On Polycosecant Numbers and Level Two Generalization of Arakawa-Kaneko Zeta Functions

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On Polycosecant Numbers and Level Two Generalization of Arakawa-Kaneko Zeta Functions

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Contents

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Chapter 1

Introduction

The multiple zeta value is one of the most fascinating number in the literature of mathematics. It is a real number associated to each index set $\mathbf{k} = (k_1, \dots, k_r)$ given by

$$\zeta(\mathbf{k}) = \zeta(k_1, k_2, \dots, k_r) = \sum_{0<m_1<m_2<\cdots<m_r} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_r^{k_r}},$$

where all k_i are integers greater than or equal to 1 and we assume $k_r \geq 2$ to make the series converges. This is a generalization of the Riemann zeta value

$$\zeta(k) = \sum_{n \geq 1} \frac{1}{n^k},$$

and Leonhard Euler was the first one to study the Riemann zeta values and the multiple zeta values in the case $r = 2$. Among his many amazing discoveries the identity $\zeta(1, 2) = \zeta(3)$ is basic and generalized in surprisingly various ways. Later, Micheal Hoffman and Don Zagier independently and almost simultaneously initiated the study of multiple zeta values for general "depth" r. In recent years, multiple zeta values have been studied by many mathematicians of various backgrounds and they have found many remarkable relations among multiple zeta values. These numbers have applications in various context in number theory, geometry, arithmetic algebraic geometry, knot theory, mathematical physics, etc.

In 1999, Tsuneo Arakawa and Masanobu Kaneko [2] introduced a new function

$$\xi(k_1, \dots, k_r; s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} Li_{k_1,\dots,k_r}(1 - e^{-t}) dt$$

where $r, k_1, \dots, k_r \in \mathbb{Z}_{\geq 1}$, $s \in \mathbb{C}$ with $Re(s) > 0$, $Li_{k_1,\dots,k_r}(z)$ is the multiple polylogarithm (see Definition 2.2 for the precise definition) as a generalization of the Riemann zeta function. Later this function is named as Arakawa-Kaneko zeta function. Arakawa-Kaneko zeta function provides a connection between multiple zeta values and poly-Bernoulli numbers. Kaneko and Hirofumi Tsumura (refer [12] and [11]) conducted a further study of the function $\xi(\mathbf{k};s)$.

In this thesis, we study the level two generalization of Arakawa-Kaneko zeta function. Kaneko and Tsumura (see [12, §5]) defined the level two analogue of $\xi(k_1,\dots,k_r;s)$ as

$$\psi(k_1,\dots,k_r;s)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{\mathrm{A}(k_1,\dots,k_r;\tanh t/2)}{\sinh(t)}dt$$

for $k_1,\dots,k_r\in\mathbb{Z}_{\geq 1}$ and $Re(s)>0$. The function $\mathrm{A}(k_1,\dots,k_r;z)$ is the level two analogue of $\mathrm{Li}_{k_1,\dots,k_r}(z)$ (see Definition 3.2). We provide several formulas analogue to those of Arakawa-Kaneko zeta function. We study the level two generalization ([12, §5]) of the zeta function. Its normalized version is defined as

$$T(k_1,\dots,k_{r-1},s)=2^r\sum_{\substack{0<m_1<\cdots<m_r\\ m_i\equiv i \pmod 2}}\frac{1}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}m_r^{s}}.$$

The values $T(k_1,\dots,k_{r-1},k_r)$, ($k_j\in\mathbb{Z}_{\geq 1}$, $k_r\geq 2$) are called the multiple T-values. We obtain several relations of ψ function and the multiple T-values.

Secondly, we introduce and study the level two analogue of poly-Bernoulli numbers which are named as polycosecant numbers $D_n^{(k)}$ (Yoshitaka Sasaki 2012 [16]; Masanobu Kaneko-M.-Hirofumi Tsumura 2019 [10]). For $k\in\mathbb{Z}$, the polycosecant numbers $D_n^{(k)}$ are defined by

$$\frac{\mathrm{A}_k(\tanh t/2)}{\sinh t}=\sum_{n=0}^\infty D_n^{(k)}\frac{t^n}{n!},$$

where $\mathrm{A}_k(z)$ is the polylogarithm function of level two defined by

$$\mathrm{A}_k(z)=2\sum_{n=0}^\infty\frac{z^{2n+1}}{(2n+1)^k}\quad(z\in\mathbb{C};\ |z|<1).$$

We give several relations among polycosecant numbers such as explicit formulas, duality relation, etc. Further, we define multi-indexed polycosecant numbers and generalize the formulas for polycosecant numbers.

This thesis consists of four main chapters. In Chapter 2, we review poly-Bernoulli numbers and Arakawa-Kaneko zeta functions. Mainly, we present the formulas that we want to generalize. In Chapter 3, we introduce the polycosecant numbers. We present our main results, obtained for the polycosecant numbers. We also give a congruence relation for finite T-values. In Chapter 4, we obtain certain formulas for Arakawa-Kaneko zeta function of level two, corresponding to the formulas of the original Arakawa-Kaneko zeta functions.

In the separately submitted reference paper [15], we study Mordell-Tornheim zeta values defined by

$$\zeta_{MT,r}(k_1,\ldots,k_r;k_{r+1}) = \sum_{m_1,\ldots,m_r>0} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}(m_1+\cdots+m_r)^{k_{r+1}}}$$

for positive integers. Mordell-Tornheim zeta values are a special class of multiple zeta values and can be written as a linear combination of multiple zeta values. In [15], we present weighted sum formulas for double Mordell-Tornheim zeta values. Moreover, we present a sum formula for the Mordell-Tornheim series of even arguments.

Chapter 2

Review of poly-Bernoulli numbers and Arakawa-Kaneko zeta functions

In this chapter, we recall some basic facts on poly-Bernoulli numbers, Arakawa-Kaneko zeta functions and multiple zeta values. Our main purpose is to generalize these theories to "level two" analogue.

2.1 Poly-Bernoulli numbers

Poly-Bernoulli numbers (Kaneko 1997; Arakawa-Kaneko 1999) have two versions, namely $B_n^{(k)}$ and $C_n^{(k)}$, which were defined by Kaneko in [7] and in Arakawa-Kaneko [2] by using generating series.

Definition 2.1 (Poly-Bernoulli numbers)**.** *For any integer $k \in \mathbb{Z}$, the sequences of rational numbers $\{B_n^{(k)}\}_{n\geq 0}$ and $\{C_n^{(k)}\}_{n\geq 0}$ are defined by*

$$\frac{Li_k(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!}$$

and

$$\frac{Li_k(1-e^{-t})}{e^t - 1} = \sum_{n=0}^{\infty} C_n^{(k)} \frac{t^n}{n!},$$

where $Li_k(z)$ is the poly-logarithm function (or rational function when $k \leq 0$) defined by

$$Li_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k} \quad (|z| < 1).$$

Since $Li_1(z) = -\log(1-z)$, the generating functions on the left-hand sides respectively become

$$\frac{te^t}{e^t-1} \quad \text{and} \quad \frac{t}{e^t-1}$$

when $k = 1$, and hence $B_n^{(1)}$ and $C_n^{(1)}$ become the usual Bernoulli numbers with $B_1^{(1)} = \frac{1}{2}$ and $C_1^{(1)} = -\frac{1}{2}$. When $k \geq 1$, we can define $B_n^{(k)}$ and $C_n^{(k)}$ in the form of iterated integrals as

$$e^t \cdot \underbrace{\frac{1}{e^t-1}\int_0^t \frac{1}{e^t-1}\int_0^t \cdots \frac{1}{e^t-1}\int_0^t}_{(k-1)-\text{times}} \frac{1}{e^t-1} dtdt\cdots dt = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!}$$

and

$$\underbrace{\frac{1}{e^t-1}\int_0^t \frac{1}{e^t-1}\int_0^t \cdots \frac{1}{e^t-1}\int_0^t}_{(k-1)-\text{times}} \frac{1}{e^t-1} dtdt\cdots dt = \sum_{n=0}^{\infty} C_n^{(k)} \frac{t^n}{n!}.$$

We recall the definitions of Stirling numbers. Stirling numbers that we use in this chapter are the second kind, but we also define Stirling numbers of the first kind, which will be needed later. Let $(x)_n = x(x-1)\cdots(x-n+1)$ and $x^{(n)} = x(x+1)\cdots(x+n-1)$ which are known as falling and rising factorial respectively. Then Stirling numbers of the first kind is defined by

$$x^{(n)} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k$$

and Stirling numbers of the second kind is defined by

$$x^n = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} (x)_k.$$

There are various properties of the poly-Bernoulli numbers. In ([7], Theorem 1 and 2) and ([9], §2) we have the explicit formulas

$$B_n^{(k)} = (-1)^n \sum_{i=0}^{n} \frac{(-1)^i i! \begin{Bmatrix} n \\ i \end{Bmatrix}}{(i+1)^k}, \qquad C_n^{(k)} = (-1)^n \sum_{i=0}^{n} \frac{(-1)^i i! \begin{Bmatrix} n+1 \\ i+1 \end{Bmatrix}}{(i+1)^k}$$

for $k \in \mathbb{Z}, n \in \mathbb{Z}_{\geq 0}$ involving Stirling numbers of the second kind, and the dualities

$$B_n^{(-k)} = B_k^{(-n)}, \qquad C_n^{(-k-1)} = C_k^{(-n-1)}$$

for $n, k \in \mathbb{Z}_{\geq 0}$.

In [6], the multi-indexed version of poly-Bernoulli numbers is defined as follows.

Definition 2.2 (Multi-poly-Bernoulli numbers). *For $k_1, \ldots, k_r \in \mathbb{Z}$,*

$$\frac{Li_{k_1,\ldots,k_r}(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k_1,\ldots,k_r)} \frac{t^n}{n!}$$

and

$$\frac{Li_{k_1,\ldots,k_r}(1-e^{-t})}{e^{t}-1} = \sum_{n=0}^{\infty} C_n^{(k_1,\ldots,k_r)} \frac{t^n}{n!},$$

where $Li_{k_1,\ldots,k_r}(z)$ is the multiple polylogarithm function defined by

$$Li_{k_1,\ldots,k_r}(z) = \sum_{0<m_1<\cdots<m_r}^{\infty} \frac{z^{m_r}}{m_1^{k_1}\cdots m_r^{k_r}} \quad (|z|< 1).$$

When $r = 1$, the numbers $B_n^{(k)}$ and $C_n^{(k)}$ are the poly-Bernoulli numbers.

The following lemma is needed to prove many results, related to multiple poly-Bernoulli numbers and Arakawa-Kaneko zeta functions.

Lemma 2.1. *([2])* (i) *For $r, k_1, \ldots, k_r \in \mathbb{Z}_{>0}$,*

$$\frac{d}{dz} Li_{k_1,\ldots,k_{r-1},k_r}(z) = \begin{cases} \frac{1}{z} Li_{k_1,\cdots,k_{r-1},k_r-1}(z) & (k_r \geq 2), \\ \frac{1}{1-z} Li_{k_1,\ldots,k_{r-1}}(z) & (k_r = 1). \end{cases}$$

(ii) *For $r \in \mathbb{Z}_{>0}$,*

$$Li_{\underbrace{1,\ldots,1}_{r\text{-times}}}(z) = \frac{(-1)^r}{r!} \log^r(1-z).$$

The following formula gives a relation between $B_n^{(k_1,\ldots,k_r)}$ and $C_n^{(k_1,\ldots,k_r)}$. The $r = 1$ case is given in [2].

Proposition 2.2. *(K. Imatomi, M. Kaneko and E. Takeda [6]) For any $r \geq 1, k_i \in \mathbb{Z}$ and $n \geq 1$, we have*

$$B_n^{(k_1,\ldots,k_r)} = C_n^{(k_1,\ldots,k_r)} + C_{n-1}^{(k_1,\ldots,k_{r-1},k_r-1)}.$$

Imatomi, Kaneko and Takeda obtained the following recurrence relation for the multi-poly-Bernoulli numbers. The $r = 1$ case is given in [2].

Proposition 2.3. *(K. Imatomi, M. Kaneko and E. Takeda [6]) For any $r \geq 1, k_i \in \mathbb{Z}$ and $n \geq 1$, we have*

$$B_n^{(k_1,\ldots,k_r)} = \frac{1}{n+1}\left(B_n^{(k_1,\ldots,k_{r-1},k_r-1)} - \sum_{m=1}^{n-1} B_n^{(k_1,\ldots,k_r)}\right)$$

and

$$C_n^{(k_1,\ldots,k_r)} = \frac{(-1)^n}{n+1}\left(\sum_{m=0}^{n}(-1)^m\binom{n}{m}B_m^{(k_1,\ldots,k_{r-1},k_r-1)} - \sum_{m=1}^{n-1}(-1)^m\binom{n}{m-1}C_n^{(k_1,\ldots,k_r)}\right),$$

where an empty sum is understood to be 0.

2.2 Multiple zeta values

In this section we mainly discuss algebric setup introduced by Hoffman. First, we define the multiple zeta values.

Definition 2.3 (Multiple zeta values)**.** *For the positive integer set $\boldsymbol{k} = (k_1, ..., k_r)$ with $k_r \geq 2$ for the convergence, the multiple zeta values (MZVs) are defined by*

$$\zeta(k_1, \ldots, k_r) := \sum_{0<m_1<\cdots<m_r} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}}.$$

The quantities $wt(\boldsymbol{k}) := k_1 + \cdots + k_r$, $dep(\boldsymbol{k}) := r$ and $ht(\boldsymbol{k}) := \#\{i | k_i \geq 2, 1 \leq i \leq r\}$ are called the weight, depth and height of the index set $\boldsymbol{k}$ of the multiple zeta values $\zeta(k_1, \ldots, k_r)$ respectively.

We say that the index $\mathbf{k}$ is admissible if $k_r \geq 2$.

Definition 2.4. *Let*

$$\mathfrak{H} := \mathbb{Q}\langle x, y\rangle$$

be the non-commutative polynomial ring in two indeterminates x and y. We refer to monomials in x and y as words. For any word w, let $l(w)$ be the number of y and $|w|$ be the total number of factors. We also define subrings,

$$\mathfrak{H}^1 := \mathbb{Q} + y\mathfrak{H}$$

and

$$\mathfrak{H}^0 := \mathbb{Q} + y\mathfrak{H}x$$

For any integer $k > 0$, put $z_k = yx^{k-1}$. Then the ring $\mathfrak{H}^1$ is freely generated by z_k $(k \geq 1)$. When $k \geq 2$, z_k is contained in $\mathfrak{H}^0$. But $\mathfrak{H}^0$ is not freely generated by $z_k (k \geq 1)$. Now let us define the *evaluation map $Z : \mathfrak{H}^0 \to \mathbb{R}$ by*

$$Z(z_{k_1} \cdots z_{k_r}) := \zeta(k_1, \ldots, k_r) \tag{2.1}$$

Since $z_{k_1} \cdots z_{k_r} \in \mathfrak{H}^0, k_r > 1$, therefore, $\zeta(k_1, \ldots, k_r)$ is finite.

