more supra usitato ita concinne referatur

$$\frac{\left[\frac{m}{n}\right]\left[\frac{\lambda}{n}\right]}{\left[\frac{\lambda+m}{n}\right]}=\frac{\lambda m}{\lambda+m}\left(\frac{\lambda}{m}\right).$$

Iam loco λ successive scribantur numeri 1, 2, 3, 4, $\cdots n$ omnesque hae aequationes, quarum numerus est $=n$, in se invicem ducantur et aequatio resultans erit

$$\left[\frac{m}{n}\right]^n\frac{\left[\frac{1}{n}\right]\left[\frac{2}{n}\right]\left[\frac{3}{n}\right]\cdots\left[\frac{n}{n}\right]}{\left[\frac{m+1}{n}\right]\left[\frac{m+2}{n}\right]\left[\frac{m+3}{n}\right]\cdots\left[\frac{m+n}{n}\right]}$$

$$=m^n\frac{1}{m+1}\cdot\frac{2}{m+2}\cdot\frac{3}{m+3}\cdots\frac{n}{m+n}\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right)$$

$$=m^n\frac{1\cdot 2\cdot 3\cdots m}{(n+1)(n+2)(n+3)\cdots(n+m)}\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right).$$

Simili autem modo pars prior transformetur, ut sit

$$\left[\frac{m}{n}\right]^n\frac{\left[\frac{1}{n}\right]\left[\frac{2}{n}\right]\left[\frac{3}{n}\right]\cdots\left[\frac{m}{n}\right]}{\left[\frac{n+1}{n}\right]\left[\frac{n+2}{n}\right]\left[\frac{n+3}{n}\right]\cdots\left[\frac{n+m}{n}\right]},$$

cuius convenientia cum forma praecedente multiplicando per crucem, ut aiunt, sponte se prodit. Cum vero ex natura harum formularum sit

$$\left[\frac{n+1}{n}\right]=\frac{n+1}{n}\left[\frac{1}{n}\right],\quad\left[\frac{n+2}{n}\right]=\frac{n+2}{n}\left[\frac{2}{n}\right],\quad\left[\frac{n+3}{n}\right]=\frac{n+3}{n}\left[\frac{3}{n}\right]\quad\text{etc.},$$

ob harum formularum numerum $=m$ evadet haec prior pars

$$\left[\frac{m}{n}\right]^n \frac{n^m}{(n+1)(n+2)(n+3)\cdots(n+m)};$$

quae cum aequalis sit parti alteri ante exhibitae

$$m^n \frac{1\cdot 2\cdot 3\cdots m}{(n+1)(n+2)(n+3)\cdots(n+m)}\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right),$$

adipiscimur hanc aequationem

$$\left[\frac{m}{n}\right]^n = \frac{m^n}{n^m}\,1\cdot 2\cdot 3\cdots m\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right),$$

ita ut sit

$$\left[\frac{m}{n}\right] = m\sqrt[n]{\frac{1\cdot 2\cdot 3\cdots m}{n^m}\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right)},$$

quae cum proposita in § 53 ob $\left(\frac{n}{m}\right)=\frac{1}{m}$ omnino congruit, ex quo eius veritas nunc quidem ex principiis certissimis est evicta.

DEMONSTRATIO THEOREMATIS § 59 PROPOSITI

Etiam hoc theorema firmiori demonstratione indiget, quam ex aequalitate ante stabilita

$$\frac{\left[\frac{m}{n}\right]\left[\frac{\lambda}{n}\right]}{\left[\frac{\lambda+m}{n}\right]} = \frac{\lambda m}{\lambda+m}\left(\frac{\lambda}{m}\right)$$

ita adorno. Existente α communi divisore numerorum m et n loco λ successive scribantur numeri α, 2α, 3α etc. usque ad n, quorum multitudo est $=\frac{n}{\alpha}$, atque omnes aequalitates hoc modo resultantes in se invicem ducantur, ut prodeat haec aequatio

$$\left[\frac{m}{n}\right]^{\frac{n}{\alpha}} \frac{\left[\frac{\alpha}{n}\right]\left[\frac{2\alpha}{n}\right]\left[\frac{3\alpha}{n}\right]\cdots\left[\frac{n}{n}\right]}{\left[\frac{m+\alpha}{n}\right]\left[\frac{m+2\alpha}{n}\right]\left[\frac{m+3\alpha}{n}\right]\cdots\left[\frac{m+n}{n}\right]}$$

$$= m^{\frac{n}{\alpha}}\frac{\alpha}{m+\alpha}\cdot\frac{2\alpha}{m+2\alpha}\cdot\frac{3\alpha}{m+3\alpha}\cdots\frac{n}{m+n}\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\left(\frac{n}{m}\right).$$

Iam prior pars in hanc formam ipsi aequalem transmutetur

$$\left[\frac{m}{n}\right]^{\frac{n}{\alpha}} \frac{\left[\frac{\alpha}{n}\right]\left[\frac{2\alpha}{n}\right]\left[\frac{3\alpha}{n}\right]\cdots\left[\frac{m}{n}\right]}{\left[\frac{n+\alpha}{n}\right]\left[\frac{n+2\alpha}{n}\right]\left[\frac{n+3\alpha}{n}\right]\cdots\left[\frac{n+m}{n}\right]},$$

quae ob $\left[\frac{n+\alpha}{n}\right]=\frac{n+\alpha}{n}\left[\frac{\alpha}{n}\right]$ sicque de ceteris reducitur ad hanc

$$\left[\frac{m}{n}\right]^{\frac{n}{\alpha}} \frac{n}{n+\alpha}\cdot\frac{n}{n+2\alpha}\cdot\frac{n}{n+3\alpha}\cdots\frac{n}{n+m}.$$

Posterior vero aequationis pars simili modo transformatur in

$$m^{\frac{n}{\alpha}}\frac{\alpha}{n+\alpha}\cdot\frac{2\alpha}{n+2\alpha}\cdot\frac{3\alpha}{n+3\alpha}\cdots\frac{m}{n+m}\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\left(\frac{n}{m}\right),$$

unde enascitur haec aequatio

$$\left[\frac{m}{n}\right]^{\frac{n}{\alpha}} n^{\frac{m}{\alpha}} = m^{\frac{n}{\alpha}}\,\alpha\cdot 2\alpha\cdot 3\alpha\cdots m\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\left(\frac{n}{m}\right)$$

hincque

$$\left[\frac{m}{n}\right] = m\sqrt[n]{\frac{1}{n^m}\left(\alpha\cdot 2\alpha\cdot 3\alpha\cdots m\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\left(\frac{n}{m}\right)\right)^{\alpha}},$$

quae expressio cum praecedente comparata praebet hanc aequationem

$$\left(\alpha\cdot 2\alpha\cdot 3\alpha\cdots m\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\left(\frac{n}{m}\right)\right)^{\alpha} = 1\cdot 2\cdot 3\cdots m\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right),$$

quod de omnibus divisoribus communibus binorum numerorum m et n est intelligendum.

DE VALORE FORMULAE INTEGRALIS

$$\int \frac{z^{m-1} \pm z^{n-m-1}}{1 \pm z^n} dz$$

CASU QUO POST INTEGRATIONEM PONITUR $z = 1$

Commentatio 462 indicis Enestroemiani
Novi commentarii academiae scientiarum Petropolitanae 19 (1774), 1775, p. 3—29
Summarium ibidem p. 5—8

SUMMARIUM

In hac dissertatione Illustr. Eulero propositum est binorum insignium theorematum demonstrationem ex principiis calculi integralis adornare, ad quae theoremata consideratio arcuum circularium, qui eundem habent vel sinum vel tangentem, iam dudum[1]) ipsum perduxerat. Possunt vero haec theoremata ita enunciari, ut valor formulae integralis supra propositae, si I° signa superiora adhibeantur, statuatur esse $= \frac{\pi}{n \sin. \frac{m\pi}{n}}$, tum vero II° si signa inferiora in usum vocentur, adfirmetur esse $= \frac{\pi}{n \operatorname{tang.} \frac{m\pi}{n}}$ integratione a termino $z = 0$ usque ad $z = 1$ instituta et designante π semiperipheriam circuli, cuius radius $= 1$. Occurrunt quidem eorundem theorematum demonstrationes in *Calculo Integrali* Illustr. Auctoris; quum vero subsidia integrationis ex alio Eiusdem opere, *Introductione* nimirum *in Analysin infinitorum*, petantur, hoc loco integrationem formularum ita perficiendam existimavit, ut simul principia, quibus illa innititur, succincte complecteretur; tum vero pro casu, quo post integrationem ponitur $z = 1$, singularia artificia, quibus summatio serierum absolvitur, dilucide exponenda indicavit.

Antequam formulae integralis propositae integratio suscipiatur, formulae hae integrales simpliciores

$$\int \frac{z^{m-1} dz}{1 + z^n} \quad \text{et} \quad \int \frac{z^{m-1} dz}{1 - z^n}$$

1) Vide notam 1 p. 360. A. G.

evolvendae sunt, ubi quidem ante omnia fractio $\frac{z^{m-1}}{1 \pm z^n}$ in suas fractiones simplices partiales resolvenda est. Ad hoc autem perficiendum necessum est, ut denominatores $1 + z^n$ et $1 - z^n$ in suos factores simplices reales et imaginarios resolvantur. Prior scilicet $1 + z^n$ casu tantum, quo n numerus impar, unum factorem habet realem $1 + z$, caeterum omnes sunt imaginarii et bini eorum semper in hac forma continebuntur $1 - 2z \cos. \varphi + z^2$. Tum vero $\cos. \varphi$ ita accipi debet, ut fiat $\cos. n\varphi = -1$ et $\sin. n\varphi = 0$, ideoque, quum anguli, quorum cosinus $= -1$, sint π, 3π, 5π, 7π etc., pro φ hinc sequentes deducentur valores $\frac{\pi}{n}$, $\frac{3\pi}{n}$, $\frac{5\pi}{n}$ etc. Posterior $1 - z^n$ factorem semper habet realem $1 - z$, praeterea casu n numeri paris $1 + z$, reliqui vero factores semper sunt imaginarii sub ista forma $1 - 2z \cos. \varphi + z^2$ comprehensi. Tum autem ita φ accipi debet, ut fiat $\cos. n\varphi = 1$, ita ut habeantur pro φ huiusmodi valores $\frac{0\pi}{n}$, $\frac{2\pi}{n}$, $\frac{4\pi}{n}$, $\frac{6\pi}{n}$ etc. Inventis factoribus denominatorum simplicibus pro fractionibus partialibus ex illis oriundis quaeri debet numerator, quod hunc in modum perficitur. Sit fractio partialis $\frac{\alpha}{z - f}$ facileque demonstrabitur casu $z = f$ esse $\alpha = \frac{z^m - f z^{m-1}}{1 + z^n}$, at hoc in casu tam numerator quam denominator evanescit; erit ergo

$$\alpha = \frac{m z^{m-1} - (m-1) f z^{m-2}}{n z^{n-1}},$$

unde, posito $z = f$, $\alpha = \frac{1}{n} f^{m-n}$. Inventis fractionibus partialibus reliquum est, ut integratio instituatur et post integrationem ponatur $z = 0$, ex quo, quum integrale evanescere debeat, valor constantis adiiciendae innotescet. Hoc igitur modo Illustr. Auctor invenit binas illas formulas $\frac{z^{m-1} dz}{1 + z^n}$, $\frac{z^{m-1} dz}{1 - z^n}$ integratas exhibere valores integrales partim logarithmicos, partim etiam qui arcus circulares involvunt.

Dum autem ad integrationem formulae propositae propius accedit Illustr. Eulerus, primum generatim considerat formulas

$$\frac{z^{m-1} + z^{\mu-1}}{1 + z^n} dz \quad \text{et} \quad \frac{z^{m-1} - z^{\mu-1}}{1 - z^n} dz$$

existente $m + \mu = n$; pro his scilicet formulis integralia logarithmica se destruent, ita ut sola remaneant quae arcus circulares involvunt. Denique si pro his formulis integratis post integrationem ponatur $z = 1$, integralia per binas progressiones sinuum ex arcubus in arithmetica progressione exprimentur, quarum progressionum summationes singularia requirunt artificia, quibus absolutis ista obtinetur conclusio, quod formulae integralis valor signis adhibitis superioribus sit $\frac{\pi}{n \sin. \frac{m\pi}{n}}$ et signis inferioribus $\frac{\pi}{n \operatorname{tang.} \frac{m\pi}{n}}$.

1. Hic mihi propositum est duo insignia theoremata, ad quae iam dudum[1]) ex consideratione arcuum circularium, qui vel eundem habent sinum vel tangentem, fueram perductus, ex ipsis principiis calculi integralis demonstrare; duo autem illa theoremata ita se habent:

$$\text{I.} \quad \int \frac{z^{m-1} + z^{n-m-1}}{1 + z^n} dz = \frac{\pi}{n \sin. \frac{m\pi}{n}},$$

$$\text{II.} \quad \int \frac{z^{m-1} - z^{n-m-1}}{1 - z^n} dz = \frac{\pi}{n \operatorname{tang.} \frac{m\pi}{n}},$$

siquidem integratio a termino $z = 0$ usque ad terminum $z = 1$ extendatur, ubi π denotat semiperipheriam circuli, cuius radius $= 1$. Has quidem formulas iam integratas dedi in *Calculo integrali*[2]), verum ibi subsidia integrationis, scilicet resolutionem denominatoris $1 \pm z^n$, tum vero etiam resolutionem ipsius fractionis in fractiones partiales ex mea *Introductione in analysin infinitorum*[3]) petivi; nunc autem, ne opus sit haec adminicula aliunde conquirere, in ipsa integratione omnia principia, quibus innititur, succincte complectar; inprimis autem reductio ad casum, quo post integrationem ponitur $z = 1$, peculiaria artificia circa summationem serierum postulat, quae etiam in sequentibus dilucide sum expositurus; quae tractatio eo maioris momenti videtur, quod similis integratio etiam in his formulis multo latius patentibus succedit, cuiusmodi sunt

$$\int \frac{z^{m-1} \pm z^{n-m-1}}{1 \pm z^n} dz (lz)^{\mu},$$

siquidem exponens μ numeros integros denotet, quemadmodum alia occasione[4]) fusius explicabo.

1) Vide Commentationem 59 huius voluminis. A. G.

2) Vide *Institutionum calculi integralis* vol. I, § 77 et seq.; *Leonhardi Euleri Opera omnia*, series I, vol. 11. A. G.

3) Vide *Introductionis in analysin infinitorum* vol. I, cap. 2 et 9; *Leonhardi Euleri Opera omnia*, series I, vol. 8. A. G.

4) Vide Commentationes 463 et 464 huius voluminis. A. G.

PROBLEMA

2. *Formulam differentialem* $\frac{z^{m-1} dz}{1 + z^n}$ *integrare, ubi scilicet esse debet* $m < n$.

SOLUTIO

Hic igitur denominator $1 + z^n$ in suos factores simplices resolvi debet; ubi vero ante omnia notandum est, si n fuerit numerus impar, unum factorem fore $1 + z$; pro reliquis factoribus imaginariis bini contineantur in hac forma

$$pp - 2pz \cos. \varphi + zz,$$

quae posita nihilo aequalis praebet

$$z = p(\cos. \varphi \pm \sqrt{-1} \cdot \sin. \varphi).$$

Iisdem igitur casibus ipse denominator $1 + z^n$ evanescere debet. Cum igitur sit

$$z = p(\cos. \varphi \pm \sqrt{-1} \cdot \sin. \varphi),$$

erit

$$zz = pp(\cos. 2\varphi \pm \sqrt{-1} \cdot \sin. 2\varphi),$$

$$z^3 = p^3(\cos. 3\varphi \pm \sqrt{-1} \cdot \sin. 3\varphi)$$

et

$$z^n = p^n(\cos. n\varphi \pm \sqrt{-1} \cdot \sin. n\varphi);$$

hoc igitur duplici valore loco z^n substituto fiet

$$\text{I.} \quad 1 + z^n = 1 + p^n \cos. n\varphi + p^n \sqrt{-1} \cdot \sin. n\varphi = 0,$$

$$\text{II.} \quad 1 + z^n = 1 + p^n \cos. n\varphi - p^n \sqrt{-1} \cdot \sin. n\varphi = 0,$$

quarum aequationum summa praebet

$$2 + 2p^n \cos. n\varphi = 0,$$

differentia vero earundem

$$2p^n \sqrt{-1} \cdot \sin. n\varphi = 0;$$

ex posteriore sequitur

$$\sin. n\varphi = 0,$$

ex priore vero

$$1 + p^n \cos. n\varphi = 0,$$

id quod fieri nequit in rationalibus, nisi sit $p=1$ et $\cos. n\varphi = -1$, quo ipso fit $\sin. n\varphi = 0$, uti conditio ex posteriore postulat; omnes autem anguli, quorum cosinus est $=-1$, sunt

$$\pi, \quad 3\pi, \quad 5\pi, \quad 7\pi \quad \text{etc.},$$

quibus ergo angulus $n\varphi$ aequari potest; unde sequentes pro φ obtinebimus valores

$$\frac{\pi}{n}, \quad \frac{3\pi}{n}, \quad \frac{5\pi}{n}, \quad \frac{7\pi}{n} \quad \text{etc.},$$

ex quibus tot capi debent, donec denominator resultet $1+z^n$, quemadmodum ex singulis casibus facile iudicatur:

I. Si $n=1$, erit $\varphi = \pi$ hincque

$$1+z = 1+z;$$

II. si $n=2$, erit $\varphi = 90^0$ hincque

$$1+zz = 1+zz;$$

III. si $n=3$, erit $\varphi = 60^0$ et $=180^0$, hinc

$$1+z^3 = (1+z)(1-z+zz);$$

IV. si $n=4$, erit $\varphi = 45^0$ et $=135^0$, hinc

$$1+z^4 = (1-z\sqrt{2}+zz)(1+z\sqrt{2}+zz);$$

V. si $n=5$, erit $\varphi = 36^0$ et $=108^0$ et $=180^0$ hincque

$$1+z^5 = (1+z)(1+2z\cos. 72^0+zz)(1-2z\cos. 36^0+zz).$$

Cum igitur in genere denominatoris $1+z^n$ unus factor duplex sit

$$1-2z\cos.\varphi+zz,$$

siquidem angulo φ debitos tribuamus valores, fractio $\frac{z^{m-1}}{1+z^n}$ fractionem involvet partialem huius formae

$$\frac{A+Bz}{1-2z\cos.\varphi+zz},$$

ubi totum negotium redit ad coefficientes A et B determinandos. Hi autem facilius reperientur, si factores contemplemur simplices imaginarios, qui sunt

$$\text{I.} \quad z-\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi,$$

$$\text{II.} \quad z-\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi;$$

tum enim fractio proposita tales involvet fractiones partiales

$$\frac{\alpha}{z-\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi}+\frac{\beta}{z-\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi}.$$

Iam pro coefficiente α inveniendo statuatur

$$\frac{z^{m-1}}{1+z^n} = \frac{\alpha}{z-\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi}+R,$$

ubi R complectitur omnes reliquas fractiones partiales; sit autem brevitatis ergo

$$\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi = f,$$

ut habeamus

$$\frac{z^{m-1}}{1+z^n} = \frac{\alpha}{z-f}+R$$

seu multiplicando per $z-f$

$$\frac{z^m-fz^{m-1}}{1+z^n} = \alpha+R(z-f),$$

indeque capiendo $z=f$ habebimus

$$\alpha = \frac{z^m-fz^{m-1}}{1+z^n}$$

casu $z=f$. Hoc autem casu tam numerator quam denominator evanescit; erit ergo

$$\alpha = \frac{mz^{m-1}-(m-1)fz^{m-2}}{nz^{n-1}}$$

et posito iterum $z=f$

$$\alpha = \frac{f^{m-1}}{nf^{n-1}} = \frac{1}{n}f^{m-n}.$$

Cum igitur sit $f = \cos.\varphi + \sqrt{-1}\cdot\sin.\varphi$, erit

$$f^{m-n} = \cos.(m-n)\varphi + \sqrt{-1}\cdot\sin.(m-n)\varphi$$

hincque

$$\alpha = \frac{1}{n}(\cos.(m-n)\varphi + \sqrt{-1}\cdot\sin.(m-n)\varphi)$$

et

$$\beta = \frac{1}{n}(\cos.(m-n)\varphi - \sqrt{-1}\cdot\sin.(m-n)\varphi);$$

quibus valoribus inventis binae nostrae fractiones partiales erunt

$$\frac{\alpha}{z - \cos.\varphi - \sqrt{-1}\cdot\sin.\varphi} + \frac{\beta}{z - \cos.\varphi + \sqrt{-1}\cdot\sin.\varphi},$$

quae ad eandem denominationem perductae dant

$$\frac{(\alpha+\beta)z - (\alpha+\beta)\cos.\varphi + (\alpha-\beta)\sqrt{-1}\cdot\sin.\varphi}{1 - 2z\cos.\varphi + zz}$$

seu loco α et β valores inventos substituendo

$$\frac{\frac{2z}{n}\cos.(m-n)\varphi - \frac{2}{n}\cos.\varphi\cos.(m-n)\varphi - \frac{2}{n}\sin.\varphi\sin.(m-n)\varphi}{1 - 2z\cos.\varphi + zz};$$

hacque fractione partiali cum supra posita

$$\frac{A + Bz}{1 - 2z\cos.\varphi + zz}$$

comparata colligimus

$$A = -\frac{2}{n}\cos.\varphi\cos.(m-n)\varphi - \frac{2}{n}\sin.\varphi\sin.(m-n)\varphi = -\frac{2}{n}\cos.(m-n-1)\varphi$$

et

$$B = \frac{2}{n}\cos.(m-n)\varphi;$$

cum autem sit

$$\sin.n\varphi = 0 \quad \text{et} \quad \cos.n\varphi = -1,$$

erit

$$\cos.(m-n)\varphi = -\cos.m\varphi \quad \text{et} \quad \sin.(m-n)\varphi = -\sin.m\varphi$$

ideoque

$$A = \frac{2}{n}\cos.(m-1)\varphi \quad \text{et} \quad B = -\frac{2}{n}\cos.m\varphi;$$

consequenter ex hac fractione partiali nascitur integrale

$$Bl\sqrt{(1 - 2z\cos.\varphi + zz)} + \frac{A + B\cos.\varphi}{\sin.\varphi}\,\text{A tang.}\frac{z - \cos.\varphi}{\sin.\varphi},$$

ubi si loco A et B valores substituantur, erit hoc integrale

$$-\frac{2}{n}\cos.m\varphi\, l\sqrt{(1 - 2z\cos.\varphi + zz)} + \frac{2}{n}\sin.m\varphi\,\text{A tang.}\frac{z - \cos.\varphi}{\sin.\varphi} + C,$$

quae constans ex termino $z = 0$ definita praebet integrale hoc determinatum

$$-\frac{2}{n}\cos.m\varphi\, l\sqrt{(1 - 2z\cos.\varphi + zz)} + \frac{2}{n}\sin.m\varphi\,\text{A tang.}\frac{z\sin.\varphi}{1 - z\cos.\varphi},$$

ubi tantum opus est loco φ debitos suos valores scribere indeque omnia integralia partialia iunctim sumere.

Praeterea vero casibus, quibus denominator $1 + z^n$ factorem habet $1 + z$, quod evenit, si n fuerit numerus impar, pars integralis inde oriunda adiici debet, quae ita invenitur. Statuatur

$$\frac{z^{m-1}}{1 + z^n} = \frac{\alpha}{1 + z} + R,$$

unde fit

$$\frac{z^{m-1} + z^m}{1 + z^n} = \alpha + R(1 + z),$$

factoque $z = -1$ prodit

$$\alpha = \frac{z^{m-1} + z^m}{1 + z^n};$$

quia autem hoc casu tam numerator quam denominator evanescit, loco utriusque suum differentiale ponatur fietque

$$\alpha = \frac{(m-1)z^{m-2} + mz^{m-1}}{nz^{n-1}},$$

ubi numerator $z^{m-2}(m - 1 + mz)$ posito $z = -1$ abit in $-(-1)^m$ et denominator in $+n$ adeoque $\alpha = \frac{-(-1)^m}{n}$; pars igitur integralis hinc nata erit $\frac{-(-1)^m}{n}l(1 + z)$; casibus igitur, ubi m est numerus par, hoc integrale erit $-\frac{1}{n}l(1 + z)$, sin autem m est numerus impar, fit illud $+\frac{1}{n}l(1 + z)$. Quodsi iam loco φ substituamus suos valores

$$\frac{\pi}{n}, \quad \frac{3\pi}{n}, \quad \frac{5\pi}{n}, \quad \frac{7\pi}{n} \quad \text{etc.},$$

integrale quaesitum erit

$$\int \frac{z^{m-1}dz}{1+z^n} = -\frac{2}{n}\cos.\frac{m\pi}{n}\, l\sqrt{1-2z\cos.\frac{\pi}{n}+zz} + \frac{2}{n}\sin.\frac{m\pi}{n}\,\text{A tang.}\,\frac{z\sin.\frac{\pi}{n}}{1-z\cos.\frac{\pi}{n}}$$

$$-\frac{2}{n}\cos.\frac{3m\pi}{n}\, l\sqrt{1-2z\cos.\frac{3\pi}{n}+zz} + \frac{2}{n}\sin.\frac{3m\pi}{n}\,\text{A tang.}\,\frac{z\sin.\frac{3\pi}{n}}{1-z\cos.\frac{3\pi}{n}}$$

$$-\frac{2}{n}\cos.\frac{5m\pi}{n}\, l\sqrt{1-2z\cos.\frac{5\pi}{n}+zz} + \frac{2}{n}\sin.\frac{5m\pi}{n}\,\text{A tang.}\,\frac{z\sin.\frac{5\pi}{n}}{1-z\cos.\frac{5\pi}{n}}$$

etc.,

quibus insuper casu, quo n sit numerus impar, adiungi debet

$$\frac{-(-1)^m}{n}\, l(1+z).$$

SCHOLION

3. Ne opus sit integrationem formulae

$$\int \frac{(A+Bz)dz}{1-2z\cos.\varphi+zz}$$

aliunde repetere, resolvatur numerator $A+Bz$ in has partes

$$-B\cos.\varphi + Bz \quad \text{et} \quad A + B\cos.\varphi$$

atque ex priore manifesto oritur integrale

$$Bl\sqrt{1-2z\cos.\varphi+zz};$$

pro altera autem parte cum sit

$$\int \frac{dz\sin.\varphi}{1-2z\cos.\varphi+zz} = \text{Arc. tang.}\,\frac{z\sin.\varphi}{1-z\cos.\varphi},^{1)}$$

1) In editione principe praeter $\sqrt{(\ldots)}$ etiam scribitur $\sqrt{\ldots}$ pariterque Arc. tang. et Angl. tang. praeter A tang. Id quod conservandum esse putavimus. A. G.

altera pars huius integralis

$$(A+B\cos.\varphi)\int \frac{dz}{1-2z\cos.\varphi+zz}$$

fiet

$$\frac{A+B\cos.\varphi}{\sin.\varphi}\,\text{Arc. tang.}\,\frac{z\sin.\varphi}{1-z\cos.\varphi}$$

sicque illius formulae integratio ita se habebit

$$\int \frac{(A+Bz)dz}{1-2z\cos.\varphi+zz} = Bl\sqrt{1-2z\cos.\varphi+zz} + \frac{A+B\cos.\varphi}{\sin.\varphi}\,\text{Arc. tang.}\,\frac{z\sin.\varphi}{1-z\cos.\varphi},$$

quod integrale iam evanescit posito $z=0$, ita ut constantis additione non sit opus.

PROBLEMA

4. *Formulam differentialem* $\frac{z^{m-1}dz}{1-z^n}$ *integrare, ubi scilicet esse debet* $m<n$.

