CAPUT XI

DE ALIIS ARCUUM ATQUE SINUUM EXPRESSIONIBUS INFINITIS

184. Quoniam supra (§ 158) denotante z arcum circuli quemcunque vidimus esse

$$\sin. z = z\left(1-\frac{zz}{\pi\pi}\right)\left(1-\frac{zz}{4\pi\pi}\right)\left(1-\frac{zz}{9\pi\pi}\right)\left(1-\frac{zz}{16\pi\pi}\right) \text{ etc.}$$

et

$$\cos. z = \left(1-\frac{4zz}{\pi\pi}\right)\left(1-\frac{4zz}{9\pi\pi}\right)\left(1-\frac{4zz}{25\pi\pi}\right)\left(1-\frac{4zz}{49\pi\pi}\right) \text{ etc.},$$

ponamus esse arcum $z=\frac{m\pi}{n}$; erit

$$\sin. \frac{m\pi}{n} = \frac{m\pi}{n}\left(1-\frac{mm}{nn}\right)\left(1-\frac{mm}{4nn}\right)\left(1-\frac{mm}{9nn}\right)\left(1-\frac{mm}{16nn}\right) \text{ etc.}$$

et

$$\cos. \frac{m\pi}{n} = \left(1-\frac{4mm}{nn}\right)\left(1-\frac{4mm}{9nn}\right)\left(1-\frac{4mm}{25nn}\right)\left(1-\frac{4mm}{49nn}\right) \text{ etc.}$$

Vel ponatur $2n$ loco n, ut prodeant hae expressiones

$$\sin. \frac{m\pi}{2n} = \frac{m\pi}{2n}\cdot\frac{4nn-mm}{4nn}\cdot\frac{16nn-mm}{16nn}\cdot\frac{36nn-mm}{36nn}\cdot \text{etc.},$$

$$\cos. \frac{m\pi}{2n} = \frac{nn-mm}{nn}\cdot\frac{9nn-mm}{9nn}\cdot\frac{25nn-mm}{25nn}\cdot\frac{49nn-mm}{49nn}\cdot \text{etc.},$$

quae in factores simplices resolutae dant

$$\sin. \frac{m\pi}{2n} = \frac{m\pi}{2n}\cdot\frac{2n-m}{2n}\cdot\frac{2n+m}{2n}\cdot\frac{4n-m}{4n}\cdot\frac{4n+m}{4n}\cdot\frac{6n-m}{6n}\cdot \text{etc.},$$

$$\cos. \frac{m\pi}{2n} = \frac{n-m}{n}\cdot\frac{n+m}{n}\cdot\frac{3n-m}{3n}\cdot\frac{3n+m}{3n}\cdot\frac{5n-m}{5n}\cdot\frac{5n+m}{5n}\cdot \text{etc.}$$

Ponatur $n-m$ loco m; quia est

$$\sin. \frac{(n-m)\pi}{2n} = \cos. \frac{m\pi}{2n} \quad \text{et} \quad \cos. \frac{(n-m)\pi}{2n} = \sin. \frac{m\pi}{2n},$$

provenient hae expressiones

$$\cos. \frac{m\pi}{2n} = \frac{(n-m)\pi}{2n}\cdot\frac{n+m}{2n}\cdot\frac{3n-m}{2n}\cdot\frac{3n+m}{4n}\cdot\frac{5n-m}{4n}\cdot\frac{5n+m}{6n}\cdot \text{etc.},$$

$$\sin. \frac{m\pi}{2n} = \frac{m}{n}\cdot\frac{2n-m}{n}\cdot\frac{2n+m}{3n}\cdot\frac{4n-m}{3n}\cdot\frac{4n+m}{5n}\cdot\frac{6n-m}{5n}\cdot \text{etc.}$$

185. Cum igitur pro sinu et cosinu anguli $\frac{m\pi}{2n}$ binae habeantur expressiones, si eae inter se comparentur dividendo, erit

$$1 = \frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdot\frac{7}{8}\cdot\frac{9}{8}\cdot \text{etc.}$$

ideoque

$$\frac{\pi}{2} = \frac{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8\cdot10\cdot10\cdot12\cdot12\cdot\text{etc.}}{1\cdot3\cdot3\cdot5\cdot5\cdot7\cdot7\cdot9\cdot9\cdot11\cdot11\cdot13\cdot\text{etc.}},$$

quae est expressio pro peripheria circuli, quam WALLISIUS invenit in *Arithmetica infinitorum.*[1]) Similes autem huic innumeras expressiones exhibere licet ope primae expressionis pro sinu; ex ea enim deducitur fore

$$\frac{\pi}{2} = \frac{n}{m}\sin. \frac{m\pi}{2n}\cdot\frac{2n}{2n-m}\cdot\frac{2n}{2n+m}\cdot\frac{4n}{4n-m}\cdot\frac{4n}{4n+m}\cdot\frac{6n}{6n-m}\cdot \text{etc.},$$

1) J. WALLIS, *Arithmetica infinitorum sive nova methodus inquirendi in curvilineorum quadraturam aliaque difficiliora Matheseos problemata*, Oxoniae 1655; *Opera mathematica*, t. I, Oxoniae 1695, p. 355, imprimis p. 469. A. K.

quae posito $\frac{m}{n} = 1$ praebet illam ipsam WALLISII formulam. Sit ergo $\frac{m}{n} = \frac{1}{2}$; ob $\sin. \frac{1}{4}\pi = \frac{1}{\sqrt{2}}$ erit

$$\frac{\pi}{2} = \frac{\sqrt{2}}{1} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdot \frac{12}{11} \cdot \frac{12}{13} \cdot \frac{16}{15} \cdot \frac{16}{17} \cdot \text{etc.}$$

Sit $\frac{m}{n} = \frac{1}{3}$; ob $\sin. \frac{1}{6}\pi = \frac{1}{2}$ erit

$$\frac{\pi}{2} = \frac{3}{2} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{12}{11} \cdot \frac{12}{13} \cdot \frac{18}{17} \cdot \frac{18}{19} \cdot \frac{24}{23} \cdot \text{etc.}$$

Quodsi expressio WALLISIANA dividatur per illam, ubi $\frac{m}{n} = \frac{1}{2}$, erit

$$\sqrt{2} = \frac{2 \cdot 2 \cdot 6 \cdot 6 \cdot 10 \cdot 10 \cdot 14 \cdot 14 \cdot 18 \cdot 18 \cdot \text{etc.}}{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot 15 \cdot 17 \cdot 19 \cdot \text{etc.}}.$$

186. Quoniam tangens cuiusque anguli aequatur sinui per cosinum diviso, tangens quoque per huiusmodi factores infinitos exprimi poterit. Quodsi autem prima sinus expressio dividatur per alteram cosinus expressionem, erit

$$\text{tang.} \frac{m\pi}{2n} = \frac{m}{n-m} \cdot \frac{2n-m}{n+m} \cdot \frac{2n+m}{3n-m} \cdot \frac{4n-m}{3n+m} \cdot \frac{4n+m}{5n-m} \cdot \text{etc.},$$

$$\text{cot.} \frac{m\pi}{2n} = \frac{n-m}{m} \cdot \frac{n+m}{2n-m} \cdot \frac{3n-m}{2n+m} \cdot \frac{3n+m}{4n-m} \cdot \frac{5n-m}{4n+m} \cdot \text{etc.}$$

Simili modo autem secantes et cosecantes exprimentur

$$\text{sec.} \frac{m\pi}{2n} = \frac{n}{n-m} \cdot \frac{n}{n+m} \cdot \frac{3n}{3n-m} \cdot \frac{3n}{3n+m} \cdot \frac{5n}{5n-m} \cdot \frac{5n}{5n+m} \cdot \text{etc.},$$

$$\text{cosec.} \frac{m\pi}{2n} = \frac{n}{m} \cdot \frac{n}{2n-m} \cdot \frac{3n}{2n+m} \cdot \frac{3n}{4n-m} \cdot \frac{5n}{4n+m} \cdot \frac{5n}{6n-m} \cdot \text{etc.}$$

Sin autem alterae sinuum et cosinuum formulae combinentur, erit

$$\text{tang.} \frac{m\pi}{2n} = \frac{\pi}{2} \cdot \frac{m}{n-m} \cdot \frac{1(2n-m)}{2(n+m)} \cdot \frac{3(2n+m)}{2(3n-m)} \cdot \frac{3(4n-m)}{4(3n+m)} \cdot \text{etc.},$$

$$\text{cot.} \frac{m\pi}{2n} = \frac{\pi}{2} \cdot \frac{n-m}{m} \cdot \frac{1(n+m)}{2(2n-m)} \cdot \frac{3(3n-m)}{2(2n+m)} \cdot \frac{3(3n+m)}{4(4n-m)} \cdot \text{etc.},$$

$$\text{sec.} \frac{m\pi}{2n} = \frac{2}{\pi} \cdot \frac{n}{n-m} \cdot \frac{2n}{n+m} \cdot \frac{2n}{3n-m} \cdot \frac{4n}{3n+m} \cdot \frac{4n}{5n-m} \cdot \text{etc.},$$

$$\text{cosec.} \frac{m\pi}{2n} = \frac{2}{\pi} \cdot \frac{n}{m} \cdot \frac{2n}{2n-m} \cdot \frac{2n}{2n+m} \cdot \frac{4n}{4n-m} \cdot \frac{4n}{4n+m} \cdot \text{etc.}$$

187. Si loco m scribatur k similique modo anguli $\frac{k\pi}{2n}$ sinus et cosinus definiantur ac per has expressiones illae priores dividantur, prodibunt istae formulae

$$\frac{\sin. \frac{m\pi}{2n}}{\sin. \frac{k\pi}{2n}} = \frac{m}{k} \cdot \frac{2n-m}{2n-k} \cdot \frac{2n+m}{2n+k} \cdot \frac{4n-m}{4n-k} \cdot \frac{4n+m}{4n+k} \cdot \text{etc.},$$

$$\frac{\sin. \frac{m\pi}{2n}}{\cos. \frac{k\pi}{2n}} = \frac{m}{n-k} \cdot \frac{2n-m}{n+k} \cdot \frac{2n+m}{3n-k} \cdot \frac{4n-m}{3n+k} \cdot \frac{4n+m}{5n-k} \cdot \text{etc.},$$

$$\frac{\cos. \frac{m\pi}{2n}}{\cos. \frac{k\pi}{2n}} = \frac{n-m}{n-k} \cdot \frac{n+m}{n+k} \cdot \frac{3n-m}{3n-k} \cdot \frac{3n+m}{3n+k} \cdot \frac{5n-m}{5n-k} \cdot \text{etc.},$$

$$\frac{\cos. \frac{m\pi}{2n}}{\sin. \frac{k\pi}{2n}} = \frac{n-m}{k} \cdot \frac{n+m}{2n-k} \cdot \frac{3n-m}{2n+k} \cdot \frac{3n+m}{4n-k} \cdot \frac{5n-m}{4n+k} \cdot \text{etc.}$$

Sumpto ergo pro $\frac{k\pi}{2n}$ eiusmodi angulo, cuius sinus et cosinus dentur, per hos licebit alius cuiuscunque anguli $\frac{m\pi}{2n}$ sinum et cosinum determinare.

188. Vicissim igitur huiusmodi expressionum, quae ex factoribus infinitis constant, valores veri vel per circuli peripheriam vel per sinus et cosinus angulorum datorum assignari possunt, quod ipsum non parvi est momenti, cum etiamnunc aliae methodi non constent, quarum ope huiusmodi productorum infinitorum valores exhiberi queant. Ceterum vero huiusmodi expressiones parum utilitatis afferunt ad valores cum ipsius π tum sinuum cosinuumve angulorum $\frac{m\pi}{2n}$ per approximationem eruendos. Quanquam enim isti factores

$$\frac{\pi}{2} = 2\left(1 - \frac{1}{9}\right)\left(1 - \frac{1}{25}\right)\left(1 - \frac{1}{49}\right) \text{ etc.}$$

in fractionibus decimalibus non difficulter in se multiplicantur, tamen nimis multi termini in computum duci deberent, si valorem ipsius π ad decem tantum figuras iustum invenire vellemus.

189. Praecipuus autem usus huiusmodi expressionum etsi infinitarum in inventione logarithmorum versatur, in quo negotio factorum utilitas tanta est, ut sine illis logarithmorum supputatio esset difficillima. Ac primo quidem, cum sit

$$\pi = 4\left(1-\frac{1}{9}\right)\left(1-\frac{1}{25}\right)\left(1-\frac{1}{49}\right) \text{ etc.},$$

erit sumendis logarithmis

$$l\pi = l4 + l\left(1-\frac{1}{9}\right) + l\left(1-\frac{1}{25}\right) + l\left(1-\frac{1}{49}\right) + \text{etc.}$$

vel

$$l\pi = l2 - l\left(1-\frac{1}{4}\right) - l\left(1-\frac{1}{16}\right) - l\left(1-\frac{1}{36}\right) - \text{etc.},$$

sive logarithmi communes sive hyperbolici sumantur. Quoniam vero ex logarithmis hyperbolicis vulgares facile reperiuntur, insigne compendium adhiberi poterit ad logarithmum hyperbolicum ipsius π inveniendum.

190. Cum igitur logarithmis hyperbolicis sumendis sit

$$l(1-x) = -x - \frac{xx}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \text{etc.},$$

si hoc modo singuli termini evolvantur, erit

$$\begin{aligned} l\pi = l4 &- \frac{1}{9} - \frac{1}{2\cdot 9^2} - \frac{1}{3\cdot 9^3} - \frac{1}{4\cdot 9^4} - \text{etc.} \\ &- \frac{1}{25} - \frac{1}{2\cdot 25^2} - \frac{1}{3\cdot 25^3} - \frac{1}{4\cdot 25^4} - \text{etc.} \\ &- \frac{1}{49} - \frac{1}{2\cdot 49^2} - \frac{1}{3\cdot 49^3} - \frac{1}{4\cdot 49^4} - \text{etc.} \\ &\text{etc.} \end{aligned}$$

In his seriebus numero infinitis verticaliter descendendo eiusmodi prodeunt series, quarum summas supra [§ 169, 170] iam invenimus; quare, si brevitatis gratia ponamus

$$\begin{aligned} A &= 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \text{etc.}, \\ B &= 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \text{etc.}, \\ C &= 1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \text{etc.}, \\ D &= 1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{9^8} + \text{etc.} \\ &\text{etc.}, \end{aligned}$$

erit

$$l\pi = l4 - (A-1) - \frac{1}{2}(B-1) - \frac{1}{3}(C-1) - \frac{1}{4}(D-1) - \text{etc.}$$

Est vero summis supra inventis proxime exprimendis

$A = 1{,}23370\,05501\,36169\,82735\,431,$

$B = 1{,}01467\,80316\,04192\,05454\,625,$

$C = 1{,}00144\,70766\,40942\,12190\,647,$

$D = 1{,}00015\,51790\,25296\,11930\,298,$

$E = 1{,}00001\,70413\,63044\,82548\,818$ [1]),

$F = 1{,}00000\,18858\,48583\,11957\,590,$

$G = 1{,}00000\,02092\,40519\,21150\,010,$

$H = 1{,}00000\,00232\,37157\,37915\,670,$

$I = 1{,}00000\,00025\,81437\,55665\,977,$

$K = 1{,}00000\,00002\,86807\,69745\,558,$

$L = 1{,}00000\,00000\,31866\,77514\,044,$

$M = 1{,}00000\,00000\,03540\,72294\,392,$

$N = 1{,}00000\,00000\,00393\,41246\,691,$

$O = 1{,}00000\,00000\,00043\,71244\,859,$

$P = 1{,}00000\,00000\,00004\,85693\,682,$

$Q = 1{,}00000\,00000\,00000\,53965\,957,$

$R = 1{,}00000\,00000\,00000\,05996\,217,$

1) In editione principe quinque ultimae figurae sunt 50816. Correxit A. K.

$$S = 1{,}00000\,00000\,00000\,00666\,246,$$
$$T = 1{,}00000\,00000\,00000\,00074\,027,$$
$$V = 1{,}00000\,00000\,00000\,00008\,225,$$
$$W = 1{,}00000\,00000\,00000\,00000\,914\ ^{1}),$$
$$X = 1{,}00000\,00000\,00000\,00000\,102\ ^{2}).$$

Hinc sine taedioso calculo reperitur logarithmus hyperbolicus ipsius π

$$= 1{,}14472\,98858\,49400\,17414\,345\ ^{3});$$

qui si multiplicetůr per 0,43429 etc., prodit logarithmus vulgaris ipsius π

$$= 0{,}49714\,98726\,94133\,85435\,128\ ^{4}).$$

191. Quia porro tam sinum quam cosinum anguli $\frac{m\pi}{2n}$ expressum habemus per factores numero infinitos, utriusque logarithmum commode exprimere poterimus. Erit autem ex formulis primo [§ 184] inventis

$$l\sin.\frac{m\pi}{2n} = l\pi + l\frac{m}{2n} + l\left(1-\frac{mm}{4nn}\right) + l\left(1-\frac{mm}{16nn}\right) + l\left(1-\frac{mm}{36nn}\right) + \text{etc.},$$
$$l\cos.\frac{m\pi}{2n} = l\left(1-\frac{mm}{nn}\right) + l\left(1-\frac{mm}{9nn}\right) + l\left(1-\frac{mm}{25nn}\right) + l\left(1-\frac{mm}{49nn}\right) + \text{etc.}$$

Hinc primum logarithmi hyperbolici ut ante per series maxime convergentes facile exprimuntur. Ne autem praeter necessitatem series infinitas multiplicemus, terminos priores actu in logarithmis involutos relinquamus eritque

$$\begin{aligned} l\sin.\frac{m\pi}{2n} = {} & l\pi + lm + l(2n-m) + l(2n+m) - l8 - 3ln \\ & - \frac{mm}{16nn} - \frac{m^4}{2\cdot 16^2n^4} - \frac{m^6}{3\cdot 16^3n^6} - \frac{m^8}{4\cdot 16^4n^8} - \text{etc.} \\ & - \frac{mm}{36nn} - \frac{m^4}{2\cdot 36^2n^4} - \frac{m^6}{3\cdot 36^3n^6} - \frac{m^8}{4\cdot 36^4n^8} - \text{etc.} \\ & - \frac{mm}{64nn} - \frac{m^4}{2\cdot 64^2n^4} - \frac{m^6}{3\cdot 64^3n^6} - \frac{m^8}{4\cdot 64^4n^8} - \text{etc.} \\ & \text{etc.}, \end{aligned}$$

1) In editione principe ultima figura est 3. 2) In editione principe ultima figura est 1. 3) In editione principe ultima figura est 2. 4) In editione principe ultima figura est 6. Correxit A. K.

$$\begin{aligned} l\cos.\frac{m\pi}{2n} = {} & l(n-m) + l(n+m) - 2ln \\ & - \frac{mm}{9nn} - \frac{m^4}{2\cdot 9^2n^4} - \frac{m^6}{3\cdot 9^3n^6} - \frac{m^8}{4\cdot 9^4n^8} - \text{etc.} \\ & - \frac{mm}{25nn} - \frac{m^4}{2\cdot 25^2n^4} - \frac{m^6}{3\cdot 25^3n^6} - \frac{m^8}{4\cdot 25^4n^8} - \text{etc.} \\ & - \frac{mm}{49nn} - \frac{m^4}{2\cdot 49^2n^4} - \frac{m^6}{3\cdot 49^3n^6} - \frac{m^8}{4\cdot 49^4n^8} - \text{etc.} \\ & \text{etc.} \end{aligned}$$

192. Occurrunt ergo in his seriebus singulae potestates pares ipsius $\frac{m}{n}$, quae sunt multiplicatae per series, quarum summas iam supra assignavimus. Erit nempe

$$\begin{aligned} l\sin.\frac{m\pi}{2n} = {} & lm + l(2n-m) + l(2n+m) - 3ln + l\pi - l8 \\ & - \frac{mm}{nn}\left(\frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \frac{1}{10^2} + \frac{1}{12^2} + \text{etc.}\right) \\ & - \frac{m^4}{2n^4}\left(\frac{1}{4^4} + \frac{1}{6^4} + \frac{1}{8^4} + \frac{1}{10^4} + \frac{1}{12^4} + \text{etc.}\right) \\ & - \frac{m^6}{3n^6}\left(\frac{1}{4^6} + \frac{1}{6^6} + \frac{1}{8^6} + \frac{1}{10^6} + \frac{1}{12^6} + \text{etc.}\right) \\ & - \frac{m^8}{4n^8}\left(\frac{1}{4^8} + \frac{1}{6^8} + \frac{1}{8^8} + \frac{1}{10^8} + \frac{1}{12^8} + \text{etc.}\right) \\ & \text{etc.}, \end{aligned}$$

$$\begin{aligned} l\cos.\frac{m\pi}{2n} = {} & l(n-m) + l(n+m) - 2ln \\ & - \frac{mm}{nn}\left(\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \text{etc.}\right) \\ & - \frac{m^4}{2n^4}\left(\frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \text{etc.}\right) \\ & - \frac{m^6}{3n^6}\left(\frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \text{etc.}\right) \\ & - \frac{m^8}{4n^8}\left(\frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{9^8} + \text{etc.}\right) \\ & \text{etc.} \end{aligned}$$

Serierum posteriorum modo ante (§ 190) summae sunt exhibitae; priores series quidem ex his derivari possent, at, quo facilius ad usum transferri queant, earum summas pariter hic adiiciam.

193. Quodsi ergo brevitatis gratia ponamus

$$\alpha = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \text{etc.},$$

$$\beta = \frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \frac{1}{8^4} + \text{etc.},$$

$$\gamma = \frac{1}{2^6} + \frac{1}{4^6} + \frac{1}{6^6} + \frac{1}{8^6} + \text{etc.},$$

$$\delta = \frac{1}{2^8} + \frac{1}{4^8} + \frac{1}{6^8} + \frac{1}{8^8} + \text{etc.}$$

etc.,

erunt summae in numeris proxime expressae hae:

$\alpha = 0{,}41123\,35167\,12056\,60911\,810$,

$\beta = 0{,}06764\,52021\,06946\,13696\,975$,

$\gamma = 0{,}01589\,59853\,43507\,01780\,804$,

$\delta = 0{,}00392\,21771\,72648\,22007\,570$,

$\varepsilon = 0{,}00097\,75337\,64773\,25984\,896$ [1]),

$\zeta = 0{,}00024\,42007\,04724\,92872\,273$ [2]),

$\eta = 0{,}00006\,10388\,94539\,49332\,915$,

$\theta = 0{,}00001\,52590\,22251\,27271\,502$ [3]),

$\iota = 0{,}00000\,38147\,11827\,44318\,008$,

$\varkappa = 0{,}00000\,09536\,75226\,17534\,053$,

$\lambda = 0{,}00000\,02384\,18635\,95259\,255$ [4]),

1) In editione principe ultima figura est 8. Correxit A. K.
2) In editione principe ultima figura est 4. Correxit A. K.
3) In editione principe quinque ultimae figurae sunt 69977. Correxit A. K.
4) In editione principe tres ultimae figurae sunt 154. Correxit A. K.

$\mu = 0{,}00000\,00596\,04648\,32831\,556$ [1]),

$\nu = 0{,}00000\,00149\,01161\,41589\,813$,

$\xi = 0{,}00000\,00037\,25290\,31233\,986$,

$o = 0{,}00000\,00009\,31322\,57548\,284$,

$\pi = 0{,}00000\,00002\,32830\,64370\,808$ [1]),

$\varrho = 0{,}00000\,00000\,58207\,66091\,686$ [1]),

$\sigma = 0{,}00000\,00000\,14551\,91522\,858$,

$\tau = 0{,}00000\,00000\,03637\,97880\,710$,

$\upsilon = 0{,}00000\,00000\,00909\,49470\,177$,

$\varphi = 0{,}00000\,00000\,00227\,37367\,544$,

$\chi = 0{,}00000\,00000\,00056\,84341\,886$,

$\psi = 0{,}00000\,00000\,00014\,21085\,472$ [1]),

$\omega = 0{,}00000\,00000\,00003\,55271\,368$ [1]).

Reliquae summae in ratione quadrupla decrescunt.

194. His ergo in subsidium vocatis erit

$$l\sin.\frac{m\pi}{2n} = lm + l(2n-m) + l(2n+m) - 3ln + l\pi - l8$$
$$-\frac{mm}{nn}\left(\alpha - \frac{1}{2^2}\right) - \frac{m^4}{2n^4}\left(\beta - \frac{1}{2^4}\right) - \frac{m^6}{3n^6}\left(\gamma - \frac{1}{2^6}\right) - \text{etc.},$$

$$l\cos.\frac{m\pi}{2n} = l(n-m) + l(n+m) - 2ln$$
$$-\frac{mm}{nn}(A-1) - \frac{m^4}{2n^4}(B-1) - \frac{m^6}{3n^6}(C-1) - \text{etc.};$$

quoniam igitur logarithmi $l\pi$ et $l8$ dantur, erit

1) In editione principe ultima figura est unitate minor. Correxit A. K.

Logarithmus hyperbolicus sinus anguli $\frac{m}{n}90^\circ$

$= lm + l(2n-m) + l(2n+m) - 3ln$

$- \quad 0{,}93471\,16558\,30435\,75411$ [1])

$- \frac{m^2}{n^2}\cdot 0{,}16123\,35167\,12056\,60912$ [1])

$- \frac{m^4}{n^4}\cdot 0{,}00257\,26010\,53473\,06848$

$- \frac{m^6}{n^6}\cdot 0{,}00009\,03284\,47835\,67260$

$- \frac{m^8}{n^8}\cdot 0{,}00000\,39817\,93162\,05502$ [1])

$- \frac{m^{10}}{n^{10}}\cdot 0{,}00000\,01942\,52954\,65197$ [1])

$- \frac{m^{12}}{n^{12}}\cdot 0{,}00000\,00100\,13287\,48812$

$- \frac{m^{14}}{n^{14}}\cdot 0{,}00000\,00005\,34041\,35619$ [1])

$- \frac{m^{16}}{n^{16}}\cdot 0{,}00000\,00000\,29148\,59659$ [1])

$- \frac{m^{18}}{n^{18}}\cdot 0{,}00000\,00000\,01617\,97980$ [2])

$- \frac{m^{20}}{n^{20}}\cdot 0{,}00000\,00000\,00090\,97691$ [1])

$- \frac{m^{22}}{n^{22}}\cdot 0{,}00000\,00000\,00005\,16828$ [1])

$- \frac{m^{24}}{n^{24}}\cdot 0{,}00000\,00000\,00000\,29608$ [1])

$- \frac{m^{26}}{n^{26}}\cdot 0{,}00000\,00000\,00000\,01708$

$- \frac{m^{28}}{n^{28}}\cdot 0{,}00000\,00000\,00000\,00099$

$- \frac{m^{30}}{n^{30}}\cdot 0{,}00000\,00000\,00000\,00006$ [1]).

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1) In editione principe ultima figura est unitate minor. Correxit A. K.
2) In editione principe duae ultimae figurae sunt 79. Correxit A. K.

At *Logarithmus hyperbolicus cosinus anguli* $\frac{m}{n}90^\circ$

$= l(n-m) + l(n+m) - 2ln$

$- \frac{m^2}{n^2}\cdot 0{,}23370\,05501\,36169\,82735$

$- \frac{m^4}{n^4}\cdot 0{,}00733\,90158\,02096\,02727$

$- \frac{m^6}{n^6}\cdot 0{,}00048\,23588\,80314\,04064$ [1])

$- \frac{m^8}{n^8}\cdot 0{,}00003\,87947\,56324\,02983$ [1])

$- \frac{m^{10}}{n^{10}}\cdot 0{,}00000\,34082\,72608\,96510$

$- \frac{m^{12}}{n^{12}}\cdot 0{,}00000\,03143\,08097\,18660$ [2])

$- \frac{m^{14}}{n^{14}}\cdot 0{,}00000\,00298\,91502\,74450$

$- \frac{m^{16}}{n^{16}}\cdot 0{,}00000\,00029\,04644\,67239$

$- \frac{m^{18}}{n^{18}}\cdot 0{,}00000\,00002\,86826\,39518$

$- \frac{m^{20}}{n^{20}}\cdot 0{,}00000\,00000\,28680\,76975$ [1])

$- \frac{m^{22}}{n^{22}}\cdot 0{,}00000\,00000\,02896\,97956$

$- \frac{m^{24}}{n^{24}}\cdot 0{,}00000\,00000\,00295\,06025$ [1])

$- \frac{m^{26}}{n^{26}}\cdot 0{,}00000\,00000\,00030\,26250$ [3])

$- \frac{m^{28}}{n^{28}}\cdot 0{,}00000\,00000\,00003\,12232$

$- \frac{m^{30}}{n^{30}}\cdot 0{,}00000\,00000\,00000\,32380$ [4])

$- \frac{m^{32}}{n^{32}}\cdot 0{,}00000\,00000\,00000\,03373$

$- \frac{m^{34}}{n^{34}}\cdot 0{,}00000\,00000\,00000\,00353$ [1])

$- \frac{m^{36}}{n^{36}}\cdot 0{,}00000\,00000\,00000\,00037$

$- \frac{m^{38}}{n^{38}}\cdot 0{,}00000\,00000\,00000\,00004.$

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1) In editione principe ultima figura est unitate minor. Correxit A. K.
2) In editione principe duae ultimae figurae sunt 59. Correxit A. K.
3) In editione principe duae ultimae figurae sunt 49. Correxit A. K.
4) In editione principe duae ultimae figurae sunt 79. Correxit A. K.

195. Si isti sinuum et cosinuum logarithmi hyperbolici multiplicentur per 0,43429 44819 etc., prodibunt eorundem logarithmi vulgares ad radium $= 1$ relati. Quoniam vero in tabulis logarithmus sinus totius statui solet $= 10$, quo logarithmi tabulares sinuum et cosinuum obtineantur, post multiplicationem addi debet 10. Hinc erit

Logarithmus tabularis sinus anguli $\frac{m}{n}90^\circ$

$= lm + l(2n - m) + l(2n + m) - 3ln$

$+ \quad 9{,}59405\,98857\,02190$

$- \frac{m^2}{n^2}\cdot 0{,}07002\,28266\,05902$ [1]

$- \frac{m^4}{n^4}\cdot 0{,}00111\,72664\,41662$ [1]

$- \frac{m^6}{n^6}\cdot 0{,}00003\,92291\,46454$ [1]

$- \frac{m^8}{n^8}\cdot 0{,}00000\,17292\,70798$

$- \frac{m^{10}}{n^{10}}\cdot 0{,}00000\,00843\,62986$

$- \frac{m^{12}}{n^{12}}\cdot 0{,}00000\,00043\,48715$

$- \frac{m^{14}}{n^{14}}\cdot 0{,}00000\,00002\,31931$

$- \frac{m^{16}}{n^{16}}\cdot 0{,}00000\,00000\,12659$

$- \frac{m^{18}}{n^{18}}\cdot 0{,}00000\,00000\,00703$ [1]

$- \frac{m^{20}}{n^{20}}\cdot 0{,}00000\,00000\,00040$ [2].

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1) In editione principe ultima figura est unitate minor. Correxit A. K.
2) In editione principe duae ultimae figurae sunt 39. Correxit A. K.

Logarithmus tabularis cosinus anguli $\frac{m}{n}90^\circ$

$= l(n - m) + l(n + m) - 2ln$

$+ \quad 10{,}00000\,00000\,00000$

$- \frac{m^2}{n^2}\cdot 0{,}10149\,48593\,41893$ [1]

$- \frac{m^4}{n^4}\cdot 0{,}00318\,72940\,65451$

$- \frac{m^6}{n^6}\cdot 0{,}00020\,94858\,00017$

$- \frac{m^8}{n^8}\cdot 0{,}00001\,68483\,48598$ [1]

$- \frac{m^{10}}{n^{10}}\cdot 0{,}00000\,14801\,93987$ [1]

$- \frac{m^{12}}{n^{12}}\cdot 0{,}00000\,01365\,02272$

$- \frac{m^{14}}{n^{14}}\cdot 0{,}00000\,00129\,81715$

$- \frac{m^{16}}{n^{16}}\cdot 0{,}00000\,00012\,61471$

$- \frac{m^{18}}{n^{18}}\cdot 0{,}00000\,00001\,24567$

$- \frac{m^{20}}{n^{20}}\cdot 0{,}00000\,00000\,12456$

$- \frac{m^{22}}{n^{22}}\cdot 0{,}00000\,00000\,01258$

$- \frac{m^{24}}{n^{24}}\cdot 0{,}00000\,00000\,00128$

$- \frac{m^{26}}{n^{26}}\cdot 0{,}00000\,00000\,00013.$

196. Harum ergo formularum ope inveniri possunt logarithmi sinuum et cosinuum quorumvis angulorum tam hyperbolici quam vulgares etiam ignoratis ipsis sinibus et cosinibus. Ex logarithmis autem sinuum et cosinuum

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1) In editione principe ultima figura est unitate minor. A. K.

per solam subtractionem inveniuntur logarithmi tangentium, cotangentium et secantium cosecantiumque, quamobrem pro his peculiaribus formulis non erit opus. Ceterum notandum est numerorum m, n, $n-m$, $n+m$ etc. logarithmos hyperbolicos accipi oportere, cum logarithmi hyperbolici sinuum cosinuumque quaeruntur, vulgares autem, cum tales ope posteriorum formularum sunt indagandi. Praeterea $m:n$ denotat rationem, quam angulus propositus habet ad angulum rectum; sicque cum sinus angulorum semirecto maiorum aequentur cosinibus angulorum semirecto minorum ac vicissim, fractio $\frac{m}{n}$ nunquam maior accipienda erit quam $\frac{1}{2}$ hancque ob rem termini illi multo magis convergent, ut semissis instituto sufficere possit.

197. Antequam hoc argumentum relinquamus, aptiorem aperiamus modum tangentes et secantes quorumvis angulorum inveniendi, quam caput praecedens suppeditat. Quanquam enim tangentes et secantes per sinus et cosinus determinantur, tamen hoc fit per divisionem, quae operatio in tantis numeris nimis est operosa. Ac tangentes quidem et cotangentes iam supra (§ 135) exhibuimus, verum illo loco rationem formularum reddere non licuit, quam huic capiti reservavimus.

198. Ex § 181 ergo primum expressionem pro tangente anguli $\frac{m\pi}{2n}$ elicimus. Cum enim sit

$$\frac{1}{nn-mm}+\frac{1}{9nn-mm}+\frac{1}{25nn-mm}+\text{etc.}=\frac{\pi}{4mn}\operatorname{tang.}\frac{m\pi}{2n},$$

erit

$$\operatorname{tang.}\frac{m\pi}{2n}=\frac{4mn}{\pi}\left(\frac{1}{nn-mm}+\frac{1}{9nn-mm}+\frac{1}{25nn-mm}+\text{etc.}\right).$$

Cum deinde sit

$$\frac{1}{nn-mm}+\frac{1}{4nn-mm}+\frac{1}{9nn-mm}+\text{etc.}=\frac{1}{2mm}-\frac{\pi}{2mn}\cot.\frac{m\pi}{n},$$

si pro n scribamus $2n$, erit

$$\cot.\frac{m\pi}{2n}=\frac{2n}{m\pi}-\frac{4mn}{\pi}\left(\frac{1}{4nn-mm}+\frac{1}{16nn-mm}+\frac{1}{36nn-mm}+\text{etc.}\right).$$

Convertantur hae fractiones praeter primas, quippe quae facile in computum ducuntur, in series infinitas; erit

$$\begin{aligned}\operatorname{tang.}\frac{m\pi}{2n}=\frac{mn}{nn-mm}\cdot\frac{4}{\pi}&+\frac{4}{\pi}\left(\frac{m}{3^2n}+\frac{m^3}{3^4n^3}+\frac{m^5}{3^6n^5}+\text{etc.}\right)\\&+\frac{4}{\pi}\left(\frac{m}{5^2n}+\frac{m^3}{5^4n^3}+\frac{m^5}{5^6n^5}+\text{etc.}\right)\\&+\frac{4}{\pi}\left(\frac{m}{7^2n}+\frac{m^3}{7^4n^3}+\frac{m^5}{7^6n^5}+\text{etc.}\right)\\&\qquad\text{etc.},\end{aligned}$$

$$\begin{aligned}\cot.\frac{m\pi}{2n}=\frac{n}{m}\cdot\frac{2}{\pi}-\frac{mn}{4nn-mm}\cdot\frac{4}{\pi}&-\frac{4}{\pi}\left(\frac{m}{4^2n}+\frac{m^3}{4^4n^3}+\frac{m^5}{4^6n^5}+\text{etc.}\right)\\&-\frac{4}{\pi}\left(\frac{m}{6^2n}+\frac{m^3}{6^4n^3}+\frac{m^5}{6^6n^5}+\text{etc.}\right)\\&-\frac{4}{\pi}\left(\frac{m}{8^2n}+\frac{m^3}{8^4n^3}+\frac{m^5}{8^6n^5}+\text{etc.}\right)\\&\qquad\text{etc.}\end{aligned}$$

198[a][1]. At ex valore ipsius π cognito reperitur

$$\frac{1}{\pi}=0{,}31830\,98861\,83790\,67153\,77675\,26745\,028724\,{}^{2)},$$

deinde hic eaedem series occurrunt, quas supra [§ 190 et 193] litteris A, B, C, D etc. et α, β, γ, δ etc. indicavimus. His ergo notatis erit

$$\operatorname{tang.}\frac{m\pi}{2n}=\frac{mn}{nn-mm}\cdot\frac{4}{\pi}$$

$$+\frac{m}{n}\cdot\frac{4}{\pi}(A-1)+\frac{m^3}{n^3}\cdot\frac{4}{\pi}(B-1)+\frac{m^5}{n^5}\cdot\frac{4}{\pi}(C-1)+\frac{m^7}{n^7}\cdot\frac{4}{\pi}(D-1)+\text{etc.}$$

1) In editione principe numerus 198 per errorem iteratur. A. K.

2) In editione principe figura vicesima quinta est 9. Correxit A. K.

Deinde erit pro cotangente

$$\cot.\frac{m\pi}{2n} = \frac{n}{m}\cdot\frac{2}{\pi} - \frac{4mn}{4nn-mm}\cdot\frac{1}{\pi}$$
$$-\frac{m}{n}\cdot\frac{4}{\pi}\left(\alpha-\frac{1}{2^2}\right)-\frac{m^3}{n^3}\cdot\frac{4}{\pi}\left(\beta-\frac{1}{2^4}\right)-\frac{m^5}{n^5}\cdot\frac{4}{\pi}\left(\gamma-\frac{1}{2^6}\right)-\text{etc.},$$

atque ex his formulis natae sunt expressiones, quas supra (§ 135) pro tangente et cotangente dedimus; simul vero (§ 137) ostendimus, quomodo ex tangentibus et cotangentibus inventis per solam additionem et subtractionem secantes et cosecantes reperiantur. Harum ergo regularum ope universus canon sinuum, tangentium et secantium, eorumque logarithmorum multo facilius supputari posset, quam quidem hoc a primis conditoribus est factum.

CAPUT XII

DE REALI FUNCTIONUM FRACTARUM EVOLUTIONE

199. Iam supra, in capite secundo, methodus est tradita functionem quamcunque fractam in tot partes resolvendi, quot eius denominator habeat factores simplices; hi enim praebent denominatores fractionum illarum partialium. Ex quo manifestum est, si denominator quos habeat factores simplices imaginarios, fractiones quoque inde ortas fore imaginarias; his ergo casibus parum iuvabit fractionem realem in imaginarias resolvisse. Cum igitur ostendissem [cap. IX] omnem functionem integram, qualis est denominator cuiusvis fractionis, quantumvis factoribus simplicibus imaginariis scateat, tamen in factores duplices, seu secundae dimensionis, reales semper resolvi posse, hoc modo in resolutione fractionum quantitates imaginariae evitari poterunt, si pro denominatoribus fractionum partialium non factores denominatoris principalis simplices, sed duplices reales assumamus.

200. Sit igitur proposita haec functio fracta $\frac{M}{N}$, ex qua tot fractiones simplices secundum methodum supra [cap. II] expositam eliciantur, quot denominator N habuerit factores simplices reales. Sit autem loco imaginariorum haec expressio

$$pp-2pqz\cos.\varphi+qqzz$$

factor ipsius N, et quoniam in hoc negotio numeratorem et denominatorem in forma evoluta contemplari oportet, sit haec fractio proposita

$$\frac{A+Bz+Cz^2+Dz^3+Ez^4+\text{etc.}}{(pp-2pqz\cos.\varphi+qqzz)(\alpha+\beta z+\gamma zz+\delta z^3+\text{etc.})}$$

ac ponatur fractio partialis ex denominatoris factore $pp - 2pqz\cos.\varphi + qqzz$ oriunda haec

$$\frac{\mathfrak{A} + \mathfrak{a}z}{pp - 2pqz\cos.\varphi + qqzz};$$

quoniam enim variabilis z in denominatore duas habet dimensiones, in numeratore unam habere poterit, non vero plures; alias enim integra functio contineretur, quam seorsim elici oportet.

201. Sit brevitatis gratia numerator

$$A + Bz + Cz^2 + \text{etc.} = M$$

et alter denominatoris factor

$$\alpha + \beta z + \gamma z^2 + \text{etc.} = Z;$$

ponatur altera pars ex denominatoris factore Z oriunda $= \frac{Y}{Z}$ eritque

$$Y = \frac{M - \mathfrak{A}Z - \mathfrak{a}zZ}{pp - 2pqz\cos.\varphi + qqzz},$$

quae expressio functio integra ipsius z esse debet, ideoque necesse est, ut

$$M - \mathfrak{A}Z - \mathfrak{a}zZ$$

divisibile sit per $pp - 2pqz\cos.\varphi + qqzz$. Evanescet ergo $M - \mathfrak{A}Z - \mathfrak{a}zZ$, si ponatur

$$pp - 2pqz\cos.\varphi + qqzz = 0,$$

hoc est, si ponatur [§ 146] tam

$$z = \frac{p}{q}(\cos.\varphi + \sqrt{-1}\cdot\sin.\varphi)$$

quam

$$z = \frac{p}{q}(\cos.\varphi - \sqrt{-1}\cdot\sin.\varphi);$$

sit $\frac{p}{q} = f$ eritque [§ 133]

$$z^n = f^n(\cos.n\varphi \pm \sqrt{-1}\cdot\sin.n\varphi).$$

Duplex ergo hic valor pro z substitutus duplicem dabit aequationem, unde ambas incognitas constantes $\mathfrak{A}$ et $\mathfrak{a}$ definire licet.

202. Facta ergo hac substitutione aequatio

$$M = \mathfrak{A}Z + \mathfrak{a}zZ$$

evoluta hanc duplicem dabit aequationem

$$\begin{aligned}
&A + Bf\cos.\varphi + Cff\cos.2\varphi + Df^3\cos.3\varphi + \text{etc.}\\
&\quad \pm (Bf\sin.\varphi + Cff\sin.2\varphi + Df^3\sin.3\varphi + \text{etc.})\sqrt{-1}\\
&= \mathfrak{A}(\alpha + \beta f\cos.\varphi + \gamma ff\cos.2\varphi + \delta f^3\cos.3\varphi + \text{etc.})\\
&\quad \pm \mathfrak{A}(\beta f\sin.\varphi + \gamma ff\sin.2\varphi + \delta f^3\sin.3\varphi + \text{etc.})\sqrt{-1}\\
&\quad + \mathfrak{a}(\alpha f\cos.\varphi + \beta ff\cos.2\varphi + \gamma f^3\cos.3\varphi + \text{etc.})\\
&\quad \pm \mathfrak{a}(\alpha f\sin.\varphi + \beta ff\sin.2\varphi + \gamma f^3\sin.3\varphi + \text{etc.})\sqrt{-1}.
\end{aligned}$$

Sit ad calculum abbreviandum

$$\begin{aligned}
A + Bf\cos.\varphi + Cff\cos.2\varphi + Df^3\cos.3\varphi + \text{etc.} &= \mathfrak{P},\\
Bf\sin.\varphi + Cff\sin.2\varphi + Df^3\sin.3\varphi + \text{etc.} &= \mathfrak{p},\\
\alpha + \beta f\cos.\varphi + \gamma ff\cos.2\varphi + \delta f^3\cos.3\varphi + \text{etc.} &= \mathfrak{Q},\\
\beta f\sin.\varphi + \gamma ff\sin.2\varphi + \delta f^3\sin.3\varphi + \text{etc.} &= \mathfrak{q},\\
\alpha f\cos.\varphi + \beta ff\cos.2\varphi + \gamma f^3\cos.3\varphi + \text{etc.} &= \mathfrak{R},\\
\alpha f\sin.\varphi + \beta ff\sin.2\varphi + \gamma f^3\sin.3\varphi + \text{etc.} &= \mathfrak{r}
\end{aligned}$$

eritque his positis

$$\mathfrak{P} \pm \mathfrak{p}\sqrt{-1} = \mathfrak{A}\mathfrak{Q} \pm \mathfrak{A}\mathfrak{q}\sqrt{-1} + \mathfrak{a}\mathfrak{R} \pm \mathfrak{a}\mathfrak{r}\sqrt{-1}.$$

203. Ob signorum ambiguitatem hae duae oriuntur aequationes

$$\begin{aligned}
\mathfrak{P} &= \mathfrak{A}\mathfrak{Q} + \mathfrak{a}\mathfrak{R},\\
\mathfrak{p} &= \mathfrak{A}\mathfrak{q} \;\, + \mathfrak{a}\mathfrak{r},
\end{aligned}$$

ex quibus incognitae $\mathfrak{A}$ et $\mathfrak{a}$ ita definiuntur, ut sit

$$\mathfrak{A} = \frac{\mathfrak{P}\mathfrak{r} - \mathfrak{p}\mathfrak{R}}{\mathfrak{Q}\mathfrak{r} - \mathfrak{q}\mathfrak{R}} \quad \text{et} \quad \mathfrak{a} = \frac{\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q}}{\mathfrak{q}\mathfrak{R} - \mathfrak{Q}\mathfrak{r}}.$$

Proposita ergo fractione

$$\frac{M}{(pp - 2pqz \cos.\varphi + qqzz)Z}$$

per sequentem regulam fractio partialis ex ea oriunda

$$\frac{\mathfrak{A} + \mathfrak{a}z}{pp - 2pqz \cos.\varphi + qqzz}$$

definietur. Posito $f = \frac{p}{q}$ et evolutis singulis terminis fiat, ut sequitur:

Posito $z^n = f^n \cos. n\varphi$ sit $M = \mathfrak{P}$,
$z^n = f^n \sin. n\varphi$ sit $M = \mathfrak{p}$,
$z^n = f^n \cos. n\varphi$ sit $Z = \mathfrak{Q}$,
$z^n = f^n \sin. n\varphi$ sit $Z = \mathfrak{q}$,
$z^n = f^n \cos. n\varphi$ sit $zZ = \mathfrak{R}$,
$z^n = f^n \sin. n\varphi$ sit $zZ = \mathfrak{r}$.

Inventis hoc modo valoribus $\mathfrak{P}$, $\mathfrak{Q}$, $\mathfrak{R}$, $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$ erit

$$\mathfrak{A} = \frac{\mathfrak{P}\mathfrak{r} - \mathfrak{p}\mathfrak{R}}{\mathfrak{Q}\mathfrak{r} - \mathfrak{q}\mathfrak{R}} \quad \text{et} \quad \mathfrak{a} = \frac{\mathfrak{p}\mathfrak{Q} - \mathfrak{P}\mathfrak{q}}{\mathfrak{Q}\mathfrak{r} - \mathfrak{q}\mathfrak{R}}.$$

EXEMPLUM 1

Sit proposita haec functio fracta

$$\frac{zz}{(1 - z + zz)(1 + z^4)},$$

ex qua partem a denominatoris factore $1 - z + zz$ oriundam definire oporteat, quae sit

$$\frac{\mathfrak{A} + \mathfrak{a}z}{1 - z + zz}.$$

Ac primo quidem hic factor cum forma generali $pp - 2pqz \cos.\varphi + qqzz$ comparatus dat

$$p = 1, \quad q = 1 \quad \text{et} \quad \cos.\varphi = \frac{1}{2},$$

unde fit

$$\varphi = 60^0 = \frac{\pi}{3}.$$

Quia itaque est

$$M = zz, \quad Z = 1 + z^4 \quad \text{et} \quad f = 1,$$

erit

$$\mathfrak{P} = \cos.\frac{2\pi}{3} = -\frac{1}{2}, \quad \mathfrak{p} = \frac{\sqrt{3}}{2},$$

$$\mathfrak{Q} = 1 + \cos.\frac{4\pi}{3} = \frac{1}{2}, \quad \mathfrak{q} = -\frac{\sqrt{3}}{2},$$

$$\mathfrak{R} = \cos.\frac{\pi}{3} + \cos.\frac{5\pi}{3} = 1, \quad \mathfrak{r} = 0.$$

Ex his invenitur

$$\mathfrak{A} = -1 \quad \text{et} \quad \mathfrak{a} = 0$$

ideoque fractio quaesita est

$$\frac{-1}{1 - z + zz}$$

huiusque complementum erit

$$\frac{1 + z + zz}{1 + z^4};$$

cuius denominator $1 + z^4$ cum habeat factores $1 + z\sqrt{2} + zz$ et $1 - z\sqrt{2} + zz$, resolutio denuo suscipi potest; fit autem $\varphi = \frac{\pi}{4}$ et priori casu $f = -1$, posteriori $f = +1$.