Definition 2.5. *We define the shuffle product $ш$ on $\mathfrak{H}$ inductively by*

$$\begin{aligned} 1 ш w &= w ш 1 = w \\ u_1 w_1 ш u_2 w_2 &= u_1(w_1 ш u_2 w_2) + u_2(u_1 w_1 ш w_2) \end{aligned}$$

for any words $w, w_1, w_2 \in \mathfrak{H}$ and $u_1, u_2 \in \{x, y\}$, with $\mathbb{Q}$-biliniarity.

The shuffle product is commutative and associative. We denote the commutative $\mathbb{Q}$-algebra $\mathfrak{H}$ equipped with multiplication $ш$ by $\mathfrak{H}_ш$. Then we have that the subspaces $\mathfrak{H}^0$ and $\mathfrak{H}^1$ of $\mathfrak{H}$ are closed under $ш$ and become subalgebras of $\mathfrak{H}_ш$ denoted by $\mathfrak{H}^0_ш$ and $\mathfrak{H}^1_ш$.

The map $\zeta : \mathfrak{H}^0 \to \mathbb{R}$ is a $\mathbb{Q}$-algebra homomorphism on $\mathfrak{H}^0_ш$, that is,

$$\zeta(\mathbf{k} ш \mathbf{l}) = \zeta(\mathbf{k})\zeta(\mathbf{l}) \tag{2.2}$$

for any admissible indices $\mathbf{k}$ and $\mathbf{l}$. This is called the shuffle product of MZVs.

Example 2.1. Let us find the shuffle product of the indices $(1, 1)$ and (2). Then we can write

$$\begin{aligned} y^2 ш yx &= (y^2 ш y)x + (y ш yx)y \\ &= (3y^3)x + (2y^2x + yxy)y \\ &= 3y^3x + 2y^2xy + yxy^2. \end{aligned}$$

Then, we have

$$(1, 1) ш (2) = 3(1, 1, 2) + 2(1, 2, 1) + (2, 1, 1).$$

2.3 Arakawa-Kaneko zeta function

In this section, mainly, we discuss some formulas of Arakawa-Kaneko zeta function that we want to generalize as the level two analogue.

In their research, Arakawa and Kaneko [2] studied the single variable function

$$\zeta(k_1, \ldots, k_{r-1}; s) = \sum_{0<m_1<\cdots<m_{r-1}<m_r} \frac{1}{m_1^{k_1} \cdots m_{r-1}^{k_{r-1}} m_r^s}$$

for the purpose of establishing the connection between multiple zeta values and poly-Bernoulli numbers. This is absolutely convergent for $Re(s) > 1$. They have shown that the poly-Bernoulli numbers can be expressed as special values at negative arguments of certain combinations of these functions. Corresponding to these functions, Arakawa and Kaneko [2] defined the following zeta function which is known as Arakawa-Kaneko zeta function as follows.

Definition 2.6 (Arakawa-Kaneko zeta function)**.** *For $r, k_1, \ldots, k_r \in \mathbb{Z}_{\geq 1}$, $s \in \mathbb{C}$ with $Re(s) > 0$, we write*

$$\xi(k_1, \ldots, k_r; s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} Li_{k_1,\ldots,k_r}(1 - e^{-t}) dt.$$

This function can be analytically continued to an entire function of the complex variable $s \in \mathbb{C}$ for $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 1}$ ([2], §3 and §4). For $r = 1$ we denote $\xi(k; s)$ by $\xi_k(s)$. Note that $\xi_1(s) = s\zeta(s+1)$. Poly-Bernoulli numbers appear as special values of Arakawa-Kaneko zeta function at negative agreements. The following theorem illustrates these facts on Arakawa-Kaneko zeta function.

Theorem 2.4. *[2] The function $\xi(k_1, \ldots, k_r; s)$ is continued analytically to $\mathbb{C}$ as an entire function and satisfies*

$$\xi(k_1, \ldots, k_r; -m) = (-1)^m C_m^{(k_1,\ldots,k_r)} \quad (m \in \mathbb{Z}_{\geq 0})$$

for $(k_1, \ldots, k_r) \in \mathbb{Z}^r_{\geq 1}$.

Now, we define the dual index.

Definition 2.7. *Let $\mathbf{k} = (k_1, \ldots, k_r)$ be an admissible index of weight k. We write*

$$\mathbf{k} = (\underbrace{1, \ldots, 1}_{a_1-1}, b_1 + 1, \ldots, \underbrace{1, \ldots, 1}_{a_h-1}, b_h + 1),$$

with (uniquely determined) integers $h \geq 1$, $a_i, b_i \geq 1$ $(1 \leq i \leq h)$. Then the dual index of $\mathbf{k}$ is given by

$$\mathbf{k}^* = (\underbrace{1, \ldots, 1}_{b_h-1}, a_h + 1, \ldots, \underbrace{1, \ldots, 1}_{b_1-1}, a_1 + 1).$$

In [11], Kaneko and Tsumura obtained an explicit expression for $\xi(\mathbf{k}; m)$ in terms of multiple zeta values. We introduce the following notations which will be needed in stating the following theorem and for the later results. For $\mathbf{j} = (j_1, \ldots, j_r) \in \mathbb{Z}^r_{\geq 0}$, we set $|\mathbf{j}| = j_1 + \cdots + j_r$ and $d(\mathbf{j}) = r$ and call them the weight and the depth of $\mathbf{j}$ respectively. For two such indices $\mathbf{k}$ and $\mathbf{j}$ of the same depth, we denote by $\mathbf{k} + \mathbf{j}$ the index obtained by the component-wise addition, $\mathbf{k} + \mathbf{j} = (k_1 + j_1, \ldots, k_r + j_r)$, and by $b(\mathbf{k}; \mathbf{j})$ the quantity given by

$$b(\mathbf{k}; \mathbf{j}) := \prod_{i=1}^r \binom{k_i + j_i - 1}{j_i}.$$

For any index set $\mathbf{k} = (k_1, \ldots, k_r) \in \mathbb{Z}_{\geq 1}^r$, let

$$\mathbf{k}_+ = (k_1, \ldots, k_{r-1}, k_r + 1).$$

Theorem 2.5. *([11], Theorem 2.5) For any index set $\mathbf{k} = (k_1, \ldots, k_r) \in \mathbb{Z}_{\geq 1}^r$ and any $m \in \mathbb{Z}_{\geq 1}$, we have*

$$\xi(\mathbf{k}; m) = \sum_{|\mathbf{j}|=m-1,\ d(\mathbf{j})=n} b\left((\mathbf{k}_+)^*; \mathbf{j}\right) \zeta\left((\mathbf{k}_+)^* + \mathbf{j}\right),$$

where the sum is over all $\mathbf{j} \in \mathbb{Z}_{\geq 0}^r$ of weight $m-1$ and depth $n := d(\mathbf{k}_+^*) = |\mathbf{k}|+1 - d(\mathbf{k})$.

In particular, we have

$$\xi(\mathbf{k}; 1) = \zeta(\mathbf{k}_+).$$

Kaneko and Tsumura proved an Euler-type connection formula for the multiple polylogarithm.

Theorem 2.6. *(Kaneko and Tsumura [11]) Let $\mathbf{k}$ be any index set. Then we have*

$$Li_{\mathbf{k}}(1-z) = \sum_{\mathbf{k}', j \geq 0} C_{\mathbf{k}}(\mathbf{k}'; j) Li_{\underbrace{1, \ldots, 1}_{j-\text{times}}}(1-z) Li_{\mathbf{k}'}(z),$$

where the sum on the right runs over indices $\mathbf{k}'$ and integers $j \geq 0$ that satisfy $|\mathbf{k}'|+j \leq |\mathbf{k}|$, and $C_{\mathbf{k}}(\mathbf{k}'; j)$ is a $\mathbb{Q}-$linear combination of MZVs of weight $|\mathbf{k}| - |\mathbf{k}'| - j$. For the empty index $\emptyset$, we understand $Li_{\emptyset} = 1$ and $|\emptyset|= 0$ and the constant 1 is interpreted as a multiple zeta value of weight 0.

With this formula Kaneko and Tsumura established the following formula.

Theorem 2.7. *(Kaneko and Tsumura [11]) Let $\mathbf{k}$ be any index set. The function $\xi(\mathbf{k}; s)$ can be written in terms of multiple zeta functions as*

$$\xi(\mathbf{k}; s) = \sum_{\mathbf{k}', j \geq 0} C_{\mathbf{k}}(\mathbf{k}'; j) \binom{s+j-1}{j} \zeta(\mathbf{k}'; s+j),$$

where the sum on the right runs over indices $\mathbf{k}'$ and integers $j \geq 0$ that satisfy $|\mathbf{k}'|+j \leq |\mathbf{k}|$, and $C_{\mathbf{k}}(\mathbf{k}'; j)$ is a $\mathbb{Q}-$linear combination of MZVs of weight $|\mathbf{k}| - |\mathbf{k}'| - j$. When the index $\mathbf{k}'$ is $\emptyset$, we set $\zeta(\emptyset; s+j) = \zeta(s+j)$.

Chapter 3

Polycosecant numbers

In this chapter, we present our main results on polycosecant numbers. Each section consists of newly obtained results such as duality, recurrence formulas, explicit formulas, etc.

In our research, we study the level two analogue of poly-Bernoulli numbers, which we also call the polycosecant numbers (Kaneko-M.-Tsumura 2020 [10]) $D_n^{(k)}$.

Definition 3.1 (Polycosecant numbers)**.** *For $k \in \mathbb{Z}$, the polycosecant numbers $D_n^{(k)}$ are defined by*

$$\frac{\mathrm{A}_k(\tanh t/2)}{\sinh t} = \sum_{n=0}^{\infty} D_n^{(k)} \frac{t^n}{n!},$$

where $\mathrm{A}_k(z)$ is the polylogarithm function of level two defined by

$$\mathrm{A}_k(z) = 2\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)^k} \quad (z \in \mathbb{C};\ |z| < 1).$$

In particular, for $k = 1$, we have $\mathrm{A}_1(z) = 2\tanh^{-1}(z)$. In this case, $D_n^{(1)}$ becomes the ordinary cosecant number D_n defined by

$$\frac{t}{\sinh t} = \sum_{n=0}^{\infty} D_n \frac{t^n}{n!}.$$

Note that $D_{2n+1}^{(k)} = 0$ for any k and $n \geq 0$. In [16, Definition 5], Y. Sasaki introduced a slightly different case of a generalization of the poly-Bernoulli numbers.

3.1 The duality relation

We present a duality relation for polycosecant numbers (see [10, Theorem 3.1]). Two types of proofs are given in [10]. Here, we present the proof by using a generating function.

Theorem 3.1. *(Kaneko-M.-Tsumura [10]) For $n, k \in \mathbb{Z}_{\geq 0}$, we have*

$$D_{2n}^{(-2k-1)} = D_{2k}^{(-2n-1)}.$$

Proof. To prove the theorem, we define the generating function of $D_{2n}^{(-2k-1)}$ by

$$F(x,y) := \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} D_{2n}^{(-2k-1)} \frac{x^{2n}}{(2n)!}\frac{y^{2k}}{(2k)!}, \tag{3.1}$$

and show $F(x,y)$ is symmetric in x and y (i.e. $F(x,y) = F(y,x)$). This is ensured by the following closed formula for $F(x,y)$. ■

Proposition 3.2. *(Kaneko-M.-Tsumura [10]) Let*

$$G(x,y) = \frac{e^{x+y}}{(1+e^x+e^y-e^{x+y})^2}.$$

Then one finds

$$F(x,y) = G(x,y) + G(x,-y) + G(-x,y) + G(-x,-y).$$

Proof. We first compute the generating function of all $D_n^{(-k)}$,

$$f(x,y) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} D_n^{(-k)} \frac{x^n}{n!}\frac{y^k}{k!}. \tag{3.2}$$

We claim that the formula

$$f(x,y) = \frac{e^x(e^y-1)}{1+e^x+e^y-e^{x+y}} + \frac{e^{-x}(e^y-1)}{1+e^{-x}+e^y-e^{-x+y}} \tag{3.3}$$

holds. To prove this, we first observe that, by definition,

$$\begin{aligned} f(x,y) &= \sum_{k=0}^{\infty} \frac{\mathrm{A}_{-k}(\tanh(x/2))}{\sinh x}\frac{y^k}{k!} \\ &= \frac{2}{\sinh x}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(2n+1)^k(\tanh(x/2))^{2n+1}\frac{y^k}{k!}. \end{aligned}$$

Noting that

$$2\sum_{n=0}^{\infty}(2n+1)^k t^{2n+1} = 2\left(t\frac{d}{dt}\right)^k \frac{t}{1-t^2} = \left(t\frac{d}{dt}\right)^k\left(\frac{1}{1-t}-\frac{1}{1+t}\right), \tag{3.4}$$

and by the standard formula (*cf.*, *e.g.*, [1, Proposition 2.6 (4)])

$$\left(t\frac{d}{dt}\right)^k = \sum_{m=1}^{k}\left\{ {k \atop m} \right\} t^m \left(\frac{d}{dt}\right)^m,$$

we find that the right-hand side of (3.4) becomes

$$\sum_{m=1}^{k} \left\{ {k \atop m} \right\} t^m \left(\frac{d}{dt}\right)^m \left(\frac{1}{1-t} - \frac{1}{1+t}\right) = \sum_{m=1}^{k} \left\{ {k \atop m} \right\} m! \left(\frac{t^m}{(1-t)^{m+1}} - \frac{(-t)^m}{(1+t)^{m+1}}\right).$$

Therefore, by setting $t = \tanh(x/2)$ and noting $t/(1-t) = (e^x - 1)/2$, $-t/(1+t) = (e^{-x} - 1)/2$, $(\sinh x)(1-t) = e^{-x}(e^x - 1)$, $(\sinh x)(1+t) = e^x - 1$, we obtain

$$\begin{aligned}
f(x,y) &= \frac{1}{\sinh x} \sum_{k=0}^{\infty} \sum_{m=1}^{k} \left\{ {k \atop m} \right\} m! \left(\frac{t^m}{(1-t)^{m+1}} - \frac{(-t)^m}{(1+t)^{m+1}}\right) \frac{y^k}{k!} \\
&= \sum_{k=0}^{\infty} \sum_{m=1}^{k} \left\{ {k \atop m} \right\} m! \left(\frac{e^x}{e^x - 1}\left(\frac{e^x - 1}{2}\right)^m - \frac{1}{e^x - 1}\left(\frac{e^{-x} - 1}{2}\right)^m\right) \frac{y^k}{k!} \\
&= \sum_{m=1}^{\infty} (e^y - 1)^m \left(\frac{e^x}{e^x - 1}\left(\frac{e^x - 1}{2}\right)^m - \frac{1}{e^x - 1}\left(\frac{e^{-x} - 1}{2}\right)^m\right) \\
&= \frac{e^x}{e^x - 1} \cdot \frac{(e^y - 1)(e^x - 1)}{2 - (e^y - 1)(e^x - 1)} - \frac{1}{e^x - 1} \cdot \frac{(e^y - 1)(e^{-x} - 1)}{2 - (e^y - 1)(e^{-x} - 1)} \\
&= \frac{e^x(e^y - 1)}{1 + e^x + e^y - e^{x+y}} + \frac{e^{-x}(e^y - 1)}{1 + e^{-x} + e^y - e^{-x+y}}.
\end{aligned}$$

This proves the identity (3.3). From (3.3) we see that $f(x,y)$ is even in x, and so we have

$$\frac{f(x,y) - f(x,-y)}{2} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} D_{2n}^{(-2k-1)} \frac{x^{2n}}{(2n)!} \frac{y^{2k+1}}{(2k+1)!}.$$

Our generating function $F(x,y)$ is the derivative of this relation with respect to y, and Proposition 3.2 follows from a straightforward calculation. And by the symmetry of $F(x,y)$ in x and y, Theorem 3.1 is proved. ∎

Remark 3.1.1. *Hiroyuki Ochiai suggested a simpler method of proving the duality relation by considering the exponential generating function*

$$H(x,y) := \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} D_{2n}^{(-2k-1)} \frac{x^{2n+1}}{(2n+1)!} \frac{y^{2k+1}}{(2k+1)!}.$$

This method also leads to obtain a similar formula for $C_n^{(k)}$.