SOLUTIO

Hic observandum est denominatorem semper factorem habere $1-z$; tum vero, quoties n fuerit numerus par, etiam factor aderit $1+z$, reliqui autem factores simplices omnes erunt imaginarii, quorum bini talem constituunt factorem duplicem

$$pp-2pz\cos.\varphi+zz;$$

qui cum evanescat posito vel

$$z = p(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi)$$

vel

$$z = p(\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi),$$

iisdem casibus ipse denominator $1-z^n$ evanescet; tum autem erit

$$z^n = p^n(\cos.n\varphi \pm \sqrt{-1}\cdot\sin.n\varphi)$$

ideoque denominator fiet

$$1-p^n(\cos.n\varphi \pm \sqrt{-1}\cdot\sin.n\varphi);$$

qui cum evanescere debeat, fieri oportet

$$\text{I. } 1 - p^n \cos. n\varphi = 0 \quad \text{et} \quad \text{II. } p^n \sqrt{-1} \cdot \sin. n\varphi = 0,$$

ex quo concludimus

$$\sin. n\varphi = 0 \quad \text{et} \quad \cos. n\varphi = \pm 1;$$

ut autem fiat

$$1 - p^n \cos. n\varphi = 0,$$

capi debet

$$\cos. n\varphi = +1$$

eritque $p = 1$, ita ut factor duplex sit

$$1 - 2z \cos. \varphi + zz.$$

Loco $n\varphi$ igitur omnes arcus sumi possunt, quorum cosinus $= +1$, qui sunt

$$0\pi, \quad 2\pi, \quad 4\pi, \quad 6\pi, \quad 8\pi \quad \text{etc.},$$

valoresque anguli ipsi φ erunt

$$\frac{0\pi}{n}, \quad \frac{2\pi}{n}, \quad \frac{4\pi}{n}, \quad \frac{6\pi}{n} \quad \text{etc.}$$

et factores simplices denominatoris hinc oriundi erunt

$$z - \cos \varphi \pm \sqrt{-1} \cdot \sin. \varphi.$$

Ponamus brevitatis gratia

$$f = \cos. \varphi \pm \sqrt{-1} \cdot \sin. \varphi,$$

ita ut f geminum valorem involvat, et factor simplex erit $z - f$; statuatur ergo fractio partialis hinc oriunda $= \frac{\alpha}{z-f}$ ponaturque

$$\frac{z^{m-1}}{1-z^n} = \frac{\alpha}{z-f} + R$$

et per $z - f$ multiplicando erit

$$\frac{z^m - fz^{m-1}}{1-z^n} = \alpha + R(z-f),$$

hinc sumto $z = f$ invenitur

$$\alpha = \frac{z^m - fz^{m-1}}{1-z^n}.$$

Casu autem $z = f$ tam numerator quam denominator simul evanescunt ideoque loco utriusque differentiale capi debet reperiturque $\alpha = -\frac{1}{n} f^{m-n}$; cum autem sit $f = \cos. \varphi \pm \sqrt{-1} \cdot \sin. \varphi$, erit

$$f^{m-n} = \cos. (m-n)\varphi \pm \sqrt{-1} \cdot \sin. (m-n)\varphi$$

sive ob

$$\sin. n\varphi = 0, \quad \cos. n\varphi = 1, \quad \cos. (m-n)\varphi = \cos. m\varphi \quad \text{et} \quad \sin. (m-n)\varphi = \sin. m\varphi$$

erit

$$f^{m-n} = \cos. m\varphi \pm \sqrt{-1} \cdot \sin. m\varphi,$$

ex quo duplici factore imaginario hae duae oriuntur fractiones partiales

$$-\frac{1}{n} \cdot \frac{\cos. m\varphi + \sqrt{-1} \cdot \sin. m\varphi}{z - \cos. \varphi - \sqrt{-1} \cdot \sin. \varphi} - \frac{1}{n} \cdot \frac{\cos. m\varphi - \sqrt{-1} \cdot \sin. m\varphi}{z - \cos. \varphi + \sqrt{-1} \cdot \sin. \varphi},$$

quae contrahuntur in hanc

$$-\frac{2}{n} \cdot \frac{z \cos. m\varphi - \cos. \varphi \cos. m\varphi - \sin. \varphi \sin. m\varphi}{1 - 2z \cos. \varphi + zz};$$

hinc igitur pars integralis nascitur

$$-\frac{2}{n} \int \frac{z dz \cos. m\varphi - dz \cos. \varphi \cos. m\varphi - dz \sin. \varphi \sin. m\varphi}{1 - 2z \cos. \varphi + zz},$$

cuius integrale erit

$$-\frac{2}{n} \cos. m\varphi \, l \sqrt{1 - 2z \cos. \varphi + zz} + \frac{2}{n} \sin. m\varphi \text{ Angl. tang. } \frac{z - \cos. \varphi}{\sin. \varphi} + C,$$

seu constante definita hocce nanciscimur integrale determinatum

$$-\frac{2}{n} \cos. m\varphi \, l \sqrt{1 - 2z \cos. \varphi + zz} + \frac{2}{n} \sin. m\varphi \text{ Angl. tang. } \frac{z \sin. \varphi}{1 - z \cos. \varphi};$$

casu igitur, quo $\varphi = 0$, erit hoc integrale $= -\frac{2}{n} l(1-z)$, cuius autem tantum semissis sumi debet

$$= -\frac{1}{n} l(1-z);$$

casibus autem, quibus n est numerus par et $\varphi = \pi$, haec producitur pars integralis $-\frac{2}{n}\cos. m\pi\, l\sqrt{(1+2z+zz)}$, cuius autem iterum tantum semissis

$$-\frac{1}{n}\cos. m\pi\, l(1+z)$$

capi oportet, ubi notandum, si m sit numerus par, [fore $\cos. m\pi = +1$, sin autem m sit numerus impar] fore $\cos. m\pi = -1$; consequenter integrale quaesitum sequenti modo exprimetur

$$\int \frac{z^{m-1}dz}{1-z^n} = -\frac{1}{n}l(1-z)$$

$$-\frac{1}{n}\cos.\frac{2m\pi}{n}\, l\sqrt{1-2z\cos.\frac{2\pi}{n}+zz} + \frac{2}{n}\sin.\frac{2m\pi}{n}\text{ A tang. }\frac{z\sin.\frac{2\pi}{n}}{1-z\cos.\frac{2\pi}{n}}$$

$$-\frac{1}{n}\cos.\frac{4m\pi}{n}\, l\sqrt{1-2z\cos.\frac{4\pi}{n}+zz} + \frac{2}{n}\sin.\frac{4m\pi}{n}\text{ A tang. }\frac{z\sin.\frac{4\pi}{n}}{1-z\cos.\frac{4\pi}{n}}$$

$$-\frac{1}{n}\cos.\frac{6m\pi}{n}\, l\sqrt{1-2z\cos.\frac{6\pi}{n}+zz} + \frac{2}{n}\sin.\frac{6m\pi}{n}\text{ A tang. }\frac{z\sin.\frac{6\pi}{n}}{1-z\cos.\frac{6\pi}{n}}$$

etc.

PROBLEMA

5. *Formulae differentialis*

$$\frac{z^{m-1}+z^{\mu-1}}{1+z^n}dz$$

integrale invenire existente $m+\mu=n$, ita tamen, ut tam m quam μ sint numeri positivi.

SOLUTIO

Hic igitur nil aliud opus est, nisi ut termini integrales formulae $\int\frac{z^{m-1}dz}{1+z^n}$ supra inventi geminentur, dum altera vice loco m scribitur μ; cum sit pro terminis logarithmicis $\frac{m\pi}{n}+\frac{\mu\pi}{n}=\pi$, erit

$$\cos.\frac{\mu\pi}{n} = -\cos.\frac{m\pi}{n},\quad \cos.\frac{3\mu\pi}{n} = -\cos.\frac{3m\pi}{n},\quad \cos.\frac{5\mu\pi}{n} = -\cos.\frac{5m\pi}{n}\quad \text{etc.},$$

unde patet omnes terminos logarithmicos se invicem destruere. Porro vero pro arcubus circularibus cum sit

$$\sin.\frac{\mu\pi}{n} = \sin.\frac{m\pi}{n},\quad \sin.\frac{3\mu\pi}{n} = \sin.\frac{3m\pi}{n}\quad \text{etc.},$$

hi termini duplicabuntur, ita ut integrale quaesitum proditurum sit

$$\frac{4}{n}\sin.\frac{\mu\pi}{n}\text{ A tang. }\frac{z\sin.\frac{\pi}{n}}{1-z\cos.\frac{\pi}{n}} + \frac{4}{n}\sin.\frac{3\mu\pi}{n}\text{ A tang. }\frac{z\sin.\frac{3\pi}{n}}{1-z\cos.\frac{3\pi}{n}}$$

$$+\frac{4}{n}\sin.\frac{5\mu\pi}{n}\text{ A tang. }\frac{z\sin.\frac{5\pi}{n}}{1-z\cos.\frac{5\pi}{n}} + \text{etc.},$$

quorum terminorum, si i denotet numerum quemcunque imparem, forma generalis erit

$$\frac{4}{n}\sin.\frac{i\mu\pi}{n}\text{ A tang. }\frac{z\sin.\frac{i\pi}{n}}{1-z\cos.\frac{i\pi}{n}};$$

terminos autem eousque continuare oportet, quoad numerus non superet exponentem n, ita ut, si n fuerit numerus impar, ultimus terminus contineat $i=n$, sin autem n sit numerus par, valor futurus ipsius i sit $i=n-1$.

COROLLARIUM

6. Cum casus $n=1$ hinc excludatur, casu $n=2$ integrale erit

$$2\sin.\frac{m\pi}{2}\text{ A tang } z.$$

Casu $n=3$ integrale erit

$$\frac{4}{3}\sin.\frac{m\pi}{3}\text{ A tang. }\frac{z\sqrt{3}}{2-z}$$

et casu $n=4$ erit integrale

$$\sin.\frac{m\pi}{4}\text{ A tang. }\frac{z}{\sqrt{2}-z} + \sin.\frac{3m\pi}{4}\text{ A tang. }\frac{z}{\sqrt{2}+z}.$$

PROBLEMA

7. *Formulae differentialis praecedentis integrale assignare casu, quo* $z = 1$, *quandoquidem superius integrale ita est sumtum, ut evanescat posito* $z = 0$.

SOLUTIO

Cum integralis quaesiti quaelibet pars hanc habeat formam

$$\frac{4}{n} \sin. \frac{im\pi}{n} \text{ A tang.} \frac{z \sin. \frac{i\pi}{n}}{1 - z \cos. \frac{i\pi}{n}},$$

haec forma posito $z = 1$ abit in hanc

$$\frac{4}{n} \sin. \frac{im\pi}{n} \text{ A tang.} \frac{\sin. \frac{i\pi}{n}}{1 - \cos. \frac{i\pi}{n}};$$

iam vero est

$$\frac{\sin. \frac{i\pi}{n}}{1 - \cos. \frac{i\pi}{n}} = \cot. \frac{i\pi}{2n} = \text{tang.} \left(\frac{\pi}{2} - \frac{i\pi}{2n} \right)$$

ideoque

$$\text{A tang.} \frac{\sin. \frac{i\pi}{n}}{1 - \cos. \frac{i\pi}{n}} = \frac{\pi}{2} - \frac{i\pi}{2n},$$

unde generatim pars integralis erit

$$\frac{4}{n} \sin. \frac{im\pi}{n} \left(\frac{\pi}{2} - \frac{i\pi}{2n} \right) = \frac{2\pi}{n} \sin. \frac{im\pi}{n} - \frac{2i\pi}{nn} \sin. \frac{im\pi}{n};$$

integrale ergo quaesitum per binas sequentes progressiones exprimitur

$$\frac{2\pi}{n} \left(\sin. \frac{m\pi}{n} + \sin. \frac{3m\pi}{n} + \sin. \frac{5m\pi}{n} + \cdots + \sin. \frac{im\pi}{n} \right)$$

$$- \frac{2\pi}{nn} \left(1 \sin. \frac{m\pi}{n} + 3 \sin. \frac{3m\pi}{n} + 5 \sin. \frac{5m\pi}{n} + \cdots + i \sin. \frac{im\pi}{n} \right);$$

ubi si brevitatis gratia scribamus $\frac{m\pi}{n} = \vartheta$, erit integrale istud commodius expressum ita

$$\frac{2\pi}{n} (\sin. \vartheta + \sin. 3\vartheta + \sin. 5\vartheta + \cdots + \sin. i\vartheta)$$

$$- \frac{2\pi}{nn} (1 \sin. \vartheta + 3 \sin. 3\vartheta + 5 \sin. 5\vartheta + \cdots + i \sin. i\vartheta),$$

ubi, quoties n fuerit numerus impar, erit $i = n$, sin autem n numerus par, erit $i = n - 1$.

Cum igitur totum negotium huc redeat, ut hae duae series summentur, statuamus

$$s = \sin. \vartheta + \sin. 3\vartheta + \sin. 5\vartheta + \cdots + \sin. i\vartheta$$

et

$$t = 1 \sin. \vartheta + 3 \sin. 3\vartheta + 5 \sin. 5\vartheta + \cdots + i \sin. i\vartheta,$$

ita ut nostrum integrale sit

$$\frac{2\pi}{n} s - \frac{2\pi}{nn} t;$$

pro priore serie cum sit

$$2 \sin. \vartheta \sin. i\vartheta = \cos. (i-1)\vartheta - \cos. (i+1)\vartheta,$$

erit

$$2s \sin. \vartheta = \cos. 0\vartheta - \cos. 2\vartheta - \cos. 4\vartheta - \cos. 6\vartheta - \cdots - \cos. (i+1)\vartheta$$
$$+ \cos. 2\vartheta + \cos. 4\vartheta + \cos. 6\vartheta + \cdots,$$

ita ut sit $2s \sin. \vartheta = 1 - \cos. (i+1)\vartheta$, ergo

$$s = \frac{1}{2 \sin. \vartheta} - \frac{\cos. (i+1)\vartheta}{2 \sin. \vartheta}.$$

Pro altera autem serie spectemus primum angulum ϑ ut variabilem, et cum sit $d. \cos. i\vartheta = - i d\vartheta \sin. i\vartheta$, erit

$$\int i d\vartheta \sin. i\vartheta = - \cos. i\vartheta,$$

quo notato reperietur

$$\int t d\vartheta = - \cos. \vartheta - \cos. 3\vartheta - \cos. 5\vartheta - \cdots - \cos. i\vartheta,$$

quae series multiplicetur per $2 \sin. \vartheta$, et cum sit

$$2 \sin. \vartheta \cos. i\vartheta = - \sin. (i-1)\vartheta + \sin. (i+1)\vartheta,$$

erit

$$2 \sin. \vartheta \int t d\vartheta = - \sin. (i+1)\vartheta,$$

quocirca habebimus

$$\int t d\vartheta = -\frac{\sin.(i+1)\vartheta}{2 \sin.\vartheta}$$

hincque

$$t = -\frac{(i+1)\cos.(i+1)\vartheta}{2\sin.\vartheta} + \frac{\sin.(i+1)\vartheta \cos.\vartheta}{2\sin.\vartheta^2},$$

quibus valoribus inventis integrale nostrum ita se habebit

$$\frac{\pi}{n \sin.\vartheta} - \frac{\pi \cos.(i+1)\vartheta}{n \sin.\vartheta} + \frac{\pi(i+1)\cos.(i+1)\vartheta}{nn \sin.\vartheta} - \frac{\pi \sin.(i+1)\vartheta \cos.\vartheta}{nn \sin.\vartheta^2};$$

cum nunc sit vel $i = n-1$ vel $i = n$, prout n fuerit numerus par vel impar, utrumque casum seorsim evolvamus.

I. Si n sit numerus par, erit $i = n-1$ et $i+1 = n$, et quia $\vartheta = \frac{m\pi}{n}$, erit

$$(i+1)\vartheta = m\pi,$$

hinc

$$\sin.(i+1)\vartheta = 0 \quad \text{et} \quad \cos.(i+1)\vartheta = \pm 1;$$

quocirca formula nostra erit $-\frac{\pi}{n \sin.\vartheta}$, consequenter integrale quaesitum hoc casu erit

$$\frac{\pi}{n \sin.\frac{m\pi}{n}}.$$

II. At si sit $i = n$ ideoque $i+1 = n+1$, erit angulus

$$(i+1)\vartheta = (n+1)\frac{m\pi}{n} = m\pi + \frac{m\pi}{n} = m\pi + \vartheta,$$

unde fit

$$\cos.(i+1)\vartheta = \pm \cos.\vartheta \quad \text{et} \quad \sin.(i+1)\vartheta = \pm \sin.\vartheta,$$

quibus valoribus substitutis formula evadet

$$\frac{\pi}{n \sin.\vartheta} \mp \frac{\pi \cos.\vartheta}{n \sin.\vartheta} \pm \frac{(n+1)\pi \cos.\vartheta}{nn \sin.\vartheta} \mp \frac{\pi \cos.\vartheta}{nn \sin.\vartheta},$$

quae contrahitur in

$$\frac{\pi}{n \sin.\vartheta} = \frac{\pi}{n \sin.\frac{m\pi}{n}}.$$

Consequenter, sive n sit numerus par sive impar, erit

$$\int \frac{z^{m-1} + z^{\mu-1}}{1+z^n} dz = \frac{\pi}{n \sin.\frac{m\pi}{n}}.$$

COROLLARIUM 1

8. Si ergo fuerit $m + \mu = n$ et post integrationem ita institutam, ut integrale evanescat posito $z = 0$, capiatur $z = 1$, semper fiet

$$\int \frac{z^{m-1} + z^{\mu-1}}{1+z^n} dz = \frac{\pi}{n \sin.\frac{m\pi}{n}}.$$

COROLLARIUM 2

9. Cum per seriem infinitam sit

$$\frac{1}{1+z^n} = 1 - z^n + z^{2n} - z^{3n} + z^{4n} - z^{5n} + \text{etc.},$$

nostrae formulae integrale in genere erit

$$\begin{aligned} &+ \frac{z^m}{m} - \frac{z^{m+n}}{m+n} + \frac{z^{m+2n}}{m+2n} - \frac{z^{m+3n}}{m+3n} + \frac{z^{m+4n}}{m+4n} - \text{etc.} \\ &+ \frac{z^\mu}{\mu} - \frac{z^{\mu+n}}{\mu+n} + \frac{z^{\mu+2n}}{\mu+2n} - \frac{z^{\mu+3n}}{\mu+3n} + \frac{z^{\mu+4n}}{\mu+4n} - \text{etc.}, \end{aligned}$$

unde posito $z = 1$ sequentis seriei infinitae summatio habebitur

$$\frac{\pi}{n \sin.\frac{m\pi}{n}} = \begin{cases} + \frac{1}{m} - \frac{1}{m+n} + \frac{1}{m+2n} - \frac{1}{m+3n} + \frac{1}{m+4n} - \text{etc.} \\ + \frac{1}{\mu} - \frac{1}{\mu+n} + \frac{1}{\mu+2n} - \frac{1}{\mu+3n} + \frac{1}{\mu+4n} - \text{etc.} \end{cases}$$

vel ob $n = m + \mu$ huius

$$\frac{\pi}{(m+\mu)\sin.\frac{m\pi}{m+\mu}} = \begin{cases} + \frac{1}{m} - \frac{1}{2m+\mu} + \frac{1}{3m+2\mu} - \frac{1}{4m+3\mu} + \text{etc.} \\ + \frac{1}{\mu} - \frac{1}{2\mu+m} + \frac{1}{3\mu+2m} - \frac{1}{4\mu+3m} + \text{etc.} \end{cases}$$

EXEMPLA

I. Si $m = 1$ et $\mu = 1$, erit

$$\frac{\pi}{2} = \begin{cases} +1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \text{etc.} \\ +1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \text{etc.} \end{cases}$$

ideoque

$$\frac{\pi}{4} = +1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \text{etc.}$$

II. Si $m = 1$ et $\mu = 2$, erit

$$m + \mu = 3 \quad \text{et} \quad \sin.\frac{m\pi}{m+\mu} = \frac{\sqrt{3}}{2}$$

ideoque

$$\frac{2\pi}{3\sqrt{3}} = \begin{cases} +1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \frac{1}{13} - \frac{1}{16} + \text{etc.} \\ +\frac{1}{2} - \frac{1}{5} + \frac{1}{8} - \frac{1}{11} + \frac{1}{14} - \frac{1}{17} + \text{etc.} \end{cases}$$

sive

$$\frac{2\pi}{3\sqrt{3}} = 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \frac{1}{10} - \frac{1}{11} + \frac{1}{13} + \text{etc.}$$

III. Si $m = 1$ et $\mu = 3$, erit

$$\mu + m = 4 \quad \text{et} \quad \sin.\frac{m\pi}{m+\mu} = \frac{1}{\sqrt{2}}$$

ideoque

$$\frac{\pi}{2\sqrt{2}} = \begin{cases} +1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \frac{1}{17} - \frac{1}{21} + \text{etc.} \\ +\frac{1}{3} - \frac{1}{7} + \frac{1}{11} - \frac{1}{15} + \frac{1}{19} - \frac{1}{23} + \text{etc.} \end{cases}$$

seu

$$\frac{\pi}{2\sqrt{2}} = +1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \text{etc.}$$

Addatur huic series exemplo I inventa prodibitque

$$\frac{\pi}{4} + \frac{\pi}{2\sqrt{2}} = 2 - \frac{2}{7} + \frac{2}{9} - \frac{2}{15} + \frac{2}{17} - \text{etc.}$$

sive

$$\frac{\pi}{8} + \frac{\pi}{4\sqrt{2}} = 1 - \frac{1}{7} + \frac{1}{9} - \frac{1}{15} + \frac{1}{17} - \frac{1}{23} + \text{etc.},$$

ubi termini positivi in forma $8a + 1$, negativi vero in forma $8a - 1$ continentur.

PROBLEMA

10. *Formulam integralem*

$$\int \frac{z^{m-1} - z^{\mu-1}}{1 - z^n} dz$$

existente $m + \mu = n$ *integrare.*

SOLUTIO

Cum igitur a formula integrali $\int \frac{z^{m-1} dz}{1 - z^n}$ haec formula $\int \frac{z^{\mu-1} dz}{1 - z^n}$ subtrahi debeat, primi logarithmi se destruunt; id quod ob

$$\frac{2m\pi}{n} + \frac{2\mu\pi}{n} = 2\pi$$

et hinc ob

$$\cos.\frac{2m\pi}{n} = \cos.\frac{2\mu\pi}{n}$$

etiam de secundis valet; pro tertiis idem evenit, quia

$$\frac{4m\pi}{n} + \frac{4\mu\pi}{n} = 4\pi$$

et hinc quia

$$\cos.\frac{4m\pi}{n} = \cos.\frac{4\mu\pi}{n};$$

atque hoc modo omnes logarithmi plane se destruent; arcus vero circulares, quia

$$\sin.\frac{2\mu\pi}{n} = -\sin.\frac{2m\pi}{n} \quad \text{et} \quad \sin.\frac{4\mu\pi}{n} = -\sin.\frac{4m\pi}{n},$$

omnes manifesto duplicabuntur; unde integrale quaesitum per meros arcus circulares exprimitur eritque

$$\int \frac{z^{m-1} - z^{\mu-1}}{1 - z^n} dz$$

$$-\frac{4}{n}\sin.\frac{2m\pi}{n}\,\text{A tang.}\frac{z\sin.\frac{2\pi}{n}}{1 - z\cos.\frac{2\pi}{n}} + \frac{4}{n}\sin.\frac{4m\pi}{n}\,\text{A tang.}\frac{z\sin.\frac{4\pi}{n}}{1 - z\cos.\frac{4\pi}{n}}$$

$$+\frac{4}{n}\sin.\frac{6m\pi}{n}\,\text{A tang.}\frac{z\sin.\frac{6\pi}{n}}{1 - z\cos.\frac{6\pi}{n}} + \frac{4}{n}\sin.\frac{8m\pi}{n}\,\text{A tang.}\frac{z\sin.\frac{8\pi}{n}}{1 - z\cos.\frac{8\pi}{n}}$$

etc.,

unde, si i denotet numerum parem quemcunque, singuli hi termini in hac forma generali continebuntur

$$\frac{4}{n} \sin. \frac{im\pi}{n} \text{Arc. tang.} \frac{z \sin. \frac{i\pi}{n}}{1 - z \cos. \frac{i\pi}{n}}.$$

Has autem formulas eousque continuari oportet, quamdiu i non superet exponentem n; quare si n sit numerus par, ultimus valor erit $i = n$, sin autem n sit impar, ultimus ille valor erit $i = n - 1$. Caeterum notasse iuvabit totum hoc integrale evanescere sumto $z = 0$.

PROBLEMA

11. *Praecedentis formulae integralis valorem investigare pro casu, quo ponitur* $z = 1$.

SOLUTIO

Cum omnium partium forma generalis hoc casu abeat in hanc

$$\frac{4}{n} \sin. \frac{im\pi}{n} \text{A tang.} \frac{\sin. \frac{i\pi}{n}}{1 - \cos. \frac{i\pi}{n}},$$

est vero, uti ante iam vidimus,

$$\frac{\sin. \frac{i\pi}{n}}{1 - \cos. \frac{i\pi}{n}} = \cot. \frac{i\pi}{2n} = \text{tang.} \left(\frac{\pi}{2} - \frac{i\pi}{2n}\right),$$

unde iste arcus erit

$$\frac{\pi}{2} - \frac{i\pi}{2n}$$

ideoque tota forma

$$\frac{2\pi}{n} \sin. \frac{im\pi}{n} - \frac{2i\pi}{nn} \sin. \frac{im\pi}{n},$$

ponamus brevitatis gratia $\frac{m\pi}{n} = \vartheta$, ut habeamus hanc formulam

$$\frac{2\pi}{n} \sin. i\vartheta - \frac{2i\pi}{nn} \sin. i\vartheta;$$

quodsi iam loco i successive scribamus numeros 2, 4, 6, 8 etc. usque ad ultimum i, qui est vel n vel $n - 1$, valor integralis quaesitus per has duas series exprimetur

$$\frac{2\pi}{n}(\sin. 2\vartheta + \sin. 4\vartheta + \sin. 6\vartheta + \cdots + \sin. i\vartheta),$$

$$-\frac{2\pi}{nn}(2 \sin. 2\vartheta + 4 \sin. 4\vartheta + 6 \sin. 6\vartheta + \cdots + i \sin. i\vartheta);$$

statuamus igitur ut supra

$$s = \sin. 2\vartheta + \sin. 4\vartheta + \sin. 6\vartheta + \cdots + \sin. i\vartheta,$$

$$t = 2 \sin. 2\vartheta + 4 \sin. 4\vartheta + 6 \sin. 6\vartheta + \cdots + i \sin. i\vartheta,$$

ita ut valor, quem quaerimus, futurus sit

$$\frac{2\pi}{n} s - \frac{2\pi}{nn} t.$$

Iam seriem priorem multiplicemus per $2 \sin. \vartheta$, et cum sit

$$2 \sin. \vartheta \sin. i\vartheta = \cos. (i-1)\vartheta - \cos. (i+1)\vartheta,$$

erit

$$2s \sin. \vartheta = \cos. \vartheta - \cos. 3\vartheta - \cos. 5\vartheta - \cos. 7\vartheta - \cdots - \cos. (i+1)\vartheta$$
$$+ \cos. 3\vartheta + \cos. 5\vartheta + \cos. 7\vartheta + \cdots$$

seu

$$2s \sin. \vartheta = \cos. \vartheta - \cos. (i+1)\vartheta,$$

ergo

$$s = \frac{\cos. \vartheta}{2 \sin. \vartheta} - \frac{\cos. (i+1)\vartheta}{2 \sin. \vartheta}.$$

Altera series multiplicetur per $d\vartheta$, et cum sit

$$\int i d\vartheta \sin. i\vartheta = -\cos. i\vartheta,$$

prodibit integrando

$$\int t d\vartheta = -\cos. 2\vartheta - \cos. 4\vartheta - \cos. 6\vartheta - \cdots - \cos. i\vartheta,$$

quae denuo multiplicata per $2 \sin. \vartheta$ ob

$$2 \sin. \vartheta \cos. i\vartheta = \sin. (i+1)\vartheta - \sin. (i-1)\vartheta$$

praebet

$$2 \sin. \vartheta \int t d\vartheta = \sin. \vartheta - \sin. 3\vartheta - \sin. 5\vartheta - \sin. 7\vartheta - \cdots - \sin. (i+1)\vartheta$$
$$+ \sin. 3\vartheta + \sin. 5\vartheta + \sin. 7\vartheta + \cdots;$$

hinc per $2 \sin. \vartheta$ dividendo fit

$$\int t d\vartheta = + \frac{1}{2} - \frac{\sin. (i+1)\vartheta}{2 \sin. \vartheta},$$

unde colligimus

$$t = - \frac{(i+1) \cos. (i+1)\vartheta}{2 \sin. \vartheta} + \frac{\sin. (i+1)\vartheta \cos. \vartheta}{2 \sin. \vartheta^2}.$$

His igitur valoribus s et t inventis integrale quaesitum erit

$$\frac{\pi \cos. \vartheta}{n \sin. \vartheta} - \frac{\pi \cos. (i+1)\vartheta}{n \sin. \vartheta} + \frac{\pi (i+1) \cos. (i+1)\vartheta}{nn \sin. \vartheta} - \frac{\pi \sin. (i+1)\vartheta \cos. \vartheta}{nn \sin. \vartheta^2};$$

cum nunc sit $\vartheta = \frac{m\pi}{n}$, duo casus evolvendi supersunt, alter, quo n est numerus par et $i = n$, alter vero, quo n est numerus impar et $i = n - 1$.

I. Si $i = n$, erit

$$(i + 1)\vartheta = m\pi + \frac{m\pi}{n} = m\pi + \vartheta,$$

unde ob $\sin. m\pi = 0$ erit

$$\cos. (i + 1)\vartheta = \cos. m\pi \cos. \vartheta \quad \text{et} \quad \sin. (i + 1)\vartheta = \cos. m\pi \sin. \vartheta,$$

quibus substitutis habebimus $\frac{\pi \cos. \vartheta}{n \sin. \vartheta}$; reliqua scilicet membra se mutuo destruunt, ita ut valor quaesitus sit $\frac{\pi \cos. \vartheta}{n \sin. \vartheta} = \frac{\pi}{n \operatorname{tang.} \vartheta}$.

II. Si $i = n - 1$ ideoque $i + 1 = n$, erit

$$(i + 1)\vartheta = m\pi \quad \text{et} \quad \cos. (i + 1)\vartheta = \cos. m\pi, \quad \text{at} \quad \sin. (i + 1)\vartheta = 0,$$

unde formula nostra fiet $\frac{\pi \cos. \vartheta}{n \sin. \vartheta}$, ubi scilicet reliqui termini praeter hunc sese mutuo destruxerunt.