EXEMPLUM 2

Sit igitur proposita haec fractio resolvenda

$$\frac{1 + z + zz}{(1 + z\sqrt{2} + zz)(1 - z\sqrt{2} + zz)}$$

et erit

$$M = 1 + z + zz;$$

et pro priore factore habebitur

$$f = -1, \quad \varphi = \frac{\pi}{4} \quad \text{et} \quad Z = 1 - z\sqrt{2} + zz,$$

unde erit

$$\mathfrak{P} = 1 - \cos.\frac{\pi}{4} + \cos.\frac{2\pi}{4} = \frac{\sqrt{2}-1}{\sqrt{2}},$$

$$\mathfrak{p} = \quad -\sin.\frac{\pi}{4} + \sin.\frac{2\pi}{4} = \frac{\sqrt{2}-1}{\sqrt{2}},$$

$$\mathfrak{Q} = 1 + \sqrt{2}\cdot\cos.\frac{\pi}{4} + \cos.\frac{2\pi}{4} = 2,$$

$$\mathfrak{q} = \quad +\sqrt{2}\cdot\sin.\frac{\pi}{4} + \sin.\frac{2\pi}{4} = 2,$$

$$\mathfrak{R} = -\cos.\frac{\pi}{4} - \sqrt{2}\cdot\cos.\frac{2\pi}{4} - \cos.\frac{3\pi}{4} = 0,$$

$$\mathfrak{r} = -\sin.\frac{\pi}{4} - \sqrt{2}\cdot\sin.\frac{2\pi}{4} - \sin.\frac{3\pi}{4} = -2\sqrt{2}.$$

Ex his reperitur

$$\mathfrak{Q}\mathfrak{r} - \mathfrak{q}\mathfrak{R} = -4\sqrt{2}$$

et

$$\mathfrak{A} = \frac{\sqrt{2}-1}{2\sqrt{2}} \quad \text{et} \quad \mathfrak{a} = 0,$$

unde ex denominatoris factore $1 + z\sqrt{2} + zz$ haec orietur fractio partialis

$$\frac{(\sqrt{2}-1):2\sqrt{2}}{1 + z\sqrt{2} + zz}.$$

Alter autem factor dabit simili modo hanc

$$\frac{(\sqrt{2}+1):2\sqrt{2}}{1 - z\sqrt{2} + zz}.$$

Hinc functio primum proposita

$$\frac{zz}{(1 - z + zz)(1 + z^4)}$$

resolvitur in has

$$\frac{-1}{1 - z + zz} + \frac{(\sqrt{2}-1):2\sqrt{2}}{1 + z\sqrt{2} + zz} + \frac{(\sqrt{2}+1):2\sqrt{2}}{1 - z\sqrt{2} + zz}.$$

EXEMPLUM 3

Sit proposita haec fractio resolvenda

$$\frac{1 + 2z + zz}{(1 - \frac{8}{5}z + zz)(1 + 2z + 3zz)}.$$

Pro factore denominatoris $1 - \frac{8}{5}z + zz$ oriatur ista fractio

$$\frac{\mathfrak{A} + \mathfrak{a}z}{1 - \frac{8}{5}z + zz}$$

eritque

$$p = 1, \quad q = 1, \quad \cos.\varphi = \frac{4}{5},$$

unde

$$f = 1, \quad M = 1 + 2z + zz, \quad Z = 1 + 2z + 3zz.$$

Quia vero hic ratio anguli φ ad rectum non constat, sinus et cosinus eius multiplorum seorsim debent investigari. Cum sit

$$\cos.\varphi = \frac{4}{5}, \qquad \text{erit} \qquad \sin.\varphi = \frac{3}{5},$$

$$\cos.2\varphi = \frac{7}{25}, \qquad \sin.2\varphi = \frac{24}{25},$$

$$\cos.3\varphi = -\frac{44}{125}, \qquad \sin.3\varphi = \frac{117}{125};$$

hinc fit

$$\mathfrak{P} = 1 + 2\cdot\frac{4}{5} + \frac{7}{25} = \frac{72}{25},$$

$$\mathfrak{p} = 2\cdot\frac{3}{5} + \frac{24}{25} = \frac{54}{25},$$

$$\mathfrak{Q} = 1 + 2\cdot\frac{4}{5} + 3\cdot\frac{7}{25} = \frac{86}{25},$$

$$\mathfrak{q} = 2\cdot\frac{3}{5} + 3\cdot\frac{24}{25} = \frac{102}{25},$$

$$\mathfrak{R} = \frac{4}{5} + 2\cdot\frac{7}{25} - 3\cdot\frac{44}{125} = \frac{38}{125},$$

$$\mathfrak{r} = \frac{3}{5} + 2\cdot\frac{24}{25} + 3\cdot\frac{117}{125} = \frac{666}{125}$$

ideoque

$$\mathfrak{Q}\mathfrak{r} - \mathfrak{q}\mathfrak{R} = \frac{53400}{25\cdot 125} = \frac{2136}{125}.$$

Ergo

$$\mathfrak{A} = \frac{1836}{2136} = \frac{153}{178}, \quad \mathfrak{a} = -\frac{540}{2136} = -\frac{45}{178}.$$

Quare fractio ex factore $1 - \frac{8}{5}z + zz$ oriunda erit

$$\frac{9(17-5z):178}{1-\frac{8}{5}z+zz}.$$

Quaeramus simili modo fractionem alteri factori respondentem; erit

$$p = 1, \quad q = -\sqrt{3} \quad \text{et} \quad \cos.\varphi = \frac{1}{\sqrt{3}},$$

ergo

$$f = -\frac{1}{\sqrt{3}}, \quad M = 1 + 2z + zz \quad \text{et} \quad Z = 1 - \frac{8}{5}z + zz.$$

Fiet autem ob

$$\cos.\varphi = \frac{1}{\sqrt{3}} \qquad \sin.\varphi = \frac{\sqrt{2}}{\sqrt{3}},$$

$$\cos.2\varphi = -\frac{1}{3}, \qquad \sin.2\varphi = \frac{2\sqrt{2}}{3},$$

$$\cos.3\varphi = -\frac{5}{3\sqrt{3}}, \qquad \sin.3\varphi = \frac{\sqrt{2}}{3\sqrt{3}},$$

consequenter

$$\mathfrak{P} = 1 - \frac{2}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}} + \frac{1}{3}\cdot -\frac{1}{3} = \frac{2}{9},$$

$$\mathfrak{p} = -\frac{2}{\sqrt{3}}\cdot\frac{\sqrt{2}}{\sqrt{3}} + \frac{1}{3}\cdot\frac{2\sqrt{2}}{3} = -\frac{4\sqrt{2}}{9},$$

$$\mathfrak{Q} = 1 + \frac{8}{5\sqrt{3}}\cdot\frac{1}{\sqrt{3}} + \frac{1}{3}\cdot -\frac{1}{3} = \frac{64}{45},$$

$$\mathfrak{q} = +\frac{8}{5\sqrt{3}}\cdot\frac{\sqrt{2}}{\sqrt{3}} + \frac{1}{3}\cdot\frac{2\sqrt{2}}{3} = \frac{34\sqrt{2}}{45},$$

$$\mathfrak{R} = -\frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}} - \frac{8}{5\cdot 3}\cdot -\frac{1}{3} - \frac{1}{3\sqrt{3}}\cdot -\frac{5}{3\sqrt{3}} = \frac{4}{135},$$

$$\mathfrak{r} = -\frac{1}{\sqrt{3}}\cdot\frac{\sqrt{2}}{\sqrt{3}} - \frac{8}{5\cdot 3}\cdot\frac{2\sqrt{2}}{3} - \frac{1}{3\sqrt{3}}\cdot\frac{\sqrt{2}}{3\sqrt{3}} = -\frac{98\sqrt{2}}{135}$$

ideoque

$$\mathfrak{Q}\mathfrak{r} - \mathfrak{q}\mathfrak{R} = -\frac{712\sqrt{2}}{675};$$

fiet ergo

$$\mathfrak{A} = \frac{100}{712} = \frac{25}{178}, \quad \mathfrak{a} = \frac{540}{712} = \frac{135}{178}.$$

Fractio ergo proposita

$$\frac{1+2z+zz}{(1-\frac{8}{5}z+zz)(1+2z+3zz)}$$

resolvitur in

$$\frac{9(17-5z):178}{1-\frac{8}{5}z+zz} + \frac{5(5+27z):178}{1+2z+3zz}.$$

204. Possunt autem valores litterarum $\mathfrak{R}$ et $\mathfrak{r}$ ex litteris $\mathfrak{Q}$ et $\mathfrak{q}$ definiri. Cum enim sit

$$\mathfrak{Q} = \alpha + \beta f\cos.\varphi + \gamma f^2\cos.2\varphi + \delta f^3\cos.3\varphi + \text{etc.},$$

$$\mathfrak{q} = \quad \beta f\sin.\varphi + \gamma f^2\sin.2\varphi + \delta f^3\sin.3\varphi + \text{etc.},$$

erit

$$\mathfrak{Q}\cos.\varphi - \mathfrak{q}\sin.\varphi = \alpha\cos.\varphi + \beta f\cos.2\varphi + \gamma f^2\cos.3\varphi + \text{etc.}$$

ideoque

$$\mathfrak{R} = f(\mathfrak{Q}\cos.\varphi - \mathfrak{q}\sin.\varphi).$$

Deinde erit

$$\mathfrak{Q}\sin.\varphi + \mathfrak{q}\cos.\varphi = \alpha\sin.\varphi + \beta f\sin.2\varphi + \gamma f^2\sin.3\varphi + \text{etc.},$$

ergo

$$\mathfrak{r} = f(\mathfrak{Q}\sin.\varphi + \mathfrak{q}\cos.\varphi).$$

Ex his porro fit

$$\mathfrak{Q}\mathfrak{r} - \mathfrak{q}\mathfrak{R} = (\mathfrak{Q}\mathfrak{Q} + \mathfrak{q}\mathfrak{q})f\sin.\varphi,$$

$$\mathfrak{P}\mathfrak{r} - \mathfrak{p}\mathfrak{R} = (\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q})f\sin.\varphi + (\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q})f\cos.\varphi$$

eritque consequenter

$$\mathfrak{A} = \frac{\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q}}{\mathfrak{Q}\mathfrak{Q} + \mathfrak{q}\mathfrak{q}} + \frac{\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q}}{\mathfrak{Q}\mathfrak{Q} + \mathfrak{q}\mathfrak{q}}\cdot\frac{\cos.\varphi}{\sin.\varphi},$$

$$\mathfrak{a} = \frac{-\mathfrak{P}\mathfrak{q} + \mathfrak{p}\mathfrak{Q}}{(\mathfrak{Q}\mathfrak{Q} + \mathfrak{q}\mathfrak{q})f\sin.\varphi}.$$

Quare ex denominatoris factore $pp - 2pqz\cos.\varphi + qqzz$ nascitur ista fractio partialis

$$\frac{(\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q})f\sin.\varphi + (\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q})(f\cos.\varphi - z)}{(pp - 2pqz\cos.\varphi + qqzz)(\mathfrak{Q}\mathfrak{Q} + \mathfrak{q}\mathfrak{q})f\sin.\varphi}$$

seu ob $f = \frac{p}{q}$ haec

$$\frac{(\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q})p\sin.\varphi + (\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q})(p\cos.\varphi - qz)}{(pp - 2pqz\cos.\varphi + qqzz)(\mathfrak{Q}\mathfrak{Q} + \mathfrak{q}\mathfrak{q})p\sin.\varphi}.$$

205. Oritur ergo haec fractio partialis ex functionis propositae

$$\frac{M}{(pp - 2pqz\cos.\varphi + qqzz)Z}$$

factore denominatoris $pp - 2pqz\cos.\varphi + qqzz$ atque litterae $\mathfrak{P}$, $\mathfrak{p}$, $\mathfrak{Q}$ et $\mathfrak{q}$ sequenti modo ex functionibus M et Z inveniuntur: Posito

$$z^n = \frac{p^n}{q^n}\cos.n\varphi \quad \text{sit} \quad M = \mathfrak{P} \quad \text{et} \quad Z = \mathfrak{Q}$$

et posito

$$z^n = \frac{p^n}{q^n}\sin.n\varphi \quad \text{sit} \quad M = \mathfrak{p} \quad \text{et} \quad Z = \mathfrak{q},$$

ubi notandum est functiones M et Z, antequam haec substitutio fiat, omnino evolvi debere, ut huiusmodi habeant formas

$$M = A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc.}$$

et

$$Z = \alpha + \beta z + \gamma z^2 + \delta z^3 + \varepsilon z^4 + \text{etc.};$$

eritque ideo

$$\mathfrak{P} = A + B\frac{p}{q}\cos.\varphi + C\frac{p^2}{q^2}\cos.2\varphi + D\frac{p^3}{q^3}\cos.3\varphi + \text{etc.},$$

$$\mathfrak{p} = \quad B\frac{p}{q}\sin.\varphi + C\frac{p^2}{q^2}\sin.2\varphi + D\frac{p^3}{q^3}\sin.3\varphi + \text{etc.},$$

$$\mathfrak{Q} = \alpha + \beta\frac{p}{q}\cos.\varphi + \gamma\frac{p^2}{q^2}\cos.2\varphi + \delta\frac{p^3}{q^3}\cos.3\varphi + \text{etc.},$$

$$\mathfrak{q} = \quad \beta\frac{p}{q}\sin.\varphi + \gamma\frac{p^2}{q^2}\sin.2\varphi + \delta\frac{p^3}{q^3}\sin.3\varphi + \text{etc.}$$

206. Ex praecedentibus autem intelligitur hanc resolutionem locum habere non posse, si functio Z eundem factorem $pp - 2pqz\cos.\varphi + qqzz$ adhuc in se complectatur; hoc enim casu in aequatione

$$M = \mathfrak{A}Z + \mathfrak{a}zZ$$

facta substitutione

$$z^n = f^n(\cos.n\varphi \pm \sqrt{-1}\cdot\sin.n\varphi)$$

ipsa quantitas Z evanesceret nihilque propterea colligi posset. Quamobrem si functionis fractae $\frac{M}{N}$ denominator habeat factorem $(pp - 2pqz\cos.\varphi + qqzz)^2$ vel altiorem potestatem, peculiari opus erit resolutione. Sit igitur

$$N = (pp - 2pqz\cos.\varphi + qqzz)^2 Z$$

atque ex denominatoris factore $(pp - 2pqz\cos.\varphi + qqzz)^2$ orientur huiusmodi duae fractiones partiales

$$\frac{\mathfrak{A} + \mathfrak{a}z}{(pp - 2pqz\cos.\varphi + qqzz)^2} + \frac{\mathfrak{B} + \mathfrak{b}z}{pp - 2pqz\cos.\varphi + qqzz},$$

ubi litteras constantes $\mathfrak{A}$, $\mathfrak{a}$, $\mathfrak{B}$, $\mathfrak{b}$ determinari oportet.

207. His positis debebit ista expressio

$$\frac{M - (\mathfrak{A} + \mathfrak{a}z)Z - (\mathfrak{B} + \mathfrak{b}z)Z(pp - 2pqz\cos.\varphi + qqzz)}{(pp - 2pqz\cos.\varphi + qqzz)^2}$$

esse functio integra et hanc ob rem numerator divisibilis erit per denominatorem [§ 43]. Primum ergo haec expressio

$$M - \mathfrak{A}Z - \mathfrak{a}zZ$$

divisibilis esse debet per $pp - 2pqz\cos.\varphi + qqzz$; qui cum sit casus praecedens, eodem quoque modo litterae $\mathfrak{A}$ et $\mathfrak{a}$ determinabuntur.

Quare posito

$$z^n = \frac{p^n}{q^n}\cos.n\varphi \quad \text{sit} \quad M = \mathfrak{P} \quad \text{et} \quad Z = \mathfrak{N}$$

et posito

$$z^n = \frac{p^n}{q^n}\sin.n\varphi \quad \text{sit} \quad M = \mathfrak{p} \quad \text{et} \quad Z = \mathfrak{n}.$$

Hisque factis secundum regulam supra datam erit

$$\mathfrak{A}=\frac{\mathfrak{P}\mathfrak{N}+\mathfrak{p}\mathfrak{n}}{\mathfrak{N}^2+\mathfrak{n}^2}+\frac{\mathfrak{P}\mathfrak{n}-\mathfrak{p}\mathfrak{N}}{\mathfrak{N}^2+\mathfrak{n}^2}\cdot\frac{\cos.\varphi}{\sin.\varphi},$$

$$\mathfrak{a}=\frac{-\mathfrak{P}\mathfrak{n}+\mathfrak{p}\mathfrak{N}}{\mathfrak{N}^2+\mathfrak{n}^2}\cdot\frac{q}{p\sin.\varphi}.$$

208. Inventis ergo hoc modo $\mathfrak{A}$ et $\mathfrak{a}$ fiet

$$\frac{M-(\mathfrak{A}+\mathfrak{a}z)Z}{pp-2pqz\cos.\varphi+qqzz}$$

functio integra, quae sit $=P$, atque superest, ut

$$P-\mathfrak{B}Z-\mathfrak{b}zZ$$

divisibile evadat per $pp-2pqz\cos.\varphi+qqzz$; quae expressio cum similis sit praecedenti, si posito

$$z^n=\frac{p^n}{q^n}\cos.n\varphi \quad \text{vocetur} \quad P=\mathfrak{R}$$

et posito

$$z^n=\frac{p^n}{q^n}\sin.n\varphi \quad \text{vocetur} \quad P=\mathfrak{r},$$

erit

$$\mathfrak{B}=\frac{\mathfrak{R}\mathfrak{N}+\mathfrak{r}\mathfrak{n}}{\mathfrak{N}^2+\mathfrak{n}^2}+\frac{\mathfrak{R}\mathfrak{n}-\mathfrak{r}\mathfrak{N}}{\mathfrak{N}^2+\mathfrak{n}^2}\cdot\frac{\cos.\varphi}{\sin.\varphi},$$

$$\mathfrak{b}=\frac{-\mathfrak{R}\mathfrak{n}+\mathfrak{r}\mathfrak{N}}{\mathfrak{N}^2+\mathfrak{n}^2}\cdot\frac{q}{p\sin.\varphi}.$$

209. Hinc iam generaliter concludere licet, quomodo resolutio institui debeat, si denominator functionis propositae $\frac{M}{N}$ factorem habeat

$$(pp-2pqz\cos.\varphi+qqzz)^k.$$

Sit enim

$$N=(pp-2pqz\cos.\varphi+qqzz)^k Z,$$

ita ut haec resolvenda sit functio fracta

$$\frac{M}{(pp-2pqz\cos.\varphi+qqzz)^k Z}.$$

Praebeat ergo factor denominatoris $(pp-2pqz\cos.\varphi+qqzz)^k$ has partes

$$\frac{\mathfrak{A}+\mathfrak{a}z}{(pp-2pqz\cos.\varphi+qqzz)^k}+\frac{\mathfrak{B}+\mathfrak{b}z}{(pp-2pqz\cos.\varphi+qqzz)^{k-1}}$$
$$+\frac{\mathfrak{C}+\mathfrak{c}z}{(pp-2pqz\cos.\varphi+qqzz)^{k-2}}+\frac{\mathfrak{D}+\mathfrak{d}z}{(pp-2pqz\cos.\varphi+qqzz)^{k-3}}+\text{etc.}$$

Iam posito

$$z^n=\frac{p^n}{q^n}\cos.n\varphi \quad \text{sit} \quad M=\mathfrak{M} \quad \text{et} \quad Z=\mathfrak{N}$$

et posito

$$z^n=\frac{p^n}{q^n}\sin.n\varphi \quad \text{sit} \quad M=\mathfrak{m} \quad \text{et} \quad Z=\mathfrak{n};$$

erit

$$\mathfrak{A}=\frac{\mathfrak{M}\mathfrak{N}+\mathfrak{m}\mathfrak{n}}{\mathfrak{N}^2+\mathfrak{n}^2}+\frac{\mathfrak{M}\mathfrak{n}-\mathfrak{m}\mathfrak{N}}{\mathfrak{N}^2+\mathfrak{n}^2}\cdot\frac{\cos.\varphi}{\sin.\varphi},$$

$$\mathfrak{a}=\frac{-\mathfrak{M}\mathfrak{n}+\mathfrak{m}\mathfrak{N}}{\mathfrak{N}^2+\mathfrak{n}^2}\cdot\frac{q}{p\sin.\varphi}.$$

Deinde vocetur

$$\frac{M-(\mathfrak{A}+\mathfrak{a}z)Z}{pp-2pqz\cos.\varphi+qqzz}=P$$

atque posito

$$z^n=\frac{p^n}{q^n}\cos.n\varphi \quad \text{sit} \quad P=\mathfrak{P}$$

et posito

$$z^n=\frac{p^n}{q^n}\sin.n\varphi \quad \text{sit} \quad P=\mathfrak{p};$$

erit

$$\mathfrak{B}=\frac{\mathfrak{P}\mathfrak{N}+\mathfrak{p}\mathfrak{n}}{\mathfrak{N}^2+\mathfrak{n}^2}+\frac{\mathfrak{P}\mathfrak{n}-\mathfrak{p}\mathfrak{N}}{\mathfrak{N}^2+\mathfrak{n}^2}\cdot\frac{\cos.\varphi}{\sin.\varphi},$$

$$\mathfrak{b}=\frac{-\mathfrak{P}\mathfrak{n}+\mathfrak{p}\mathfrak{N}}{\mathfrak{N}^2+\mathfrak{n}^2}\cdot\frac{q}{p\sin.\varphi}.$$

Tum vocetur

$$\frac{P-(\mathfrak{B}+\mathfrak{b}z)Z}{pp-2pqz\cos.\varphi+qqzz}=Q$$

atque posito

$$z^n=\frac{p^n}{q^n}\cos.n\varphi \quad \text{sit} \quad Q=\mathfrak{Q}$$

et posito

$$z^n=\frac{p^n}{q^n}\sin.n\varphi \quad \text{sit} \quad Q=\mathfrak{q};$$

erit

$$\mathfrak{C} = \frac{\mathfrak{Q}\mathfrak{N} + \mathfrak{q}\mathfrak{n}}{\mathfrak{N}^2 + \mathfrak{n}^2} + \frac{\mathfrak{Q}\mathfrak{n} - \mathfrak{q}\mathfrak{N}}{\mathfrak{N}^2 + \mathfrak{n}^2} \cdot \frac{\cos.\varphi}{\sin.\varphi},$$

$$\mathfrak{c} = \frac{-\mathfrak{Q}\mathfrak{n} + \mathfrak{q}\mathfrak{N}}{\mathfrak{N}^2 + \mathfrak{n}^2} \cdot \frac{q}{p\sin.\varphi}.$$

Porro vocetur

$$\frac{Q - (\mathfrak{C} + \mathfrak{c}z)Z}{pp - 2pqz\cos.\varphi + qqzz} = R$$

atque posito

$$z^n = \frac{p^n}{q^n}\cos.n\varphi \quad \text{sit} \quad R = \mathfrak{R}$$

et posito

$$z^n = \frac{p^n}{q^n}\sin.n\varphi \quad \text{sit} \quad R = \mathfrak{r};$$

erit

$$\mathfrak{D} = \frac{\mathfrak{R}\mathfrak{N} + \mathfrak{r}\mathfrak{n}}{\mathfrak{N}^2 + \mathfrak{n}^2} + \frac{\mathfrak{R}\mathfrak{n} - \mathfrak{r}\mathfrak{N}}{\mathfrak{N}^2 + \mathfrak{n}^2} \cdot \frac{\cos.\varphi}{\sin.\varphi},$$

$$\mathfrak{d} = \frac{-\mathfrak{R}\mathfrak{n} + \mathfrak{r}\mathfrak{N}}{\mathfrak{N}^2 + \mathfrak{n}^2} \cdot \frac{q}{p\sin.\varphi}.$$

Hocque modo progrediendum est, donec ultimae fractionis, cuius denominator est $pp - 2pqz\cos.\varphi + qqzz$, numerator fuerit determinatus.

EXEMPLUM

Sit ista proposita functio fracta

$$\frac{z - z^3}{(1 + zz)^4(1 + z^4)},$$

ex cuius denominatoris factore $(1 + zz)^4$ oriantur hae fractiones partiales

$$\frac{\mathfrak{A} + \mathfrak{a}z}{(1 + zz)^4} + \frac{\mathfrak{B} + \mathfrak{b}z}{(1 + zz)^3} + \frac{\mathfrak{C} + \mathfrak{c}z}{(1 + zz)^2} + \frac{\mathfrak{D} + \mathfrak{d}z}{1 + zz}.$$

Comparatione ergo instituta erit

$$p = 1, \quad q = 1, \quad \cos.\varphi = 0 \quad \text{ideoque} \quad \varphi = \frac{\pi}{2}$$

porroque

$$M = z - z^3 \quad \text{et} \quad Z = 1 + z^4.$$

Hinc erit

$$\mathfrak{M} = 0, \quad \mathfrak{m} = 2, \quad \mathfrak{N} = 2, \quad \mathfrak{n} = 0 \quad \text{et} \quad \sin.\varphi = 1.$$

Hinc itaque invenitur

$$\mathfrak{A} = -\tfrac{4}{4}\cdot 0 = 0 \quad \text{et} \quad \mathfrak{a} = 1,$$

ergo

$$\mathfrak{A} + \mathfrak{a}z = z$$

hincque

$$P = \frac{z - z^3 - z - z^5}{1 + zz} = -z^3$$

et

$$\mathfrak{P} = 0, \quad \mathfrak{p} = 1,$$

unde reperitur

$$\mathfrak{B} = 0 \quad \text{et} \quad \mathfrak{b} = \tfrac{1}{2}.$$

Ergo

$$\mathfrak{B} + \mathfrak{b}z = \tfrac{1}{2}z$$

et

$$Q = \frac{-z^3 - \frac{1}{2}z - \frac{1}{2}z^5}{1 + zz} = -\frac{1}{2}z - \frac{1}{2}z^3,$$

unde

$$\mathfrak{Q} = 0 \quad \text{et} \quad \mathfrak{q} = 0,$$

ergo

$$\mathfrak{C} = 0 \quad \text{et} \quad \mathfrak{c} = 0.$$

Hincque

$$R = \frac{-\frac{1}{2}z - \frac{1}{2}z^3}{1 + zz} = -\frac{1}{2}z,$$

ergo

$$\mathfrak{R} = 0 \quad \text{et} \quad \mathfrak{r} = -\tfrac{1}{2},$$

unde fit

$$\mathfrak{D} = 0 \quad \text{et} \quad \mathfrak{d} = -\tfrac{1}{4}.$$

Quamobrem fractiones quaesitae sunt hae

$$\frac{z}{(1 + zz)^4} + \frac{z}{2(1 + zz)^3} - \frac{z}{4(1 + zz)}.$$

Reliquae vero fractionis numerator est

$$S = \frac{R - (\mathfrak{D} + \mathfrak{d}z)Z}{1 + zz} = -\frac{1}{4}z + \frac{1}{4}z^3,$$

quae ergo erit

$$= \frac{-z + z^3}{4(1 + z^4)}.$$

210. Hac ergo methodo simul innotescit fractio complementi, quae cum inventis coniuncta producat fractionem propositam ipsam. Scilicet si fractionis

$$\frac{M}{(pp-2pqz\cos.\varphi+qqzz)^k Z}$$

inventae fuerint omnes fractiones partiales ex factore $(pp-2pqz\cos.\varphi+qqzz)^k$ oriundae, pro quibus formati sunt valores functionum P, Q, R, S, T, si harum litterarum series ulterius continuetur, erit ea, quae ultimam, qua opus est ad numeratores inveniendos, sequitur, numerator reliquae fractionis denominatorem Z habentis; nempe, si $k=1$, erit reliqua fractio $\frac{P}{Z}$; si $k=2$, erit reliqua fractio $\frac{Q}{Z}$; si $k=3$, erit ea $\frac{R}{Z}$, et ita porro. Inventa autem hac reliqua fractione denominatorem Z habente ea per has regulas ulterius resolvi poterit.

CAPUT XIII

DE SERIEBUS RECURRENTIBUS

211. Ad hoc serierum genus, quas MOIVREUS[1]) *recurrentes* vocare solet, hic refero omnes series, quae ex evolutione functionis cuiusque fractae per divisionem actualem instituta nascuntur. Supra [cap. IV] enim iam ostendimus has series ita esse comparatas, ut quivis terminus ex aliquot praecedentibus secundum legem quandam constantem determinetur, quae lex a denominatore functionis fractae pendet. Cum autem nunc functionem quamcunque fractam in alias simpliciores resolvere docuerim, hinc series quoque recurrens in alias simpliciores resolvetur. In hoc igitur capite propositum est serierum recurrentium cuiusvis gradus resolutionem in simpliciores exponere.

212. Sit proposita ista functio fracta genuina

$$\frac{a+bz+czz+dz^3+\text{etc.}}{1-\alpha z-\beta zz-\gamma z^3-\delta z^4-\text{etc.}},$$

quae per divisionem evolvatur in hanc seriem recurrentem

$$A+Bz+Cz^2+Dz^3+Ez^4+Fz^5+\text{etc.};$$

cuius coefficientes quemadmodum progrediantur, supra est ostensum. Quodsi iam functio illa fracta resolvatur in fractiones suas simplices et unaquaeque in seriem recurrentem evolvatur, manifestum est summam omnium harum

1) Vide notam p. 79. F. R.

serierum ex fractionibus partialibus ortarum aequalem esse debere seriei recurrenti

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \text{etc.}$$

Fractiones ergo partiales, quas supra [cap. II] invenire docuimus, dabunt series partiales, quarum indoles ob simplicitatem facile perspicitur; omnes autem series partiales iunctim sumptae producent seriem recurrentem propositam, unde et huius natura penitius cognoscetur.

213. Sint series recurrentes ex singulis fractionibus partialibus ortae hae

$$\begin{array}{l} a + bz + czz + dz^3 + ez^4 + \text{etc.}, \\ a' + b'z + c'zz + d'z^3 + e'z^4 + \text{etc.}, \\ a'' + b''z + c''zz + d''z^3 + e''z^4 + \text{etc.}, \\ a''' + b'''z + c'''zz + d'''z^3 + e'''z^4 + \text{etc.} \\ \text{etc.} \end{array}$$

Quoniam hae series iunctim sumptae aequales esse debent huic

$$A + Bz + Czz + Dz^3 + Ez^4 + \text{etc.},$$

necesse est, ut sit

$$\begin{array}{l} A = a + a' + a'' + a''' + \text{etc.}, \\ B = b + b' + b'' + b''' + \text{etc.}, \\ C = c + c' + c'' + c''' + \text{etc.}, \\ D = d + d' + d'' + d''' + \text{etc.} \\ \text{etc.} \end{array}$$

Hinc, si singularum serierum ex fractionibus partialibus ortarum definiri queant coefficientes potestatis z^n, horum summa dabit coefficientem potestatis z^n in serie recurrente $A + Bz + Cz^2 + Dz^3 + \text{etc.}$

214. Dubium hic suboriri posset, an, si duae huiusmodi series fuerint inter se aequales

$$A + Bz + Cz^2 + Dz^3 + \text{etc.} = \mathfrak{A} + \mathfrak{B}z + \mathfrak{C}z^2 + \mathfrak{D}z^3 + \text{etc.},$$

necessario inde sequatur coefficientes similium potestatum ipsius z inter se esse aequales, seu an sit $A = \mathfrak{A}$, $B = \mathfrak{B}$, $C = \mathfrak{C}$, $D = \mathfrak{D}$ etc. Hoc autem dubium facile tolletur, si perpendamus hanc aequalitatem subsistere debere, quemcunque valorem obtineat variabilis z. Sit igitur $z = 0$ atque manifestum est fore $A = \mathfrak{A}$. His ergo terminis aequalibus utrinque sublatis ac reliqua aequatione per z divisa habebitur

$$B + Cz + Dz^2 + \text{etc.} = \mathfrak{B} + \mathfrak{C}z + \mathfrak{D}z^2 + \text{etc.},$$

unde sequitur fore $B = \mathfrak{B}$; simili autem modo ostendetur esse $C = \mathfrak{C}$, $D = \mathfrak{D}$ et ita porro in infinitum.[1])

215. Contemplemur ergo series, quae ex fractionibus partialibus, in quas fractio quaepiam proposita resolvitur, oriuntur. Ac primo quidem patet fractionem

$$\frac{\mathfrak{A}}{1 - pz}$$

dare seriem

$$\mathfrak{A} + \mathfrak{A}pz + \mathfrak{A}p^2z^2 + \mathfrak{A}p^3z^3 + \text{etc.},$$

cuius terminus generalis est

$$\mathfrak{A}p^nz^n;$$

haec enim expressio vocari solet *terminus generalis*, quoniam ex ea loco n numeros omnes successive substituendo omnes seriei termini nascuntur. Deinde ex fractione

$$\frac{\mathfrak{A}}{(1 - pz)^2}$$

oritur series

$$\mathfrak{A} + 2\mathfrak{A}pz + 3\mathfrak{A}p^2z^2 + 4\mathfrak{A}p^3z^3 + \text{etc.},$$

cuius terminus generalis est

$$(n + 1)\mathfrak{A}p^nz^n.$$

1) Confer EULERI Commentationem 130 (indicis ENESTROEMIANI): *De seriebus quibusdam considerationes*, Comment. acad. sc. Petrop. 12 (1740), 1750, p. 53, imprimis p. 61; *LEONHARDI EULERI Opera omnia*, series I, vol. 14. F. R.

Tum ex fractione

$$\frac{\mathfrak{A}}{(1-pz)^3}$$

oritur series

$$\mathfrak{A}+3\mathfrak{A}pz+6\mathfrak{A}p^2z^2+10\mathfrak{A}p^3z^3+\text{etc.},$$

cuius terminus generalis est

$$\frac{(n+1)(n+2)}{1\cdot 2}\mathfrak{A}p^nz^n.$$

Generatim autem fractio

$$\frac{\mathfrak{A}}{(1-pz)^k}$$

praebet seriem hanc

$$\mathfrak{A}+k\mathfrak{A}pz+\frac{k(k+1)}{1\cdot 2}\mathfrak{A}p^2z^2+\frac{k(k+1)(k+2)}{1\cdot 2\cdot 3}\mathfrak{A}p^3z^3+\text{etc.},$$

cuius terminus generalis est

$$\frac{(n+1)(n+2)(n+3)\cdots(n+k-1)}{1\cdot 2\cdot 3\cdots(k-1)}\mathfrak{A}p^nz^n.$$

Ex ipsa autem seriei progressione colligitur hic idem terminus

$$=\frac{k(k+1)(k+2)\cdots(k+n-1)}{1\cdot 2\cdot 3\cdots n}\mathfrak{A}p^nz^n.$$

Haec vero expressio illi est aequalis, id quod multiplicatione per crucem instituta patebit; fiet enim

$$1\cdot 2\cdot 3\cdots n(n+1)\cdots(n+k-1)=1\cdot 2\cdot 3\cdots(k-1)k\cdots(k+n-1),$$

quae est aequatio identica.

216. Quoties ergo in resolutione functionum fractarum ad huiusmodi fractiones partiales $\frac{\mathfrak{A}}{(1-pz)^k}$ pervenitur, toties seriei recurrentis ex illa functione fracta ortae

$$A+Bz+Cz^2+Dz^3+\text{etc.}$$

terminus generalis assignari poterit, quippe qui erit summa terminorum generalium serierum, quae ex fractionibus partialibus nascuntur.

EXEMPLUM 1

Invenire terminum generalem seriei recurrentis, quae ex hac fractione

$$\frac{1-z}{1-z-2zz}$$

nascitur.

Series hinc nata est

$$1+0z+2zz+2z^3+6z^4+10z^5+22z^6+42z^7+86z^8+\text{etc.}$$

Ad coefficientem potestatis generalis z^n inveniendum fractio $\frac{1-z}{1-z-2zz}$ resolvatur in

$$\frac{\frac{2}{3}}{1+z}+\frac{\frac{1}{3}}{1-2z},$$

unde oritur terminus generalis quaesitus

$$\left(\frac{2}{3}(-1)^n+\frac{1}{3}\cdot 2^n\right)z^n=\frac{2^n\pm 2}{3}z^n,$$

ubi signum $+$ valet, si n sit numerus par, signum $-$, si n sit impar.

EXEMPLUM 2

Invenire terminum generalem seriei recurrentis, quae oritur ex fractione

$$\frac{1-z}{1-5z+6zz},$$

seu seriei huius

$$1+4z+14z^2+46z^3+146z^4+454z^5+\text{etc.}$$

Ob denominatorem $=(1-2z)(1-3z)$ resolvitur fractio in has

$$\frac{-1}{1-2z}+\frac{2}{1-3z},$$

ex quibus fit terminus generalis

$$2\cdot 3^nz^n-2^nz^n=(2\cdot 3^n-2^n)z^n.$$

EXEMPLUM 3

Invenire terminum generalem seriei huius

$$1+3z+4z^2+7z^3+11z^4+18z^5+29z^6+47z^7+\text{etc.},$$

quae oritur ex evolutione fractionis

$$\frac{1+2z}{1-z-zz}.$$

Ob denominatoris factores

$$1-\frac{1+\sqrt{5}}{2}z \quad \text{et} \quad 1-\frac{1-\sqrt{5}}{2}z$$

per resolutionem prodeunt

$$\frac{\frac{1+\sqrt{5}}{2}}{1-\frac{1+\sqrt{5}}{2}z}+\frac{\frac{1-\sqrt{5}}{2}}{1-\frac{1-\sqrt{5}}{2}z},$$

unde erit terminus generalis

$$\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}z^n+\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}z^n.$$

EXEMPLUM 4

Invenire terminum generalem seriei huius

$$a+(\alpha a+b)z+(\alpha^2 a+\alpha b+\beta a)z^2+(\alpha^3 a+\alpha^2 b+2\alpha\beta a+\beta b)z^3+\text{etc.},$$

quae oritur ex evolutione fractionis

$$\frac{a+bz}{1-\alpha z-\beta zz}.$$

Per resolutionem oriuntur hae duae fractiones

$$\frac{\left(a\left(\sqrt{(\alpha\alpha+4\beta)}+\alpha\right)+2b\right):2\sqrt{(\alpha\alpha+4\beta)}}{1-\frac{\alpha+\sqrt{(\alpha\alpha+4\beta)}}{2}z}+\frac{\left(a\left(\sqrt{(\alpha\alpha+4\beta)}-\alpha\right)-2b\right):2\sqrt{(\alpha\alpha+4\beta)}}{1-\frac{\alpha-\sqrt{(\alpha\alpha+4\beta)}}{2}z};$$

hinc terminus generalis erit

$$\frac{a\left(\sqrt{(\alpha\alpha+4\beta)}+\alpha\right)+2b}{2\sqrt{(\alpha\alpha+4\beta)}}\left(\frac{\alpha+\sqrt{(\alpha\alpha+4\beta)}}{2}\right)^n z^n$$
$$+\frac{a\left(\sqrt{(\alpha\alpha+4\beta)}-\alpha\right)-2b}{2\sqrt{(\alpha\alpha+4\beta)}}\left(\frac{\alpha-\sqrt{(\alpha\alpha+4\beta)}}{2}\right)^n z^n.$$

Ex quo omnium serierum recurrentium, quarum quisque terminus per duos praecedentes determinatur, termini generales expedite definiri poterunt.

EXEMPLUM 5

Invenire terminum generalem huius seriei

$$1+z+2z^2+2z^3+3z^4+3z^5+4z^6+4z^7+\text{etc.},$$

quae oritur ex fractione

$$\frac{1}{1-z-zz+z^3}=\frac{1}{(1-z)^2(1+z)}.$$

Quanquam lex progressionis primo intuitu ita est manifesta, ut explicatione non indigeat, tamen fractiones per resolutionem ortae

$$\frac{\frac{1}{2}}{(1-z)^2}+\frac{\frac{1}{4}}{1-z}+\frac{\frac{1}{4}}{1+z}$$

dant hunc terminum generalem

$$\frac{1}{2}(n+1)z^n+\frac{1}{4}z^n+\frac{1}{4}(-1)^n z^n=\frac{2n+3\pm 1}{4}z^n,$$

ubi signum superius valet, si n fuerit numerus par, inferius, si n fuerit impar.

217. Hoc pacto omnium serierum recurrentium termini generales exhiberi possunt, quoniam omnes fractiones in huiusmodi fractiones partiales simplices resolvere licet. Quodsi autem expressiones imaginarias vitare velimus, saepenumero ad huiusmodi fractiones partiales pervenietur

$$\frac{\mathfrak{A}+\mathfrak{B}pz}{1-2pz\cos.\varphi+ppzz},\quad \frac{\mathfrak{A}+\mathfrak{B}pz}{(1-2pz\cos.\varphi+ppzz)^2},\quad\ldots\quad \frac{\mathfrak{A}+\mathfrak{B}pz}{(1-2pz\cos.\varphi+ppzz)^k};$$

ex quarum evolutione cuiusmodi series nascantur, videndum est. Ac primo quidem ob

$$\cos.n\varphi = 2\cos.\varphi\cos.(n-1)\varphi - \cos.(n-2)\varphi$$

fractio

$$\frac{\mathfrak{A}}{1-2pz\cos.\varphi+ppzz}$$

evoluta dabit [§ 61]

$$\begin{aligned}\mathfrak{A} + 2\mathfrak{A}pz\cos.\varphi + 2\mathfrak{A}ppzz\cos.2\varphi &+ 2\mathfrak{A}p^3z^3\cos.3\varphi + 2\mathfrak{A}p^4z^4\cos.4\varphi + \text{etc.}\\ + \mathfrak{A}ppzz &+ 2\mathfrak{A}p^3z^3\cos.\varphi + 2\mathfrak{A}p^4z^4\cos.2\varphi + \text{etc.}\\ &+ \mathfrak{A}p^4z^4 + \text{etc.}\\ &\text{etc.},\end{aligned}$$

cuius seriei terminus generalis non tam facile apparet.