3.2 Recurrence and explicit formulas for polycosecant numbers

In this section, we obtain recurrence and two explicit formulas for polycosecant numbers. The following proposition gives a recurrence formula for $D_n^{(k)}$ which can be derived in two ways by using the definition and the iterated integral expression of the generating function. Here we only consider the proof by definition.

Note that since $\mathrm{A}_0(\tanh(t/2)) = \sinh(t)$, $D_0^{(0)} = 1$ and $D_n^{(0)} = 0$ for all $n \geq 1$.

Proposition 3.3. *(Kaneko-M.-Tsumura [10]) For any integers k and $n \geq 0$,*

$$D_n^{(k-1)} = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2m+1} D_{n-2m}^{(k)}.$$

Proof. By the definition of polycosecant numbers we have that,

$$\mathrm{A}_k(\tanh(t/2)) = \sinh t \sum_{n=0}^{\infty} D_n^{(k)} \frac{t^n}{n!}.$$

Differentiate with respect to t,

$$\frac{\mathrm{A}_{k-1}(\tanh t/2)}{\sinh t} = \cosh t \sum_{n=0}^{\infty} D_n^{(k)} \frac{t^n}{n!} + \sinh t \sum_{n=1}^{\infty} D_n^{(k)} \frac{t^{n-1}}{(n-1)!}.$$

By using the definitions we can write the above equation as,

$$\begin{aligned}
\sum_{n=0}^{\infty} D_n^{(k-1)} \frac{t^n}{n!} &= \sum_{m=0}^{\infty} \frac{t^{2m}}{(2m)!} \sum_{n=0}^{\infty} D_n^{(k)} \frac{t^n}{n!} + \sum_{m=0}^{\infty} \frac{t^{2m+1}}{(2m+1)!} \sum_{n=1}^{\infty} D_n^{(k)} \frac{t^{n-1}}{(n-1)!} \\
&= \sum_{m=0}^{\infty} \sum_{n=2m}^{\infty} D_{n-2m}^{(k)} \frac{t^n}{(2m)!\,(n-2m)!} \\
&\quad + \sum_{m=0}^{\infty} \sum_{n=2m}^{\infty} D_{n-2m}^{(k)} \frac{t^n}{(2m+1)!\,(n-2m-1)!} \quad ;(n = n+2m) \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2m} D_{n-2m}^{(k)} \frac{t^n}{n!} + \sum_{n=0}^{\infty} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2m+1} D_{n-2m}^{(k)} \frac{t^n}{n!}.
\end{aligned}$$

By equating the coefficients of $\frac{t^n}{n!}$ we can get the desired result. ■

When $k > 0$, we may want to write this as

$$(n+1) D_n^{(k)} = D_n^{(k-1)} - \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2m+1} D_{n-2m}^{(k)} \quad (n > 0).$$

Note that $D_0^{(k)} = 1$ for all $k \in \mathbb{Z}$.

In the following theorem we obtain two explicit formulas for $D_n^{(k)}$. The first formula involves Bernoulli numbers and Stirling numbers. The second formula involves Stirling numbers of the second kind.

Theorem 3.4. *(Kaneko-M.-Tsumura [10]) For any $k \in \mathbb{Z}$ and $n \geq 0$, we have*

1.

$$D_n^{(k)} = 4\sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{(2m+1)^{k+1}} \sum_{p=1}^{2m+1} \sum_{q=0}^{n} (-1)^n (2^{p+q+1}-1) \binom{n}{q} \left\{ {n-q \atop 2m} \right\} \left[{2m+1 \atop p} \right] \frac{B_{p+q+1}}{p+q+1}$$

and

2.

$$D_n^{(k)} = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{(2m+1)^{k+1}} \sum_{p=2m+1}^{n+1} \frac{(-1)^{p+1} p!}{2^{p-1}} \binom{p-1}{2m} \left\{ {n+1 \atop p} \right\}.$$

To prove the first formula of Theorem 3.4, we prepare the following lemma.

Lemma 3.5. *For $n \geq 1$ we have,*

$$x^n \left(\frac{d}{dx}\right)^n = \sum_{m=1}^{n} (-1)^{n-m} \left[{n \atop m} \right] \left(x\frac{d}{dx}\right)^m.$$

Proof. We can prove this by induction on n. For $n=1$ both sides equal to $x\frac{d}{dx}$.

Suppose the formula is true for n. Then,

$$\begin{aligned} x^{n+1}\left(\frac{d}{dx}\right)^{n+1} &= x^{n+1}\left(\frac{d}{dx}\right)\left(\frac{d}{dx}\right)^n \\ &= x^{n+1}\frac{d}{dx}\left[\sum_{m=1}^{n} \frac{(-1)^{n-m}}{x^n} \left[{n \atop m} \right] \left(x\frac{d}{dx}\right)^m\right] \\ &= \sum_{m=1}^{n} (-1)^{n-m} \left[{n \atop m} \right] \left[-n\left(x\frac{d}{dx}\right)^m + \left(x\frac{d}{dx}\right)^{m+1}\right] \\ &= \sum_{m=1}^{n+1} (-1)^{n-m+1} \left(n\left[{n \atop m} \right] + \left[{n \atop m-1} \right]\right)\left(x\frac{d}{dx}\right)^m \\ &= \sum_{m=1}^{n+1} (-1)^{n-m+1} \left[{n+1 \atop m} \right] \left(x\frac{d}{dx}\right)^m. \end{aligned}$$

Here we have used $\left[{n \atop 0} \right] = 0$ $(n \neq 0)$ and $\left[{n \atop n+1} \right] = 0$. This shows the formula is true for $n+1$. Therefore the formula holds. ∎

Now we give the proof for the first formula of Theorem 3.4.

Proof of Theorem 3.4-(1). We write

$$\begin{aligned}\sum_{n=0}^{\infty} D_n^{(k)} \frac{t^n}{n!} &= \frac{\mathrm{A}_k(\tanh(t/2))}{\sinh t} \\ &= 2\sum_{m=0}^{\infty} \frac{(\tanh(t/2))^{2m+1}}{(2m+1)^k} \frac{1}{\sinh t} \\ &= 4\sum_{m=0}^{\infty} \frac{1}{(2m+1)^k} \frac{e^t(e^t-1)^{2m}}{(e^t+1)^{2m+2}}. \end{aligned} \tag{3.5}$$

Since

$$\frac{1}{(x+1)^{n+1}} = \frac{(-1)^n}{n!}\left(\frac{d}{dx}\right)^n \frac{1}{x+1}, \tag{3.6}$$

we see by setting $x = e^t$ and using Lemma 3.5 that

$$\frac{e^{nt}}{(e^t+1)^{n+1}} = \frac{1}{n!}\sum_{p=1}^{n}(-1)^p \begin{bmatrix} n \\ p \end{bmatrix} \left(\frac{d}{dt}\right)^p \frac{1}{e^t+1}. \tag{3.7}$$

Hence we consider $B_n = C_n^{(1)}$ $(n \geq 0)$ defined by

$$\frac{t}{e^t-1} = \sum_{q=0}^{\infty} B_q \frac{t^q}{q!}.$$

Since

$$\frac{1}{e^t+1} = \frac{1}{e^t-1} - \frac{2}{e^{2t}-1},$$

we have

$$\frac{1}{e^t+1} = \sum_{q=0}^{\infty}(1-2^q)B_q\frac{t^{q-1}}{q!}.$$

By taking the p-th derivative of both sides, we get

$$\left(\frac{d}{dt}\right)^p \left(\frac{1}{e^t+1}\right) = \sum_{q=p+1}^{\infty}(1-2^q)\frac{B_q}{q}\frac{t^{q-p-1}}{(q-p-1)!} = \sum_{q=0}^{\infty}(1-2^{p+q+1})\frac{B_{p+q+1}}{p+q+1}\frac{t^q}{q!}$$

and we substitute this in (3.7) to obtain

$$\frac{e^{nt}}{(e^t+1)^{n+1}} = \frac{1}{n!}\sum_{p=1}^{n}(-1)^p \begin{bmatrix} n \\ p \end{bmatrix} \sum_{q=0}^{\infty}(1-2^{p+q+1})\frac{B_{p+q+1}}{p+q+1}\frac{t^q}{q!}$$

$$\frac{1}{n!}\sum_{q=0}^{\infty}\sum_{p=1}^{n}(-1)^p\begin{bmatrix}n\\p\end{bmatrix}(1-2^{p+q+1})\frac{B_{p+q+1}}{p+q+1}\frac{t^q}{q!}.$$

From this, we have

$$\begin{aligned}\frac{e^t}{(e^t+1)^{2m+2}}&=\frac{e^{-(2m+1)t}}{(e^{-t}+1)^{2m+2}}\\&=\frac{1}{(2m+1)!}\sum_{q=0}^{\infty}\sum_{p=1}^{2m+1}(-1)^{p+q}\begin{bmatrix}2m+1\\p\end{bmatrix}(1-2^{p+q+1})\frac{B_{p+q+1}}{p+q+1}\frac{t^q}{q!}.\end{aligned}$$

Together with the well-known generating series ([1, Proposition 2.6 (7)], note that $\left\{\begin{matrix}s\\2m\end{matrix}\right\}=0$ if $s<2m$)

$$(e^t-1)^{2m}=(2m)!\sum_{s=0}^{\infty}\left\{\begin{matrix}s\\2m\end{matrix}\right\}\frac{t^s}{s!},$$

we obtain

$$\begin{aligned}&\frac{e^t(e^t-1)^{2m}}{(e^t+1)^{2m+2}}\\&=\frac{1}{2m+1}\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}\sum_{p=1}^{2m+1}(-1)^{p+q}(1-2^{p+q+1})\begin{bmatrix}2m+1\\p\end{bmatrix}\left\{\begin{matrix}s\\2m\end{matrix}\right\}\frac{B_{p+q+1}}{p+q+1}\frac{t^{q+s}}{q!\,s!}\\&=\frac{1}{2m+1}\sum_{n=0}^{\infty}\sum_{q=0}^{n}\sum_{p=1}^{2m+1}(-1)^{p+q}(1-2^{p+q+1})\binom{n}{q}\begin{bmatrix}2m+1\\p\end{bmatrix}\left\{\begin{matrix}n-q\\2m\end{matrix}\right\}\frac{B_{p+q+1}}{p+q+1}\frac{t^n}{n!}.\end{aligned}$$

Substituting this into (3.5), we have

$$\begin{aligned}&\sum_{n=0}^{\infty}D_n^{(k)}\frac{t^n}{n!}\\&=4\sum_{m=0}^{\infty}\frac{1}{(2m+1)^{k+1}}\sum_{n=0}^{\infty}\sum_{q=0}^{n}\sum_{p=1}^{2m+1}(-1)^{p+q}(1-2^{p+q+1})\binom{n}{q}\begin{bmatrix}2m+1\\p\end{bmatrix}\left\{\begin{matrix}n-q\\2m\end{matrix}\right\}\frac{B_{p+q+1}}{p+q+1}\frac{t^n}{n!}\\&=4\sum_{n=0}^{\infty}\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{(2m+1)^{k+1}}\sum_{p=1}^{2m+1}\sum_{q=0}^{n-2m}(2^{p+q+1}-1)\binom{n}{q}\begin{bmatrix}2m+1\\p\end{bmatrix}\left\{\begin{matrix}n-q\\2m\end{matrix}\right\}\frac{B_{p+q+1}}{p+q+1}\frac{t^n}{n!}.\end{aligned}$$

(We have used the facts that $B_{p+q+1}=0$ if $p+q\geq 1$ is even and $\left\{\begin{matrix}n-q\\2m\end{matrix}\right\}=0$ if $n-q<2m$.) By equating the coefficients of $t^n/n!$ on both sides, we obtain the desired result. ■

We can easily prove the second formula of Theorem 3.4 by using the definition of the n-th tangent numbers of order k, $T_{n,k}$, for non negative integers n and k, by the generating relation (see [3, P. 259]).

$$\frac{\tan^k t}{k!} = \sum_{n=k}^{\infty} T_{n,k} \frac{t^n}{n!}, \tag{3.8}$$

and the formula in [4, Proposition 9]

$$T_{n,k} = (-1)^{\frac{n-k}{2}} (-1)^n \sum_{m=k}^{n} (-1)^m 2^{n-m} \left\{ {n \atop m} \right\} \binom{m-1}{k-1} \frac{m!}{k!} \tag{3.9}$$

for $n \geq 1$ and $k \geq 0$.

Note that both $T_{n,k}$ and the sum on the right side of (3.9) are zero if $n \not\equiv k(2)$.

Proof of Theorem 3.4-(2). From the definition we have

$$\begin{aligned} \sum_{n=0}^{\infty} D_n^{(k)} \frac{t^n}{n!} &= \frac{\mathrm{A}_k(\tanh(t/2))}{\sinh t} = \frac{d}{dt} \mathrm{A}_{k+1}(\tanh(t/2)) \\ &= 2 \frac{d}{dt} \sum_{m=0}^{\infty} \frac{(\tanh(t/2))^{2m+1}}{(2m+1)^{k+1}}. \end{aligned} \tag{3.10}$$

By using $\tanh t = -i \tan(it)$ and equations (3.8) and (3.9), we can write

$$\begin{aligned} (\tanh(t/2))^m &= (-i)^m m! \sum_{n=m}^{\infty} T_{n,m} \frac{i^n}{2^n} \frac{t^n}{n!} \\ &= (-i)^m \sum_{n=m}^{\infty} (-1)^{\frac{n-m}{2}} \sum_{p=m}^{n} (-2)^{n-p} p! \binom{p-1}{m-1} \left\{ {n \atop p} \right\} \frac{i^n}{2^n} \frac{t^n}{n!} \\ &= (-1)^m \sum_{n=m}^{\infty} \sum_{p=m}^{n} (-1)^p \frac{p!}{2^p} \binom{p-1}{m-1} \left\{ {n \atop p} \right\} \frac{t^n}{n!}. \end{aligned}$$

We therefore have

$$\begin{aligned} \sum_{n=0}^{\infty} D_n^{(k)} \frac{t^n}{n!} &= \sum_{m=0}^{\infty} \frac{1}{(2m+1)^{k+1}} \sum_{n=2m}^{\infty} \sum_{p=2m}^{n} (-1)^p \frac{(p+1)!}{2^p} \binom{p}{2m} \left\{ {n+1 \atop p+1} \right\} \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{(2m+1)^{k+1}} \sum_{p=2m}^{n} \frac{(-1)^p (p+1)!}{2^p} \binom{p}{2m} \left\{ {n+1 \atop p+1} \right\} \frac{t^n}{n!}. \end{aligned}$$

By equating the coefficients of $t^n/n!$, we complete the proof of the theorem. ∎

3.3 Multi-indexed analogue

In this section, we obtain formulas for the multi-indexed version of the polycosecant numbers. We define the multi-indexed version as follows.