Unde patet, sive exponens n fuerit par sive impar, utroque casu valorem integralis quaesiti esse

$$= \frac{\pi}{n \operatorname{tang.} \frac{m\pi}{n}}.$$

COROLLARIUM 1

12. Si ergo fuerit $m + \mu = n$ et post integrationem ita institutam, ut integrale evanescat posito $z = 0$, capiatur $z = 1$, semper fiet

$$\int \frac{z^{m-1} - z^{\mu-1}}{1 - z^n} dz = \frac{\pi}{n \operatorname{tang.} \frac{m\pi}{n}}.$$

COROLLARIUM 2

13. Cum per seriem infinitam sit

$$\frac{1}{1 - z^n} = 1 + z^n + z^{2n} + z^{3n} + z^{4n} + z^{5n} + \text{etc.},$$

integrale nostrae formulae erit in genere

$$\begin{aligned} &\frac{z^m}{m} + \frac{z^{m+n}}{m+n} + \frac{z^{m+2n}}{m+2n} + \frac{z^{m+3n}}{m+3n} + \text{etc.} \\ &- \frac{z^\mu}{\mu} - \frac{z^{\mu+n}}{\mu+n} - \frac{z^{\mu+2n}}{\mu+2n} - \frac{z^{\mu+3n}}{\mu+3n} - \text{etc.}, \end{aligned}$$

unde posito $z = 1$ sequentis seriei infinitae summatio habebitur

$$\frac{\pi}{n \operatorname{tang.} \frac{m\pi}{n}} = \begin{cases} \frac{1}{m} + \frac{1}{m+n} + \frac{1}{m+2n} + \frac{1}{m+3n} + \frac{1}{m+4n} + \text{etc.} \\ - \frac{1}{\mu} - \frac{1}{\mu+n} - \frac{1}{\mu+2n} - \frac{1}{\mu+3n} - \frac{1}{\mu+4n} - \text{etc.}, \end{cases}$$

quae series a superiori tantum ratione signorum discrepat; vel cum sit $n = m + \mu$, erit

$$\frac{\pi}{(m + \mu) \operatorname{tang.} \frac{m\pi}{m+\mu}} = \begin{cases} \frac{1}{m} + \frac{1}{2m+\mu} + \frac{1}{3m+2\mu} + \frac{1}{4m+3\mu} + \text{etc.} \\ - \frac{1}{\mu} - \frac{1}{2\mu+m} - \frac{1}{3\mu+2m} - \frac{1}{4\mu+3m} - \text{etc.} \end{cases}$$

EXEMPLA

I. Quia hae duae series se mutuo destruunt casu $\mu = m$, hoc casu fiet

$$\frac{\pi}{2m \operatorname{tang.} \frac{\pi}{2}} = 0.$$

II. Sumamus $m = 1$ et $\mu = 2$ colligiturque

$$\frac{\pi}{3\sqrt{3}} = \begin{cases} + 1 + \frac{1}{4} + \frac{1}{7} + \frac{1}{10} + \frac{1}{13} + \frac{1}{16} + \frac{1}{19} + \text{etc.} \\ - \frac{1}{2} - \frac{1}{5} - \frac{1}{8} - \frac{1}{11} - \frac{1}{14} - \frac{1}{17} - \frac{1}{20} - \text{etc.} \end{cases}$$

sive

$$\frac{\pi}{3\sqrt{3}} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \frac{1}{10} - \frac{1}{11} + \frac{1}{13} - \text{etc.};$$

si ergo hanc seriem per 2 multiplicemus, habebimus

$$\frac{2\pi}{3\sqrt{3}} = \frac{2}{1} - \frac{2}{2} + \frac{2}{4} - \frac{2}{5} + \frac{2}{7} - \frac{2}{8} + \frac{2}{10} - \text{etc.},$$

supra autem (§ 9) inveneramus

$$\frac{2\pi}{3\sqrt{3}} = 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \frac{1}{10} - \frac{1}{11} + \text{etc.};$$

hinc, si ab illa serie hanc subtrahamus, prodibit

$$0 = 1 - \frac{3}{2} + \frac{3}{4} - \frac{1}{5} + \frac{1}{7} - \frac{3}{8} + \frac{3}{10} - \frac{1}{11} + \frac{1}{13} - \frac{3}{14} + \text{etc.},$$

quae ita commode in periodos distribuitur

$$0 = \begin{cases} + \frac{1}{1} - \frac{3}{2} + \frac{3}{4} - \frac{1}{5} \\ + \frac{1}{7} - \frac{3}{8} + \frac{3}{10} - \frac{1}{11} \\ + \frac{1}{13} - \frac{3}{14} + \frac{3}{16} - \frac{1}{17} \\ \qquad \text{etc.,} \end{cases}$$

unde sequitur fore

$$1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{17} + \frac{1}{19} - \text{etc.} = 3\left(\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{10} + \frac{1}{14} - \text{etc.}\right).$$

SCHOLION

14. Aequalitas harum duarum serierum eo magis est notatu digna, quod eius veritas non parum abstrusa videtur; rem igitur sequenti modo tentemus. Ponamus pro priore

$$s = \frac{z}{1} - \frac{z^5}{5} + \frac{z^7}{7} - \frac{z^{11}}{11} + \frac{z^{13}}{13} - \frac{z^{17}}{17} + \text{etc.}$$

eritque differentiando

$$\frac{ds}{dz} = 1 - z^4 + z^6 - z^{10} + z^{12} - z^{16} + z^{18} - \text{etc.} = \frac{1 - z^4}{1 - z^6},$$

unde fit

$$s = \int \frac{(1 - z^4)\,dz}{1 - z^6},$$

in quo integrali poni debet $z = 1$; qua forma cum problemate postremo comparata fit $m = 1$, $\mu = 5$ et $n = 6$, ita ut sit $m + \mu = n$; hinc ergo colligitur

$$s = \frac{\pi}{6 \operatorname{tang.} \frac{\pi}{6}} = \frac{\pi}{2\sqrt{3}}.$$

Pro altera serie ponamus

$$t = \frac{z^2}{2} - \frac{z^4}{4} + \frac{z^8}{8} - \frac{z^{10}}{10} + \frac{z^{14}}{14} - \text{etc.},$$

ut posito $z = 1$ fieri debeat $s = 3t$; erit ergo differentiando

$$\frac{dt}{dz} = z - z^3 + z^7 - z^9 + z^{13} - z^{15} + z^{19} - \text{etc.} = \frac{z - z^3}{1 - z^6},$$

unde fit

$$t = \int \frac{(z - z^3)\,dz}{1 - z^6},$$

qua aequatione cum problemate ultimo comparata ob $m = 2$, $\mu = 4$, $n = 6$ positoque $z = 1$ prodit

$$t = \frac{\pi}{6 \operatorname{tang.} \frac{\pi}{3}} = \frac{\pi}{6\sqrt{3}},$$

quocirca erit $3t = \frac{\pi}{2\sqrt{3}}$ hincque

$$s = 3t = \frac{\pi}{2\sqrt{3}}.$$

DE VALORE FORMULAE INTEGRALIS

$$\int \frac{z^{\lambda-\omega} \pm z^{\lambda+\omega}}{1 \pm z^{2\lambda}} \cdot \frac{dz}{z} (lz)^{\mu}$$

CASU QUO POST INTEGRATIONEM PONITUR $z = 1$

Commentatio 463 indicis Enestroemiani
Novi commentarii academiae scientiarum Petropolitanae 19 (1774), 1775, p. 30—65
Summarium ibidem p. 8—13

SUMMARIUM

Ex consideratione arcuum circularium, qui eundem habent vel sinum vel tangentem, iam olim[1]) invenit Illustr. huius dissertationis Auctor designantibus m et n numeros quoscunque esse

$$\frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \frac{1}{3n-m} - \text{etc.} = \frac{\pi}{n \sin. \frac{m\pi}{n}}$$

et

$$\frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m} + \text{etc.} = \frac{\pi}{n \operatorname{tang.} \frac{m\pi}{n}},$$

tum vero easdem quoque series oriri ex evolutione formularum integralium in priori dissertatione consideratarum, si quidem post integrationem ponitur $z = 1$. Quodsi loco n adhibeatur 2λ et m ponatur $= \lambda - \omega$, prodibit

$$\frac{1}{\lambda-\omega} + \frac{1}{\lambda+\omega} - \frac{1}{3\lambda-\omega} - \frac{1}{3\lambda+\omega} + \frac{1}{5\lambda-\omega} + \frac{1}{5\lambda+\omega} - \text{etc.} = \frac{\pi}{2\lambda \cos. \frac{\pi\omega}{2\lambda}} = \int \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z}$$

et

$$\frac{1}{\lambda-\omega} - \frac{1}{\lambda+\omega} + \frac{1}{3\lambda-\omega} - \frac{1}{3\lambda+\omega} + \frac{1}{5\lambda-\omega} - \frac{1}{5\lambda+\omega} + \text{etc.} = \frac{\pi}{2\lambda} \operatorname{tang.} \frac{\pi\omega}{2\lambda} = \int \frac{z^{\lambda-\omega} - z^{\lambda+\omega}}{1-z^{2\lambda}} \cdot \frac{dz}{z}.$$

1) Vide notam p. 387. A. G.

In hac igitur dissertatione Illustr. Auctori propositum est, ut ostendat, quomodo ex valoribus harum formularum integralium cognitis illi derivari queant, qui respondent formulis integralibus in fronte huius dissertationis propositis, quod novo et singulari artificio perficit. Posito nimirum

$$\frac{\pi}{2\lambda \cos. \frac{\pi\omega}{2\lambda}} = S \quad \text{et} \quad \frac{\pi}{2\lambda} \operatorname{tang.} \frac{\pi\omega}{2\lambda} = T$$

tam in his valoribus quam formulis integralibus, quibus aequantur, non solum quantitatem z, sed etiam ω tamquam variabilem spectat, unde primum differentiando posita sola z variabili colligitur

$$\left(\frac{dS}{dz}\right) = \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{1}{z},$$

tum vero denuo differentiando posita sola littera ω variabili erit

$$\left(\frac{ddS}{dz\,d\omega}\right) = \frac{-z^{\lambda-\omega} + z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{1}{z}\, lz,$$

ex quo vicissim integratione ita instituta, ut sola z habeatur pro variabili, deducitur

$$\left(\frac{dS}{d\omega}\right) = \int \frac{-z^{\lambda-\omega} + z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z}\, lz = \frac{\pi\pi \sin. \frac{\pi\omega}{2\lambda}}{4\lambda\lambda \cos. \frac{\pi\omega^2}{2\lambda}}.^{1)}$$

Simili modo ex altera formula colligitur

$$\left(\frac{dT}{d\omega}\right) = \frac{\pi\pi}{4\lambda\lambda \cos \frac{\pi\omega^2}{2\lambda}} = -\int \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1-z^{2\lambda}} \cdot \frac{dz}{z}\, lz.$$

Si nunc ponatur

$$\left(\frac{dS}{d\omega}\right) = S' \quad \text{et} \quad \left(\frac{dT}{d\omega}\right) = T',$$

idem ratiocinium prosequendo obtinebitur

$$\left(\frac{dS'}{d\omega}\right) = \frac{\pi^3}{8\lambda^3}\left(\frac{2}{\cos. \frac{\pi\omega^3}{2\lambda}} - \frac{1}{\cos. \frac{\pi\omega}{2\lambda}}\right) = \int \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z} (lz)^2,$$

$$\left(\frac{dT'}{d\omega}\right) = \frac{\pi^3}{8\lambda^3} \cdot \frac{2 \sin. \frac{\pi\omega}{2\lambda}}{\cos. \frac{\pi\omega^3}{2\lambda}} = \int \frac{z^{\lambda-\omega} - z^{\lambda+\omega}}{1-z^{2\lambda}} \cdot \frac{dz}{z} (lz)^2$$

similique modo pertingere licet ad formulas integrales, quae $(lz)^3$ vel altiores potestates

1) $\cos \varphi^2$, $\cos \varphi^3$ etc. hic et in sequentibus formulis idem atque $\cos^2 \varphi$, $\cos^3 \varphi$ etc. significant. A. G.

ipsius lz involvunt. Sufficit vero heic observasse esse in genere

$$\int \frac{\pm z^{\lambda-\omega} + z^{\lambda+\omega}}{1 + z^{2\lambda}} \cdot \frac{dz}{z} (lz)^{\nu} = \left(\frac{d^{\nu} S}{d\omega^{\nu}}\right)$$

posito $S = \frac{\pi}{2\lambda \cos. \frac{\pi\omega}{2\lambda}}$; accipiendum vero est signum superius, si fuerit ν numerus par, inferius autem, si impar.

Simili ratione erit

$$\int \frac{\pm z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z} (lz)^{\nu} = \left(\frac{d^{\nu} T}{d\omega^{\nu}}\right)$$

posito $T = \frac{\pi}{2\lambda} \operatorname{tang.} \frac{\pi\omega}{2\lambda}$ heicque signum $+$ valebit pro ν numero pari, $-$ vero pro ν numero impari. Inventio igitur valorum pro formulis propositis reducitur ad successivas evolutiones differentialium

$$\left(\frac{dS}{d\omega}\right), \quad \left(\frac{ddS}{d\omega^2}\right), \quad \left(\frac{d^3 S}{d\omega^3}\right) \quad \text{etc.}, \quad \left(\frac{dT}{d\omega}\right), \quad \left(\frac{ddT}{d\omega^2}\right), \quad \left(\frac{d^3 T}{d\omega^3}\right) \quad \text{etc.}$$

Deinde si in seriebus, quibus S et T aequantur, variabilitas ipsius ω spectetur, consequemur quoque per successivas differentiationes valores ipsorum $\left(\frac{dS}{d\omega}\right)$, $\left(\frac{ddS}{d\omega^2}\right)$ etc., ubi quidem observare convenit generatim haberi

$$\left(\frac{d^{\nu} S}{d\omega^{\nu}}\right) = \nu(\nu-1)(\nu-2)\cdots 1 \left\{ \begin{array}{l} \frac{1}{(\lambda-\omega)^{\nu+1}} \mp \frac{1}{(\lambda+\omega)^{\nu+1}} - \frac{1}{(3\lambda-\omega)^{\nu+1}} \pm \frac{1}{(3\lambda+\omega)^{\nu+1}} \\ \qquad + \frac{1}{(5\lambda-\omega)^{\nu+1}} \mp \frac{1}{(5\lambda+\omega)^{\nu+1}} - \text{etc.} \end{array} \right\};$$

et signa superiora valent, si ν numerus impar, inferiora vero, si ν numerus par. Eadem ratione erit

$$\left(\frac{d^{\nu} T}{d\omega^{\nu}}\right) = \nu(\nu-1)(\nu-2)\cdots 1 \left\{ \begin{array}{l} \frac{1}{(\lambda-\omega)^{\nu+1}} \pm \frac{1}{(\lambda+\omega)^{\nu+1}} + \frac{1}{(3\lambda-\omega)^{\nu+1}} \pm \frac{1}{(3\lambda+\omega)^{\nu+1}} \\ \qquad + \frac{1}{(5\lambda-\omega)^{\nu+1}} \pm \frac{1}{(5\lambda+\omega)^{\nu+1}} + \text{etc.} \end{array} \right\};$$

de signis eadem regula valet ac supra.

Quemadmodum valores formularum integralium iam propositarum per continuam differentiationem formularum S et T eliciuntur, ita per integrationem earundem formularum, dum in $d\omega$ ductae supponuntur, aliarum formularum integralium valores exhibentur, cuius rei specimen Illustr. Auctor in Additamento dissertationi praesenti subiuncto exponit. Nam si ponatur

$$\frac{z^{\lambda-\omega} \pm z^{\lambda+\omega}}{1 \pm z^{2\lambda}} = V,$$

ubi praeter variabilem z etiam ω ut variabilis consideratur, per naturam formularum integralium duas variabiles involventium erit $\int S d\omega = \int \frac{dz}{z} \int V d\omega$, ubi in integralibus $\int S d\omega$, $\int V d\omega$ sola ω ut variabilis tractatur, tum vero in integratione $\int \frac{dz}{z} \int V d\omega$ sola z ut variabilis spectatur. Ex hoc igitur principio consequitur esse

$$\int S d\omega = l \operatorname{tang.} \frac{\pi(\lambda+\omega)}{4\lambda} = \int \frac{-z^{\lambda-\omega} + z^{\lambda+\omega}}{1 + z^{2\lambda}} \cdot \frac{dz}{z\, lz}$$

similique modo

$$\int T d\omega = -l \cos. \frac{\pi\omega}{2\lambda} = \int \frac{-z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z\, lz},$$

quibus aequari debent expressiones ex seriebus deductae

$$l \frac{(\lambda+\omega)(3\lambda-\omega)(5\lambda+\omega)(7\lambda-\omega) \text{ etc.}}{(\lambda-\omega)(3\lambda+\omega)(5\lambda-\omega)(7\lambda+\omega) \text{ etc.}}$$

et

$$l \frac{\lambda\lambda}{\lambda\lambda - \omega\omega} \cdot \frac{9\lambda\lambda}{9\lambda\lambda - \omega\omega} \cdot \frac{25\lambda\lambda}{25\lambda\lambda - \omega\omega} \cdot \text{etc.}$$

Casuum particularium evolutiones, quae ab Illustr. Auctore propositae sunt, lectores rerum mathematicarum curiosi ex ipsa dissertatione haurire non intermittent.

1. Ex consideratione innumerabilium arcuum circularium, qui communem habent vel sinum vel tangentem, iam olim[1]) summationem duarum serierum infinitarum deduxi, quae ob summam generalitatem maxime memoratu dignae videbantur. Si enim litterae m et n numeros quoscunque denotant, posita diametri ratione ad peripheriam ut 1 ad π illae duae summationes hoc modo se habebant

$$\frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \frac{1}{3n-m} - \text{etc.} = \frac{\pi}{n \sin. \frac{m\pi}{n}}$$

et

$$\frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m} + \text{etc.} = \frac{\pi}{n \operatorname{tang.} \frac{m\pi}{n}}$$

atque ex his duabus seriebus iam tum temporis elicueram summationes omnium serierum illarum, quarum denominatores secundum potestates numerorum naturalium progrediuntur, quemadmodum in *Introductione in analysin*

1) Vide Commentationem 59 huius voluminis. A. G.

infinitorum et alibi[1]) fusius exposui. Nunc autem eaedem series me perduxerunt ad integrationem formulae in titulo expressae, quae eo magis attentione digna videtur, quod huiusmodi integrationes aliis methodis neutiquam exsequi liceat.

2. Statim autem patet has duas series infinitas oriri ex evolutione quarundam formularum integralium, si post integrationem quantitati variabili certus valor, veluti unitas, tribuatur; ita prior series deducitur ex evolutione huius formulae integralis

$$\int \frac{z^{m-1} + z^{n-m-1}}{1 + z^n} dz,$$

posterior vero ex evolutione istius

$$\int \frac{z^{m-1} - z^{n-m-1}}{1 - z^n} dz,$$

siquidem post integrationem statuatur $z = 1$. Deinceps autem ex ipsis principiis calculi integralis demonstravi valorem integralis prioris harum duarum formularum, siquidem ponatur $z = 1$, reduci ad hanc formulam simplicem

$$\frac{\pi}{n \sin. \frac{m\pi}{n}},$$

integrale autem posterius eodem casu $z = 1$ ad istam

$$\frac{\pi}{n \operatorname{tang.} \frac{m\pi}{n}},$$

ta ut ex ipsis calculi integralis principiis certum sit esse

$$\int \frac{z^{m-1} + z^{n-m-1}}{1 + z^n} dz = \frac{\pi}{n \sin. \frac{m\pi}{n}},$$

$$\int \frac{z^{m-1} - z^{n-m-1}}{1 - z^n} dz = \frac{\pi}{n \operatorname{tang.} \frac{m\pi}{n}},$$

1) Vide *Introductionis in analysin infinitorum* vol. I, cap. 10, LEONHARDI EULERI *Opera omnia*, series I, vol. 8; vide porro notam p. 392. A. G.

siquidem post integrationem ita institutam, ut integrale evanescat posito $z = 0$, statuatur $z = 1$.

3. Quo iam hanc duplicem integrationem ad formam propositam reducamus, faciamus $n = 2\lambda$ et $m = \lambda - \omega$, unde binae illae series infinitae hanc induent formam

$$\frac{1}{\lambda - \omega} + \frac{1}{\lambda + \omega} - \frac{1}{3\lambda - \omega} - \frac{1}{3\lambda + \omega} + \frac{1}{5\lambda - \omega} + \frac{1}{5\lambda + \omega} - \text{etc.}$$

et

$$\frac{1}{\lambda - \omega} - \frac{1}{\lambda + \omega} + \frac{1}{3\lambda - \omega} - \frac{1}{3\lambda + \omega} + \frac{1}{5\lambda - \omega} - \frac{1}{5\lambda + \omega} + \text{etc.};$$

harum igitur serierum prioris summa erit

$$\frac{\pi}{2\lambda \sin. \frac{\pi(\lambda - \omega)}{2\lambda}} = \frac{\pi}{2\lambda \cos. \frac{\pi\omega}{2\lambda}},$$

posterioris vero summa erit

$$\frac{\pi}{2\lambda \operatorname{tang.} \frac{\pi(\lambda - \omega)}{2\lambda}} = \frac{\pi}{2\lambda \operatorname{cotang.} \frac{\pi\omega}{2\lambda}} = \frac{\pi}{2\lambda} \operatorname{tang.} \frac{\pi\omega}{2\lambda}.$$

Quodsi ergo brevitatis gratia ponamus

$$\frac{\pi}{2\lambda \cos. \frac{\pi\omega}{2\lambda}} = S \quad \text{et} \quad \frac{\pi}{2\lambda} \operatorname{tang.} \frac{\pi\omega}{2\lambda} = T,$$

habebimus sequentes duas integrationes

$$\int \frac{z^{\lambda - \omega} + z^{\lambda + \omega}}{1 + z^{2\lambda}} \cdot \frac{dz}{z} = S \quad \text{et} \quad \int \frac{z^{\lambda - \omega} - z^{\lambda + \omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z} = T.$$

4. Circa has binas integrationes ante omnia observo eas perinde locum habere, sive pro litteris λ et ω accipiantur numeri integri sive fracti. Sint enim λ et ω numeri fracti quicunque, qui evadant integri, si multiplicentur per α, quo posito fiat $z = x^{\alpha}$, eritque $\frac{dz}{z} = \frac{\alpha dx}{x}$ et potestas quaecunque $z^{\vartheta} = x^{\alpha\vartheta}$; prior igitur formula erit

$$\int \frac{x^{\alpha(\lambda - \omega)} + x^{\alpha(\lambda + \omega)}}{1 + x^{2\alpha\lambda}} \cdot \frac{\alpha dx}{x};$$

ubi cum iam omnes exponentes sint numeri integri, valor huius formulae posito post integrationem $x=1$, quandoquidem tunc etiam sit $z=1$, a praecedente eo tantum differt, quod hic habeamus $\alpha\lambda$ et $\alpha\omega$ loco λ et ω ac praeterea hic adsit factor α, quocirca valor istius formulae erit

$$\alpha\frac{\pi}{2\alpha\lambda\cos.\frac{\pi\omega}{2\lambda}}=\frac{\pi}{2\lambda\cos.\frac{\pi\omega}{2\lambda}},$$

qui ergo valor est $=S$ prorsus ut ante; quae identitas etiam manifesto est in altera formula, unde patet, etiamsi pro λ et ω fractiones quaecunque accipiantur, integrationem hic exhibitam nihilo minus locum esse habituram; quae circumstantia probe notari meretur, quoniam in sequentibus litteram ω tanquam variabilem sumus tractaturi.

5. Postquam igitur binae istae formulae integrales litteris S et T indicatae fuerint integratae ita, ut evanescant posito $z=0$, integralia spectari poterunt non solum ut functiones quantitatis z, sed etiam ut functiones binarum variabilium z et ω, quandoquidem numerum ω tanquam quantitatem variabilem tractare licet; quin etiam exponentem λ pro quantitate variabili habere liceret; sed quia hinc formulae integrales alius generis essent proditurae, atque hic contemplari constitui, solam quantitatem ω praeter ipsam variabilem z hic ut quantitatem variabilem sum tractaturus.

6. Cum igitur sit

$$S=\int\frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{z},$$

in qua integratione sola z ut variabilis spectatur, erit utique secundum signandi morem iam satis usu receptum

$$\left(\frac{dS}{dz}\right)=\frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{1}{z};$$

haec iam formula denuo differentietur posita sola littera ω variabili eritque

$$\left(\frac{ddS}{dz\,d\omega}\right)=\frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{1}{z}lz,$$

quae formula ducta in dz ac denuo integrata sola z habita pro variabili dabit

$$\int dz\left(\frac{ddS}{dz\,d\omega}\right)=\int\frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{z}lz,$$

ubi notetur esse

$$S=\frac{\pi}{2\lambda\cos.\frac{\pi\omega}{2\lambda}},$$

ita ut hinc deducamus

$$\left(\frac{dS}{d\omega}\right)=\frac{\pi\pi\sin.\frac{\pi\omega}{2\lambda}}{4\lambda\lambda\cos.\frac{\pi\omega}{2\lambda}^2};$$

hoc igitur valore substituto nanciscimur hanc integrationem

$$\int\frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{z}lz=\frac{\pi\pi\sin.\frac{\pi\omega}{2\lambda}}{4\lambda\lambda\cos.\frac{\pi\omega}{2\lambda}^2}.$$

7. Quodsi iam altera formula simili modo tractetur, cum sit

$$T=\frac{\pi}{2\lambda}\operatorname{tang.}\frac{\pi\omega}{2\lambda},$$

erit

$$\left(\frac{dT}{d\omega}\right)=\frac{\pi\pi}{4\lambda\lambda\cos.\frac{\pi\omega}{2\lambda}^2},$$

ex formula autem integrali erit

$$\left(\frac{dT}{d\omega}\right)=\int\frac{-z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}}\cdot\frac{dz}{z}lz,$$

unde colligimus sequentem integrationem

$$\int\frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1-z^{2\lambda}}\cdot\frac{dz}{z}lz=\frac{-\pi\pi}{4\lambda\lambda\cos.\frac{\pi\omega}{2\lambda}^2}.$$

8. Quoniam litteras S et T etiam per series expressas dedimus, erit etiam per similes series

$$\left(\frac{dS}{d\omega}\right) = \frac{1}{(\lambda-\omega)^2} - \frac{1}{(\lambda+\omega)^2} - \frac{1}{(3\lambda-\omega)^2} + \frac{1}{(3\lambda+\omega)^2} + \frac{1}{(5\lambda-\omega)^2} - \text{etc.}$$

$$= \frac{\pi\pi \sin. \frac{\pi\omega}{2\lambda}}{4\lambda\lambda \cos. \frac{\pi\omega^2}{2\lambda}}.$$

Similique modo etiam pro altera serie

$$\left(\frac{dT}{d\omega}\right) = \frac{1}{(\lambda-\omega)^2} + \frac{1}{(\lambda+\omega)^2} + \frac{1}{(3\lambda-\omega)^2} + \frac{1}{(3\lambda+\omega)^2} + \frac{1}{(5\lambda-\omega)^2} + \text{etc.}$$

$$= \frac{\pi\pi}{4\lambda\lambda \cos. \frac{\pi\omega^2}{2\lambda}}$$

sicque summas harum serierum quoque duplici modo repraesentavimus, scilicet per formulam evolutam quantitatem π involventem, tum vero etiam per formulam integralem, quae ita est comparata, ut eius integrale nulla methodo adhuc consueta assignari possit.

9. Applicemus has integrationes ad aliquot casus particulares; ac primo quidem sumamus $\omega = 0$, quo quidem casu prior integratio sponte in oculos incurrit, at posterior praebet

$$\int \frac{2z^{\lambda}}{1-z^{2\lambda}} \cdot \frac{dz}{z} lz = -\frac{\pi\pi}{4\lambda\lambda}$$

sive

$$\int \frac{z^{\lambda-1} dz lz}{1-z^{2\lambda}} = -\frac{\pi\pi}{8\lambda\lambda}$$

hincque simul istam summationem adipiscimur

$$\frac{1}{\lambda\lambda} + \frac{1}{\lambda\lambda} + \frac{1}{9\lambda\lambda} + \frac{1}{9\lambda\lambda} + \frac{1}{25\lambda\lambda} + \frac{1}{25\lambda\lambda} + \text{etc.} = \frac{\pi\pi}{4\lambda\lambda}$$

sive

$$1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \text{etc.} = \frac{\pi\pi}{8},$$

id quod iam dudum[1]) a me est demonstratum.

1) Vide L. EULERI Commentationem 41 (indicis ENESTROEMIANI): *De summis serierum reciprocarum*, Comment. acad. sc. Petrop. 7 (1734/5), 1740, p. 123; *LEONHARDI EULERI Opera omnia*, series I, vol. 14. Vide etiam P. STÄCKEL, *Eine vergessene Abhandlung LEONHARD EULERS über die Summe der reziproken Quadrate der natürlichen Zahlen*, Biblioth. Mathem. 8_3, 1907, p. 37. A. G.