218. Quo igitur ad scopum perveniamus, consideremus has duas series

$$Ppz\sin.\varphi + Pp^2z^2\sin.2\varphi + Pp^3z^3\sin.3\varphi + Pp^4z^4\sin.4\varphi + \text{etc.},$$

$$Q + Qpz\cos.\varphi + Qp^2z^2\cos.2\varphi + Qp^3z^3\cos.3\varphi + Qp^4z^4\cos.4\varphi + \text{etc.},$$

quae duae series utique nascuntur ex evolutione fractionis, cuius denominator est

$$1 - 2pz\cos.\varphi + ppzz.$$

Ac prior quidem oritur ex hac fractione

$$\frac{Ppz\sin.\varphi}{1-2pz\cos.\varphi+ppzz},$$

posterior vero ex hac

$$\frac{Q-Qpz\cos.\varphi}{1-2pz\cos.\varphi+ppzz}.$$

Addantur hae duae fractiones atque summa

$$\frac{Q+Ppz\sin.\varphi-Qpz\cos.\varphi}{1-2pz\cos.\varphi+ppzz}$$

dabit seriem, cuius terminus generalis erit

$$(P\sin.n\varphi + Q\cos.n\varphi)p^nz^n.$$

Fiat autem haec fractio propositae

$$\frac{\mathfrak{A}+\mathfrak{B}pz}{1-2pz\cos.\varphi+ppzz}$$

aequalis; erit

$$Q = \mathfrak{A} \quad\text{et}\quad P = \mathfrak{A}\cot.\varphi + \mathfrak{B}\operatorname{cosec}.\varphi.$$

Seriei ergo ex hac fractione

$$\frac{\mathfrak{A}+\mathfrak{B}pz}{1-2pz\cos.\varphi+ppzz}$$

ortae terminus generalis erit

$$\frac{\mathfrak{A}\cos.\varphi\sin.n\varphi + \mathfrak{B}\sin.n\varphi + \mathfrak{A}\sin.\varphi\cos.n\varphi}{\sin.\varphi}p^nz^n = \frac{\mathfrak{A}\sin.(n+1)\varphi + \mathfrak{B}\sin.n\varphi}{\sin.\varphi}p^nz^n.$$

219. Ad terminum generalem inveniendum, si denominator fractionis fuerit potestas ut

$$(1-2pz\cos.\varphi+ppzz)^k,$$

conveniet hanc fractionem resolvi in duas etsi imaginarias

$$\frac{a}{(1-(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi)pz)^k} + \frac{b}{(1-(\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi)pz)^k},$$

quarum simul sumptarum terminus generalis seriei ex ipsis ortae erit

$$\begin{aligned}&\frac{(n+1)(n+2)(n+3)\cdots(n+k-1)}{1\cdot2\cdot3\cdots(k-1)}(\cos.n\varphi+\sqrt{-1}\cdot\sin.n\varphi)ap^nz^n\\ +&\frac{(n+1)(n+2)(n+3)\cdots(n+k-1)}{1\cdot2\cdot3\cdots(k-1)}(\cos.n\varphi-\sqrt{-1}\cdot\sin.n\varphi)bp^nz^n.\end{aligned}$$

Sit

$$a+b=f,\quad a-b=\frac{g}{\sqrt{-1}},$$

ut sit

$$a=\frac{f\sqrt{-1}+g}{2\sqrt{-1}} \quad\text{et}\quad b=\frac{f\sqrt{-1}-g}{2\sqrt{-1}},$$

eritque haec expressio

$$\frac{(n+1)(n+2)(n+3)\cdots(n+k-1)}{1\cdot 2\cdot 3\cdots(k-1)}(f\cos.n\varphi+g\sin.n\varphi)p^nz^n$$

terminus generalis seriei, quae oritur ex his fractionibus

$$\frac{\frac{1}{2}f+\frac{1}{2\sqrt{-1}}g}{(1-(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi)pz)^k}+\frac{\frac{1}{2}f-\frac{1}{2\sqrt{-1}}g}{(1-(\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi)pz)^k},$$

seu quae oritur ex hac fractione una

$$\frac{\left\{\begin{array}{l} f-kfpz\cos.\varphi+\frac{k(k-1)}{1\cdot 2}fp^2z^2\cos.2\varphi-\frac{k(k-1)(k-2)}{1\cdot 2\cdot 3}fp^3z^3\cos.3\varphi+\text{etc.}\\ \quad +kgpz\sin.\varphi-\frac{k(k-1)}{1\cdot 2}gp^2z^2\sin.2\varphi+\frac{k(k-1)(k-2)}{1\cdot 2\cdot 3}gp^3z^3\sin.3\varphi-\text{etc.}\end{array}\right\}}{(1-2pz\cos.\varphi+ppzz)^k}.$$

220. Posito ergo $k=2$ erit seriei ex hac fractione

$$\frac{f-2pz(f\cos.\varphi-g\sin.\varphi)+ppzz(f\cos.2\varphi-g\sin.2\varphi)}{(1-2pz\cos.\varphi+ppzz)^2}$$

ortae terminus generalis

$$(n+1)(f\cos.n\varphi+g\sin.n\varphi)p^nz^n.$$

At seriei ex hac fractione [§ 218]

$$\frac{a}{1-2pz\cos.\varphi+ppzz}$$

seu hac

$$\frac{a-2apz\cos.\varphi+appzz}{(1-2pz\cos.\varphi+ppzz)^2}$$

ortae terminus generalis est

$$\frac{a\sin.(n+1)\varphi}{\sin.\varphi}p^nz^n.$$

Addantur hae fractiones invicem ac ponatur

$$a+f=\mathfrak{A},$$

$$2a\cos.\varphi+2f\cos.\varphi-2g\sin.\varphi=-\mathfrak{B}$$

et

$$a+f\cos.2\varphi-g\sin.2\varphi=0;$$

hinc erit

$$g=\frac{\mathfrak{B}+2\mathfrak{A}\cos.\varphi}{2\sin.\varphi},$$

$$a=\frac{\mathfrak{A}+\mathfrak{B}\cos.\varphi}{1-\cos.2\varphi}=\frac{\mathfrak{A}+\mathfrak{B}\cos.\varphi}{2(\sin.\varphi)^2}$$

et

$$f=\frac{-\mathfrak{A}\cos.2\varphi-\mathfrak{B}\cos.\varphi}{2(\sin.\varphi)^2}$$

et

$$g=\frac{\mathfrak{B}\sin.\varphi+\mathfrak{A}\sin.2\varphi}{2(\sin.\varphi)^2}.$$

Hanc ob rem seriei ex hac fractione

$$\frac{\mathfrak{A}+\mathfrak{B}pz}{(1-2pz\cos.\varphi+ppzz)^2}$$

ortae terminus generalis est

$$\frac{\mathfrak{A}+\mathfrak{B}\cos.\varphi}{2(\sin.\varphi)^3}\sin.(n+1)\varphi\cdot p^nz^n$$

$$+(n+1)\frac{(\mathfrak{B}\sin.\varphi\sin.n\varphi+\mathfrak{A}\sin.2\varphi\sin.n\varphi-\mathfrak{B}\cos.\varphi\cos.n\varphi-\mathfrak{A}\cos.2\varphi\cos.n\varphi}{2(\sin.\varphi)^2}p^nz^n$$

$$=-\frac{(n+1)(\mathfrak{A}\cos.(n+2)\varphi+\mathfrak{B}\cos.(n+1)\varphi)}{2(\sin.\varphi)^2}p^nz^n+\frac{(\mathfrak{A}+\mathfrak{B}\cos.\varphi)\sin.(n+1)\varphi}{2(\sin.\varphi)^3}p^nz^n$$

$$=\frac{\frac{1}{2}(n+3)\sin.(n+1)\varphi-\frac{1}{2}(n+1)\sin.(n+3)\varphi}{2(\sin.\varphi)^3}\mathfrak{A}p^nz^n$$

$$+\frac{\frac{1}{2}(n+2)\sin.n\varphi-\frac{1}{2}n\sin.(n+2)\varphi}{2(\sin.\varphi)^3}\mathfrak{B}p^nz^n.$$

Est ergo iste terminus generalis quaesitus

$$=\frac{(n+3)\sin.(n+1)\varphi-(n+1)\sin.(n+3)\varphi}{4(\sin.\varphi)^3}\mathfrak{A}p^nz^n+\frac{(n+2)\sin.n\varphi-n\sin.(n+2)\varphi}{4(\sin.\varphi)^3}\mathfrak{B}p^nz^n$$

seriei, quae oritur ex fractione

$$\frac{\mathfrak{A}+\mathfrak{B}pz}{(1-2pz\cos.\varphi+ppzz)^2}.$$

221. Sit $k=3$ eritque seriei ex hac fractione ortae

$$\frac{f-3pz(f\cos.\varphi-g\sin.\varphi)+3ppzz(f\cos.2\varphi-g\sin.2\varphi)-p^3z^3(f\cos.3\varphi-g\sin.3\varphi)}{(1-2pz\cos.\varphi+ppzz)^3}$$

terminus generalis

$$=\frac{(n+1)(n+2)}{1\cdot 2}(f\cos.n\varphi+g\sin.n\varphi)p^nz^n.$$

Deinde seriei ex fractione

$$\frac{a+bpz}{(1-2pz\cos.\varphi+ppzz)^2}$$

seu ex hac

$$\frac{a-(2a\cos.\varphi-b)pz+(a-2b\cos.\varphi)ppzz+bp^3z^3}{(1-2pz\cos.\varphi+ppzz)^3}$$

ortae terminus generalis est

$$\frac{(n+3)\sin.(n+1)\varphi-(n+1)\sin.(n+3)\varphi}{4(\sin.\varphi)^3}ap^nz^n+\frac{(n+2)\sin.n\varphi-n\sin.(n+2)\varphi}{4(\sin.\varphi)^3}bp^nz^n$$

Addantur hae fractiones ac ponatur numerator $=\mathfrak{A}$; erit

$$a+f=\mathfrak{A},$$

$$3f\cos.\varphi-3g\sin.\varphi+2a\cos.\varphi-b=0,$$

$$3f\cos.2\varphi-3g\sin.2\varphi+a-2b\cos.\varphi=0$$

et

$$b=f\cos.3\varphi-g\sin.3\varphi;$$

hinc erit

$$a=\frac{f\cos.3\varphi-g\sin.3\varphi-3f\cos.\varphi+3g\sin.\varphi}{2\cos.\varphi}$$

$$=2g(\sin.\varphi)^2\operatorname{tang}.\varphi-f-2f(\sin.\varphi)^2.$$

Deinde reperitur

$$\frac{f}{g}=\frac{\sin.5\varphi-2\sin.3\varphi+\sin.\varphi}{\cos.5\varphi-2\cos.3\varphi+\cos.\varphi}$$

et

$$a+f=\mathfrak{A}=2g(\sin.\varphi)^2\operatorname{tang}.\varphi-2f(\sin.\varphi)^2,$$

ergo

$$\frac{\mathfrak{A}}{2(\sin.\varphi)^2}=\frac{g\sin.\varphi-f\cos.\varphi}{\cos.\varphi};$$

ex quibus tandem oritur

$$f=\frac{\mathfrak{A}(\sin.\varphi-2\sin.3\varphi+\sin.5\varphi)}{16(\sin.\varphi)^5},$$

$$g=\frac{\mathfrak{A}(\cos.\varphi-2\cos.3\varphi+\cos.5\varphi)}{16(\sin.\varphi)^5}.$$

Ob

$$16(\sin.\varphi)^5=\sin.5\varphi-5\sin.3\varphi+10\sin.\varphi$$

erit

$$a=\frac{\mathfrak{A}(9\sin.\varphi-3\sin.3\varphi)}{16(\sin.\varphi)^5}$$

et

$$b=\frac{\mathfrak{A}(-\sin.2\varphi+\sin.2\varphi)}{16(\sin.\varphi)^5}=0.$$

Est autem

$$3\sin.\varphi-\sin.3\varphi=4(\sin.\varphi)^3,$$

ergo

$$a=\frac{3\mathfrak{A}}{4(\sin.\varphi)^2}.$$

Quocirca erit terminus generalis

$$\frac{(n+1)(n+2)}{1\cdot 2}\mathfrak{A}p^nz^n\frac{\sin.(n+1)\varphi-2\sin.(n+3)\varphi+\sin.(n+5)\varphi}{16(\sin.\varphi)^5}$$

$$+3\mathfrak{A}p^nz^n\frac{(n+3)\sin.(n+1)\varphi-(n+1)\sin.(n+3)\varphi}{16(\sin.\varphi)^5}$$

$$=\frac{\mathfrak{A}p^nz^n}{16(\sin.\varphi)^5}\left\{\begin{aligned}&\frac{(n+4)(n+5)}{1\cdot 2}\sin.(n+1)\varphi-\frac{2(n+1)(n+5)}{1\cdot 2}\sin.(n+3)\varphi\\&\qquad+\frac{(n+1)(n+2)}{1\cdot 2}\sin.(n+5)\varphi\end{aligned}\right\}.$$

222. Seriei ergo, quae oritur ex hac fractione

$$\frac{\mathfrak{A}+\mathfrak{B}pz}{(1-2pz\cos.\varphi+ppzz)^3},$$

terminus generalis erit hic

$$\frac{\mathfrak{A}p^n z^n}{16(\sin.\varphi)^5}\left\{\begin{array}{l}\frac{(n+5)(n+4)}{1\cdot 2}\sin.(n+1)\varphi-\frac{2(n+1)(n+5)}{1\cdot 2}\sin.(n+3)\varphi\\ \qquad +\frac{(n+1)(n+2)}{1\cdot 2}\sin.(n+5)\varphi\end{array}\right\}$$

$$+\frac{\mathfrak{B}p^n z^n}{16(\sin.\varphi)^5}\left\{\begin{array}{l}\frac{(n+4)(n+3)}{1\cdot 2}\sin.n\varphi-\frac{2n(n+4)}{1\cdot 2}\sin.(n+2)\varphi\\ \qquad +\frac{n(n+1)}{1\cdot 2}\sin.(n+4)\varphi\end{array}\right\}.$$

Atque ulterius progrediendo seriei, quae oritur ex hac fractione

$$\frac{\mathfrak{A}+\mathfrak{B}pz}{(1-2pz\cos.\varphi+ppzz)^4},$$

terminus generalis erit hic

$$\frac{\mathfrak{A}p^n z^n}{64(\sin.\varphi)^7}\left\{\begin{array}{l}\frac{(n+7)(n+6)(n+5)}{1\cdot 2\cdot 3}\sin.(n+1)\varphi-\frac{3(n+1)(n+7)(n+6)}{1\cdot 2\cdot 3}\sin.(n+3)\varphi\\ +\frac{3(n+1)(n+2)(n+7)}{1\cdot 2\cdot 3}\sin.(n+5)\varphi-\frac{(n+1)(n+2)(n+3)}{1\cdot 2\cdot 3}\sin.(n+7)\varphi\end{array}\right\}$$

$$+\frac{\mathfrak{B}p^n z^n}{64(\sin.\varphi)^7}\left\{\begin{array}{l}\frac{(n+6)(n+5)(n+4)}{1\cdot 2\cdot 3}\sin.n\varphi-\frac{3n(n+6)(n+5)}{1\cdot 2\cdot 3}\sin.(n+2)\varphi\\ +\frac{3n(n+1)(n+6)}{1\cdot 2\cdot 3}\sin.(n+4)\varphi-\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}\sin.(n+6)\varphi\end{array}\right\}.$$

Ex his autem expressionibus facile intelligitur, quemadmodum formae terminorum generalium pro altioribus dignitatibus progrediantur. Ad naturam vero harum expressionum penitius inspiciendam notari convenit esse[1])

$$\begin{aligned}\sin.\varphi &= \sin.\varphi,\\ 4(\sin.\varphi)^3 &= 3\sin.\varphi-\sin.3\varphi,\\ 16(\sin.\varphi)^5 &= 10\sin.\varphi-5\sin.3\varphi+\sin.5\varphi,\\ 64(\sin.\varphi)^7 &= 35\sin.\varphi-21\sin.3\varphi+7\sin.5\varphi-\sin.7\varphi,\\ 256(\sin.\varphi)^9 &= 126\sin.\varphi-84\sin.3\varphi+36\sin.5\varphi-9\sin.7\varphi+\sin.9\varphi\\ &\text{etc.}\end{aligned}$$

1) Confer § 262. F. R.

223. Cum igitur hoc pacto omnes functiones fractae in fractiones partiales reales resolvi queant, simul omnium serierum recurrentium termini generales per expressiones reales exhiberi poterunt. Quod quo clarius appareat, exempla sequentia adiuncta sunt.

EXEMPLUM 1

Ex fractione

$$\frac{1}{(1-z)(1-zz)(1-z^3)}=\frac{1}{1-z-zz+z^4+z^5-z^6}$$

oritur ista series recurrens

$$1+z+2z^2+3z^3+4z^4+5z^5+7z^6+8z^7+10z^8+12z^9+\text{etc.},$$

cuius terminus generalis desideratur.

Fractio proposita secundum factores ordinata fit

$$-\frac{1}{(1-z)^3(1+z)(1+z+zz)},$$

quae resolvitur in has fractiones

$$\frac{1}{6(1-z)^3}+\frac{1}{4(1-z)^2}+\frac{17}{72(1-z)}+\frac{1}{8(1+z)}+\frac{2+z}{9(1+z+zz)}.$$

Harum prima $\frac{1}{6(1-z)^3}$ dat terminum generalem

$$\frac{(n+1)(n+2)}{1\cdot 2}\cdot\frac{1}{6}z^n=\frac{nn+3n+2}{12}z^n,$$

secunda $\frac{1}{4(1-z)^2}$ dat

$$\frac{n+1}{4}z^n,$$

tertia $\frac{17}{72(1-z)}$ dat

$$\frac{17}{72}z^n,$$

quarta $\frac{1}{8(1+z)}$ dat

$$\frac{1}{8}(-1)^n z^n.$$

Quinta vero $\frac{2+z}{9(1+z+zz)}$ comparata cum forma (§ 218)

$$\frac{\mathfrak{A}+\mathfrak{B}pz}{1-2pz\cos.\varphi+ppzz}$$

dat

$$p=-1, \quad \varphi=\frac{\pi}{3}=60^0, \quad \mathfrak{A}=+\frac{2}{9} \quad \text{et} \quad \mathfrak{B}=-\frac{1}{9},$$

unde oritur terminus generalis

$$\frac{2\sin.(n+1)\varphi-\sin.n\varphi}{9\sin.\varphi}(-1)^n z^n=\frac{4\sin.(n+1)\varphi-2\sin.n\varphi}{9\sqrt{3}}(-1)^n z^n$$

$$=\frac{4\sin.\frac{(n+1)\pi}{3}-2\sin.\frac{n\pi}{3}}{9\sqrt{3}}(-1)^n z^n.$$

Colligantur hae expressiones omnes in unam summam ac prodibit seriei propositae terminus generalis quaesitus

$$=\left(\frac{nn}{12}+\frac{n}{2}+\frac{47}{72}\right)z^n\pm\frac{1}{8}z^n\pm\frac{4\sin.\frac{(n+1)\pi}{3}-2\sin.\frac{n\pi}{3}}{9\sqrt{3}}z^n,$$

ubi signa superiora valent, si n numerus par, inferiora, sin impar. Ubi notandum est, si fuerit n numerus formae $3m$, fore

$$\frac{4\sin.\frac{(n+1)\pi}{3}-2\sin.\frac{n\pi}{3}}{9\sqrt{3}}=\pm\frac{2}{9};$$

si fuerit $n=3m+1$, erit haec expressio $=\mp\frac{1}{9}$; at si $n=3m+2$, erit ista expressio [iterum] $=\mp\frac{1}{9}$; [semper] prout n fuerit numerus vel par vel impar. Ex his natura seriei ita explicari potest, ut,

si fuerit	terminus generalis futurus sit
$n=6m+0$	$\left(\frac{nn}{12}+\frac{n}{2}+1\right)z^n$
$n=6m+1$	$\left(\frac{nn}{12}+\frac{n}{2}+\frac{5}{12}\right)z^n$
$n=6m+2$	$\left(\frac{nn}{12}+\frac{n}{2}+\frac{2}{3}\right)z^n$
$n=6m+3$	$\left(\frac{nn}{12}+\frac{n}{2}+\frac{3}{4}\right)z^n$
$n=6m+4$	$\left(\frac{nn}{12}+\frac{n}{2}+\frac{2}{3}\right)z^n$
$n=6m+5$	$\left(\frac{nn}{12}+\frac{n}{2}+\frac{5}{12}\right)z^n$

Sic si fuerit $n=50$, valet forma $n=6m+2$ eritque terminus seriei $=234z^{50}$.

EXEMPLUM 2

Ex fractione

$$\frac{1+z+zz}{1-z-z^4+z^5}$$

oritur haec series recurrens

$$1+2z+3zz+3z^3+4z^4+5z^5+6z^6+6z^7+7z^8+\text{etc.},$$

cuius terminum generalem invenire oportet.

Fractio proposita ad hanc formam reducitur

$$\frac{1+z+zz}{(1-z)^2(1+z)(1+zz)},$$

quae propterea resolvitur in has fractiones partiales

$$\frac{3}{4(1-z)^2}+\frac{3}{8(1-z)}+\frac{1}{8(1+z)}+\frac{-1+z}{4(1+zz)}.$$

Harum prima $\frac{3}{4(1-z)^2}$ dat terminum generalem

$$\frac{3(n+1)}{4}z^n,$$

secunda $\frac{3}{8(1-z)}$ dat

$$\frac{3}{8}z^n,$$

tertia $\frac{1}{8(1+z)}$ dat

$$\frac{1}{8}(-1)^n z^n$$

et quarta $\frac{-1+z}{4(1+zz)}$ comparata cum forma

$$\frac{\mathfrak{A}+\mathfrak{B}pz}{1-2pz\cos.\varphi+ppzz}$$

dat

$$p=1,\quad \cos.\varphi=0 \quad \text{et} \quad \varphi=\frac{\pi}{2},\quad \mathfrak{A}=-\frac{1}{4},\quad \mathfrak{B}=+\frac{1}{4},$$

unde fit terminus generalis

$$=\left(-\frac{1}{4}\sin.\frac{(n+1)\pi}{2}+\frac{1}{4}\sin.\frac{n\pi}{2}\right)z^n.$$

Quare colligendo erit terminus generalis quaesitus

$$-\left(\frac{3}{4}n+\frac{9}{8}\right)z^n \pm \frac{1}{8}z^n - \frac{1}{4}\left(\sin.\frac{(n+1)\pi}{2}-\sin.\frac{n\pi}{2}\right)z^n.$$

Hinc

si fuerit	erit terminus generalis
$n=4m+0$	$\left(\frac{3}{4}n+1\right)z^n$
$n=4m+1$	$\left(\frac{3}{4}n+\frac{5}{4}\right)z^n$
$n=4m+2$	$\left(\frac{3}{4}n+\frac{3}{2}\right)z^n$
$n=4m+3$	$\left(\frac{3}{4}n+\frac{3}{4}\right)z^n$

Ita si $n=50$, valebit $n=4m+2$ eritque terminus $=39z^{50}$.

224. Proposita ergo serie recurrente, quoniam illa fractio, unde oritur, facile cognoscitur, eius terminus generalis secundum praecepta data reperietur. Ex lege autem seriei recurrentis, qua quisque terminus ex praecedentibus definitur, statim innotescit denominator fractionis huiusque factores praebe-

bunt formam termini generalis; per numeratorem enim tantum coefficientes determinantur. Sit nempe proposita haec series recurrens

$$A+Bz+Cz^2+Dz^3+Ez^4+Fz^5+\text{etc.},$$

cuius lex progressionis, qua unusquisque terminus ex aliquot praecedentibus determinatur, praebeat hunc fractionis denominatorem

$$1-\alpha z-\beta z^2-\gamma z^3,$$

ita ut sit

$$D=\alpha C+\beta B+\gamma A,\quad E=\alpha D+\beta C+\gamma B,\quad F=\alpha E+\beta D+\gamma C \quad \text{etc.},$$

qui multiplicatores $+\alpha$, $+\beta$, $+\gamma$ a MOIVREO *scalam relationis*[1]) constituere dicuntur. Lex ergo progressionis posita est in scala relationis atque scala relationis statim praebet denominatorem fractionis, ex cuius resolutione proposita series recurrens oritur.

225. Ad terminum ergo generalem seu coefficientem potestatis indefinitae z^n inveniendum quaeri debent denominatoris $1-\alpha z-\beta z^2-\gamma z^3$ factores vel simplices vel duplices, si imaginarios vitare velimus. Sint primo factores simplices omnes inter se inaequales et reales hi

$$(1-pz)(1-qz)(1-rz)$$

atque fractio generans seriem propositam resolvetur in

$$\frac{\mathfrak{A}}{1-pz}+\frac{\mathfrak{B}}{1-qz}+\frac{\mathfrak{C}}{1-rz};$$

unde seriei terminus generalis erit

$$(\mathfrak{A}p^n+\mathfrak{B}q^n+\mathfrak{C}r^n)z^n.$$

1) Accuratius *indicem seu scalam relationis*; vide p. 27 libri, qui inscribitur *Miscellanea analytica*, nota p. 79 laudati. In dissertatione enim *De fractionibus algebraicis* eadem nota laudata tantum significatio *index relationis* invenitur. F. R.

Si duo factores fuerint aequales, nempe $q = p$, tum terminus generalis huiusmodi erit[1])

$$\left((\mathfrak{A}(n+1) + \mathfrak{B})p^n + \mathfrak{C}r^n\right)z^n,$$

et si insuper fuerit $r = q = p$, erit terminus generalis[2])

$$\left(\mathfrak{A}\frac{(n+1)(n+2)}{1\cdot 2} + \mathfrak{B}(n+1) + \mathfrak{C}\right)p^n z^n.$$

Quodsi vero denominator $1 - \alpha z - \beta z^2 - \gamma z^3$ duplicem habeat factorem, ut sit

$$= (1 - pz)(1 - 2qz \cos.\varphi + qqzz),$$

tum terminus generalis erit

$$= \left(\mathfrak{A}p^n + \frac{\mathfrak{B}\sin.(n+1)\varphi + \mathfrak{C}\sin.n\varphi}{\sin.\varphi}q^n\right)z^n.$$

Cum igitur positis pro n successive numeris 0, 1, 2 prodire debeant termini A, Bz, Cz^2, hinc valores litterarum $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ determinabuntur.

226. Sit scala relationis bimembris seu determinetur quisque terminus per duos praecedentes, ita ut sit

$$C = \alpha B - \beta A, \quad D = \alpha C - \beta B, \quad E = \alpha D - \beta C \quad \text{etc.},$$

atque manifestum est seriem hanc recurrentem, quae sit

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + \cdots + Pz^n + Qz^{n+1} + \text{etc.},$$

oriri ex fractione, cuius denominator sit

$$1 - \alpha z + \beta zz.$$

Sint huius denominatoris factores

$$(1 - pz)(1 - qz);$$

erit

$$p + q = \alpha \quad \text{et} \quad pq = \beta$$

1) Editio princeps: $((\mathfrak{A}n + \mathfrak{B})p^n + \mathfrak{C}r^n)z^n$. Correxit F. R.
2) Editio princeps: $(\mathfrak{A}n^2 + \mathfrak{B}n + \mathfrak{C})p^n z^n$. Correxit F. R.

atque seriei terminus generalis erit

$$(\mathfrak{A}p^n + \mathfrak{B}q^n)z^n.$$

Hinc facto $n = 0$ erit

$$A = \mathfrak{A} + \mathfrak{B}$$

et facto $n = 1$ erit

$$B = \mathfrak{A}p + \mathfrak{B}q,$$

unde fit

$$Aq - B = \mathfrak{A}(q - p)$$

et

$$\mathfrak{A} = \frac{Aq - B}{q - p} \quad \text{et} \quad \mathfrak{B} = \frac{Ap - B}{p - q}.$$

Inventis autem valoribus $\mathfrak{A}$ et $\mathfrak{B}$ erit

$$P = \mathfrak{A}p^n + \mathfrak{B}q^n \quad \text{et} \quad Q = \mathfrak{A}p^{n+1} + \mathfrak{B}q^{n+1}.$$

Tum vero erit

$$\mathfrak{A}\mathfrak{B} = \frac{BB - \alpha AB + \beta AA}{4\beta - \alpha\alpha}.$$

227. Hinc deduci potest modus quemvis terminum ex unico praecedente formandi, cum ad hoc per legem progressionis duo requirantur. Cum enim sit

$$P = \mathfrak{A}p^n + \mathfrak{B}q^n \quad \text{et} \quad Q = \mathfrak{A}p \cdot p^n + \mathfrak{B}q \cdot q^n,$$

erit

$$Pq - Q = \mathfrak{A}(q - p)p^n \quad \text{et} \quad Pp - Q = \mathfrak{B}(p - q)q^n.$$

Multiplicentur hae expressiones in se invicem eritque

$$P^2 pq - (p + q)PQ + QQ + \mathfrak{A}\mathfrak{B}(p - q)^2 p^n q^n = 0.$$

At est

$$p + q = \alpha, \quad pq = \beta, \quad (p - q)^2 = (p + q)^2 - 4pq = \alpha\alpha - 4\beta \quad \text{et} \quad p^n q^n = \beta^n.$$

Quibus substitutis habebitur

$$\beta P^2 - \alpha PQ + QQ = (\beta AA - \alpha AB + BB)\beta^n$$

seu

$$\frac{QQ - \alpha PQ + \beta PP}{BB - \alpha AB + \beta AA} = \beta^n,$$

quae est insignis proprietas serierum recurrentium, quarum quisque terminus per duos praecedentes determinatur. At cognito quovis termino P erit sequens

$$Q = \tfrac{1}{2}\alpha P + \sqrt{\left(\left(\tfrac{1}{4}\alpha^2 - \beta\right)P^2 + \left(BB - \alpha AB + \beta AA\right)\beta^n\right)},$$

quae expressio, etsi speciem irrationalitatis prae se fert, tamen semper est rationalis, propterea quod termini irrationales in serie non occurrunt.

228. Ex datis porro duobus terminis contiguis quibusvis Pz^n et Qz^{n+1} commode assignari potest terminus multo magis remotus Xz^{2n}. Ponatur enim

$$X = fP^2 + gPQ - h\mathfrak{A}\mathfrak{B}\beta^n.$$

Quoniam est

$$P = \mathfrak{A}p^n + \mathfrak{B}q^n \quad \text{et} \quad Q = \mathfrak{A}p\cdot p^n + \mathfrak{B}q\cdot q^n \quad \text{atque} \quad X = \mathfrak{A}p^{2n} + \mathfrak{B}q^{2n},$$

erit ut sequitur:

$$\begin{array}{rl} fP^2 = & f\mathfrak{A}^2p^{2n} + f\mathfrak{B}^2q^{2n} + 2f\mathfrak{A}\mathfrak{B}\beta^n \\ gPQ = & g\mathfrak{A}^2p\cdot p^{2n} + g\mathfrak{B}^2q\cdot q^{2n} + g\mathfrak{A}\mathfrak{B}\alpha\beta^n \\ -h\mathfrak{A}\mathfrak{B}\beta^n = & \qquad\qquad\qquad - h\mathfrak{A}\mathfrak{B}\beta^n \\ \hline X = & \mathfrak{A}p^{2n} + \mathfrak{B}q^{2n} \end{array}$$

Fiet ergo

$$f + gp = \frac{1}{\mathfrak{A}}, \quad f + gq = \frac{1}{\mathfrak{B}} \quad \text{et} \quad h = 2f + g\alpha,$$

unde

$$g = \frac{\mathfrak{B} - \mathfrak{A}}{\mathfrak{A}\mathfrak{B}(p-q)} \quad \text{et} \quad f = \frac{\mathfrak{A}p - \mathfrak{B}q}{\mathfrak{A}\mathfrak{B}(p-q)}.$$

At est

$$\mathfrak{B} - \mathfrak{A} = \frac{\alpha A - 2B}{p-q}, \quad \mathfrak{A}p - \mathfrak{B}q = \frac{\alpha B - 2A\beta}{p-q}.$$

Ergo

$$f = \frac{\alpha B - 2A\beta}{\mathfrak{A}\mathfrak{B}(\alpha\alpha - 4\beta)} \quad \text{et} \quad g = \frac{\alpha A - 2B}{\mathfrak{A}\mathfrak{B}(\alpha\alpha - 4\beta)}$$

seu

$$f = \frac{2A\beta - \alpha B}{BB - \alpha AB + \beta AA} \quad \text{et} \quad g = \frac{2B - \alpha A}{BB - \alpha AB + \beta AA};$$

ideoque

$$h = \frac{(4\beta - \alpha\alpha)A}{BB - \alpha AB + \beta AA}.$$

Eritque ergo

$$X = \frac{(2A\beta - \alpha B)P^2 + (2B - \alpha A)PQ}{BB - \alpha AB + \beta AA} - A\beta^n.$$

Simili vero modo reperitur

$$X = \frac{(\alpha\beta A - (\alpha\alpha - 2\beta)B)P^2 + (2B - \alpha A)Q^2}{\alpha(BB - \alpha AB + \beta AA)} - \frac{2B\beta^n}{\alpha}.$$

His coniungendis per eliminationem termini β^n reperitur

$$X = \frac{(\beta A - \alpha B)P^2 + 2BPQ - AQQ}{BB - \alpha AB + \beta AA}.\text{[1]}$$

229. Simili modo si statuantur termini sequentes

$$A + Bz + Cz^2 + \cdots + Pz^n + Qz^{n+1} + Rz^{n+2} + \cdots + Xz^{2n} + Yz^{2n+1} + Zz^{2n+2} + \text{etc.},$$

erit

$$Z = \frac{(\beta A - \alpha B)Q^2 + 2BQR - ARR}{BB - \alpha AB + \beta AA}$$

et ob $R = \alpha Q - \beta P$ erit

$$Z = \frac{-\beta\beta AP^2 + 2\beta(\alpha A - B)PQ + (\alpha B - (\alpha\alpha - \beta)A)Q^2}{BB - \alpha AB + \beta AA}.$$

At est $Z = \alpha Y - \beta X$, ergo $Y = \frac{Z + \beta X}{\alpha}$; unde fit

$$Y = \frac{-\beta BP^2 + 2\beta APQ + (B - \alpha A)QQ}{BB - \alpha AB + \beta AA}.$$

Sic igitur porro ex X et Y definiri poterunt simili modo coefficientes potestatum z^{4n} et z^{4n+1} hincque ipsarum z^{8n}, z^{8n+1}, et ita porro.

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1) Quae resolutio statim iam ex prima aequatione pro X inventa elicitur ponendo (§ 227) $\beta^n = \frac{QQ - \alpha PQ + \beta PP}{BB - \alpha AB + \beta AA}$. F. R.

EXEMPLUM

Sit proposita ista series recurrens

$$1 + 3z + 4z^2 + 7z^3 + 11z^4 + 18z^5 + \cdots + Pz^n + Qz^{n+1} + \text{etc.};$$

cuius cum quilibet coefficiens sit summa duorum praecedentium, erit denominator fractionis hanc seriem producentis

$$1 - z - zz;$$

ideoque

$$\alpha = 1, \quad \beta = -1 \quad \text{et} \quad A = 1, \quad B = 3,$$

unde fit

$$BB - \alpha AB + \beta AA = 5.$$

Ex quo orietur primum

$$Q = \frac{P + \sqrt{(5PP + 20(-1)^n)}}{2} = \frac{P + \sqrt{(5PP \pm 20)}}{2},$$

ubi signum superius valet, si n sit numerus par, inferius, si impar. Sic si $n = 4$, ob $P = 11$ erit

$$Q = \frac{11 + \sqrt{(5 \cdot 121 + 20)}}{2} = \frac{11 + 25}{2} = 18.$$

Si porro coefficiens termini z^{2n} sit X, erit

$$X = \frac{-4PP + 6PQ - QQ}{5};$$

ergo potestatis z^8 coefficiens erit

$$= \frac{-4 \cdot 121 + 6 \cdot 198 - 324}{5} = 76.$$

Cum autem sit

$$Q = \frac{P + \sqrt{(5PP \pm 20)}}{2},$$

erit

$$QQ = \frac{3PP \pm 10 + P\sqrt{(5PP \pm 20)}}{2}$$

ideoque

$$X = \frac{-PP \mp 2 + P\sqrt{(5PP \pm 20)}}{2}.$$

Ex termino ergo seriei quocunque Pz^n obtinentur hi

$$\frac{P + \sqrt{(5PP \pm 20)}}{2} z^{n+1} \quad \text{et} \quad \frac{-PP \mp 2 + P\sqrt{(5PP \pm 20)}}{2} z^{2n}.$$

230. Simili modo in seriebus recurrentibus, quarum quilibet terminus ex tribus antecedentibus determinatur, quivis terminus ex duobus antecedentibus definiri potest. Sit enim series huiusmodi recurrens

$$A + Bz + Cz^2 + Dz^3 + \cdots + Pz^n + Qz^{n+1} + Rz^{n+2} + \text{etc.},$$

cuius scala relationis sit $\alpha, -\beta, +\gamma$ seu quae oriatur ex fractione, cuius denominator

$$= 1 - \alpha z + \beta z^2 - \gamma z^3.$$

Quodsi iam termini P, Q, R eodem modo per factores huius denominatoris, qui sint

$$(1 - pz)(1 - qz)(1 - rz),$$

exprimantur, ut sit

$$P = \mathfrak{A}p^n + \mathfrak{B}q^n + \mathfrak{C}r^n,$$

$$Q = \mathfrak{A}p \cdot p^n + \mathfrak{B}q \cdot q^n + \mathfrak{C}r \cdot r^n$$

et

$$R = \mathfrak{A}p^2 \cdot p^n + \mathfrak{B}q^2 \cdot q^n + \mathfrak{C}r^2 \cdot r^n,$$

ob

$$p + q + r = \alpha, \quad pq + pr + qr = \beta \quad \text{et} \quad pqr = \gamma$$

reperietur haec proportio[1])

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1) Quae proportio reperietur computando valores ipsorum $\mathfrak{A}p^n$, $\mathfrak{B}q^n$, $\mathfrak{C}r^n$ ex aequationibus

$$\mathfrak{A}p^n + \mathfrak{B}q^n + \mathfrak{C}r^n = P,$$
$$\mathfrak{A}p^n \cdot p + \mathfrak{B}q^n \cdot q + \mathfrak{C}r^n \cdot r = Q,$$
$$\mathfrak{A}p^n \cdot p^2 + \mathfrak{B}q^n \cdot q^2 + \mathfrak{C}r^n \cdot r^2 = R$$

atque simili modo valores ipsorum $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ ex aequationibus correspondentibus

$$\begin{array}{rl} R^3 - 2\alpha Q R^2 + (\alpha\alpha + \beta) Q^2 R - (\alpha\beta - \gamma) Q^3 : \gamma^n \\ + \beta P - (\alpha\beta + 3\gamma) PQ + (\alpha\gamma + \beta\beta) PQ^2 \\ + \alpha\gamma P^2 - 2\beta\gamma P^2 Q \\ + \gamma\gamma P^3 \\ = C^3 - 2\alpha B C^2 + (\alpha\alpha + \beta) B^2 C - (\alpha\beta - \gamma) B^3 : 1. \\ + \beta A - (\alpha\beta + 3\gamma) AB + (\alpha\gamma + \beta\beta) AB^2 \\ + \alpha\gamma A^2 - 2\beta\gamma A^2 B \\ + \gamma\gamma A^3 \end{array}$$

Pendet ergo inventio termini R ex duobus praecedentibus P et Q a resolutione aequationis cubicae.

231. His de terminis generalibus serierum recurrentium notatis superest, ut earundem serierum summas investigemus. Ac primo quidem manifestum est summam seriei recurrentis in infinitum extensae aequalem esse fractioni, ex qua oritur; cuius fractionis cum denominator ex ipsa progressionis lege pateat, reliquum est, ut numeratorem definiamus. Sit itaque proposita haec series

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + Gz^6 + \text{etc.},$$

cuius lex progressionis praebeat hunc denominatorem

$$1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4.$$

$$\begin{aligned} \mathfrak{A} + \mathfrak{B} + \mathfrak{C} &= A, \\ \mathfrak{A}p + \mathfrak{B}q + \mathfrak{C}r &= B, \\ \mathfrak{A}p^2 + \mathfrak{B}q^2 + \mathfrak{C}r^2 &= C. \end{aligned}$$

Solutiones inter se multiplicando obtinetur

$$\begin{aligned} &(Pqr - Q(q+r) + R)(Prp - Q(r+p) + R)(Ppq - Q(p+q) + R) \\ &: (Aqr - B(q+r) + C)(Arp - B(r+p) + C)(Apq - B(p+q) + C) \\ &= p^n q^n r^n : 1 = \gamma^n : 1, \end{aligned}$$

quae aequatio secundum potestates ipsorum R et C evoluta ope relationum $p + q + r = \alpha$, $qr + rp + pq = \beta$, $pqr = \gamma$ proportionem illam praebet. F. R.

Sumamus fractionem summae seriei in infinitum [extensae] aequalem esse

$$= \frac{a + bz + cz^2 + dz^3}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4};$$

ex qua cum series proposita oriri debeat, erit comparando

$$\begin{aligned} a &= A, \\ b &= B - \alpha A, \\ c &= C - \alpha B + \beta A, \\ d &= D - \alpha C + \beta B - \gamma A. \end{aligned}$$

Hinc erit summa quaesita

$$\frac{A + (B - \alpha A)z + (C - \alpha B + \beta A)z^2 + (D - \alpha C + \beta B - \gamma A)z^3}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}.$$

232. Hinc facile intelligitur, quemadmodum seriei recurrentis summa ad datum terminum usque inveniri debeat. Quaeratur scilicet seriei modo assumptae summa ad terminum Pz^n atque ponatur

$$s = A + Bz + Cz^2 + Dz^3 + Ez^4 + \cdots + Pz^n.$$

Quoniam huius seriei summa in infinitum constat, quaeratur summa terminorum ultimum Pz^n in infinitum sequentium, qui sint

$$t = Qz^{n+1} + Rz^{n+2} + Sz^{n+3} + Tz^{n+4} + \text{etc.};$$

haec series per z^{n+1} divisa dat seriem recurrentem propositae aequalem, cuius propterea summa erit

$$t = \frac{Qz^{n+1} + (R - \alpha Q)z^{n+2} + (S - \alpha R + \beta Q)z^{n+3} + (T - \alpha S + \beta R - \gamma Q)z^{n+4}}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}.$$

Unde orietur summa quaesita

$$s = \frac{A + (B - \alpha A)z + (C - \alpha B + \beta A)z^2 + (D - \alpha C + \beta B - \gamma A)z^3}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}$$

$$- \frac{Qz^{n+1} + (R - \alpha Q)z^{n+2} + (S - \alpha R + \beta Q)z^{n+3} + (T - \alpha S + \beta R - \gamma Q)z^{n+4}}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}.$$

233. Quodsi ergo scala relationis fuerit bimembris $\alpha, -\beta$, seriei

$$A + Bz + Cz^2 + Dz^3 + \cdots + Pz^n,$$

quae oritur ex fractione

$$\frac{A + (B - \alpha A)z}{1 - \alpha z + \beta zz},$$

summa erit

$$\frac{A + (B - \alpha A)z - Qz^{n+1} - (R - \alpha Q)z^{n+2}}{1 - \alpha z + \beta zz}.$$

At est ex natura seriei

$$R = \alpha Q - \beta P,$$

unde prodibit summa

$$\frac{A + (B - \alpha A)z - Qz^{n+1} + \beta Pz^{n+2}}{1 - \alpha z + \beta zz}.$$

EXEMPLUM

Sit proposita series

$$1 + 3z + 4z^2 + 7z^3 + \cdots + Pz^n,$$

ubi est

$$\alpha = 1, \quad \beta = -1, \quad A = 1, \quad B = 3;$$

erit huius summa

$$\frac{1 + 2z - Qz^{n+1} - Pz^{n+2}}{1 - z - zz}.$$

Posito vero $z = 1$ erit summa seriei

$$1 + 3 + 4 + 7 + 11 + \cdots + P$$
$$= P + Q - 3.$$

Summa ergo termini ultimi et sequentis ternario excedit summam seriei. Quia vero est

$$Q = \frac{P + \sqrt{(5PP \pm 20)}}{2},$$

erit summa seriei

$$1 + 3 + 4 + 7 + 11 + \cdots + P$$
$$= \frac{3P - 6 + \sqrt{(5PP \pm 20)}}{2}.$$

Ex solo ergo termino ultimo summa potest exhiberi.

CAPUT XIV

DE MULTIPLICATIONE AC DIVISIONE ANGULORUM

234. Sit angulus vel arcus in circulo, cuius radius $= 1$, quicunque $= z$, eius sinus $= x$, cosinus $= y$ et tangens $= t$; erit

$$xx + yy = 1 \quad \text{et} \quad t = \frac{x}{y}.$$

Cum igitur, uti supra [§ 129] vidimus, tam sinus quam cosinus angulorum z, $2z$, $3z$, $4z$, $5z$ etc. constituant seriem recurrentem, cuius scala relationis est $2y$, -1, primum sinus horum arcuum ita se habebunt:

$$\begin{aligned}
\sin. 0z &= 0,\\
\sin. 1z &= x,\\
\sin. 2z &= 2xy,\\
\sin. 3z &= 4xy^2 - x,\\
\sin. 4z &= 8xy^3 - 4xy,\\
\sin. 5z &= 16xy^4 - 12xy^2 + x,\\
\sin. 6z &= 32xy^5 - 32xy^3 + 6xy,\\
\sin. 7z &= 64xy^6 - 80xy^4 + 24xy^2 - x,\\
\sin. 8z &= 128xy^7 - 192xy^5 + 80xy^3 - 8xy.
\end{aligned}$$

Hinc concluditur fore

$$\sin. nz = x\left\{\begin{array}{c} 2^{n-1}y^{n-1} - (n-2)2^{n-3}y^{n-3} \\ + \dfrac{(n-3)(n-4)}{1\cdot 2}2^{n-5}y^{n-5} - \dfrac{(n-4)(n-5)(n-6)}{1\cdot 2\cdot 3}2^{n-7}y^{n-7} \\ + \dfrac{(n-5)(n-6)(n-7)(n-8)}{1\cdot 2\cdot 3\cdot 4}2^{n-9}y^{n-9} - \text{etc.} \end{array}\right\}.$$

235. Si ponamus arcum $nz = s$, erit

$$\sin. nz = \sin. s = \sin. (\pi - s) = \sin. (2\pi + s) = \sin. (3\pi - s) \text{ etc.};$$

hi enim sinus omnes sunt inter se aequales. Hinc obtinemus plures valores pro x, qui erunt

$$\sin. \frac{s}{n}, \quad \sin. \frac{\pi - s}{n}, \quad \sin. \frac{2\pi + s}{n}, \quad \sin. \frac{3\pi - s}{n}, \quad \sin. \frac{4\pi + s}{n} \quad \text{etc.},$$

qui ergo omnes aequationi inventae aeque conveniunt. Tot autem prodibunt diversi pro x valores, quot numerus n continet unitates, qui propterea erunt radices aequationis inventae. Cavendum ergo est, ne valores aequales pro iisdem habeantur, quod fiet, dum alternae tantum expressiones assumantur. Cognitis igitur radicibus aequationis a posteriori, earum comparatio cum terminis aequationis notatu dignas praebebit proprietates. Quoniam autem ad hoc aequatio, in qua tantum x tamquam incognita insit, requiritur, pro y suus valor $\sqrt{(1 - xx)}$ substitui debet; unde duplex operatio instituenda erit, prout n fuerit vel numerus par vel impar.

236. Sit n numerus impar; quia arcuum $-z$, $+z$, $+3z$, $+5z$ etc. differentia est $2z$ huiusque cosinus $= 1 - 2xx$, erit progressionis sinuum scala relationis haec $2 - 4xx$, -1. Hinc erit

$$\begin{aligned}
\sin. -z &= -x,\\
\sin. z &= x,\\
\sin. 3z &= 3x - 4x^3,\\
\sin. 5z &= 5x - 20x^3 + 16x^5,\\
\sin. 7z &= 7x - 56x^3 + 112x^5 - 64x^7,\\
\sin. 9z &= 9x - 120x^3 + 432x^5 - 576x^7 + 256x^9.
\end{aligned}$$

Ergo

$$\sin. nz = nx - \frac{n(nn-1)}{1\cdot 2\cdot 3}x^3 + \frac{n(nn-1)(nn-9)}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^5 - \frac{n(nn-1)(nn-9)(nn-25)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}x^7 + \text{etc.},$$

siquidem n fuerit numerus impar. Huiusque aequationis radices sunt

$$\sin. z,\quad \sin.\left(\frac{2\pi}{n}+z\right),\quad \sin.\left(\frac{4\pi}{n}+z\right),\quad \sin.\left(\frac{6\pi}{n}+z\right),\quad \sin.\left(\frac{8\pi}{n}+z\right)\ \text{etc.},$$

quarum numerus est n.

237. Huius ergo aequationis

$$0 = 1 - \frac{nx}{\sin. nz} + \frac{n(nn-1)x^3}{1\cdot 2\cdot 3\sin. nz} - \frac{n(nn-1)(nn-9)x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5\sin. nz} + \cdots \pm \frac{2^{n-1}x^n}{\sin. nz}$$

(ubi signum superius valet, si n unitate deficiat a multiplo quaternarii, contra inferius) factores sunt

$$\left(1-\frac{x}{\sin. z}\right)\left(1-\frac{x}{\sin.\left(\frac{2\pi}{n}+z\right)}\right)\left(1-\frac{x}{\sin.\left(\frac{4\pi}{n}+z\right)}\right)\ \text{etc.},$$

ex quibus concluditur fore

$$\frac{n}{\sin. nz} = \frac{1}{\sin. z} + \frac{1}{\sin.\left(\frac{2\pi}{n}+z\right)} + \frac{1}{\sin.\left(\frac{4\pi}{n}+z\right)} + \frac{1}{\sin.\left(\frac{6\pi}{n}+z\right)} + \text{etc.},$$

donec habeantur n termini. Tum vero productum omnium erit

$$\mp\frac{2^{n-1}}{\sin. nz} = \frac{1}{\sin. z\sin.\left(\frac{2\pi}{n}+z\right)\sin.\left(\frac{4\pi}{n}+z\right)\sin.\left(\frac{6\pi}{n}+z\right)\ \text{etc.}}$$

seu

$$\sin. nz = \mp 2^{n-1}\sin. z\sin.\left(\frac{2\pi}{n}+z\right)\sin.\left(\frac{4\pi}{n}+z\right)\sin.\left(\frac{6\pi}{n}+z\right)\ \text{etc.}$$

Et, quia terminus penultimus deest, erit

$$0 = \sin. z + \sin.\left(\frac{2\pi}{n}+z\right) + \sin.\left(\frac{4\pi}{n}+z\right) + \sin.\left(\frac{6\pi}{n}+z\right) + \text{etc.}$$

EXEMPLUM 1

Si ergo fuerit $n = 3$, prodibunt hae aequalitates

$$0 = \sin. z + \sin.(120+z) + \sin.(240+z) = \sin. z + \sin.(60-z) - \sin.(60+z),$$

$$\frac{3}{\sin. 3z} = \frac{1}{\sin. z} + \frac{1}{\sin.(120+z)} + \frac{1}{\sin.(240+z)} = \frac{1}{\sin. z} + \frac{1}{\sin.(60-z)} - \frac{1}{\sin.(60+z)},$$

$$\sin. 3z = -4\sin. z\sin.(120+z)\sin.(240+z) = 4\sin. z\sin.(60-z)\sin.(60+z).$$

Erit ergo, uti iam supra [§ 131] notavimus,

$$\sin.(60+z) = \sin. z + \sin.(60-z)$$

et

$$3\,\text{cosec.}\,3z = \text{cosec.}\,z + \text{cosec.}(60-z) - \text{cosec.}(60+z).$$

EXEMPLUM 2

Ponamus esse $n = 5$ atque prodibunt hae aequationes

$$0 = \sin. z + \sin.\left(\frac{2\pi}{5}+z\right) + \sin.\left(\frac{4\pi}{5}+z\right) + \sin.\left(\frac{6\pi}{5}+z\right) + \sin.\left(\frac{8\pi}{5}+z\right)$$

seu

$$0 = \sin. z + \sin.\left(\frac{2\pi}{5}+z\right) + \sin.\left(\frac{\pi}{5}-z\right) - \sin.\left(\frac{\pi}{5}+z\right) - \sin.\left(\frac{2\pi}{5}-z\right)$$

seu

$$0 = \sin. z + \sin.\left(\frac{\pi}{5}-z\right) - \sin.\left(\frac{\pi}{5}+z\right) - \sin.\left(\frac{2\pi}{5}-z\right) + \sin.\left(\frac{2\pi}{5}+z\right).$$

Deinde erit

$$\frac{5}{\sin. 5z} = \frac{1}{\sin. z} + \frac{1}{\sin. \left(\frac{\pi}{5} - z\right)} - \frac{1}{\sin. \left(\frac{\pi}{5} + z\right)} - \frac{1}{\sin. \left(\frac{2\pi}{5} - z\right)} + \frac{1}{\sin. \left(\frac{2\pi}{5} + z\right)},$$

$$\sin. 5z = 16 \sin. z \sin. \left(\frac{\pi}{5} - z\right) \sin. \left(\frac{\pi}{5} + z\right) \sin. \left(\frac{2\pi}{5} - z\right) \sin. \left(\frac{2\pi}{5} + z\right).$$

EXEMPLUM 3

Hoc modo, si ponamus $n = 2m + 1$, erit

$$\begin{aligned} 0 = \sin. z &+ \sin. \left(\frac{\pi}{n} - z\right) - \sin. \left(\frac{\pi}{n} + z\right) \\ &- \sin. \left(\frac{2\pi}{n} - z\right) + \sin. \left(\frac{2\pi}{n} + z\right) \\ &+ \sin. \left(\frac{3\pi}{n} - z\right) - \sin. \left(\frac{3\pi}{n} + z\right) \\ &\vdots \\ &\pm \sin. \left(\frac{m\pi}{n} - z\right) \mp \sin. \left(\frac{m\pi}{n} + z\right), \end{aligned}$$

ubi signa superiora valent, si m sit numerus impar, inferiora, si sit par. Altera aequatio erit haec

$$\begin{aligned} \frac{n}{\sin. nz} = \frac{1}{\sin. z} &+ \frac{1}{\sin. \left(\frac{\pi}{n} - z\right)} - \frac{1}{\sin. \left(\frac{\pi}{n} + z\right)} \\ &- \frac{1}{\sin. \left(\frac{2\pi}{n} - z\right)} + \frac{1}{\sin. \left(\frac{2\pi}{n} + z\right)} \\ &+ \frac{1}{\sin. \left(\frac{3\pi}{n} - z\right)} - \frac{1}{\sin. \left(\frac{3\pi}{n} + z\right)} \\ &\vdots \\ &\pm \frac{1}{\sin. \left(\frac{m\pi}{n} - z\right)} \mp \frac{1}{\sin. \left(\frac{m\pi}{n} + z\right)}, \end{aligned}$$

quae ad cosecantes commode transfertur. Tertio habetur hoc productum

$$\begin{aligned} \sin. nz = 2^{2m} \sin. z &\sin. \left(\frac{\pi}{n} - z\right) \sin. \left(\frac{\pi}{n} + z\right) \\ &\sin. \left(\frac{2\pi}{n} - z\right) \sin. \left(\frac{2\pi}{n} + z\right) \\ &\sin. \left(\frac{3\pi}{n} - z\right) \sin. \left(\frac{3\pi}{n} + z\right) \\ &\vdots \\ &\sin. \left(\frac{m\pi}{n} - z\right) \sin. \left(\frac{m\pi}{n} + z\right). \end{aligned}$$

238. Sit n nunc numerus par, et quoniam est

$$y = \sqrt{(1 - xx)} \quad \text{et} \quad \cos. 2z = 1 - 2xx,$$

ita ut seriei sinuum sit scala relationis ut ante $2 - 4xx, -1$, erit

$$\begin{aligned} &\sin. 0z = 0, \\ &\sin. 2z = 2x\sqrt{(1 - xx)}, \\ &\sin. 4z = (4x - 8x^3)\sqrt{(1 - xx)}, \\ &\sin. 6z = (6x - 32x^3 + 32x^5)\sqrt{(1 - xx)}, \\ &\sin. 8z = (8x - 80x^3 + 192x^5 - 128x^7)\sqrt{(1 - xx)} \end{aligned}$$

et generaliter

$$\sin. nz = \left\{ \begin{array}{l} nx - \frac{n(nn-4)}{1 \cdot 2 \cdot 3} x^3 + \frac{n(nn-4)(nn-16)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} x^5 \\ - \frac{n(nn-4)(nn-16)(nn-36)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} x^7 + \cdots \pm 2^{n-1} x^{n-1} \end{array} \right\} \sqrt{(1 - xx)}$$

denotante n numerum quemcunque parem.