Definition 3.2 (Multi-polycosecant numbers). *For $k_1, \ldots, k_r \in \mathbb{Z}$, we define*

$$\frac{\mathrm{A}(k_1, \ldots, k_r; \tanh(t/2))}{\sinh t} = \sum_{n=0}^{\infty} D_n^{(k_1,\ldots,k_r)} \frac{t^n}{n!},$$

where the function

$$\mathrm{A}(k_1, \ldots, k_r; z) = 2^r \sum_{\substack{0<m_1<\cdots<m_r \\ m_i \equiv i \bmod 2}} \frac{z^{m_r}}{m_1^{k_1} \cdots m_r^{k_r}}$$

is known as the level two analogue of multiple polylogarithm.

Here, $\mathrm{A}(k_1, \ldots, k_r; z)$ is 2^r times $\mathrm{Ath}(k_1, \ldots, k_r; z)$ which was introduced in [12, §5]. (Our $\mathrm{A}_k(z)$ is $\mathrm{A}(k; z)$). We can regard $D_n^{(k_1,\ldots,k_r)}$ as a level two analogue of the multiple poly-Bernoulli numbers $B_n^{(k_1,\ldots,k_r)}$ and $C_n^{(k_1,\ldots,k_r)}$.

The following lemma will be needed in proving our main results and also in later discussion. We denote $\underbrace{1, \cdots, 1}_{r-\text{times}}$ by $\{1\}_r$ for the convenience.

Lemma 3.6. *[12, Lemma 5.1]*

1. *For $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 1}$,*

$$\frac{d}{dt}\mathrm{A}(k_1, \ldots, k_r; z) = \begin{cases} \frac{1}{z}\mathrm{A}(k_1, \ldots, k_{r-1}, k_r - 1; z) & (k_r \geq 2), \\ \frac{2}{1-z^2}\mathrm{A}(k_1, \ldots, k_{r-1}; z) & (k_r = 1). \end{cases}$$

2. $\mathrm{A}(\{1\}_r; z) = \frac{2^r}{r!}\left(\mathrm{A}_1(z)\right)^r = \frac{(-1)^r}{r!}\log^r\left(\frac{1-z}{1+z}\right)$.

We can obtain a recurrence formula for multi-polycosecant numbers as follows.

Proposition 3.7. *For any index set $(k_1, \ldots, k_r)$ and $n \geq 0$,*

$$D_n^{(k_1,\ldots,k_{r-1},k_r-1)} = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2m+1} D_{n-2m}^{(k_1,\ldots,k_r)}.$$

The proof of the proposition is similar to the single-indexed case. So, we omit the proof.

Now let us discuss the process of the recurrence formula. By using the definition of $\mathrm{A}(\mathbf{k}; z)$, we have

$$\begin{aligned}
\mathrm{A}(k_1, \ldots, k_{r-1}, 0; z) &= 2^r \sum_{\substack{0<m_1<\cdots<m_r \\ m_i \equiv i \bmod 2}} \frac{z^{m_r}}{m_1^{k_1} \cdots m_r^{k_{r-1}}} \\
&= \frac{2^r z}{1-z^2} \sum_{\substack{0<m_1<\cdots<m_{r-1} \\ m_i \equiv i \bmod 2}} \frac{z^{m_{r-1}}}{m_1^{k_1} \cdots m_r^{k_{r-1}}} \\
&= \frac{2z}{1-z^2} \mathrm{A}(k_1, \ldots, k_{r-1}; z).
\end{aligned}$$

By using the definition of multi-polycosecant numbers, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} D_n^{(k_1,\ldots,k_{r-1},0)} \frac{t^n}{n!} &= \frac{\mathrm{A}(k_1, \ldots, k_{r-1}, 0; \tanh t/2)}{\sinh t} \\
&= \sinh t \sum_{m=0}^{\infty} D_m^{(k_1,\ldots,k_{r-1})} \frac{t^m}{m!} \\
&= \sum_{q=0}^{\infty} \frac{t^{2q+1}}{(2q+1)!} \sum_{m=0}^{\infty} D_m^{(k_1,\ldots,k_{r-1})} \frac{t^m}{m!} \\
&= \sum_{q=0}^{\infty} \sum_{l=2q+1}^{\infty} \binom{l}{2q+1} D_{l-2q-1}^{(k_1,\ldots,k_{r-1})} \frac{t^l}{l!} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2m+1} D_{n-2m-1}^{(k_1,\ldots,k_{r-1})} \frac{t^n}{n!}.
\end{aligned}$$

By equating the coefficients of $t^n/n!$, we get

$$D_n^{(k_1,\ldots,k_{r-1},0)} = \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2m+1} D_{n-2m-1}^{(k_1,\ldots,k_{r-1})}$$

From this we can see that the $D_n^{(k_1,\ldots,k_{r-1},0)}$ can be written as a simple linear combination of $D_m^{(k_1,\ldots,k_{r-1})}$ for $0 \leq m \leq n-1$.

Let us look at the algorithm to compute the $D_n^{(k_1,\ldots,k_r)}$:

- *Step 1*: Set the initial conditions:
  $D_0^{(0)} = 1$, $D_n^{(0)} = 0$ for all $n \geq 1$, $D_0^{(k)} = 1$, for all $k > 0$ and $D_0^{(\mathbf{k})} = 0$, if $\mathrm{dep}(\mathbf{k}) > 1$.
- *Step 2*: For $n > 0$, we use the following recurrence relation to calculate $D_n^{(\mathbf{k})}$.

$$(n+1) D_n^{(k_1,\ldots,k_r)} = D_n^{(k_1,\ldots,k_{r-1},k_r-1)} - \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2m+1} D_{n-2m}^{(k_1,\ldots,k_r)} \quad . \tag{3.11}$$

This gives a way to express $D_n^{(\mathbf{k})}$ in terms of multi-polycosecant numbers with equal or lower weights and n.

- *Step 3*: We will use the above formula repeatedly until we get all the numbers satisfying the initial conditions or in the form of $D_m^{(k_1,\ldots,k_{r-1},0)}$.

- *Step 4*: Once we obtain a number in the form of $D_m^{(k_1,\ldots,k_{r-1},0)}$, we will use the following formula.
$$D_n^{(k_1,\ldots,k_{r-1},0)} = \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2m+1} D_{n-2m-1}^{(k_1,\ldots,k_{r-1})}. \tag{3.12}$$
This will give a way to write $D_n^{(k_1,\ldots,k_{r-1},0)}$ in terms of a simple linear combination of $D_m^{(k_1,\ldots,k_{r-1})}$ with $0 \le m \le n-1$.

- *Step 5*: If we get all numbers satisfying the initial conditions, we use the backward substitution to obtain the desired result. Otherwise repeat the process.

Example 3.1. Let us find the $D_2^{(2,1,3)}$ by using the above algorithm.

By using equation (3.11), we have

$$\begin{aligned}
D_2^{(2,1,3)} &= \frac{1}{3}\left(D_2^{(2,1,2)} - D_0^{(2,1,3)}\right) \\
&= \frac{1}{3}\left(\frac{1}{3}\left(D_2^{(2,1,1)} - D_0^{(2,1,2)}\right) - D_0^{(2,1,3)}\right) \\
&= \frac{1}{3}\left(\frac{1}{3}\left(\frac{1}{3}\left(D_2^{(2,1,0)} - D_0^{(2,1,1)}\right) - D_0^{(2,1,2)}\right) - D_0^{(2,1,3)}\right).
\end{aligned}$$

By using equation (3.12)

$$D_2^{(2,1,0)} = 2D_1^{(2,1)} = D_1^{(2,0)} = D_0^{(2)}$$

together with the initial values of $D_0^{(\mathbf{k})}$ gives

$$D_2^{(2,1,3)} = \frac{1}{27}.$$

We obtain explicit formulas for the multi-index case as below. We write the multiple polycosecant numbers as finite sums involving Stirling numbers.

Theorem 3.8. *1. For any index set* $\mathbf{k}$ *and* $n \ge 0$,

$$D_n^{(\mathbf{k})} = 2^{r+1} \sum_{\substack{0<m_1<\cdots<m_{r-1}<m_r<n+2 \\ m_i \equiv i \pmod 2}} \frac{1}{m_1^{k_1}\cdots m_{r-1}^{r-1} m_r^{k_r+1}} \sum_{p=1}^{m_r} \sum_{q=0}^{n-m_r+1} (-1)^n (2^{p+q+1}-1)\binom{n}{q}$$

$$\times \left\{ {n-q \atop m_r-1} \right\} \left[{m_r \atop p} \right] \frac{B_{p+q+1}}{p+q+1}.$$

2. For any index set $\mathbf{k}$ *and* $n \geq 0$*,*

$$D_n^{(\mathbf{k})} = \sum_{\substack{0<m_1<\cdots<m_{r-1}<m_r<n+2 \\ m_i \equiv i \pmod 2}} \frac{1}{m_1^{k_1} \cdots m_{r-1}^{k_{r-1}} m_r^{k_r+1}} \sum_{j=m_r}^{n+1} \frac{(-1)^{j+m_r} j!}{2^{j-r}} \binom{j-1}{m_r-1} \left\{ {n+1 \atop j} \right\}.$$

The proof of the formulas are similar to the single-indexed case. Here, we omit the proof of the first explicit formula. To get an idea about the multi-indexed case we only give the proof of the second formula.

Proof of Theorem 3.8 (Second Formula). Recall the notation $\mathbf{k}_+ = (k_1, \ldots, k_{r-1}, k_r + 1)$, for any index set $\mathbf{k} = (k_1, \ldots, k_r) \in \mathbb{Z}_{\geq 1}^r$. By the definition, we have

$$\begin{aligned} \sum_{n=0}^{\infty} D_n^{(\mathbf{k})} \frac{t^n}{n!} &= \frac{\mathrm{A}(\mathbf{k}; \tanh(t/2))}{\sinh t} = \frac{d}{dt} \mathrm{A}(\mathbf{k}_+; \tanh(t/2)) \\ &= 2^r \frac{d}{dt} \sum_{\substack{0<m_1<\cdots<m_{r-1}<m_r \\ m_i \equiv i \bmod 2}} \frac{(\tanh(t/2))^{m_r}}{m_1^{k_1} \cdots m_{r-1}^{k_{r-1}} m_r^{k_r+1}}. \end{aligned} \tag{3.13}$$

By using $\tanh t = -i \tan(it)$ and formulas in [3, p.259]) and [4, Proposition 9], we can write

$$\begin{aligned} (\tanh(t/2))^{m_r} &= (-i)^{m_r} m_r! \sum_{n=m_r}^{\infty} T_{n,m_r} \frac{i^n}{2^n} \frac{t^n}{n!} \\ &= (-i)^{m_r} \sum_{n=m_r}^{\infty} (-1)^{\frac{n-m_r}{2}} \sum_{j=m_r}^{n} (-2)^{n-j} j! \binom{j-1}{m_r-1} \left\{ {n \atop j} \right\} \frac{i^n}{2^n} \frac{t^n}{n!} \\ &= (-1)^{m_r} \sum_{n=m_r}^{\infty} \sum_{j=m_r}^{n} (-1)^j \frac{j!}{2^j} \binom{j-1}{m_r-1} \left\{ {n \atop j} \right\} \frac{t^n}{n!}. \end{aligned}$$

We therefore have

$$\begin{aligned} \sum_{n=0}^{\infty} D_n^{(\mathbf{k})} \frac{t^n}{n!} = &\sum_{\substack{0<m_1<\cdots<m_{r-1}<m_r \\ m_i \equiv i \bmod 2}} \frac{(-1)^{m_r}}{m_1^{k_1} \cdots m_{r-1}^{k_{r-1}} m_r^{k_r+1}} \\ &\times \sum_{n=m_r-1}^{\infty} \sum_{j=m_r}^{n+1} (-1)^j \frac{(j+1)!}{2^j} \binom{j-1}{m_r-1} \left\{ {n+1 \atop j} \right\} \frac{t^n}{n!} \end{aligned}$$

$$= \sum_{n=r-1}^{\infty} \sum_{\substack{0<m_1<\cdots<m_{r-1}<m_r\le n+1\\ m_i\equiv i \bmod 2}} \frac{1}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}} m_r^{k_r+1}}$$
$$\times \sum_{j=m_r}^{n+1} \frac{(-1)^{j+m_r} j!}{2^j} \binom{j-1}{m_r-1} \left\{ {n+1 \atop j} \right\} \frac{t^n}{n!}.$$

By equating the coefficients of $t^n/n!$, we complete the proof of the theorem. ∎

In the following proposition, we give a formula for the special case $k_1 = \cdots = k_r = 1$. This gives a connection between multi-polycosecant numbers and Bernoulli numbers.

