On Polycosecant Numbers and Level Two Generalization of Arakawa-Kaneko Zeta Functions

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(多重余割数と，レベル 2 荒川ー金子ゼータ関数について)
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論 文 内 容 の 要 旨
Thesis Summary

The multiple zeta value is a real number associated to each index set $\boldsymbol{k} = (k_1, \ldots, k_r)$ given by

$$\zeta(\boldsymbol{k}) = \zeta(k_1, \ldots, k_r) = \sum_{0<m_1<\cdots<m_r} \frac{1}{m_1^{k_1} \ldots m_r^{k_r}},$$

where all k_i are integers greater than or equal to 1 and we assume $k_r \geq 2$ to make the series converges. This is a generalization of the Riemann zeta value

$$\zeta(k) = \sum_{n\geq 1} \frac{1}{n^k},$$

and Leonhard Euler was the first one to study the Riemann zeta values and the multiple zeta values in the case $r = 2$. Among his many amazing discoveries the identity $\zeta(1,2) = \zeta(3)$ is basic and generalized in surprisingly various ways. Later, Micheal Hoffman and Don Zagier independently and almost simultaneously initiated the study of multiple zeta values for general "depth" r. In recent years, multiple zeta values have been studied by many mathematicians of various backgrounds and they have found many remarkable relations among multiple zeta values. These numbers have applications in various context in number theory, geometry, arithmetic algebraic geometry, knot theory, mathematical physics, etc.

In 1999, Tsuneo Arakawa and Masanobu Kaneko [1] introduced a new function

$$\xi(k_1, \ldots, k_r; s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} Li_{k_1,\ldots,k_r}(1 - e^{-t})\, dt,$$

where $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 1},\ s \in \mathbb{C}$ with $Re(s) > 0$, as a generalization of the Riemann zeta function. Later this function is named as Arakawa-Kaneko zeta function. Arakawa-Kaneko zeta function provides a connection between multiple zeta values and poly-Bernoulli numbers. Kaneko and Hirofumi Tsumura (refer [3] and [4]) conducted a further study of the function $\xi(\boldsymbol{k}; s)$.

In our thesis, we study the level two generalization of Arakawa-Kaneko zeta function. Kaneko and Tsumura (see [5]) defined the level two analogue of $\xi(\boldsymbol{k}; s)$ as

$$\psi(k_1, \ldots, k_r; s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{\sinh t} A(k_1, \ldots, k_r; \tanh t/2)\, dt$$

for $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 1},\ s \in \mathbb{C}$ and $Re(s) > 0$. We provide several formulas analogue to those of Arakawa-Kaneko zeta function. We study the level two generalization [4] of the zeta function. Its normalized version is defined as

$$T(k_1, \ldots, k_{r-1}; s) = 2^r \sum_{\substack{0<m_1<\cdots<m_r \\ m_i \equiv i \bmod 2}} \frac{1}{m_1^{k_1} \cdots m_{r-1}^{k_{r-1}} m_r^s}.$$

The values $T(k_1, \ldots, k_{r-1}, k_r)$, ($k_j \in \mathbb{Z}_{\geq 1},\ k_r \geq 2$: admissible) are called the multiple T-values. We obtain several relations of ψ function and the multiple T-values.

Secondly, we introduce and study the level two analogue of poly-Bernoulli numbers which are named as polycosecant numbers $D_n^{(k)}$ (Sasaki 2012 [6]; Kaneko-M.-Tsumura [2]). For $k \in \mathbb{Z}$, polycosecant numbers are defined by

$$\frac{A_k(\tanh t/2)}{\sinh t} = \sum_{n=0}^{\infty} D_n^{(k)} \frac{t^n}{n!},$$

where $A_k(z)$ is the polylogarithm function of level two defined by

$$A_k(z) = 2\sum_{n=0}^{\infty} \frac{z^{(2n+1)}}{(2n+1)!} \qquad (z \in \mathbb{C};\ |z| < 1).$$

We give several relations among polycosecant numbers such as explicit formula, duality relation, etc. Further, we define multi-indexed polycosecant numbers and generalize the formulas for polycosecant numbers.

Our thesis consists of four main chapters. In Chapter 2, we review poly-Bernoulli numbers and Arakawa-Kaneko zeta functions. Mainly, we present the formulas that we want to generalize. In Chapter 3, we introduce the polycosecant numbers and the level two generalization of Arakawa-Kaneko zeta functions. We present our main results, obtained for the polycosecant numbers. In Chapter 4, we obtain several formulas for Arakawa-Kaneko zeta function of level two, corresponding to the formulas of the Arakawa-Kaneko zeta functions.

In the separately, submitted reference paper [5], we studied a slightly different type of sum called Mordell-Tornheim zeta values, which are defined by

$$\zeta_{MT,r}(k_1, k_2, \dots, k_r; k_{r+1}) = \sum_{m_1, m_2, \dots, m_r > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_r^{k_r} (m_1 + m_2 + \cdots + m_r)^{k_{r+1}}}.$$

Here, all k_i are positive integers. The Tornheim double sum was first introduced by Leonard Tornheim [7] in 1949 which is called. In 2002, K. Matsumoto defined the Mordell-Tornheim r −ple-zeta function. We present weighted sum formulas for double Mordell-Tornheim zeta values. Moreover, we present a sum formula for the Mordell-Tornheim series of even arguments.

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