239. Ad aequationem hanc rationalem efficiendam sumantur utrinque quadrata ac prodibit huiusmodi aequatio

$$(\sin. nz)^2 = nnxx + Px^4 + Qx^6 + \cdots - 2^{2n-2} x^{2n}$$

seu

$$x^{2n} - \cdots - \frac{nn}{2^{2n-2}} xx + \frac{1}{2^{2n-2}} (\sin. nz)^2 = 0,$$

cuius aequationis radices erunt tam affirmativae quam negativae, scilicet

$$\pm \sin. z,\ \pm \sin.\left(\frac{\pi}{n} - z\right),\ \pm \sin.\left(\frac{2\pi}{n} + z\right),\ \pm \sin.\left(\frac{3\pi}{n} - z\right),\ \pm \sin.\left(\frac{4\pi}{n} + z\right) \text{ etc.}$$

sumendo omnino n huiusmodi expressiones. Cum igitur ultimus terminus sit productum omnium harum radicum, extrahendo utrinque radicem quadratam erit

$$\sin. nz = \pm 2^{n-1} \sin. z \sin.\left(\frac{\pi}{n} - z\right) \sin.\left(\frac{2\pi}{n} + z\right) \sin.\left(\frac{3\pi}{n} - z\right) \cdots;$$

ubi quibus casibus utrumvis signum valeat, ex casibus particularibus erit dispiciendum.

EXEMPLUM

Substituendo autem pro n successive numeros 2, 4, 6 etc. et eligendo n sinus diversos erit

$$\sin. 2z = 2 \sin. z \sin.\left(\frac{\pi}{2} - z\right),$$

$$\sin. 4z = 8 \sin. z \sin.\left(\frac{\pi}{4} - z\right) \sin.\left(\frac{\pi}{4} + z\right) \sin.\left(\frac{2\pi}{4} - z\right),$$

$$\sin. 6z = 32 \sin. z \sin.\left(\frac{\pi}{6} - z\right) \sin.\left(\frac{\pi}{6} + z\right) \sin.\left(\frac{2\pi}{6} - z\right) \sin.\left(\frac{2\pi}{6} + z\right) \sin.\left(\frac{3\pi}{6} - z\right),$$

$$\sin. 8z = 128 \sin. z \sin.\left(\frac{\pi}{8} - z\right) \sin.\left(\frac{\pi}{8} + z\right) \sin.\left(\frac{2\pi}{8} - z\right) \sin.\left(\frac{2\pi}{8} + z\right) \sin.\left(\frac{3\pi}{8} - z\right) \sin.\left(\frac{3\pi}{8} + z\right) \sin.\left(\frac{4\pi}{8} - z\right).$$

240. Patet ergo fore generatim

$$\sin. nz = 2^{n-1} \sin. z \sin.\left(\frac{\pi}{n} - z\right) \sin.\left(\frac{\pi}{n} + z\right) \sin.\left(\frac{2\pi}{n} - z\right) \sin.\left(\frac{2\pi}{n} + z\right) \sin.\left(\frac{3\pi}{n} - z\right) \sin.\left(\frac{3\pi}{n} + z\right) \cdots \sin.\left(\frac{\pi}{2} - z\right),$$

si n fuerit numerus par. Quodsi autem haec cum superiori, ubi n erat numerus impar, comparetur, tanta similitudo adesse deprehenditur, ut utramque in unam redigere liceat. Erit ergo, sive n fuerit numerus par sive impar,

$$\sin. nz = 2^{n-1} \sin. z \sin.\left(\frac{\pi}{n} - z\right) \sin.\left(\frac{\pi}{n} + z\right) \sin.\left(\frac{2\pi}{n} - z\right) \sin.\left(\frac{2\pi}{n} + z\right) \sin.\left(\frac{3\pi}{n} - z\right) \sin.\left(\frac{3\pi}{n} + z\right) \text{ etc.},$$

donec tot habeantur factores, quot numerus n continet unitates.

241. Expressiones istae, quibus sinus angulorum multiplorum per factores exponuntur, non parum utilitatis afferre possunt ad logarithmos sinuum angulorum multiplorum inveniendos itemque ad plures expressiones sinuum per factores, quales supra (§ 184) dedimus, reperiendas. Erit autem

$$\sin. z = 1 \sin. z,$$

$$\sin. 2z = 2 \sin. z \sin.\left(\frac{\pi}{2} - z\right),$$

$$\sin. 3z = 4 \sin. z \sin.\left(\frac{\pi}{3} - z\right) \sin.\left(\frac{\pi}{3} + z\right),$$

$$\sin. 4z = 8 \sin. z \sin. \left(\frac{\pi}{4} - z\right) \sin. \left(\frac{\pi}{4} + z\right)$$
$$\sin. \left(\frac{2\pi}{4} - z\right),$$

$$\sin. 5z = 16 \sin. z \sin. \left(\frac{\pi}{5} - z\right) \sin. \left(\frac{\pi}{5} + z\right)$$
$$\sin. \left(\frac{2\pi}{5} - z\right) \sin. \left(\frac{2\pi}{5} + z\right),$$

$$\sin. 6z = 32 \sin. z \sin. \left(\frac{\pi}{6} - z\right) \sin. \left(\frac{\pi}{6} + z\right)$$
$$\sin. \left(\frac{2\pi}{6} - z\right) \sin. \left(\frac{2\pi}{6} + z\right)$$
$$\sin. \left(\frac{3\pi}{6} - z\right)$$

etc.

242. Cum deinde sit $\frac{\sin. 2nz}{\sin. nz} = 2 \cos. nz$, cosinus angulorum multiplorum simili modo per factores exprimentur:

$$\cos. z = 1 \sin. \left(\frac{\pi}{2} - z\right),$$

$$\cos. 2z = 2 \sin. \left(\frac{\pi}{4} - z\right) \sin. \left(\frac{\pi}{4} + z\right),$$

$$\cos. 3z = 4 \sin. \left(\frac{\pi}{6} - z\right) \sin. \left(\frac{\pi}{6} + z\right)$$
$$\sin. \left(\frac{3\pi}{6} - z\right),$$

$$\cos. 4z = 8 \sin. \left(\frac{\pi}{8} - z\right) \sin. \left(\frac{\pi}{8} + z\right)$$
$$\sin. \left(\frac{3\pi}{8} - z\right) \sin. \left(\frac{3\pi}{8} + z\right),$$

$$\cos. 5z = 16 \sin. \left(\frac{\pi}{10} - z\right) \sin. \left(\frac{\pi}{10} + z\right)$$
$$\sin. \left(\frac{3\pi}{10} - z\right) \sin. \left(\frac{3\pi}{10} + z\right)$$
$$\sin. \left(\frac{5\pi}{10} - z\right)$$

et generaliter

$$\cos. nz = 2^{n-1} \sin. \left(\frac{\pi}{2n} - z\right) \sin. \left(\frac{\pi}{2n} + z\right)$$
$$\sin. \left(\frac{3\pi}{2n} - z\right) \sin. \left(\frac{3\pi}{2n} + z\right)$$
$$\sin. \left(\frac{5\pi}{2n} - z\right) \sin. \left(\frac{5\pi}{2n} + z\right)$$

etc.,

quoad tot habeantur factores, quot numerus n continet unitates.

243. Eaedem expressiones prodibunt ex consideratione cosinuum arcuum multiplorum. Si enim fuerit $\cos. z = y$, erit [§ 129], ut sequitur:

$$\cos. 0z = 1,$$
$$\cos. 1z = y,$$
$$\cos. 2z = 2y^2 - 1,$$
$$\cos. 3z = 4y^3 - 3y,$$
$$\cos. 4z = 8y^4 - 8y^2 + 1,$$
$$\cos. 5z = 16y^5 - 20y^3 + 5y,$$
$$\cos. 6z = 32y^6 - 48y^4 + 18y^2 - 1,$$
$$\cos. 7z = 64y^7 - 112y^5 + 56y^3 - 7y$$

et generaliter

$$\cos. nz = 2^{n-1} y^n - \frac{n}{1} 2^{n-3} y^{n-2} + \frac{n(n-3)}{1 \cdot 2} 2^{n-5} y^{n-4} - \frac{n(n-4)(n-5)}{1 \cdot 2 \cdot 3} 2^{n-7} y^{n-6}$$
$$+ \frac{n(n-5)(n-6)(n-7)}{1 \cdot 2 \cdot 3 \cdot 4} 2^{n-9} y^{n-8} - \text{etc.},$$

cuius aequationis, cum sit

$$\cos. nz = \cos. (2\pi - nz) = \cos. (2\pi + nz) = \cos. (4\pi \pm nz) = \cos. (6\pi \pm nz) \text{ etc.},$$

erunt radices ipsius y hae

$$\cos. z, \quad \cos. \left(\frac{2\pi}{n} \pm z\right), \quad \cos. \left(\frac{4\pi}{n} \pm z\right), \quad \cos. \left(\frac{6\pi}{n} \pm z\right) \quad \text{etc.},$$

quarum formularum tot diversae sunt pro y eligendae, quot dantur; dantur autem tot, quot n continet unitates.

244. Primum igitur patet ob terminum secundum deficientem excepto casu $n = 1$ fore summam harum radicum omnium $= 0$. Erit ergo

$$0 = \cos. z + \cos.\left(\frac{2\pi}{n} - z\right) + \cos.\left(\frac{2\pi}{n} + z\right)$$
$$+ \cos.\left(\frac{4\pi}{n} - z\right) + \cos.\left(\frac{4\pi}{n} + z\right)$$
$$+ \text{etc.}$$

sumendo tot terminos, quot n continet unitates. Haec autem aequalitas sponte se offert, si n sit numerus par, cum quivis terminus ab alio sui negativo destruatur. Contemplemur ergo numeros impares unitate exclusa eritque ob $\cos. v = -\cos.(\pi - v)$

$$0 = \cos. z - \cos.\left(\frac{\pi}{3} - z\right) - \cos.\left(\frac{\pi}{3} + z\right),$$

$$0 = \cos. z - \cos.\left(\frac{\pi}{5} - z\right) - \cos.\left(\frac{\pi}{5} + z\right)$$
$$+ \cos.\left(\frac{2\pi}{5} - z\right) + \cos.\left(\frac{2\pi}{5} + z\right),$$

$$0 = \cos. z - \cos.\left(\frac{\pi}{7} - z\right) - \cos.\left(\frac{\pi}{7} + z\right)$$
$$+ \cos.\left(\frac{2\pi}{7} - z\right) + \cos.\left(\frac{2\pi}{7} + z\right)$$
$$- \cos.\left(\frac{3\pi}{7} - z\right) - \cos.\left(\frac{3\pi}{7} + z\right)$$

et generaliter, si fuerit n numerus impar quicunque, erit

$$0 = \cos. z - \cos.\left(\frac{\pi}{n} - z\right) - \cos.\left(\frac{\pi}{n} + z\right)$$
$$+ \cos.\left(\frac{2\pi}{n} - z\right) + \cos.\left(\frac{2\pi}{n} + z\right)$$
$$- \cos.\left(\frac{3\pi}{n} - z\right) - \cos.\left(\frac{3\pi}{n} + z\right)$$
$$+ \cos.\left(\frac{4\pi}{n} - z\right) + \cos.\left(\frac{4\pi}{n} + z\right)$$
$$- \text{etc.}$$

sumendo tot terminos, quot numerus n continet unitates. Oportet autem n esse numerum imparem unitate maiorem, uti iam monuimus.

245. Quod ad productum ex omnibus attinet, variae quidem prodeunt expressiones, prout n fuerit numerus vel impar vel impariter par vel pariter par. Omnes autem comprehenduntur in expressione generali (§ 242) inventa, si singuli sinus in cosinus transmutentur. Erit scilicet

$$\cos. z = 1 \cos. z,$$

$$\cos. 2z = 2\cos.\left(\frac{\pi}{4} + z\right)\cos.\left(\frac{\pi}{4} - z\right),$$

$$\cos. 3z = 4\cos.\left(\frac{2\pi}{6} + z\right)\cos.\left(\frac{2\pi}{6} - z\right)$$
$$\cos. z,$$

$$\cos. 4z = 8\cos.\left(\frac{3\pi}{8} + z\right)\cos.\left(\frac{3\pi}{8} - z\right)$$
$$\cos.\left(\frac{\pi}{8} + z\right)\cos.\left(\frac{\pi}{8} - z\right),$$

$$\cos. 5z = 16\cos.\left(\frac{4\pi}{10} + z\right)\cos.\left(\frac{4\pi}{10} - z\right)$$
$$\cos.\left(\frac{2\pi}{10} + z\right)\cos.\left(\frac{2\pi}{10} - z\right)$$
$$\cos. z$$

et generaliter

$$\cos. nz = 2^{n-1}\cos.\left(\frac{n-1}{2n}\pi + z\right)\cos.\left(\frac{n-1}{2n}\pi - z\right)$$
$$\cos.\left(\frac{n-3}{2n}\pi + z\right)\cos.\left(\frac{n-3}{2n}\pi - z\right)$$
$$\cos.\left(\frac{n-5}{2n}\pi + z\right)\cos.\left(\frac{n-5}{2n}\pi - z\right)$$
$$\cos.\left(\frac{n-7}{2n}\pi + z\right)\cos.\left(\frac{n-7}{2n}\pi - z\right)$$
$$\text{etc.}$$

sumptis tot factoribus, quot numerus n continet unitates.

246. Sit n numerus impar atque aequatio incipiatur ab unitate; erit

$$0 = 1 \mp \frac{ny}{\cos. nz} \pm \text{etc.},$$

ubi signum superius valet, si n fuerit numerus impar formae $4m+1$, inferius, si $n = 4m-1$. Hinc erit

$$+\frac{1}{\cos. z} = \frac{1}{\cos. z},$$

$$-\frac{3}{\cos. 3z} = \frac{1}{\cos. z} - \frac{1}{\cos.\left(\frac{\pi}{3}-z\right)} - \frac{1}{\cos.\left(\frac{\pi}{3}+z\right)},$$

$$+\frac{5}{\cos. 5z} = \frac{1}{\cos. z} - \frac{1}{\cos\left(\frac{\pi}{5}-z\right)} - \frac{1}{\cos.\left(\frac{\pi}{5}+z\right)} + \frac{1}{\cos.\left(\frac{2\pi}{5}-z\right)} + \frac{1}{\cos.\left(\frac{2\pi}{5}+z\right)}$$

et generaliter posito $n = 2m+1$ erit

$$\frac{n}{\cos. nz} = \frac{2m+1}{\cos.(2m+1)z} = \frac{1}{\cos.\left(\frac{m}{n}\pi+z\right)} + \frac{1}{\cos.\left(\frac{m}{n}\pi-z\right)}$$
$$- \frac{1}{\cos.\left(\frac{m-1}{n}\pi+z\right)} - \frac{1}{\cos.\left(\frac{m-1}{n}\pi-z\right)}$$
$$+ \frac{1}{\cos.\left(\frac{m-2}{n}\pi+z\right)} + \frac{1}{\cos.\left(\frac{m-2}{n}\pi-z\right)}$$
$$- \frac{1}{\cos.\left(\frac{m-3}{n}\pi+z\right)} - \frac{1}{\cos.\left(\frac{m-3}{n}\pi-z\right)}$$
$$+ \text{etc.}$$

sumendis tot terminis, quot n continet unitates.

247. Cum ergo sit $\frac{1}{\cos. v} = \sec. v$, hinc pro secantibus insignes proprietates deducuntur; erit nempe

$$\sec. z = \sec. z,$$

$$3 \sec. 3z = \sec.\left(\frac{\pi}{3}+z\right) + \sec.\left(\frac{\pi}{3}-z\right) - \sec.\left(\frac{0\pi}{3}+z\right),$$

$$5 \sec. 5z = \sec.\left(\frac{2\pi}{5}+z\right) + \sec.\left(\frac{2\pi}{5}-z\right)$$
$$- \sec.\left(\frac{\pi}{5}+z\right) - \sec.\left(\frac{\pi}{5}-z\right)$$
$$+ \sec.\left(\frac{0\pi}{5}+z\right),$$

$$7 \sec. 7z = \sec.\left(\frac{3\pi}{7}+z\right) + \sec.\left(\frac{3\pi}{7}-z\right)$$
$$- \sec.\left(\frac{2\pi}{7}+z\right) - \sec.\left(\frac{2\pi}{7}-z\right)$$
$$+ \sec.\left(\frac{\pi}{7}+z\right) + \sec.\left(\frac{\pi}{7}-z\right)$$
$$- \sec.\left(\frac{0\pi}{7}+z\right)$$

et generaliter posito $n = 2m+1$ erit

$$n \sec. nz = \sec.\left(\frac{m}{n}\pi+z\right) + \sec.\left(\frac{m}{n}\pi-z\right)$$
$$- \sec.\left(\frac{m-1}{n}\pi+z\right) - \sec.\left(\frac{m-1}{n}\pi-z\right)$$
$$+ \sec.\left(\frac{m-2}{n}\pi+z\right) + \sec.\left(\frac{m-2}{n}\pi-z\right)$$
$$- \sec.\left(\frac{m-3}{n}\pi+z\right) - \sec.\left(\frac{m-3}{n}\pi-z\right)$$
$$+ \sec.\left(\frac{m-4}{n}\pi+z\right) + \sec.\left(\frac{m-4}{n}\pi-z\right)$$
$$\vdots$$
$$\pm \sec. z.$$

248. Pro cosecantibus autem erit ex § 237

$$\operatorname{cosec}. z = \operatorname{cosec}. z,$$

$$3 \operatorname{cosec}. 3z = \operatorname{cosec}. z + \operatorname{cosec}.\left(\frac{\pi}{3}-z\right) - \operatorname{cosec}.\left(\frac{\pi}{3}+z\right),$$

$$5\,\text{cosec.}\,5z = \text{cosec.}\,z + \text{cosec.}\left(\frac{\pi}{5} - z\right) - \text{cosec.}\left(\frac{\pi}{5} + z\right)$$
$$- \text{cosec.}\left(\frac{2\pi}{5} - z\right) + \text{cosec.}\left(\frac{2\pi}{5} + z\right),$$

$$7\,\text{cosec.}\,7z = \text{cosec.}\,z + \text{cosec.}\left(\frac{\pi}{7} - z\right) - \text{cosec.}\left(\frac{\pi}{7} + z\right)$$
$$- \text{cosec.}\left(\frac{2\pi}{7} - z\right) + \text{cosec.}\left(\frac{2\pi}{7} + z\right)$$
$$+ \text{cosec.}\left(\frac{3\pi}{7} - z\right) - \text{cosec.}\left(\frac{3\pi}{7} + z\right)$$

et generaliter ponendo $n = 2m + 1$ erit

$$n\,\text{cosec.}\,nz = \text{cosec.}\,z + \text{cosec.}\left(\frac{\pi}{n} - z\right) - \text{cosec.}\left(\frac{\pi}{n} + z\right)$$
$$- \text{cosec.}\left(\frac{2\pi}{n} - z\right) + \text{cosec.}\left(\frac{2\pi}{n} + z\right)$$
$$+ \text{cosec.}\left(\frac{3\pi}{n} - z\right) - \text{cosec.}\left(\frac{3\pi}{n} + z\right)$$
$$\vdots$$
$$\mp \text{cosec.}\left(\frac{m\pi}{n} - z\right) \pm \text{cosec.}\left(\frac{m\pi}{n} + z\right),$$

ubi signa superiora valent, si m fuerit numerus par, inferiora, si m sit impar.

249. Cum sit, uti supra [§ 133] vidimus,

$$\cos.\,nz \pm \sqrt{-1}\cdot\sin.\,nz = (\cos.\,z \pm \sqrt{-1}\cdot\sin.\,z)^n,$$

erit

$$\cos.\,nz = \frac{(\cos.\,z + \sqrt{-1}\cdot\sin.\,z)^n + (\cos.\,z - \sqrt{-1}\cdot\sin.\,z)^n}{2}$$

et

$$\sin.\,nz = \frac{(\cos.\,z + \sqrt{-1}\cdot\sin.\,z)^n - (\cos.\,z - \sqrt{-1}\cdot\sin.\,z)^n}{2\sqrt{-1}},$$

ergo

$$\text{tang.}\,nz = \frac{(\cos.\,z + \sqrt{-1}\cdot\sin.\,z)^n - (\cos.\,z - \sqrt{-1}\cdot\sin.\,z)^n}{(\cos.\,z + \sqrt{-1}\cdot\sin.\,z)^n\sqrt{-1} + (\cos.\,z - \sqrt{-1}\cdot\sin.\,z)^n\sqrt{-1}}.$$

Ponamus

$$\text{tang.}\,z = \frac{\sin.\,z}{\cos.\,z} = t;$$

erit

$$\text{tang.}\,nz = \frac{(1 + t\sqrt{-1})^n - (1 - t\sqrt{-1})^n}{(1 + t\sqrt{-1})^n\sqrt{-1} + (1 - t\sqrt{-1})^n\sqrt{-1}},$$

unde oriuntur tangentes angulorum multiplorum sequentes

$$\text{tang.}\,z = t,$$
$$\text{tang.}\,2z = \frac{2t}{1 - tt},$$
$$\text{tang.}\,3z = \frac{3t - t^3}{1 - 3tt},$$
$$\text{tang.}\,4z = \frac{4t - 4t^3}{1 - 6tt + t^4},$$
$$\text{tang.}\,5z = \frac{5t - 10t^3 + t^5}{1 - 10tt + 5t^4}$$

et generaliter

$$\text{tang.}\,nz = \frac{nt - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}t^3 + \frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}t^5 - \text{etc.}}{1 - \frac{n(n-1)}{1\cdot 2}tt + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}t^4 - \text{etc.}}.$$

Cum iam sit

$$\text{tang.}\,nz = \text{tang.}\,(\pi + nz) = \text{tang.}\,(2\pi + nz) = \text{tang.}\,(3\pi + nz)\ \text{etc.},$$

erunt valores ipsius t seu radices aequationis hae

$$\text{tang.}\,z,\quad \text{tang.}\left(\frac{\pi}{n} + z\right),\quad \text{tang.}\left(\frac{2\pi}{n} + z\right),\quad \text{tang.}\left(\frac{3\pi}{n} + z\right)\quad \text{etc.},$$

quarum numerus est n.

250. Quodsi aequatio ab unitate incipiat, erit

$$0 = 1 - \frac{nt}{\text{tang.}\,nz} - \frac{n(n-1)tt}{1\cdot 2} + \frac{n(n-1)(n-2)t^3}{1\cdot 2\cdot 3\,\text{tang.}\,nz} + \text{etc.}$$

Ex comparatione ergo coefficientium cum radicibus erit

$$n \cot. nz = \cot. z + \cot. \left(\frac{\pi}{n} + z\right) + \cot. \left(\frac{2\pi}{n} + z\right) + \cot. \left(\frac{3\pi}{n} + z\right)$$

$$+ \cot. \left(\frac{4\pi}{n} + z\right) + \cdots + \cot. \left(\frac{n-1}{n}\pi + z\right).$$

Deinde erit summa quadratorum harum cotangentium omnium

$$= \frac{nn}{(\sin. nz)^2} - n$$

similique modo ulteriores potestates possunt definiri. Ponendo autem loco n numeros definitos erit

$$\cot. z \quad = \cot. z,$$

$$2 \cot. 2z = \cot. z + \cot. \left(\frac{\pi}{2} + z\right),$$

$$3 \cot. 3z = \cot. z + \cot. \left(\frac{\pi}{3} + z\right) + \cot. \left(\frac{2\pi}{3} + z\right),$$

$$4 \cot. 4z = \cot. z + \cot. \left(\frac{\pi}{4} + z\right) + \cot. \left(\frac{2\pi}{4} + z\right)$$

$$+ \cot. \left(\frac{3\pi}{4} + z\right),$$

$$5 \cot. 5z = \cot. z + \cot. \left(\frac{\pi}{5} + z\right) + \cot. \left(\frac{2\pi}{5} + z\right)$$

$$+ \cot. \left(\frac{3\pi}{5} + z\right) + \cot. \left(\frac{4\pi}{5} + z\right).$$

251. Quia vero est $\cot. v = -\cot. (\pi - v)$, erit

$$\cot. z \quad = \cot. z,$$

$$2 \cot. 2z = \cot. z - \cot. \left(\frac{\pi}{2} - z\right),$$

$$3 \cot. 3z = \cot. z - \cot. \left(\frac{\pi}{3} - z\right) + \cot. \left(\frac{\pi}{3} + z\right),$$

$$4 \cot. 4z = \cot. z - \cot. \left(\frac{\pi}{4} - z\right) + \cot. \left(\frac{\pi}{4} + z\right)$$

$$- \cot. \left(\frac{2\pi}{4} - z\right),$$

$$5 \cot. 5z = \cot. z - \cot. \left(\frac{\pi}{5} - z\right) + \cot. \left(\frac{\pi}{5} + z\right)$$

$$- \cot. \left(\frac{2\pi}{5} - z\right) + \cot. \left(\frac{2\pi}{5} + z\right)$$

et generaliter

$$n \cot. nz = \cot. z - \cot. \left(\frac{\pi}{n} - z\right) + \cot. \left(\frac{\pi}{n} + z\right)$$

$$- \cot. \left(\frac{2\pi}{n} - z\right) + \cot. \left(\frac{2\pi}{n} + z\right)$$

$$- \cot. \left(\frac{3\pi}{n} - z\right) + \cot. \left(\frac{3\pi}{n} + z\right)$$

$$- \text{etc.},$$

donec tot habeantur termini, quot numerus n continet unitates.

252. Incipiamus aequationem inventam a potestate summa, ubi primum distinguendi sunt casus, quibus n est vel numerus par vel impar. Sit n numerus impar, $n = 2m + 1$; erit

$$t - \text{tang}. z = 0,$$

$$t^3 - 3tt \,\text{tang}. 3z - 3t + \text{tang}. 3z = 0,$$

$$t^5 - 5t^4 \,\text{tang}. 5z - 10t^3 + 10tt \,\text{tang}. 5z + 5t - \text{tang}. 5z = 0$$

et generaliter

$$t^n - nt^{n-1}\,\text{tang.}\,nz - \cdots \mp \text{tang.}\,nz = 0,$$

ubi signum superius — valet, si m sit numerus par, inferius +, si m sit numerus impar. Erit ergo ex coefficiente secundi termini

$$\text{tang.}\,z = \text{tang.}\,z,$$

$$3\,\text{tang.}\,3z = \text{tang.}\,z + \text{tang.}\left(\frac{\pi}{3}+z\right) + \text{tang.}\left(\frac{2\pi}{3}+z\right),$$

$$5\,\text{tang.}\,5z = \text{tang.}\,z + \text{tang.}\left(\frac{\pi}{5}+z\right) + \text{tang.}\left(\frac{2\pi}{5}+z\right)$$
$$+ \text{tang.}\left(\frac{3\pi}{5}+z\right) + \text{tang.}\left(\frac{4\pi}{5}+z\right)$$

etc.

253. Cum igitur sit $\text{tang.}\,v = -\text{tang.}\,(\pi - v)$, anguli recto maiores ad angulos recto minores reducuntur eritque

$$\text{tang.}\,z = \text{tang.}\,z,$$

$$3\,\text{tang.}\,3z = \text{tang.}\,z - \text{tang.}\left(\frac{\pi}{3}-z\right) + \text{tang.}\left(\frac{\pi}{3}+z\right),$$

$$5\,\text{tang.}\,5z = \text{tang.}\,z - \text{tang.}\left(\frac{\pi}{5}-z\right) + \text{tang.}\left(\frac{\pi}{5}+z\right)$$
$$- \text{tang.}\left(\frac{2\pi}{5}-z\right) + \text{tang.}\left(\frac{2\pi}{5}+z\right),$$

$$7\,\text{tang.}\,7z = \text{tang.}\,z - \text{tang.}\left(\frac{\pi}{7}-z\right) + \text{tang.}\left(\frac{\pi}{7}+z\right)$$
$$- \text{tang.}\left(\frac{2\pi}{7}-z\right) + \text{tang.}\left(\frac{2\pi}{7}+z\right)$$
$$- \text{tang.}\left(\frac{3\pi}{7}-z\right) + \text{tang.}\left(\frac{3\pi}{7}+z\right)$$

et generaliter, si $n = 2m + 1$, erit

$$n\,\text{tang.}\,nz = \text{tang.}\,z - \text{tang.}\left(\frac{\pi}{n}-z\right) + \text{tang.}\left(\frac{\pi}{n}+z\right)$$
$$- \text{tang.}\left(\frac{2\pi}{n}-z\right) + \text{tang.}\left(\frac{2\pi}{n}+z\right)$$
$$- \text{tang.}\left(\frac{3\pi}{n}-z\right) + \text{tang.}\left(\frac{3\pi}{n}+z\right)$$
$$\vdots$$
$$- \text{tang.}\left(\frac{m\pi}{n}-z\right) + \text{tang.}\left(\frac{m\pi}{n}+z\right).$$

254. Tum vero productum ex his tangentibus omnibus erit $= \text{tang.}\,nz$, propterea quod per signorum negativorum numerum alternatim parem et imparem superior signorum ambiguitas tollitur. Sic erit

$$\text{tang.}\,z = \text{tang.}\,z,$$

$$\text{tang.}\,3z = \text{tang.}\,z\,\text{tang.}\left(\frac{\pi}{3}-z\right)\text{tang.}\left(\frac{\pi}{3}+z\right),$$

$$\text{tang.}\,5z = \text{tang.}\,z\,\text{tang.}\left(\frac{\pi}{5}-z\right)\text{tang.}\left(\frac{\pi}{5}+z\right)$$
$$\text{tang.}\left(\frac{2\pi}{5}-z\right)\text{tang.}\left(\frac{2\pi}{5}+z\right)$$

et generaliter, si $n = 2m + 1$, erit

$$\text{tang.}\,nz = \text{tang.}\,z\,\text{tang.}\left(\frac{\pi}{n}-z\right)\text{tang.}\left(\frac{\pi}{n}+z\right)$$
$$\text{tang.}\left(\frac{2\pi}{n}-z\right)\text{tang.}\left(\frac{2\pi}{n}+z\right)$$
$$\text{tang.}\left(\frac{3\pi}{n}-z\right)\text{tang.}\left(\frac{3\pi}{n}+z\right)$$
$$\vdots$$
$$\text{tang.}\left(\frac{m\pi}{n}-z\right)\text{tang.}\left(\frac{m\pi}{n}+z\right).$$

255. Sit iam n numerus par atque incipiendo a potestate summa erit

$$tt + 2t \cot. 2z - 1 = 0,$$
$$t^4 + 4t^3 \cot. 4z - 6tt - 4t \cot. 4z + 1 = 0$$

et generaliter, si $n = 2m$, erit

$$t^n + nt^{n-1} \cot. nz - \cdots \mp 1 = 0,$$

ubi signum superius — valet, si m sit numerus impar, inferius +, si m sit par. Comparando ergo radices cum coefficiente secundi termini erit

$$-2 \cot. 2z = \tang. z + \tang. \left(\frac{\pi}{2} + z\right),$$

$$-4 \cot. 4z = \tang. z + \tang. \left(\frac{\pi}{4} + z\right) + \tang. \left(\frac{2\pi}{4} + z\right) + \tang. \left(\frac{3\pi}{4} + z\right),$$

$$-6 \cot. 6z = \tang. z + \tang. \left(\frac{\pi}{6} + z\right) + \tang. \left(\frac{2\pi}{6} + z\right) + \tang. \left(\frac{3\pi}{6} + z\right) + \tang. \left(\frac{4\pi}{6} + z\right) + \tang. \left(\frac{5\pi}{6} + z\right)$$

etc.

256. Cum sit $\tang. v = -\tang. (\pi - v)$, sequentes formabuntur aequationes

$$2 \cot. 2z = -\tang. z + \tang. \left(\frac{\pi}{2} - z\right),$$

$$4 \cot. 4z = -\tang. z + \tang. \left(\frac{\pi}{4} - z\right) - \tang. \left(\frac{\pi}{4} + z\right) + \tang. \left(\frac{2\pi}{4} - z\right),$$

$$6 \cot. 6z = -\tang. z + \tang. \left(\frac{\pi}{6} - z\right) - \tang. \left(\frac{\pi}{6} + z\right) + \tang. \left(\frac{2\pi}{6} - z\right) - \tang. \left(\frac{2\pi}{6} + z\right) + \tang. \left(\frac{3\pi}{6} - z\right)$$

et generaliter, si $n = 2m$, erit

$$n \cot. nz = -\tang. z + \tang. \left(\frac{\pi}{n} - z\right) - \tang. \left(\frac{\pi}{n} + z\right) + \tang. \left(\frac{2\pi}{n} - z\right) - \tang. \left(\frac{2\pi}{n} + z\right) + \tang. \left(\frac{3\pi}{n} - z\right) - \tang. \left(\frac{3\pi}{n} + z\right) \; \vdots \; + \tang. \left(\frac{m\pi}{n} - z\right).$$

257. Per has formas iterum ambiguitas producti ex omnibus radicibus destruitur eritque idcirco

$$1 = \tang. z \tang. \left(\frac{\pi}{2} - z\right),$$

$$1 = \tang. z \tang. \left(\frac{\pi}{4} - z\right) \tang. \left(\frac{\pi}{4} + z\right) \tang. \left(\frac{2\pi}{4} - z\right),$$

$$1 = \tang. z \tang. \left(\frac{\pi}{6} - z\right) \tang. \left(\frac{\pi}{6} + z\right) \tang. \left(\frac{2\pi}{6} - z\right) \tang. \left(\frac{2\pi}{6} + z\right) \tang. \left(\frac{3\pi}{6} - z\right)$$

etc.

Harum vero aequationum ratio statim sponte in oculos incurrit, cum perpetuo bini anguli reperiantur, quorum alter est alterius complementum ad rectum Huiusmodi ergo binorum angulorum tangentes productum dant $= 1$ ideoque omnium productum unitati debet esse aequale.

258. Quoniam sinus et cosinus angulorum progressionem arithmeticam constituentium seriem recurrentem praebent, per caput praecedens summa huiusmodi sinuum et cosinuum quotcunque exhiberi poterit. Sint anguli in arithmetica progressione

$$a, \quad a + b, \quad a + 2b, \quad a + 3b, \quad a + 4b, \quad a + 5b \quad \text{etc.}$$

et quaeratur primo summa sinuum horum angulorum in infinitum progredientium; ponatur ergo

$$s = \sin. a + \sin. (a + b) + \sin. (a + 2b) + \sin. (a + 3b) + \text{etc.},$$

et quia haec series est recurrens, cuius scala relationis est $2 \cos. b,\ -1$, orietur haec series ex evolutione fractionis, cuius denominator est

$$1 - 2z \cos. b + zz$$

posito $z = 1$. Ipsa vero fractio erit

$$= \frac{\sin. a + z(\sin. (a + b) - 2 \sin. a \cos. b)}{1 - 2z \cos. b + zz};$$

quare facto $z = 1$ erit

$$s = \frac{\sin. a + \sin. (a + b) - 2 \sin. a \cos. b}{2 - 2 \cos. b} = \frac{\sin. a - \sin. (a - b)}{2(1 - \cos. b)}$$

ob

$$2 \sin. a \cos. b = \sin. (a + b) + \sin. (a - b).$$

Cum autem sit

$$\sin. f - \sin. g = 2 \cos. \frac{f + g}{2} \sin. \frac{f - g}{2},$$

erit

$$\sin. a - \sin. (a - b) = 2 \cos. \left(a - \tfrac{1}{2} b\right) \sin. \tfrac{1}{2} b;$$

at

$$1 - \cos. b = 2 \left(\sin. \tfrac{1}{2} b\right)^2,$$

unde erit

$$s = \frac{\cos. (a - \frac{1}{2} b)}{2 \sin. \frac{1}{2} b}.$$

259. Hinc itaque summa quotcunque sinuum, quorum arcus in arithmetica progressione incedunt, assignari poterit. Quaeratur nempe summa huius progressionis

$$\sin. a + \sin. (a + b) + \sin. (a + 2b) + \sin. (a + 3b) + \cdots + \sin. (a + nb).$$

Quia summa huius progressionis in infinitum continuatae est

$$\frac{\cos. (a - \frac{1}{2} b)}{2 \sin. \frac{1}{2} b},$$

considerentur termini ultimum sequentes in infinitum hi

$$\sin. (a + (n + 1)b) + \sin. (a + (n + 2)b) + \sin. (a + (n + 3)b) + \text{etc.};$$

quia horum sinuum summa est

$$= \frac{\cos. (a + (n + \frac{1}{2})b)}{2 \sin. \frac{1}{2} b},$$

si haec a priori subtrahatur, remanebit summa quaesita. Scilicet, si fuerit

$$s = \sin. a + \sin. (a + b) + \sin. (a + 2b) + \cdots + \sin. (a + nb),$$

erit

$$s = \frac{\cos. (a - \frac{1}{2} b) - \cos. (a + (n + \frac{1}{2})b)}{2 \sin. \frac{1}{2} b} = \frac{\sin. (a + \frac{1}{2} nb) \sin. \frac{1}{2}(n + 1) b}{\sin. \frac{1}{2} b}.$$

260. Pari modo si consideretur summa cosinuum atque ponatur

$$s = \cos. a + \cos. (a + b) + \cos. (a + 2b) + \cos. (a + 3b) + \text{etc. in infinitum},$$

erit

$$s = \frac{\cos. a + z(\cos. (a + b) - 2 \cos. a \cos. b)}{1 - 2z \cos. b + zz}$$

posito $z = 1$. Quare ob

$$2 \cos. a \cos. b = \cos. (a - b) + \cos. (a + b)$$

fiet

$$s = \frac{\cos. a - \cos. (a - b)}{2(1 - \cos. b)}.$$

At est

$$\cos. f - \cos. g = 2 \sin. \frac{f + g}{2} \sin. \frac{g - f}{2};$$

unde erit

$$\cos. a - \cos. (a - b) = -2 \sin. \left(a - \tfrac{1}{2} b\right) \sin. \tfrac{1}{2} b,$$

et ob

$$1 - \cos. b = 2 \left(\sin. \tfrac{1}{2} b\right)^2$$

erit

$$s = -\frac{\sin. (a - \frac{1}{2} b)}{2 \sin. \frac{1}{2} b}.$$

Quare, cum simili modo sit huius seriei

$$\cos. (a + (n + 1)b) + \cos. (a + (n + 2)b) + \cos. (a + (n + 3)b) + \text{etc.}$$

summa

$$= -\frac{\sin. (a + (n + \frac{1}{2})b)}{2 \sin. \frac{1}{2} b},$$

si haec ab illa subtrahatur, relinquetur summa huius seriei

$$s = \cos. a + \cos. (a + b) + \cos. (a + 2b) + \cos. (a + 3b) + \cdots + \cos. (a + nb)$$

eritque

$$s = \frac{-\sin. (a - \frac{1}{2} b) + \sin. (a + (n + \frac{1}{2}) b)}{2 \sin. \frac{1}{2} b} = \frac{\cos. (a + \frac{1}{2} nb) \sin. \frac{1}{2} (n + 1) b}{\sin. \frac{1}{2} b}.$$

261. Plurimae aliae quaestiones circa sinus et tangentes ex principiis allatis resolvi possent; cuiusmodi sunt, si quadrata altioresve potestates sinuum tangentiumve summari deberent; verum quia haec ex reliquis aequationum superiorum coefficientibus similiter derivantur, iis hic diutius non immoror. Quod autem ad has postremas summationes attinet, notandum est quamcunque sinuum cosinuumque potestatem per singulos sinus cosinusve explicari posse, quod, ut clarius perspiciatur, breviter exponamus.

262. Ad hoc expediendum iuvabit ex praecedentibus haec lemmata deprompsisse

$$2 \sin. a \sin. z = \cos. (a - z) - \cos. (a + z),$$
$$2 \cos. a \sin. z = \sin. (a + z) - \sin. (a - z),$$
$$2 \sin. a \cos. z = \sin. (a + z) + \sin. (a - z),$$
$$2 \cos. a \cos. z = \cos. (a - z) + \cos. (a + z).$$

Hinc igitur primum potestates sinuum reperiuntur:

$$\sin. z = \sin. z,$$
$$2 (\sin. z)^2 = 1 - \cos. 2z,$$
$$4 (\sin. z)^3 = 3 \sin. z - \sin. 3z,$$
$$8 (\sin. z)^4 = 3 - 4 \cos. 2z + \cos. 4z,$$
$$16 (\sin. z)^5 = 10 \sin. z - 5 \sin. 3z + \sin. 5z,$$
$$32 (\sin. z)^6 = 10 - 15 \cos. 2z + 6 \cos. 4z - \cos. 6z,$$
$$64 (\sin. z)^7 = 35 \sin. z - 21 \sin. 3z + 7 \sin. 5z - \sin. 7z,$$
$$128 (\sin. z)^8 = 35 - 56 \cos. 2z + 28 \cos. 4z - 8 \cos. 6z + \cos. 8z,$$
$$256 (\sin. z)^9 = 126 \sin. z - 84 \sin. 3z + 36 \sin. 5z - 9 \sin. 7z + \sin. 9z$$

etc.

Lex, qua hi coefficientes progrediuntur, ex unciis binomii elevati intelligitur, nisi quod numerus absolutus in potestatibus paribus semissis tantum sit eius, quem unciae praebent.

263. Pari modo potestates cosinuum definientur:

$$\cos. z = \cos. z,$$
$$2 (\cos. z)^2 = 1 + \cos. 2z,$$
$$4 (\cos. z)^3 = 3 \cos. z + \cos. 3z,$$
$$8 (\cos. z)^4 = 3 + 4 \cos. 2z + \cos. 4z,$$
$$16 (\cos. z)^5 = 10 \cos. z + 5 \cos. 3z + \cos. 5z,$$
$$32 (\cos. z)^6 = 10 + 15 \cos. 2z + 6 \cos. 4z + \cos. 6z,$$
$$64 (\cos. z)^7 = 35 \cos. z + 21 \cos. 3z + 7 \cos. 5z + \cos. 7z$$

etc.

Hic ratione legis progressionis eadem sunt monenda, quae circa sinus notavimus.

CAPUT XV

DE SERIEBUS EX EVOLUTIONE FACTORUM ORTIS

264. Sit propositum productum ex factoribus numero sive finitis sive infinitis constans huiusmodi

$$(1+\alpha z)(1+\beta z)(1+\gamma z)(1+\delta z)(1+\varepsilon z)(1+\zeta z) \text{ etc.},$$

quod, si per multiplicationem actualem evolvatur, det

$$1+Az+Bz^2+Cz^3+Dz^4+Ez^5+Fz^6+\text{etc.},$$

atque manifestum est coefficientes A, B, C, D, E etc. ita formari ex numeris α, β, γ, δ, ε, ζ etc., ut sit

$A=\alpha+\beta+\gamma+\delta+\varepsilon+\zeta+\text{etc.}=$ summae singulorum,
$B=$ summae factorum ex binis diversis,
$C=$ summae factorum ex ternis diversis,
$D=$ summae factorum ex quaternis diversis,
$E=$ summae factorum ex quinis diversis
etc.,

donec perveniatur ad productum ex omnibus.

265. Quodsi ergo ponatur $z=1$, productum hoc

$$(1+\alpha)(1+\beta)(1+\gamma)(1+\delta)(1+\varepsilon) \text{ etc.}$$

aequabitur unitati cum serie numerorum omnium, qui ex his α, β, γ, δ, ε etc. vel sumendis singulis vel duobus pluribusve diversis in se multiplicandis nascuntur. Atque si idem numerus duobus pluribusve modis resultare queat, etiam idem bis pluriesve in hac numerorum serie occurret.

266. Si ponatur $z=-1$, productum hoc

$$(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)(1-\varepsilon) \text{ etc.}$$

aequabitur unitati cum serie numerorum omnium, qui ex his α, β, γ, δ, ε etc. vel sumendis singulis vel duobus pluribusve diversis in se multiplicandis nascuntur; ut ante quidem, verum hoc discrimine, ut ii numeri, qui vel ex singulis vel ternis vel quinis vel numero imparibus nascuntur, sint negativi, illi vero, qui vel ex binis vel quaternis vel senis vel numero paribus resultant, sint affirmativi.

267. Scribantur pro α, β, γ, δ etc. numeri primi omnes

$$2,\ 3,\ 5,\ 7,\ 11,\ 13 \text{ etc.}$$

atque hoc productum

$$(1+2)(1+3)(1+5)(1+7)(1+11)(1+13) \text{ etc.}=P$$

aequabitur unitati cum serie omnium numerorum vel primorum ipsorum vel ex primis diversis per multiplicationem ortorum. Erit ergo

$$P=1+2+3+5+6+7+10+11+13+14+15+17+\text{etc.},$$

in qua serie omnes occurrunt numeri naturales exceptis potestatibus iisque, qui per quamvis potestatem sunt divisibiles. Desunt scilicet numeri 4, 8, 9, 12, 16, 18 etc., quoniam sunt vel potestates ut 4, 8, 9, 16 etc., vel per potestates divisibiles ut 12, 18 etc.