Proposition 3.9. *For any $r \ge 1$ and $n \ge r-1$, we have*

$$D_n^{(\{1\}_r)} = \frac{2(1-2^{n-r})}{n+1} \binom{n+1}{r} B_{n-r+1}. \tag{3.14}$$

Proof. By using the identity

$$\mathrm{A}(\{1\}_r; z) = \frac{2^r}{r!} (\mathrm{A}(1; z))^r,$$

and the definition of multi-polycosecant numbers, we get

$$\sum_{n=0}^{\infty} D_n^{(\{1\}_r)} \frac{t^n}{n!} = \frac{t^r}{r!} \mathrm{csch} t. \tag{3.15}$$

Let us recall the expansion of the hyperbolic cosecant function

$$\mathrm{csch} x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2(1-2^{2n-1}) B_{2n} x^{2n-1}}{(2n)!}. \tag{3.16}$$

By substituting (3.16) in (3.15), we get

$$\begin{aligned}\sum_{n=0}^{\infty} D_n^{(\{1\}_r)} \frac{t^n}{n!} &= \frac{t^{r-1}}{r!} + \sum_{m=1}^{\infty} 2(1-2^{2m-1}) B_{2m} \frac{t^{2m+r-1}}{r!\,(2m)!} \\ &= \frac{t^{r-1}}{r!} + \sum_{n=r+1}^{\infty} \frac{2(1-2^{n-r})}{(n+1)} \binom{n+1}{r} B_{n-r+1} \frac{t^n}{n!} \quad (n = 2m+r-1).\end{aligned}$$

By equating the coefficients of $t^n/n!$, we can obtain the desired results. ∎

Here is a table of multiple polycosecant numbers $D_n^{(\mathbf{k})}$ for some $\mathbf{k}$ and n.

n / $\mathbf{k}$	0	1	2	3	4	5	6	7	8
(3)	1	0	$-\frac{13}{27}$	0	$\frac{3103}{3375}$	0	$-\frac{859939}{231525}$	0	$\frac{12761501}{496125}$
$(2,3)$	0	$\frac{1}{8}$	0	$-\frac{43}{192}$	0	$\frac{10913}{12960}$	0	$-\frac{7849379}{1451520}$	0
$(3,2,1)$	0	0	$\frac{1}{12}$	0	$-\frac{59}{240}$	0	$\frac{19051}{9450}$	0	$-\frac{8348311}{595350}$
$(2,3,4,5)$	0	0	0	$\frac{1}{110592}$	0	$-\frac{140399}{1679616000}$	0	$\frac{142414471589}{147483721728000}$	0
$(1,1,1,1,1)$	0	0	0	0	$\frac{1}{5}$	0	-1	0	$\frac{98}{15}$
$(6,2,2,3,4,5)$	0	0	0	0	0	$-\frac{1}{93312000}$	0	$-\frac{25900907}{98322481152000}$	0

Table 3.1: $D_n^{(k)}$ $(0 \leq n \leq 8)$

3.4 Congruence relations of polycosecant numbers

In this section, we preset our main results related to the congruence relations of polycosecant numbers. First, we discuss the connection among the multiple polycosecant numbers and the finite multiple T-values. Corresponding to the finite multiple zeta values (we refer [8]), we introduce the finite analogue of the multiple T-values.

First, define the ring $\mathscr{A}$ by

$$\mathscr{A} := \frac{\prod_p \mathbb{Z}/p\mathbb{Z}}{\oplus_p \mathbb{Z}/p\mathbb{Z}},$$

where p runs over all prime numbers. Component-wise addition and multiplication equip $\mathscr{A}$ with the structure of a ring. Moreover, the well-defined injective map $\mathbb{Q} \in r \mapsto (r \pmod{p}) \in \mathscr{A}$ makes $\mathscr{A}$ into $\mathbb{Q}$-algebra. Alternatively, $\mathscr{A}$ is isomorphic to $\left(\prod_p \mathbb{Z}/p\mathbb{Z}\right) \otimes_{\mathbb{Z}} \mathbb{Q}$ or $\prod_p \mathbb{Z}/p\mathbb{Z}$ modulo torsion. Elements of $\mathscr{A}$ are presented by $(a_{(p)})_p$, where $a_{(p)} \in \mathbb{Z}/p\mathbb{Z}$, and two elements $(a_{(p)})_p$ and $(b_{(p)})_p$ are identified if and only if $a_{(p)} = b_{(p)}$ for all but finitely many primes p.

Definition 3.3 (Finite multiple T-values)**.** *For an index set $(k_1, \ldots, k_r)$, we define the finite multiple T-values $T_{\mathscr{A}}(k_1, \ldots, k_r) \in \mathscr{A}$ by*

$$T_{\mathscr{A}}(k_1, \ldots, k_r)_{(p)} := 2^r \sum_{\substack{0<m_1<\cdots<m_r<p \\ m_i \equiv i \pmod{2}}} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \pmod{p} \tag{3.17}$$

We obtain the following congruence relation for the multi-polycosecant numbers which is the level two analogue of Theorem 8 in [6].

Theorem 3.10. *Let $k_i \in \mathbb{Z}$ and p be an odd prime. When r is even, we have*

$$T_{\mathscr{A}}(k_1, \ldots, k_r)_{(p)} = -D_{p-2}^{(k_1,\ldots,k_{r-1},k_r-1)} \pmod{p}. \tag{3.18}$$

When r is odd, we have

$$T_{\mathscr{A}}(k_1,\ldots,k_r)_{(p)} = D_{p-2}^{(k_1,\ldots,k_{r-1},k_r+1,-1)} \pmod p. \tag{3.19}$$

Proof. By considering the explicit formula (2) in Theorem 3.8, we have

$$D_{p-2}^{(k_1,\ldots,k_{r-1},k_r-1)} = \sum_{\substack{0<m_1<\cdots<m_{r-1}<m_r<p \\ m_i\equiv i \bmod 2}} \frac{2^r}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}} m_r^{k_r}} \times \sum_{j=m_r}^{p-1} \frac{(-1)^{j+m_r} j!}{2^j} \binom{j-1}{m_r-1} \left\{ {p-1 \atop j} \right\}. \tag{3.20}$$

By using the congruence [6, §4]

$$(-1)^j j! \left\{ {p-1 \atop j} \right\} \equiv -1 \pmod p,$$

we can write the inner sum of the above equation as

$$\sum_{j=m_r}^{p-1} \frac{(-1)^{j+m_r} j!}{2^j} \binom{j-1}{m_r-1} \left\{ {p-1 \atop j} \right\} \equiv (-1)^{m_r-1} \sum_{j=m_r}^{p-1} 2^{p-1-j} \binom{j-1}{m_r-1} \pmod p. \tag{3.21}$$

Consider the inner sum $\sum_{j=m_r}^{p-1} 2^{p-1-j}\binom{j-1}{m_r-1}$ and put $m_r = p-1-i$. Then, we have

$$\begin{aligned}
\sum_{j=p-1-i}^{p-1} 2^{p-1-j}\binom{j-1}{p-2-i} &= 2^i \binom{p-2-i}{p-2-i} + 2^{i-1}\binom{p-1-i}{p-2-i} + \cdots + 2^1 \binom{p-3}{p-2-i} \\
&\quad + 2^0 \binom{p-2}{p-2-i} \\
&= 2^i + 2^{i-1}(p-1-i) + \cdots + 2\frac{(p-1-i)(p-i)\cdots(p-4)(p-3)}{(i-1)!} \\
&\quad + \frac{(p-1-i)(p-i)\cdots(p-3)(p-2)}{i!} \\
&\equiv \sum_{j=0}^{i} 2^{i-j} \frac{(p-1-i)^{(j)}}{j!} \pmod p \\
&\equiv \sum_{j=0}^{i} (-1)^j 2^{i-j} \binom{i+1}{j} \pmod p \\
&\equiv \frac{1}{2}[(-1)^i + 1] \pmod p.
\end{aligned}$$

By putting $i = p-1-m_r$, we get

$$\sum_{j=m_r}^{p-1} \frac{1}{2^j}\binom{j-1}{m_r-1} \equiv \frac{1}{2}[(-1)^{m_r}+1] \pmod{p}. \tag{3.22}$$

By using the equations (3.22) and (3.21) we can write the equation (3.20) as follows.

$$D_{p-2}^{(k_1,\dots,k_{r-1},k_r-1)} \equiv -\sum_{\substack{0<m_1<\cdots<m_{r-1}<m_r<p \\ m_i\equiv i \bmod 2}} \frac{2^r}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}} m_r^{k_r}} \times \frac{1}{2}[(-1)^{m_r}+1] \pmod{p}. \tag{3.23}$$

When r is even, m_r becomes even. Then by using the definition of the finite multiple T-values, we get the first formula.

When r is odd, we consider the index set $(k_1,\dots,k_{r-1},k_r+1,-1)$. Then, by using the explicit formula (2) in Theorem 3.8, we have

$$D_{p-2}^{(k_1,\dots,k_{r-1},k_r+1,-1)} = \sum_{\substack{0<m_1<\cdots<m_r<m_{r+1}<p \\ m_i\equiv i \pmod{2}}} \frac{2^{r+1}}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}} m_r^{k_r+1} m_{r+1}^0} \times \sum_{j=m_{r+1}}^{p-1} \frac{(-1)^{j+m_{r+1}} j!}{2^j}\binom{j-1}{m_{r+1}-1}\left\{ {p-1 \atop j} \right\}. \tag{3.24}$$

Then, m_{r+1} becomes even and the last sum is congruent to $-1 \pmod{p}$ as shown in the previous case. Now, since the number of even m_{r+1} in the range $m_r < m_{r+1} < p$ is $\frac{1}{2}(p-m_r)$ (note m_r is odd), the sum $\sum_{m_r<m_{r+1}<p} \frac{1}{m_{r+1}^0}$ is congruent to $-\frac{m_r}{2} \pmod{p}$ and the conclusion follows. ∎

We obtain several congruence relations of polycosecant numbers with negative upper index. We recall the following identity which will be needed in proving our main results.

Proposition 3.11. *[1] For any positive integers n, m, we have*

$$\left\{ {n \atop m} \right\} = \frac{(-1)^m}{m!}\sum_{l=0}^{m}(-1)^l\binom{m}{l} l^n.$$

By using Euler's Theorem, we obtain the following theorem which is an analogue of Kummer's congruence for poly-Bernoulli numbers.

Theorem 3.12. *Let n, m, k, N be natural numbers and p be a prime number satisfying $2n, 2m \geq N$, $2n \equiv 2m \pmod{p^{N-1}(p-1)}$. We have*

$$D_{2n}^{(-2k-1)} \equiv D_{2m}^{(-2k-1)} \pmod{p^N} \tag{3.25}$$

Proof. Suppose $n \geq m$ without loss of generality. By the duality (Theorem 3.1), 3.25 is equivalent to the congruence $D_{2k}^{(-2n-1)} \equiv D_{2k}^{(-2m-1)} \pmod{p^N}$ which we now prove. Then, by using the second explicit formula of $D_n^{(k)}$, we have

$$D_{2k}^{(-2n-1)} = \sum_{l=0}^{k} \sum_{q=2l}^{2k} \frac{(-1)^q (q+1)!}{2^q} \binom{q}{2l} \left\{ \begin{matrix} 2k+1 \\ q+1 \end{matrix} \right\} (2l+1)^{2n}$$

This can be written as

$$\begin{aligned} D_{2k}^{(-2n-1)} &= \sum_{\substack{l=0 \\ p \mid (2l+1)}}^{k} \sum_{q=2l}^{2k} \frac{(-1)^q (q+1)!}{2^q} \binom{q}{2l} \left\{ \begin{matrix} 2k+1 \\ q+1 \end{matrix} \right\} (2l+1)^{2n} \\ &\quad + \sum_{\substack{l=0 \\ p \nmid (2l+1)}}^{k} \sum_{q=2l}^{2k+1} \frac{(-1)^q (q+1)!}{2^q} \binom{q}{2l} \left\{ \begin{matrix} 2k+1 \\ q+1 \end{matrix} \right\} (2l+1)^{2n} \\ &\equiv \sum_{\substack{l=0 \\ p \nmid (2l+1)}}^{k} \sum_{q=2l}^{2k} \frac{(-1)^q (q+1)!}{2^q} \binom{q}{2l} \left\{ \begin{matrix} 2k+1 \\ q+1 \end{matrix} \right\} (2l+1)^{2n} \pmod{p^N}. \end{aligned}$$

By using Euler's Theorem, we have

$$D_{2k}^{(-2n-1)} \equiv \sum_{\substack{l=0 \\ p \nmid (2l+1)}}^{k} \sum_{q=2l}^{2k} \frac{(-1)^q (q+1)!}{2^q} \binom{q}{2l} \left\{ \begin{matrix} 2k+1 \\ q+1 \end{matrix} \right\} (2l+1)^{2m} \pmod{p^N}.$$

Therefore, we get $D_{2k}^{(-2n-1)} \equiv D_{2k}^{(-2m-1)} \pmod{p^N}$ and the duality relation in Theorem 3.1 gives the desired result. ■

We obtain the following theorem by using Fermat's Little Theorem.

Theorem 3.13. *Let k be an even integer and $p > 2$ be a prime number. We have*

$$D_{m(p-1)}^{(-k-1)} \equiv 1 \pmod{p} \tag{3.26}$$

for all $m \geq 1$.

Proof. We first prove the case of $m = 1$. From formula (2) of Theorem 3.4, we get

$$\begin{aligned} D_{p-1}^{(-k-1)} &= \sum_{m=0}^{\frac{p-1}{2}} \sum_{l=2m}^{p-1} \frac{(-1)^l (l+1)!}{2^l} \binom{l}{2m} \begin{Bmatrix} p \\ l+1 \end{Bmatrix} (2m+1)^k \\ &\equiv \sum_{m=0}^{\lfloor \frac{p-3}{2} \rfloor} \sum_{l=2m}^{p-2} \frac{(-1)^l (l+1)!}{2^l} \binom{l}{2m} \begin{Bmatrix} p \\ l+1 \end{Bmatrix} (2m+1)^k \pmod{p}. \end{aligned}$$

By substituting Proposition 3.11 into the above equation, we get

$$\begin{aligned} D_{p-1}^{(-k-1)} &\equiv \sum_{m=0}^{\lfloor \frac{p-3}{2} \rfloor} (2m+1)^k \sum_{l=2m}^{p-2} \frac{1}{2^l} \binom{l}{2m} \sum_{t=1}^{l+1} (-1)^{t+1} \binom{l+1}{t} t^p \pmod{p} \\ &\equiv - \sum_{m=0}^{\lfloor \frac{p-3}{2} \rfloor} (2m+1)^{k+1} \sum_{l=2m}^{p-2} \frac{1}{2^l} \binom{l+1}{2m+1} \sum_{t=1}^{l+1} (-1)^t \binom{l}{t-1} t^{p-1} \pmod{p}. \end{aligned}$$

By using Fermat's Little Theorem, we have

$$D_{p-1}^{(-k-1)} \equiv - \sum_{m=0}^{\lfloor \frac{p-2}{2} \rfloor} (2m+1)^{k+1} \sum_{l=2m}^{p-2} \frac{1}{2^l} \binom{l+1}{2m+1} \sum_{t=1}^{l+1} (-1)^t \binom{l}{t-1} \pmod{p}.$$

We have

$$\sum_{t=1}^{l+1} (-1)^t \binom{l}{t-1} = \begin{cases} 0 & l \geq 1 \\ -1 & l = 0. \end{cases}$$

We can see that, only the term with $m = 0$, $l = 0$ remains. Therefore, the conclusion follows. For general m, we see from Theorem 3.12 that

$$D_{m(p-1)}^{(-k-1)} \equiv D_{(p-1)}^{(-k-1)} \equiv 1 \pmod{p}$$

■

Chapter 4

Relations of the level two Arakawa-Kaneko zeta function to multiple T-functions and multiple T-values

4.1 Level two analogue of Arakawa-Kaneko zeta functions and single variable multiple zeta functions

Kaneko and Tsumura defined the single variable multiple zeta function of level two as follows.

Definition 4.1. *(Kaneko, Tsumura [12]) For $k_1, \ldots, k_{r-1} \in \mathbb{Z}_{\geq 1}$ and $Re(s) > 1$, we write*

$$T_0(k_1, \ldots, k_{r-1}, s) = \sum_{\substack{0<m_1<\cdots<m_r \\ m_i \equiv i \pmod 2}} \frac{1}{m_1^{k_1} \cdots m_{r-1}^{k_{r-1}} m_r^s}.$$

Furthermore, as its normalized version,

$$T(k_1, \ldots, k_{r-1}, s) = 2^r T_0(k_1, \ldots, k_{r-1}, s).$$

The values $T(k_1, \ldots, k_{r-1}, k_r)$ ($k_j \in \mathbb{Z}_{\geq 1}$, $k_r \geq 2$: admissible) are called the multiple T-values.