10. Hic statim patet perinde esse, quinam numerus pro λ accipiatur; sit igitur $\lambda = 1$ et habebitur ista integratio

$$\int \frac{dz lz}{1-z^2} = -\frac{\pi\pi}{8},$$

ex qua sequentia integralia simpliciora

$$\int \frac{dz lz}{1-z} \quad \text{et} \quad \int \frac{dz lz}{1+z}$$

derivare licet ope huius ratiocinii; statuatur

$$\int \frac{z dz lz}{1-zz} = P$$

et posito $zz = v$, ut sit $zdz = \frac{dv}{2}$ et $lz = \frac{1}{2} lv$, prodibit

$$\frac{1}{4} \int \frac{dv lv}{1-v} = P,$$

si scilicet post integrationem fiat $v = 1$, quippe quo casu etiam fit $z = 1$; sic igitur erit

$$\int \frac{dv lv}{1-v} = 4P;$$

nunc prior illa formula addatur ad inventam eritque

$$\int \frac{dz lz + z dz lz}{1-zz} = P - \frac{\pi\pi}{8},$$

haec autem formula sponte reducitur ad hanc

$$\int \frac{dz lz}{1-z} = P - \frac{\pi\pi}{8};$$

modo autem vidimus esse $\int \frac{dv lv}{1-v}$ sive $\int \frac{dz lz}{1-z} = 4P$, ita ut sit

$$4P = P - \frac{\pi\pi}{8},$$

unde manifesto fit

$$P = -\frac{\pi\pi}{24},$$

ex quo sequitur fore

$$\int \frac{dzlz}{1-z} = -\frac{\pi\pi}{6};$$

simili modo erit

$$\int \frac{dzlz - zdzlz}{1-zz} = -P - \frac{\pi\pi}{8} = -\frac{\pi\pi}{12},$$

quae supra et infra per $1-z$ dividendo praebet

$$\int \frac{dzlz}{1+z} = -\frac{\pi\pi}{12},$$

quare iam adepti sumus tres integrationes memoratu maxime dignas

$$\text{I.} \quad \int \frac{dzlz}{1+z} = -\frac{\pi\pi}{12},$$

$$\text{II.} \quad \int \frac{dzlz}{1-z} = -\frac{\pi\pi}{6},$$

$$\text{III.} \quad \int \frac{dzlz}{1-zz} = -\frac{\pi\pi}{8},$$

quibus adiungi potest

$$\text{IV.} \quad \int \frac{zdzlz}{1-zz} = -\frac{\pi\pi}{24}.$$

11. Quemadmodum igitur hae formulae ex ipsis calculi integralis principiis sunt deductae, ita etiam earum veritas per resolutionem in series facile comprobatur; cum enim sit

$$\frac{1}{1+z} = 1 - z + zz - z^3 + z^4 - z^5 + \text{etc.}$$

et in genere

$$\int z^n dzlz = \frac{z^{n+1}}{n+1} lz - \frac{z^{n+1}}{(n+1)^2},$$

qui valor posito $z=1$ reducitur ad $-\frac{1}{(n+1)^2}$, patet fore

$$\int \frac{dzlz}{1+z} = -1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} + \text{etc.} = -\frac{\pi\pi}{12}$$

sive

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \text{etc.} = \frac{\pi\pi}{12};$$

simili modo ob

$$\frac{1}{1-z} = 1 + z + zz + z^3 + z^4 + \text{etc.}$$

erit

$$\int \frac{dzlz}{1-z} = -1 - \frac{1}{4} - \frac{1}{9} - \frac{1}{16} - \frac{1}{25} - \text{etc.} = -\frac{\pi\pi}{6}$$

seu

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \text{etc.} = \frac{\pi\pi}{6};$$

tum vero ob

$$\frac{1}{1-zz} = 1 + zz + z^4 + z^6 + z^8 + \text{etc.}$$

erit

$$\int \frac{dzlz}{1-zz} = -1 - \frac{1}{9} - \frac{1}{25} - \frac{1}{49} - \frac{1}{81} - \text{etc.} = -\frac{\pi\pi}{8}$$

sive

$$1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \text{etc.} = \frac{\pi\pi}{8}.$$

Eodem modo etiam

$$\int \frac{zdzlz}{1-zz} = -\frac{1}{4} - \frac{1}{16} - \frac{1}{36} - \frac{1}{64} - \text{etc.} = -\frac{\pi\pi}{24}$$

sive

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{36} + \frac{1}{64} + \text{etc.} = \frac{\pi\pi}{24},$$

quae quidem summationes iam sunt notissimae. Neque tamen quisquam adhuc methodo directa ostendit esse

$$\int \frac{dzlz}{1+z} = -\frac{\pi\pi}{12}.$$

12. Ponamus nunc $\omega = 1$ et nostrae integrationes has induent formas

$$1^0. \quad \int \frac{-z^{\lambda-2}(1-zz)dzlz}{1+z^{2\lambda}} = \frac{\pi\pi \sin. \frac{\pi}{2\lambda}}{4\lambda\lambda \cos. \frac{\pi}{2\lambda}^2}$$

et

$$2^0. \quad \int \frac{-z^{\lambda-2}(1+zz)dzlz}{1-z^{2\lambda}} = +\frac{\pi\pi}{4\lambda\lambda \cos. \frac{\pi}{2\lambda}^2},$$

unde pro diversis valoribus ipsius λ, quos quidem binario non minores accipere licet, sequentes obtinentur integrationes:

I. Si $\lambda = 2$, erit

$$1^0.\ \int \frac{-(1-zz)dzlz}{1+z^4} = \frac{\pi\pi}{8\sqrt{2}},$$

$$2^0.\ \int \frac{-(1+zz)dzlz}{1-z^4} = +\frac{\pi\pi}{8} \quad \text{sive} \quad \int \frac{-dzlz}{1-zz} = +\frac{\pi\pi}{8}.$$

II. Si $\lambda = 3$, habebimus

$$1^0.\ \int \frac{-z(1-zz)dzlz}{1+z^6} = \frac{\pi\pi}{54} \quad \text{et} \quad 2^0.\ \int \frac{-z(1+zz)dzlz}{1-z^6} = \frac{\pi\pi}{27}.^{1)}$$

Hae autem duae formulae ponendo $zz = v$ abibunt in sequentes

$$1^0.\ \int \frac{-dv(1-v)lv}{1+v^3} = \frac{2\pi\pi}{27} \quad \text{et} \quad 2^0.\ \int \frac{-dv(1+v)lv}{1-v^3} = \frac{4\pi\pi}{27}.^{2)}$$

III. Sit $\lambda = 4$ et consequemur

$$1^0.\ \int \frac{-zz(1-zz)dzlz}{1+z^8} = \frac{\pi\pi\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}}{16(2+\sqrt{2})} = \frac{\pi\pi\sqrt{(2-\sqrt{2})}}{32(2+\sqrt{2})}$$

et

$$2^0.\ \int \frac{-zz(1+zz)dzlz}{1-z^8} = \int \frac{-zzdzlz}{(1-zz)(1+z^4)} = \frac{\pi\pi}{16(2+\sqrt{2})},$$

quae postrema forma reducitur ad hanc

$$\int \frac{-dzlz}{1-zz} + \int \frac{(1-zz)dzlz}{1+z^4} = \frac{\pi\pi}{8(2+\sqrt{2})};$$

est vero $\int \frac{-dzlz}{1-zz} = \frac{\pi\pi}{8}$, unde reperitur

$$\int \frac{dzlz(1-zz)}{1+z^4} = -\frac{\pi\pi(1+\sqrt{2})}{8(2+\sqrt{2})} = -\frac{\pi\pi}{8\sqrt{2}},$$

qui valor iam in superiori casu $\lambda = 2$ est inventus.

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1) Editio princeps atque etiam editiones 463a et 463A (indicis ENESTROEMIANI):

$$\int \frac{-z(1+zz)dzlz}{1-z^6} = \int \frac{-zdzlz}{1-zz+z^4} = \frac{\pi\pi}{27}.$$ Correxit A. G.

2) Editio princeps atque etiam editiones 463a et 463A (indicis ENESTROEMIANI):

$$\int \frac{dvlv}{1-v+vv} = \frac{4\pi\pi}{27}.$$ Correxit A. G.

13. Nihil autem impedit, quominus etiam faciamus $\lambda = 1$, dummodo integralia ita capiantur, ut evanescant posito $z = 0$; tum autem reperiemus

$$1^0.\ \int \frac{-(1-zz)dzlz}{z(1+zz)} = \infty \quad \text{et} \quad 2^0.\ \int \frac{-(1+zz)dzlz}{z(1-zz)} = \infty,$$

unde hinc nihil concludere licet. Ceterum etiam nostrae series supra inventae manifesto declarant earum summas esse infinitas, quandoquidem primus terminus utriusque $\frac{1}{(\lambda-\omega)^2}$ fit infinitus sumto, uti fecimus, $\lambda = 1$ et $\omega = 1$.

14. His casibus evolutis ulterius progrediamur ac ponamus formulas integrales inventas

$$\int \frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z} lz = S' \quad \text{et} \quad \int \frac{-z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}} \cdot \frac{dz}{z} lz = T',$$

ita ut sit

$$S' = \frac{\pi\pi \sin. \frac{\pi\omega}{2\lambda}}{4\lambda\lambda \cos. \frac{\pi\omega}{2\lambda}^2} \quad \text{et} \quad T' = \frac{\pi\pi}{4\lambda\lambda \cos. \frac{\pi\omega}{2\lambda}^2},$$

atque ut ante iam differentiemus solo numero ω pro variabili habito; quo facto sequentes nanciscimur integrationes

$$\int \frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z} (lz)^2 = \left(\frac{dS'}{d\omega}\right) \quad \text{et} \quad \int \frac{z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}} \cdot \frac{dz}{z} (lz)^2 = \left(\frac{dT'}{d\omega}\right).$$

Hunc in finem ponamus brevitatis ergo angulum $\frac{\pi\omega}{2\lambda} = \varphi$, ut sit

$$S' = \frac{\pi\pi \sin. \varphi}{4\lambda\lambda \cos. \varphi^2} = \frac{\pi\pi}{4\lambda\lambda} \cdot \frac{\sin. \varphi}{\cos. \varphi^2} \quad \text{et} \quad T' = \frac{\pi\pi}{4\lambda\lambda} \cdot \frac{1}{\cos. \varphi^2},$$

ac reperiemus

$$d. \frac{\sin. \varphi}{\cos. \varphi^2} = \frac{\cos. \varphi^2 + 2 \sin. \varphi^2}{\cos. \varphi^3} d\varphi = \frac{1 + \sin. \varphi^2}{\cos. \varphi^3} d\varphi,$$

ubi est $d\varphi = \frac{\pi d\omega}{2\lambda}$; unde colligimus

$$\left(\frac{dS'}{d\omega}\right) = \frac{\pi^3}{8\lambda^3} \left(\frac{1 + \sin. \frac{\pi\omega}{2\lambda}^2}{\cos. \frac{\pi\omega}{2\lambda}^3} \right) = \frac{\pi^3}{8\lambda^3} \left(\frac{2}{\cos. \frac{\pi\omega}{2\lambda}^3} - \frac{1}{\cos. \frac{\pi\omega}{2\lambda}} \right);$$

simili modo ob $T' = \frac{\pi\pi}{4\lambda\lambda} \cdot \frac{1}{\cos. \varphi^2}$ erit

$$d. \frac{1}{\cos. \varphi^2} = \frac{2\, d\varphi \sin. \varphi}{\cos. \varphi^3}$$

hincque

$$\left(\frac{dT'}{d\omega}\right) = \frac{\pi^3}{8\lambda^3} \cdot \frac{2 \sin. \frac{\pi\omega}{2\lambda}}{\cos. \frac{\pi\omega}{2\lambda}^3};$$

consequenter integrationes hinc natae erunt

$$\int \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1 + z^{2\lambda}} \cdot \frac{dz}{z} (lz)^2 = \frac{\pi^3}{8\lambda^3} \left(\frac{2}{\cos. \frac{\pi\omega}{2\lambda}^3} - \frac{1}{\cos. \frac{\pi\omega}{2\lambda}} \right),$$

$$\int \frac{z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z} (lz)^2 = \frac{\pi^3}{8\lambda^3} \frac{2 \sin. \frac{\pi\omega}{2\lambda}}{\cos. \frac{\pi\omega}{2\lambda}^3}.$$

15. Si iam eodem modo series § 8 inventas denuo differentiemus sumta sola ω variabili, perveniamus ad sequentes summationes

$$\frac{\pi^3}{8\lambda^3} \left\{ \frac{2}{\cos. \frac{\pi\omega}{2\lambda}^3} - \frac{1}{\cos. \frac{\pi\omega}{2\lambda}} \right\}$$

$$= \frac{2}{(\lambda-\omega)^3} + \frac{2}{(\lambda+\omega)^3} - \frac{2}{(3\lambda-\omega)^3} - \frac{2}{(3\lambda+\omega)^3} + \frac{2}{(5\lambda-\omega)^3} + \frac{2}{(5\lambda+\omega)^3} - \text{etc.},$$

$$\frac{\pi^3}{8\lambda^3} \cdot \frac{2 \sin. \frac{\pi\omega}{2\lambda}}{\cos. \frac{\pi\omega}{2\lambda}^3} = \frac{2}{(\lambda-\omega)^3} - \frac{2}{(\lambda+\omega)^3} + \frac{2}{(3\lambda-\omega)^3} - \frac{2}{(3\lambda+\omega)^3} + \frac{2}{(5\lambda-\omega)^3} - \text{etc.}$$

16. Si iam hic sumamus $\omega = 0$ et $\lambda = 1$, prior integratio hanc induit formam

$$\int \frac{2\, dz (lz)^2}{1 + zz} = \frac{\pi^3}{8} = \frac{2}{1^3} + \frac{2}{1^3} - \frac{2}{3^3} - \frac{2}{3^3} + \frac{2}{5^3} + \frac{2}{5^3} - \frac{2}{7^3} - \frac{2}{7^3} + \text{etc.},$$

ita ut sit

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + \text{etc.} = \frac{\pi^3}{32},$$

quemadmodum iam dudum[1]) demonstravi. Altera autem integratio hoc casu

1) Vide notam p. 392. A. G.

in nihilum abit. Ex priori vero integrali

$$\int \frac{dz (lz)^2}{1 + zz} = \frac{\pi^3}{16}$$

alia derivare non licet, uti supra fecimus ex formula $\int \frac{dz\, lz}{1 - zz} = -\frac{\pi\pi}{8}$, propterea quod hic denominator $1 + zz$ non habet factores reales.

17. Sumamus igitur $\lambda = 2$ et $\omega = 1$ ac prior integratio dabit

$$\int \frac{(1 + zz) dz (lz)^2}{1 + z^4} = \frac{3\pi^3}{32\sqrt{2}};$$

series autem hinc nata erit

$$\frac{2}{1^3} + \frac{2}{3^3} - \frac{2}{5^3} - \frac{2}{7^3} + \frac{2}{9^3} + \frac{2}{11^3} - \text{etc.},$$

ita ut sit

$$\frac{1}{1^3} + \frac{1}{3^3} - \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} + \frac{1}{11^3} - \text{etc.} = \frac{3\pi^3}{64\sqrt{2}},$$

quae superiori addita praebet

$$\frac{1}{1^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{15^3} + \frac{1}{17^3} - \frac{1}{23^3} + \text{etc.} = \frac{\pi^3(3 + 2\sqrt{2})}{128\sqrt{2}}.$$

Altera vero integratio hoc casu dat

$$\int \frac{dz (lz)^2}{1 + zz} = \frac{\pi^3}{16},$$

quae cum paragrapho praecedenti perfecte congruit, quemadmodum etiam series hinc nata est

$$\frac{2}{1^3} - \frac{2}{3^3} + \frac{2}{5^3} - \frac{2}{7^3} + \frac{2}{9^3} - \frac{2}{11^3} + \frac{2}{13^3} - \text{etc.}$$

18. Quo autem facilius sequentes integrationes per continuam differentiationem elicere valeamus, eas in genere repraesentemus; et cum pro priore sit

$$S = \frac{\pi}{2\lambda \cos. \frac{\pi\omega}{2\lambda}},$$

integrationes hinc ortae ita ordine procedent:

$$\text{I. } \int \frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z} = S,$$

$$\text{II. } \int \frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z}\, lz = \left(\frac{dS}{d\omega}\right),$$

$$\text{III. } \int \frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z} (lz)^2 = \left(\frac{ddS}{d\omega^2}\right),$$

$$\text{IV. } \int \frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z} (lz)^3 = \left(\frac{d^3S}{d\omega^3}\right),$$

$$\text{V. } \int \frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z} (lz)^4 = \left(\frac{d^4S}{d\omega^4}\right),$$

$$\text{VI. } \int \frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z} (lz)^5 = \left(\frac{d^5S}{d\omega^5}\right),$$

$$\text{VII. } \int \frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z} (lz)^6 = \left(\frac{d^6S}{d\omega^6}\right)$$

etc.

19. Pro his differentiationibus continuis facilius absolvendis ponamus brevitatis ergo $\frac{\pi}{2\lambda} = \alpha$, ut sit

$$S = \frac{\alpha}{\cos.\alpha\omega};$$

tum vero sit

$$\sin.\alpha\omega = p \quad \text{et} \quad \cos.\alpha\omega = q$$

eritque

$$dp = \alpha q d\omega \quad \text{et} \quad dq = -\alpha p d\omega.$$

Praeterea vero notetur esse

$$d.\frac{p^n}{q^{n+1}} = \alpha d\omega \left(\frac{np^{n-1}}{q^n} + \frac{(n+1)p^{n+1}}{q^{n+2}}\right).$$

His praemissis ob $S = \alpha \cdot \frac{1}{q}$ erit

$$\left(\frac{dS}{d\omega}\right) = \alpha\alpha\frac{p}{qq},$$

deinde

$$\left(\frac{ddS}{d\omega^2}\right) = \alpha^3\left(\frac{1}{q} + \frac{2pp}{q^3}\right),$$

porro

$$\left(\frac{d^3S}{d\omega^3}\right) = \alpha^4\left(\frac{5p}{qq} + \frac{6p^3}{q^4}\right),$$

$$\left(\frac{d^4S}{d\omega^4}\right) = \alpha^5\left(\frac{5}{q} + \frac{28pp}{q^3} + \frac{24p^4}{q^5}\right),$$

$$\left(\frac{d^5S}{d\omega^5}\right) = \alpha^6\left(\frac{61p}{qq} + \frac{180p^3}{q^4} + \frac{120p^5}{q^6}\right),$$

$$\left(\frac{d^6S}{d\omega^6}\right) = \alpha^7\left(\frac{61}{q} + \frac{662pp}{q^3} + \frac{1320p^4}{q^5} + \frac{720p^6}{q^7}\right),$$

$$\left(\frac{d^7S}{d\omega^7}\right) = \alpha^8\left(\frac{1385p}{qq} + \frac{7266p^3}{q^4} + \frac{10920p^5}{q^6} + \frac{5040p^7}{q^8}\right);$$

hi autem valores ob $pp = 1 - qq$ ad sequentes reducuntur

$$S = \alpha\frac{1}{q},$$

$$\left(\frac{dS}{d\omega}\right) = \alpha\alpha p\frac{1}{qq},$$

$$\left(\frac{ddS}{d\omega^2}\right) = \alpha^3\left(\frac{1\cdot 2}{q^3} - \frac{1}{q}\right),$$

$$\left(\frac{d^3S}{d\omega^3}\right) = \alpha^4 p\left(\frac{1\cdot 2\cdot 3}{q^4} - \frac{1}{qq}\right),$$

$$\left(\frac{d^4S}{d\omega^4}\right) = \alpha^5\left(\frac{1\cdot 2\cdot 3\cdot 4}{q^5} - \frac{20}{q^3} + \frac{1}{q}\right),$$

$$\left(\frac{d^5S}{d\omega^5}\right) = \alpha^6 p\left(\frac{1\cdot 2\cdot 3\cdot 4\cdot 5}{q^6} - \frac{60}{q^4} + \frac{1}{qq}\right),$$

$$\left(\frac{d^6S}{d\omega^6}\right) = \alpha^7\left(\frac{1\cdots 6}{q^7} - \frac{840}{q^5} + \frac{182}{q^3} - \frac{1}{q}\right).$$

20. Has posteriores formas reperire licet ope horum duorum lemmatum

$$\text{I. } d.\frac{1}{q^{n+1}} = \alpha d\omega\frac{(n+1)p}{q^{n+2}}, \quad \text{II. } d.\frac{p}{q^{n+1}} = \alpha d\omega\left(\frac{n+1}{q^{n+2}} - \frac{n}{q^n}\right);$$

hinc enim reperiemus

$$S = \alpha \frac{1}{q},$$
$$\left(\frac{dS}{d\omega}\right) = \alpha\alpha \frac{p}{qq},$$
$$\left(\frac{ddS}{d\omega^2}\right) = \alpha^3 \left(\frac{2}{q^3} - \frac{1}{q}\right),$$
$$\left(\frac{d^3S}{d\omega^3}\right) = \alpha^4 \left(\frac{2\cdot 3p}{q^4} - \frac{p}{qq}\right),$$
$$\left(\frac{d^4S}{d\omega^4}\right) = \alpha^5 \left(\frac{2\cdot 3\cdot 4}{q^5} - \frac{20}{q^3} + \frac{1}{q}\right),$$
$$\left(\frac{d^5S}{d\omega^5}\right) = \alpha^6 \left(\frac{2\cdot 3\cdot 4\cdot 5p}{q^6} - \frac{3\cdot 20p}{q^4} + \frac{p}{qq}\right),$$
$$\left(\frac{d^6S}{d\omega^6}\right) = \alpha^7 \left(\frac{2\cdot 3\cdot 4\cdot 5\cdot 6}{q^7} - \frac{840}{q^5} + \frac{182}{q^3} - \frac{1}{q}\right),$$
$$\left(\frac{d^7S}{d\omega^7}\right) = \alpha^8 \left(\frac{2\cdots 7p}{q^8} - \frac{5\cdot 840p}{q^6} + \frac{3\cdot 182p}{q^4} - \frac{p}{qq}\right).$$

21. Ipsae autem series his formulis respondentes [erunt]

$$S = \frac{1}{\lambda-\omega} + \frac{1}{\lambda+\omega} - \frac{1}{3\lambda-\omega} - \frac{1}{3\lambda+\omega} + \frac{1}{5\lambda-\omega} + \frac{1}{5\lambda+\omega} - \text{etc.},$$
$$\left(\frac{dS}{d\omega}\right) = \frac{1}{(\lambda-\omega)^2} - \frac{1}{(\lambda+\omega)^2} - \frac{1}{(3\lambda-\omega)^2} + \frac{1}{(3\lambda+\omega)^2} + \frac{1}{(5\lambda-\omega)^2} - \frac{1}{(5\lambda+\omega)^2} - \text{etc.},$$
$$\left(\frac{ddS}{d\omega^2}\right) = \frac{1\cdot 2}{(\lambda-\omega)^3} + \frac{1\cdot 2}{(\lambda+\omega)^3} - \frac{1\cdot 2}{(3\lambda-\omega)^3} - \frac{1\cdot 2}{(3\lambda+\omega)^3} + \frac{1\cdot 2}{(5\lambda-\omega)^3} + \text{etc.},$$
$$\left(\frac{d^3S}{d\omega^3}\right) = \frac{1\cdot 2\cdot 3}{(\lambda-\omega)^4} - \frac{1\cdot 2\cdot 3}{(\lambda+\omega)^4} - \frac{1\cdot 2\cdot 3}{(3\lambda-\omega)^4} + \frac{1\cdot 2\cdot 3}{(3\lambda+\omega)^4} + \frac{1\cdot 2\cdot 3}{(5\lambda-\omega)^4} - \text{etc.},$$
$$\left(\frac{d^4S}{d\omega^4}\right) = \frac{1\cdot 2\cdot 3\cdot 4}{(\lambda-\omega)^5} + \frac{1\cdot 2\cdot 3\cdot 4}{(\lambda+\omega)^5} - \frac{1\cdot 2\cdot 3\cdot 4}{(3\lambda-\omega)^5} - \frac{1\cdot 2\cdot 3\cdot 4}{(3\lambda+\omega)^5} + \frac{1\cdot 2\cdot 3\cdot 4}{(5\lambda-\omega)^5} + \text{etc.},$$
$$\left(\frac{d^5S}{d\omega^5}\right) = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5}{(\lambda-\omega)^6} - \frac{1\cdot 2\cdot 3\cdot 4\cdot 5}{(\lambda+\omega)^6} - \frac{1\cdot 2\cdot 3\cdot 4\cdot 5}{(3\lambda-\omega)^6} + \frac{1\cdot 2\cdot 3\cdot 4\cdot 5}{(3\lambda+\omega)^6} + \frac{1\cdot 2\cdot 3\cdot 4\cdot 5}{(5\lambda-\omega)^6} - \text{etc.},$$
$$\left(\frac{d^6S}{d\omega^6}\right) = \frac{1\cdots 6}{(\lambda-\omega)^7} + \frac{1\cdots 6}{(\lambda+\omega)^7} - \frac{1\cdots 6}{(3\lambda-\omega)^7} - \frac{1\cdots 6}{(3\lambda+\omega)^7} + \frac{1\cdots 6}{(5\lambda-\omega)^7} - \text{etc.},$$
$$\left(\frac{d^7S}{d\omega^7}\right) = \frac{1\cdots 7}{(\lambda-\omega)^8} - \frac{1\cdots 7}{(\lambda+\omega)^8} - \frac{1\cdots 7}{(3\lambda-\omega)^8} + \frac{1\cdots 7}{(3\lambda+\omega)^8} + \frac{1\cdots 7}{(5\lambda-\omega)^8} - \text{etc.}$$

etc.

Circa hos autem valores probe meminisse oportet esse

$$\alpha = \frac{\pi}{2\lambda}, \quad p = \sin.\alpha\omega = \sin.\frac{\pi\omega}{2\lambda} \quad \text{et} \quad q = \cos.\alpha\omega = \cos.\frac{\pi\omega}{2\lambda}.$$

22. Eodem modo expediamus valores seu formulas integrales alterius generis, pro quibus est

$$T = \frac{\pi}{2\lambda} \operatorname{tang.} \frac{\pi\omega}{2\lambda},$$

unde continuo differentiando oriuntur sequentes integrationes:

$$\text{I. } \int \frac{z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z} = T,$$
$$\text{II. } \int \frac{-z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z} lz = \left(\frac{dT}{d\omega}\right),$$
$$\text{III. } \int \frac{z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z} (lz)^2 = \left(\frac{ddT}{d\omega^2}\right),$$
$$\text{IV. } \int \frac{-z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z} (lz)^3 = \left(\frac{d^3T}{d\omega^3}\right),$$
$$\text{V. } \int \frac{z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z} (lz)^4 = \left(\frac{d^4T}{d\omega^4}\right),$$
$$\text{VI. } \int \frac{-z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z} (lz)^5 = \left(\frac{d^5T}{d\omega^5}\right),$$
$$\text{VII. } \int \frac{z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z} (lz)^6 = \left(\frac{d^6T}{d\omega^6}\right).$$

23. Ponatur iterum $\frac{\pi}{2\lambda} = \alpha$, $\sin.\alpha\omega = p$ et $\cos.\alpha\omega = q$, ut sit

$$T = \frac{\alpha p}{q},$$

quae formula secundum lemmata § 20 continuo differentiata dabit

$$T = \alpha \frac{p}{q},$$

$$\left(\frac{dT}{d\omega}\right) = \alpha\alpha \frac{1}{qq},$$

$$\left(\frac{ddT}{d\omega^2}\right) = \alpha^3 \frac{2p}{q^3},$$

$$\left(\frac{d^3T}{d\omega^3}\right) = \alpha^4 \left(\frac{6}{q^4} - \frac{4}{qq}\right),$$

$$\left(\frac{d^4T}{d\omega^4}\right) = \alpha^5 \left(\frac{24p}{q^5} - \frac{8p}{q^3}\right),$$

$$\left(\frac{d^5T}{d\omega^5}\right) = \alpha^6 \left(\frac{120}{q^6} - \frac{120}{q^4} + \frac{16}{qq}\right),$$

$$\left(\frac{d^6T}{d\omega^6}\right) = \alpha^7 \left(\frac{720p}{q^7} - \frac{480p}{q^5} + \frac{32p}{q^3}\right),$$

$$\left(\frac{d^7T}{d\omega^7}\right) = \alpha^8 \left(\frac{5040}{q^8} - \frac{6720}{q^6} + \frac{2016}{q^4} - \frac{64}{qq}\right).$$

24. Series autem infinitae, quae hinc nascuntur, erunt

$$T = \frac{1}{\lambda - \omega} - \frac{1}{\lambda + \omega} + \frac{1}{3\lambda - \omega} - \frac{1}{3\lambda + \omega} + \frac{1}{5\lambda - \omega} - \frac{1}{5\lambda + \omega} + \text{etc.},$$

$$\left(\frac{dT}{d\omega}\right) = \frac{1}{(\lambda - \omega)^2} + \frac{1}{(\lambda + \omega)^2} + \frac{1}{(3\lambda - \omega)^2} + \frac{1}{(3\lambda + \omega)^2} + \frac{1}{(5\lambda - \omega)^2} + \text{etc.},$$

$$\left(\frac{ddT}{d\omega^2}\right) = \frac{1 \cdot 2}{(\lambda - \omega)^3} - \frac{1 \cdot 2}{(\lambda + \omega)^3} + \frac{1 \cdot 2}{(3\lambda - \omega)^3} - \frac{1 \cdot 2}{(3\lambda + \omega)^3} + \frac{1 \cdot 2}{(5\lambda - \omega)^3} - \text{etc.},$$

$$\left(\frac{d^3T}{d\omega^3}\right) = \frac{1 \cdot 2 \cdot 3}{(\lambda - \omega)^4} + \frac{1 \cdot 2 \cdot 3}{(\lambda + \omega)^4} + \frac{1 \cdot 2 \cdot 3}{(3\lambda - \omega)^4} + \frac{1 \cdot 2 \cdot 3}{(3\lambda + \omega)^4} + \frac{1 \cdot 2 \cdot 3}{(5\lambda - \omega)^4} + \text{etc.},$$

$$\left(\frac{d^4T}{d\omega^4}\right) = \frac{1 \cdot 2 \cdot 3 \cdot 4}{(\lambda - \omega)^5} - \frac{1 \cdot 2 \cdot 3 \cdot 4}{(\lambda + \omega)^5} + \frac{1 \cdot 2 \cdot 3 \cdot 4}{(3\lambda - \omega)^5} - \frac{1 \cdot 2 \cdot 3 \cdot 4}{(3\lambda + \omega)^5} + \frac{1 \cdot 2 \cdot 3 \cdot 4}{(5\lambda - \omega)^5} - \text{etc.},$$

$$\left(\frac{d^5T}{d\omega^5}\right) = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{(\lambda - \omega)^6} + \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{(\lambda + \omega)^6} + \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{(3\lambda - \omega)^6} + \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{(3\lambda + \omega)^6} + \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{(5\lambda - \omega)^6} + \text{etc.},$$

$$\left(\frac{d^6T}{d\omega^6}\right) = \frac{1 \cdots 6}{(\lambda - \omega)^7} - \frac{1 \cdots 6}{(\lambda + \omega)^7} + \frac{1 \cdots 6}{(3\lambda - \omega)^7} - \frac{1 \cdots 6}{(3\lambda + \omega)^7} + \frac{1 \cdots 6}{(5\lambda - \omega)^7} - \frac{1 \cdots 6}{(5\lambda + \omega)^7} + \text{etc.}$$

25. Operae pretium erit hinc casus simplicissimos evolvere, qui oriuntur ponendo $\lambda = 1$ et $\omega = 0$, ita ut sit $\alpha = \frac{\pi}{2}$, $p = 0$ et $q = 1$, unde habebimus:

Pro ordine priore	Pro ordine posteriore
$S = \frac{\pi}{2}$	$T = 0$
$\left(\frac{dS}{d\omega}\right) = 0$	$\left(\frac{dT}{d\omega}\right) = \frac{\pi\pi}{4}$
$\left(\frac{ddS}{d\omega^2}\right) = \frac{\pi^3}{8}$	$\left(\frac{ddT}{d\omega^2}\right) = 0$
$\left(\frac{d^3S}{d\omega^3}\right) = 0$	$\left(\frac{d^3T}{d\omega^3}\right) = \frac{\pi^4}{8}$
$\left(\frac{d^4S}{d\omega^4}\right) = \frac{5\pi^5}{32}$	$\left(\frac{d^4T}{d\omega^4}\right) = 0$
$\left(\frac{d^5S}{d\omega^5}\right) = 0$	$\left(\frac{d^5T}{d\omega^5}\right) = \frac{\pi^6}{4}$
$\left(\frac{d^6S}{d\omega^6}\right) = \frac{61\pi^7}{128}$	$\left(\frac{d^6T}{d\omega^6}\right) = 0$
$\left(\frac{d^7S}{d\omega^7}\right) = 0$	$\left(\frac{d^7T}{d\omega^7}\right) = \frac{17\pi^8}{16}$
etc.	etc.

26. Hinc ergo omissis valoribus evanescentibus ex priore ordine habebimus sequentes formulas integrales cum seriebus inde natis

$$\int \frac{dz}{1+zz} = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \text{etc.},$$

$$\int \frac{dz(lz)^2}{1+zz} = \frac{\pi^3}{16} = \frac{2}{1^3} - \frac{2}{3^3} + \frac{2}{5^3} - \frac{2}{7^3} + \frac{2}{9^3} - \frac{2}{11^3} + \frac{2}{13^3} - \text{etc.},$$

$$\int \frac{dz(lz)^4}{1+zz} = \frac{5\pi^5}{64} = \frac{24}{1^5} - \frac{24}{3^5} + \frac{24}{5^5} - \frac{24}{7^5} + \frac{24}{9^5} - \frac{24}{11^5} + \frac{24}{13^5} - \text{etc.},$$

$$\int \frac{dz(lz)^6}{1+zz} = \frac{61\pi^7}{256} = \frac{720}{1^7} - \frac{720}{3^7} + \frac{720}{5^7} - \frac{720}{7^7} + \frac{720}{9^7} - \frac{720}{11^7} + \frac{720}{13^7} - \text{etc.}$$

etc.