268. Simili modo res se habebit, si pro α, β, γ, δ etc. potestates quaecunque numerorum primorum substituantur, scilicet si ponamus

$$P=\left(1+\frac{1}{2^n}\right)\left(1+\frac{1}{3^n}\right)\left(1+\frac{1}{5^n}\right)\left(1+\frac{1}{7^n}\right)\left(1+\frac{1}{11^n}\right) \text{ etc.}$$

Erit enim multiplicatione instituta

$$P = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{10^n} + \frac{1}{11^n} + \text{etc.},$$

in quibus fractionibus omnes occurrunt numeri praeter illos, qui vel ipsi sunt potestates vel per potestatem quampiam divisibiles. Cum enim omnes numeri integri sint vel primi vel ex primis per multiplicationem compositi, hic ii tantum numeri excludentur, in quorum formationem idem numerus primus bis vel pluries ingreditur.

269. Si numeri $\alpha, \beta, \gamma, \delta$ etc. negative capiantur, ut ante (§ 266) fecimus, atque ponatur

$$P = \left(1 - \frac{1}{2^n}\right)\left(1 - \frac{1}{3^n}\right)\left(1 - \frac{1}{5^n}\right)\left(1 - \frac{1}{7^n}\right)\left(1 - \frac{1}{11^n}\right) \text{ etc.},$$

erit

$$P = 1 - \frac{1}{2^n} - \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} + \frac{1}{10^n} - \frac{1}{11^n} - \frac{1}{13^n} + \frac{1}{14^n} + \frac{1}{15^n} - \text{etc.},$$

ubi iterum ut ante omnes occurrunt numeri praeter potestates ac divisibiles per potestates. Verum ipsi numeri primi et qui ex ternis, quinis numerove imparibus constant, signum habent praefixum $-$, qui autem ex binis vel quaternis vel senis vel numero paribus formantur, signum habent $+$. Sic in hac serie occurret terminus $\frac{1}{30^n}$, quia est $30 = 2 \cdot 3 \cdot 5$ neque adeo potestatem complectitur; habebit vero hic terminus $\frac{1}{30^n}$ signum $-$, quia 30 est productum ex tribus numeris primis.

270. Consideremus iam hanc expressionem

$$\frac{1}{(1-\alpha z)(1-\beta z)(1-\gamma z)(1-\delta z)(1-\varepsilon z) \text{ etc.}},$$

quae per divisionem actualem evoluta praebeat hanc seriem

$$1 + Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + Fz^6 + \text{etc.},$$

atque manifestum est coefficientes A, B, C, D, E etc. sequenti modo ex numeris $\alpha, \beta, \gamma, \delta, \varepsilon$ etc. componi, ut sit

$A =$ summae singulorum,
$B =$ summae factorum ex binis,
$C =$ summae factorum ex ternis,
$D =$ summae factorum ex quaternis
etc.

non exclusis factoribus iisdem.

271. Posito ergo $z = 1$ ista expressio

$$\frac{1}{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)(1-\varepsilon) \text{ etc.}}$$

aequabitur unitati cum serie numerorum omnium, qui ex his $\alpha, \beta, \gamma, \delta, \varepsilon$ etc. vel sumendis singulis vel duobus pluribusve in se multiplicandis oriuntur non exclusis aequalibus. Hoc ergo differt ista numerorum series ab illa, quae § 265 prodiit, quod ibi factores tantum diversi sumi debebant, hic autem idem factor bis pluriesve occurrere possit. Hic scilicet omnes numeri occurrunt, qui per multiplicationem ex his $\alpha, \beta, \gamma, \delta$ etc. provenire possunt.

272. Hanc ob rem series semper ex terminorum numero infinito constat, sive factorum numerus fuerit infinitus sive finitus. Sic erit

$$\frac{1}{1-\frac{1}{2}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \text{etc.},$$

ubi omnes numeri adsunt, qui ex binario solo per multiplicationem oriuntur, seu omnes binarii potestates. Deinde erit

$$\frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \frac{1}{16} + \frac{1}{18} + \text{etc.},$$

ubi alii numeri non occurrunt, nisi qui ex his duobus 2 et 3 per multiplicationem originem trahunt, seu qui alios divisores praeter 2 et 3 non habent.

273. Si igitur pro $\alpha, \beta, \gamma, \delta$ etc. unitas per singulos omnes numeros primos scribatur ac ponatur

$$P = \frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\left(1-\frac{1}{13}\right)\text{ etc.}},$$

fiet

$$P = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \text{etc.},$$

ubi omnes numeri, tam primi quam qui ex primis per multiplicationem nascuntur, occurrunt. Cum autem omnes numeri vel sint ipsi primi vel ex primis per multiplicationem oriundi, manifestum est hic omnes omnino numeros integros in denominatoribus adesse debere.

274. Idem evenit, si numerorum primorum potestates quaecunque accipiantur. Si enim ponatur

$$P = \frac{1}{\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\left(1-\frac{1}{11^n}\right)\text{ etc.}},$$

fiet

$$P = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{8^n} + \text{etc.},$$

ubi omnes numeri naturales nullo excepto occurrunt. Quodsi autem in factoribus ubique signum + statuatur, ut sit

$$P = \frac{1}{\left(1+\frac{1}{2^n}\right)\left(1+\frac{1}{3^n}\right)\left(1+\frac{1}{5^n}\right)\left(1+\frac{1}{7^n}\right)\left(1+\frac{1}{11^n}\right)\text{ etc.}},$$

erit

$$P = 1 - \frac{1}{2^n} - \frac{1}{3^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} - \frac{1}{8^n} + \frac{1}{9^n} + \frac{1}{10^n} - \text{etc.},$$

ubi numeri primi habent signum —; qui sunt producti ex duobus primis, sive iisdem sive diversis, signum habent +; et generatim, quorum numerorum numerus factorum primorum est par, signum habent +, qui autem ex factoribus primis numero imparibus constant, habent signum —. Sic terminus $\frac{1}{240^n}$ ob $240 = 2\cdot 2\cdot 2\cdot 2\cdot 3\cdot 5$ habebit signum +. Cuius legis ratio percipitur ex § 270, si ponatur $z = -1$.

275. Si haec cum superioribus conferantur, nascentur binae series, quarum productum unitati aequatur. Sit enim

$$P = \frac{1}{\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\left(1-\frac{1}{11^n}\right)\text{ etc.}}$$

et

$$Q = \left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\left(1-\frac{1}{11^n}\right)\text{ etc.};$$

erit

$$P = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{8^n} + \text{etc.},$$

$$Q = 1 - \frac{1}{2^n} - \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} + \frac{1}{10^n} - \frac{1}{11^n} - \text{etc.}$$

(§ 269) atque manifestum est fore $PQ = 1$.

276. Sin autem ponatur

$$P = \frac{1}{\left(1+\frac{1}{2^n}\right)\left(1+\frac{1}{3^n}\right)\left(1+\frac{1}{5^n}\right)\left(1+\frac{1}{7^n}\right)\left(1+\frac{1}{11^n}\right)\text{ etc.}}$$

et

$$Q = \left(1+\frac{1}{2^n}\right)\left(1+\frac{1}{3^n}\right)\left(1+\frac{1}{5^n}\right)\left(1+\frac{1}{7^n}\right)\left(1+\frac{1}{11^n}\right)\text{ etc.},$$

erit

$$P = 1 - \frac{1}{2^n} - \frac{1}{3^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} - \frac{1}{8^n} + \frac{1}{9^n} + \text{etc.},$$

$$Q = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{10^n} + \frac{1}{11^n} + \text{etc.}$$

similique modo habebitur $PQ = 1$. Cognita ergo alterius seriei summa simul alterius innotescet.

277. Vicissim porro ex cognitis summis harum serierum assignari poterunt valores factorum infinitorum. Sit nimirum

$$M = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \text{etc.},$$

$$N = 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \frac{1}{5^{2n}} + \frac{1}{6^{2n}} + \frac{1}{7^{2n}} + \text{etc.}$$

eritque

$$M = \frac{1}{\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\left(1-\frac{1}{11^n}\right)\text{ etc.}},$$

$$N = \frac{1}{\left(1-\frac{1}{2^{2n}}\right)\left(1-\frac{1}{3^{2n}}\right)\left(1-\frac{1}{5^{2n}}\right)\left(1-\frac{1}{7^{2n}}\right)\left(1-\frac{1}{11^{2n}}\right)\text{ etc.}}.$$

Hinc per divisionem nascitur

$$\frac{M}{N} = \left(1+\frac{1}{2^n}\right)\left(1+\frac{1}{3^n}\right)\left(1+\frac{1}{5^n}\right)\left(1+\frac{1}{7^n}\right)\left(1+\frac{1}{11^n}\right)\text{ etc.},$$

denique vero erit

$$\frac{MM}{N} = \frac{2^n+1}{2^n-1}\cdot\frac{3^n+1}{3^n-1}\cdot\frac{5^n+1}{5^n-1}\cdot\frac{7^n+1}{7^n-1}\cdot\frac{11^n+1}{11^n-1}\cdot\text{etc.}$$

Ex cognitis ergo M et N praeter valores horum productorum summae harum serierum habebuntur:

$$\frac{1}{M} = 1-\frac{1}{2^n}-\frac{1}{3^n}-\frac{1}{5^n}+\frac{1}{6^n}-\frac{1}{7^n}+\frac{1}{10^n}-\frac{1}{11^n}-\text{etc.},$$

$$\frac{1}{N} = 1-\frac{1}{2^{2n}}-\frac{1}{3^{2n}}-\frac{1}{5^{2n}}+\frac{1}{6^{2n}}-\frac{1}{7^{2n}}+\frac{1}{10^{2n}}-\frac{1}{11^{2n}}-\text{etc.},$$

$$\frac{M}{N} = 1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{6^n}+\frac{1}{7^n}+\frac{1}{10^n}+\frac{1}{11^n}+\text{etc.},$$

$$\frac{N}{M} = 1-\frac{1}{2^n}-\frac{1}{3^n}+\frac{1}{4^n}-\frac{1}{5^n}+\frac{1}{6^n}-\frac{1}{7^n}-\frac{1}{8^n}+\frac{1}{9^n}+\frac{1}{10^n}-\text{etc.},$$

ex quarum combinatione multae aliae deduci possunt.

EXEMPLUM 1

Sit $n = 1$, et quoniam supra [§ 123] demonstravimus esse

$$l\frac{1}{1-x} = x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}+\frac{x^6}{6}+\text{etc.},$$

erit posito $x = 1$

$$l\frac{1}{1-1} = l\infty = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\text{etc.}$$

At logarithmus numeri infinite magni ∞ ipse est infinite magnus, ex quo erit

$$M = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\text{etc.} = \infty.$$

Hinc ob $\frac{1}{M} = \frac{1}{\infty} = 0$ fiet

$$0 = 1-\frac{1}{2}-\frac{1}{3}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}+\frac{1}{10}-\frac{1}{11}-\frac{1}{13}+\frac{1}{14}+\frac{1}{15}-\text{etc.}$$

Tum vero in productis habebitur

$$M = \infty = \frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\text{ etc.}},$$

unde fit

$$\infty = \frac{2}{1}\cdot\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdot\frac{11}{10}\cdot\frac{13}{12}\cdot\frac{17}{16}\cdot\frac{19}{18}\cdot\text{etc.}$$

et

$$0 = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{10}{11}\cdot\frac{12}{13}\cdot\frac{16}{17}\cdot\frac{18}{19}\cdot\text{etc.}$$

Deinde per summationem serierum supra [§ 167] traditam erit

$$N = 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\text{etc.} = \frac{\pi\pi}{6}.$$

Hinc obtinentur istae summae serierum:

$$\frac{6}{\pi\pi} = 1-\frac{1}{2^2}-\frac{1}{3^2}-\frac{1}{5^2}+\frac{1}{6^2}-\frac{1}{7^2}+\frac{1}{10^2}-\frac{1}{11^2}-\text{etc.},$$

$$\infty = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{10}+\frac{1}{11}+\text{etc.},$$

$$0 = 1-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\frac{1}{9}+\frac{1}{10}-\frac{1}{11}-\text{etc.}$$

Denique pro factoribus orietur

$$\frac{\pi\pi}{6} = \frac{2^2}{2^2-1}\cdot\frac{3^2}{3^2-1}\cdot\frac{5^2}{5^2-1}\cdot\frac{7^2}{7^2-1}\cdot\frac{11^2}{11^2-1}\cdot\text{etc.}$$

seu

$$\frac{\pi\pi}{6} = \frac{4}{3}\cdot\frac{9}{8}\cdot\frac{25}{24}\cdot\frac{49}{48}\cdot\frac{121}{120}\cdot\frac{169}{168}\cdot\text{etc.}$$

et ob $\frac{M}{N} = \infty$ seu $\frac{N}{M} = 0$ habebitur

$$\infty = \frac{3}{2}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\frac{8}{7}\cdot\frac{12}{11}\cdot\frac{14}{13}\cdot\frac{18}{17}\cdot\frac{20}{19}\cdot\text{etc.}$$

seu

$$0 = \frac{2}{3}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdot\frac{11}{12}\cdot\frac{13}{14}\cdot\frac{17}{18}\cdot\frac{19}{20}\cdot\text{etc.}$$

atque

$$\infty = \frac{3}{1}\cdot\frac{4}{2}\cdot\frac{6}{4}\cdot\frac{8}{6}\cdot\frac{12}{10}\cdot\frac{14}{12}\cdot\frac{18}{16}\cdot\frac{20}{18}\cdot\text{etc.}$$

seu

$$0 = \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{6}{7}\cdot\frac{8}{9}\cdot\frac{9}{10}\cdot\text{etc.},$$

quarum fractionum (excepta prima) numeratores unitate deficiunt a denominatoribus, summae autem ex numeratoribus et denominatoribus cuiusque fractionis constanter praebent numeros primos 3, 5, 7, 11, 13, 17, 19 etc.

EXEMPLUM 2

Sit $n = 2$ eritque ex superioribus [§ 167]

$$M = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \text{etc.} = \frac{\pi\pi}{6},$$

$$N = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \frac{1}{7^4} + \text{etc.} = \frac{\pi^4}{90}.$$

Hinc primo istae series summantur:

$$\frac{6}{\pi\pi} = 1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} + \frac{1}{10^2} - \frac{1}{11^2} - \text{etc.},$$

$$\frac{90}{\pi^4} = 1 - \frac{1}{2^4} - \frac{1}{3^4} - \frac{1}{5^4} + \frac{1}{6^4} - \frac{1}{7^4} + \frac{1}{10^4} - \frac{1}{11^4} - \text{etc.},$$

$$\frac{15}{\pi^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{10^2} + \frac{1}{11^2} + \text{etc.},$$

$$\frac{\pi\pi}{15} = 1 - \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} - \frac{1}{8^2} + \frac{1}{9^2} + \frac{1}{10^2} - \text{etc.}$$

Deinde valores sequentium productorum innotescunt:

$$\frac{\pi\pi}{6} = \frac{2^2}{2^2-1}\cdot\frac{3^2}{3^2-1}\cdot\frac{5^2}{5^2-1}\cdot\frac{7^2}{7^2-1}\cdot\frac{11^2}{11^2-1}\cdot\text{etc.},$$

$$\frac{\pi^4}{90} = \frac{2^4}{2^4-1}\cdot\frac{3^4}{3^4-1}\cdot\frac{5^4}{5^4-1}\cdot\frac{7^4}{7^4-1}\cdot\frac{11^4}{11^4-1}\cdot\text{etc.},$$

$$\frac{15}{\pi\pi} = \frac{2^2+1}{2^2}\cdot\frac{3^2+1}{3^2}\cdot\frac{5^2+1}{5^2}\cdot\frac{7^2+1}{7^2}\cdot\frac{11^2+1}{11^2}\cdot\text{etc.}$$

seu

$$\frac{\pi\pi}{15} = \frac{4}{5}\cdot\frac{9}{10}\cdot\frac{25}{26}\cdot\frac{49}{50}\cdot\frac{121}{122}\cdot\frac{169}{170}\cdot\text{etc.}$$

et

$$\frac{5}{2} = \frac{2^2+1}{2^2-1}\cdot\frac{3^2+1}{3^2-1}\cdot\frac{5^2+1}{5^2-1}\cdot\frac{7^2+1}{7^2-1}\cdot\frac{11^2+1}{11^2-1}\cdot\text{etc.}$$

sive

$$\frac{5}{2} = \frac{5}{3}\cdot\frac{5}{4}\cdot\frac{13}{12}\cdot\frac{25}{24}\cdot\frac{61}{60}\cdot\frac{85}{84}\cdot\text{etc.}$$

vel

$$\frac{3}{2} = \frac{5}{4}\cdot\frac{13}{12}\cdot\frac{25}{24}\cdot\frac{61}{60}\cdot\frac{85}{84}\cdot\text{etc.}$$

In his fractionibus numeratores unitate superant denominatores, simul vero sumpti praebent quadrata numerorum primorum 3^2, 5^2, 7^2, 11^2 etc.

EXEMPLUM 3

Quia ex superioribus [§ 167] valores ipsius M tantum, si n sit numerus par, assignare licet, ponamus $n = 4$ eritque

$$M = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \text{etc.} = \frac{\pi^4}{90},$$

$$N = 1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} + \frac{1}{6^8} + \text{etc.} = \frac{\pi^8}{9450}.$$

Hinc primo sequentes series summantur:

$$\frac{90}{\pi^4} = 1 - \frac{1}{2^4} - \frac{1}{3^4} - \frac{1}{5^4} + \frac{1}{6^4} - \frac{1}{7^4} + \frac{1}{10^4} - \frac{1}{11^4} - \text{etc.},$$

$$\frac{9450}{\pi^8} = 1 - \frac{1}{2^8} - \frac{1}{3^8} - \frac{1}{5^8} + \frac{1}{6^8} - \frac{1}{7^8} + \frac{1}{10^8} - \frac{1}{11^8} - \text{etc.},$$

$$\frac{105}{\pi^4} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{6^4} + \frac{1}{7^4} + \frac{1}{10^4} + \frac{1}{11^4} + \text{etc.},$$

$$\frac{\pi^4}{105} = 1 - \frac{1}{2^4} - \frac{1}{3^4} + \frac{1}{4^4} - \frac{1}{5^4} + \frac{1}{6^4} - \frac{1}{7^4} - \frac{1}{8^4} + \frac{1}{9^4} + \text{etc.}$$

Deinde etiam valores sequentium productorum obtinentur:

$$\frac{\pi^4}{90} = \frac{2^4}{2^4-1} \cdot \frac{3^4}{3^4-1} \cdot \frac{5^4}{5^4-1} \cdot \frac{7^4}{7^4-1} \cdot \frac{11^4}{11^4-1} \cdot \text{etc.},$$

$$\frac{\pi^8}{9450} = \frac{2^8}{2^8-1} \cdot \frac{3^8}{3^8-1} \cdot \frac{5^8}{5^8-1} \cdot \frac{7^8}{7^8-1} \cdot \frac{11^8}{11^8-1} \cdot \text{etc.},$$

$$\frac{105}{\pi^4} = \frac{2^4+1}{2^4} \cdot \frac{3^4+1}{3^4} \cdot \frac{5^4+1}{5^4} \cdot \frac{7^4+1}{7^4} \cdot \frac{11^4+1}{11^4} \cdot \text{etc.}$$

et

$$\frac{7}{6} = \frac{2^4+1}{2^4-1} \cdot \frac{3^4+1}{3^4-1} \cdot \frac{5^4+1}{5^4-1} \cdot \frac{7^4+1}{7^4-1} \cdot \frac{11^4+1}{11^4-1} \cdot \text{etc.}$$

seu

$$\frac{35}{34} = \frac{41}{40} \cdot \frac{313}{312} \cdot \frac{1201}{1200} \cdot \frac{7321}{7320} \cdot \text{etc.}$$

In his factoribus numeratores unitate superant denominatores, simul vero sumpti praebent biquadrata numerorum primorum imparium 3, 5, 7, 11 etc.

278. Quoniam hic summam seriei

$$M = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \text{etc.}$$

ad factores reduximus, ad logarithmos commode progredi licebit. Nam cum sit

$$M = \frac{1}{\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\left(1-\frac{1}{11^n}\right)\text{etc.}},$$

erit

$$lM = -l\left(1-\frac{1}{2^n}\right) - l\left(1-\frac{1}{3^n}\right) - l\left(1-\frac{1}{5^n}\right) - l\left(1-\frac{1}{7^n}\right) - \text{etc.}$$

Hinc sumendis logarithmis hyperbolicis erit

$$\begin{aligned} lM = &+ 1\left(\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \text{etc.}\right) \\ &+ \frac{1}{2}\left(\frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \frac{1}{11^{2n}} + \text{etc.}\right) \\ &+ \frac{1}{3}\left(\frac{1}{2^{3n}} + \frac{1}{3^{3n}} + \frac{1}{5^{3n}} + \frac{1}{7^{3n}} + \frac{1}{11^{3n}} + \text{etc.}\right) \\ &+ \frac{1}{4}\left(\frac{1}{2^{4n}} + \frac{1}{3^{4n}} + \frac{1}{5^{4n}} + \frac{1}{7^{4n}} + \frac{1}{11^{4n}} + \text{etc.}\right) \\ &\qquad \text{etc.} \end{aligned}$$

Quodsi insuper ponamus

$$N = 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \frac{1}{5^{2n}} + \frac{1}{6^{2n}} + \text{etc.},$$

ut sit

$$N = \frac{1}{\left(1-\frac{1}{2^{2n}}\right)\left(1-\frac{1}{3^{2n}}\right)\left(1-\frac{1}{5^{2n}}\right)\left(1-\frac{1}{7^{2n}}\right)\left(1-\frac{1}{11^{2n}}\right)\text{etc.}},$$

fiet logarithmis hyperbolicis sumendis

$$\begin{aligned} lN = &+ 1\left(\frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \frac{1}{11^{2n}} + \text{etc.}\right) \\ &+ \frac{1}{2}\left(\frac{1}{2^{4n}} + \frac{1}{3^{4n}} + \frac{1}{5^{4n}} + \frac{1}{7^{4n}} + \frac{1}{11^{4n}} + \text{etc.}\right) \\ &+ \frac{1}{3}\left(\frac{1}{2^{6n}} + \frac{1}{3^{6n}} + \frac{1}{5^{6n}} + \frac{1}{7^{6n}} + \frac{1}{11^{6n}} + \text{etc.}\right) \\ &+ \frac{1}{4}\left(\frac{1}{2^{8n}} + \frac{1}{3^{8n}} + \frac{1}{5^{8n}} + \frac{1}{7^{8n}} + \frac{1}{11^{8n}} + \text{etc.}\right) \\ &\qquad \text{etc.} \end{aligned}$$

Ex his coniunctis fiet

$$\begin{aligned} lM - \tfrac{1}{2} lN = &+ 1\left(\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \text{etc.}\right) \\ &+ \frac{1}{3}\left(\frac{1}{2^{3n}} + \frac{1}{3^{3n}} + \frac{1}{5^{3n}} + \frac{1}{7^{3n}} + \frac{1}{11^{3n}} + \text{etc.}\right) \\ &+ \frac{1}{5}\left(\frac{1}{2^{5n}} + \frac{1}{3^{5n}} + \frac{1}{5^{5n}} + \frac{1}{7^{5n}} + \frac{1}{11^{5n}} + \text{etc.}\right) \\ &+ \frac{1}{7}\left(\frac{1}{2^{7n}} + \frac{1}{3^{7n}} + \frac{1}{5^{7n}} + \frac{1}{7^{7n}} + \frac{1}{11^{7n}} + \text{etc.}\right) \\ &\text{etc.} \end{aligned}$$

279. Si $n = 1$, erit

$$M = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{etc.} = l\infty$$

et

$$N = \frac{\pi\pi}{6};$$

hincque erit

$$\begin{aligned} l.l\infty - \tfrac{1}{2} l\tfrac{\pi\pi}{6} = &+ 1\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \text{etc.}\right) \\ &+ \frac{1}{3}\left(\frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \frac{1}{11^3} + \text{etc.}\right) \\ &+ \frac{1}{5}\left(\frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{5^5} + \frac{1}{7^5} + \frac{1}{11^5} + \text{etc.}\right) \\ &+ \frac{1}{7}\left(\frac{1}{2^7} + \frac{1}{3^7} + \frac{1}{5^7} + \frac{1}{7^7} + \frac{1}{11^7} + \text{etc.}\right) \\ &\text{etc.} \end{aligned}$$

Verum hae series praeter primam non solum summas habent finitas, sed etiam cunctae simul sumptae summam efficiunt finitam eamque satis parvam; unde necesse est, ut seriei primae

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \text{etc.}$$

summa sit infinite magna. Quantitate scilicet satis parva deficiet a logarithmo hyperbolico seriei

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \text{etc.}$$

280. Sit $n = 2$; erit

$$M = \frac{\pi\pi}{6} \quad \text{et} \quad N = \frac{\pi^4}{90},$$

unde fit

$$\begin{aligned} 2l\pi - l6 = 1&\left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \text{etc.}\right) \\ &+ \frac{1}{2}\left(\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{11^4} + \text{etc.}\right) \\ &+ \frac{1}{3}\left(\frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{11^6} + \text{etc.}\right) \\ &\text{etc.,} \end{aligned}$$

$$\begin{aligned} 4l\pi - l90 = &+ 1\left(\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{11^4} + \text{etc.}\right) \\ &+ \frac{1}{2}\left(\frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{11^8} + \text{etc.}\right) \\ &+ \frac{1}{3}\left(\frac{1}{2^{12}} + \frac{1}{3^{12}} + \frac{1}{5^{12}} + \frac{1}{7^{12}} + \frac{1}{11^{12}} + \text{etc.}\right) \\ &\text{etc.,} \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2} l\tfrac{5}{2} = 1&\left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \text{etc.}\right) \\ &+ \frac{1}{3}\left(\frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{11^6} + \text{etc.}\right) \\ &+ \frac{1}{5}\left(\frac{1}{2^{10}} + \frac{1}{3^{10}} + \frac{1}{5^{10}} + \frac{1}{7^{10}} + \frac{1}{11^{10}} + \text{etc.}\right) \\ &\text{etc.} \end{aligned}$$

281. Quanquam lex, qua numeri primi progrediuntur, non constat, tamen harum serierum altiorum potestatum summae non difficulter proxime assignari poterunt. Sit enim haec series

$$M = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \text{etc.}$$

et

$$S = \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \text{etc.};$$

erit

$$S = M - 1 - \frac{1}{4^n} - \frac{1}{6^n} - \frac{1}{8^n} - \frac{1}{9^n} - \frac{1}{10^n} - \text{etc.}$$

et ob

$$\frac{M}{2^n} = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} + \frac{1}{10^n} + \frac{1}{12^n} + \text{etc.}$$

erit

$$S = M - \frac{M}{2^n} - 1 + \frac{1}{2^n} - \frac{1}{9^n} - \frac{1}{15^n} - \frac{1}{21^n} - \text{etc.}$$

seu

$$S = (M-1)\left(1 - \frac{1}{2^n}\right) - \frac{1}{9^n} - \frac{1}{15^n} - \frac{1}{21^n} - \frac{1}{25^n} - \frac{1}{27^n} - \text{etc.}$$

et ob

$$M\left(1 - \frac{1}{2^n}\right)\frac{1}{3^n} = \frac{1}{3^n} + \frac{1}{9^n} + \frac{1}{15^n} + \frac{1}{21^n} + \text{etc.}$$

erit

$$S = (M-1)\left(1 - \frac{1}{2^n}\right)\left(1 - \frac{1}{3^n}\right) + \frac{1}{6^n} - \frac{1}{25^n} - \frac{1}{35^n} - \frac{1}{49^n} - \text{etc.}^{1)}.$$

Hinc ob datam summam M [§ 168] valor ipsius S commode invenitur, siquidem n fuerit numerus mediocriter magnus.

282. Inventis autem summis altiorum potestatum etiam summae potestatum minorum ex formulis inventis exhiberi possunt. Atque hac methodo sequentes prodierunt summae seriei

$$\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} + \text{etc.}:$$

1) Editio princeps: $S = \cdots + \frac{1}{6^n} - \frac{1}{25^n} - \frac{1}{35^n} - \frac{1}{45^n} - \text{etc.}$ Correxit F. R.

Si sit	erit summa seriei[1])
$n = 2$	0,452247420041065
$n = 4$	0,076993139764247
$n = 6$	0,017070086850637
$n = 8$	0,004061405366518
$n = 10$	0,000993603574437
$n = 12$	0,000246026470035
$n = 14$	0,000061244396725
$n = 16$	0,000015282026219
$n = 18$	0,000003817278703
$n = 20$	0,000000953961124
$n = 22$	0,000000238450446
$n = 24$	0,000000059608185
$n = 26$	0,000000014901555
$n = 28$	0,000000003725334
$n = 30$	0,000000000931327
$n = 32$	0,000000000232831
$n = 34$	0,000000000058208
$n = 36$	0,000000000014552

Reliquae summae parium potestatum in ratione quadrupla decrescunt.

283. Haec autem seriei

$$1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \text{etc.}$$

1) In editione principe quinque ultimae figurae harum serierum ita se habent:
($n = 2$) 41222; ($n = 4$) 64252; ($n = 6$) 50639; ($n = 8$) 66515; ($n = 10$) 73633; ($n = 12$) 70033; ($n = 18$) 78702; ($n = 20$) 61123; ($n = 24$) 08184; ($n = 28$) 25333; ($n = 30$) 31323; ($n = 32$) 32830; ($n = 34$) 58207; ($n = 36$) 14551. Correxit F. R.

in productum infinitum conversio etiam directe institui potest hoc modo. Sit

$$A = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{8^n} + \text{etc.};$$

subtrahe

$$\frac{1}{2^n} A = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} + \text{etc.};$$

erit

$$\left(1 - \frac{1}{2^n}\right) A = 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \frac{1}{11^n} + \text{etc.} = B.$$

Sic sublati sunt omnes termini per 2 divisibiles. Subtrahe

$$\frac{1}{3^n} B = \frac{1}{3^n} + \frac{1}{9^n} + \frac{1}{15^n} + \frac{1}{21^n} + \text{etc.};$$

erit

$$\left(1 - \frac{1}{3^n}\right) B = 1 + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \text{etc.} = C.$$

Sic insuper sublati sunt omnes termini per 3 divisibiles. Subtrahe

$$\frac{1}{5^n} C = \frac{1}{5^n} + \frac{1}{25^n} + \frac{1}{35^n} + \frac{1}{55^n} + \text{etc.};$$

erit

$$\left(1 - \frac{1}{5^n}\right) C = 1 + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} + \text{etc.}$$

Sic sublati etiam sunt omnes termini per 5 divisibiles. Pari modo tolluntur termini divisibiles per 7, 11 reliquosque numeros primos; manifestum autem est sublatis omnibus terminis, qui per numeros primos divisibiles sint, solam unitatem relinqui. Quare pro B, C, D, E etc. valoribus restitutis tandem orietur

$$A\left(1 - \frac{1}{2^n}\right)\left(1 - \frac{1}{3^n}\right)\left(1 - \frac{1}{5^n}\right)\left(1 - \frac{1}{7^n}\right)\left(1 - \frac{1}{11^n}\right) \text{etc.} = 1,$$

unde seriei propositae summa erit

$$A = \frac{1}{\left(1 - \frac{1}{2^n}\right)\left(1 - \frac{1}{3^n}\right)\left(1 - \frac{1}{5^n}\right)\left(1 - \frac{1}{7^n}\right)\left(1 - \frac{1}{11^n}\right) \text{etc.}}$$

seu

$$A = \frac{2^n}{2^n - 1} \cdot \frac{3^n}{3^n - 1} \cdot \frac{5^n}{5^n - 1} \cdot \frac{7^n}{7^n - 1} \cdot \frac{11^n}{11^n - 1} \cdot \text{etc.}$$

284. Haec methodus iam commode adhiberi poterit ad alias series, quarum summas supra invenimus, in producta infinita convertendas. Invenimus autem supra (§ 175) summas harum serierum

$$1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} - \frac{1}{11^n} + \frac{1}{13^n} - \text{etc.},$$

si n fuerit numerus impar. Summa enim est $= N\pi^n$ et valores ipsius N loco citato dedimus. Notandum autem est, cum hic tantum numeri impares occurrunt, eos, qui sint formae $4m+1$, habere signum $+$, reliquos formae $4m-1$ signum $-$. Sit igitur

$$A = 1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{15^n} + \text{etc.}$$

Addatur

$$\frac{1}{3^n} A = \frac{1}{3^n} - \frac{1}{9^n} + \frac{1}{15^n} - \frac{1}{21^n} + \frac{1}{27^n} - \text{etc.};$$

erit

$$\left(1 + \frac{1}{3^n}\right) A = 1 + \frac{1}{5^n} - \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} - \text{etc.} = B.$$

Subtrahatur

$$\frac{1}{5^n} B = \frac{1}{5^n} + \frac{1}{25^n} - \frac{1}{35^n} - \frac{1}{55^n} + \text{etc.};$$

erit

$$\left(1 - \frac{1}{5^n}\right) B = 1 - \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} - \text{etc.} = C,$$

ubi iam numeri per 3 et 5 divisibiles desunt. Addatur

$$\frac{1}{7^n} C = \frac{1}{7^n} - \frac{1}{49^n} - \frac{1}{77^n} + \text{etc.};$$

erit

$$\left(1 + \frac{1}{7^n}\right) C = 1 - \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} - \text{etc.} = D.$$

Sic etiam numeri per 7 divisibiles sunt sublati. Addatur

$$\frac{1}{11^n} D = \frac{1}{11^n} - \frac{1}{121^n} + \text{etc.};$$

erit

$$\left(1 + \frac{1}{11^n}\right) D = 1 + \frac{1}{13^n} + \frac{1}{17^n} - \text{etc.} = E.$$

Sic numeri per 11 divisibiles quoque sunt sublati. Auferendis autem hoc modo reliquis numeris omnibus per reliquos numeros primos divisibilibus tandem prodibit

$$A\left(1+\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1+\frac{1}{7^n}\right)\left(1+\frac{1}{11^n}\right)\left(1-\frac{1}{13^n}\right)\text{ etc.} = 1$$

seu

$$A = \frac{3^n}{3^n+1}\cdot\frac{5^n}{5^n-1}\cdot\frac{7^n}{7^n+1}\cdot\frac{11^n}{11^n+1}\cdot\frac{13^n}{13^n-1}\cdot\frac{17^n}{17^n-1}\cdot\text{etc.},$$

ubi in numeratoribus occurrunt potestates omnium numerorum primorum, quae in denominatoribus insunt unitate sive auctae sive minutae, prout numeri primi fuerint formae $4m-1$ vel $4m+1$.

285. Posito ergo $n=1$ ob $A=\frac{\pi}{4}$ [§ 175] erit

$$\frac{\pi}{4} = \frac{3}{4}\cdot\frac{5}{4}\cdot\frac{7}{8}\cdot\frac{11}{12}\cdot\frac{13}{12}\cdot\frac{17}{16}\cdot\frac{19}{20}\cdot\frac{23}{24}\cdot\text{etc.}$$

Supra [§ 277] autem invenimus esse

$$\frac{\pi\pi}{6} = \frac{4}{3}\cdot\frac{3^2}{2\cdot 4}\cdot\frac{5^2}{4\cdot 6}\cdot\frac{7^2}{6\cdot 8}\cdot\frac{11^2}{10\cdot 12}\cdot\frac{13^2}{12\cdot 14}\cdot\frac{17^2}{16\cdot 18}\cdot\frac{19^2}{18\cdot 20}\cdot\text{etc.}$$

Dividatur secunda per primam et orietur

$$\frac{2\pi}{3} = \frac{4}{3}\cdot\frac{3}{2}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdot\frac{11}{10}\cdot\frac{13}{14}\cdot\frac{17}{18}\cdot\frac{19}{18}\cdot\frac{23}{22}\cdot\text{etc.}$$

seu

$$\frac{\pi}{2} = \frac{3}{2}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdot\frac{11}{10}\cdot\frac{13}{14}\cdot\frac{17}{18}\cdot\frac{19}{18}\cdot\frac{23}{22}\cdot\text{etc.},$$

ubi numeri primi constituunt numeratores, denominatores vero sunt numeri impariter pares unitate differentes a numeratoribus. Quodsi haec denuo per primam $\frac{\pi}{4}$ dividatur, erit

$$2 = \frac{4}{2}\cdot\frac{4}{6}\cdot\frac{8}{6}\cdot\frac{12}{10}\cdot\frac{12}{14}\cdot\frac{16}{18}\cdot\frac{20}{18}\cdot\frac{24}{22}\cdot\text{etc.}$$

seu

$$2 = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{9}\cdot\frac{10}{9}\cdot\frac{12}{11}\cdot\text{etc.},$$

quae fractiones oriuntur ex numeris primis imparibus 3, 5, 7, 11, 13, 17 etc. quemque in duas partes unitate differentes dispescendo et partes pares pro numeratoribus, impares pro denominatoribus sumendo.

286. Si hae expressiones cum WALLISIANA[1]) comparentur

$$\frac{\pi}{2} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdot 10\cdot 10\cdot 12}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdot 9\cdot 11\cdot 11}\text{ etc.}$$

seu

$$\frac{4}{\pi} = \frac{3\cdot 3}{2\cdot 4}\cdot\frac{5\cdot 5}{4\cdot 6}\cdot\frac{7\cdot 7}{6\cdot 8}\cdot\frac{9\cdot 9}{8\cdot 10}\cdot\frac{11\cdot 11}{10\cdot 12}\cdot\text{etc.},$$

cum sit [§ 277]

$$\frac{\pi\pi}{8} = \frac{3\cdot 3}{2\cdot 4}\cdot\frac{5\cdot 5}{4\cdot 6}\cdot\frac{7\cdot 7}{6\cdot 8}\cdot\frac{11\cdot 11}{10\cdot 12}\cdot\frac{13\cdot 13}{12\cdot 14}\cdot\text{etc.},$$

illa per hanc divisa dabit

$$\frac{32}{\pi^3} = \frac{9\cdot 9}{8\cdot 10}\cdot\frac{15\cdot 15}{14\cdot 16}\cdot\frac{21\cdot 21}{20\cdot 22}\cdot\frac{25\cdot 25}{24\cdot 26}\cdot\text{etc.},$$

ubi in numeratoribus occurrunt omnes numeri impares non primi.

287. Sit iam $n=3$; erit $A=\frac{\pi^3}{32}$ [§ 175], unde fit

$$\frac{\pi^3}{32} = \frac{3^3}{3^3+1}\cdot\frac{5^3}{5^3-1}\cdot\frac{7^3}{7^3+1}\cdot\frac{11^3}{11^3+1}\cdot\frac{13^3}{13^3-1}\cdot\frac{17^3}{17^3-1}\cdot\text{etc.}$$

At ex serie [§ 167]

$$\frac{\pi^6}{945} = 1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\text{etc.}$$

fit [§ 277]

$$\frac{\pi^6}{945} = \frac{2^6}{2^6-1}\cdot\frac{3^6}{3^6-1}\cdot\frac{5^6}{5^6-1}\cdot\frac{7^6}{7^6-1}\cdot\frac{11^6}{11^6-1}\cdot\frac{13^6}{13^6-1}\cdot\text{etc.}$$

seu

$$\frac{\pi^6}{960} = \frac{3^6}{3^6-1}\cdot\frac{5^6}{5^6-1}\cdot\frac{7^6}{7^6-1}\cdot\frac{11^6}{11^6-1}\cdot\frac{13^6}{13^6-1}\cdot\text{etc.},$$

1) Vide notam p. 197. F. R.

quae per primam divisa dabit

$$\frac{\pi^3}{30} = \frac{3^3}{3^3-1}\cdot\frac{5^3}{5^3+1}\cdot\frac{7^3}{7^3-1}\cdot\frac{11^3}{11^3-1}\cdot\frac{13^3}{13^3+1}\cdot\frac{17^3}{17^3+1}\cdot \text{etc.}$$

Haec vero denuo per primam divisa dabit

$$\frac{16}{15} = \frac{3^3+1}{3^3-1}\cdot\frac{5^3-1}{5^3+1}\cdot\frac{7^3+1}{7^3-1}\cdot\frac{11^3+1}{11^3-1}\cdot\frac{13^3-1}{13^3+1}\cdot\frac{17^3-1}{17^3+1}\cdot \text{etc.}$$

seu

$$\frac{16}{15} = \frac{14}{13}\cdot\frac{62}{63}\cdot\frac{172}{171}\cdot\frac{666}{665}\cdot\frac{1098}{1099}\cdot \text{etc.},$$

quae fractiones formantur ex cubis numerorum primorum imparium quemque in duas partes unitate differentes dispescendo ac partes pares pro numeratoribus, impares pro denominatoribus sumendo.

288. Ex his expressionibus denuo novae series formari possunt, in quibus omnes numeri naturales denominatores constituunt. Cum enim sit [§ 285]

$$\frac{\pi}{4} = \frac{3}{3+1}\cdot\frac{5}{5-1}\cdot\frac{7}{7+1}\cdot\frac{11}{11+1}\cdot\frac{13}{13-1}\cdot \text{etc.},$$

erit

$$\frac{\pi}{6} = \frac{1}{\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1-\frac{1}{13}\right)\text{ etc.}},$$

unde per evolutionem haec series nascetur

$$\frac{\pi}{6} = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} - \text{etc.},$$

ubi ratio signorum ita est comparata, ut binarius habeat —, numeri primi formae $4m-1$ signum — et numeri primi formae $4m+1$ signum +; numeri autem compositi ea habent signa, quae ipsis ratione multiplicationis ex primis conveniunt. Sic patebit signum fractionis $\frac{1}{60}$ ob

$$60 = \overset{-}{2}\cdot\overset{-}{2}\cdot\overset{-}{3}\cdot\overset{+}{5},$$

quod erit —.

Simili modo porro erit

$$\frac{\pi}{2} = \frac{1}{\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1-\frac{1}{13}\right)\text{ etc.}},$$

unde orietur haec series

$$\frac{\pi}{2} = 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} - \text{etc.},$$

ubi binarius habet signum +, numeri primi formae $4m-1$ signum —, numeri primi formae $4m+1$ signum +; et numerus quisque compositus id habet signum, quod ipsi ratione compositionis ex primis convenit secundum regulas multiplicationis.

289. Cum deinde sit [§ 285]

$$\frac{\pi}{2} = \frac{1}{\left(1-\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\text{ etc.}},$$

erit per evolutionem

$$\frac{\pi}{2} = 1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} - \text{etc.},$$

ubi tantum numeri impares occurrunt, signa autem ita sunt comparata, ut numeri primi formae $4m-1$ signum habeant +, numeri primi formae $4m+1$ signum —, unde simul numerorum compositorum signa definiuntur.

Binae porro series hinc formari possunt, ubi omnes numeri occurrunt. Erit scilicet

$$\pi = \frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\text{ etc.}},$$

unde per evolutionem oritur

$$\pi = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \text{etc.},$$

ubi binarius signum habet $+$, numeri primi formae $4m-1$ signum $+$, numeri vero primi formae $4m+1$ signum $-$.

Tum vero etiam erit

$$\frac{\pi}{3}=\frac{1}{\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\text{ etc.}},$$

unde per evolutionem oritur

$$\frac{\pi}{3}=1-\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\text{etc.},$$

ubi binarius habet signum $-$, numeri primi formae $4m-1$ signum $+$ et numeri primi formae $4m+1$ signum $-$.

290. Possunt hinc etiam innumerabiles aliae signorum conditiones exhiberi, ita ut seriei

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \frac{1}{6},\ \frac{1}{7},\ \frac{1}{8}\ \text{etc.}$$

summa assignari queat. Cum scilicet sit

$$\frac{\pi}{2}=\frac{1}{\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\text{ etc.}},$$

multiplicetur haec expressio per $\frac{1+\frac{1}{3}}{1-\frac{1}{3}}=2$; erit

$$\pi=\frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\text{ etc.}}$$

et

$$\pi=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}-\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}-\frac{1}{11}+\text{etc.},$$

ubi binarius signum habet $+$, ternarius $+$, reliqui numeri primi omnes formae $4m-1$ signum $-$, at numeri primi formae $4m+1$ signum $+$; unde pro numeris compositis ratio signorum intelligitur.

Simili modo cum sit

$$\pi=\frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\text{ etc.}},$$

multiplicetur per $\frac{1+\frac{1}{5}}{1-\frac{1}{5}}=\frac{3}{2}$; erit

$$\frac{3\pi}{2}=\frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\left(1+\frac{1}{17}\right)\text{ etc.}},$$

unde per evolutionem oritur

$$\frac{3\pi}{2}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}-\frac{1}{13}+\text{etc.},$$

ubi binarius habet signum $+$, numeri primi formae $4m-1$ signum $+$ et numeri primi formae $4m+1$ praeter quinarium signum $-$.

291. Possunt etiam innumerabiles huiusmodi series exhiberi, quarum summa sit $=0$. Cum enim sit [§ 277]

$$0=\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdot\frac{11}{12}\cdot\frac{13}{14}\cdot\frac{17}{18}\cdot\text{etc.},$$

erit

$$0=\frac{1}{\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\text{ etc.}},$$

unde, ut supra vidimus, oritur

$$0=1-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\frac{1}{9}+\frac{1}{10}-\text{etc.},$$

ubi omnes numeri primi signum habent $-$ compositorumque numerorum signa regulam multiplicationis sequuntur.

Multiplicemus autem illam expressionem per $\frac{1+\frac{1}{2}}{1-\frac{1}{2}}=3$; erit pariter

$$0=\frac{1}{\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\text{ etc.}},$$

unde per evolutionem nascitur

$$0 = 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{1}{10} - \text{etc.},$$

ubi binarius habet signum $+$, reliqui numeri primi omnes signum $-$.

Simili modo quoque erit

$$0 = \frac{1}{\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\text{ etc.}},$$

unde oritur ista series

$$0 = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} - \text{etc.},$$

ubi omnes numeri primi praeter 3 et 5 habent signum $-$.

In genere autem notandum est, quoties omnes numeri primi exceptis tantum aliquibus habeant signum $-$, summa seriei fore $=0$, contra autem, quoties omnes numeri primi exceptis tantum aliquibus habeant signum $+$, tum summam seriei fore infinite magnam.