When $k_r > 1$, we see that

$$\mathrm{A}(k_1, \ldots, k_r; 1) = T(k_1, \ldots, k_r).$$

We obtain a level two version of [2, Proposition 2] which will be needed in proving our main results as follows.

Proposition 4.1. *1. For $Re(s) > 1$*

$$T(k_1,\dots,k_{n-1},s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}}{\sinh(t)}\mathrm{A}(k_1,\dots,k_{n-1};e^{-t})dt.$$

2. For $Re(s) > 1, n \geq 2, j \geq 0$

$$\int_0^\infty t^{s+j-1}\mathrm{A}(k_1,\dots,k_{n-1};e^{-t})dt = \Gamma(s+j)T(k_1,\dots,k_{n-2},s+j+k_{n-1}).$$

Proof. To prove (1), we use the definition

$$\begin{aligned} T(k_1,\dots,k_{n-1},s) &= 2^n \sum_{\substack{0<m_1<\cdots<m_n \\ m_i\equiv i \pmod 2}} \frac{1}{m_1^{k_1}\cdots m_{n-1}^{k_{n-1}}m_n^s} \\ &= 2^n \sum_{\substack{0<m_1<\cdots<m_{n-1} \\ m_i\equiv i \pmod 2}} \frac{1}{m_1^{k_1}\cdots m_{n-1}^{k_{n-1}}} \sum_{\substack{m_n=m_{n-1}+1 \\ m_n\not\equiv m_{n-1} \pmod 2}} \frac{1}{m_n^s}, \end{aligned}$$

and use the standard expression

$$\frac{1}{n^s} = \frac{1}{\Gamma(s)}\int_0^\infty e^{-nt}t^{s-1}dt \tag{4.1}$$

to convert the inner sum into the integral. Then we can get the desired results.

To obtain (2), we only need to use the definition

$$\mathrm{A}(k_1,\dots,k_{n-1};e^{-t}) = 2^{n-1}\sum_{\substack{0<m_1<\cdots<m_{n-1} \\ m_i\equiv i \pmod 2}} \frac{e^{-m_{n-1}t}}{m_1^{k_1}\cdots m_{n-1}^{k_{n-1}}}$$

and use equation (4.1) to obtain

$$\int_0^\infty e^{-m_{n-1}t}t^{s+j-1}dt = \frac{\Gamma(s+j)}{m_{n-1}^{s+j}}.$$

This completes the proof of the proposition. ∎

According to these functions, Kaneko and Tsumura defined a level two analogue of $\xi(k_1,\dots,k_r;s)$ as follows.

Definition 4.2. *(M. Kaneko, H. Tsumura [12, §5]) For $k_1,\dots,k_r \in \mathbb{Z}_{\geq 1}$ and $Re(s) > 0$, we write*

$$\psi(k_1,\dots,k_r;s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{\mathrm{A}(k_1,\dots,k_r;\tanh t/2)}{\sinh(t)}dt.$$

The following theorem says that the values at non-positive integers of $\psi(k_1, \dots, k_r; s)$ interpolate multi-polycosecant numbers $D_m^{(k_1,\dots,k_r)}$.

Theorem 4.2. *The function $\psi(k_1, \dots, k_r; s)$ can be continued analytically to $\mathbb{C}$ as an entire function and satisfies*

$$\psi(k_1, \dots, k_r; -m) = (-1)^m D_m^{(k_1,\dots,k_r)} \quad (m \in \mathbb{Z}_{\geq 0})$$

where $(k_1, \dots, k_r) \in \mathbb{Z}_{\geq 1}^r$.

Proof. By the definition

$$\psi(k_1, \dots, k_r; s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\mathrm{A}(k_1, \dots, k_r; \tanh t/2)}{\sinh(t)} dt,$$

we can see that $\psi(k_1, \dots, k_r; s)$ is in the form of Mellin transform of the function $\frac{\mathrm{A}(k_1,\dots,k_r;\tanh t/2)}{\sinh(t)}$. Also, we have

$$\frac{\mathrm{A}(k_1, \dots, k_r; \tanh(t/2))}{\sinh t} = \sum_{n=0}^\infty D_n^{(k_1,\dots,k_r)} \frac{t^n}{n!}.$$

Then by using a standard method (cf. [19, §4 of Part 1]), we can obtain that $\psi(k_1, \dots, k_r; s)$ is an entire function of s and the desired formula at the point $s = -m$. ■

4.2 Relations among the functions ψ and T

In this section, we present our newly obtained results on the level two analogue of Arakawa-Kaneko zeta functions. We deduce that the Arakawa-Kaneko zeta function of level two can be written as a linear combination multiple T-functions.

Let us consider the integral representation of the level two multiple polylogarithm $\mathrm{A}(\mathbf{k}; z)$ as follows.

$$\mathrm{A}(\mathbf{k}; z) = \int_{0<t_1<t_2\cdots<t_k<z} \frac{2dt_1}{1-t_1^2} \underbrace{\frac{dt_2}{t_2} \cdots \frac{dt_{k_1}}{t_{k_1}}}_{(k_1-1)-\text{times}} \cdots \frac{2dt_{k-k_r+1}}{1-t_{k-k_r+1}^2} \underbrace{\frac{dt_{k-k_r+2}}{t_{k-k_r+2}} \cdots \frac{dt_k}{t_k}}_{(k_r-1)-\text{times}},$$

where $\mathbf{k} = (k_1, \dots, k_r)$.

By using the formula

$$\int_{a<t_1\cdots<t_r<b} \underbrace{\frac{dt_1}{t_1} \cdots \frac{dt_r}{t_r}}_{r\text{-times}} = \frac{1}{r!} \left(\log \frac{b}{a} \right)^r, \tag{4.2}$$

we can write above integral expression as

$$\mathrm{A}(k_1,\dots,k_r;z)=\frac{2^r}{\prod_{i=1}^r(k_i-1)!}\int_{0<t_1<t_2\cdots<t_r<z}\frac{dt_1}{1-t_1^2}\left(\log\frac{t_2}{t_1}\right)^{k_1-1}\cdots\frac{dt_r}{1-t_r^2}\left(\log\frac{z}{t_r}\right)^{k_r-1}. \tag{4.3}$$

Change the variables in equation 4.3 by using $t_j\mapsto\frac{1-t_{r+1-j}}{1+t_{r+1-j}}$ $(j=1,2,\dots,r)$. Then, we get

$$\begin{aligned}\mathrm{A}(k_1,\dots,k_r;z)=\frac{1}{\prod_{i=1}^r(k_i-1)!}\int_{E_r(z)}&\frac{dt_r}{t_r}\left(\log\frac{(1+t_r)(1-t_{r-1})}{(1-t_r)(1+t_{r-1})}\right)^{k_1-1}\\&\cdots\frac{dt_2}{t_2}\left(\log\frac{(1+t_2)(1-t_1)}{(1-t_2)(1+t_1)}\right)^{k_{r-1}-1}\frac{dt_1}{t_1}\left(\log\frac{(1+t_1)z}{(1-t_1)}\right)^{k_r-1},\end{aligned} \tag{4.4}$$

where

$$E_r(z):=\left\{(t_1,\dots,t_r)\mid\frac{1-z}{1+z}<t_1<\cdots<t_r<1\right\}.$$

The following theorem was first obtained by Naho Kawasaki. We prove the following theorem by using the integral representations of ψ and T.

Theorem 4.3. *For any index set $\mathbf{k}=(k_1,\dots,k_r)\in\mathbb{Z}_{\geq1}^r$ and $m\in\mathbb{Z}_{\geq1}$, we have*

$$\psi(\mathbf{k};m)=\sum_{|\mathbf{j}|=m-1,\ d(\mathbf{j})=n}b\left((\mathbf{k}_+)^*;\mathbf{j}\right)T\left((\mathbf{k}_+)^*+\mathbf{j}\right),$$

where the sum runs over all $\mathbf{j}\in\mathbb{Z}_{\geq0}^r$ of weight m and depth $n:=d(\mathbf{k}_+^*)=|\mathbf{k}|+1-d(\mathbf{k})$.

Kawasaki and Ohno [13] have given an alternative proof of Theorem 2.5 by using the associated integral representations of $\xi(\mathbf{k};m)$ (see [13, Theorem 2.2]). By a similar argument as in the proof of Theorem 2.2 in [13], Kawasaki obtained Theorem 4.3. Here we give a proof by similar arguments as in [11]. To prove the theorem, we obtain some integral expressions for $\psi(k_1,\dots,k_r;s)$ in the following proposition. This formula is corresponding to Proposition 2.6 (i) in [11].

Proposition 4.4. *Let $(\mathbf{k}_+)^*=(l_1,\dots,l_n)$. Then we have,*

$$\begin{aligned}\psi(k_1,\dots,k_r;s)=&\frac{1}{\prod_{i=1}^n\Gamma(l_i)\Gamma(s)}\int_0^\infty\cdots\int_0^\infty(x_1+\cdots+x_n)^{s-1}x_1^{l_1-1}\cdots x_n^{l_n-1}\\&\times\frac{1}{\sinh(x_1+\cdots+x_n)\sinh(x_2+\cdots+x_n)\cdots\sinh(x_n)}dx_1\cdots dx_n\end{aligned}$$

for $Re(s)>1-r$.

Proof. We take the index $(k_1,\dots,k_r)$ as

$$(k_1,\dots,k_r)=(\underbrace{1,\dots,1}_{(a_1-1)\text{-times}},b_1+1,\dots,\underbrace{1,\dots,1}_{(a_h-1)\text{-times}},b_h+1),$$

for integers $h\geq 1$, $a_i\geq 1$ $(1\leq i\leq h)$, $b_i\geq 1$ $(1\leq i\leq h-1)$, and $b_h\geq 0$. By using Lemma 3.6 we can write the integral expression of $\mathrm{A}(\mathbf{k};z)$ as:

$$\begin{aligned}
&\mathrm{A}(k_1,\dots,k_r;z)\\
&=\underbrace{\int_0^z\frac{dx_h}{x_h}\int_0^{x_h}\cdots\int_0^{x_h}\frac{dx_h}{x_h}}_{b_h\text{-times}}\int_0^{x_h}\frac{1}{a_h!}\left(\log\frac{(1-x_{h-1})(1+x_h)}{(1+x_{h-1})(1-x_h)}\right)^{a_h}\frac{dx_{h-1}}{x_{h-1}}\\
&\cdot\underbrace{\int_0^{x_{h-1}}\frac{dx_{h-1}}{x_{h-1}}\int_0^{x_{h-1}}\cdots\int_0^{x_{h-1}}\frac{dx_{h-1}}{x_{h-1}}}_{(b_{h-1}-1)\text{-times}}\int_0^{x_{h-1}}\frac{1}{a_{h-1}!}\left(\log\frac{(1-x_{h-2})(1+x_{h-1})}{(1+x_{h-2})(1-x_{h-1})}\right)^{a_{h-1}}\frac{dx_{h-2}}{x_{h-2}}\\
&\cdots\underbrace{\int_0^{x_3}\frac{dx_3}{x_3}\int_0^{x_3}\cdots\int_0^{x_3}\frac{dx_3}{x_3}}_{(b_3-1)\text{-times}}\int_0^{x_3}\frac{1}{a_3!}\left(\log\frac{(1-x_2)(1+x_3)}{(1+x_2)(1-x_3)}\right)^{a_3}\frac{dx_2}{x_2}\underbrace{\int_0^{x_2}\frac{dx_2}{x_2}\int_0^{x_2}\cdots\int_0^{x_2}\frac{dx_2}{x_2}}_{(b_2-1)\text{-times}}\\
&\cdot\int_0^{x_2}\frac{1}{a_2!}\left(\log\frac{(1-x_1)(1+x_2)}{(1+x_1)(1-x_2)}\right)^{a_2}\frac{dx_1}{x_1}\underbrace{\int_0^{x_1}\frac{dx_1}{x_1}\int_0^{x_1}\cdots\int_0^{x_1}\frac{dx_1}{x_1}\int_0^{x_1}}_{(b_1-1)\text{-times}}\frac{1}{a_1!}\left(\log\frac{(1+x)}{(1-x)}\right)^{a_1}\frac{dx}{x}.
\end{aligned}$$

The paths of the integration are in the domain $\mathbb{C}\setminus[1,\infty)$, and the formula is valid for $z\in\mathbb{C}\setminus[1,\infty)$. Put $z=\tanh t/2$ in the above equation and change the variables accordingly. Then, we get

$$\begin{aligned}
&\mathrm{A}(k_1,\dots,k_r;\tanh t/2)\\
&=\int_0^t\int_0^{t_{b_1+\cdots+b_h}}\cdots\int_0^{t_2}\underbrace{\frac{1}{\sinh(t_{b_1+\cdots+b_h})\cdots\sinh(t_{b_1+\cdots+b_{h-1}+2})}}_{(b_h-1)\text{-times}}\\
&\times\frac{1}{a_h!}\frac{(t_{b_1+\cdots+b_{h-1}+1}-t_{b_1+\cdots+b_{h-1}})^{a_h}}{\sinh(t_{b_1+\cdots+b_{h-1}+1})}\cdot\underbrace{\frac{1}{\sinh(t_{b_1+\cdots+b_{h-1}})\cdots\sinh(t_{b_1+\cdots+b_{h-2}+2})}}_{(b_{h-1}-1)\text{-times}}\\
&\times\cdots\frac{1}{a_3!}\frac{(t_{b_1+b_2+1}-t_{b_1+b_2})^{a_3}}{\sinh(t_{b_1+b_2+1})}\cdot\underbrace{\frac{1}{\sinh(t_{b_1+b_2})\cdots\sinh(t_{b_1+2})}}_{(b_2-1)\text{-times}}\\
&\frac{1}{a_2!}\frac{(t_{b_1+1}-t_{b_1})^{a_2}}{\sinh(t_{b_1+1})}\cdot\underbrace{\frac{1}{\sinh(t_{b_1})\cdots\sinh(t_2)}}_{(b_1-1)\text{-times}}\cdot\frac{1}{a_1!}\frac{t_1^{a_1}}{\sinh(t_1)}dt_1dt_2\cdots dt_{b_1+\cdots+b_h}.
\end{aligned}$$

By substituting the above equation in Definition 4.2 and changing the variables as: $t=x_1+\cdots+x_n$, $t_{b_1+\cdots+b_h}=x_2+\cdots+x_n$, $t_{b_1+\cdots+b_h-1}=x_3+\cdots+x_n$, …, $t_2=x_{n-1}+x_n$, $t_1=x_n$, we obtain the desired formula. We should note that the dual index $(\mathbf{k}_+)^* = (l_1, \dots, l_n)$ is given by

$$(\mathbf{k}_+)^* = (\underbrace{1,\dots,1}_{b_h\text{-times}}, a_h+1, \underbrace{1,\dots,1}_{(b_{h-1}-1)\text{-times}}, a_{h-1}+1, \dots, \underbrace{1,\dots,1}_{(b_1-1)\text{-times}}, a_1+1)$$

and the depth $n = b_1 + \cdots + b_h + 1$. ■

Proof of Theorem 4.3. In order to obtain the formula, put $s = m$ in the above proposition and expand $(x_1 + \cdots + x_k)^{m-1}$ by the multinomial theorem. Then the theorem follows from the below lemma. ■

Lemma 4.5. *[12, Lemma 5.4] For $l_1, \dots, l_{r-1} \in \mathbb{Z}_{\geq 1}$ and $Re(s) > 1$, we have*

$$T(l_1, \dots, l_{r-1}, s) = \frac{1}{\Gamma(l_1)\cdots\Gamma(l_{r-1})\Gamma(s)} \int_0^\infty \cdots \int_0^\infty x_1^{l_1-1} \cdots x_{r-1}^{l_{r-1}-1} x_r^{s-1} \prod_{j=1}^{r} \frac{1}{\sinh(x_j + \cdots + x_r)} dx_1 \cdots dx_r.$$

We obtain the following formula which will also be needed in proving our results.