27. Ex altero autem ordine pro eodem casu oriuntur

$$\int \frac{-dzlz}{1-zz} = \frac{\pi\pi}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{13^2} + \text{etc.},$$

$$\int \frac{-dz(lz)^3}{1-zz} = \frac{\pi^4}{16} = \frac{6}{1^4} + \frac{6}{3^4} + \frac{6}{5^4} + \frac{6}{7^4} + \frac{6}{9^4} + \frac{6}{11^4} + \frac{6}{13^4} + \text{etc.},$$

$$\int \frac{-dz(lz)^5}{1-zz} = \frac{\pi^6}{8} = \frac{120}{1^6} + \frac{120}{3^6} + \frac{120}{5^6} + \frac{120}{7^6} + \frac{120}{9^6} + \frac{120}{11^6} + \frac{120}{13^6} + \text{etc.}$$

etc.

28. Quemadmodum ex primo integrali ordinis posterioris deduximus has formulas

$$\int \frac{dzlz}{1-z} = -\frac{\pi\pi}{6} \quad \text{et} \quad \int \frac{dzlz}{1+z} = -\frac{\pi\pi}{12},$$

similes quoque formulae integrales ex sequentibus deduci possunt; cum enim sit $\int \frac{dz(lz)^3}{1-zz} = -\frac{\pi^4}{16}$, ponamus esse

$$\int \frac{zdz(lz)^3}{1-zz} = P$$

eritque

$$\int \frac{dz(lz)^3}{1-z} = P - \frac{\pi^4}{16} \quad \text{et} \quad \int \frac{dz(lz)^3}{1+z} = -P - \frac{\pi^4}{16};$$

nunc vero statuatur $zz = v$, ut sit $zdz = \frac{1}{2}dv$ et $lz = \frac{1}{2}lv$ ideoque $(lz)^3 = \frac{1}{8}(lv)^3$, quibus substitutis erit

$$P = \frac{1}{16} \int \frac{dv(lv)^3}{1-v} = \frac{1}{16}\left(P - \frac{\pi^4}{16}\right),$$

unde fit

$$16P = P - \frac{\pi^4}{16} \quad \text{ideoque} \quad P = -\frac{\pi^4}{240},$$

sicque has duas habebimus integrationes novas

$$\int \frac{dz(lz)^3}{1-z} = -\frac{\pi^4}{15} \quad \text{et} \quad \int \frac{dz(lz)^3}{1+z} = -\frac{7\pi^4}{120};$$

hinc autem per series erit

$$\int \frac{-dz(lz)^3}{1-z} = +\frac{\pi^4}{15} = 6\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \frac{1}{7^4} + \frac{1}{8^4} + \text{etc.}\right)$$

et

$$\int \frac{-dz(lz)^3}{1+z} = +\frac{7\pi^4}{120} = 6\left(1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4} - \frac{1}{6^4} + \frac{1}{7^4} - \frac{1}{8^4} + \text{etc.}\right).$$

29. Porro $\int \frac{dz(lz)^5}{1-zz} = -\frac{\pi^6}{8}$; ponamus esse

$$\int \frac{zdz(lz)^5}{1-zz} = P,$$

ut hinc obtineamus

$$\int \frac{dz(lz)^5}{1-z} = P - \frac{\pi^6}{8} \quad \text{et} \quad \int \frac{dz(lz)^5}{1+z} = -P - \frac{\pi^6}{8};$$

nunc igitur statuamus $zz = v$ eritque

$$P = \frac{1}{64} \int \frac{dv(lv)^5}{1-v} = \frac{1}{64}\left(P - \frac{\pi^6}{8}\right),$$

unde fit

$$P = -\frac{\pi^6}{504},$$

novaeque integrationes hinc deductae sunt

$$\int \frac{dz(lz)^5}{1-z} = -\frac{8\pi^6}{63} \quad \text{et} \quad \int \frac{dz(lz)^5}{1+z} = -\frac{31\pi^6}{252},$$

at vero per series reperitur

$$\int \frac{dz(lz)^5}{1-z} = -\frac{8\pi^6}{63} = -120\left(1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \frac{1}{6^6} + \frac{1}{7^6} + \text{etc.}\right)$$

et

$$\int \frac{dz(lz)^5}{1+z} = -\frac{31\pi^6}{252} = -120\left(1 - \frac{1}{2^6} + \frac{1}{3^6} - \frac{1}{4^6} + \frac{1}{5^6} - \frac{1}{6^6} + \frac{1}{7^6} - \text{etc.}\right),$$

ita ut sit

$$1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \frac{1}{6^6} + \frac{1}{7^6} + \text{etc.} = \frac{\pi^6}{945}$$

et

$$1 - \frac{1}{2^6} + \frac{1}{3^6} - \frac{1}{4^6} + \frac{1}{5^6} - \frac{1}{6^6} + \frac{1}{7^6} - \frac{1}{8^6} + \text{etc.} = \frac{31\pi^6}{30240} = \frac{31\pi^6}{32 \cdot 945}.$$

30. Consideremus etiam casus, quibus $\lambda = 2$ et $\omega = 1$, ita ut sit $\alpha = \frac{\pi}{4}$ et $\alpha\omega = \frac{\pi}{4}$, hinc $p = q = \frac{1}{\sqrt{2}}$, unde pro utroque ordine sequentes habebimus valores:

Pro ordine priore	Pro ordine posteriore
$S = \frac{\pi}{2\sqrt{2}}$	$T = \frac{\pi}{4}$
$\left(\frac{dS}{d\omega}\right) = \frac{\pi\pi}{8\sqrt{2}}$	$\left(\frac{dT}{d\omega}\right) = \frac{\pi\pi}{8}$
$\left(\frac{ddS}{d\omega^2}\right) = \frac{3\pi^3}{32\sqrt{2}}$	$\left(\frac{ddT}{d\omega^2}\right) = \frac{\pi^3}{16}$
$\left(\frac{d^3S}{d\omega^3}\right) = \frac{11\pi^4}{128\sqrt{2}}$	$\left(\frac{d^3T}{d\omega^3}\right) = \frac{\pi^4}{16}$
$\left(\frac{d^4S}{d\omega^4}\right) = \frac{57\pi^5}{512\sqrt{2}}$	$\left(\frac{d^4T}{d\omega^4}\right) = \frac{5\pi^5}{64}$
$\left(\frac{d^5S}{d\omega^5}\right) = \frac{361\pi^6}{2048\sqrt{2}}$	$\left(\frac{d^5T}{d\omega^5}\right) = \frac{\pi^6}{8}$
$\left(\frac{d^6S}{d\omega^6}\right) = \frac{2763\pi^7}{8192\sqrt{2}}$	$\left(\frac{d^6T}{d\omega^6}\right) = \frac{61\pi^7}{256}$
$\left(\frac{d^7S}{d\omega^7}\right) = \frac{24611\pi^8}{32768\sqrt{2}}$	$\left(\frac{d^7T}{d\omega^7}\right) = \frac{17\pi^8}{32}$
etc.	etc.

31. Hinc igitur sequentes integrationes cum seriebus respondentibus resultant; ac primo quidem ex ordine primo

$$\int \frac{(1+zz)dz}{1+z^4} = \frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \text{etc.},$$

$$\int \frac{-(1-zz)dz\,lz}{1+z^4} = \frac{\pi\pi}{2\sqrt{2}} = 1 - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} - \frac{1}{13^2} + \text{etc.},$$

$$\int \frac{(1+zz)dz(lz)^2}{1+z^4} = \frac{3\pi^3}{32\sqrt{2}} = \frac{2}{1^3} + \frac{2}{3^3} - \frac{2}{5^3} - \frac{2}{7^3} + \frac{2}{9^3} + \frac{2}{11^3} - \frac{2}{13^3} - \text{etc.},$$

$$\int \frac{-(1-zz)dz(lz)^3}{1+z^4} = \frac{11\pi^4}{128\sqrt{2}} = \frac{6}{1^4} - \frac{6}{3^4} - \frac{6}{5^4} + \frac{6}{7^4} + \frac{6}{9^4} - \frac{6}{11^4} - \frac{6}{13^4} + \text{etc.},$$

$$\int \frac{(1+zz)dz(lz)^4}{1+z^4} = \frac{57\pi^5}{512\sqrt{2}} = \frac{24}{1^5} + \frac{24}{3^5} - \frac{24}{5^5} - \frac{24}{7^5} + \frac{24}{9^5} + \frac{24}{11^5} - \frac{24}{13^5} - \text{etc.},$$

$$\int \frac{-(1-zz)dz(lz)^5}{1+z^4} = \frac{361\pi^6}{2048\sqrt{2}} = \frac{120}{1^6} - \frac{120}{3^6} - \frac{120}{5^6} + \frac{120}{7^6} + \frac{120}{9^6} - \frac{120}{11^6} - \frac{120}{13^6} + \text{etc.},$$

$$\int \frac{(1+zz)dz(lz)^6}{1+z^4} = \frac{2763\pi^7}{8192\sqrt{2}} = \frac{720}{1^7} + \frac{720}{3^7} - \frac{720}{5^7} - \frac{720}{7^7} + \frac{720}{9^7} + \frac{720}{11^7} - \frac{720}{13^7} - \text{etc.},$$

$$\int \frac{-(1-zz)dz(lz)^7}{1+z^4} = \frac{24611\pi^8}{32768\sqrt{2}} = \frac{5040}{1^8} - \frac{5040}{3^8} - \frac{5040}{5^8} + \frac{5040}{7^8} + \frac{5040}{9^8} - \frac{5040}{11^8} - \frac{5040}{13^8} + \text{etc.}$$

etc.

32. Eodem modo integrationes alterius ordinis cum seriebus erunt

$$\int \frac{dz}{1+zz} = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \text{etc.},$$

$$\int \frac{-dz\,lz}{1-zz} = \frac{\pi\pi}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{13^2} + \text{etc.},$$

$$\int \frac{dz(lz)^2}{1+zz} = \frac{\pi^3}{16} = \frac{2}{1^3} - \frac{2}{3^3} + \frac{2}{5^3} - \frac{2}{7^3} + \frac{2}{9^3} - \frac{2}{11^3} + \frac{2}{13^3} - \text{etc.},$$

$$\int \frac{-dz(lz)^3}{1-zz} = \frac{\pi^4}{16} = \frac{6}{1^4} + \frac{6}{3^4} + \frac{6}{5^4} + \frac{6}{7^4} + \frac{6}{9^4} + \frac{6}{11^4} + \frac{6}{13^4} + \text{etc.},$$

$$\int \frac{dz(lz)^4}{1+zz} = \frac{5\pi^5}{64} = \frac{24}{1^5} - \frac{24}{3^5} + \frac{24}{5^5} - \frac{24}{7^5} + \frac{24}{9^5} - \frac{24}{11^5} + \frac{24}{13^5} - \text{etc.},$$

$$\int \frac{-dz(lz)^5}{1-zz} = \frac{\pi^6}{8} = \frac{120}{1^6} + \frac{120}{3^6} + \frac{120}{5^6} + \frac{120}{7^6} + \frac{120}{9^6} + \frac{120}{11^6} + \frac{120}{13^6} + \text{etc.},$$

$$\int \frac{dz(lz)^6}{1+zz} = \frac{61\pi^7}{256} = \frac{720}{1^7} - \frac{720}{3^7} + \frac{720}{5^7} - \frac{720}{7^7} + \frac{720}{9^7} - \frac{720}{11^7} + \frac{720}{13^7} - \text{etc.},$$

$$\int \frac{-dz(lz)^7}{1-zz} = \frac{17\pi^8}{32} = \frac{5040}{1^8} + \frac{5040}{3^8} + \frac{5040}{5^8} + \frac{5040}{7^8} + \frac{5040}{9^8} + \frac{5040}{11^8} + \frac{5040}{13^8} + \text{etc.}$$

etc.

Hae autem series sunt eae ipsae, quas iam supra (§ 26 et 27) sumus consecuti.

33. Praeterea autem ii casus imprimis notari merentur, quibus formulae integrales in formas simpliciores resolvi possunt. Haec autem resolutio tantum spectat ad fractionem

$$\pm \frac{z^{\lambda-\omega} \pm z^{\lambda+\omega}}{1 \pm z^{2\lambda}}$$

omisso factore $\frac{dz}{z}(lz)^{\mu}$, ad quod ostendendum sumamus primo $\lambda = 3$ et $\omega = 1$, unde fit $\alpha = \frac{\pi}{6}$, $p = \sin.\frac{\pi}{6}$ et $q = \cos.\frac{\pi}{6}$; tum autem in priori ordine occurrunt alternatim sequentes fractiones

$$\text{I.} \quad \frac{zz(1+zz)}{1+z^6} = \frac{zz}{1-zz+z^4},$$

quae posito $zz = v$ abit in $\frac{v}{1-v+vv}$; ergo cum sit $\frac{dz}{z} = \frac{1}{2}\frac{dv}{v}$ et $lz = \frac{1}{2} lv$, hinc talis forma

$$\frac{1}{2^{2i+1}} \int \frac{dv(lv)^{2i}}{1-v+vv}$$

integrari poterit, casu scilicet $v = 1$;

$$\text{II.} \quad -\frac{zz(1-zz)}{1+z^6} = +\frac{2}{3(1+zz)} - \frac{2-zz}{3(1-zz+z^4)},$$

quae posito $zz = v$ abit in

$$\frac{2}{3(1+v)} - \frac{2-v}{3(1-v+vv)},$$

quae ergo forma ducta in $\frac{dz}{z}(lz)^{2i+1}$ vel in $\frac{1}{2^{2i+2}} \cdot \frac{dv}{v}(lv)^{2i+1}$ semper integrari potest posito $v = 1$.

34. Eodem casu ordo posterior sequentes suppeditat resolutiones

$$\text{I.} \quad \frac{zz(1-zz)}{1-z^6} = \frac{zz}{1+zz+z^4} = \frac{v}{1+v+vv},$$

quae in $\frac{dz}{z}(lz)^{2i}$ vel in $\frac{1}{2^{2i+1}}\frac{dv}{v}(lv)^{2i}$ ducta semper est integrabilis;

$$\text{II.} \quad \frac{-zz(1+zz)}{1-z^6} = \frac{-2}{3(1-zz)} + \frac{2+zz}{3(1+zz+z^4)},$$

quae facto $zz = v$ fit

$$\frac{-2}{3(1-v)} + \frac{2+v}{3(1+v+vv)},$$

quae ergo formulae in $\frac{dv}{v}(lv)^{2i+1}$ ductae fiunt integrabiles; quia autem in hac resolutione numeratores per z vel v dividere non licet, alia resolutione est opus, quae reperitur

$$\frac{-zz(1+zz)}{1-z^6} = \frac{-2zz}{3(1-zz)} - \frac{zz(1+2zz)}{3(1+zz+z^4)}$$

sive

$$\frac{-2v}{3(1-v)} - \frac{v(1+2v)}{3(1+v+vv)},$$

quae formulae ductae in $\frac{dz}{z}(lz)^{2i+1}$ vel in $\frac{1}{2^{2i+2}} \cdot \frac{dv}{v}(lv)^{2i+1}$ integrationem quoque admittunt.

35. Porro manente $\lambda = 3$ sumatur $\omega = 2$, ut sit

$$\alpha = \frac{\pi}{6}, \quad p = \sin.\frac{\pi}{3} \quad \text{et} \quad q = \cos.\frac{\pi}{3},$$

et ex ordine priore orientur sequentes reductiones

$$\text{I.} \quad \frac{z(1+z^4)}{1+z^6} = \frac{2z}{3(1+zz)} + \frac{z(1+zz)}{3(1-zz+z^4)},$$

unde multiplicando per $\frac{dz}{z}(lz)^{2i}$ oriuntur formulae integrationem admittentes casu $z = 1$;

$$\text{II.} \quad \frac{-z(1-z^4)}{1+z^6} = -\frac{z(1-zz)}{1-zz+z^4},$$

quae per $\frac{dz}{z}(lz)^{2i+1}$ multiplicata integrari poterit casu $z = 1$.

Ex ordine vero posteriore sequentes prodibunt reductiones

$$\text{I.} \quad \frac{z(1-z^4)}{1-z^6} = \frac{z(1+zz)}{1+zz+z^4},$$

quae ducta in $\frac{dz}{z}(lz)^{2i}$ fit integrabilis;

$$\text{II.} \quad \frac{-z(1+z^4)}{1-z^6} = \frac{-2z}{3(1-zz)} - \frac{z(1-zz)}{3(1+zz+z^4)},$$

quae formulae in $\frac{dz}{z}(lz)^{2i+1}$ ductae fiunt integrabiles.

36. Operae iam erit pretium haec integralia actu evolvere, quare ex § 33 eiusque numero I nanciscimur sequentes integrationes

$$1^0. \quad \frac{1}{2}\int \frac{dv}{1-v+vv} = \alpha\frac{1}{q} = \frac{\pi}{3\sqrt{3}},$$

$$2^0. \quad \frac{1}{8}\int \frac{dv(lv)^2}{1-v+vv} = \alpha^3\left(\frac{2}{q^3} - \frac{1}{q}\right) = \frac{5\pi^3}{324\sqrt{3}},$$

deinde vero ex eiusdem paragraphi numero II, ubi etiam haec reductio locum habet

$$-\frac{zz(1-zz)}{1+z^6} = -\frac{2zz}{3(1+zz)} - \frac{zz(1-2zz)}{3(1-zz+z^4)} = -\frac{2v}{3(1+v)} - \frac{v(1-2v)}{3(1-v+vv)},$$

quae ducta in $\frac{1}{4} \cdot \frac{dv}{v} lv$ dabit

$$-\frac{1}{6}\int \frac{dv\,lv}{1+v} - \frac{1}{12}\int \frac{dv(1-2v)lv}{1-v+vv} = \alpha\alpha \frac{p}{qq} = \frac{\pi\pi}{54},$$

quarum formularum prior integrationem admittit; est enim

$$\int \frac{dv\,lv}{1+v} = -\frac{\pi\pi}{12},$$

unde invenitur posterior

$$\int \frac{dv(1-2v)lv}{1-v+vv} = -\frac{\pi\pi}{18}.$$

37. Ex § 34 eiusque numero I sequitur

$$1^{\circ}.\quad \frac{1}{2}\int \frac{dv}{1+v+vv} = \frac{\alpha p}{q} = \frac{\pi}{6\sqrt{3}},$$

$$2^{\circ}.\quad \frac{1}{8}\int \frac{dv(lv)^2}{1+v+vv} = \alpha^3 \frac{2p}{q^3} = \frac{\pi^3}{81\sqrt{3}},$$

deinde vero ex numero II fit

$$-\frac{1}{6}\int \frac{dv\,lv}{1-v} - \frac{1}{12}\int \frac{dv(1+2v)lv}{1+v+vv} = \alpha\alpha \frac{1}{qq} = \frac{\pi\pi}{27};$$

supra autem invenimus esse

$$\int \frac{dv\,lv}{1-v} = -\frac{\pi\pi}{6},$$

quo valore substituto fit

$$\int \frac{dv(1+2v)lv}{1+v+vv} = -\frac{\pi\pi}{9};$$

maxime igitur operae pretium est visum has postremas integrationes evolvisse.

38. Quodsi ambae formulae integrales

$$\int \frac{dv(1-2v)lv}{1-v+vv} \quad \text{et} \quad \int \frac{dv(1+2v)lv}{1+v+vv}$$

in series convertantur, reperitur

$$\int \frac{dv(1-2v)lv}{1-v+vv} = -1 + \frac{1}{4} + \frac{2}{9} + \frac{1}{16} - \frac{1}{25} - \frac{2}{36} - \frac{1}{49} + \text{etc.}$$

et

$$\int \frac{dv(1+2v)lv}{1+v+vv} = -1 - \frac{1}{4} + \frac{2}{9} - \frac{1}{16} - \frac{1}{25} + \frac{2}{36} - \frac{1}{49} + \text{etc.},$$

unde has duas summationes attentione nostra non indignas assequimur

$$\text{I.}\quad 1 - \frac{1}{4} - \frac{2}{9} - \frac{1}{16} + \frac{1}{25} + \frac{2}{36} + \frac{1}{49} - \frac{1}{64} - \frac{2}{81} - \frac{1}{100} + \text{etc.} = \frac{\pi\pi}{18},$$

$$\text{II.}\quad 1 + \frac{1}{4} - \frac{2}{9} + \frac{1}{16} + \frac{1}{25} - \frac{2}{36} + \frac{1}{49} + \frac{1}{64} - \frac{2}{81} + \frac{1}{100} + \text{etc.} = \frac{\pi\pi}{9},$$

quarum prior a posteriore ablata praebet

$$\frac{2}{4} + \frac{2}{16} - \frac{4}{36} + \frac{2}{64} + \frac{2}{100} - \text{etc.} = \frac{\pi\pi}{18},$$

cuius duplum perducit ad hanc

$$1 + \frac{1}{4} - \frac{2}{9} + \frac{1}{16} + \frac{1}{25} - \frac{2}{36} + \text{etc.} = \frac{\pi\pi}{9};$$

quae quoniam cum secunda congruit, veritas utriusque summationis satis confirmatur; quodsi vero secunda a duplo primae subtrahatur, remanebit ista series memorabilis

$$1 - \frac{3}{4} - \frac{2}{9} - \frac{3}{16} + \frac{1}{25} + \frac{6}{36} + \frac{1}{49} - \frac{3}{64} - \frac{2}{81} - \frac{3}{100} + \text{etc.} = 0,$$

quae in periodos 6 terminos complectentes distributa manifestum ordinem in numeratoribus declarat, quippe qui sunt $1, -3, -2, -3, +1, +6$.

ADDITAMENTUM

39. Quemadmodum superiores integrationes per continuam differentiationem formularum S et T deduximus, ita etiam per integrationem alias et prorsus singulares integrationes impetrabimus; si enim ut supra [§ 3] fuerit

$$S = \int \frac{P\,dz}{z}\,^{1)}$$

1) Editio princeps: $S = \int \frac{T\,dz}{z}$. A. G.

existente P formula illa

$$\frac{z^{\lambda-\omega} \pm z^{\lambda+\omega}}{1 \pm z^{2\lambda}},$$

quae praeter z etiam exponentem variabilem ω involvere concipitur, erit per naturam integralium duas variabiles involventium

$$\int S d\omega = \int \frac{dz}{z} \int P d\omega,$$

ubi in priore formula integrali $\int S d\omega$, ubi z pro constanti habetur, statim scribi potest $z = 1$; hoc igitur lemmate praemisso, quia est

$$\int P d\omega = \frac{-z^{\lambda-\omega} \pm z^{\lambda+\omega}}{(1 \pm z^{2\lambda}) lz},$$

ambas formulas supra tractatas, nempe S et T, hoc modo evolvamus, et quia utramque triplici modo expressam dedimus, primo scilicet per seriem infinitam, secundo per formulam finitam ac tertio per formulam integralem, etiam quantitates, quae pro integralibus $\int S d\omega$ et $\int T d\omega$ resultabunt, erunt inter se aequales.

40. Incipiamus a formula S, et cum per seriem fuerit

$$S = \frac{1}{\lambda - \omega} + \frac{1}{\lambda + \omega} - \frac{1}{3\lambda - \omega} - \frac{1}{3\lambda + \omega} + \frac{1}{5\lambda - \omega} + \frac{1}{5\lambda + \omega} - \text{etc.},$$

erit

$$\int S d\omega = -l(\lambda - \omega) + l(\lambda + \omega) + l(3\lambda - \omega) - l(3\lambda + \omega) - \text{etc.} + C,$$

quam constantem ita definire decet, ut integrale evanescat posito $\omega = 0$, quo facto erit

$$\int S d\omega = l\frac{\lambda + \omega}{\lambda - \omega} + l\frac{3\lambda - \omega}{3\lambda + \omega} + l\frac{5\lambda + \omega}{5\lambda - \omega} + l\frac{7\lambda - \omega}{7\lambda + \omega} + \text{etc.},$$

quae expressio reducitur ad sequentem

$$\int S d\omega = l\frac{(\lambda + \omega)(3\lambda - \omega)(5\lambda + \omega)(7\lambda - \omega)(9\lambda + \omega) \cdot \text{etc.}}{(\lambda - \omega)(3\lambda + \omega)(5\lambda - \omega)(7\lambda + \omega)(9\lambda - \omega) \cdot \text{etc.}}.$$

Deinde quia per formulam finitam erat $S = \frac{\pi}{2\lambda \cos. \frac{\pi\omega}{2\lambda}}$, erit

$$\int S d\omega = \int \frac{\pi d\omega}{2\lambda \cos. \frac{\pi\omega}{2\lambda}},$$

ubi si brevitatis gratia ponatur $\frac{\pi\omega}{2\lambda} = \varphi$, ut sit $d\omega = \frac{2\lambda d\varphi}{\pi}$, erit

$$\int S d\omega = \int \frac{d\varphi}{\cos. \varphi};$$

quia igitur novimus esse

$$\int \frac{d\vartheta}{\sin. \vartheta} = l \operatorname{tang.} \frac{1}{2}\vartheta,$$

sumamus $\sin. \vartheta = \cos. \varphi$ sive $\vartheta = 90^\circ - \varphi = \frac{\pi}{2} - \varphi$ eritque $d\vartheta = -d\varphi$, unde fit

$$\int \frac{-d\varphi}{\cos. \varphi} = l \operatorname{tang.} \left(\frac{\pi}{4} - \frac{1}{2}\varphi\right);$$

quoniam autem est $\varphi = \frac{\pi\omega}{2\lambda}$, erit

$$\frac{\pi}{4} - \frac{1}{2}\varphi = \frac{\pi(\lambda - \omega)}{4\lambda},$$

unde nostrum integrale erit

$$\int S d\omega = -l \operatorname{tang.} \frac{\pi(\lambda - \omega)}{4\lambda} = +l \operatorname{tang.} \frac{\pi(\lambda + \omega)}{4\lambda};$$

ex tertia autem formula integrali

$$S = \int \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1 + z^{2\lambda}} \cdot \frac{dz}{z}$$

colligitur fore

$$\int S d\omega = \int \frac{-z^{\lambda-\omega} + z^{\lambda+\omega}}{1 + z^{2\lambda}} \cdot \frac{dz}{z lz},$$

quod integrale a termino $z = 0$ usque ad terminum $z = 1$ extendi assumitur; sicque tres isti valores inventi inter se erunt aequales. Ac ne ob constantes forte addendas ullum dubium supersit, singulae istae expressiones sponte evanescunt casu $\omega = 0$.

41. Consideremus hinc primo aequalitatem inter formulam primam et secundam, et quia utraque est logarithmus, erit

$$\operatorname{tang.} \frac{\pi(\lambda+\omega)}{4\lambda} = \frac{(\lambda+\omega)(3\lambda-\omega)(5\lambda+\omega)(7\lambda-\omega)\cdot \text{etc.}}{(\lambda-\omega)(3\lambda+\omega)(5\lambda-\omega)(7\lambda+\omega)\cdot \text{etc.}};$$

cum igitur huius fractionis numerator evanescat casibus vel $\omega = -\lambda$ vel $\omega = +3\lambda$ vel $\omega = -5\lambda$ vel $\omega = +7\lambda$ etc., evidens est iisdem casibus quoque tangentem fieri $= 0$; denominator vero evanescit casibus vel $\omega = \lambda$ vel $\omega = -3\lambda$ vel $\omega = +5\lambda$ vel $\omega = -7\lambda$ etc., quibus ergo casibus tangens in infinitum excrescere debet, id quod etiam pulcherrime evenit. Ceterum haec expressio congruit cum ea, quam iam dudum inveni et in *Introductione* exposui.[1])

42. Productum autem istud infinitum per principia alibi[2]) stabilita ad formulas integrales reduci potest ope huius lemmatis latissime patentis

$$\frac{a(c+b)(a+k)(c+b+k)(a+2k)(c+b+2k)\cdot \text{etc.}}{b(c+a)(b+k)(c+a+k)(b+2k)(c+a+2k)\cdot \text{etc.}} = \frac{\int z^{c-1}dz(1-z^k)^{\frac{b-k}{k}}}{\int z^{c-1}dz(1-z^k)^{\frac{a-k}{k}}},$$

siquidem post utramque integrationem fiat $z = 1$. Nostro igitur casu erit

$$a = \lambda+\omega, \quad b = \lambda-\omega, \quad c = 2\lambda \quad \text{et} \quad k = 4\lambda,$$

unde valor nostri producti erit

$$\frac{\int z^{2\lambda-1}dz(1-z^{4\lambda})^{\frac{-3\lambda-\omega}{4\lambda}}}{\int z^{2\lambda-1}dz(1-z^{4\lambda})^{\frac{-3\lambda+\omega}{4\lambda}}} = \operatorname{tang.} \frac{\pi(\lambda+\omega)}{4\lambda};$$

formulae autem istae integrales concinniores evadunt statuendo $z^{2\lambda} = y$; tum enim erit

$$\operatorname{tang.} \frac{\pi(\lambda+\omega)}{4\lambda} = \frac{\int dy(1-yy)^{\frac{-3\lambda-\omega}{4\lambda}}}{\int dy(1-yy)^{\frac{-3\lambda+\omega}{4\lambda}}},$$

1) *Introductio in analysin infinitorum*, t. I cap. XI, § 186; *Leonhardi Euleri Opera omnia*, series I, vol. 8. A. G.