292. Supra etiam (§ 176) summam dedimus seriei

$$A = 1 - \frac{1}{2^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{7^n} - \frac{1}{8^n} + \frac{1}{10^n} - \frac{1}{11^n} + \frac{1}{13^n} - \text{etc.},$$

si fuerit n numerus impar. Erit ergo

$$\frac{1}{2^n}A = \frac{1}{2^n} - \frac{1}{4^n} + \frac{1}{8^n} - \frac{1}{10^n} + \frac{1}{14^n} - \text{etc.},$$

quae addita dat

$$B = \left(1+\frac{1}{2^n}\right)A = 1 - \frac{1}{5^n} + \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{17^n} + \frac{1}{19^n} - \frac{1}{23^n} + \frac{1}{25^n} - \text{etc.}$$

Addatur

$$\frac{1}{5^n}B = \frac{1}{5^n} - \frac{1}{25^n} + \frac{1}{35^n} - \frac{1}{55^n} + \text{etc.};$$

erit

$$C = \left(1+\frac{1}{5^n}\right)B = 1 + \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{17^n} + \frac{1}{19^n} - \frac{1}{23^n} - \text{etc.}$$

Subtrahatur

$$\frac{1}{7^n}C = \frac{1}{7^n} + \frac{1}{49^n} - \frac{1}{77^n} + \text{etc.};$$

erit

$$D = \left(1-\frac{1}{7^n}\right)C = 1 - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{17^n} + \frac{1}{19^n} - \text{etc.}$$

Ex his tandem fiet

$$A\left(1+\frac{1}{2^n}\right)\left(1+\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\left(1+\frac{1}{11^n}\right)\left(1-\frac{1}{13^n}\right)\text{ etc.} = 1,$$

ubi numeri primi unitate excedentes multipla senarii habent signum $-$, deficientes autem signum $+$. Eritque

$$A = \frac{2^n}{2^n+1}\cdot\frac{5^n}{5^n+1}\cdot\frac{7^n}{7^n-1}\cdot\frac{11^n}{11^n+1}\cdot\frac{13^n}{13^n-1}\cdot\text{etc.}$$

293. Consideremus casum $n=1$, quo $A = \frac{\pi}{3\sqrt{3}}$, eritque

$$\frac{\pi}{3\sqrt{3}} = \frac{2}{3}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdot\frac{11}{12}\cdot\frac{13}{12}\cdot\frac{17}{18}\cdot\frac{19}{18}\cdot\text{etc.},$$

ubi in numeratoribus post 3 occurrunt omnes numeri primi, denominatores vero a numeratoribus unitate discrepant suntque omnes per 6 divisibiles. Cum iam sit [§ 277]

$$\frac{\pi\pi}{6} = \frac{4}{3}\cdot\frac{9}{8}\cdot\frac{5\cdot 5}{4\cdot 6}\cdot\frac{7\cdot 7}{6\cdot 8}\cdot\frac{11\cdot 11}{10\cdot 12}\cdot\frac{13\cdot 13}{12\cdot 14}\cdot\text{etc.},$$

erit hac expressione per illam divisa

$$\frac{\pi\sqrt{3}}{2} = \frac{9}{4}\cdot\frac{5}{4}\cdot\frac{7}{8}\cdot\frac{11}{10}\cdot\frac{13}{14}\cdot\frac{17}{16}\cdot\frac{19}{20}\cdot\text{etc.},$$

ubi denominatores non sunt per 6 divisibiles. Vel erit

$$\frac{\pi}{2\sqrt{3}} = \frac{5}{6}\cdot\frac{7}{6}\cdot\frac{11}{12}\cdot\frac{13}{12}\cdot\frac{17}{18}\cdot\frac{19}{18}\cdot\frac{23}{24}\cdot\text{etc.},$$

$$\frac{2\pi}{3\sqrt{3}} = \frac{5}{4}\cdot\frac{7}{8}\cdot\frac{11}{10}\cdot\frac{13}{14}\cdot\frac{17}{16}\cdot\frac{19}{20}\cdot\frac{23}{22}\cdot\text{etc.},$$

quarum haec per illam divisa dat

$$\frac{4}{3} = \frac{6}{4} \cdot \frac{6}{8} \cdot \frac{12}{10} \cdot \frac{12}{14} \cdot \frac{18}{16} \cdot \frac{18}{20} \cdot \frac{24}{22} \cdot \text{etc.}$$

seu

$$\frac{4}{3} = \frac{3}{2} \cdot \frac{3}{4} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{9}{8} \cdot \frac{9}{10} \cdot \frac{12}{11} \cdot \text{etc.},$$

ubi singulae fractiones ex numeris primis 5, 7, 11 etc. formantur singulos numeros primos in duas partes unitate differentes dispescendo et partes per 3 divisibiles constanter pro numeratoribus sumendo.

294. Quoniam vero supra [§ 285] vidimus esse

$$\frac{\pi}{4} = \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \text{etc.}$$

seu

$$\frac{\pi}{3} = \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{20} \cdot \text{etc.},$$

si superiores $\frac{\pi}{2\sqrt{3}}$ et $\frac{2\pi}{3\sqrt{3}}$ per hanc dividantur, orietur

$$\frac{\sqrt{3}}{2} = \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{8}{9} \cdot \frac{10}{9} \cdot \frac{14}{15} \cdot \frac{16}{15} \cdot \text{etc.},$$

$$\frac{2}{\sqrt{3}} = \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{12}{11} \cdot \frac{18}{19} \cdot \frac{24}{23} \cdot \frac{30}{29} \cdot \text{etc.}$$

In priori expressione fractiones formantur ex numeris primis formae $12m+6\pm1$, in posteriore ex numeris primis formae $12m\pm1$, singulos in duas partes unitate discrepantes dispescendo et partes pares pro numeratoribus, impares vero pro denominatoribus sumendo.

295. Contemplemur adhuc seriem supra (§ 179) inventam, quae ita progrediebatur

$$\frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \text{etc.} = A;$$

erit

$$\frac{1}{3}A = \frac{1}{3} + \frac{1}{9} - \frac{1}{15} - \frac{1}{21} + \frac{1}{27} + \frac{1}{33} - \text{etc.}$$

Subtrahatur [eritque]

$$\left(1-\frac{1}{3}\right)A = 1 - \frac{1}{5} - \frac{1}{7} + \frac{1}{11} - \frac{1}{13} + \frac{1}{17} + \frac{1}{19} - \text{etc.} = B.$$

Addatur

$$\frac{1}{5}B = \frac{1}{5} - \frac{1}{25} - \frac{1}{35} + \frac{1}{55} - \text{etc.};$$

erit

$$\left(1+\frac{1}{5}\right)B = 1 - \frac{1}{7} + \frac{1}{11} - \frac{1}{13} + \frac{1}{17} + \frac{1}{19} - \text{etc.} = C.$$

Sicque progrediendo tandem pervenietur ad

$$\frac{\pi}{2\sqrt{2}}\left(1-\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1+\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\left(1-\frac{1}{17}\right)\left(1-\frac{1}{19}\right)\text{etc.} = 1,$$

ubi signa ita se habent, ut numerorum primorum formae $8m+1$ vel $8m+3$ signa sint $-$, numerorum primorum vero formae $8m+5$ vel $8m+7$ signa sint $+$. Hinc itaque erit

$$\frac{\pi}{2\sqrt{2}} = \frac{3}{2} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{16} \cdot \frac{19}{18} \cdot \frac{23}{24} \cdot \text{etc.},$$

ubi omnes denominatores vel divisibiles sunt per 8 vel tantum sunt numeri impariter pares. Cum igitur sit [§ 285]

$$\frac{\pi}{4} = \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{20} \cdot \frac{23}{24} \cdot \text{etc.},$$

$$\frac{\pi}{2} = \frac{3}{2} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{18} \cdot \frac{19}{18} \cdot \frac{23}{22} \cdot \text{etc.},$$

ergo

$$\frac{\pi\pi}{8} = \frac{3\cdot3}{2\cdot4} \cdot \frac{5\cdot5}{4\cdot6} \cdot \frac{7\cdot7}{6\cdot8} \cdot \frac{11\cdot11}{10\cdot12} \cdot \frac{13\cdot13}{12\cdot14} \cdot \text{etc.},$$

erit

$$\frac{\pi}{2\sqrt{2}} = \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{18} \cdot \frac{19}{20} \cdot \frac{23}{22} \cdot \text{etc.},$$

ubi nulli denominatores per 8 divisibiles occurrunt, pariter pares vero adsunt, quoties unitate differunt a numeratoribus. Prima vero per ultimam divisa dat

$$1 = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{9}{8} \cdot \frac{10}{9} \cdot \frac{11}{12} \cdot \text{etc.},$$

quae fractiones formantur ex numeris primis singulos in duas partes unitate discrepantes dispescendo et partes pares (nisi sint pariter pares) pro numeratoribus sumendo.

296. Simili modo reliquae series, quas supra pro expressione arcuum circularium invenimus (§ 179 et sq.) in factores transformari possunt, qui ex numeris primis constituantur. Sicque multae aliae insignes proprietates tam huiusmodi factorum quam serierum infinitarum erui poterunt. Quoniam vero praecipuas hic iam commemoravi, pluribus evolvendis hic non immorabor. Sed ad aliud huic affine argumentum procedam. Quemadmodum scilicet in hoc capite numeri, quatenus per multiplicationem oriuntur, sunt considerati, ita in sequenti generatio numerorum per additionem perpendetur.

CAPUT XVI

DE PARTITIONE NUMERORUM[1]

297. Proposita sit ista expressio

$$(1+x^{\alpha}z)(1+x^{\beta}z)(1+x^{\gamma}z)(1+x^{\delta}z)(1+x^{\varepsilon}z) \text{ etc.};$$

quae cuiusmodi induat formam, si per multiplicationem evolvatur, inquiramus. Ponamus prodire

$$1+Pz+Qz^2+Rz^3+Sz^4+\text{etc.}$$

atque manifestum est P fore summam potestatum

$$x^{\alpha}+x^{\beta}+x^{\gamma}+x^{\delta}+x^{\varepsilon}+\text{etc.}$$

1) Confer hoc cum capite L. EULERI Commentationes 158, 191, 394 (indicis ENESTROEMIANI): *Observationes analyticae variae de combinationibus*, Comment. acad. sc. Petrop. **13** (1741/3), 1751, p. 64, *De partitione numerorum*, Novi comment. acad. sc. Petrop. **3** (1750/1), 1753, p. 125, *De partitione numerorum in partes tam numero quam specie datas*, Novi comment. acad. sc. Petrop. **14** (1769): I, 1770, p. 168; *LEONHARDI EULERI Opera omnia*, series I, vol. 2, p. 163 et 254 (vide etiam Prooemium huius voluminis p. XVIII—XX), vol. 3, p. 131 (vide etiam Prooemium huius voluminis p. XIX—XX).

Vide porro epistolam a PH. NAUDÉ minore (1684—1745) ad EULERUM datam a. d. IV. Calendas Septembres 1740; *LEONHARDI EULERI Opera omnia*, series III. Qua epistola NAUDÉ haec duo problemata EULERO proposuerat:

Invenire, quot variis modis datus numerus produci queat ex additione aliquot numerorum integrorum inter se inaequalium, quorum numerus detur.

Invenire, quot variis modis datus numerus m partiri possit in μ partes tam aequales quam inaequales, sive invenire, quot variis modis datus numerus m per additionem μ numerorum integrorum, sive aequalium sive inaequalium, produci queat.

Resolutionem horum problematum EULERUS primum dedit in Commentatione 158 supra laudata. F. R.

Deinde Q est summa factorum ex binis potestatibus diversis seu Q erit aggregatum plurium potestatum ipsius x, quarum exponentes sunt summae duorum terminorum diversorum huius seriei

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta,\ \eta \text{ etc.}$$

Simili modo R erit aggregatum potestatum ipsius x, quarum exponentes sunt summae trium terminorum diversorum. Atque S erit aggregatum potestatum ipsius x, quarum exponentes sunt summae quatuor terminorum diversorum eiusdem seriei α, β, γ, δ, ε etc., et ita porro.

298. Singulae hae potestates ipsius x, quae in valoribus litterarum P, Q, R, S etc. insunt, unitatem pro coefficiente habebunt, siquidem earum exponentes unico modo ex α, β, γ, δ etc. formari queant; sin autem eiusdem potestatis exponens pluribus modis possit esse summa duorum, trium pluriumve terminorum seriei α, β, γ, δ, ε etc., tum etiam potestas illa coefficientem habebit, qui unitatem toties in se complectatur. Sic si in valore ipsius Q reperiatur Nx^n, indicio hoc erit numerum n esse N diversis modis summam duorum terminorum diversorum seriei α, β, γ, δ etc. Atque si in evolutione factorum propositorum occurrat terminus $Nx^n z^m$, eius coefficiens N indicabit, quot variis modis numerus n possit esse summa m terminorum diversorum seriei α, β, γ, δ, ε, ζ etc.

299. Quodsi ergo productum propositum

$$(1+x^{\alpha}z)(1+x^{\beta}z)(1+x^{\gamma}z)(1+x^{\delta}z) \text{ etc.}$$

per multiplicationem veram evolvatur, ex expressione resultante statim apparebit, quot variis modis datus numerus possit esse summa tot terminorum diversorum seriei

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta \text{ etc.},$$

quot quis voluerit. Scilicet, si quaeratur, quot variis modis numerus n possit esse summa m terminorum illius seriei diversorum, in expressione evoluta quaeri debet terminus $x^n z^m$ eiusque coefficiens indicabit numerum quaesitum.

300. Quo haec fiant planiora, sit propositum hoc productum ex factoribus constans infinitis

$$(1+xz)(1+x^2z)(1+x^3z)(1+x^4z)(1+x^5z) \text{ etc.},$$

quod per multiplicationem actualem evolutum dat

$$\begin{aligned}
1 &+ z\,(x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + \text{etc.})\\
&+ z^2(x^3 + x^4 + 2x^5 + 2x^6 + 3x^7 + 3x^8 + 4x^9 + 4x^{10} + 5x^{11} + \text{etc.})\\
&+ z^3(x^6 + x^7 + 2x^8 + 3x^9 + 4x^{10} + 5x^{11} + 7x^{12} + 8x^{13} + 10x^{14} + \text{etc.})\\
&+ z^4(x^{10} + x^{11} + 2x^{12} + 3x^{13} + 5x^{14} + 6x^{15} + 9x^{16} + 11x^{17} + 15x^{18} + \text{etc.})\\
&+ z^5(x^{15} + x^{16} + 2x^{17} + 3x^{18} + 5x^{19} + 7x^{20} + 10x^{21} + 13x^{22} + 18x^{23} + \text{etc.})\\
&+ z^6(x^{21} + x^{22} + 2x^{23} + 3x^{24} + 5x^{25} + 7x^{26} + 11x^{27} + 14x^{28} + 20x^{29} + \text{etc.})\\
&+ z^7(x^{28} + x^{29} + 2x^{30} + 3x^{31} + 5x^{32} + 7x^{33} + 11x^{34} + 15x^{35} + 21x^{36} + \text{etc.})\\
&+ z^8(x^{36} + x^{37} + 2x^{38} + 3x^{39} + 5x^{40} + 7x^{41} + 11x^{42} + 15x^{43} + 22x^{44} + \text{etc.})\\
&\qquad\text{etc.}
\end{aligned}$$

Ex his ergo seriebus statim definire licet, quot variis modis propositus numerus ex dato terminorum diversorum huius seriei

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8 \text{ etc.}$$

numero oriri queat. Sic si quaeratur, quot variis modis numerus 35 possit esse summa septem terminorum diversorum seriei 1, 2, 3, 4, 5, 6, 7, 8 etc., quaeratur in serie z^7 multiplicante potestas x^{35} eiusque coefficiens 15 indicabit numerum propositum 35 quindecim variis modis esse summam septem terminorum seriei 1, 2, 3, 4, 5, 6, 7, 8 etc.

301. Quodsi autem ponatur $z=1$ et similes potestates ipsius x in unam summam coniiciantur seu, quod eodem redit, si evolvatur haec expressio infinita

$$(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6) \text{ etc.},$$

quo facto orietur haec series

$$1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + 6x^8 + \text{etc.},$$

ubi quivis coefficiens indicat, quot variis modis exponens potestatis ipsius x coniunctae ex terminis diversis seriei 1, 2, 3, 4, 5, 6, 7 etc. per additionem emergere possit. Sic apparet numerum 8 sex modis per additionem diversorum numerorum produci, qui sunt

$8 = 8$	$8 = 5 + 3$
$8 = 7 + 1$	$8 = 5 + 2 + 1$
$8 = 6 + 2$	$8 = 4 + 3 + 1$

ubi notandum est numerum propositum ipsum simul computari debere, quia numerus terminorum non definitur ideoque unitas inde non excluditur.

302. Hinc igitur intelligitur, quomodo quisque numerus per additionem diversorum numerorum producatur. Conditio autem diversitatis omittetur, si factores illos in denominatorem transponamus. Sit igitur proposita haec expressio

$$\frac{1}{(1-x^{\alpha}z)(1-x^{\beta}z)(1-x^{\gamma}z)(1-x^{\delta}z)(1-x^{\varepsilon}z)\text{ etc.}},$$

quae per divisionem evoluta det

$$1 + Pz + Qz^2 + Rz^3 + Sz^4 + \text{etc.}$$

Atque manifestum est fore P aggregatum potestatum ipsius x, quarum exponentes contineantur in hac serie

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta,\ \eta\ \text{etc.}$$

Deinde Q erit aggregatum potestatum ipsius x, quarum exponentes sint summae duorum terminorum huius seriei, sive eorundem sive diversorum. Tum erit R summa potestatum ipsius x, quarum exponentes ex additione trium terminorum illius seriei oriantur, et S summa potestatum, quarum exponentes ex additione quatuor terminorum in illa serie contentorum formantur, et ita porro.

303. Si igitur tota expressio per singulos terminos explicetur et termini similes coniunctim exprimantur, intelligetur, quot variis modis propositus numerus n per additionem m terminorum, sive diversorum sive non diversorum, seriei

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta,\ \eta\ \text{etc.}$$

produci queat. Quaeratur scilicet in expressione evoluta terminus $x^n z^m$ eiusque coefficiens, qui sit N, ita ut totus terminus sit $= Nx^n z^m$, atque coefficiens N indicabit, quot variis modis numerus n per additionem m terminorum in serie $\alpha, \beta, \gamma, \delta, \varepsilon$ etc. contentorum produci queat. Hoc igitur pacto quaestio priori, quam ante sumus contemplati, similis resolvetur.

304. Accommodemus haec ad casum inprimis notatu dignum sitque proposita haec expressio

$$\frac{1}{(1-xz)(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)\text{ etc.}},$$

quae per divisionem evoluta dabit

$$\begin{aligned}
1 &+ z\,(x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + \text{etc.})\\
&+ z^2(x^2 + x^3 + 2x^4 + 2x^5 + 3x^6 + 3x^7 + 4x^8 + 4x^9 + 5x^{10} + \text{etc.})\\
&+ z^3(x^3 + x^4 + 2x^5 + 3x^6 + 4x^7 + 5x^8 + 7x^9 + 8x^{10} + 10x^{11} + \text{etc.})\\
&+ z^4(x^4 + x^5 + 2x^6 + 3x^7 + 5x^8 + 6x^9 + 9x^{10} + 11x^{11} + 15x^{12} + \text{etc.})\\
&+ z^5(x^5 + x^6 + 2x^7 + 3x^8 + 5x^9 + 7x^{10} + 10x^{11} + 13x^{12} + 18x^{13} + \text{etc.})\\
&+ z^6(x^6 + x^7 + 2x^8 + 3x^9 + 5x^{10} + 7x^{11} + 11x^{12} + 14x^{13} + 20x^{14} + \text{etc.})\\
&+ z^7(x^7 + x^8 + 2x^9 + 3x^{10} + 5x^{11} + 7x^{12} + 11x^{13} + 15x^{14} + 21x^{15} + \text{etc.})\\
&+ z^8(x^8 + x^9 + 2x^{10} + 3x^{11} + 5x^{12} + 7x^{13} + 11x^{14} + 15x^{15} + 22x^{16} + \text{etc.})\\
&\qquad\qquad\text{etc.}
\end{aligned}$$

Ex his ergo seriebus statim definire licet, quot variis modis propositus numerus per additionem ex dato terminorum huius seriei

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7\ \text{etc.}$$

numero produci queat. Sic si quaeratur, quot variis modis numerus 13 oriri possit per additionem quinque numerorum integrorum, spectari debebit terminus $x^{13}z^5$, cuius coefficiens 18 indicat numerum propositum 13 ex quinque numerorum additione octodecim modis oriri posse.

305. Si ponatur $z=1$ atque similes potestates ipsius x coniunctim exprimantur, haec expressio

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)\text{ etc.}}$$

evolvetur in hanc seriem

$$1+x+2x^2+3x^3+5x^4+7x^5+11x^6+15x^7+22x^8+\text{etc.};$$

in qua quilibet coefficiens indicat, quot variis modis exponens potestatis adiunctae per additionem produci queat ex numeris integris, sive aequalibus sive inaequalibus. Scilicet ex termino $11x^6$ cognoscitur numerum 6 undecim modis per additionem numerorum integrorum produci posse, qui sunt

$6=6$	$6=3+1+1+1$
$6=5+1$	$6=2+2+2$
$6=4+2$	$6=2+2+1+1$
$6=4+1+1$	$6=2+1+1+1+1$
$6=3+3$	$6=1+1+1+1+1+1$
$6=3+2+1$	

ubi quoque notari debet ipsum numerum propositum, cum in serie numerorum 1, 2, 3, 4, 5, 6 etc. proposita contineatur, unum modum praebere.

306. His in genere expositis diligentius inquiramus in modum hanc compositionum multitudinem inveniendi. Ac primo quidem consideremus eam ex numeris integris compositionem, in qua numeri tantum diversi admittuntur, quam prius commemoravimus. Sit igitur in hunc finem proposita haec expressio

$$Z=(1+xz)(1+x^2z)(1+x^3z)(1+x^4z)(1+x^5z)\text{ etc.},$$

quae evoluta et secundum potestates ipsius z digesta praebeat

$$Z=1+Pz+Qz^2+Rz^3+Sz^4+Tz^5+\text{etc.},$$

ubi methodus desideratur has ipsius x functiones P, Q, R, S, T etc. expedite inveniendi; hoc enim pacto quaestioni propositae convenientissime satisfiet.

307. Patet autem, si loco z ponatur xz, prodire

$$(1+x^2z)(1+x^3z)(1+x^4z)(1+x^5z)\text{ etc.}=\frac{Z}{1+xz}.$$

Ergo posito xz loco z valor producti, qui erat Z, abibit in $\frac{Z}{1+xz}$; sicque, cum sit

$$Z=1+Pz+Qz^2+Rz^3+Sz^4+\text{etc.},$$

erit

$$\frac{Z}{1+xz}=1+Pxz+Qx^2z^2+Rx^3z^3+Sx^4z^4+\text{etc.}$$

Multiplicetur ergo actu per $1+xz$ atque prodibit

$$Z=1+Pxz+Qx^2z^2+Rx^3z^3+Sx^4z^4+\text{etc.}$$
$$\quad +xz+Px^2z^2+Qx^3z^3+Rx^4z^4+\text{etc.},$$

qui valor ipsius Z cum superiori comparatus dabit

$$P=\frac{x}{1-x},\quad Q=\frac{Px^2}{1-x^2},\quad R=\frac{Qx^3}{1-x^3},\quad S=\frac{Rx^4}{1-x^4}\quad \text{etc.}$$

Sequentes ergo pro P, Q, R, S etc. obtinentur valores:

$$P=\frac{x}{1-x},$$
$$Q=\frac{x^3}{(1-x)(1-x^2)},$$
$$R=\frac{x^6}{(1-x)(1-x^2)(1-x^3)},$$
$$S=\frac{x^{10}}{(1-x)(1-x^2)(1-x^3)(1-x^4)},$$
$$T=\frac{x^{15}}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)}$$
etc.

308. Sic igitur seorsim unamquamque seriem potestatum ipsius x exhibere possumus, ex qua definire licet, quot variis modis propositus numerus ex dato partium integrarum numero per additionem formari possit. Manifestum autem

porro est has singulas series esse recurrentes, quia ex evolutione functionis fractae ipsius x nascuntur. Prima scilicet expressio

$$P = \frac{x}{1-x}$$

dat seriem geometricam

$$x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \text{etc.},$$

ex qua quidem manifestum est quemvis numerum semel in serie numerorum integrorum contineri.

309. Expressio secunda

$$\frac{x^3}{(1-x)(1-xx)}$$

dat hanc seriem

$$x^3 + x^4 + 2x^5 + 2x^6 + 3x^7 + 3x^8 + 4x^9 + 4x^{10} + \text{etc.},$$

in qua cuiusvis termini coefficiens indicat, quot modis exponens ipsius x in duas partes inaequales dispertiri possit. Sic terminus $4x^9$ indicat numerum 9 quatuor modis in duas partes inaequales secari posse. Quodsi hanc seriem per x^3 dividamus, prodibit series, quam praebet ista fractio

$$\frac{1}{(1-x)(1-x^2)},$$

quae erit

$$1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + 4x^6 + 4x^7 + \text{etc.},$$

cuius terminus generalis sit $= Nx^n$; atque ex genesi huius seriei intelligitur coefficientem N indicare, quot variis modis exponens n ex numeris 1 et 2 per additionem nasci queat. Cum igitur prioris seriei terminus generalis sit $= Nx^{n+3}$, deducitur hinc istud theorema:

Quot variis modis numerus n per additionem ex numeris 1 *et* 2 *produci potest, totidem variis modis numerus $n+3$ in duas partes inaequales secari poterit.*

310. Expressio tertia

$$\frac{x^6}{(1-x)(1-x^2)(1-x^3)}$$

in seriem evoluta dabit

$$x^6 + x^7 + 2x^8 + 3x^9 + 4x^{10} + 5x^{11} + 7x^{12} + 8x^{13} + \text{etc.},$$

in qua cuiusvis termini coefficiens indicat, quot variis modis exponens potestatis x adiunctae in tres partes inaequales dispertiri possit. Quodsi autem haec fractio

$$\frac{1}{(1-x)(1-x^2)(1-x^3)}$$

evolvatur, prodibit haec series

$$1 + x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 7x^6 + 8x^7 + \text{etc.};$$

cuius terminus generalis si ponatur $= Nx^n$, coefficiens N indicabit, quot variis modis numerus n ex numeris 1, 2, 3 per additionem produci possit. Cum igitur prioris seriei terminus generalis sit Nx^{n+6}, sequetur hinc istud theorema:

Quot variis modis numerus n per additionem ex numeris 1, 2, 3 *produci potest, totidem variis modis numerus $n+6$ in tres partes inaequales secari poterit.*

311. Expressio quarta

$$\frac{x^{10}}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$$

in seriem recurrentem evoluta dabit

$$x^{10} + x^{11} + 2x^{12} + 3x^{13} + 5x^{14} + 6x^{15} + 9x^{16} + 11x^{17} + \text{etc.},$$

in qua cuiusvis termini coefficiens indicabit, quot variis modis exponens potestatis x adiunctae in quatuor partes inaequales dispertiri possit. Quodsi autem haec expressio

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$$

evolvatur, prodibit superior series per x^{10} divisa, nempe

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 6x^5 + 9x^6 + 11x^7 + \text{etc.},$$

cuius terminum generalem ponamus $= Nx^n$; atque hinc patebit coefficientem N indicare, quot variis modis numerus n per additionem oriri possit ex his quatuor numeris 1, 2, 3, 4. Cum igitur prioris seriei terminus generalis futurus sit $= Nx^{n+10}$, deducitur hoc theorema:

Quot variis modis numerus n per additionem produci potest ex numeris 1, 2, 3, 4, totidem variis modis numerus $n+10$ in quatuor partes inaequales secari poterit.

312. Generaliter ergo, si haec expressio

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^m)}$$

in seriem evolvatur eiusque terminus generalis fuerit

$$= Nx^n,$$

coefficiens N indicabit, quot variis modis numerus n per additionem produci possit ex his numeris 1, 2, 3, 4, ... m. Quodsi autem haec expressio

$$\frac{x^{\frac{m(m+1)}{2}}}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^m)}$$

in seriem evolvatur, erit eius terminus generalis

$$= Nx^{n+\frac{m(m+1)}{2}}$$

atque hic coefficiens N indicat, quot variis modis numerus $n+\frac{m(m+1)}{1\cdot 2}$ in m partes inaequales secari possit, unde hoc habetur theorema:

Quot variis modis numerus n per additionem produci potest ex numeris 1, 2, 3, 4, ... m, totidem modis numerus $n+\frac{m(m+1)}{1\cdot 2}$ in m partes inaequales secari poterit.

313. Ex posita partitione numerorum in partes inaequales perpendamus quoque partitionem in partes, ubi aequalitas partium non excluditur; quae partitio ex hac expressione originem habet

$$Z = \frac{1}{(1-xz)(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)\text{ etc.}}.$$

Ponamus evolutione per divisionem instituta prodire

$$Z = 1 + Pz + Qz^2 + Rz^3 + Sz^4 + Tz^5 + \text{etc.}$$

Perspicuum autem est, si loco z ponatur xz, prodire

$$\frac{1}{(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)\text{ etc.}} = (1-xz)Z.$$

Facta ergo in serie evoluta eadem mutatione fiet

$$(1-xz)Z = 1 + Pxz + Qx^2z^2 + Rx^3z^3 + Sx^4z^4 + \text{etc.}$$

Multiplicetur ergo superior series pariter per $(1-xz)$ eritque

$$\begin{aligned}(1-xz)Z = 1 + Pz + {}& Qz^2 + Rz^3 + Sz^4 + \text{etc.}\\ - xz - {}& Pxz^2 - Qxz^3 - Rxz^4 - \text{etc.}\end{aligned}$$

Comparatione ergo instituta orietur

$$P = \frac{x}{1-x},\quad Q = \frac{Px}{1-x^2},\quad R = \frac{Qx}{1-x^3},\quad S = \frac{Rx}{1-x^4}\quad \text{etc.},$$

unde pro P, Q, R, S etc. sequentes valores proveniunt:

$$P = \frac{x}{1-x},$$

$$Q = \frac{x^2}{(1-x)(1-x^2)},$$

$$R = \frac{x^3}{(1-x)(1-x^2)(1-x^3)},$$

$$S = \frac{x^4}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$$

etc.

314. Expressiones istae a superioribus aliter non discrepant, nisi quod numeratores hic minores habeant exponentes quam casu praecedente. Atque hanc ob rem series, quae per evolutionem nascuntur, ratione coefficientium omnino convenient, quae convenientia iam ex comparatione § 300 et 304

perspicitur, nunc vero demum eius ratio intelligitur. Hinc ergo omnino similia theoremata consequentur, quae sunt:

Quot variis modis numerus n per additionem produci potest ex numeris 1, 2, *totidem modis numerus* $n+2$ *in duas partes dispertiri poterit.*

Quot variis modis numerus n per additionem produci potest ex numeris 1, 2, 3, *totidem modis numerus* $n+3$ *in tres partes dispertiri poterit.*

Quot variis modis numerus n per additionem produci potest ex numeris 1, 2, 3, 4, *totidem modis numerus* $n+4$ *in quatuor partes dispertiri poterit.*

Atque generaliter habebitur hoc theorema:

Quot variis modis numerus n per additionem produci potest ex numeris 1, 2, 3, ... m, *totidem modis numerus* $n+m$ *in m partes dispertiri poterit.*

315. Sive ergo quaeratur, quot modis datus numerus in m partes inaequales, sive in m partes aequalibus non exclusis dispertiri possit, utraque quaestio resolvetur, si cognoscatur, quot modis quisque numerus per additionem produci possit ex numeris 1, 2, 3, 4, ... m, quemadmodum hoc patebit ex sequentibus theorematis, quae ex superioribus sunt derivata:

Numerus n tot modis in m partes inaequales dispertiri potest, quot modis numerus $n-\frac{m(m+1)}{2}$ *per additionem produci potest ex numeris* 1, 2, 3, 4, ... m.

Numerus n tot modis in m partes, sive aequales sive inaequales, dispertiri potest, quot modis numerus $n-m$ *per additionem produci potest ex numeris* 1, 2, 3, ... m.

Hinc porro sequuntur haec theoremata:

Numerus n totidem modis in m partes inaequales secari potest, quot modis numerus $n-\frac{m(m-1)}{2}$ *in m partes, sive aequales sive inaequales, dispertitur.*

Numerus n totidem modis in m partes, sive inaequales sive aequales, secari potest, quot modis numerus $n+\frac{m(m-1)}{2}$ *in m partes inaequales dispertiri potest.*

316. Per formationem autem serierum recurrentium inveniri poterit, quot variis modis datus numerus n per additionem produci possit ex numeris 1, 2, 3, ... m. Ad hoc enim inveniendum evolvi debebit fractio

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^m)}$$

atque series recurrens continuari debebit usque ad terminum Nx^n, cuius coefficiens N indicabit, quot modis numerus n per additionem produci possit ex numeris 1, 2, 3, 4, ... m. At vero hic solvendi modus non parum habebit difficultatis, si numeri m et n sint modice magni; scala enim relationis, quam praebet denominator per multiplicationem evolutus, ex pluribus terminis constat, unde operosum erit seriem ad plures terminos continuare.

317. Haec autem disquisitio minus erit molesta, si casus simpliciores primum expediantur; ex his enim facile erit ad casus magis compositos progredi. Sit seriei, quae ex hac fractione oritur

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^m)},$$

terminus generalis $= Nx^n$; at seriei ex hac forma

$$\frac{x^m}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^m)}$$

ortae terminus generalis sit Mx^n, ubi coefficiens M indicabit, quot variis modis numerus $n-m$ per additionem produci possit ex numeris 1, 2, 3, ... m. Subtrahatur posterior expressio a priori ac remanebit

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^{m-1})}$$

atque manifestum est seriei hinc ortae terminum generalem futurum esse $(N-M)x^n$; quare coefficiens $N-M$ indicabit, quot variis modis numerus n per additionem produci possit ex numeris 1, 2, 3, ... $m-1$.

318. Hinc ergo sequentem regulam nanciscimur:

Sit L numerus modorum, quibus numerus n per additionem produci potest ex numeris 1, 2, 3, ... $m-1$,

sit M numerus modorum, quibus numerus $n-m$ *per additionem produci potest ex numeris* 1, 2, 3, ... m,

sitque N numerus modorum, quibus numerus n per additionem produci potest ex numeris 1, 2, 3, ... m;

his positis erit, ut vidimus,

$$L = N - M$$

ideoque

$$N = L + M.$$

Quodsi ergo iam invenerimus, quot variis modis numeri n et $n - m$ per additionem produci queant, ille ex numeris 1, 2, 3, ... $m - 1$, hic vero ex numeris 1, 2, 3, ... m, hinc addendo cognoscemus, quot variis modis numerus n per additionem produci queat ex numeris 1, 2, 3, ... m. Ope huius theorematis a casibus simplicioribus, qui nihil habent difficultatis, continuo ad magis compositos progredi licebit hocque modo tabula hic annexa[1]) est computata, cuius usus ita se habet:

Si quaeratur, quot variis modis numerus 50 in 7 partes inaequales dispertiri possit, sumatur in prima columna verticali numerus $50 - \frac{7 \cdot 8}{2} = 22$, in horizontali autem suprema numerus romanus VII; atque numerus in angulo positus 522 indicabit modorum numerum quaesitum.

Sin autem quaeratur, quot variis modis numerus 50 in 7 partes, sive aequales sive inaequales, dispertiri possit, in prima columna verticali sumatur numerus $50 - 7 = 43$, cui in columna septima respondebit numerus quaesitus 8946.

1) Confer tabulam correspondentem, quae continetur in Commentatione 191 nota 1 pag. 313 laudata. Confer imprimis paragraphum 33 huius Commentationis 191, ubi expositum est, quomodo haec tabula per solam continuam additionem ratione satis perspicua construi possit. F. R.

TABULA

n	I	II	III	IV	V	VI	VII	VIII	IX	X	XI
1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2	2	2
3	1	2	3	3	3	3	3	3	3	3	3
4	1	3	4	5	5	5	5	5	5	5	5
5	1	3	5	6	7	7	7	7	7	7	7
6	1	4	7	9	10	11	11	11	11	11	11
7	1	4	8	11	13	14	15	15	15	15	15
8	1	5	10	15	18	20	21	22	22	22	22
9	1	5	12	18	23	26	28	29	30	30	30
10	1	6	14	23	30	35	38	40	41	42	42
11	1	6	16	27	37	44	49	52	54	55	56
12	1	7	19	34	47	58	65	70	73	75	76
13	1	7	21	39	57	71	82	89	94	97	99
14	1	8	24	47	70	90	105	116	123	128	131
15	1	8	27	54	84	110	131	146	157	164	169
16	1	9	30	64	101	136	164	186	201	212	219
17	1	9	33	72	119	163	201	230	252	267	278
18	1	10	37	84	141	199	248	288	318	340	355
19	1	10	40	94	164	235	300	352	393	423	445
20	1	11	44	108	192	282	364	434	488	530	560
21	1	11	48	120	221	331	436	525	598	653	695
22	1	12	52	136	255	391	522	638	732	807	863
23	1	12	56	150	291	454	618	764	887	984	1060
24	1	13	61	169	333	532	733	919	1076	1204	1303
25	1	13	65	185	377	612	860	1090	1291	1455	1586
26	1	14	70	206	427	709	1009	1297	1549	1761	1930
27	1	14	75	225	480	811	1175	1527	1845	2112	2331
28	1	15	80	249	540	931	1367	1801	2194	2534	2812
29	1	15	85	270	603	1057	1579	2104	2592	3015	3370
30	1	16	91	297	674	1206	1824	2462	3060	3590	4035

n	I	II	III	IV	V	VI	VII	VIII	IX	X	XI
31	1	16	96	321	748	1360	2093	2857	3589	4242	4802
32	1	17	102	351	831	1540	2400	3319	4206	5013	5708[1]
33	1	17	108	378	918	1729	2738	3828	4904	5888	6751
34	1	18	114	411	1014	1945	3120	4417	5708	6912	7972
35	1	18	120	441	1115	2172	3539	5066	6615	8070	9373
36	1	19	127	478	1226	2432	4011	5812	7657	9418	11004
37	1	19	133	511	1342	2702	4526	6630	8824	10936	12866
38	1	20	140	551	1469	3009	5102	7564	10156	12690	15021
39	1	20	147	588	1602	3331	5731	8588	11648	14663	17475
40	1	21	154	632	1747	3692	6430	9749	13338	16928	20298
41	1	21	161	672	1898	4070	7190	11018	15224	19466	23501
42	1	22	169	720	2062	4494	8033	12450	17354	22367	27169
43	1	22	176	764	2233	4935	8946	14012	19720	25608	31316
44	1	23	184	816	2418	5427	9953	15765	22380	29292	36043
45	1	23	192	864	2611	5942	11044	17674	25331	33401	41373
46	1	24	200	920	2818	6510	12241	19805	28629	38047	47420
47	1	24	208	972	3034	7104	13534	22122	32278	43214	54218
48	1	25	217	1033	3266	7760	14950	24699	36347	49037	61903
49	1	25	225	1089	3507	8442	16475	27493	40831	55494	70515
50	1	26	234	1154	3765	9192	18138	30588	45812	62740	80215
51	1	26	243	1215	4033	9975	19928	33940	51294	70760	91058
52	1	27	252	1285	4319	10829	21873	37638	57358	79725	103226
53	1	27	261	1350	4616	11720	23961	41635	64015	89623	116792
54	1	28	271	1425	4932	12692	26226	46031	71362	100654	131970
55	1	28	280	1495	5260	13702	28652	50774	79403	112804	148847
56	1	29	290	1575	5608	14800	31275	55974	88252	126299	167672
57	1	29	300	1650	5969	15944	34082	61575	97922	141136	188556
58	1	30	310	1735	6351	17180	37108	67696	108527	157564	211782
59	1	30	320	1815	6747	18467	40340	74280	120092	175586	237489
60	1	31	331	1906	7166	19858	43819	81457	132751	195491	266006
61	1	31	341	1991	7599	21301	47527	89162	146520	217280	297495
62	1	32	352	2087	8056	22856	51508	97539	161554	241279	332337
63	1	32	363	2178	8529	24473	55748	106522	177884	267507	370733
64	1	33	374	2280	9027	26207	60289	116263	195666	296320	413112
65	1	33	385	2376	9542	28009	65117	126692	214944	327748	459718
66	1	34	397	2484	10083	29941	70281	137977	235899	362198	511045
67	1	34	408	2586	10642	31943	75762	150042	258569	399705	567377
68	1	35	420	2700	11229	34085	81612	163069	283161	440725	629281
69	1	35	432	2808	11835	36308	87816	176978	309729	485315	697097

1) Editio princeps: 5788. Correxit F. R.

319. Series huius tabulae verticales, etsi sunt recurrentes, tamen ingentem habent connexionem cum numeris naturalibus, trigonalibus, pyramidalibus et sequentibus, quam paucis exponere operae pretium erit. Quoniam enim ex fractione

$$\frac{1}{(1-x)(1-xx)}$$

oritur series

$$1+x+2x^2+2x^3+3x^4+3x^5+\text{etc.}$$

ac proinde ex fractione

$$\frac{x}{(1-x)(1-xx)}$$

haec

$$x+x^2+2x^3+2x^4+3x^5+3x^6+\text{etc.},$$

si duae hae series addantur, nascitur ista

$$1+2x+3x^2+4x^3+5x^4+6x^5+7x^6+\text{etc.},$$

quae per divisionem oritur ex fractione

$$\frac{1+x}{(1-x)(1-xx)}=\frac{1}{(1-x)^2};$$

unde patet seriei postremae terminos numericos seriem numerorum naturalium constituere. Hinc ex serie tabulae secunda addendo binos terminos proveniet series numerorum naturalium posito $x=1$:

$$1+1+2+2+3+3+4+4+5+\ 5+\ 6+\ 6+\text{etc.}$$
$$1+2+3+4+5+6+7+8+9+10+11+12+\text{etc.}$$

Vicissim ergo ex serie numerorum naturalium superior invenitur subtrahendo quemque terminum seriei superioris a termino inferioris sequente.

320. Series verticalis tertia oritur ex fractione

$$\frac{1}{(1-x)(1-xx)(1-x^3)}.$$

Cum autem sit

$$\frac{1}{(1-x)^3}=\frac{(1+x)(1+x+xx)}{(1-x)(1-xx)(1-x^3)},$$

manifestum est, si primo seriei illius terni termini addantur, tum bini huius novae seriei denuo addantur, prodire debere numeros trigonales; id quod ex schemate sequente apparebit:

$$1+1+2+3+4+5+7+8+10+12+14+16+19+\text{etc.}$$
$$1+2+4+6+9+12+16+20+25+30+36+42+49+\text{etc.}$$
$$1+3+6+10+15+21+28+36+45+55+66+78+91+\text{etc.}$$

Vicissim autem apparet, quomodo ex serie trigonalium erui debeat series superior.

321. Simili modo, quia series quarta oritur ex fractione

$$\frac{1}{(1-x)(1-xx)(1-x^3)(1-x^4)},$$

erit

$$\frac{(1+x)(1+x+xx)(1+x+xx+x^3)}{(1-x)(1-xx)(1-x^3)(1-x^4)}=\frac{1}{(1-x)^4}.$$

Si in serie quarta primum quaterni termini addantur, tum in serie resultante terni, denique in hac bini, prodibit series numerorum pyramidalium, uti ex sequenti calculo videre licet:

$$1+1+2+3+5+6+9+11+15+18+23+27+\text{etc.}$$
$$1+2+4+7+11+16+23+31+41+53+67+83+\text{etc.}$$
$$1+3+7+13+22+34+50+70+95+125+161+203+\text{etc.}$$
$$1+4+10+20+35+56+84+120+165+220+286+364+\text{etc.}$$

Simili autem modo series quinta deducet ad numeros pyramidales secundi ordinis, sexta ad tertii ordinis, et ita porro.

322. Vicissim igitur ex numeris figuratis illae ipsae series, quae in tabula occurrunt, formari poterunt per operationes, quae ex inspectione calculi sequentis sponte elucebunt.

$$1+2+3+4+5+6+7+8+9+10+\text{etc.}$$
$$1+1+2+2+3+3+4+4+5+5+\text{etc.} \qquad \text{II}$$

$$1+3+6+10+15+21+28+36+45+55+\text{etc.}$$
$$1+2+4+6+9+12+16+20+25+30+\text{etc.}$$
$$1+1+2+3+4+5+7+8+10+12+\text{etc.} \qquad \text{III}$$

$$1+4+10+20+35+56+84+120+165+220+\text{etc.}$$
$$1+3+7+13+22+34+50+70+95+125+\text{etc.}$$
$$1+2+4+7+11+16+23+31+41+53+\text{etc.}$$
$$1+1+2+3+5+6+9+11+15+18+\text{etc.} \qquad \text{IV}$$

$$1+5+15+35+70+126+210+330+495+715+\text{etc.}$$
$$1+4+11+24+46+80+130+200+295+420+\text{etc.}$$
$$1+3+7+14+25+41+64+95+136+189+\text{etc.}$$
$$1+2+4+7+12+18+27+38+53+71+\text{etc.}$$
$$1+1+2+3+5+7+10+13+18+23+\text{etc.} \qquad \text{V}$$

etc.

In his ordinibus primae series sunt numeri figurati, unde subtrahendo quemvis terminum seriei secundae a termino primae sequente formatur series secunda. Tum seriei tertiae bini termini coniunctim subtrahantur a termino sequente seriei secundae sicque oritur series tertia. Hocque modo subtrahendo ulterius summam trium, quatuor et ita porro terminorum a termino superioris seriei sequente formabuntur reliquae series, donec perveniatur ad seriem, quae incipit ab $1+1+2+$ etc., haecque erit series in tabula exhibita.

323. Series verticales tabulae omnes similiter incipiunt continuoque plures habent terminos communes; ex quo intelligitur in infinitum has series inter se fore congruentes. Prodibit autem series, quae oritur ex hac fractione

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)\text{ etc.}};$$

quae cum sit recurrens, primum denominator spectari debet, ut hinc scala relationis habeatur. Quodsi autem factores denominatoris continuo in se multiplicentur, prodibit

$$1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + x^{51} + \text{etc.};$$

quae series si attentius consideretur, aliae potestates ipsius x adesse non deprehenduntur, nisi quarum exponentes contineantur in hac formula $\frac{3nn \pm n}{2}$, atque si n sit numerus impar, potestates erunt negativae, affirmativae autem, si n fuerit numerus par.[1])

324. Cum igitur scala relationis sit

$$+1, +1, 0, 0, -1, 0, -1, 0, 0, 0, 0, +1, 0, 0, +1, 0, 0 \text{ etc.},$$

series recurrens ex evolutione fractionis

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7) \text{ etc.}}$$

oriunda erit haec

1) Haec celebris series apud EULERUM primum in Commentatione 158 (exhib. 6. Apr. 1741) nota 1 p. 313 laudata invenitur. Evolutionis demonstrationem EULERUS dedit in Commentatione 244 (indicis ENESTROEMIANI): *Demonstratio theorematis circa ordinem in summis divisorum observatum*, Novi comment. acad. sc. Petrop. **5** (1754/5), 1760, p. 75; *LEONHARDI EULERI Opera omnia*, series I, vol. 2, p. 390.

Series illa eo magis digna est, quae consideretur, quod iam exemplum praebet illarum functionum, quas centum fere abhinc annos C. G. J. JACOBI ut fundamenta theoriae functionum ellipticarum in analysin introduxit et hoc charactere ϑ significavit; confer C. G. J. JACOBI, *Elementarer Beweis einer merkwürdigen analytischen Formel etc.*, Journal f. d. reine u. angew. Mathem. **21**, 1840, p. 13; *Ges. Werke* **6**, 1891, p. 281.

Formula EULERIANA

$$(1-x)(1-x^2)(1-x^3)\cdots = 1 - x - x^2 + x^5 + x^7 - x^{12} - \cdots$$

invenitur etiam apud JACOBI in libro, qui inscribitur *Fundamenta nova theoriae functionum ellipticarum*, Regiomonti 1829, § 66, formula (6), *Ges. Werke* **1**, 1881, p. 237.

Notandum autem est series, quarum exponentes seriem arithmeticam secundi ordinis formant, iam sexaginta annis ante EULERUM inveniri apud JAC. BERNOULLI et G. LEIBNIZ; confer G. ENESTRÖM, *JAKOB BERNOULLI und die JACOBISCHEN Thetafunktionen*, Biblioth. Mathem. 9_3, 1908—1909, p. 206. De cetero vide notam adiectam ad paragraphum 36 illius Commentationis 158; *LEONHARDI EULERI Opera omnia*, series I, vol. 2, p. 191. F. R.

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + 30x^9 + 42x^{10} + 56x^{11}$$
$$+ 77x^{12} + 101x^{13} + 135x^{14} + 176x^{15} + 231x^{16} + 297x^{17} + 385x^{18} + 490x^{19} + 627x^{20}$$
$$+ 792x^{21} + 1002x^{22} + 1255x^{23} + 1575x^{24} + \text{etc.}$$ [1])

In hac ergo serie coefficiens quisque indicat, quot variis modis exponens ipsius x per additionem ex numeris integris oriri queat. Sic numerus 7 quindecim modis per additionem oriri potest:

$7 = 7$	$7 = 4 + 2 + 1$	$7 = 3 + 1 + 1 + 1 + 1$
$7 = 6 + 1$	$7 = 4 + 1 + 1 + 1$	$7 = 2 + 2 + 2 + 1$
$7 = 5 + 2$	$7 = 3 + 3 + 1$	$7 = 2 + 2 + 1 + 1 + 1$
$7 = 5 + 1 + 1$	$7 = 3 + 2 + 2$	$7 = 2 + 1 + 1 + 1 + 1 + 1$
$7 = 4 + 3$	$7 = 3 + 2 + 1 + 1$	$7 = 1 + 1 + 1 + 1 + 1 + 1 + 1$

325. Quodsi autem hoc productum

$$(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6) \text{ etc.}$$

evolvatur, sequens prodibit series

$$1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + 6x^8 + 8x^9 + 10x^{10} + \text{etc.},$$

in qua quisque coefficiens indicat, quot variis modis exponens ipsius x per additionem numerorum inaequalium oriri possit. Sic numerus 9 octo variis modis per additionem ex numeris inaequalibus formari potest:

$9 = 9$	$9 = 6 + 2 + 1$
$9 = 8 + 1$	$9 = 5 + 4$
$9 = 7 + 2$	$9 = 5 + 3 + 1$
$9 = 6 + 3$	$9 = 4 + 3 + 2$

1) Editio princeps: $1 + x + 2x^2 + \cdots + 1002x^{22} + 1250x^{23} + 1570x^{24}$ etc. Correxit F. R.