Lemma 4.6. *1. For $k_1, \dots, k_r \in \mathbb{Z}_{\geq 1}$*

$$\frac{d}{dt}\mathrm{A}\left(k_1, \dots, k_r; \frac{1-z}{1+z}\right) = \begin{cases} -\frac{2}{1-z^2}\mathrm{A}\left(k_1, \dots, k_{r-1}, k_r - 1; \frac{1-z}{1+z}\right) & (k_r \geq 2) \\ -\frac{1}{z}\mathrm{A}\left(k_1, \dots, k_{r-1}; \frac{1-z}{1+z}\right) & (k_r = 1). \end{cases}$$

2. For $r \geq 1$

$$\mathrm{A}\left(\{1\}_r; \frac{1-z}{1+z}\right) = \frac{1}{r!}\left(\mathrm{A}\left(1; \frac{1-z}{1+z}\right)\right)^r = \frac{(-1)^r}{r!}\log^r z.$$

The proof of the above lemma is similar to the proof of Lemma 3.6.

In order to prove the next theorem, we establish the following lemma.

Lemma 4.7. *For any index $\mathbf{k}$, we have*

$$\frac{2}{1-z^2}\mathrm{A}\left(\{1\}_j; \frac{1-z}{1+z}\right)\mathrm{A}(\mathbf{k}; z) = \frac{d}{dz}\left(\sum_{i=0}^{j}\mathrm{A}\left(\{1\}_{j-i}; \frac{1-z}{1+z}\right)\mathrm{A}(\mathbf{k}, i+1; z)\right). \tag{4.5}$$

Proof. By using Lemma 3.6 and Lemma 4.6, we can easily obtain the desired result by induction on j. ■

Now we present the following Euler-type connection formula associated with the multiple polylogarithm functions of level two, corresponding to Theorem 2.6 in Chapter 2.

Theorem 4.8. *Let* $\mathbf{k}$ *be any index. Then we have*

$$\mathrm{A}\left(\mathbf{k};\frac{1-z}{1+z}\right)=\sum_{\mathbf{k}',j\geq 0}C_{\mathbf{k}}(\mathbf{k}';j)\mathrm{A}\left(\{1\}_j;\frac{1-z}{1+z}\right)\mathrm{A}(\mathbf{k}';z),$$

where the sum on the right runs over indices $\mathbf{k}'$ *and integers* $j\geq 0$ *that satisfy* $|\mathbf{k}'|+j\leq|\mathbf{k}|$*, and* $C_{\mathbf{k}}(\mathbf{k}';j)$ *is a* $\mathbb{Q}-$*linear combination of multiple* $T-$*values of weight* $|\mathbf{k}|-|\mathbf{k}'|-j$*. We understand* $\mathrm{Ath}_{\emptyset}(z)=1$ *and* $|\emptyset|=0$ *for the empty index* $\emptyset$*, and the constant* 1 *is regarded as a multiple* T*-value of weight* 0*.*

Proof. We prove this by induction on the weight $\mathbf{k}$. When $\mathbf{k}=(1)$, the trivial identity

$$\mathrm{A}_1\left(\frac{1-z}{1+z}\right)=\mathrm{A}_1\left(\frac{1-z}{1+z}\right)$$

itself gives the desired form, thus $C_{(1)}(\emptyset;0)=C_{(1)}((1);0)=0$ and $C_{(1)}(\emptyset;1)=1$. Suppose the weight $|\mathbf{k}|>1$ and assume the statement holds for any index of weight less than $|\mathbf{k}|$.

For $\mathbf{k}=(k_1,\ldots,k_r)$, set $\mathbf{k}_-=(k_1,\ldots,k_{r-1},k_r-1)$.

First, assume that $\mathbf{k}$ is admissible. Then by the differential relation and the induction hypothesis, we get

$$\begin{aligned}\frac{d}{dz}\mathrm{A}\left(\mathbf{k};\frac{1-z}{1+z}\right)&=-\frac{2}{1-z^2}\mathrm{A}\left(\mathbf{k}_-;\frac{1-z}{1+z}\right)\\&=-\frac{2}{1-z^2}\sum_{\mathbf{l},j\geq 0}C_{\mathbf{k}_-}(\mathbf{l};j)\mathrm{A}\left(\{1\}_j;\frac{1-z}{1+z}\right)\mathrm{A}(\mathbf{l};z).\end{aligned}\tag{4.6}$$

Let the depth of $\mathbf{l}$ be s. By substituting (4.5) from Lemma 4.7 into (4.6) and integrating, we get

$$\mathrm{A}\left(\mathbf{k};\frac{1-z}{1+z}\right)=-\sum_{\mathbf{l},j\geq 0}C_{\mathbf{k}_-}(\mathbf{l};j)\left(\sum_{i=0}^{j}\mathrm{A}\left(\{1\}_{j-i};\frac{1-z}{1+z}\right)\mathrm{A}(\mathbf{l},i+1;z)\right)+C,$$

where C is a constant. Since

$$\lim_{z\to 0}\mathrm{A}\left(\{1\}_{j-i};\frac{1-z}{1+z}\right)\mathrm{A}(\mathbf{l},i+1;z)=0,$$

we have $C=T(\mathbf{k})$. Now we can obtain the desired result.

In order to prove the non-admissible case, we recall that $\mathrm{A}\left(\mathbf{k};\frac{1-z}{1+z}\right)$ satisfies the shuffle relation (cf. [5]). Suppose $\mathbf{k}$ is not admissible. Then, we can write $\mathrm{A}\left(\mathbf{k};\frac{1-z}{1+z}\right)$ as a polynomial of $\mathrm{A}\left(1;\frac{1-z}{1+z}\right)$ with each coefficient of $\mathrm{A}^i\left(1;\frac{1-z}{1+z}\right)$ being a linear combination of $\mathrm{A}\left(\mathbf{k}';\frac{1-z}{1+z}\right)$, $\mathbf{k}'$: admissible. Write this polynomial as

$$\mathrm{A}\left(\mathbf{k}; \frac{1-z}{1+z}\right) = \sum_{j=0}^{m} a_i \cdot \mathrm{A}^j \left(1; \frac{1-z}{1+z}\right).$$

Then a_i can be written in the desired form (admissible case). We know that

$$\mathrm{A}^j \left(1; \frac{1-z}{1+z}\right) = j!\,\mathrm{A}\left(\{1\}_j; \frac{1-z}{1+z}\right)$$

and

$$\mathrm{A}\left(\{1\}_i; \frac{1-z}{1+z}\right) \mathrm{A}\left(\{1\}_j; \frac{1-z}{1+z}\right) = \binom{i+j}{i} \mathrm{A}\left(\{1\}_{i+j}; \frac{1-z}{1+z}\right).$$

Hence $a_i \cdot \mathrm{A}^j \left(1; \frac{1-z}{1+z}\right)$ can be written in the claimed form, and the proof is done. ■

From Theorem 4.8, we can obtain formulas expressing $\psi(\mathbf{k}; s)$ in terms of multiple zeta functions.

Theorem 4.9. *Let* $\mathbf{k}$ *be any index set. The function* $\psi(\mathbf{k}; s)$ *can be written in terms of multiple* $T-$*functions as*

$$\psi(\mathbf{k}; s) = \sum_{\mathbf{k}', j \geq 0} C_{\mathbf{k}}(\mathbf{k}'; j) \binom{s+j-1}{j} T(\mathbf{k}'; s+j)$$

Here, the sum is over indices $\mathbf{k}'$ *and integers* $j \geq 0$ *that satisfy* $|\mathbf{k}'|+j \leq |\mathbf{k}|$, *and* $C_{\mathbf{k}}(\mathbf{k}'; j)$ *is the same as in Theorem 4.8.*

Proof. Let r, l be the depths of $\mathbf{k}$ and $\mathbf{k}'$ respectively. Put $z = e^{-t}$ in the above lemma.

$$\mathrm{A}\left(\mathbf{k}; \frac{1-e^{-t}}{1+e^{-t}}\right) = \sum_{\mathbf{k}', j \geq 0} C_{\mathbb{K}}(\mathbf{k}'; j) \mathrm{A}\left(\{1\}_j; \frac{1-e^{-t}}{1+e^{-t}}\right) \mathrm{A}(\mathbf{k}'; e^{-t}).$$

By using Lemma 3.6 we can write the above equation as

$$\mathrm{A}\left(\mathbf{k}; \tanh t/2\right) = \sum_{\mathbf{k}', j \geq 0} C_{\mathbf{k}}(\mathbf{k}'; j) \frac{t^j}{j!} \mathrm{A}(\mathbf{k}'; e^{-t}). \tag{4.7}$$

Recall the definition

$$\psi(\mathbf{k}; s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\mathrm{A}(\mathbf{k}; \tanh t/2)}{\sinh(t)} dt,$$

and we substitute equation (4.7) into the above equation and apply Proposition 4.1 to obtain the desired formula for $\psi(\mathbf{k}; s)$. ■

Example 4.1. We give examples for the identity in Theorem 4.8 up to weight 5.