2) Vide Lemma 4 Commentationis 59 et notam 2 p. 21. A. G.

quae expressio utique omni attentione digna videtur. Denique ex formula integrali inventa erit quoque

$$\int \frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z\,lz} = l \operatorname{tang.} \frac{\pi(\lambda+\omega)}{4\lambda}.$$

43. Operae erit pretium etiam aliquot casus particulares evolvere. Sit igitur primo $\lambda = 2$ et $\omega = 1$ ac per expressionem infinitam erit

$$\int S d\omega = l \frac{3\cdot 5}{1\cdot 7} \cdot \frac{11\cdot 13}{9\cdot 15} \cdot \frac{19\cdot 21}{17\cdot 23} \cdot \frac{27\cdot 29}{25\cdot 31} \cdot \frac{35\cdot 37}{33\cdot 39} \cdot \text{etc.},$$

deinde per expressionem finitam habebimus

$$\int S d\omega = l \operatorname{tang.} \frac{3\pi}{8}$$

ac per formulam integralem

$$\int S d\omega = \int \frac{-(1-zz)}{1+z^4} \cdot \frac{dz}{lz},$$

tum vero ex aequalitate duarum priorum expressionum

$$\operatorname{tang.} \frac{3\pi}{8} = \frac{3\cdot 5}{1\cdot 7} \cdot \frac{11\cdot 13}{9\cdot 15} \cdot \frac{19\cdot 21}{17\cdot 23} \cdot \text{etc.}$$

hincque per binas formulas integrales

$$\operatorname{tang.} \frac{3\pi}{8} = \frac{\int dy(1-yy)^{-\frac{7}{8}}}{\int dy(1-yy)^{-\frac{5}{8}}}.$$

44. Ponamus nunc esse $\lambda = 3$ et $\omega = 1$ ac per expressionem infinitam erit

$$\int S d\omega = l \frac{2\cdot 4}{1\cdot 5} \cdot \frac{8\cdot 10}{7\cdot 11} \cdot \frac{14\cdot 16}{13\cdot 17} \cdot \frac{20\cdot 22}{19\cdot 23} \cdot \text{etc.},$$

secundo per expressionem finitam

$$\int S d\omega = l \operatorname{tang.} \frac{\pi}{3} = l\sqrt{3} = \frac{1}{2} l3,$$

ita ut futurum sit

$$\sqrt{3} = \frac{2\cdot 4}{1\cdot 5} \cdot \frac{8\cdot 10}{7\cdot 11} \cdot \frac{14\cdot 16}{13\cdot 17} \cdot \text{etc.},$$

huiusque producti valor per formulas integrales erit

$$\frac{\int dy(1-yy)^{-\frac{5}{6}}}{\int dy(1-yy)^{-\frac{2}{3}}}.$$

Denique formula integralis praebebit

$$\int S d\omega = \int \frac{-z(1-zz)}{1+z^6} \cdot \frac{dz}{lz}.$$

45. Eodem modo etiam evolvamus alteram formulam T, cuius valor per seriem erat

$$T = \frac{1}{\lambda-\omega} - \frac{1}{\lambda+\omega} + \frac{1}{3\lambda-\omega} - \frac{1}{3\lambda+\omega} + \frac{1}{5\lambda-\omega} - \frac{1}{5\lambda+\omega} + \text{etc.};$$

unde fit

$$\int T d\omega = -l(\lambda-\omega) - l(\lambda+\omega) - l(3\lambda-\omega) - l(3\lambda+\omega) - \text{etc.};$$

quae expressio ut evanescat posito $\omega = 0$, erit

$$\int T d\omega = l \frac{\lambda\lambda}{\lambda\lambda-\omega\omega} \cdot \frac{9\lambda\lambda}{9\lambda\lambda-\omega\omega} \cdot \frac{25\lambda\lambda}{25\lambda\lambda-\omega\omega} \cdot \text{etc.};$$

deinde vero cum per formulam finitam fuerit $T = \frac{\pi}{2\lambda} \operatorname{tang.} \frac{\pi\omega}{2\lambda}$, erit

$$\int T d\omega = \int \frac{\pi d\omega}{2\lambda} \operatorname{tang.} \frac{\pi\omega}{2\lambda},$$

ubi posito $\frac{\pi\omega}{2\lambda} = \varphi$ erit

$$\int T d\omega = \int d\varphi \operatorname{tang.} \varphi = -l \cos. \varphi,$$

ita ut sit

$$\int T d\omega = -l \cos. \frac{\pi\omega}{2\lambda},$$

cuius valor casu $\omega = 0$ fit sponte $= 0$; denique per formulam integralem habebimus

$$\int T d\omega = \int \frac{-z^{\lambda-\omega} - z^{\lambda+\omega}}{1-z^{2\lambda}} \cdot \frac{dz}{z\,lz};$$

integrale itidem a termino $z = 0$ usque ad terminum $z = 1$ extendi debet.

46. Iam comparatio duorum priorum valorum hanc praebet aequationem

$$\frac{1}{\cos. \frac{\pi\omega}{2\lambda}} = \frac{\lambda\lambda}{\lambda\lambda-\omega\omega} \cdot \frac{9\lambda\lambda}{9\lambda\lambda-\omega\omega} \cdot \frac{25\lambda\lambda}{25\lambda\lambda-\omega\omega} \cdot \frac{49\lambda\lambda}{49\lambda\lambda-\omega\omega} \cdot \text{etc.},$$

$$\cos. \frac{\pi\omega}{2\lambda} = \left(1 - \frac{\omega\omega}{\lambda\lambda}\right)\left(1 - \frac{\omega\omega}{9\lambda\lambda}\right)\left(1 - \frac{\omega\omega}{25\lambda\lambda}\right)\left(1 - \frac{\omega\omega}{49\lambda\lambda}\right) \cdot \text{etc.},$$

vel si factores singuli iterum in simplices evolvantur, erit

$$\cos. \frac{\pi\omega}{2\lambda} = \frac{\lambda+\omega}{\lambda} \cdot \frac{\lambda-\omega}{\lambda} \cdot \frac{3\lambda+\omega}{3\lambda} \cdot \frac{3\lambda-\omega}{3\lambda} \cdot \frac{5\lambda+\omega}{5\lambda} \cdot \frac{5\lambda-\omega}{5\lambda} \cdot \text{etc.},$$

quae formula cum reductione generali supra [§ 42] allata comparata dat $a = \lambda + \omega$, $b = \lambda$, $c = -\omega$ et $k = 2\lambda$, unde colligimus

$$\cos. \frac{\pi\omega}{2\lambda} = \frac{\int z^{-\omega-1} dz (1-z^{2\lambda})^{-\frac{1}{2}}}{\int z^{-\omega-1} dz (1-z^{2\lambda})^{\frac{\omega-\lambda}{2\lambda}}}.$$

Ut autem exponentes negativos $z^{-\omega-1}$ evitemus, superius productum ita repraesentemus

$$\cos. \frac{\pi\omega}{2\lambda} = \frac{\lambda-\omega}{\lambda} \cdot \frac{\lambda+\omega}{\lambda} \cdot \frac{3\lambda-\omega}{3\lambda} \cdot \frac{3\lambda+\omega}{3\lambda} \cdot \text{etc.}$$

eritque facta comparatione $a = \lambda - \omega$, $b = \lambda$, $c = +\omega$ et $k = 2\lambda$ sicque per formulas integrales erit

$$\cos. \frac{\pi\omega}{2\lambda} = \frac{\int z^{\omega-1} dz (1-z^{2\lambda})^{-\frac{1}{2}}}{\int z^{\omega-1} dz (1-z^{2\lambda})^{\frac{-\lambda-\omega}{2\lambda}}},$$

quae expressio ad simpliciorem formam reduci nequit.

47. Sit nunc etiam $\lambda = 2$ et $\omega = 1$ eruntque ternae nostrae expressiones

$$\text{I. } \int T d\omega = l \frac{4}{3} \cdot \frac{36}{35} \cdot \frac{100}{99} \cdot \frac{196}{195} \cdot \text{etc.} \quad \text{sive} \quad \int T d\omega = l \frac{2\cdot 2}{1\cdot 3} \cdot \frac{6\cdot 6}{5\cdot 7} \cdot \frac{10\cdot 10}{9\cdot 11} \cdot \frac{14\cdot 14}{13\cdot 15} \cdot \text{etc.};$$

$$\text{II. } \int T d\omega = -l \cos. \frac{\pi}{4} = +\frac{1}{2} l2,$$

ita ut sit

$$\frac{1}{2}\sqrt{2} = \frac{2\cdot 2}{1\cdot 3} \cdot \frac{6\cdot 6}{5\cdot 7} \cdot \frac{10\cdot 10}{9\cdot 11} \cdot \frac{14\cdot 14}{13\cdot 15} \cdot \text{etc.},$$

quod productum per formulas integrales ita exprimitur

$$\frac{\int dz(1-z^4)^{-\frac{3}{4}}}{\int dz(1-z^4)^{-\frac{1}{2}}} = \sqrt{2};$$

$$\text{III. } \int T d\omega = \int \frac{-(1+zz)}{1-z^4} \cdot \frac{dz}{lz} = \int \frac{-dz}{(1-zz)lz},$$

quod ergo integrale a termino $z = 0$ usque ad $z = 1$ extensum praebet eundem valorem $+\frac{1}{2}l2$, cuius aequalitatis ratio utique difficillime patet.

48. Sit denique ut supra $\lambda = 3$ et $\omega = 1$ ac ternae formulae ita se habebunt

$$\text{I. } \int T d\omega = l\,\frac{9}{8} \cdot \frac{81}{80} \cdot \frac{225}{224} \cdot \text{etc.} = l\,\frac{3\cdot 3}{2\cdot 4} \cdot \frac{9\cdot 9}{8\cdot 10} \cdot \frac{15\cdot 15}{14\cdot 16} \cdot \frac{21\cdot 21}{20\cdot 22} \cdot \text{etc.};$$

$$\text{II. } \int T d\omega = -l\cos.\frac{\pi}{6} = -l\frac{\sqrt{3}}{2} = +l\frac{2}{\sqrt{3}},$$

ita ut sit

$$\frac{2}{\sqrt{3}} = \frac{3\cdot 3}{2\cdot 4} \cdot \frac{9\cdot 9}{8\cdot 10} \cdot \frac{15\cdot 15}{14\cdot 16} \cdot \frac{21\cdot 21}{20\cdot 22} \cdot \text{etc.}$$

ideoque per binas formulas integrales

$$\frac{2}{\sqrt{3}} = \frac{\int dz(1-z^6)^{-\frac{1}{2}}}{\int dz(1-z^6)^{-\frac{2}{3}}};$$

$$\text{III. } \int T d\omega = \int \frac{-(1+zz)}{1-z^6} \cdot \frac{dz}{lz},$$

quae posito $zz = v$ abit in hanc

$$\int T d\omega = \int \frac{-dv(1+v)}{(1-v^3)lv}.$$

Hinc igitur patet hac methodo plane nova perveniri ad formulas integrales, quas per methodos adhuc cognitas nullo modo evolvere vel saltem inter se comparare licuit.

NOVA METHODUS QUANTITATES INTEGRALES DETERMINANDI

Commentatio 464 indicis Enestroemiani
Novi commentarii academiae scientiarum Petropolitanae 19 (1774), 1775, p. 66—102
Summarium ibidem p. 13—17

SUMMARIUM

Refert Illustr. huius dissertationis Auctor, dum saepius sibi occurrissent formulae integrales, quae per logarithmum quantitatis variabilis erant divisae, se nunquam perspicere potuisse, ad quodnam genus quantitatum essent referendae. De simplicissima quidem formula huius generis $\int \frac{dz}{lz}$ constabat eam a termino $z = 0$ ad $z = 1$ integratam infinite magnum exhibere. Nunc vero Illustr. Auctori successit evolutio plurium huiusmodi formularum $\int \frac{P dz}{lz}$, quae posito post integrationem $z = 1$ valores finitae magnitudinis sortiuntur. Inter simpliciores istarum est haec $\int \frac{(z-1)dz}{lz}$, cuius valorem primum quantitatem finitae magnitudinis, deinde reapse ipsi $l2$ aequalem singulari ratiocinio heic ostendit Illustr. Auctor, quo eodem ratiocinio quoque ostendi potest esse $\int \frac{(z^m-1)dz}{lz} = l(m+1)$. Verum quum hoc ratiocinium per quantitates infinite parvas procedat, Illustr. Auctor de planiori methodo sollicitus erat, quam methodum ipsi suppeditavit consideratio functionum binas variabiles involventium. Scopus autem huius dissertationis praecipuus is est, ut methodus ista perspicue explicetur. Ex natura functionum, quae binas variabiles z et u involvunt, colligitur, quodsi P aliqua huiusmodi fuerit functio sitque $\int P dz = S$, tum esse

$$\int dz \left(\frac{dP}{du}\right) = \left(\frac{dS}{du}\right)$$

similique modo ulterius procedendo

$$\left(\frac{ddS}{du^2}\right) = \int dz \left(\frac{ddP}{du^2}\right), \quad \left(\frac{d^3S}{du^3}\right) = \int dz \left(\frac{d^3P}{du^3}\right) \quad \text{etc.}$$

Hoc principium pro praesenti negotio utile evadit, quando functio P ita est comparata, ut casu particulari, quo post integrationem ipsi z certus valor, utpote $z=a$, tribuitur, $S=\int Pdz$ abeat in functionem solius variabilis u; tum enim integrationes supra memoratae locum habebunt, modo post singulas ponatur $z=a$. Simili quoque modo ex natura functionum binas variabiles involventium deducitur, quod sit

$$\int Sdu = \int dz \int Pdu,$$

ubi in integralibus $\int Sdu$, $\int Pdu$ variabilitas solius u spectatur, tum vero in integrali $\int dz \int Pdu$ variabilitas solius z. Hae vero integrationes repeti quoque possunt, ut sit

$$\int du \int Sdu = \int dz \int du \int Pdu \quad \text{et} \quad \int du \int du \int Sdu = \int dz \int du \int du \int Pdu.$$

Quodsi nunc P eiusmodi sit functio variabilium z et u, ut formulae integralis $\int Pdz$ valor certo casu, puta $z=a$, commode exhiberi queat per quantitatem S, quae sit functio solius variabilis u, pro eodem casu $z=a$ formularum integralium $\int dz \int Pdu$, $\int dz \int du \int Pdu$ valores determinari possunt, modo formulae $\int Sdu$, $\int du \int Sdu$ integrationem admittant. Usum et applicationem binorum horum principiorum Illustr. Auctor variis exemplis illustravit et pro priori statuendo

$$P = \frac{z^{n-u-1} \pm z^{n+u-1}}{1 \pm z^{2n}}$$

integrationes illae prodeunt, quas in dissertatione praecedenti contemplatus erat. Pro posteriori statuendo $P = z^u$ invenitur

$$\int Sdu - l(u+1) = \int \frac{z^u dz}{lz},$$

ubi addi debet constans C, cuius valor intelligitur infinitus ob $\frac{z^u}{lz}$ infinitum, dum ponitur $z=1$. At quum valor ipsius C non dependeat ab u, C eundem retinebit valorem, quicquid sit u, ideoque habebitur

$$\int \frac{z^m dz}{lz} - l(m+1) + C \quad \text{et} \quad \int \frac{z^n dz}{lz} = l(n+1) + C$$

hincque

$$\int \frac{(z^m - z^n)dz}{lz} = l\frac{m+1}{n+1}.$$

Ascendendo ad alteram integrationem fiet

$$\int du \int Pdu = \frac{z^u}{(lz)^2} \quad \text{ideoque} \quad \int du \int Sdu = (u+1)(l(u+1)-1) + Cu + D = \int \frac{z^u dz}{(lz)^3};$$

hinc tribus huiusmodi integralibus coniunctis elidi possunt constantes C et D. Deinde si ponatur

$$P = \frac{z^{n-u-1} \pm z^{n+u-1}}{1 \pm z^{2n}},$$

prodibunt formulae integrales, quas Illustr. EULERUS in Additamento prioris dissertationis fusius contemplatus erat. Praecipue vero heic occupatus est in eo, ut principii iam stabiliti adplicationem faciat ad formulas integrales $\int \frac{Pdz}{z}$, $\int Pdu$ atque $\int \frac{Qdz}{z}$, $\int Qdu$, dum scilicet ponitur

$$P = z \cos. u + z^2 \cos. 2u + z^3 \cos. 3u + z^4 \cos. 4u + \text{etc.}$$

et

$$Q = z \sin. u + z^2 \sin. 2u + z^3 \sin. 3u + z^4 \sin. 4u + \text{etc.}$$

Relationes autem istae horum integralium inveniuntur, ut sit

$$\int \frac{Pdz}{z} = -\int Qdu, \quad \int \frac{dz}{z} \int \frac{Pdz}{z} = -\int du \int Pdu, \quad \int \frac{dz}{z} \int \frac{dz}{z} \int \frac{Pdz}{z} = +\int du \int du \int Qdu \quad \text{etc.}$$

nec non

$$\int \frac{Qdz}{z} = +\int Pdu, \quad \int \frac{dz}{z} \int \frac{Qdz}{z} = -\int du \int Qdu, \quad \int \frac{dz}{z} \int \frac{dz}{z} \int \frac{Qdz}{z} = -\int du \int du \int Pdu \quad \text{etc.}$$

Tum vero si illi valores integralium desiderentur, quos consequuntur posito $z=1$, in formulis integralibus, ubi solus angulus u pro variabili habetur, ante integrationes iam statuere licebit $z=1$, ex quo fiet

$$P = \frac{\cos. u - 1}{2(1-\cos. u)} = -\frac{1}{2}, \quad Q = \frac{1}{2} \cot. \frac{1}{2} u,$$

$$\int Pdu = A - \frac{1}{2} u, \quad \int du \int Pdu = B + Au - \frac{1}{4} u^2,$$

$$\int du \int du \int Pdu = C + Bu + \frac{1}{2} Au^2 - \frac{1}{12} u^3 \quad \text{etc.}$$

nec non

$$\int Qdu = l \sin. \frac{1}{2} u, \quad \int du \int Qdu - \int du\, l \sin. \frac{1}{2} u,$$

cuius integralis evolutio non constat. Pro formulis variabilem z involventibus colligitur

$$\int \frac{dz}{z} \int \frac{Pdz}{z} = -\int \frac{Pdz}{z} lz, \quad \int \frac{dz}{z} \int \frac{dz}{z} \int \frac{Pdz}{z} = +\int \frac{Pdz}{z} (lz)^2 \quad \text{etc.}$$

posito nimirum $z=1$, ita ut hinc valores integralium

$$\int \frac{Pdz}{z} (lz)^m \quad \text{vel} \quad \int \frac{Qdz}{z} (lz)^m$$

per arcum u expressos assignari liceat casu, quo post integrationem ponitur $z=1$.

1. Cum mihi saepius[1]) occurrissent formulae differentiales, quae per logarithmum quantitatis variabilis erant divisae, veluti $\frac{Pdz}{lz}$, nunquam perspicere potui, ad quodnam genus quantitatum earum integralia sint referenda, quin etiam maxime difficile videbatur eorum valores saltem vero proxime assignare. Quod quidem ad formulam integralem simplicissimam huius generis $\int\frac{dz}{lz}$ attinet, facile patet, si eam ita integrari concipiam, ut evanescat posito $z=0$, tum vero statuatur $z=1$, quantitatem infinite magnam esse prodituram; quodsi enim variabilis z iam proxime ad unitatem accesserit, ut sit $z=1-u$ existente u quantitate infinite parva, tum ob

$$dz=-du \quad \text{et} \quad lz=l(1-u)=-u$$

haec formula erit $\int\frac{du}{u}$, cuius valor utique fit infinitus. At vero dantur omnino huiusmodi formulae integrales $\int\frac{Pdz}{lz}$, quae, etiamsi ponatur $z=1$, tamen valores finitae magnitudinis sortiuntur; quod determinasse eo magis operae pretium videtur, quod nulla adhuc cognita est via istos valores investigandi.

2. Consideremus exempli gratia hanc formulam satis simplicem

$$\int\frac{(z-1)dz}{lz},$$

quae memorata lege integrata valorem finitum habere facile ostendi potest. Posito enim

$$\frac{z-1}{lz}=y,$$

ut formula nostra fiat $\int ydz$ ideoque exprimat aream curvae pro abscissa z applicatam habentis $=y$, ista area a termino $z=0$ usque ad terminum $z=1$ extensa utique valorem finitum non multo maiorem quam $\frac{1}{2}$ repraesentabit; posita enim abscissa $z=0$ fiet etiam applicata $y=0$, at sumta $z=1$ pro applicata $y=\frac{z-1}{lz}$ tam numerator quam denominator evanescit, ergo eorum loco substitutis suis differentialibus fiet $y=z=1$. Pro abscissis autem mediis ponamus $z=e^{-n}$ existente e numero, cuius logarithmus hyperbolicus est unitas; erit

$$y=\frac{e^{-n}-1}{-n}=\frac{e^{n}-1}{ne^{n}},$$

1) Vide *Institutionum calculi integralis* vol. 1, § 215 et seq. LEONHARDI EULERI *Opera omnia*, series I, vol. 11, p. 120. A. G.

quae, si n fuerit numerus valde magnus, ut abscissa z fiat minima, applicata erit proxime $y=\frac{1}{n}$; qui ergo valor multo maior erit quam abscissa z; forma scilicet huius curvae similis erit figurae adiectae, ubi AP (Fig. 1) denotat abscissam z et PM applicatam y; abscissae vero $AB=1$ respondet applicata $BC=1$; qua curva descripta eius area $AMCB$ non multum superabit aream trianguli ABC, quae est $=\frac{1}{2}$.

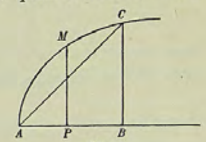

Fig. 1.

3. Nuper autem in aliis investigationibus[1]) occupatus praeter expectationem inveni hanc aream aequalem esse logarithmo hyperbolico binarii, ita ut ea per fractiones decimales sit $l2=0{,}69314\,71805$; sequenti autem ratiocinio huc sum perductus. Cum revera sit $lz=\frac{z^0-1}{0}$, quia differentiando utrinque prodit $\frac{dz}{z}=\frac{dz}{z}$ et sumto $z=1$ utraque expressio evanescit, loco 0 scribo $\frac{1}{i}$ denotante i numerum infinitum eritque $lz=i(z^{\frac{1}{i}}-1)$ hincque applicata

$$y=\frac{z-1}{i(z^{\frac{1}{i}}-1)}=\frac{1-z}{i(1-z^{\frac{1}{i}})}$$

et formula integralis $\int\frac{(1-z)dz}{i(1-z^{\frac{1}{i}})}$. Nunc igitur statuo $z^{\frac{1}{i}}=x$, ut fiat $z=x^i$, ubi notetur pro utroque integrationis termino $z=0$ et $z=1$ etiam fore $x=0$ et $x=1$; quia igitur hinc fit $dz=ix^{i-1}dx$, formula integralis evadit

$$\int\frac{x^{i-1}dx(1-x^i)}{1-x},$$

quam ergo integrari oportet a termino $x=0$ usque ad terminum $x=1$.

4. Spectemus nunc i ut numerum valde magnum et fractio $\frac{1-x^i}{1-x}$ resolvitur in hanc progressionem geometricam

$$1+x+x^2+x^3+x^4+x^5+x^6+\cdots+x^{i-1},$$

1) Vide Commentationem 463 huius voluminis. A. G.

cuius singuli termini in $x^{i-1}dx$ ducti et integrati praebent hanc seriem

$$\frac{x^i}{i}+\frac{x^{i+1}}{i+1}+\frac{x^{i+2}}{i+2}+\frac{x^{i+3}}{i+3}+\cdots+\frac{x^{2i-1}}{2i-1},$$

quae utique evanescit facto $x=0$. Nunc igitur sumatur $x=1$ et valor quaesitus nostrae formulae integralis erit

$$\frac{1}{i}+\frac{1}{i+1}+\frac{1}{i+2}+\frac{1}{i+3}+\cdots+\frac{1}{2i-1},$$

ubi quidem littera i denotat numerum infinite magnum, ita ut numerus horum terminorum sit revera infinitus. Nihilo vero minus, quia singuli termini sunt infinite parvi, haec series summam habebit finitam, quam sequenti modo ad seriem ordinariam reducere licet.

5. Series inventa spectari potest tanquam differentia inter binas sequentes progressiones harmonicas

$$A=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots+\frac{1}{2i-1},$$

$$B=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots+\frac{1}{i-1},$$

quandoquidem differentia $A-B$ ipsam seriem inventam exhibet; quia autem numerus terminorum seriei A est $2i-1$, seriei vero $B=i-1$, ille duplo maior est quam hic, quocirca, ut seriem regularem obtineamus, singulos terminos seriei B per saltum a seriei A termino secundo, quarto, sexto, octavo etc. auferamus, quo pacto simul ad finem utriusque pervenietur eritque

$$A-B=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\text{etc. in infinitum},$$

cuius ergo valor est $l2$, ita ut nunc quidem solide sit demonstratum formulae integralis propositae $\int\frac{(z-1)dz}{lz}$ casu $z=1$ valorem revera esse $=l2$.

6. Simile ratiocinium etiam ad formulam integralem generaliorem

$$\int\frac{(z^m-1)dz}{lz}$$

accommodari potest ac tandem reperietur casu $z=1$ eius valorem fore

$l(m+1)$; quia igitur pari modo erit $\int\frac{(z^n-1)dz}{lz}=l(n+1)$, si hanc ab illa subtrahamus, prodit sequens integratio

$$\int\frac{(z^m-z^n)dz}{lz}=l\frac{m+1}{n+1},$$

si scilicet integratio a termino $z=0$ usque ad terminum $z=1$ extendatur.

7. Quia autem haec demonstratio per quantitates infinitas et infinite parvas procedit, merito aliam methodum planam et consuetam desideramus, quae ad easdem summas perducere valeat; quae quidem investigatio maxime ardua videbitur. Interim tamen, cum nuper[1]) consideratio functionum duas variabiles involventium me ad integrationem formularum differentialium prorsus singularium perduxisset, quae aliis methodis frustra tentantur, ex eodem principio quoque integrationes hic exhibitas derivandas esse intellexi. Hanc igitur methodum tanquam fontem prorsus novum, ex quo integrationes aliis methodis inaccessas haurire liceat, clare et perspicue explicabo, cui negotio istam disquisitionem praecipue destinavi.

LEMMA 1

8. *Si P fuerit functio quaecunque duarum variabilium z et u ac ponatur $\int Pdz=S$, ut etiam S sit functio binarum variabilium z et u, tum erit*

$$\int dz\left(\frac{dP}{du}\right)=\left(\frac{dS}{du}\right).$$

DEMONSTRATIO

Cum in integratione formulae $\int Pdz$ sola z ut variabilis spectetur, erit $\left(\frac{dS}{dz}\right)=P$, quae formula denuo differentiata sola u pro variabili habita praebet $\left(\frac{ddS}{dudz}\right)=\left(\frac{dP}{du}\right)$, quae in dz ducta et integrata producit

$$\left(\frac{dS}{du}\right)=\int dz\left(\frac{dP}{du}\right),$$

quandoquidem ex principiis calculi integralis est $\int dz\left(\frac{ddS}{dzdu}\right)=\left(\frac{dS}{du}\right)$. Q. E. D.

1) Vide Commentationem 463 huius voluminis. A. G.

COROLLARIUM 1

9. Eodem modo per huiusmodi differentialia, ubi tantum u pro variabili spectatur, ulterius progredi licet, unde sequentes oriuntur integrationes

$$\left(\frac{ddS}{du^2}\right) = \int dz \left(\frac{ddP}{du^2}\right)$$

et

$$\left(\frac{d^3S}{du^3}\right) = \int dz \left(\frac{d^3P}{du^3}\right)$$

etc.

COROLLARIUM 2

10. Quodsi ergo formula $\int Pdz$ fuerit integrabilis, ita ut eius integrale S exhiberi possit, tum etiam omnes istae formulae integrales

$$\int dz \left(\frac{dP}{du}\right), \quad \int dz \left(\frac{ddP}{du^2}\right), \quad \int dz \left(\frac{d^3P}{du^3}\right) \quad \text{etc.}$$

integrationem admittent atque adeo ipsa integralia exhiberi poterunt.

SCHOLION

11. Ex his quidem formulis, si in genere tractentur, parum utilitatis in calculum integralem redundat. At si functio P ita fuerit comparata, ut integrale $\int Pdz$ casu saltem particulari, quo post integrationem variabili z certus quidam valor, puta $z = a$, tribuitur, commode exhiberi possit, ut hoc casu quantitas S abeat in functionem solius variabilis u satis simplicem, tum integrationes memoratae perinde locum habebunt, si quidem post singulas integrationes ponatur $z = a$, atque hinc ad eiusmodi integrationes plerumque pervenitur, quas aliis methodis vix ac ne vix quidem perficere liceat; atque hinc oritur

PRIMUM PRINCIPIUM INTEGRATIONUM

12. Si P eiusmodi fuerit functio binarum variabilium z et u, ut valor integralis $\int Pdz$ saltem casu certo $z = a$ commode exprimi queat, qui valor sit $= S$, functio scilicet ipsius u tantum, tum etiam sequentia integralia, si

quidem post integrationem pariter statuatur $z = a$, commode exhiberi poterunt, scilicet

$$\int Pdz = S,$$

$$\int dz \left(\frac{dP}{du}\right) = \left(\frac{dS}{du}\right),$$

$$\int dz \left(\frac{ddP}{du^2}\right) = \left(\frac{ddS}{du^2}\right),$$

$$\int dz \left(\frac{d^3P}{du^3}\right) = \left(\frac{d^3S}{du^3}\right),$$

$$\int dz \left(\frac{d^4P}{du^4}\right) = \left(\frac{d^4S}{du^4}\right)$$

etc.

EXEMPLUM 1

13. Si fuerit $P = z^u$, erit quidem in genere

$$\int Pdz = \frac{z^{u+1}}{u+1},$$

unde casu $z = 1$ hic valor satis simplex nascitur $\frac{1}{u+1}$, ita ut sit $S = \frac{1}{u+1}$; cum deinde per differentiationes continuas, dum sola u pro variabili habetur, prodeat

$$\left(\frac{dP}{du}\right) = z^u lz,$$

tum vero

$$\left(\frac{ddP}{du^2}\right) = z^u (lz)^2,$$

porro

$$\left(\frac{d^3P}{du^3}\right) = z^u (lz)^3, \quad \left(\frac{d^4P}{du^4}\right) = z^u (lz)^4 \quad \text{etc.},$$

hinc sequentes obtinentur valores integrales, si quidem post singulas integrationes statuatur $z = 1$:

$$\int z^u dz = + \frac{1}{u+1} \qquad \int z^u dz (lz)^4 = + \frac{1 \cdot 2 \cdot 3 \cdot 4}{(u+1)^5}$$

$$\int z^u dz lz = - \frac{1}{(u+1)^2} \qquad \int z^u dz (lz)^5 = - \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{(u+1)^6}$$

$$\int z^u dz (lz)^2 = + \frac{1 \cdot 2}{(u+1)^3} \qquad \int z^u dz (lz)^6 = + \frac{1 \cdots 6}{(u+1)^7}$$

$$\int z^u dz (lz)^3 = - \frac{1 \cdot 2 \cdot 3}{(u+1)^4} \qquad \int z^u dz (lz)^7 = - \frac{1 \cdots 7}{(u+1)^8}$$

unde concludimus generaliter fore

$$\int z^u dz(lz)^n = \pm\frac{1\cdot 2\cdot 3\cdot 4\cdots n}{(u+1)^{n+1}},$$

ubi signum $+$ valet, si n sit numerus par, alterum vero $-$, si n sit numerus impar. Hae quidem integrationes iam aliunde satis sunt notae, id quod mirum non est, quoniam tam simplicem formulam pro P assumsimus; breviter igitur repetamus eos casus, quos iam nuper[1]) expedivi.