326. Ut comparationem inter has formas instituamus, sit

$$P = (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6) \text{ etc.}$$

et

$$Q = (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6) \text{ etc.};$$

erit

$$PQ = (1-x^2)(1-x^4)(1-x^6)(1-x^8)(1-x^{10})(1-x^{12}) \text{ etc.};$$

qui factores cum omnes in P contineantur, dividatur P per PQ; erit

$$\frac{1}{Q} = (1-x)(1-x^3)(1-x^5)(1-x^7)(1-x^9) \text{ etc.}$$

ideoque

$$Q = \frac{1}{(1-x)(1-x^3)(1-x^5)(1-x^7)(1-x^9) \text{ etc.}};$$

quae fractio si evolvatur, prodibit series, in qua quisque coefficiens indicabit, quot variis modis exponens ipsius x per additionem ex numeris imparibus produci possit. Cum igitur haec expressio aequalis sit illi, quam in paragrapho praecedente contemplati sumus, sequitur hinc istud theorema:

Quot modis datus numerus per additionem formari potest ex omnibus numeris integris inter se inaequalibus, totidem modis idem numerus formari poterit per additionem ex numeris tantum imparibus, sive aequalibus sive inaequalibus.

327. Cum igitur, ut ante vidimus, sit

$$P = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + \text{etc.},$$

erit scribendo xx loco x

$$PQ = 1 - x^2 - x^4 + x^{10} + x^{14} - x^{24} - x^{30} + x^{44} + x^{52} - \text{etc.}$$

Quocirca erit hanc per illam dividendo

$$Q = \frac{1 - x^2 - x^4 + x^{10} + x^{14} - x^{24} - x^{30} + \text{etc.}}{1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \text{etc.}};$$

erit ergo series Q pariter recurrens atque ex serie $\frac{1}{P}$ oritur hanc per

$$1 - x^2 - x^4 + x^{10} + x^{14} - x^{24} - \text{etc.}$$

multiplicando. Nempe, cum sit ex § 324

$$\frac{1}{P} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + 30x^9 + \text{etc.},$$

si is multiplicetur per

$$1 - x^2 - x^4 + x^{10} + x^{14} - \text{etc.},$$

fiet

$$\begin{aligned} 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + 30x^9 + \text{etc.} \\ - x^2 - x^3 - 2x^4 - 3x^5 - 5x^6 - 7x^7 - 11x^8 - 15x^9 - \text{etc.} \\ - x^4 - x^5 - 2x^6 - 3x^7 - 5x^8 - 7x^9 - \text{etc.} \end{aligned}$$

aut

$$1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + 6x^8 + 8x^9 + \text{etc.} = Q.$$

Hinc ergo, si formatio numerorum per additionem numerorum, sive aequalium sive inaequalium, constet, deducetur formatio numerorum per additionem numerorum inaequalium hincque porro formatio numerorum per additionem numerorum imparium tantum.

328. Restant in hoc genere casus quidam memorabiles, quorum evolutio non omni utilitate carebit in numerorum natura cognoscenda. Consideretur nempe haec expressio

$$(1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})(1+x^{32}) \text{ etc.},$$

in qua exponentes ipsius x in ratione dupla progrediuntur. Haec expressio si evolvatur, reperietur quidem haec series

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + \text{etc.},$$

quoniam vero dubium esse potest, utrum haec series in infinitum hac lege geometrica progrediatur, hanc ipsam seriem investigemus. Sit igitur

$$P = (1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16}) \text{ etc.}$$

ac ponatur series per evolutionem orienda

$$P = 1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \zeta x^6 + \eta x^7 + \theta x^8 + \text{etc.}$$

Patet autem, si loco x scribatur xx, tum prodire productum

$$(1+xx)(1+x^4)(1+x^8)(1+x^{16})(1+x^{32})\text{ etc.} = \frac{P}{1+x}.$$

Facta ergo in serie eadem substitutione erit

$$\frac{P}{1+x} = 1 + \alpha x^2 + \beta x^4 + \gamma x^6 + \delta x^8 + \varepsilon x^{10} + \zeta x^{12} + \text{etc.}$$

Multiplicetur ergo per $1+x$ eritque

$$P = 1 + x + \alpha x^2 + \alpha x^3 + \beta x^4 + \beta x^5 + \gamma x^6 + \gamma x^7 + \delta x^8 + \delta x^9 + \text{etc.};$$

qui valor ipsius P si cum superiori comparetur, habebitur

$$\alpha = 1,\ \ \beta = \alpha,\ \ \gamma = \alpha,\ \ \delta = \beta,\ \ \varepsilon = \beta,\ \ \zeta = \gamma,\ \ \eta = \gamma\ \ \text{etc.};$$

erunt ergo omnes coefficientes $= 1$ ideoque productum propositum P evolutum dabit seriem geometricam

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + \text{etc.}$$

329. Cum igitur hic omnes ipsius x potestates singulaeque semel occurrant, ex forma producti

$$(1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})(1+x^{32})\text{ etc.}$$

sequitur omnem numerum integrum ex terminis progressionis geometricae duplae

$$1,\ 2,\ 4,\ 8,\ 16,\ 32\ \text{etc.}$$

diversis per additionem formari posse hocque unico modo.

Nota est haec proprietas in praxi ponderandi. Si enim habeantur pondera 1, 2, 4, 8, 16, 32 etc. librarum, his solis ponderibus omnia onera ponderari poterunt, nisi partes librae requirant.[1]) Sic his decem ponderibus, nempe

1) Id quod iam docuit LEONARDO PISANO in libro, qui inscribitur *Liber abbaci* (1202), ed. B. BONCOMPAGNI, Roma 1857, p. 297. Vide porro M. STIFEL (1486—1567), *Die Coss CHRISTOFFS RUDOLFFS*, Königsperg 1553, fol. 11ʳ, et FR. v. SCHOOTEN (1615—1660), *Exercitationum mathematicarum libri quinque*, Lugd. Batav. 1657, lib. V sectio VIII, p. 410. Confer denique G. ENESTRÖM, *Über die ältere Geschichte der Zerfällung ganzer Zahlen in Summen kleinerer Zahlen*, Biblioth. Mathem. 13_3, 1912—1913, p. 352. F. R.

1 ℔, 2 ℔, 4 ℔, 8 ℔, 16 ℔, 32 ℔, 64 ℔, 128 ℔, 256 ℔, 512 ℔,

omnia pondera usque ad 1024 ℔ librari possunt, et si unum pondus 1024 ℔ addatur, omnibus oneribus usque ad 2048 ℔ ponderandis sufficient.

330. Ostendi autem insuper solet in praxi ponderandi paucioribus ponderibus, quae scilicet in ratione geometrica tripla progrediantur, nempe

$$1,\ 3,\ 9,\ 27,\ 81\ \text{etc.}$$

librarum, pariter omnia onera ponderari posse, nisi opus sit fractionibus. In hac autem praxi pondera non solum uni lanci, sed ambabus, uti necessitas exigit, imponi debent.[1]) Nititur ergo ista praxis hoc fundamento, quod ex terminis progressionis geometricae triplae 1, 3, 9, 27, 81 etc. diversis semper sumendis per additionem ac subtractionem omnes omnino numeri produci queant; erit scilicet

$1 = 1$	$5 = 9 - 3 - 1$	$9 = 9$
$2 = 3 - 1$	$6 = 9 - 3$	$10 = 9 + 1$
$3 = 3$	$7 = 9 - 3 + 1$	$11 = 9 + 3 - 1$
$4 = 3 + 1$	$8 = 9 - 1$	$12 = 9 + 3$

etc.

331. Ad hanc veritatem ostendendam considero hoc productum infinitum

$$(x^{-1}+1+x^1)(x^{-3}+1+x^3)(x^{-9}+1+x^9)(x^{-27}+1+x^{27})\text{ etc.} = P,$$

quod evolutum alias non dabit potestates ipsius x, nisi quarum exponentes formari possint ex numeris 1, 3, 9, 27, 81 etc., sive addendo sive subtrahendo. Num vero omnes potestates prodeant singulaeque semel, sic exploro. Sit

$$P = \text{etc.} + cx^{-3} + bx^{-2} + ax^{-1} + 1 + \alpha x^1 + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \text{etc.}$$

Manifestum vero est, si x^3 loco x scribatur, tum prodire

$$\frac{P}{x^{-1}+1+x^1} = \text{etc.} + bx^{-6} + ax^{-3} + 1 + \alpha x^3 + \beta x^6 + \gamma x^9 + \text{etc.}$$

1) Etiam hoc invenitur apud LEONARDO PISANO; vide notam praecedentem. F. R.

Hinc igitur reperitur

$$P = \text{etc.} + ax^{-4} + ax^{-3} + ax^{-2} + x^{-1} + 1 + x + \alpha x^2 + \alpha x^3 + \alpha x^4 + \beta x^5 + \beta x^6 + \beta x^7 + \text{etc.},$$

quae expressio cum assumpta comparata dabit

$$\alpha = 1, \quad \beta = \alpha, \quad \gamma = \alpha, \quad \delta = \alpha, \quad \varepsilon = \beta, \quad \zeta = \beta \quad \text{etc.}$$

et

$$a = 1, \quad b = a, \quad c = a, \quad d = a, \quad e = b \quad \text{etc.}$$

Hinc itaque erit

$$P = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \text{etc.}$$
$$+ x^{-1} + x^{-2} + x^{-3} + x^{-4} + x^{-5} + x^{-6} + x^{-7} + \text{etc.},$$

unde patet omnes ipsius x potestates, tam affirmativas quam negativas, hic occurrere atque adeo omnes numeros ex terminis progressionis geometricae triplae vel addendo vel subtrahendo formari posse et unumquemque numerum unico tantum modo.

CAPUT XVII

DE USU SERIERUM RECURRENTIUM IN RADICIBUS AEQUATIONUM INDAGANDIS

332. Indicavit Vir Celeb. DANIEL BERNOULLI[1]) insignem usum serierum recurrentium in investigandis radicibus aequationum cuiusvis gradus in Comment. Acad. Petropol. Tomo III, ubi ostendit, quemadmodum cuiusque aequationis algebraicae, quotcunque fuerit dimensionum, valores radicum veris proximi ope serierum recurrentium assignari queant. Quae inventio cum saepenumero maximam afferat utilitatem, eam hic diligentius explicare constitui, ut intelligatur, quibus casibus adhiberi possit. Interdum enim praeter expectationem evenit, ut nulla aequationis radix ope huius methodi cognosci queat. Quocirca, ut vis huius methodi clarius perspiciatur, ex proprietatibus serierum recurrentium totum fundamentum, quo nititur, contemplemur.

333. Quoniam omnis series recurrens ex evolutione cuiusdam fractionis rationalis oritur, sit ista fractio

$$= \frac{a + bz + cz^2 + dz^3 + ez^4 + \text{etc.}}{1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \text{etc.}},$$

unde oriatur sequens series recurrens

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \text{etc.},$$

1) D. BERNOULLI, *Observationes de seriebus recurrentibus*, Comment. acad. sc. Petrop. 3 (1728), 1732, p. 85. F. R.

cuius coefficientes A, B, C, D etc. ita determinantur, ut sit

$$\begin{aligned} A &= a, \\ B &= \alpha A + b, \\ C &= \alpha B + \beta A + c, \\ D &= \alpha C + \beta B + \gamma A + d, \\ E &= \alpha D + \beta C + \gamma B + \delta A + e \\ &\text{etc.} \end{aligned}$$

Terminus autem generalis seu coefficiens potestatis z^n invenitur ex resolutione fractionis propositae in fractiones simplices, quarum denominatores sint factores denominatoris

$$1 - \alpha z - \beta zz - \gamma z^3 - \text{etc.},$$

uti cap. XIII est ostensum.

334. Forma autem termini generalis potissimum pendet ab indole factorum simplicium denominatoris, utrum sint reales an imaginarii, et utrum sint inter se inaequales an eorum bini pluresve aequales. Quos varios casus ut ordine percurramus, ponamus primum omnes denominatoris factores simplices cum reales esse tum inter se inaequales. Sint ergo factores simplices denominatoris omnes

$$(1 - pz)(1 - qz)(1 - rz)(1 - sz) \text{ etc.},$$

ex quibus fractio proposita in sequentes fractiones simplices resolvatur

$$\frac{\mathfrak{A}}{1 - pz} + \frac{\mathfrak{B}}{1 - qz} + \frac{\mathfrak{C}}{1 - rz} + \frac{\mathfrak{D}}{1 - sz} + \text{etc.}$$

Quibus cognitis erit seriei recurrentis terminus generalis [§ 215]

$$= z^n(\mathfrak{A}p^n + \mathfrak{B}q^n + \mathfrak{C}r^n + \mathfrak{D}s^n + \text{etc.}),$$

quem statuamus $= Pz^n$; sit scilicet P coefficiens potestatis z^n sequentiumque Q, R etc., ita ut series recurrens fiat

$$A + Bz + Cz^2 + Dz^3 + \cdots + Pz^n + Qz^{n+1} + Rz^{n+2} + \text{etc.}$$

335. Ponamus iam n esse numerum maximum seu seriem recurrentem ad plurimos terminos esse continuatam. Quoniam numerorum inaequalium potestates eo magis fiunt inaequales, quo fuerint altiores, tanta erit diversitas in potestatibus $\mathfrak{A}p^n$, $\mathfrak{B}q^n$, $\mathfrak{C}r^n$ etc., ut ea, quae oritur ex maximo numerorum p, q, r etc., reliquas magnitudine longe superet prae eaque reliquae penitus evanescant, si n fuerit numerus plane infinite magnus. Cum igitur numeri p, q, r etc. sint inter se inaequales, ponamus inter eos p esse maximum. Ac propterea, si n sit numerus infinitus, fiet

$$P = \mathfrak{A}p^n;$$

sin autem n sit numerus vehementer magnus, erit tantum proxime $P = \mathfrak{A}p^n$. Simili vero modo erit

$$Q = \mathfrak{A}p^{n+1}$$

ideoque

$$\frac{Q}{P} = p.$$

Unde patet, si series recurrens iam longe fuerit producta, coefficientem cuiusque termini per praecedentem divisum proxime esse exhibiturum valorem maximae litterae p.

336. Si igitur in fractione proposita

$$\frac{a + bz + cz^2 + dz^3 + \text{etc.}}{1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \text{etc.}}$$

denominator habeat omnes factores simplices reales et inter se inaequales, ex serie recurrente inde orta cognosci poterit unus factor simplex, is scilicet $1 - pz$, in quo littera p omnium maximum habet valorem. Neque in hoc negotio coefficientes numeratoris a, b, c, d etc. in computum ingrediuntur, sed quicunque ii statuantur, tamen denique idem verus valor litterae maximae p invenitur. Verus quidem valor ipsius p tum demum innotescit, quando series in infinitum fuerit continuata; interim tamen, si iam plures eius termini fuerint formati, eo propius valor ipsius p cognoscetur, quo maior fuerit terminorum numerus et quo magis littera ista p excedat reliquas q, r, s etc. Perinde vero est, utrum haec maxima littera p fuerit signo + an signo — affecta, quoniam eius potestates aeque increscunt.

337. Quemadmodum nunc haec investigatio ad inventionem radicum aequationis cuiusvis algebraicae accommodari possit, satis est perspicuum. Ex factoribus enim denominatoris

$$1 - \alpha z - \beta zz - \gamma z^3 - \delta z^4 - \text{etc.}$$

cognitis facile assignantur radices aequationis huius

$$1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \text{etc.} = 0,$$

ita ut, si factor fuerit $1 - pz$, huius aequationis radix una futura sit $z = \frac{1}{p}$. Cum igitur ex serie recurrente reperiatur maximus numerus p, indidem obtinebitur minima radix aequationis

$$1 - \alpha z - \beta z^2 - \gamma z^3 - \text{etc.} = 0.$$

Vel si ponatur $z = \frac{1}{x}$, ut prodeat haec aequatio

$$x^m - \alpha x^{m-1} - \beta x^{m-2} - \gamma x^{m-3} - \text{etc.} = 0,$$

eiusdem methodi ope eruitur maxima huius aequationis radix $x = p$.

338. Si igitur proponatur aequatio haec

$$x^m - \alpha x^{m-1} - \beta x^{m-2} - \gamma x^{m-3} - \text{etc.} = 0,$$

quae omnes radices habeat reales et inter se inaequales, harum radicum maxima sequenti modo reperietur. Formetur ex coefficientibus huius aequationis fractio

$$\frac{a + bz + cz^2 + dz^3 + \text{etc.}}{1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \text{etc.}}$$

Hincque formetur series recurrens assumendo pro arbitrio numeratorem seu, quod eodem redit, assumendo pro lubitu terminos initiales. Quae sit

$$A + Bz + Cz^2 + Dz^3 + \cdots + Pz^n + Qz^{n+1} + \text{etc.}$$

dabitque fractio $\frac{Q}{P}$ valorem radicis maximae x pro aequatione proposita eo propius, quo maior fuerit numerus n.

EXEMPLUM 1

Sit proposita ista aequatio

$$xx - 3x - 1 = 0,$$

cuius maximam radicem inveniri oporteat.

Formetur fractio

$$\frac{a + bz}{1 - 3z - zz},$$

unde positis duobus primis terminis 1, 2 orietur ista series recurrens

1, 2, 7, 23, 76, 251, 829, 2738 etc.

Erit ergo

$$\frac{2738}{829}$$

proxime aequalis radici aequationis propositae maximae. Valor autem huius fractionis in partibus decimalibus expressus est

3,3027744;

aequationis vero radix maxima est

$$= \frac{3 + \sqrt{13}}{2} = 3{,}3027756,$$

quae inventam superat tantum una parte millionesima. Ceterum notandum est fractiones $\frac{Q}{P}$ alternatim vera radice esse maiores et minores.

EXEMPLUM 2

Proposita sit ista aequatio

$$3x - 4x^3 = \frac{1}{2},$$

cuius radices exhibent sinus trium arcuum, quorum triplorum sinus est $= \frac{1}{2}$.

Aequatione perducta ad hanc formam

$$0 = 1 - 6x \times + 8x^3$$

quaeratur huius, ut in numeris integris maneamus, radix minima, ita ut non opus sit pro x ponere $\frac{1}{z}$. Formetur ergo haec fractio

$$\frac{a+bx+cxx}{1-6x\times+8x^3},$$

ex qua sumendis pro lubitu tribus terminis initialibus 0, 0, 1, quia hoc modo calculus facillime expeditur, orietur haec series recurrens omittendis potestatibus ipsius x, quia tantum coefficientibus opus est,

$$0,\ 0,\ 1,\ 6,\ 36,\ 208,\ 1200,\ 6912,\ 39808,\ 229248.$$

Erit ergo proxime aequationis radix minima

$$= \frac{39808}{229248} = \frac{311}{1791} = 0{,}1736460\,^{1)},$$

quae propterea esse deberet sinus anguli 10°; hic autem ex tabulis est 0,1736482, qui superat radicem inventam parte $\frac{22}{10000000}$.

Facilius autem haec eadem radix inveniri potest ponendo $x = \frac{1}{2}y$, ut prodeat aequatio

$$1-3y\times+y^3=0,$$

ex qua simili modo tractata oritur series

$$0,\ 0,\ 1,\ 3,\ 9,\ 26,\ 75,\ 216,\ 622,\ 1791,\ 5157 \text{ etc.}$$

Erit ergo proxime aequationis radix minima

$$y = \frac{1791}{5157} = \frac{199}{573} = 0{,}3472949,$$

unde fit

$$x = \frac{1}{2}y = 0{,}1736475\,^{2)},$$

qui valor fere ter propius accedit quam praecedens.

1) Editio princeps: $\frac{311}{1791} = 0{,}1736515$, *quae propterea esse deberet sinus anguli* 10°; *hic autem ex tabulis est* 0,1736482, *quem superat radix inventa parte* $\frac{33}{10000000}$. Correxit F. R.

2) Editio princeps: $x = \frac{1}{2}y = 0{,}1736479$, *qui valor decies propius accedit quam praecedens.* Correxit F. R.

EXEMPLUM 3

Si desideretur eiusdem aequationis propositae

$$0 = 1-6x\times+8x^3$$

radix maxima, ponatur $x = \frac{y}{2}$ eritque

$$y^3\times-3y+1=0.$$

Cuius aequationis radix maxima reperietur per seriem recurrentem, cuius scala relationis est 0, 3, −1, unde ergo oritur sumptis tribus terminis initialibus pro arbitrio

$$1,\ 1,\ 1,\ 2,\ 2,\ 5,\ 4,\ 13,\ 7,\ 35,\ 8,\ 98,\ -11 \text{ etc.};$$

in qua serie cum ad terminos negativos perveniatur, id indicio est maximam radicem esse negativam; est enim

$$x = -\sin. 70^\circ = -0{,}9396926.$$

Quare huius ratio in terminis initialibus est habenda hoc modo

$$1,\ -2,\ +4,\ -7,\ +14,\ -25,\ +49,\ -89,\ +172,\ -316,\ +605 \text{ etc.},$$

ex qua erit

$$y = \frac{-605}{316} \quad \text{et} \quad x = \frac{-605}{632} = -0{,}957,$$

quae a veritate vehementer abludit.

339. Ratio huius dissensus potissimum est, quod aequationis propositae radices sint

$$\sin. 10^\circ,\quad \sin. 50^\circ \quad \text{et} \quad -\sin. 70^\circ,$$

quarum binae maximae tam parum a se invicem discrepant, ut in potestatibus, ad quas seriem continuavimus, secunda radix sin. 50° adhuc notabilem teneat rationem ad radicem maximam ideoque prae ea non evanescat. Hincque etiam saltus pendet, quod alternatim valores inventi fiant nimis magni et nimis parvi. Sic sumendo

$$y = \frac{-316}{172}$$

fit

$$x = \frac{-158}{172} = \frac{-79}{86} = -0{,}919.$$

Nam quoniam potestates radicis maximae alternatim fiunt affirmativae et negativae, alternatim quoque potestates secundae radicis adduntur et tolluntur; quamobrem, quo haec discrepantia fiat insensibilis, series vehementer ulterius debet continuari.

340. Aliud vero remedium huic incommodo afferri potest transmutando aequationem ope idoneae substitutionis in aliam formam, cuius radices sibi non amplius sint tam vicinae. Sic si in aequatione

$$0 = 1 - 6x + 8x^3,$$

cuius radices sunt $-\sin. 70^\circ$, $+\sin. 50^\circ$, $+\sin. 10^\circ$, ponatur $x = y - 1$, aequationis

$$0 = 8y^3 - 24yy + 18y - 1$$

radices erunt $1 - \sin. 70^\circ$, $1 + \sin. 50^\circ$, $1 + \sin. 10^\circ$ ideoque eius radix minima erit $1 - \sin. 70^\circ$, cum tamen haec $\sin. 70^\circ$ esset radix maxima aequationis praecedentis, atque $1 + \sin. 50^\circ$ nunc est radix maxima, cum $\sin. 50^\circ$ ante esset media. Atque hoc modo quaevis radix per substitutionem in maximam minimamve radicem novae aequationis transmutari ideoque per methodum hic traditam inveniri poterit. Quia praeterea in hoc exemplo radix $1 - \sin. 70^\circ$ multo minor est quam binae reliquae, etiam facile per seriem recurrentem proxime cognoscetur.

EXEMPLUM 4

Invenire radicem minimam aequationis

$$0 = 8y^3 - 24yy + 18y - 1,$$

quae ab unitate subtracta relinquet sinum anguli 70°.

Ponatur $y = \frac{1}{2}z$, ut sit

$$0 = z^3 - 6zz + 9z - 1,$$

cuius radix minima invenietur per seriem recurrentem, cuius scala relationis est 9, -6, $+1$; pro radice autem maxima invenienda scala relationis sumi deberet 6, -9, $+1$. Pro minima ergo formetur haec series

1, 1, 1, 4, 31, 256, 2122, 17593, 145861 etc.

Erit ergo proxime

$$z = \frac{17593}{145861} = 0{,}12061483$$

et

$$y = 0{,}06030741$$

atque

$$\sin. 70^\circ = 1 - y = 0{,}93969258,$$

quae a veritate ne in ultima quidem figura discrepat. Ex hoc ergo exemplo intelligitur, quantam utilitatem idonea transformatio aequationis ope substitutionis ad inventionem radicum afferat et quod hoc pacto methodus tradita non solum ad maximas minimasve radices adstringatur, sed etiam omnes radices exhibere queat.

341. Cognita ergo iam quacunque aequationis propositae radice proxime ita ut verbi gratia numerus k quam minime a quapiam radice differat, ponatur $x - k = y$ seu $x = y + k$ hocque modo prodibit aequatio, cuius radix minima erit $= x - k$; quae igitur si per series recurrentes indagetur, quod facillime fiet, quia haec radix multo minor erit quam ceterae, si ea ad k addatur, habebitur radix vera ipsius x pro aequatione proposita. Hoc vero artificium tam late patet, ut, etiamsi aequatio contineat radices imaginarias, usum suum retineat.

342. Imprimis autem sine hoc artificio radix cognosci nequit, cui datur alia aequalis, sed signo contrario affecta. Scilicet, si aequatio, cuius maxima radix p, eadem radicem habeat $-p$, tum, etiamsi series recurrens in infinitum

continuetur, tamen radix haec p nunquam obtinebitur. Sit, ut hoc exemplo illustremus, proposita aequatio

$$x^3 - x^2 - 5x + 5 = 0,$$

cuius maxima radix est $\sqrt{5}$, praeter quam vero inest quoque $-\sqrt{5}$. Si igitur modo ante praescripto pro radice maxima invenienda utamur atque seriem recurrentem formemus ex scala relationis 1, $+5$, -5, erit haec

$$1, \ 2, \ 3, \ 8, \ 13, \ 38, \ 63, \ 188, \ 313, \ 938, \ 1563 \text{ etc.},$$

ubi ad nullam rationem constantem pervenitur. Termini vero alterni rationem aequabilem induunt; quorum si quisque per praecedentem dividatur, reperietur quadratum maximae radicis; sic enim est proxime

$$5 = \frac{1563}{313} = \frac{938}{188} = \frac{313}{63}.$$

Quoties ergo termini tantum alterni sese ad rationem constantem componunt, toties quadratum radicis quaesitae proxime obtinetur. Ipsa autem radix $x = \sqrt{5}$ invenitur ponendo $x = y + 2$, unde fit

$$1 - 3y - 5yy - y^3 = 0,$$

cuius radix minima cognoscetur ex serie

$$1, \ 1, \ 1, \ 9, \ 33, \ 145, \ 609, \ 2585, \ 10945 \text{ etc.};$$

erit enim proxime

$$= \frac{2585}{10945} = 0{,}2361;$$

at 2,2361 est proxime $= \sqrt{5}$, quae est radix maxima aequationis.

343. Quanquam numerator fractionis, ex qua series recurrens formatur, a nostro arbitrio pendet, tamen idonea eius constitutio plurimum confert, ut valor radicis cito vero proxime exhibeatur. Cum enim assumptis ut supra factoribus denominatoris (§ 334) sit terminus generalis seriei recurrentis

$$= z^n(\mathfrak{A}p^n + \mathfrak{B}q^n + \mathfrak{C}r^n + \text{etc.}),$$

isti coefficientes $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc. per numeratorem fractionis determinantur, unde fieri potest, ut $\mathfrak{A}$ sive magnum sive parvum valorem obtineat; priori casu radix maxima p cito reperitur, posteriore vero tarde. Quin etiam numerator ita accipi potest, ut $\mathfrak{A}$ prorsus evanescat, quo casu, etiamsi series in infinitum continuetur, tamen nunquam radicem maximam p praebebit. Hoc autem evenit, si numerator ita accipiatur, ut ipse eundem habeat factorem $1 - pz$; sic enim ex computo penitus tolletur. Sic si proponatur aequatio

$$x^3 - 6xx + 10x - 3 = 0,$$

cuius maxima radix est $= 3$, indeque formetur fractio

$$\frac{1 - 3z}{1 - 6z + 10z^2 - 3z^3},$$

ut seriei recurrentis sit scala relationis 6, -10, $+3$, [series erit haec]

$$1, \ 3, \ 8, \ 21, \ 55, \ 144, \ 377 \text{ etc.},$$

cuius termini prorsus non convergunt ad rationem $1:3$. Eadem enim series oritur ex fractione

$$\frac{1}{1 - 3z + zz}$$

ac propterea maximam radicem aequationis

$$x^2 - 3x + 1 = 0$$

exhibet.

344. Quin etiam numerator ita assumi potest, ut per seriem recurrentem quaevis radix aequationis reperiatur, quod fiet, si numerator fuerit productum ex omnibus factoribus denominatoris praeter eum, cui respondet radix, quam velimus. Sic si in priori exemplo sumatur numerator $1 - 3z + zz$, fractio

$$\frac{1 - 3z + zz}{1 - 6z + 10z^2 - 3z^3}$$

dabit hanc seriem recurrentem

$$1, \ 3, \ 9, \ 27, \ 81, \ 243 \text{ etc.},$$

quae, cum sit geometrica, statim monstrat radicem $x = 3$. Fractio enim illa aequalis est huic simplici

$$\frac{1}{1-3z}.$$

Hinc apparet, si termini initiales, quos pro lubitu assumere licet, ita accipiantur, ut progressionem geometricam constituant, cuius exponens aequetur uni radici aequationis, tum totam seriem recurrentem fore geometricam ideoque eam ipsam radicem esse exhibituram, etiamsi neque sit maxima neque minima.

345. Ne igitur, dum quaerimus radicem vel maximam vel minimam, praeter expectationem nobis alia radix per seriem recurrentem exhibeatur, eiusmodi numerator debet eligi, qui cum denominatore nullum factorem habeat communem, quod fiet, si pro numeratore unitas accipiatur, unde terminus primus seriei erit $= 1$, ex quo solo secundum scalam relationis sequentes omnes definiantur. Hocque modo semper certe radix aequationis vel maxima vel minima, prout fuerit propositum, eruetur. Sic proposita aequatione

$$y^3 \ast - 3y + 1 = 0,$$

cuius radix maxima desideratur, ex scala relationis $0, +3, -1$ incipiendo ab unitate sequens oritur series recurrens

$$1, -0, +3, -1, +9, -6, +28, -27, +90, -109, +297, -417,$$
$$+1000, -1548, +3417, -5644 \text{ etc.}^{1)},$$

quae ad rationem constantem convergit ostenditque radicem maximam esse negativam atque proxime

$$y = \frac{-5644}{3417} = -1{,}651741,$$

1) Editio princeps: $1 - 0 + 3 - 1 + \cdots + 297 - 517 + 1000 - 1848 + 3517 - 6544 + \text{etc.}$, *quae manifesto ad rationem constantem convergit, ostenditque radicem maximam esse negativam, atque proxime* $y = \frac{-6544}{3517} = -1{,}860676$, *quae esse debebat* $= -1{,}86793852$. Correxit F. R.

quae esse debebat $= -1{,}8793852$. Ratio autem supra [§ 330] est allata, cur tam lente ad verum valorem appropinquetur, propterea quod altera radix non multo sit minor maxima simulque sit affirmativa.

346. His probe perpensis, quae cum in genere tum ad exempla allata monuimus, summa utilitas huius methodi ad investigandas aequationum radices luculenter perspicietur. Artificia vero, quibus operatio contrahi eoque promptior reddi queat, satis quoque sunt indicata, ita ut nihil insuper addendum esset, nisi casus, quibus aequatio vel radices habet aequales vel imaginarias, evolvendi superessent. Ponamus ergo denominatorem fractionis

$$\frac{a + bz + cz^2 + dz^3 + \text{etc.}}{1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \text{etc.}}$$

habere factorem $(1-pz)^2$ reliquis factoribus existentibus $1 - qz$, $1 - rz$ etc. Seriei ergo recurrentis hinc natae terminus generalis erit

$$= z^n((n+1)\mathfrak{A}p^n + \mathfrak{B}p^n + \mathfrak{C}q^n + \text{etc.});$$

quae cuiusmodi valorem sit adeptura, si n fuerit numerus vehementer magnus, duo casus sunt distinguendi, alter, quo p est numerus maior reliquis q, r etc., alter, quo p non praebet radicem maximam. Casu priori, quo p simul est radix maxima, ob coefficientem $n+1$ reliqui termini $\mathfrak{B}p^n$, $\mathfrak{C}q^n$ etc. non tam cito prae eo evanescent quam ante; sin autem q fuerit $> p$, tum quoque tarde terminus $(n+1)\mathfrak{A}p^n$ prae $\mathfrak{C}q^n$ evanescet ideoque investigatio radicis maximae admodum evadet molesta.

EXEMPLUM 1

Sit proposita aequatio

$$x^3 - 3xx + 4 = 0,$$

cuius maxima radix 2 bis occurrit.

Quaeratur ergo maxima radix haec modo ante exposito per evolutionem fractionis

$$\frac{1}{1 - 3z \ast + 4z^3},$$

quae dabit hanc seriem recurrentem

$$1,\ 3,\ 9,\ 23,\ 57,\ 135,\ 313,\ 711,\ 1593 \text{ etc.},$$

ubi quidem quivis terminus per praecedentem divisus dat quotum binario maiorem. Cuius ratio ex termino generali facillime patet. Reiectis enim in eo terminis $\mathfrak{C}q^n$ etc. erit terminus potestati z^n respondens

$$=(n+1)\mathfrak{A}p^n+\mathfrak{B}p^n,$$

sequens

$$=(n+2)\mathfrak{A}p^{n+1}+\mathfrak{B}p^{n+1},$$

qui per illum divisus dat

$$\frac{(n+2)\mathfrak{A}+\mathfrak{B}}{(n+1)\mathfrak{A}+\mathfrak{B}}p>p,$$

nisi n iam in infinitum excreverit.

EXEMPLUM 2

Sit iam proposita aequatio

$$x^3-xx-5x-3=0,$$

cuius maxima radix $=3$, reliquae duae aequales $=-1$.

Quaeratur maxima radix ope seriei recurrentis, cuius scala relationis est $1,\ +5,\ +3$; unde oritur

$$1,\ 1,\ 6,\ 14,\ 47,\ 135,\ 412,\ 1228 \text{ etc.},$$

quae ideo satis cito valorem 3 exhibet, quod potestates minoris radicis -1, etiamsi multiplicentur per $n+1$, tamen mox prae potestatibus ipsius 3 evanescant.

EXEMPLUM 3

Sin autem proponeretur aequatio

$$x^3+xx-8x-12=0,$$

cuius radices sunt $3,\ -2,\ -2$, multo tardius maxima sese prodet.

Orietur enim haec series

$$1,\ -1,\ 9,\ -5,\ 65,\ 3,\ 457,\ 347,\ 3345,\ 4915 \text{ etc.},$$

quae adhuc longissime continuari deberet, antequam pateret radicem inde oriundam esse $=3$.

347. Simili modo si três factores essent aequales, ita ut denominatoris factor unus esset $(1-pz)^3$, reliqui $1-qz$, $1-rz$ etc., seriei recurrentis terminus generalis erit

$$=z^n\left(\frac{(n+1)(n+2)}{1\cdot 2}\mathfrak{A}p^n+(n+1)\mathfrak{B}p^n+\mathfrak{C}p^n+\mathfrak{D}q^n+\mathfrak{E}r^n+\text{etc.}\right).$$

Si ergo p fuerit maxima radix atque n fuerit numerus tantus, ut potestates q^n, r^n etc. prae p^n evanescant, tum ex serie recurrente orietur radix

$$=\frac{\frac{1}{2}(n+2)(n+3)\mathfrak{A}+(n+2)\mathfrak{B}+\mathfrak{C}}{\frac{1}{2}(n+1)(n+2)\mathfrak{A}+(n+1)\mathfrak{B}+\mathfrak{C}}p,$$

quae, nisi sit n numerus maximus et quasi infinitus, verum ipsius p valorem [non] indicabit. Erit autem iste radicis valor

$$=p+\frac{(n+2)\mathfrak{A}+\mathfrak{B}}{\frac{1}{2}(n+1)(n+2)\mathfrak{A}+(n+1)\mathfrak{B}+\mathfrak{C}}p.$$

Quodsi autem p non fuerit radix maxima, tum inventio maximae multo magis adhuc impedietur; unde sequitur aequationes, quae contineant radices aequales, hac methodo per series recurrentes multo difficilius resolvi, quam si omnes radices essent inter se inaequales.

348. Videamus nunc, quomodo series recurrens in infinitum continuata debeat esse comparata, quando denominator fractionis habet factores imaginarios. Sint igitur fractionis

$$\frac{a+bz+cz^2+dz^3+\text{etc.}}{1-\alpha z-\beta z^2-\gamma z^3-\delta z^4-\text{etc.}}$$

factores denominatoris reales

$$1-qz,\quad 1-rz \quad \text{etc.}$$

insuperque factor trinomialis

$$1-2pz\cos.\varphi+ppzz$$

continens duos factores simplices imaginarios. Quodsi ergo series recurrens ex illa fractione orta fuerit

$$A+Bz+Cz^2+Dz^3+\cdots+Pz^n+Qz^{n+1}+\text{etc.},$$

erit per ea, quae supra [§ 218] exposuimus, coefficiens P

$$=\frac{\mathfrak{A}\sin.(n+1)\varphi+\mathfrak{B}\sin.n\varphi}{\sin.\varphi}p^n+\mathfrak{C}q^n+\mathfrak{D}r^n+\text{etc.}$$

Si igitur numerus p minor fuerit quam unus ceterorum q, r etc., ita ut maxima radix aequationis

$$x^m-\alpha x^{m-1}-\beta x^{m-2}-\gamma x^{m-3}-\text{etc.}=0$$

sit realis, tum ea per series recurrentes aeque reperietur, ac si nullae radices inessent imaginariae.

349. Inventio ergo maximae radicis realis per radices imaginarias non perturbabitur, si hae ita fuerint comparatae, ut binarum, quae factorem realem componunt, productum non sit maius quadrato radicis maximae. Sin autem binae eiusmodi insint radices imaginariae, ut earum productum adaequet vel adeo superet quadratum maximae radicis realis, tum investigatio ante exposita nihil declarabit, propterea quod potestas p^n prae simili potestate radicis maximae nunquam evanescit, etiamsi series in infinitum continuetur. Cuius exempla illustrationis causa hic adiicere visum est.

EXEMPLUM 1

Sit proposita aequatio

$$x^3-2x-4=0,$$

cuius radicem maximam investigari oporteat.

Resolvitur haec aequatio in duos factores

$$(x-2)(xx+2x+2);$$

unde unam habet radicem realem 2 et duas reliquas imaginarias, quarum productum est 2, minus quam quadratum radicis realis. Quamobrem ea per modum hactenus traditum cognosci poterit. Formetur ergo series recurrens ex scala relationis 0, $+2$, $+4$, quae erit

$$1,\ 0,\ 2,\ 4,\ 4,\ 16,\ 24,\ 48,\ 112,\ 192,\ 416,\ 832\ \text{etc.},$$

unde satis luculenter radix realis 2 cognosci potest.

EXEMPLUM 2

Proposita sit aequatio

$$x^3-4xx+8x-8=0,$$

cuius radix una realis est 2, binarum imaginariarum productum vero $=4$ ideoque aequale quadrato radicis realis 2.

Quaeramus ergo radicem per seriem recurrentem; quod quo facilius fieri queat, ponamus $x=2y$, ut habeatur

$$y^3-2yy+2y-1=0,$$

unde formetur series recurrens

$$1,\ 2,\ 2,\ 1,\ 0,\ 0,\ 1,\ 2,\ 2,\ 1,\ 0,\ 0,\ 1,\ 2,\ 2,\ 1\ \text{etc.};$$

in qua cum iidem termini perpetuo revertantur, nihil inde aliud colligi potest, nisi radicem maximam vel non esse realem vel dari imaginarias, quarum productum aequale sit aut superet quadratum radicis realis.

EXEMPLUM 3

Sit iam proposita aequatio

$$x^3-3xx+4x-2=0,$$

cuius radix realis est 1, imaginariarum vero productum $=2$.

Formetur ergo ex scala relationis 3, —4, +2 series

$$1,\ 3,\ 5,\ 5,\ 1,\ -7,\ -15,\ -15,\ +1,\ 33,\ 65,\ 65,\ 1 \text{ etc.};$$

in qua cum termini modo fiant affirmativi modo negativi, radix realis 1 inde nullo modo cognosci poterit. Huiusmodi vero revolutiones semper ostendunt radicem, quam series praebere debebat, esse imaginariam; hic enim radices imaginariae potestate sunt maiores quam realis 1.

350. Sit igitur in fractione generali productum binarum radicum imaginariarum pp maius quam ullius radicis realis quadratum, ita ut prae p^n reliquae potestates q^n, r^n etc. evanescant, si n sit numerus infinitus. Hoc ergo casu fiet

$$P = \frac{\mathfrak{A} \sin.(n+1)\varphi + \mathfrak{B} \sin. n\varphi}{\sin.\varphi} p^n$$

et

$$Q = \frac{\mathfrak{A} \sin.(n+2)\varphi + \mathfrak{B} \sin.(n+1)\varphi}{\sin.\varphi} p^{n+1}$$

ideoque

$$\frac{Q}{P} = \frac{\mathfrak{A} \sin.(n+2)\varphi + \mathfrak{B} \sin.(n+1)\varphi}{\mathfrak{A} \sin.(n+1)\varphi + \mathfrak{B} \sin. n\varphi} p.$$

Quae expressio nunquam valorem constantem induet, etiamsi n sit numerus infinitus. Sinus enim angulorum perpetuo maxime manent mutabiles, ita ut mox sint affirmativi mox negativi.

351. Interim tamen si fractiones sequentes $\frac{R}{Q}$, $\frac{S}{R}$ simili modo sumantur indeque litterae $\mathfrak{A}$ et $\mathfrak{B}$ eliminentur, simul numerus n ex calculo egredietur; reperietur enim[1])

$$Ppp + R = 2Qp \cos.\varphi,$$

unde fit

$$\cos.\varphi = \frac{Ppp + R}{2Qp};$$

similiter vero erit

$$\cos.\varphi = \frac{Qpp + S}{2Rp},$$

1) Vide paragraphum sequentem. F. R.

ex quorum duorum valorum comparatione fit

$$p = \sqrt{\frac{RR - QS}{QQ - PR}}$$

atque

$$\cos.\varphi = \frac{QR - PS}{2\sqrt{(Q^2 - PR)(R^2 - QS)}}.$$

Quamobrem si series recurrens iam eousque fuerit continuata, ut prae p^n reliquarum radicum potestates evanescant, tum hoc modo factor trinomialis $1 - 2pz\cos.\varphi + ppzz$ poterit inveniri.

352. Quoniam iste calculus non satis exercitatis molestiam creare posset, eum totum hic apponam. Ex valore ipsius $\frac{Q}{P}$ invento oritur

$$\mathfrak{A}Pp \sin.(n+2)\varphi + \mathfrak{B}Pp \sin.(n+1)\varphi = \mathfrak{A}Q \sin.(n+1)\varphi + \mathfrak{B}Q \sin. n\varphi,$$

unde fit

$$\frac{\mathfrak{A}}{\mathfrak{B}} = \frac{Q \sin. n\varphi - Pp \sin.(n+1)\varphi}{Pp \sin.(n+2)\varphi - Q \sin.(n+1)\varphi}.$$

Pari ratione erit

$$\frac{\mathfrak{A}}{\mathfrak{B}} = \frac{R \sin.(n+1)\varphi - Qp \sin.(n+2)\varphi}{Qp \sin.(n+3)\varphi - R \sin.(n+2)\varphi}.$$

Aequatis his duobus valoribus fiet

$$\begin{aligned} 0 = {} & QQp \sin. n\varphi \sin.(n+3)\varphi - QR \sin. n\varphi \sin.(n+2)\varphi \\ & - PQpp \sin.(n+1)\varphi \sin.(n+3)\varphi - QQp \sin.(n+1)\varphi \sin.(n+2)\varphi \\ & + QR \sin.(n+1)\varphi \sin.(n+1)\varphi + PQpp \sin.(n+2)\varphi \sin.(n+2)\varphi. \end{aligned}$$

Cum autem sit

$$\sin. a \sin. b = \tfrac{1}{2}\cos.(a-b) - \tfrac{1}{2}\cos.(a+b),$$

fiet

$$0 = \tfrac{1}{2} QQp(\cos. 3\varphi - \cos.\varphi) + \tfrac{1}{2} QR(1 - \cos. 2\varphi) + \tfrac{1}{2} PQpp(1 - \cos. 2\varphi),$$

quae per $\frac{1}{2}Q$ divisa dat

$$(Ppp + R)(1 - \cos.2\varphi) = Qp(\cos.\varphi - \cos.3\varphi).$$

At est

$$\cos.\varphi = \cos.2\varphi\cos.\varphi + \sin.2\varphi\sin.\varphi$$

et

$$\cos.3\varphi = \cos.2\varphi\cos.\varphi - \sin.2\varphi\sin.\varphi,$$

unde

$$\cos.\varphi - \cos.3\varphi = 2\sin.2\varphi\sin.\varphi = 4\sin.\varphi^2\cos.\varphi,$$

et

$$1 - \cos.2\varphi = 2\sin.\varphi^2,$$

ex quo erit

$$Ppp + R = 2Qp\cos.\varphi$$

et

$$\cos.\varphi = \frac{Ppp + R}{2Qp}$$

atque

$$\cos.\varphi = \frac{Qpp + S}{2Rp},$$

unde superiores valores prodeunt, scilicet

$$p = \sqrt{\frac{RR - QS}{QQ - PR}}$$

et

$$\cos.\varphi = \frac{QR - PS}{2\sqrt{(Q^2 - PR)(RR - QS)}}.$$

353. Si denominator fractionis, ex qua series recurrens formatur, plures habeat factores trinomiales inter se aequales, tum spectata forma termini generalis supra [§ 219 et sq.] data patebit inventionem radicum multo magis fieri incertam. Interim tamen si una quaecunque radix realis iam proxime fuerit detecta, tum aequationis transformatione semper valor eiusdem radicis multo propior eruetur. Ponatur enim x aequalis valori illi iam detecto $+y$ atque novae aequationis quaeratur minima radix pro y, quae addita ad illum valorem praebebit verum ipsius x valorem.

EXEMPLUM

Sit proposita ista aequatio

$$x^3 - 3xx + 5x - 4 = 0;$$

cuius unam radicem fere esse $= 1$ inde constat, quod posita $x = 1$ prodit

$$x^3 - 3xx + 5x - 4 = -1.$$

Ponatur ergo $x = 1 + y$ fietque

$$1 - 2y - y^3 = 0,$$

unde pro radice minima invenienda formetur series recurrens, cuius scala relationis 2, 0, $+1$, quae erit

1, 2, 4, 9, 20, 44, 97, 214, 472, 1041, 2296 etc.,

unde radix minima ipsius y erit proxime

$$\frac{1041}{2296} = 0{,}453397,$$

ita ut sit

$$x = 1{,}453397,$$

qui valor tam prope vix alia methodo aeque facile obtineri poterit.