EXEMPLUM 2

14. Si fuerit

$$P = \frac{z^{n-u-1}+z^{n+u-1}}{1+z^{2n}},$$

iam dudum[2]) demonstravi formulae $\int Pdz$ valorem integralem casu, quo post integrationem ponitur $z=1$, esse

$$S = \frac{\pi}{2n\cos.\frac{\pi u}{2n}};$$

hinc ergo, cum sit

$$\left(\frac{dP}{du}\right) = \frac{-z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}\,lz,$$

tum vero

$$\left(\frac{ddP}{du^2}\right) = \frac{z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}\,(lz)^2$$

et

$$\left(\frac{d^3P}{du^3}\right) = \frac{-z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}\,(lz)^3$$

etc.,

ex cognito valore S sequentes nacti sumus integrationes

1) Vide Commentationem 463 huius voluminis. A. G.
2) Vide p. 29. A. G.

$$\text{I.}\quad \int\frac{z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}\,dz = S = \frac{\pi}{2n\cos.\frac{\pi u}{2n}},$$

$$\text{II.}\quad \int\frac{-z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}\,dzlz = \left(\frac{dS}{du}\right),$$

$$\text{III.}\quad \int\frac{z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}\,dz(lz)^2 = \left(\frac{ddS}{du^2}\right),$$

$$\text{IV.}\quad \int\frac{-z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}\,dz(lz)^3 = \left(\frac{d^3S}{du^3}\right),$$

$$\text{V.}\quad \int\frac{z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}\,dz(lz)^4 = \left(\frac{d^4S}{du^4}\right)$$

etc.

EXEMPLUM 3

15. Si fuerit

$$P = \frac{z^{n-u-1}-z^{n+u-1}}{1-z^{2n}},$$

simili modo demonstravi valorem formulae integralis $\int Pdz$ casu, quo post integrationem ponitur $z=1$, fore

$$S = \frac{\pi}{2n}\,\text{tang.}\,\frac{\pi u}{2n};$$

atque hinc sequentes integrationes pro eodem casu $z=1$ fuerunt deductae

$$\text{I.}\quad \int\frac{z^{n-u-1}-z^{n+u-1}}{1-z^{2n}}\,dz = S = \frac{\pi}{2n}\,\text{tang.}\,\frac{\pi u}{2n},$$

$$\text{II.}\quad \int\frac{-z^{n-u-1}-z^{n+u-1}}{1-z^{2n}}\,dzlz = \left(\frac{dS}{du}\right),$$

$$\text{III.}\quad \int\frac{z^{n-u-1}-z^{n+u-1}}{1-z^{2n}}\,dz(lz)^2 = \left(\frac{ddS}{du^2}\right),$$

$$\text{IV.}\quad \int\frac{-z^{n-u-1}-z^{n+u-1}}{1-z^{2n}}\,dz(lz)^3 = \left(\frac{d^3S}{du^3}\right),$$

$$\text{V.}\quad \int\frac{z^{n-u-1}-z^{n+u-1}}{1-z^{2n}}\,dz(lz)^4 = \left(\frac{d^4S}{du^4}\right)$$

etc.

SCHOLION

16. Quo igitur uberiores fructus ex hoc principio expectare queamus, praecipuum negotium huc redit, ut eiusmodi functiones binarum variabilium z et u pro P investigemus, ita ut valor formulae integralis saltem certo quodam casu, puta $z=1$, succincte assignari possit, quemadmodum in allatis exemplis fieri licuit. Quemadmodum autem hoc principium ex continua differentiatione est deductum, ita eodem modo continua integratio ad usum nostrum accommodari poterit.

LEMMA 2

17. *Si P fuerit functio duarum variabilium z et u ac ponatur $\int Pdz = S$, ut etiam S sit functio duarum variabilium z et u, tum erit $\int Sdu = \int dz \int Pdu$, ubi in integralibus formulis $\int Pdu$ et $\int Sdu$ sola u pro variabili habetur, in formula autem $\int dz \int Pdu$ sola z.*

DEMONSTRATIO

Ponatur $\int Sdu = V$, ut sit $S = \left(\frac{dV}{du}\right)$ ideoque $\left(\frac{dV}{du}\right) = \int Pdz$, eritque $\left(\frac{ddV}{dz\,du}\right) = P$; unde per du multiplicando et integrando erit $\left(\frac{dV}{dz}\right) = \int Pdu$, ex quo sequitur $V = \int dz \int Pdu = \int Sdu$. Q. E. D.

COROLLARIUM 1

18. Hoc modo etiam integratio repeti potest; unde orietur talis aequatio $\int du \int Sdu = \int dz \int du \int Pdu$; hinc autem plerumque parum utilitatis expectari potest, nisi forte istae integrationes commode succedant.

COROLLARIUM 2

19. Quodsi ergo formula $\int Pdz$ fuerit integrabilis, scilicet $= S$, altera hinc deducta $\int dz \int Pdu$ eatenus tantum integrari poterit, quatenus integrale $\int Sdu$ integrare licet.

SECUNDUM PRINCIPIUM INTEGRATIONUM

20. Si P eiusmodi fuerit functio duarum variabilium z et u, ut formulae integralis $\int Pdz$ valor certo saltem casu, puta $z = a$, commode exhiberi queat, ita ut hoc casu quantitas S fiat functio solius variabilis u, tum etiam pro eodem casu $z = a$ huius formulae integralis $\int dz \int Pdu$ valor assignari poterit, si modo formulam $\int Sdu$ integrare licuerit.

EXEMPLUM 1

21. Sumamus $P = z^u$ eritque $\int Pdz = \frac{z^{u+1}}{u+1}$, quae formula casu $z=1$ abit in $\frac{1}{u+1}$, quod ergo loco S scribatur. Tum vero quia est

$$\int Pdu = \int z^u du = \frac{z^u}{lz}$$

et quia $\int Sdu = l(u+1)$, erit

$$\int \frac{z^u dz}{lz} = l(u+1),$$

si quidem post illam integrationem ponatur $z=1$. Quia autem omnis integratio additionem constantis postulat, hic potius statui oportebit

$$\int \frac{z^u dz}{lz} = l(u+1) + C$$

atque hic quidem facile intelligitur hanc constantem C esse debere infinitam, quoniam in formula integrali fractio $\frac{z^u}{lz}$ posito $z=1$ fit infinita, ita ut hinc parum pro instituto nostro sequi videatur.

COROLLARIUM 1

22. Quoniam autem haec constans C non a variabili u pendet, ea retinebit eundem valorem, quicunque numeri determinati pro u accipiantur. Sumamus igitur primo $u = m$, tum vero etiam $u = n$, ut habeamus istos valores

$$\text{I.} \quad \int \frac{z^m dz}{lz} = l(m+1) + C$$

et

$$\text{II.} \quad \int \frac{z^n dz}{lz} = l(n+1) + C,$$

quarum altera ab altera subtracta relinquet istam integrationem notatu dignissimam

$$\int \frac{(z^m - z^n)\,dz}{lz} = l\frac{m+1}{n+1},$$

quemadmodum iam supra [§ 6] ex longe aliis principiis demonstravimus.

COROLLARIUM 2

23. Si ad alteram integrationem ascendamus, quia est $\int P du = \frac{z^u}{lz}$, erit

$$\int du \int P du = \frac{z^u}{(lz)^2};$$

tum vero ob $\int S du = l(u+1) + C$ erit

$$\int du \int S du = (u+1)(l(u+1) - 1) + Cu + D$$

sicque habebimus

$$\int \frac{z^u\,dz}{(lz)^2} = (u+1)(l(u+1) - 1) + Cu + D,$$

ubi constantes C et D non ab u pendent; quare ut eas eliminemus, tres casus determinatos evolvamus

$$\text{I.}\quad \int \frac{z^m\,dz}{(lz)^2} = (m+1)\,l(m+1) - m - 1 + Cm + D,$$

$$\text{II.}\quad \int \frac{z^n\,dz}{(lz)^2} = (n+1)\,l(n+1) - n - 1 + Cn + D,$$

$$\text{III.}\quad \int \frac{z^k\,dz}{(lz)^2} = (k+1)\,l(k+1) - k - 1 + Ck + D$$

eritque

$$\text{I} - \text{III} = (m+1)l(m+1) - (k+1)l(k+1) + k - m + C(m-k)$$

et

$$\text{II} - \text{III} = (n+1)l(n+1) - (k+1)l(k+1) + k - n + C(n-k)$$

hincque deducimus

$$(\text{I} - \text{III})(n-k) - (\text{II} - \text{III})(m-k) = \begin{cases} (m+1)(n-k)l(m+1) \\ -(k+1)(n-k)l(k+1) + (k-m)(n-k) \\ -(n+1)(m-k)l(n+1) - (k-n)(m-k) \\ +(k+1)(m-k)l(k+1) \end{cases}$$

atque hinc deducimus sequentem integrationem

$$\int \frac{dz((n-k)z^m - (m-k)z^n + (m-n)z^k)}{(lz)^2} = \begin{cases} +(m+1)(n-k)l(m+1) \\ -(n+1)(m-k)l(n+1) \\ +(k+1)(m-n)l(k+1) \end{cases}$$

COROLLARIUM 3

24. Operae pretium erit aliquot casus evolvere, ubi quidem numeros m, n et k inter se inaequales accipi convenit, quia aliter omnes termini se destruerent.

I. Sit igitur $m = 2$, $n = 1$ et $k = 0$; erit

$$\int \frac{(z-1)^2\,dz}{(lz)^2} = 3l3 - 4l2 = l\frac{27}{16}.$$

II. Sit $m = 3$, $n = 1$ et $k = 0$ eritque

$$\int \frac{(z^3 - 3z + 2)\,dz}{(lz)^2} = \int \frac{dz\,(z-1)^2(z+2)}{(lz)^2} = 4l4 - 6l2 = 2l2 = l4.$$

III. Sit $m = 3$, $n = 2$ et $k = 0$ et erit

$$\int \frac{(2z^3 - 3zz + 1)\,dz}{(lz)^2} = \int \frac{dz(z-1)^2(2z+1)}{(lz)^2} = 8l4 - 9l3 = l\frac{4^8}{3^9}.$$

IV. Sit $m = 3$, $n = 2$ et $k = 1$ et prodit

$$\int \frac{(z^3 - 2zz + z)\,dz}{(lz)^2} = \int \frac{z\,dz(z-1)^2}{(lz)^2} = 4l4 - 6l3 + 2l2 = l\frac{2^{10}}{3^6}.$$

COROLLARIUM 4

25. In his casibus notatu dignum occurrit, quod numerator in formulis integralibus factorem habet $(z-1)^2$, quod ideo necessario usu venit, ne valores integralium evadant infiniti. Quia enim denominator $(lz)^2$ evanescit casu $z = 1$, si ponamus $z = 1 - \omega$ existente ω infinite parvo, erit

$$lz = -\omega \quad \text{et} \quad (lz)^2 = +\omega\omega.$$

Necesse ergo est, ut in numeratore adsit factor, qui casu $z = 1 - \omega$ itidem praebeat $\omega\omega$, quod evenit, si ibi factor fuerit $(z-1)^2$.

SCHOLION

26. Integratio, quam in corollario primo sumus nacti, ideo omni digna videtur attentione, quod valores integrales inde nati casu $z=1$ nullo adhuc modo assignare potuerim, etiamsi tam simpliciter per logarithmos exprimantur. At vero integrationes in corollario secundo inventae, etiamsi multo magis arduae videantur, tamen ex prioribus ope reductionum cognitarum non difficulter derivari possunt; id quod pro unico casu ostendisse sufficiet. Ponamus

$$\int\frac{dz(z-1)^2}{(lz)^2}=\frac{p}{lz}+\int\frac{qdz}{lz}$$

eritque differentiando

$$\frac{dz(z-1)^2}{(lz)^2}=\frac{dp}{lz}-\frac{pdz}{z(lz)^2}+\frac{qdz}{lz},$$

unde aequatis terminis seorsim vel per $(lz)^2$ vel per lz divisis habebimus has duas aequalitates

$$(z-1)^2=-\frac{p}{z}\quad\text{et}\quad dp=-qdz,$$

ex quarum priore oritur $p=-z(z-1)^2$ hincque

$$\frac{dp}{dz}=-3zz+4z-1\quad\text{ideoque}\quad q=3zz-4z+1,$$

ita ut sit

$$\int\frac{dz(z-1)^2}{(lz)^2}=\frac{-z(z-1)^2}{lz}+\int\frac{(3zz-4z+1)dz}{lz};$$

hic autem prius membrum posito $z=1$ sponte evanescit; posito enim $z=1-\omega$, ut sit $lz=-\omega$, erit $p=-\omega\omega(1-\omega)$ ideoque $\frac{p}{lz}=\omega(1-\omega)=0$ ob $\omega=0$; posterius vero membrum in has partes discerpi potest

$$3\int\frac{(zz-z)dz}{lz}-\int\frac{(z-1)dz}{lz}.$$

Prioris autem partis integrale est $3l\frac{3}{2}$, posterioris vero $-1l2$ sicque totum hoc integrale erit

$$3l\frac{3}{2}-l2=3l3-4l2=l\frac{27}{16},$$

prorsus uti invenimus. Hoc igitur modo si in genere statuamus

$$\int\frac{Vdz}{(lz)^2}=\frac{p}{lz}+\int\frac{qdz}{lz},$$

erit differentiando

$$\frac{Vdz}{(lz)^2}=\frac{dp}{lz}-\frac{pdz}{z(lz)^2}+\frac{qdz}{lz},$$

unde istae duae fluunt aequalitates

$$p=-Vz\quad\text{et}\quad q=-\frac{dp}{dz}.$$

Iam ut terminus $\frac{p}{lz}$ evanescat posito $z=1$, numerator p factorem habere debet $(z-1)^2$, qui ergo etiam factor esse debet quantitatis V. Sit igitur

$$V=\frac{U(z-1)^2}{z}$$

eritque

$$p=-U(z-1)^2,$$

unde fit

$$dp=-dU(z-1)^2-2Udz(z-1)=(z-1)(-dU(z-1)-2Udz),$$

hincque fit

$$qdz=(z-1)(2Udz+dU(z-1));$$

quia ergo q factorem habet $z-1$, formula $\int\frac{qdz}{lz}$ semper in partes resolvi potest, quarum integralia per corollarium primum assignare licet, si modo U fuerit aggregatum ex quotcunque potestatibus ipsius z; unde sequens deducitur theorema.

THEOREMA

27. *Si fuerit*

$$P=Az^\alpha+Bz^\beta+Cz^\gamma+Dz^\delta+\text{etc.}$$

ita, ut summa coefficientium

$$A+B+C+D+\text{etc.}=0,$$

tum erit

$$\int\frac{Pdz}{lz}=Al(\alpha+1)+Bl(\beta+1)+Cl(\gamma+1)+Dl(\delta+1)+\text{etc.},$$

siquidem post integrationem statuatur $z=1$.

DEMONSTRATIO

Cum hoc ipso casu, quo post integrationem ponitur $z=1$, sit

$$\int\frac{z^n dz}{lz}=l(n+1)+\varDelta$$

denotante $\varDelta$ illam constantem infinitam integratione ingressam, erit

$$A\int\frac{z^\alpha dz}{lz}=Al(\alpha+1)+A\varDelta$$

eodemque modo

$$B\int\frac{z^\beta dz}{lz}=Bl(\beta+1)+B\varDelta$$

etc.;

si haec integralia omnia in unam summam colligantur, erit

$$(A+B+C+D+\text{etc.})\varDelta=0$$

hincque erit integrale quaesitum

$$\int\frac{Pdz}{lz}=Al(\alpha+1)+Bl(\beta+1)+Cl(\gamma+1)+Dl(\delta+1)+\text{etc.}$$

Q. E. D.

COROLLARIUM 1

28. Quia supponimus

$$A+B+C+D+\text{etc.}=0,$$

evidens est formulam

$$P=Az^\alpha+Bz^\beta+Cz^\gamma+Dz^\delta+\text{etc.}$$

factorem habere $z-1$, quemadmodum iam ante notavimus.

COROLLARIUM 2

29. Quia est

$$(z-1)^n=z^n-\frac{n}{1}z^{n-1}+\frac{n(n-1)}{1\cdot 2}z^{n-2}-\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}z^{n-3}+\text{etc.},$$

hoc valore loco P posito erit

$$A=1 \quad\text{et}\quad \alpha=n,$$

deinde

$$B=-\frac{n}{1} \quad\text{et}\quad \beta=n-1,$$

porro

$$C=\frac{n(n-1)}{1\cdot 2} \quad\text{et}\quad \gamma=n-2 \quad\text{etc.};$$

hinc igitur erit

$$\int\frac{(z-1)^n dz}{lz}=l(n+1)-\frac{n}{1}ln+\frac{n(n-1)}{1\cdot 2}l(n-1)-\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}l(n-2)$$
$$+\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}l(n-3)-\text{etc.},$$

si modo exponens n fuerit nihilo maior vel saltem unitate non minor, quia alioquin casu $z=1$ fractio $\frac{(z-1)^n}{lz}$ fieret infinita; hoc autem non obstante area supra considerata fiet finita, ita ut sufficiat, dummodo sit $n>0$.

EXEMPLUM 2

30. Sit $P=\frac{z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}$; erit

$$\int Pdz=\frac{\pi}{2n\cos.\frac{\pi u}{2n}},$$

siquidem post integrationem ponatur $z=1$, quem ergo valorem litterae S tribuimus. Nunc spectata z ut constante erit

$$\int Pdu=\frac{1}{1+z^{2n}}\left(\int z^{n-u-1}du+\int z^{n+u-1}du\right)$$

ideoque

$$\int Pdu=\frac{-z^{n-u-1}+z^{n+u-1}}{(1+z^{2n})lz},$$

unde fiet

$$\int Sdu=\int\frac{-z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}\cdot\frac{dz}{lz};$$

cum igitur sit $\cos.\frac{\pi u}{2n} = \sin.\frac{\pi(n-u)}{2n}$, erit

$$\int S du = \int \frac{\pi du}{2n \sin.\frac{\pi(n-u)}{2n}};$$

hinc, si ponamus $\frac{\pi(n-u)}{2n} = \varphi$, erit $d\varphi = -\frac{\pi du}{2n}$ ideoque

$$\int S du = -\int \frac{d\varphi}{\sin.\varphi} = -l\,\mathrm{tang.}\frac{1}{2}\varphi,$$

quocirca habebimus

$$\int S du = -l\,\mathrm{tang.}\frac{\pi(n-u)}{4n},$$

ita ut posito post integrationem $z=1$ assecuti simus hanc integrationem

$$\int \frac{-z^{n-u-1}+z^{n+u-1}}{1+z^{2n}} \cdot \frac{dz}{lz} = -l\,\mathrm{tang.}\frac{\pi(n-u)}{4n} = +l\,\mathrm{tang.}\frac{\pi(n+u)}{4n}.$$

EXEMPLUM 3

31. Sit $P = \frac{z^{n-u-1}-z^{n+u-1}}{1-z^{2n}}$; erit

$$\int P dz = \frac{\pi}{2n}\,\mathrm{tang.}\frac{\pi u}{2n} = S,$$

unde fit

$$\int S du = -l\cos.\frac{\pi u}{2n};$$

hinc, cum sit

$$\int P du = \frac{-z^{n-u-1}-z^{n+u-1}}{(1-z^{2n})\,lz},$$

nanciscimur sequentem integrationem, siquidem integrale a termino $z=0$ usque ad terminum $z=1$ extendatur,

$$\int \frac{z^{n-u-1}+z^{n+u-1}}{1-z^{2n}} \cdot \frac{dz}{lz} = +l\cos.\frac{\pi u}{2n}.$$

Haec quidem duo posteriora exempla iam ante[1]) fusius expedivi; unde iis magis evolvendis non immoror, sed ad sequens problema progredior.

1) Vide Additamentum Commentationis 463 huius voluminis. A. G.

PROBLEMA

32. *Si proponantur hae duae series infinitae*

$$P = z\cos.u + z^2\cos.2u + z^3\cos.3u + z^4\cos.4u + z^5\cos.5u + \text{etc.}$$

et

$$Q = z\sin.u + z^2\sin.2u + z^3\sin.3u + z^4\sin.4u + z^5\sin.5u + \text{etc.},$$

quae binas variabiles z et u involvunt, invenire relationes inter formulas integrales $\int\frac{Pdz}{z}$, $\int P du$ et $\int\frac{Qdz}{z}$, $\int Q du$ aliasque formulas integrales per continuam integrationem inde natas.

SOLUTIO

Cum utraque series sit recurrens, reperitur per formulas finitas

$$P = \frac{z\cos.u - zz}{1-2z\cos.u+zz} \quad \text{et} \quad Q = \frac{z\sin.u}{1-2z\cos.u+zz},$$

unde fit

$$\int\frac{Pdz}{z} = \int\frac{dz\cos.u - zdz}{1-2z\cos.u+zz} = -l\sqrt{1-2z\cos.u+zz}$$

et

$$\int Q du = \int\frac{zdu\sin.u}{1-2z\cos.u+zz} = +l\sqrt{1-2z\cos.u+zz},$$

ita ut sit $\int\frac{Pdz}{z} = -\int Q du$; tum vero etiam erit

$$\int\frac{Qdz}{z} = \int\frac{dz\sin.u}{1-2z\cos.u+zz} = \mathrm{A\,tang.}\frac{z\sin.u}{1-z\cos.u};$$

at si iste arcus differentietur sumto solo angulo u variabili, erit

$$\frac{d}{du}\mathrm{A\,tang.}\frac{z\sin.u}{1-z\cos.u} = \frac{z\cos.u - zz}{1-2z\cos.u+zz},$$

ita ut sit $\int\frac{Qdz}{z} = \int P du$.

33. Verum eaedem relationes facilius ex ipsis seriebus derivantur. Cum enim sit

$$P = z\cos.u + z^2\cos.2u + z^3\cos.3u + z^4\cos.4u + \text{etc.},$$

erit

$$\int\frac{Pdz}{z}=\frac{z\cos.u}{1}+\frac{zz\cos.2u}{2}+\frac{z^3\cos.3u}{3}+\text{etc.}$$

et

$$\int Pdu=\frac{z\sin.u}{1}+\frac{zz\sin.2u}{2}+\frac{z^3\sin.3u}{3}+\text{etc.},$$

et quia est

$$Q=z\sin.u+zz\sin.2u+z^3\sin.3u+\text{etc.},$$

erit

$$\int\frac{Qdz}{z}=\frac{z\sin.u}{1}+\frac{zz\sin.2u}{2}+\frac{z^3\sin.3u}{3}+\text{etc.}$$

et

$$\int Qdu=-\frac{z\cos.u}{1}-\frac{zz\cos.2u}{2}-\frac{z^3\cos.3u}{3}+\text{etc.},$$

unde manifestum est fore

$$\int\frac{Pdz}{z}=-\int Qdu \quad\text{et}\quad \int\frac{Qdz}{z}=\int Pdu.$$

34. Quo hoc modo ulterius progredi liceat, statuamus brevitatis gratia

$$P'=\frac{z\cos.u}{1}+\frac{zz\cos.2u}{2}+\frac{z^3\cos.3u}{3}+\text{etc.},$$

$$P''=\frac{z\cos.u}{1^2}+\frac{zz\cos.2u}{2^2}+\frac{z^3\cos.3u}{3^2}+\text{etc.},$$

$$P'''=\frac{z\cos.u}{1^3}+\frac{zz\cos.2u}{2^3}+\frac{z^3\cos.3u}{3^3}+\text{etc.},$$

$$P''''=\frac{z\cos.u}{1^4}+\frac{zz\cos.2u}{2^4}+\frac{z^3\cos.3u}{3^4}+\text{etc.}$$

etc.

et

$$Q'=\frac{z\sin.u}{1}+\frac{zz\sin.2u}{2}+\frac{z^3\sin.3u}{3}+\text{etc.},$$

$$Q''=\frac{z\sin.u}{1^2}+\frac{zz\sin.2u}{2^2}+\frac{z^3\sin.3u}{3^2}+\text{etc.},$$

$$Q'''=\frac{z\sin.u}{1^3}+\frac{zz\sin.2u}{2^3}+\frac{z^3\sin.3u}{3^3}+\text{etc.},$$

$$Q''''=\frac{z\sin.u}{1^4}+\frac{zz\sin.2u}{2^4}+\frac{z^3\sin.3u}{3^4}+\text{etc.}$$

etc.

et hinc comparationes ante inventae continuabuntur

$$P'=\int\frac{Pdz}{z}=-\int Qdu,\qquad Q'=\int\frac{Qdz}{z}=\int Pdu,$$

$$P''=\int\frac{P'dz}{z}=-\int Q'du,\qquad Q''=\int\frac{Q'dz}{z}=\int P'du,$$

$$P'''=\int\frac{P''dz}{z}=-\int Q''du,\qquad Q'''=\int\frac{Q''dz}{z}=\int P''du,$$

$$P''''=\int\frac{P'''dz}{z}=-\int Q'''du\qquad Q''''=\int\frac{Q'''dz}{z}=\int P'''du$$

etc., etc.,

unde plures insignes relationes deduci possunt.

35. Maxime autem notatu dignae et ad nostrum institutum accommodatae sunt eae relationes, ubi formulae integrales, in quibus sola z est variabilis, reducuntur ad alias formulas integrales, in quibus sola u est variabilis, cuiusmodi sunt quae sequuntur:

$$P'=\int\frac{Pdz}{z}=-\int Qdu,$$

$$P''=\int\frac{dz}{z}\int\frac{Pdz}{z}=-\int du\int Pdu,$$

$$P'''=\int\frac{dz}{z}\int\frac{dz}{z}\int\frac{Pdz}{z}=+\int du\int du\int Qdu,$$

$$P''''=\int\frac{dz}{z}\int\frac{dz}{z}\int\frac{dz}{z}\int\frac{Pdz}{z}=+\int du\int du\int du\int Pdu,$$

$$P^{V}=\int\frac{dz}{z}\int\frac{dz}{z}\int\frac{dz}{z}\int\frac{dz}{z}\int\frac{Pdz}{z}=-\int du\int du\int du\int du\int Qdu$$

etc.

Similique modo pro altero genere

$$Q' = \int \frac{Qdz}{z} = +\int Pdu,$$

$$Q'' = \int \frac{dz}{z} \int \frac{Qdz}{z} = -\int du \int Qdu,$$

$$Q''' = \int \frac{dz}{z} \int \frac{dz}{z} \int \frac{Qdz}{z} = -\int du \int du \int Pdu,$$

$$Q'''' = \int \frac{dz}{z} \int \frac{dz}{z} \int \frac{dz}{z} \int \frac{Qdz}{z} = +\int du \int du \int du \int Qdu,$$

$$Q^{\mathrm{V}} = \int \frac{dz}{z} \int \frac{dz}{z} \int \frac{dz}{z} \int \frac{dz}{z} \int \frac{Qdz}{z} = +\int du \int du \int du \int du \int Pdu$$

etc.

36. Quodsi iam nostrarum serierum sive, quod eodem redit, quantitatum

$$P, P', P'', P''', P'''' \text{ etc. et } Q', Q'', Q''', Q'''' \text{ etc.}$$

eos tantum valores desideremus, quos adipiscuntur posito $z = 1$, hoc commodi assequimur, ut in formulis integralibus, ubi solus angulus u pro variabili habetur, statim ante integrationes ponere liceat $z = 1$; hoc autem facto erit

$$P = \frac{\cos. u - 1}{2 - 2\cos. u} = -\frac{1}{2} \quad \text{et} \quad Q = \frac{\sin. u}{2 - 2\cos. u} = \frac{1}{2}\cot. \frac{1}{2}u,$$

tum vero porro

$$\int Pdu = A - \frac{1}{2}u,$$

$$\int du \int Pdu = B + Au - \frac{1}{4}uu,$$

$$\int du \int du \int Pdu = C + Bu + \frac{1}{2}Auu - \frac{1}{12}u^3,$$

$$\int du \int du \int du \int Pdu = D + Cu + \frac{1}{2}Buu + \frac{1}{6}Au^3 - \frac{1}{48}u^4;$$

at pro formulis, ubi est Q, calculus non tam concinne succedit; erit enim

$$Q = \frac{1}{2}\cot. \frac{1}{2}u,$$

$$\int Qdu = l \sin. \frac{1}{2}u,$$

$$\int du \int Qdu = \int dul \sin. \frac{1}{2}u;$$

quae formula cum omnem integrationem respuat, vix ulterius progredi licet; interim tamen erit

$$\int du \int du \int Qdu = \int du \int dul \sin. \frac{1}{2}u,$$

$$\int du \int du \int du \int Qdu = \int du \int du \int dul \sin. \frac{1}{2}u.$$

37. Quod ad priores formulas variabilem z involventes attinet, per notas reductiones elicitur

$$\int \frac{P'dz}{z} = \int \frac{dz}{z} \int \frac{Pdz}{z} = lz \int \frac{Pdz}{z} - \int \frac{Pdz}{z} lz,$$

ubi prius membrum $lz \int Pdz$ evanescit posito $z = 1$, tum vero

$$\int \frac{dz}{z} \int \frac{P'dz}{z} = \int \frac{dz}{z} \int \frac{dz}{z} \int \frac{Pdz}{z} = +\int \frac{Pdz}{z} \frac{(lz)^2}{2},$$

quibus expressionibus ulterius exhibitis colligimus fore

$P' = +\int \frac{Pdz}{z}$	$Q' = +\int \frac{Qdz}{z}$
$P'' = -\int \frac{Pdz}{z} lz$	$Q'' = -\int \frac{Qdz}{z} lz$
$P''' = +\int \frac{Pdz}{z} \frac{(lz)^2}{1 \cdot 2}$	$Q''' = +\int \frac{Qdz}{z} \frac{(lz)^2}{1 \cdot 2}$
$P^{\mathrm{IV}} = -\int \frac{Pdz}{z} \frac{(lz)^3}{1 \cdot 2 \cdot 3}$	$Q^{\mathrm{IV}} = -\int \frac{Qdz}{z} \frac{(lz)^3}{1 \cdot 2 \cdot 3}$
etc.	etc.

38. Ex his igitur sequentium formularum integralium valores assignare possumus casu, quo $z = 1$.

$$P = -\frac{1}{2},$$

$$P' = \int \frac{Pdz}{z} = -l\sin.\frac{1}{2}u,$$

$$P'' = -\int \frac{Pdz}{z} lz = -B - Au + \frac{1}{4}uu,$$

$$P''' = +\int \frac{Pdz}{z}\cdot\frac{(lz)^2}{1\cdot 2} = \int du \int du\, l\sin.\frac{1}{2}u,$$

$$P^{\text{IV}} = -\int \frac{Pdz}{z}\cdot\frac{(lz)^3}{1\cdot 2\cdot 3} = D + Cu + \frac{1}{2}Buu + \frac{1}{6}Au^3 - \frac{1}{48}u^4,$$

$$P^{\text{V}} = +\int \frac{Pdz}{z}\cdot\frac{(lz)^4}{1\cdot 2\cdot 3\cdot 4} - \int du \int du \int du \int du\, l\sin.\frac{1}{2}u$$

etc.

Eodemque modo

$$Q = \frac{1}{2}\cot.\frac{1}{2}u,$$

$$Q' = \int \frac{Qdz}{z} - A - \frac{1}{2}u,$$

$$Q'' = -\int \frac{Qdz}{z}\cdot\frac{lz}{1} = -\int du\, l\sin.\frac{1}{2}u,$$

$$Q''' - +\int \frac{Qdz}{z}\cdot\frac{(lz)^2}{2} = -C - Bu - \frac{1}{2}Auu + \frac{1}{12}u^3,$$

$$Q^{\text{IV}} = -\int \frac{Qdz}{z}\cdot\frac{(lz)^3}{6} = \int du \int du \int du\, l\sin.\frac{1}{2}u,$$

$$Q^{\text{V}} = +\int \frac{Qdz}{z}\cdot\frac{(lz)^4}{24} = E + Du + \frac{1}{2}Cuu + \frac{1}{6}Bu^3 + \frac{1}{24}Au^4 - \frac{1}{240}u^5$$

etc.

39. Cum igitur sit

$$P = \frac{z\cos.u - zz}{1 - 2z\cos.u + zz} \quad \text{et} \quad Q = \frac{z\sin.u}{1 - 2z\cos.u + zz},$$

hactenus id sumus assecuti, ut harum duarum formularum integralium

$$\int \frac{dz(\cos.u - z)}{1 - 2z\cos.u + zz}(lz)^n \quad \text{et} \quad \int \frac{dz\sin.u}{1 - 2z\cos.u + zz}(lz)^n$$

valores casu $z = 1$ commode per angulum u assignare valeamus, si modo constaret, quo facto quantitates A, B, C, D etc. determinari oporteat, id quod vix alio modo nisi per ipsas series, unde hae quantitates sunt natae, fieri posse videtur.

40. Omissis igitur formulis integralibus, quae quantitatem Q involvunt, quippe quarum integratio minus succedit, alteras tantum consideremus et posito statim $z = 1$, ubi fit $P = -\frac{1}{2}$, ita ut sit

$$\cos.u + \cos.2u + \cos.3u + \cos.4u + \text{etc.} = -\frac{1}{2},$$

si per du multiplicemus et integremus, habebimus

$$Q' = \frac{\sin.u}{1} + \frac{\sin.2u}{2} + \frac{\sin.3u}{3} + \frac{\sin.4u}{4} + \frac{\sin.5u}{5} + \text{etc.} = A - \frac{1}{2}u,$$

quae constans nihilo aequalis videri potest, quia posito $u = 0$ summa seriei evanescere videtur; at sumto angulo u infinite parvo series praebebit

$$u + u + u + u + u + u + \text{etc. in infinitum};$$

notum autem est talem seriem summam finitam habere posse, unde hoc casu omisso statuamus $u = \pi$ seu potius $u = \pi + \omega$ prodibitque haec series existente ω angulo infinite parvo

$$-\omega + \omega - \omega + \omega - \omega + \omega - \omega + \text{etc.};$$

ubi quia signa alternantur, nullum est dubium, quin summa seriei evanescat; quae cum esse debeat $A - \frac{\pi}{2}$, evidens est fieri constantem $A = \frac{1}{2}\pi$, ita ut iam habeamus

$$Q' = \frac{\sin.u}{1} + \frac{\sin.2u}{2} + \frac{\sin.3u}{3} + \frac{\sin.4u}{4} + \frac{\sin.5u}{5} + \text{etc.} = \frac{\pi - u}{2}.$$

Hoc modo constantem determinandi Illustr. DANIEL BERNOULLI[1]) primus est usus, qui praeterea multa praeclara circa indolem harum serierum annotavit.

1) D. BERNOULLI, *De indole singulari serierum infinitarum, quas sinus vel cosinus angulorum arithmetice progredientium formant, earumque summatione et usu.* Novi Comment. acad. sc. Petrop. 17 (1772), 1773, p. 3. A. G.

41. Multiplicemus porro hanc ultimam seriem per $-du$ et integratio dabit

$$P'' = \frac{\cos. u}{1^2} + \frac{\cos. 2u}{2^2} + \frac{\cos. 3u}{3^2} + \frac{\cos. 4u}{4^2} + \text{etc.} = B - \frac{\pi u}{2} + \frac{uu}{4},$$

ad quam constantem inveniendam ponamus primo $u = 0$ fietque

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \text{etc.} = B.$$

Cuius seriei summam iam pridem[1]) primus demonstravi esse $= \frac{\pi\pi}{6}$; verum si haec veritas nobis esset ignota, egregia illa methodo a magno BERNOULLIO adhibita utamur ac ponamus $u = \pi$ eritque

$$-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{6^2} - \text{etc.} = B - \frac{\pi\pi}{2} + \frac{\pi\pi}{4} = B - \frac{\pi\pi}{4};$$

ambae hae series additae dabunt

$$\frac{2}{2^2} + \frac{2}{4^2} + \frac{2}{6^2} + \frac{2}{8^2} + \text{etc.} = 2B - \frac{\pi\pi}{4},$$

cuius duplum praebet

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \text{etc.} = 4B - \frac{\pi\pi}{2} = B,$$

unde colligitur $B = \frac{\pi\pi}{6}$, ita ut sit

$$P'' = \frac{\cos. u}{1^2} + \frac{\cos. 2u}{2^2} + \frac{\cos. 3u}{3^2} + \frac{\cos. 4u}{4^2} + \text{etc.} = \frac{\pi\pi}{6} - \frac{\pi u}{2} + \frac{uu}{4}.$$

42. Eodem modo ulterius progrediamur et denuo per du multiplicando et integrando adipiscimur

$$Q''' = \frac{\sin. u}{1^3} + \frac{\sin. 2u}{2^3} + \frac{\sin. 3u}{3^3} + \frac{\sin. 4u}{4^3} + \text{etc.} = C + \frac{\pi\pi u}{6} - \frac{\pi uu}{4} + \frac{u^3}{12};$$

ubi si statuatur $u = 0$, summa seriei manifesto evanescit; prodiret enim posito $u = \omega$

$$\frac{\omega}{1^2} + \frac{\omega}{2^2} + \frac{\omega}{3^2} + \frac{\omega}{4^2} + \text{etc.} = \frac{\omega\pi\pi}{6},$$

1) Vide notam p. 392. A. G.

quae ob $\omega = 0$ fit $= 0$, sicque erit $C = 0$ ideoque

$$Q''' = \frac{\sin. u}{1^3} + \frac{\sin. 2u}{2^3} + \frac{\sin. 3u}{3^3} + \frac{\sin. 4u}{4^3} + \text{etc.} = \frac{\pi\pi u}{6} - \frac{\pi uu}{4} + \frac{u^3}{12}.$$

43. Ducatur haec series in $-du$ et integratio praebebit

$$P^{\text{IV}} = \frac{\cos. u}{1^4} + \frac{\cos. 2u}{2^4} + \frac{\cos. 3u}{3^4} + \frac{\cos. 4u}{4^4} + \text{etc.} = D - \frac{\pi\pi uu}{12} + \frac{\pi u^3}{12} - \frac{u^4}{48};$$

hinc sumto $u = 0$ fiet

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \text{etc.} = D;$$

nunc vero fiat etiam $u = \pi$ fietque

$$-\frac{1}{1^4} + \frac{1}{2^4} - \frac{1}{3^4} + \frac{1}{4^4} - \frac{1}{5^4} + \text{etc.} = D - \frac{\pi^4}{48};$$

hae autem ambae series additae dant

$$\frac{2}{2^4} + \frac{2}{4^4} + \frac{2}{6^4} + \frac{2}{8^4} + \text{etc.} = 2D - \frac{\pi^4}{48},$$

quae octies sumta, ut numeratores fiant $= 2^4$, praebebit

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \text{etc.} = 16D - \frac{\pi^4}{6},$$

unde oritur $D = \frac{\pi^4}{90}$, quae est eadem summa seriei

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \text{etc.},$$

quam iam dudum[1]) inveneram; habebimus iam

$$P'''' = \frac{\cos. u}{1^4} + \frac{\cos. 2u}{2^4} + \frac{\cos. 3u}{3^4} + \frac{\cos. 4u}{4^4} + \text{etc.} = \frac{\pi^4}{90} - \frac{\pi^2 u^2}{12} + \frac{\pi u^3}{12} - \frac{u^4}{48}.$$

1) Vide notam p. 392. A. G.

44. Multiplicando iterum per du et integrando consequimur

$$Q^{\mathrm{v}} = \frac{\sin. u}{1^5} + \frac{\sin. 2u}{2^5} + \frac{\sin. 3u}{3^5} + \frac{\sin. 4u}{4^5} + \text{etc.} = E + \frac{\pi^4 u}{90} - \frac{\pi^2 u^3}{36} + \frac{\pi u^4}{48} - \frac{u^5}{240},$$

ubi uti in casu penultimo constans E iterum fit $=0$, ita ut habeamus

$$Q^{\mathrm{v}} = \frac{\sin. u}{1^5} + \frac{\sin. 2u}{2^5} + \frac{\sin. 3u}{3^5} + \frac{\sin. 4u}{4^5} + \text{etc.} = \frac{\pi^4 u}{90} - \frac{\pi^2 u^3}{36} + \frac{\pi u^4}{48} - \frac{u^5}{240}.$$

45. Multiplicemus denuo per $-du$ prodibitque integrando

$$P^{\mathrm{vi}} = \frac{\cos. u}{1^6} + \frac{\cos. 2u}{2^6} + \frac{\cos. 3u}{3^6} + \frac{\cos. 4u}{4^6} + \text{etc.}$$
$$= F - \frac{\pi^4}{90} \cdot \frac{uu}{2} + \frac{\pi\pi}{6} \cdot \frac{u^4}{24} - \frac{\pi}{2} \cdot \frac{u^5}{120} + \frac{1}{2} \cdot \frac{u^6}{720},$$

ubi ad constantem determinandam ponatur $u=0$ eritque

$$\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \text{etc.} = F;$$

tum vero sumatur $u = \pi$ et fiet

$$-\frac{1}{1^6} + \frac{1}{2^6} - \frac{1}{3^6} + \frac{1}{4^6} - \text{etc.} = F - \frac{\pi^6}{480},$$

quae additae dant

$$\frac{2}{2^6} + \frac{2}{4^6} + \frac{2}{6^6} + \frac{2}{8^6} + \text{etc.} = 2F - \frac{\pi^6}{480},$$

quae multiplicetur per 32, ut omnes numeratores fiant $64 = 2^6$, et orietur

$$\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \text{etc.} = 64F - \frac{\pi^6}{15} = F,$$

unde colligitur $F = \frac{\pi^6}{945}$, ita ut sit

$$P^{\mathrm{vi}} = \frac{\cos. u}{1^6} + \frac{\cos. 2u}{2^6} + \frac{\cos. 3u}{3^6} + \frac{\cos. 4u}{4^6} + \text{etc.}$$
$$= \frac{\pi^6}{945} - \frac{\pi^4}{90} \cdot \frac{u^2}{2} + \frac{\pi^2}{6} \cdot \frac{u^4}{24} - \frac{\pi}{2} \cdot \frac{u^5}{120} + \frac{1}{2} \cdot \frac{u^6}{720}.$$

46. Has series ulterius continuare superfluum foret, cum lex progressionis iam satis sit manifesta, praecipue si in subsidium vocentur summationes potestatum reciprocarum parium, quas olim[1]) usque ad potestatem trigesimam supputatas dedi. Quod quo clarius perspiciatur, istas summas sequenti modo repraesentemus

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \text{etc.} = \alpha\pi\pi, \quad \text{ut sit} \quad \alpha = \frac{1}{6},$$
$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \text{etc.} = \beta\pi^4, \quad \text{ut sit} \quad \beta = \frac{1}{90},$$
$$\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \frac{1}{6^6} + \text{etc.} = \gamma\pi^6, \quad \text{ut sit} \quad \gamma = \frac{1}{945},$$
$$\frac{1}{1^8} + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} + \frac{1}{6^8} + \text{etc.} = \delta\pi^8, \quad \text{ut sit} \quad \delta = \frac{1}{9450},$$
etc.

atque his positis sequentes habebimus integrationes, pro casu scilicet $z=1$,

$$Q' = \int \frac{dz \sin. u}{1 - 2z\cos. u + zz} = \frac{1}{2}\pi - \frac{1}{2}u = \text{A tang.} \frac{\sin. u}{1 - \cos. u},$$
$$P'' = -\int \frac{dz(\cos. u - z)}{1 - 2z\cos. u + zz} \cdot \frac{lz}{1} = \alpha\pi\pi - \frac{1}{2}\pi u + \frac{1}{2} \cdot \frac{uu}{2},$$
$$Q''' = +\int \frac{dz \sin. u}{1 - 2z\cos. u + zz} \cdot \frac{(lz)^2}{2} = \alpha\pi\pi\frac{u}{1} - \frac{1}{2}\pi\frac{uu}{2} + \frac{1}{2} \cdot \frac{u^3}{6},$$
$$P^{\mathrm{iv}} = -\int \frac{dz(\cos. u - z)}{1 - 2z\cos. u + zz} \cdot \frac{(lz)^3}{6} = \beta\pi^4 - \alpha\pi\pi\frac{uu}{2} + \frac{1}{2}\pi\frac{u^3}{6} - \frac{1}{2} \cdot \frac{u^4}{24},$$
$$Q^{\mathrm{v}} = +\int \frac{dz \sin. u}{1 - 2z\cos. u + zz} \cdot \frac{(lz)^4}{24} = \beta\pi^4\frac{u}{1} - \alpha\pi\pi\frac{u^3}{6} + \frac{1}{2}\pi\frac{u^4}{24} - \frac{1}{2} \cdot \frac{u^5}{120},$$
$$P^{\mathrm{vi}} = -\int \frac{dz(\cos. u - z)}{1 - 2z\cos. u + zz} \cdot \frac{(lz)^5}{120} = \gamma\pi^6 - \beta\pi^4\frac{uu}{2} + \alpha\pi\pi\frac{u^4}{24} - \frac{1}{2}\pi\frac{u^5}{120} + \frac{1}{2} \cdot \frac{u^6}{720},$$
$$Q^{\mathrm{vii}} = +\int \frac{dz \sin. u}{1 - 2z\cos. u + zz} \cdot \frac{(lz)^6}{720} = \gamma\pi^6\frac{u}{1} - \beta\pi^4\frac{u^3}{6} + \alpha\pi\pi\frac{u^5}{120} - \frac{1}{2}\pi\frac{u^6}{720} + \frac{1}{2} \cdot \frac{u^7}{5040}$$
etc.

1) *Introductio in analysin infinitorum*, t. I cap. 15, ubi EULERUS summationes adeo usque ad potestatem trigesimam sextam supputatas dedit; *LEONHARDI EULERI Opera omnia*, series I, vol. 8. A. G.

47. Operae pretium erit aliquos casus, quibus angulo u datus valor tribuitur, ob oculos exponere. Ponamus igitur $u=0$, quo casu formulae nostrae alternatim evanescunt, reliquae vero praebebunt

$$-\int \frac{dz}{1-z}\, lz \quad = \alpha\pi\pi = \frac{\pi\pi}{6},$$

$$-\int \frac{dz}{1-z}\cdot\frac{(lz)^3}{6} = \beta\pi^4 = \frac{\pi^4}{90},$$

$$-\int \frac{dz}{1-z}\cdot\frac{(lz)^5}{120} = \gamma\pi^6 = \frac{\pi^6}{945};$$

his affines sunt formulae, quae oriuntur ex positione $u=\pi$, ubi iterum abeunt alternae sinum u involventes et remanebunt sequentes

$$\int \frac{dz}{1+z}\, lz \quad = -\frac{\pi\pi}{12} = -\frac{1}{2}\alpha\pi\pi,$$

$$\int \frac{dz}{1+z}\cdot\frac{(lz)^3}{6} = -\frac{7\pi^4}{720} = -\frac{7}{8}\beta\pi^4,$$

$$\int \frac{dz}{1+z}\cdot\frac{(lz)^5}{120} = -\frac{31}{32}\gamma\pi^6,$$

$$\int \frac{dz}{1+z}\cdot\frac{(lz)^7}{5040} = -\frac{127}{128}\delta\pi^8.$$

48. Hic notatu dignum occurrit, quod valores alterni, quos hic omisimus, etiam evanescant posito $u=\pi$; deinde non minus notatu dignum est easdem formulas quoque evanescere posito $u=2\pi$, sola prima excepta, quippe quae etiam non evanescit posito $u=0$; reliquae vero, scilicet tertia, quinta, septima etc., certe evanescunt casibus $u=0$ et $u=\pi$, quin etiam $u=2\pi$. Quod quo clarius appareat, has formulas per factores repraesentemus eritque tertiae valor

$$=\frac{1}{12}u(\pi-u)(2\pi-u),$$

quintae vero valor reperitur

$$\frac{u}{720}(\pi-u)(2\pi-u)(4\pi\pi+6\pi u-3uu),$$

quod etiam in sequentibus usu venit. In genere autem observari meretur omnes nostras formulas sola prima excepta eosdem sortiri valores, sive ponatur $u=0$ sive $u=2\pi$, quippe quibus tam idem sinus quam cosinus respondet. Videtur quidem eundem consensum locum habere debere, si ponatur $u=4\pi$ et $u=6\pi$; verum Illustr. BERNOULLIUS iam luculenter ostendit angulum u in his valoribus non ultra quatuor rectos augeri posse. Huiusmodi autem anomalia etiam in omnibus vulgaribus seriebus, quibus arcus exprimuntur, occurrit atque adeo in LEIBNIZIANA[1]), in qua est

$$u=\frac{\text{tang.}\,u}{1}-\frac{(\text{tang.}\,u)^3}{3}+\frac{(\text{tang.}\,u)^5}{5}-\frac{(\text{tang.}\,u)^7}{7}+\frac{(\text{tang.}\,u)^9}{9}-\text{etc.},$$

angulum u non ultra 180° augere licet. Si enim poneremus $u=180^\circ+u$, foret utique $\text{tang.}\,u=\text{tang.}\,u$ neque tamen series illa exprimeret arcum $\pi+u$, sed tantum arcum u, cuiusmodi phaenomena etiam in aliis similibus seriebus locum habent. Quod autem prima series hinc plerumque excipi debeat, ratio in eo est sita, quod in formula integrali posito $u=0$ denominator fiat $1-z$, qui casu $z=1$ evanescit, ideoque formula in infinitum excrescit, id quod in sequentibus, quae per lz sunt multiplicatae, non amplius evenit, quia $\frac{lz}{1-z}$ casu $z=1$ non amplius fit infinitus, sed tantum $=-1$, et si maior potestas logarithmi adsit, fit adeo $=0$.

49. Ponamus nunc etiam $u=90^\circ$ seu $u=\frac{\pi}{2}$, ut sit $\cos.\,u=0$ et $\sin.\,u=1$, hocque casu omnes formulae generales sequentes obtinebunt valores

$$\int \frac{dz}{1+zz}=\frac{\pi}{4},$$

$$\int \frac{zdz}{1+zz}\, lz=-\frac{\pi\pi}{48},$$

$$\int \frac{dz}{1+zz}\cdot\frac{(lz)^2}{2}=\frac{\pi^3}{32},$$

$$\int \frac{zdz}{1+zz}\cdot\frac{(lz)^3}{6}=-\frac{7\pi^4}{90\cdot 128}.$$

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1) Vide epistolas scriptas ab OLDENBURG ad LEIBNIZ d. 12. Apr. 1675, a LEIBNIZ ad OLDENBURG d. 27. Aug. 1676 et a NEWTON ad LEIBNIZ d. 24. Oct. 1676; *LEIBNIZ mathematische Schriften* 1. Abt. Bd. 1 (1849), p. 62, 114, 144; vide porro ibidem Bd. 5 (1858), p. 81: *De quadratura arithmetica circuli, ellipseos et hyperbolae.* A. G.

50. Consideremus etiam casum $u = 60^\circ$ sive $u = \frac{\pi}{3}$, ut sit $\cos. u = \frac{1}{2}$ et $\sin. u = \frac{\sqrt{3}}{2}$, et formulae generales perducent ad sequentia integralia

$$\frac{\sqrt{3}}{2}\int\frac{dz}{1-z+zz} = \frac{\pi}{3},$$

$$-\frac{1}{2}\int\frac{dz(1-2z)}{1-z+zz}lz = \frac{\pi\pi}{36},$$

$$\frac{\sqrt{3}}{2}\int\frac{dz}{1-z+zz}\cdot\frac{(lz)^2}{2} = \frac{5\pi^3}{162}.$$

Simili modo si ponamus $u = 120^\circ = \frac{2\pi}{3}$, ut sit $\cos. u = -\frac{1}{2}$ et $\sin. u = \frac{\sqrt{3}}{2}$, sequentes integrationes istis affines prodibunt

$$\frac{\sqrt{3}}{2}\int\frac{dz}{1+z+zz} = \frac{\pi}{6},$$

$$\frac{1}{2}\int\frac{dz(1+2z)}{1+z+zz}lz = -\frac{\pi\pi}{18},$$

$$\frac{\sqrt{3}}{2}\int\frac{dz}{1+z+zz}\cdot\frac{(lz)^2}{2} = \frac{2\pi^3}{81};$$

sicque pro lubitu numerus huiusmodi integrationum specialium augeri poterit.

51. Quemadmodum istae integrationes memorabiles ex priore serie nostra P posito $z = 1$ sunt deductae, ita eodem modo alteram seriem Q pertractemus. Cum igitur sit

$$Q = \sin. u + \sin. 2u + \sin. 3u + \sin. 4u + \text{etc.} = \frac{1}{2}\cot.\frac{1}{2}u,$$

si per $-du$ multiplicemus et integremus, reperitur series

$$P' = \frac{\cos. u}{1} + \frac{\cos. 2u}{2} + \frac{\cos. 3u}{3} + \frac{\cos. 4u}{4} + \text{etc.} = -l\sin.\frac{1}{2}u + A,$$

pro qua constante determinanda ponatur $u = \pi$, ut sit

$$-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \text{etc.} = A,$$

quocirca fit $A = -l2$, ita ut habeamus

$$P' = \frac{\cos. u}{1} + \frac{\cos. 2u}{2} + \frac{\cos. 3u}{3} + \frac{\cos. 4u}{4} + \text{etc.} = -l2\sin.\frac{1}{2}u,$$

pro quo valore scribamus brevitatis gratia $\varDelta : u$, siquidem eum spectamus tanquam certam ipsius u functionem, ita ut sit $P' = \varDelta : u$.

52. Multiplicando porro per du et integrando nanciscimur hanc seriem

$$Q' = \frac{\sin. u}{1^2} + \frac{\sin. 2u}{2^2} + \frac{\sin. 3u}{3^2} + \frac{\sin. 4u}{4^2} + \text{etc.} = \int du\varDelta : u = \varDelta' : u,$$

ubi haec formula integralis involvet certam constantem, quam facile definire licet ex casu $u = 0$; quia enim series evanescit, fieri debet $\varDelta' : 0 = 0$ sicque integratio plene determinatur.

53. Si eodem modo ulterius progrediamur multiplicando per $-du$, prodibit haec series

$$P''' = \frac{\cos. u}{1^3} + \frac{\cos. 2u}{2^3} + \frac{\cos. 3u}{3^3} + \frac{\cos. 4u}{4^3} + \text{etc.} = -\int du\varDelta' : u = \varDelta'' : u.$$

Iam ad constantem, quae in hac expressione continetur, definiendam sit I° $u = 0$ eritque

$$\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \text{etc.} = \varDelta'' : 0;$$

II° sit $u = \pi$ et fiet

$$-\frac{1}{1^3} + \frac{1}{2^3} - \frac{1}{3^3} + \frac{1}{4^3} - \frac{1}{5^3} + \text{etc.} = \varDelta'' : \pi,$$

quibus additis prodit

$$\frac{2}{2^3} + \frac{2}{4^3} + \frac{2}{6^3} + \frac{2}{8^3} + \text{etc.} = \varDelta''0 + \varDelta'' : \pi,$$

hacque quater sumta erit

$$\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \text{etc.} = 4\varDelta'' : 0 + 4\varDelta'' : \pi = \varDelta'' : 0,$$

unde oritur $3\varDelta'' : 0 + 4\varDelta'' : \pi = 0$, ex qua constans in formulam nostram integralem $\varDelta'' : u = -\int du\varDelta' : u$ ingressa determinari debet.

54. Multiplicemus denuo per du et integremus prodibitque

$$Q^{\mathrm{IV}} = \frac{\sin. u}{1^4} + \frac{\sin. 2u}{2^4} + \frac{\sin. 3u}{3^4} + \frac{\sin. 4u}{4^4} + \text{etc.} = \int du\,\Delta'' : u = \Delta''' : u$$

atque haec functio $\Delta''' : u$ ita debet determinari, ut evanescat sumto $u = 0$ sive ut fiat $\Delta''' : 0 = 0$. Eodem modo ulterius progrediendo fiet

$$P^{\mathrm{V}} = \frac{\cos. u}{1^5} + \frac{\cos. 2u}{2^5} + \frac{\cos. 3u}{3^5} + \frac{\cos. 4u}{4^5} + \text{etc.} = -\int du\,\Delta''' : u = \Delta^{\mathrm{IV}} : u$$

huiusque functionis indoles sequenti modo determinabitur. Ponatur scilicet ut hactenus $u = 0$ et $u = \pi$ eritque

$$\frac{1}{1^5} + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \frac{1}{5^5} + \text{etc.} = \Delta^{\mathrm{IV}} : 0$$

et

$$-\frac{1}{1^5} + \frac{1}{2^5} - \frac{1}{3^5} + \frac{1}{4^5} - \frac{1}{5^5} + \text{etc.} = \Delta^{\mathrm{IV}} : \pi,$$

hinc addendo

$$\frac{2}{2^5} + \frac{2}{4^5} + \frac{2}{6^5} + \frac{2}{8^5} + \text{etc.} = \Delta^{\mathrm{IV}} : 0 + \Delta^{\mathrm{IV}} : \pi$$

et multiplicando per 16

$$\frac{1}{1^5} + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \text{etc.} = 16\Delta^{\mathrm{IV}} : 0 + 16\Delta^{\mathrm{IV}} : \pi = \Delta^{\mathrm{IV}} : 0$$

sicque fieri debebit $15\Delta^{\mathrm{IV}} : 0 + 16\Delta^{\mathrm{IV}} : \pi = 0$ etc.

55. Hinc igitur sequentes adipiscemur integrationes pro casu $z = 1$

$$\text{I.}\quad -\int \frac{dz(\cos. u - z)}{1 - 2z\cos. u + zz} = -l2\sin. \frac{1}{2}u = \Delta : u,$$

$$\text{II.}\quad \int \frac{dz \sin. u}{1 - 2z\cos. u + zz}\, lz = \int du\,\Delta : u = \Delta' : u,$$

$$\text{III.}\quad -\int \frac{dz(\cos. u - z)}{1 - 2z\cos. u + zz} \cdot \frac{(lz)^2}{2} = -\int du\,\Delta' : u = \Delta'' : u,$$

$$\text{IV.}\quad \int \frac{dz \sin. u}{1 - 2z\cos. u + zz} \cdot \frac{(lz)^3}{6} = \int du\,\Delta'' : u = \Delta''' : u,$$

$$\text{V.}\quad -\int \frac{dz(\cos. u - z)}{1 - 2z\cos. u + zz} \cdot \frac{(lz)^4}{24} = -\int du\,\Delta''' : u = \Delta^{\mathrm{IV}} : u,$$

$$\text{VI.}\quad \int \frac{dz \sin. u}{1 - 2z\cos. u + zz} \cdot \frac{(lz)^5}{120} = \int du\,\Delta^{\mathrm{IV}} : u = \Delta^{\mathrm{V}} : u$$

etc.

Has autem expressiones facile, quousque libuerit, continuare licet, si modo integratio cuiusque integralis rite instituatur; conditiones autem, quas impleri oportet, sequenti modo referri possunt.

$\Delta' : 0 = 0$	$3\Delta'' : 0 + 4\Delta'' : \pi = 0$
$\Delta''' : 0 = 0$	$15\Delta^{\mathrm{IV}} : 0 + 16\Delta^{\mathrm{IV}} : \pi = 0$
$\Delta^{\mathrm{V}} : 0 = 0$	$63\Delta^{\mathrm{VI}} : 0 + 64\Delta^{\mathrm{VI}} : \pi = 0$
$\Delta^{\mathrm{VII}} : 0 = 0$	$255\Delta^{\mathrm{VIII}} : 0 + 256\Delta^{\mathrm{VIII}} : \pi = 0$
etc.	etc.

Caeterum quia posteriores integrationes absolvere non licet, hinc parum utilitatis exspectare possumus.

56. Caeterum methodus, qua hic sumus usi ad constantes per quamque integrationem ingressas determinandas, a celeberrimo BERNOULLIO[1]) primum est adhibita atque eo maiori attentione digna est aestimanda, quod eius ope summationes meae serierum reciprocarum potestatum obtineri possunt, quandoquidem credideram eas non aliter nisi ex consideratione infinitorum arcuum, qui vel eodem sinu vel cosinu gaudent, demonstrari posse.

1) Vide notam p. 447. A. G.