354. Quodsi autem series quaecunque recurrens tandem tam prope ad progressionem geometricam convergat, tum ex ipsa lege progressionis statim facile cognosci poterit, cuiusnam aequationis radix sit futura quotus, qui ex divisione unius termini per praecedentem oritur. Sint

$$P,\ Q,\ R,\ S,\ T \text{ etc.}$$

termini seriei recurrentis a principio iam longissime remoti, ita ut cum progressione geometrica confundantur, sitque

$$T = \alpha S + \beta R + \gamma Q + \delta P$$

seu scala relationis $\alpha, +\beta, +\gamma, +\delta$. Ponatur valor fractionis $\frac{Q}{P} = x$; erit

$$\frac{R}{P} = xx, \quad \frac{S}{P} = x^3 \quad \text{et} \quad \frac{T}{P} = x^4,$$

qui in superiori aequatione substituti dabunt

$$x^4 = \alpha x^3 + \beta x^2 + \gamma x + \delta,$$

unde patet quotum $\frac{Q}{P}$ tandem praebere radicem unam aequationis inventae. Hoc vero et praecedens methodus indicat, praeterea vero docet fractionem $\frac{Q}{P}$ dare maximam aequationis radicem.

355. Potest quoque haec methodus investigandarum radicum saepenumero utiliter adhiberi, si aequatio sit infinita. Ad quod ostendendum proposita sit aequatio

$$\frac{1}{2} = z - \frac{z^3}{6} + \frac{z^5}{120} - \frac{z^7}{5040} + \text{etc.},$$

cuius radix minima z exhibet arcum 30° seu semiperipheriae circuli sextantem. Perducatur ergo aequatio ad hanc formam

$$1 - 2z + \frac{z^3}{3} - \frac{z^5}{60} + \frac{z^7}{2520} - \text{etc.} = 0.$$

Hinc ergo formetur series recurrens, cuius scala relationis est infinita, scilicet

$$2,\ 0,\ -\frac{1}{3},\ 0,\ +\frac{1}{60},\ 0,\ -\frac{1}{2520},\ 0 \ \text{etc.},$$

eritque series recurrens

$$1,\ 2,\ 4,\ \frac{23}{3},\ \frac{44}{3},\ \frac{1681}{60},\ \frac{2408}{45} \ \text{etc.};$$

erit ergo proxime

$$z = \frac{1681 \cdot 45}{2408 \cdot 60} = \frac{1681 \cdot 3}{2408 \cdot 4} = \frac{5043}{9632} = 0{,}52356.$$

At ex proportione peripheriae ad diametrum cognita debebat esse $z = 0{,}523598$, ita ut radix inventa tantum parte $\frac{3}{100000}$ a vero discrepet.[1]) Hoc autem in hac aequatione commode usu venit, quod eius omnes radices sint reales atque a minima reliquae satis notabiliter discrepent. Quae conditio cum rarissime in aequationibus infinitis locum habeat, huic methodo ad eas resolvendas parum usus relinquitur.

1) Valores accuratiores sunt $z = 0{,}523\,567$ et $z = 0{,}523\,599$, quorum tamen differentia itidem est $\frac{3}{100000}$. F. R.

CAPUT XVIII

DE FRACTIONIBUS CONTINUIS

356. Quoniam in praecedentibus capitibus plura cum de seriebus infinitis tum de productis ex infinitis factoribus conflatis disserui, non incongruum fore visum est, si etiam nonnulla de tertio quodam expressionum infinitarum genere addidero, quod continuis fractionibus vel divisionibus continetur. Quanquam enim hoc genus parum adhuc est excultum, tamen non dubitamus, quin ex eo amplissimus usus in analysin infinitorum aliquando sit redundaturus. Exhibui enim iam aliquoties[1]) eiusmodi specimina, quibus haec expectatio non parum probabilis redditur. Imprimis vero ad ipsam arithmeticam et algebram communem non contemnenda subsidia affert ista speculatio, quae hoc capite breviter indicare atque exponere constitui.

357. Fractionem autem continuam voco eiusmodi fractionem, cuius denominator constat ex numero integro cum fractione, cuius denominator denuo est aggregatum ex integro et fractione, quae porro simili modo sit comparata, sive ista affectio in infinitum progrediatur sive alicubi sistatur. Huiusmodi ergo fractio continua erit sequens expressio

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\cfrac{1}{f+\text{etc.}}}}}} \qquad \text{vel} \qquad a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\cfrac{\varepsilon}{f+\text{etc.}}}}}}$$

1) Vide L. EULERI Commentationes 71 et 123 (indicis ENESTROEMIANI): *De fractionibus continuis*, Comment. acad. sc. Petrop. 9 (1737), 1744, p. 98, et *De fractionibus continuis observationes*, Comment. acad. sc. Petrop. 11 (1739), 1750, p. 32; LEONHARDI EULERI *Opera omnia*, series I, vol. 14. F. R.

in quarum forma priori omnes fractionum numeratores sunt unitates, quam potissimum hic contemplabor, in altera vero forma sunt numeratores numeri quicunque.

358. Exposita ergo fractionum harum continuarum forma primum videndum est, quemadmodum earum significatio consueto more expressa inveniri queat. Quae ut facilius inveniri possit, progrediamur per gradus abrumpendo illas fractiones primo in prima, tum in secunda, post in tertia et ita porro fractione; quo facto patebit fore

$$a = a,$$

$$a+\frac{1}{b} = \frac{ab+1}{b},$$

$$a+\cfrac{1}{b+\cfrac{1}{c}} = \frac{abc+a+c}{bc+1},$$

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}}} = \frac{abcd+ab+ad+cd+1}{bcd+b+d},$$

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e}}}} = \frac{abcde+abe+ade+cde+abc+a+c+e}{bcde+be+de+bc+1}$$

etc.

359. Etsi in his fractionibus ordinariis non facile lex, secundum quam numerator ac denominator ex litteris a, b, c, d etc. componantur, perspicitur, tamen attendenti statim patebit, quemadmodum quaelibet fractio ex praecedentibus formari queat. Quilibet enim numerator est aggregatum ex numeratore ultimo per novam litteram multiplicato et ex numeratore penultimo simplici; eademque lex in denominatoribus observatur. Scriptis ergo ordine litteris a, b, c, d etc. ex iis fractiones inventae facile formabuntur hoc modo

$$\begin{array}{cccccc} & a & b & c & d & e \\ \frac{1}{0}, & \frac{a}{1}, & \frac{ab+1}{b}, & \frac{abc+a+c}{bc+1}, & \frac{abcd+ab+ad+cd+1}{bcd+b+d} & \text{etc.,} \end{array}$$

ubi quilibet numerator invenitur, si praecedentium ultimus per indicem supra scriptum multiplicetur atque ad productum antepenultimus addatur; quae eadem lex pro denominatoribus valet. Quo autem hac lege ab ipso initio uti liceat, praefixi fractionem $\frac{1}{0}$, quae, etiamsi e fractione continua non oriatur, tamen progressionis legem clariorem efficit. Quaelibet autem fractio exhibet valorem fractionis continuae usque ad eam litteram, quae antecedenti imminet, inclusive continuatae.

360. Simili modo altera fractionum continuarum forma

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\cfrac{\varepsilon}{f+\text{etc.}}}}}}$$

dabit, prout aliis aliisque locis abrumpitur, sequentes valores

$$a = a,$$

$$a+\frac{\alpha}{b} = \frac{ab+\alpha}{b},$$

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c}} = \frac{abc+\beta a+\alpha c}{bc+\beta},$$

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d}}} = \frac{abcd+\beta ad+\alpha cd+\gamma ab+\alpha\gamma}{bcd+\beta d+\gamma b}$$

etc.,

quarum fractionum quaeque ex binis praecedentibus sequentem in modum invenietur:

$$\begin{array}{cccccc} a & b & c & d & e & \\ \frac{1}{0}, & \frac{a}{1}, & \frac{ab+\alpha}{b}, & \frac{abc+\beta a+\alpha c}{bc+\beta}, & \frac{abcd+\beta ad+\alpha cd+\gamma ab+\alpha\gamma}{bcd+\beta d+\gamma b} & \text{etc.} \\ \alpha & \beta & \gamma & \delta & \varepsilon & \end{array}$$

361. Fractionibus scilicet formandis supra inscribantur indices a, b, c, d etc., infra autem subscribantur indices $\alpha, \beta, \gamma, \delta$ etc. Prima fractio iterum constituatur $\frac{1}{0}$, secunda $\frac{a}{1}$. Tum sequentium quaevis formabitur, si antecedentium ultimae numerator per indicem supra scriptum, penultimae vero numerator per indicem infra scriptum multiplicetur et ambo producta addantur; aggregatum erit numerator fractionis sequentis. Simili modo eius denominator erit aggregatum ex ultimo denominatore per indicem supra scriptum et ex penultimo denominatore per indicem infra scriptum multiplicatis. Quaelibet vero fractio hoc modo inventa praebebit valorem fractionis continuae ad eum usque denominatorem, qui fractioni antecedenti est inscriptus, continuatae inclusive.

362. Quodsi ergo hae fractiones eousque continuentur, quoad fractio continua indices suppeditet, tum ultima fractio verum dabit valorem fractionis continuae. Praecedentes fractiones vero continuo propius ad hunc valorem accedent ideoque perquam idoneam appropinquationem suggerent. Ponamus enim verum valorem fractionis continuae

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\text{etc.}}}}}$$

esse $=x$ atque manifestum est fractionem primam $\frac{1}{0}$ esse maiorem quam x, secunda vero $\frac{a}{1}$ minor erit quam x, tertia $a+\frac{\alpha}{b}$ iterum vero valore erit maior, quarta denuo minor, atque ita porro hae fractiones alternatim erunt maiores et minores quam x. Porro autem perspicuum est quamlibet fractionem propius accedere ad verum valorem x quam ulla praecedentium[1]), unde hoc pacto citissime et commodissime valor ipsius x proxime obtinetur, etiamsi fractio continua in infinitum progrediatur, dummodo numeratores $\alpha, \beta, \gamma, \delta$ etc. non nimis crescant; sin autem omnes isti numeratores fuerint unitates, tum appropinquatio nulli incommodo est obnoxia.

363. Quo ratio huius appropinquationis ad verum fractionis continuae valorem melius percipiatur, consideremus fractionum inventarum differentias.

1) Hoc autem veritati non semper consentaneum est, nisi sit $\alpha=\beta=\gamma=\cdots=1$. Sic si sit $x=1+\frac{1}{1+\frac{3}{2}}$, fractionum $\frac{a}{1}=1$, $\frac{ab+\alpha}{b}=2$, $\frac{abc+\beta a+\alpha c}{bc+\beta}=\frac{7}{5}$ prima propius accedit ad verum valorem $\frac{7}{5}$ quam secunda. F. R.

Ac prima quidem $\frac{1}{0}$ praetermissa differentia inter secundam ac tertiam est

$$=\frac{\alpha}{b},$$

quarta a tertia subtracta relinquit

$$\frac{\alpha\beta}{b(bc+\beta)},$$

quarta a quinta subtracta relinquit

$$\frac{\alpha\beta\gamma}{(bc+\beta)(bcd+\beta d+\gamma b)}$$

etc. Hinc exprimetur valor fractionis continuae per seriem terminorum consuetam hoc modo, ut sit

$$x=a+\frac{\alpha}{b}-\frac{\alpha\beta}{b(bc+\beta)}+\frac{\alpha\beta\gamma}{(bc+\beta)(bcd+\beta d+\gamma b)}-\text{etc.},$$

quae series toties abrumpitur, quoties fractio continua non in infinitum progreditur.

364. Modum ergo invenimus fractionem continuam quamcunque in seriem terminorum, quorum signa alternantur, convertendi, siquidem prima littera a evanescat. Si enim fuerit

$$x=\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\cfrac{\varepsilon}{f+\text{etc.}}}}}}$$

erit per ea, quae modo invenimus,

$$x=\frac{\alpha}{b}-\frac{\alpha\beta}{b(bc+\beta)}+\frac{\alpha\beta\gamma}{(bc+\beta)(bcd+\beta d+\gamma b)}$$
$$-\frac{\alpha\beta\gamma\delta}{(bcd+\beta d+\gamma b)(bcde+\beta de+\gamma be+\delta bc+\beta\delta)}+\text{etc.}$$

Unde, si α, β, γ, δ etc. fuerint numeri non crescentes, uti omnes unitates, denominatores vero a, b, c, d etc. numeri integri quicunque affirmativi, valor fractionis continuae exprimetur per seriem terminorum maxime convergentem.

365. His probe consideratis poterit vicissim series quaecunque terminorum alternantium in fractionem continuam converti seu fractio continua inveniri, cuius valor aequalis sit summae seriei propositae. Sit enim proposita haec series

$$x=A-B+C-D+E-F+\text{etc.};$$

erit singulis terminis cum serie ex fractione continua orta comparandis

$$A=\frac{\alpha}{b}, \qquad \text{unde fit} \quad \alpha=Ab,$$

$$\frac{B}{A}=\frac{\beta}{bc+\beta}, \qquad \beta=\frac{Bbc}{A-B},$$

$$\frac{C}{B}=\frac{\gamma b}{bcd+\beta d+\gamma b}, \qquad \gamma=\frac{Cd(bc+\beta)}{b(B-C)},$$

$$\frac{D}{C}=\frac{\delta(bc+\beta)}{bcde+\beta de+\gamma be+\delta bc+\beta\delta} \qquad \delta=\frac{De(bcd+\beta d+\gamma b)}{(bc+\beta)(C-D)}$$

etc., etc.

At cum sit $\beta=\frac{Bbc}{A-B}$, erit

$$bc+\beta=\frac{Abc}{A-B},$$

unde

$$\gamma=\frac{ACcd}{(A-B)(B-C)}.$$

Porro fit

$$bcd+\beta d+\gamma b=(bc+\beta)d+\gamma b=\frac{Abcd}{A-B}+\frac{ACbcd}{(A-B)(B-C)}=\frac{ABbcd}{(A-B)(B-C)},$$

unde erit

$$\frac{bcd+\beta d+\gamma b}{bc+\beta}=\frac{Bd}{B-C}$$

et

$$\delta=\frac{BDde}{(B-C)(C-D)}.$$

Simili modo reperietur

$$\varepsilon=\frac{CEef}{(C-D)(D-E)}$$

et ita porro.

366. Quo ista lex clarius appareat, ponamus esse

$$P = b,$$
$$Q = bc + \beta,$$
$$R = bcd + \beta d + \gamma b,$$
$$S = bcde + \beta de + \gamma be + \delta bc + \beta\delta,$$
$$T = bcdef + \text{etc.},$$
$$V = bcdefg + \text{etc.}$$
$$\text{etc.};$$

erit ex lege harum expressionum

$$Q = Pc + \beta,$$
$$R = Qd + \gamma P,$$
$$S = Re + \delta Q,$$
$$T = Sf + \varepsilon R,$$
$$V = Tg + \zeta S$$
$$\text{etc.}$$

Cum igitur his adhibendis litteris fit

$$x = \frac{\alpha}{P} - \frac{\alpha\beta}{PQ} + \frac{\alpha\beta\gamma}{QR} - \frac{\alpha\beta\gamma\delta}{RS} + \frac{\alpha\beta\gamma\delta\varepsilon}{ST} - \text{etc.}$$

367. Quoniam ergo ponimus esse

$$x = A - B + C - D + E - F + \text{etc.},$$

erit

$$A = \frac{\alpha}{P}, \quad \alpha = AP,$$
$$\frac{B}{A} = \frac{\beta}{Q}, \quad \beta = \frac{BQ}{A},$$
$$\frac{C}{B} = \frac{\gamma P}{R}, \quad \gamma = \frac{CR}{BP},$$
$$\frac{D}{C} = \frac{\delta Q}{S}, \quad \delta = \frac{DS}{CQ},$$
$$\frac{E}{D} = \frac{\varepsilon R}{T}, \quad \varepsilon = \frac{ET}{DR}$$
$$\text{etc.}$$

Porro vero differentiis sumendis habebitur

$$A - B = \frac{\alpha(Q-\beta)}{PQ} = \frac{\alpha c}{Q} = \frac{APc}{Q},$$
$$B - C = \frac{\alpha\beta(R-\gamma P)}{PQR} = \frac{\alpha\beta d}{PR} = \frac{BQd}{R},$$
$$C - D = \frac{\alpha\beta\gamma(S-\delta Q)}{QRS} = \frac{\alpha\beta\gamma e}{QS} = \frac{CRe}{S},$$
$$D - E = \frac{\alpha\beta\gamma\delta(T-\varepsilon R)}{RST} = \frac{\alpha\beta\gamma\delta f}{RT} = \frac{DSf}{T}$$
$$\text{etc.}$$

Si bini igitur in se invicem ducantur, fiet

$$(A-B)(B-C) = ABcd\,\frac{P}{R} \quad \text{et} \quad \frac{R}{P} = \frac{ABcd}{(A-B)(B-C)},$$
$$(B-C)(C-D) = BCde\,\frac{Q}{S} \quad \text{et} \quad \frac{S}{Q} = \frac{BCde}{(B-C)(C-D)},$$
$$(C-D)(D-E) = CDef\,\frac{R}{T} \quad \text{et} \quad \frac{T}{R} = \frac{CDef}{(C-D)(D-E)}$$
$$\text{etc.}$$

Unde, cum sit $P = b$, $Q = \frac{\alpha c}{A - B} = \frac{Abc}{A - B}$, erit

$$\alpha = Ab,$$

$$\beta = \frac{Bbc}{A - B},$$

$$\gamma = \frac{ACcd}{(A - B)(B - C)},$$

$$\delta = \frac{BDde}{(B - C)(C - D)},$$

$$\varepsilon = \frac{CEef}{(C - D)(D - E)}$$

etc.

368. Inventis ergo valoribus numeratorum $\alpha, \beta, \gamma, \delta$ etc. denominatores b, c, d, e etc. arbitrio nostro relinquuntur; ita autem eos assumi convenit, ut cum ipsi sint numeri integri, tum valores integros pro $\alpha, \beta, \gamma, \delta$ etc. exhibeant. Hoc vero pendet quoque a natura numerorum A, B, C etc., utrum sint integri an fracti. Ponamus esse numeros integros atque quaesito satisfiet statuendo

$$\begin{array}{lll} b = 1, & \text{unde fit} & \alpha = A, \\ c = A - B, & & \beta = B, \\ d = B - C, & & \gamma = AC, \\ e = C - D, & & \delta = BD, \\ f = D - E & & \varepsilon = CE \\ \text{etc.,} & & \text{etc.} \end{array}$$

Quocirca si fuerit

$$x = A - B + C - D + E - F + \text{etc.},$$

idem ipsius x valor per fractionem continuam ita exprimi poterit, ut sit

$$x = \cfrac{A}{1 + \cfrac{B}{A - B + \cfrac{AC}{B - C + \cfrac{BD}{C - D + \cfrac{CE}{D - E + \text{etc.}}}}}}$$

369. Sin autem omnes termini seriei sint numeri fracti, ita ut fuerit

$$x = \frac{1}{A} - \frac{1}{B} + \frac{1}{C} - \frac{1}{D} + \frac{1}{E} - \text{etc.},$$

habebuntur pro $\alpha, \beta, \gamma, \delta$ etc. sequentes valores

$$\alpha = \frac{b}{A},$$

$$\beta = \frac{Abc}{B - A},$$

$$\gamma = \frac{B^2cd}{(B - A)(C - B)},$$

$$\delta = \frac{C^2de}{(C - B)(D - C)},$$

$$\varepsilon = \frac{D^2ef}{(D - C)(E - D)}$$

etc.

Ponatur ergo, ut sequitur,

$$\begin{array}{lll} b = A, & \text{unde fit} & \alpha = 1, \\ c = B - A, & & \beta = AA, \\ d = C - B, & & \gamma = BB, \\ e = D - C & & \delta = CC \\ \text{etc.,} & & \text{etc.} \end{array}$$

eritque per fractionem continuam

$$x = \cfrac{1}{A + \cfrac{AA}{B - A + \cfrac{BB}{C - B + \cfrac{CC}{D - C + \text{etc.}}}}}$$

EXEMPLUM 1

Transformetur haec series infinita

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \text{etc.}$$

in fractionem continuam.

Erit ergo

$$A = 1, \quad B = 2, \quad C = 3, \quad D = 4 \quad \text{etc.},$$

atque cum seriei propositae valor sit $= l2$, erit

$$l2 = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{4}{1 + \cfrac{9}{1 + \cfrac{16}{1 + \cfrac{25}{1 + \text{etc.}}}}}}}$$

EXEMPLUM 2

Transformetur haec series infinita [§ 140]

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \text{etc.},$$

ubi π denotat peripheriam circuli, cuius diameter $= 1$, in fractionem continuam.

Substitutis loco A, B, C, D etc. numeris 1, 3, 5, 7 etc. orietur

$$\frac{\pi}{4} = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \text{etc.}}}}}}$$

hincque invertendo fractionem erit

$$\frac{4}{\pi} = 1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \text{etc.}}}}}$$

quae est expressio, quam BROUNCKERUS[1]) primum pro quadratura circuli protulit.

1) Hanc celebrem fractionem continuam W. BROUNCKER (1620—1684) epistola cum J. WALLIS sine demonstratione communicaverat. Vide J. WALLIS, *Arithmetica infinitorum*, Oxoniae 1655, p. 182; *Opera mathematica*, t. I, Oxoniae 1695, p. 355, imprimis p. 469. Vide etiam *LEONHARDI EULERI Opera omnia*, series I, vol. 1, p. 507. F. R.

EXEMPLUM 3

Sit proposita ista series infinita

$$x = \frac{1}{m} - \frac{1}{m+n} + \frac{1}{m+2n} - \frac{1}{m+3n} + \text{etc.},$$

quae ob

$$A = m, \quad B = m + n, \quad C = m + 2n \quad \text{etc.}$$

in hanc fractionem continuam mutatur

$$x = \cfrac{1}{m + \cfrac{mm}{n + \cfrac{(m+n)^2}{n + \cfrac{(m+2n)^2}{n + \cfrac{(m+3n)^2}{n + \text{etc.}}}}}}$$

ex qua fit invertendo

$$\frac{1}{x} - m = \cfrac{mm}{n + \cfrac{(m+n)^2}{n + \cfrac{(m+2n)^2}{n + \cfrac{(m+3n)^2}{n + \text{etc.}}}}}$$

EXEMPLUM 4

Quoniam supra (§ 178) invenimus esse

$$\frac{\pi \cos. \frac{m\pi}{n}}{n \sin. \frac{m\pi}{n}} = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \text{etc.},$$

erit pro fractione continuanda

$$A = m, \quad B = n - m, \quad C = n + m, \quad D = 2n - m \quad \text{etc.},$$

unde fiet

$$\frac{\pi \cos. \frac{m\pi}{n}}{n \sin. \frac{m\pi}{n}} = \cfrac{1}{m + \cfrac{mm}{n - 2m + \cfrac{(n-m)^2}{2m + \cfrac{(n+m)^2}{n - 2m + \cfrac{(2n-m)^2}{2m + \cfrac{(2n+m)^2}{n - 2m + \text{etc.}}}}}}}$$

370. Si series proposita per continuos factores progrediatur, ut sit

$$x = \frac{1}{A} - \frac{1}{AB} + \frac{1}{ABC} - \frac{1}{ABCD} + \frac{1}{ABCDE} - \text{etc.},$$

tum prodibunt sequentes determinationes

$$\alpha = \frac{b}{A},$$

$$\beta = \frac{bc}{B-1},$$

$$\gamma = \frac{Bcd}{(B-1)(C-1)},$$

$$\delta = \frac{Cde}{(C-1)(D-1)},$$

$$\varepsilon = \frac{Def}{(D-1)(E-1)}$$

etc.

Fiat ergo, ut sequitur,

$$b = A, \quad \text{unde fit} \quad \alpha = 1,$$
$$c = B - 1, \qquad \beta = A,$$
$$d = C - 1, \qquad \gamma = B,$$
$$e = D - 1, \qquad \delta = C,$$
$$f = E - 1 \qquad \varepsilon = D$$
$$\text{etc.}, \qquad \text{etc.},$$

unde consequenter fiet

$$x = \cfrac{1}{A + \cfrac{A}{B - 1 + \cfrac{B}{C - 1 + \cfrac{C}{D - 1 + \cfrac{D}{E - 1 + \text{etc.}}}}}}$$

EXEMPLUM 1

Quoniam posito e numero, cuius logarithmus est $=1$, supra [§ 123] invenimus esse

$$\frac{1}{e} = 1 - \frac{1}{1} + \frac{1}{1\cdot 2} - \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} - \text{etc.}$$

seu

$$1 - \frac{1}{e} = \frac{1}{1} - \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} - \frac{1}{1\cdot 2\cdot 3\cdot 4} + \text{etc.},$$

haec series in fractionem continuam convertetur ponendo

$$A = 1, \quad B = 2, \quad C = 3, \quad D = 4 \quad \text{etc.};$$

quo ergo facto habebitur

$$1 - \frac{1}{e} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + \text{etc.}}}}}}}$$

unde asymmetria initio reiecta erit

$$\frac{1}{e-1} = \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + \text{etc.}}}}}}$$

EXEMPLUM 2

Invenimus quoque arcus, qui radio aequalis sumitur, cosinum esse [§ 134]

$$= 1 - \frac{1}{2} + \frac{1}{2\cdot 12} - \frac{1}{2\cdot 12\cdot 30} + \frac{1}{2\cdot 12\cdot 30\cdot 56} - \text{etc.}$$

Si ergo fiat

$$A = 1, \quad B = 2, \quad C = 12, \quad D = 30, \quad E = 56 \quad \text{etc.}$$

atque cosinus arcus, qui radio aequatur, ponatur $= x$, erit

$$x = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{2}{11 + \cfrac{12}{29 + \cfrac{30}{55 + \text{etc.}}}}}}$$

seu

$$\frac{1}{x} - 1 = \cfrac{1}{1 + \cfrac{2}{11 + \cfrac{12}{29 + \cfrac{30}{55 + \text{etc.}}}}}$$

371. Sit series insuper cum geometrica coniuncta, scilicet

$$x = A - Bz + Cz^2 - Dz^3 + Ez^4 - Fz^5 + \text{etc.};$$

erit

$$\alpha = Ab,$$

$$\beta = \frac{Bbcz}{A - Bz},$$

$$\gamma = \frac{ACcdz}{(A - Bz)(B - Cz)},$$

$$\delta = \frac{BDdez}{(B - Cz)(C - Dz)},$$

$$\varepsilon = \frac{CEefz}{(C - Dz)(D - Ez)}$$

etc.

Ponatur nunc

$$\begin{array}{ll} b = 1, & \text{erit} \quad \alpha = A, \\ c = A - Bz, & \beta = Bz, \\ d = B - Cz, & \gamma = ACz, \\ e = C - Dz & \delta = BDz \\ \text{etc.,} & \text{etc.,} \end{array}$$

unde fiet

$$x = \cfrac{A}{1 + \cfrac{Bz}{A - Bz + \cfrac{ACz}{B - Cz + \cfrac{BDz}{C - Dz + \text{etc.}}}}}$$

372. Quo autem hoc negotium generalius absolvamus, ponamus esse

$$x = \frac{A}{L} - \frac{By}{Mz} + \frac{Cy^2}{Nz^2} - \frac{Dy^3}{Oz^3} + \frac{Ey^4}{Pz^4} - \text{etc.}$$

fietque comparatione instituta

$$\alpha = \frac{Ab}{L},$$

$$\beta = \frac{BLbcy}{AMz - BLy},$$

$$\gamma = \frac{ACM^2cdyz}{(AMz - BLy)(BNz - CMy)},$$

$$\delta = \frac{BDN^2deyz}{(BNz - CMy)(COz - DNy)}$$

etc.

Statuantur valores b, c, d etc. sequenti modo

$$\begin{array}{ll} b = L, & \text{erit} \quad \alpha = A, \\ c = AMz - BLy, & \beta = BLLy, \\ d = BNz - CMy, & \gamma = ACM^2yz, \\ e = COz - DNy, & \delta = BDN^2yz, \\ f = DPz - EOy & \varepsilon = CEO^2yz \\ \text{etc.;} & \text{etc.,} \end{array}$$

unde series proposita per sequentem fractionem continuam exprimetur

$$x = \cfrac{A}{L + \cfrac{BLLy}{AMz - Bly + \cfrac{ACMMyz}{BNz - CMy + \cfrac{BDNNyz}{COz - DNy + \text{etc.}}}}}$$

373. Habeat denique series proposita huiusmodi formam

$$x = \frac{A}{L} - \frac{ABy}{LMz} + \frac{ABCy^2}{LMNz^2} - \frac{ABCDy^3}{LMNOz^3} + \text{etc.}$$

atque sequentes valores prodibunt

$$\alpha = \frac{Ab}{L},$$

$$\beta = \frac{Bbcy}{Mz - By},$$

$$\gamma = \frac{CMcdyz}{(Mz - By)(Nz - Cy)},$$

$$\delta = \frac{DNdeyz}{(Nz - Cy)(Oz - Dy)},$$

$$\varepsilon = \frac{EOefyz}{(Oz - Dy)(Pz - Ey)}$$

etc.

Ad valores ergo integros inveniendos fiat

$$\begin{array}{lll} b = Lz, & \text{erit} & \alpha = Az, \\ c = Mz - By, & & \beta = BLyz, \\ d = Nz - Cy, & & \gamma = CMyz, \\ e = Oz - Dy, & & \delta = DNyz, \\ f = Pz - Ey & & \varepsilon = EOyz \\ \text{etc.}; & & \text{etc.}, \end{array}$$

unde valor seriei propositae ita exprimetur, ut sit

$$x = \cfrac{Az}{Lz + \cfrac{BLyz}{Mz - By + \cfrac{CMyz}{Nz - Cy + \cfrac{DNyz}{Oz - Dy + \text{etc.}}}}}$$

vel, ut lex progressionis statim a principio fiat manifesta, erit

$$\frac{Az}{x} - Ay = Lz - Ay + \cfrac{BLyz}{Mz - By + \cfrac{CMyz}{Nz - Cy + \cfrac{DNyz}{Oz - Dy + \text{etc.}}}}$$

374. Hoc modo innumerabiles inveniri poterunt fractiones continuae in infinitum progredientes, quarum valor verus exhiberi queat. Cum enim ex supra traditis infinitae series, quarum summae constent, ad hoc negotium accommodari queant, unaquaeque transformari poterit in fractionem continuam, cuius adeo valor summae illius seriei est aequalis. Exempla, quae iam hic sunt allata, sufficiunt ad hunc usum ostendendum. Verumtamen optandum esset, ut methodus detegeretur, cuius beneficio, si proposita fuerit fractio continua quaecunque, eius valor immediate inveniri posset. Quanquam enim fractio continua transmutari potest in seriem infinitam, cuius summa per methodos cognitas investigari queat, tamen plerumque istae series tantopere fiunt intricatae, ut earum summa, etiamsi sit satis simplex, vix ac ne vix quidem obtineri possit.

375. Quo autem clarius perspiciatur dari eiusmodi fractiones continuas, quarum valor aliunde facile assignari queat, etiamsi ex seriebus infinitis, in quas convertuntur, nihil admodum colligere liceat, consideremus hanc fractionem continuam

$$x = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \text{etc.}}}}}$$

cuius omnes denominatores sunt inter se aequales. Si enim hinc modo supra exposito fractiones formemus

$$\begin{array}{ccccccc} 0 & 2 & 2 & 2 & 2 & 2 & 2 \\ \frac{1}{0}, & \frac{0}{1}, & \frac{1}{2}, & \frac{2}{5}, & \frac{5}{12}, & \frac{12}{29}, & \frac{29}{70} \end{array} \text{ etc.},$$

oritur haec series

$$x = 0 + \frac{1}{2} - \frac{1}{2 \cdot 5} + \frac{1}{5 \cdot 12} - \frac{1}{12 \cdot 29} + \frac{1}{29 \cdot 70} - \text{etc.}$$

vel, si bini termini coniungantur, erit

$$x = \frac{2}{1 \cdot 5} + \frac{2}{5 \cdot 29} + \frac{2}{29 \cdot 169} + \text{etc.}$$

vel

$$x = \frac{1}{2} - \frac{2}{2 \cdot 12} - \frac{2}{12 \cdot 70} - \text{etc.}$$

Quin etiam, cum sit

$$x=\frac{1}{4}-\frac{1}{2\cdot 2\cdot 5}+\frac{1}{2\cdot 5\cdot 12}-\frac{1}{2\cdot 12\cdot 29}+\text{etc.}$$

$$+\frac{1}{4}-\frac{1}{2\cdot 2\cdot 5}+\frac{1}{2\cdot 5\cdot 12}-\frac{1}{2\cdot 12\cdot 29}+\text{etc.},$$

erit

$$x=\frac{1}{4}+\frac{1}{1\cdot 5}-\frac{1}{2\cdot 12}+\frac{1}{5\cdot 29}-\frac{1}{12\cdot 70}+\text{etc.};$$

quae series etiamsi vehementer convergant, tamen vera earum summa ex earum forma colligi nequit.

376. Pro huiusmodi autem fractionibus continuis, in quibus denominatores omnes vel sunt aequales vel iidem revertuntur, ita ut ea fractio, si ab initio aliquot terminis truncetur, toti adhuc sit aequalis, facilis habetur modus earum summas explorandi. In exemplo enim proposito cum sit

$$x=\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\text{etc.}}}}}$$

erit

$$x=\frac{1}{2+x}$$

ideoque

$$xx+2x=1$$

et

$$x+1=\sqrt{2},$$

ita ut valor huius fractionis continuae sit

$$=\sqrt{2}-1.$$

Fractiones vero ex fractione continua ante erutae continuo propius ad hunc valorem accedunt idque tam cito, ut vix promptior modus ad valorem hunc irrationalem per numeros rationales proxime exprimendum inveniri queat.

Est enim $\sqrt{2}-1$ tam prope $\frac{29}{70}$, ut error sit insensibilis; namque radicem extrahendo erit

$$\sqrt{2}-1=0{,}41421356237$$

atque

$$\frac{29}{70}=0{,}41428571428,$$

ita ut error tantum in partibus centesimis millesimis consistat.

377. Quemadmodum ergo fractiones continuae commodissimum suppeditant modum ad valorem $\sqrt{2}$ appropinquandi, ita indidem facillima via aperitur ad radices aliorum numerorum proxime investigandas. Ponamus hunc in finem

$$x=\cfrac{1}{a+\cfrac{1}{a+\cfrac{1}{a+\cfrac{1}{a+\cfrac{1}{a+\text{etc.}}}}}}$$

erit

$$x=\frac{1}{a+x}$$

et

$$xx+ax=1,$$

unde fit

$$x=-\frac{1}{2}a+\sqrt{\left(1+\frac{1}{4}aa\right)}=\frac{\sqrt{(aa+4)}-a}{2}.$$

Haec ergo fractio continua inserviet valori radicis quadratae ex numero $aa+4$ inveniendo. Hincque adeo substituendo loco a successive numeros 1, 2, 3, 4 etc. reperientur $\sqrt{5}$, $\sqrt{2}$, $\sqrt{13}$, $\sqrt{5}$, $\sqrt{29}$, $\sqrt{10}$, $\sqrt{53}$ etc., perductis scilicet his radicibus ad formam simplicissimam. Erit ergo

$$\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ \frac{0}{1}, & \frac{1}{1}, & \frac{1}{2}, & \frac{2}{3}, & \frac{3}{5}, & \frac{5}{8} \end{array} \text{ etc.} = \frac{\sqrt{5}-1}{2},$$

$$\begin{array}{cccccc} 2 & 2 & 2 & 2 & 2 & 2 \\ \frac{0}{1}, & \frac{1}{2}, & \frac{2}{5}, & \frac{5}{12}, & \frac{12}{29}, & \frac{29}{70} \end{array} \text{ etc.} = \sqrt{2}-1,$$

$$\begin{array}{cccccc} 3 & 3 & 3 & 3 & 3 & 3 \\ \frac{0}{1}, & \frac{1}{3}, & \frac{3}{10}, & \frac{10}{33}, & \frac{33}{109}, & \frac{109}{360} \end{array} \text{ etc.} = \frac{\sqrt{13}-3}{2},$$

$$\begin{array}{cccccc} 4 & 4 & 4 & 4 & 4 & 4 \\ \frac{0}{1}, & \frac{1}{4}, & \frac{4}{17}, & \frac{17}{72}, & \frac{72}{305}, & \frac{305}{1292} \end{array} \text{ etc.} = \sqrt{5}-2$$

etc.

Notandum autem eo promptiorem esse approximationem, quo maior fuerit numerus a. Sic in ultimo exemplo erit

$$\sqrt{5} = 2\,\frac{305}{1292},$$

ut error minor sit quam $\frac{1}{1292 \cdot 5473}$, ubi 5473 est denominator sequentis fractionis $\frac{1292}{5473}$.

378. Hoc vero modo aliorum numerorum radices exhiberi nequeunt, nisi qui sint summa duorum quadratorum. Ut igitur haec approximatio ad alios numeros extendatur, ponamus esse

$$x = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{a + \cfrac{1}{b + \text{etc.}}}}}}}$$

Erit

$$x = \cfrac{1}{a + \cfrac{1}{b + x}} = \frac{b + x}{ab + 1 + ax}$$

ideoque

$$axx + abx = b$$

et

$$x = -\frac{1}{2}b \pm \sqrt{\left(\frac{1}{4}bb + \frac{b}{a}\right)} = \frac{-ab + \sqrt{(aabb + 4ab)}}{2a}.$$

Unde iam omnium numerorum radices inveniri poterunt. Sit verbi gratia $a = 2$, $b = 7$; erit

$$x = \frac{-14 + \sqrt{14 \cdot 18}}{4} = \frac{-7 + 3\sqrt{7}}{2}.$$

At valorem ipsius x proxime exhibebunt sequentes fractiones

$$\begin{array}{cccccc} 2 & 7 & 2 & 7 & 2 & 7 \\ \frac{0}{1}, & \frac{1}{2}, & \frac{7}{15}, & \frac{15}{32}, & \frac{112}{239}, & \frac{239}{510} \end{array} \text{ etc.}$$

Erit ergo proxime

$$\frac{-7 + 3\sqrt{7}}{2} = \frac{239}{510}$$

et

$$\sqrt{7} = \frac{2024}{765} = 2{,}64575163;$$

at revera est

$$\sqrt{7} = 2{,}64575131;$$

ita ut error minor sit quam $\frac{33}{100000000}$.

379. Progrediamur autem ulterius ponendo

$$x = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{a + \text{etc.}}}}}}}}$$

Erit

$$x = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + x}}} = \cfrac{1}{a + \cfrac{c + x}{bx + bc + 1}} = \frac{bx + bc + 1}{(ab + 1)x + abc + a + c},$$

unde

$$(ab+1)xx+(abc+a-b+c)x=bc+1$$

atque

$$x=\frac{-abc-a+b-c+\sqrt{((abc+a+b+c)^2+4)}}{2(ab+1)},$$

ubi quantitas post signum radicale posita iterum est summa duorum quadratorum; neque ergo haec forma radicibus ex aliis numeris extrahendis inservit, nisi ad quos prima forma iam suffecerat. Simili modo si quatuor litterae a, b, c, d continuo repetitae denominatores fractionis continuae constituant, tum ea plus non inserviet quam secunda, quae duas tantum litteras continebat, et ita porro.

380. Cum igitur fractiones continuae tam utiliter ad extractionem radicis quadratae adhiberi queant, simul inservient aequationibus quadratis resolvendis; quod quidem ex ipso calculo est manifestum, dum x per aequationem quadraticam affectam determinatur. Potest autem vicissim facile cuiusque aequationis quadratae radix per fractionem continuam hoc modo exprimi. Sit proposita ista aequatio

$$xx=ax+b;$$

ex qua cum sit

$$x=a+\frac{b}{x},$$

substituatur in ultimo termino loco x valor idem iam inventus eritque

$$x=a+\cfrac{b}{a+\cfrac{b}{x}}$$

simili ergo modo procedendo erit per fractionem continuam infinitam

$$x=a+\cfrac{b}{a+\cfrac{b}{a+\cfrac{b}{a+\text{etc.}}}}$$

quae autem, cum numeratores b non sint unitates, non tam commode adhiberi potest.

381. Ut autem usus in arithmetica ostendatur, primum notandum est omnem fractionem ordinariam in fractionem continuam converti posse. Sit enim proposita fractio

$$x=\frac{A}{B},$$

in qua sit $A>B$; dividatur A per B sitque quotus $=a$ et residuum C; tum per hoc residuum C dividatur praecedens divisor B prodeatque quotus b et relinquatur residuum D, per quod denuo praecedens divisor C dividatur; sicque haec operatio, quae vulgo ad maximum communem divisorem numerorum A et B investigandum usurpari solet, continuetur, donec ipsa finiatur, sequenti modo:

$$\begin{array}{l} B)\ \underline{A}\ (a \\ \quad C)\ \underline{B}\ (b \\ \qquad D)\ \underline{C}\ (c \\ \qquad\quad E)\ \underline{D}\ (d \\ \qquad\qquad\quad F \ \text{etc.} \end{array}$$

Eritque per naturam divisionis

$$\begin{array}{lll} A=aB+C, \quad \text{unde} & \dfrac{A}{B}=a+\dfrac{C}{B}, & \\ B=bC+D, & \dfrac{B}{C}=b+\dfrac{D}{C}, & \dfrac{C}{B}=\dfrac{1}{b+\frac{D}{C}}, \\ C=cD+E, & \dfrac{C}{D}=c+\dfrac{E}{D}, & \dfrac{D}{C}=\dfrac{1}{c+\frac{E}{D}}, \\ D=dE+F & \dfrac{D}{E}=d+\dfrac{F}{E}, & \dfrac{E}{D}=\dfrac{1}{d+\frac{F}{E}} \\ \text{etc.}, & \text{etc.} & \end{array}$$

Hinc sequentes valores in praecedentibus substituendo erit

$$x=\frac{A}{B}=a+\frac{C}{B}=a+\cfrac{1}{b+\cfrac{D}{C}}=a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{E}{D}}}$$

unde tandem x per meros quotos inventos a, b, c, d etc. sequentem in modum exprimetur, ut sit

$$x = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e + \cfrac{1}{f + \text{etc.}}}}}}$$

EXEMPLUM 1

Sit proposita ista fractio $\frac{1461}{59}$, quae sequenti modo in fractionem continuam transmutabitur, cuius omnes numeratores erunt unitates.

Instituatur scilicet eadem operatio, qua maximus communis divisor numerorum 59 et 1461 quaeri solet:

```
59) 1461 (24
    118
    ---
     281
     236
     ---
      45) 59 (1
          45
          --
          14) 45 (3
              42
              --
               3) 14 (4
                  12
                  --
                   2) 3 (1
                      2
                      -
                      1) 2 (2
                         2
                         -
                         0
```

Hinc ergo ex quotis fiet

$$\frac{1461}{59} = 24 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{2}}}}}$$

EXEMPLUM 2

Fractiones quoque decimales eodem modo transmutari poterunt.

Sit enim proposita

$$\sqrt{2} = 1{,}41421356 = \frac{141421356}{100000000},$$

unde haec operatio instituatur

100000000	141421356	1
82842712	100000000	2
17157288	41421356	2
14213560	34314576	2
2943728	7106780	2
2438648	5887456	2
505080	1219324	2
418328	1010160	2
86752	209164	

etc.

Ex qua operatione iam patet omnes denominatores esse 2 atque adeo esse

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\text{etc.}}}}}}$$

cuius expressionis ratio iam ex superioribus patet.

EXEMPLUM 3

Imprimis vero etiam hic attentione dignus est numerus e, cuius logarithmus est $= 1$, qui est

$$e = 2{,}718281828459.$$

Unde oritur

$$\frac{e-1}{2} = 0{,}8591409142295,$$

quae fractio decimalis, si superiori modo tractetur, dabit quotos sequentes

8591409142295	10000000000000	1
8451545146224	8591409142295	6
139863996071	1408590857704[1]	10
139312557916	1398639960710	14
551438155	9950896994	18
550224488	9925886790	22
1213667	25010204	

etc.

Si iste calculus exactius adhuc assumpto valore ipsius e ulterius continuetur, tum prodibunt isti quoti

1, 6, 10, 14, 18, 22, 26, 30, 34 etc.,

qui dempto primo progressionem arithmeticam constituunt, unde patet fore

$$\frac{e-1}{2} = \cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\cfrac{1}{18+\cfrac{1}{22+\cfrac{1}{\text{etc.}}}}}}}}$$

cuius fractionis ratio ex Calculo infinitesimali dari potest.[2])

382. Cum igitur ex huiusmodi expressionibus fractiones erui queant, quae quam citissime ad verum valorem expressionis deducant, haec methodus adhiberi poterit ad fractiones decimales per ordinarias fractiones, quae ad ipsas proxime accedant, exprimendas. Quin etiam, si fractio fuerit proposita, cuius numerator et denominator sint numeri valde magni, fractiones ex minoribus numeris constantes inveniri poterunt, quae, etiamsi propositae non sint penitus aequales, tamen ab ea quam minime discrepent. Hincque problema a Wallisio[1]) olim tractatum facile resolvi potest, quo quaeruntur fractiones minoribus numeris expressae, quae tam prope exhauriant valorem fractionis cuiuspiam in numeris maioribus propositae, quantum fieri poterit numeris non maioribus. Fractiones autem nostra hac methodo ortae tam prope ad valorem fractionis continuae, ex qua eliciuntur, accedunt, ut nullae numeris non maioribus constantes dentur, quae propius accedant.

1) Revera oritur 1408590857705, quo valore assumpto calculus sequens non mediocriter mutatur. F. R.

2) Fractionem istam Eulerus primum in Commentatione 71 nota p. 362 laudata exposuit. F. R.

EXEMPLUM 1

Exprimatur ratio diametri ad peripheriam numeris tam exiguis, ut accuratior exhiberi nequeat, nisi numeri maiores adhibeantur.

Si fractio decimalis cognita

3,1415926535 etc.

modo exposito per divisionem continuam evolvatur, reperientur sequentes quoti

3, 7, 15, 1, 292, 1, 1 etc.,

ex quibus sequentes fractiones formabuntur

$$\frac{1}{0},\ \frac{3}{1},\ \frac{22}{7},\ \frac{333}{106},\ \frac{355}{113},\ \frac{103993}{33102}\ \text{etc.}$$

Secunda fractio iam ostendit esse diametrum ad peripheriam ut 1 : 3 neque certe numeris non maioribus accuratius dari poterit. Tertia fractio dat rationem Archimedeam[2]) 7 : 22, at quinta Metianam[3]), quae ad verum tam prope accedit, ut error minor sit parte $\frac{1}{113 \cdot 33102}$. Ceterum hae fractiones alternatim vero sunt maiores minoresque.

1) J. Wallis, *Opera mathematica* t. II, Oxoniae 1693, cap. 98—99, p. 418—429. Cf. quoque cap. 56—61, p. 232—250. F. R.

2) Archimedes (287—212 a. Chr. n.), *Opera* (ed. J. L. Heiberg), vol. I, Lipsiae 1880, p. 257. F. R.

3) Adriaan Antoniszoon, vulgo cognomine Metius appellatus (1527—1607). Vide exempli gratia filii Adriani Metii (1571—1635) librum, qui inscribitur *Arithmeticae libri duo et Geometriae libri VI*, Lugd. Batav. 1626, Geometriae pars prior, cap. X. F. R.

EXEMPLUM 2

Exprimatur ratio diei ad annum solarem medium in numeris minimis proxime.

Cum annus iste sit $365^d\,5^h\,48'\,55''$, continebit in fractione annus unus

$$365\,\frac{20935}{86400}$$

dies. Tantum ergo opus est, ut haec fractio evolvatur, quae dabit sequentes quotos

$$4,\ 7,\ 1,\ 6,\ 1,\ 2,\ 2,\ 4,$$

unde istae eliciuntur fractiones

$$\frac{0}{1},\ \frac{1}{4},\ \frac{7}{29},\ \frac{8}{33},\ \frac{55}{227},\ \frac{63}{260},\ \frac{181}{747}\ \text{etc.}$$

Horae ergo cum minutis primis et secundis, quae supra 365 dies adsunt, quatuor annis unum diem circiter faciunt, unde calendarium JULIANUM originem habet. Exactius autem 33 annis 8 dies implentur vel 747 annis 181 dies; unde sequitur quadringentis annis abundare 97 dies. Quare, cum hoc intervallo calendarium JULIANUM inserat 100 dies, GREGORIANUM quaternis seculis tres annos bissextiles in communes convertit.

FINIS TOMI PRIMI.

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