$$
\begin{aligned}
\mathrm{A}\left(2;\frac{1-z}{1+z}\right) &= -\mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(1;z) - \mathrm{A}(2;z) + T(2)\\
\mathrm{A}\left(3;\frac{1-z}{1+z}\right) &= \mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(1,1;z) + \mathrm{A}\left(1,2;z\right) + \mathrm{A}(2,1;z) - T(2)\mathrm{A}(1;z) + T(3),\\
\mathrm{A}\left(1,2;\frac{1-z}{1+z}\right) &= -\mathrm{A}\left(1,1;\frac{1-z}{1+z}\right)\mathrm{A}(1;z) - \mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(2;z) - \mathrm{A}(3;z) + T(1,2),\\
\mathrm{A}\left(2,1;\frac{1-z}{1+z}\right) &= \mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(2;z) + T(2)\mathrm{A}\left(1;\frac{1-z}{1+z}\right) + 2\mathrm{A}(3;z) - 2T(1,2),\\
\mathrm{A}\left(4;\frac{1-z}{1+z}\right) &= -\mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(1,1,1;z) - \mathrm{A}(1,1,2;z) - \mathrm{A}(1,2,1;z) - \mathrm{A}(2,1,1;z)\\
&\quad + T(2)\mathrm{A}(1,1;z) - T(3)\mathrm{A}(1;z) + T(4),\\
\mathrm{A}\left(1,3;\frac{1-z}{1+z}\right) &= \mathrm{A}\left(1,1;\frac{1-z}{1+z}\right)\mathrm{A}(1,1;z) + \mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(1,2;z) + \mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(2,1;z)\\
&\quad + \mathrm{A}(1,3;z) + \mathrm{A}(2,2;z) + \mathrm{A}(3,1;z) - T(1,2)\mathrm{A}(1;z) + T(1,3),\\
\mathrm{A}\left(2,2;\frac{1-z}{1+z}\right) &= -\mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(2,1;z) - T(2)\mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(1;z)\\
&\quad - \mathrm{A}(2,2;z) - 2\mathrm{A}(3,1;z) - T(2)\mathrm{A}(2;z) + 2T(1,2)\mathrm{A}(1;z) + T(2,2),\\
\mathrm{A}\left(3,1;\frac{1-z}{1+z}\right) &= -\mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(1,2;z) + T(3)\mathrm{A}\left(1;\frac{1-z}{1+z}\right) - 2\mathrm{A}(1,3;z) - \mathrm{A}(2,2;z)\\
&\quad + T(2)\mathrm{A}(2;z) - 2T(1,3) - T(2,2),\\
\mathrm{A}\left(1,1,2;\frac{1-z}{1+z}\right) &= -\mathrm{A}\left(1,1,1;\frac{1-z}{1+z}\right)\mathrm{A}(1;z) - \mathrm{A}\left(1,1;\frac{1-z}{1+z}\right)\mathrm{A}(2;z)\\
&\quad - \mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(3;z) - \mathrm{A}(4;z) + T(1,1,2),\\
\mathrm{A}\left(1,2,1;\frac{1-z}{1+z}\right) &= \mathrm{A}\left(1,1;\frac{1-z}{1+z}\right)\mathrm{A}(2;z) + 2\mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(3;z) + T(1,2)\mathrm{A}\left(1;\frac{1-z}{1+z}\right)\\
&\quad + 3\mathrm{A}(4;z) - 3T(1,1,2),\\
\mathrm{A}\left(2,1,1;\frac{1-z}{1+z}\right) &= T(2)\mathrm{A}\left(1,1;\frac{1-z}{1+z}\right) - \mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(3;z) - 2T(1,2)\mathrm{A}\left(1;\frac{1-z}{1+z}\right)\\
&\quad - 3\mathrm{A}(4;z) + 3T(1,1,2),\\
\mathrm{A}\left(5;\frac{1-z}{1+z}\right) &= \mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(1,1,1,1;z) + \mathrm{A}(1,1,1,2;z) + \mathrm{A}(1,1,2,1;z) + \mathrm{A}(1,2,1,1;z)\\
&\quad + \mathrm{A}(2,1,1,1;z) - T(2)\mathrm{A}(1,1,1;z) + T(3)\mathrm{A}(1,1;z) - T(4)\mathrm{A}(1;z) + T(5),\\
\mathrm{A}\left(1,4;\frac{1-z}{1+z}\right) &= -\mathrm{A}\left(1,1;\frac{1-z}{1+z}\right)\mathrm{A}(1,1,1;z) - \mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(1,1,2;z)\\
&\quad - \mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(1,2,1;z) - \mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(2,1,1;z) - \mathrm{A}(1,1,3;z)\\
&\quad - \mathrm{A}(1,2,2;z) - \mathrm{A}(1,3,1;z) - \mathrm{A}(2,1,2;z) - \mathrm{A}(2,2,1;z) - \mathrm{A}(3,1,1;z)\\
&\quad + T(1,2)\mathrm{A}(1,1;z) - T(1,3)\mathrm{A}(1;z) + T(1,4),
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{A}\left(2,3;\frac{1-z}{1+z}\right) &= \mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(2,1,1;z) + T(2)\mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(1,1;z) + \mathrm{A}(2,1,2;z) \\
&\quad + \mathrm{A}(2,2,1;z) + 2\mathrm{A}(3,1,1;z) + T(2)\mathrm{A}(1,2;z) + T(2)\mathrm{A}(2,1;z) \\
&\quad - 2T(1,2)\mathrm{A}(1,1;z) - T(2,2)\mathrm{A}(1;z) + T(2,3), \\
\mathrm{A}\left(3,2;\frac{1-z}{1+z}\right) &= \mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(1,2,1;z) - T(3)\mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(1;z) + \mathrm{A}(1,2,2;z) \\
&\quad + 2\mathrm{A}(1,3,1;z) + \mathrm{A}(2,2,1;z) - T(2)\mathrm{A}(2,1;z) - T(3)\mathrm{A}(2;z) \\
&\quad + 2T(1,3)\mathrm{A}(1;z) + T(2,2)\mathrm{A}(1;z) + T(3,2), \\
\mathrm{A}\left(4,1;\frac{1-z}{1+z}\right) &= \mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(1,1,2;z) + T(4)\mathrm{A}\left(1;\frac{1-z}{1+z}\right) + 2\mathrm{A}(1,1,3;z) \\
&\quad + \mathrm{A}(1,2,2;z) + \mathrm{A}(2,1,2;z) - T(2)\mathrm{A}(1,2;z) + T(3)\mathrm{A}(2;z) \\
&\quad - 2T(1,4) - T(2,3) - T(3,2), \\
\mathrm{A}\left(1,1,3;\frac{1-z}{1+z}\right) &= \mathrm{A}\left(1,1,1;\frac{1-z}{1+z}\right)\mathrm{A}(1,1;z) + \mathrm{A}\left(1,1;\frac{1-z}{1+z}\right)\mathrm{A}(1,2;z) \\
&\quad + \mathrm{A}\left(1,1;\frac{1-z}{1+z}\right)\mathrm{A}(2,1;z) + \mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(1,3;z) \\
&\quad + \mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(2,2;z) + \mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(3,1;z) + \mathrm{A}(1,4;z) \\
&\quad + \mathrm{A}(2,3;z) + \mathrm{A}(3,2;z) + \mathrm{A}(4,1;z) - T(1,1,2)\mathrm{A}(1;z) + T(1,1,3), \\
\mathrm{A}\left(1,2,2;\frac{1-z}{1+z}\right) &= -\mathrm{A}\left(1,1;\frac{1-z}{1+z}\right)\mathrm{A}(2,1;z) - \mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(2,2;z) \\
&\quad - 2\mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(3,1;z) - T(1,2)\mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(1;z) - \mathrm{A}(2,3;z) \\
&\quad - 2\mathrm{A}(3,2;z) + 3T(1,1,2)\mathrm{A}(1;z) - 3\mathrm{A}(4,1;z) - T(1,2)\mathrm{A}(2;z) \\
&\quad + T(1,2,2), \\
\mathrm{A}\left(1,3,1;\frac{1-z}{1+z}\right) &= -\mathrm{A}\left(1,1;\frac{1-z}{1+z}\right)\mathrm{A}(1,2;z) - 2\mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(1,3;z) \\
&\quad - \mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(2,2;z) + T(1,3)\mathrm{A}\left(1;\frac{1-z}{1+z}\right) + T(1,2)\mathrm{A}(2;z) \\
&\quad - 3\mathrm{A}(1,4;z) - 2\mathrm{A}(2,3;z) - \mathrm{A}(3,2;z) - 3T(1,1,3) - T(1,2,2), \\
\mathrm{A}\left(2,1,2;\frac{1-z}{1+z}\right) &= -T(2)\mathrm{A}\left(1,1;\frac{1-z}{1+z}\right)\mathrm{A}(1;z) + \mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(3,1;z) \\
&\quad - T(2)\mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(2;z) + 2T(1,2)\mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(1;z) + \mathrm{A}(3,2;z) \\
&\quad + 3\mathrm{A}(4,1;z) - T(2)\mathrm{A}(3;z) + 2T(1,2)\mathrm{A}(2;z) - 3T(1,1,2)\mathrm{A}(1;z) + T(2,1,2), \\
\mathrm{A}\left(2,2,1;\frac{1-z}{1+z}\right) &= \mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(2,2;z) + T(2)\mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(2;z) + T(2,2)\mathrm{A}\left(1;\frac{1-z}{1+z}\right) \\
&\quad + 2\mathrm{A}(2,3;z) + 2\mathrm{A}(3,2;z) + 2T(2)\mathrm{A}(3;z) - 2T(1,2)\mathrm{A}(2;z) \\
&\quad - 2T(1,2,2) - 2T(2,1,2),
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{A}\left(3,1,1;\frac{1-z}{1+z}\right) &= T(3)\mathrm{A}\left(1,1;\frac{1-z}{1+z}\right)+\mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(1,3;z)-2T(1,3)\mathrm{A}\left(1;\frac{1-z}{1+z}\right)\\
&\quad -T(2,2)\mathrm{A}\left(1;\frac{1-z}{1+z}\right)+3\mathrm{A}(1,4;z)+\mathrm{A}(2,3;z)-T(2)\mathrm{A}(3;z)\\
&\quad +3T(1,1,3)+2T(1,2,2)+T(2,1,2),\\
\mathrm{A}\left(1,1,1,2;\frac{1-z}{1+z}\right) &= -\mathrm{A}\left(1,1,1,1;\frac{1-z}{1+z}\right)\mathrm{A}(1;z)-\mathrm{A}\left(1,1,1;\frac{1-z}{1+z}\right)\mathrm{A}(2;z)\\
&\quad -\mathrm{A}\left(1,1;\frac{1-z}{1+z}\right)\mathrm{A}(3;z)-\mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(4;z)-\mathrm{A}(5;z)\\
&\quad +T(1,1,1,2),\\
\mathrm{A}\left(1,1,2,1;\frac{1-z}{1+z}\right) &= \mathrm{A}\left(1,1,1;\frac{1-z}{1+z}\right)\mathrm{A}(2;z)+2\mathrm{A}\left(1,1;\frac{1-z}{1+z}\right)\mathrm{A}(3;z)+3\mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(4;z)\\
&\quad +T(1,1,2)\mathrm{A}\left(1;\frac{1-z}{1+z}\right)+4\mathrm{A}(5;z)-4T(1,1,1,2),\\
\mathrm{A}\left(1,2,1,1;\frac{1-z}{1+z}\right) &= -\mathrm{A}\left(1,1;\frac{1-z}{1+z}\right)\mathrm{A}(3;z)+T(1,2)\mathrm{A}\left(1,1;\frac{1-z}{1+z}\right)-3\mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(4;z)\\
&\quad -3T(1,1,2)\mathrm{A}\left(1;\frac{1-z}{1+z}\right)-6\mathrm{A}(5;z)+6T(1,1,1,2),\\
\mathrm{A}\left(2,1,1,1;\frac{1-z}{1+z}\right) &= T(2)\mathrm{A}\left(1,1,1;\frac{1-z}{1+z}\right)-2T(1,2)\mathrm{A}\left(1,1;\frac{1-z}{1+z}\right)\\
&\quad +\mathrm{A}\left(1;\frac{1-z}{1+z}\right)\mathrm{A}(4;z)+3T(1,1,2)\mathrm{A}\left(1;\frac{1-z}{1+z}\right)+4\mathrm{A}(5;z)-4T(1,1,1,2).
\end{aligned}
$$

Accordingly, we have

$$
\begin{aligned}
\psi(2;s) &= -T(2,s)-sT(1,s+1)+T(2)T(s),\\
\psi(3;s) &= T(1,2,s)+T(2,1,s)+sT(1,1,s+1)-T(2)T(1,s)+T(3)T(s),\\
\psi(1,2;s) &= -T(3,s)-sT(2,s+1)-\frac{s(s+1)}{2}T(1,s+2)+T(1,2)T(s),\\
\psi(2,1;s) &= 2T(3,s)+sT(2,s+1)+sT(2)T(s+1)-2T(1,2)T(s),\\
\psi(4;s) &= -T(1,1,2,s)-T(1,2,1,s)-T(2,1,1,s)-sT(1,1,1,s+1)+T(2)T(1,1,s)\\
&\quad -T(3)T(1,s)+T(4)T(s),\\
\psi(1,3;s) &= T(1,3,s)+T(3,1,s)+T(2,2,s)+sT(1,2,s+1)+sT(2,1,s+1)\\
&\quad +\frac{s(s+1)}{2}T(1,1,s+2)-T(1,2)T(1,s)+T(1,3)T(s),\\
\psi(2,2;s) &= -2T(3,1,s)-T(2,2,s)-sT(2,1,s+1)-sT(2)T(1,s+1)-T(2)T(2,s)\\
&\quad +2T(1,2)T(1,s)+T(2,2)T(s),\\
\psi(3,1;s) &= -2T(1,3,s)-T(2,2;s)-sT(1,2,s+1)+T(2)T(2,s)+sT(3)T(s+1)\\
&\quad -2T(1,3)T(s)-T(2,2)T(s),
\end{aligned}
$$

$$
\begin{aligned}
\psi(1,1,2;s) &= -T(4,s) - sT(3,s+1) - \frac{s(s+1)}{2}T(2,s+2) - \frac{s(s+1)(s+2)}{6}T(1,s+3) \\
&\quad + T(1,1,2)T(s), \\
\psi(1,2,1;s) &= 3T(4,s) + 2sT(3,s+1) + \frac{s(s+1)}{2}T(2,s+2) + sT(1,2)T(s+1) - 3T(1,1,2)T(s), \\
\psi(2,1,1;s) &= -3T(4,s) - sT(3,s+1) + \frac{s(s+1)}{2}T(2)T(s+2) - 2sT(1,2)T(s+1) \\
&\quad + 3T(1,1,2)T(s), \\
\psi(5;s) &= T(1,1,1,2,s) + T(1,1,2,1,s) + T(1,2,1,1,s) + T(2,1,1,1,s) + sT(1,1,1,1,s+1) \\
&\quad - T(2)T(1,1,1,s) + T(3)T(1,1,s) - T(4)T(1,s) + T(5)T(s), \\
\psi(1,4;s) &= -T(1,3,1,s) - T(3,1,1,s) - T(2,2,1,s) - T(1,2,2,s) - T(2,1,2,s) \\
&\quad - T(1,1,3,s) - sT(1,2,1,s+1) - sT(2,1,1,s+1) \\
&\quad - \frac{s(s+1)}{2}T(1,1,1,s+2) - sT(1,1,2,s+1) \\
&\quad + T(1,2)T(1,1,s) - T(1,3)T(1,s) + T(1,4)T(s), \\
\psi(2,3;s) &= 2T(3,1,1,s) + T(2,2,1,s) + T(2,1,2,s) + sT(2,1,1,s+1) + T(2)sT(1,1,s+1) \\
&\quad + T(2)T(2,1,s) + T(2)T(1,2,s) - 2T(1,2)T(1,1,s) \\
&\quad - T(2,2)T(1,s) + T(2,3)T(s), \\
\psi(3,2;s) &= 2T(1,3,1,s) + T(2,2,1,s) + T(1,2,2,s) + sT(1,2,1,s+1) \\
&\quad - T(3)sT(1,s+1) - T(2)T(2,1,s) - T(3)T(2,s) \\
&\quad + 2T(1,3)T(1,s) + T(2,2)T(1,s) + T(3,2)T(s), \\
\psi(4,1;s) &= T(1,2,2,s) + T(2,1,2,s) + 2T(1,1,3,s) + sT(1,1,2,s+1) \\
&\quad + T(4)sT(s+1) - T(2)T(1,2,s) + T(3)T(2,s) \\
&\quad - 2T(1,4)T(s) - T(2,3)T(s) - T(3,2)T(s), \\
\psi(1,1,3;s) &= T(1,4,s) + T(4,1,s) + T(3,2,s) + T(2,3,s) + sT(3,1,s+1) \\
&\quad + \frac{s(s+1)}{2}T(2,1,s+2) + sT(2,2,s+1) \\
&\quad + \frac{s(s+1)}{2}T(1,2,s+2) + sT(1,3,s+1) \\
&\quad + \frac{s(s+1)(s+2)}{6}T(1,1,s+3) - T(1,1,2)T(1,s) + T(1,1,3)T(s), \\
\psi(1,2,2;s) &= -3T(4,1;z) - 2T(3,2;z) - T(2,3;z) - 2sT(3,1,s+1) \\
&\quad - \frac{s(s+1)}{2}T(2,1,s+2) - sT(2,2,s+1) - T(1,2)T(2,s) \\
&\quad - T(1,2)sT(1,s+1) + 3T(1,1,2)T(1,s) + T(1,2,2)T(s), \\
\psi(1,3,1;s) &= -T(3,2;z) - 2T(2,3;z) - 3T(1,4;z) - \frac{s(s+1)}{2}T(1,2,s+2) \\
&\quad - 2sT(1,3,s+1) - sT(2,2,s+1) + T(1,3)sT(s+1) \\
&\quad + T(1,2)T(2,s) - 3T(1,1,3)T(s) - T(1,2,2)T(s),
\end{aligned}
$$

$$\begin{aligned}
\psi(2,1,2;s) &= 3T(4,1;z) + T(3,2;z) + sT(3,1,s+1) - T(2)\frac{s(s+1)}{2}T(1,s+2) \\
&\quad - T(2)sT(2,s+1) + 2T(1,2)sT(1,s+1) - T(2)T(3,s) \\
&\quad + 2T(1,2)T(2,s) - 3T(1,1,2)T(1,s) + T(2,1,2)T(s), \\
\psi(2,2,1;s) &= 2T(2,3;z) + 2T(3,2;z) + T(2)sT(2,s+1) + T(2,2)sT(s+1) \\
&\quad + 2T(2)T(3,s) - 2T(1,2,2)T(s) + sT(2,2,s+1) \\
&\quad - 2T(1,2)T(2,s) - 2T(2,1,2)T(s), \\
\psi(3,1,1;s) &= T(2,3;s) + 3T(1,4;s) \\
&\quad - T(2)T(3,s) - 2T(1,3)sT(s+1) - T(2,2)sT(s+1) \\
&\quad + sT(1,3,s+1) + T(3)\frac{s(s+1)}{2}T(s+2) \\
&\quad + 2T(1,2,2)T(s) + T(2,1,2)T(s) + 3T(1,1,3)T(s), \\
\psi(1,1,1,2;s) &= -T(5;z) - sT(4,s+1) - \frac{s(s+1)}{2}T(3,s+2) \\
&\quad - \frac{s(s+1)(s+2)}{6}T(2,s+3) - \frac{s(s+1)(s+2)(s+3)}{24}T(1,s+4) \\
&\quad + T(1,1,1,2)T(s), \\
\psi(1,1,2,1;s) &= 4T(5;z) + 3sT(4,s+1) + s(s+1)T(3,s+2) \\
&\quad \frac{s(s+1)(s+2)}{6}T(2,s+3) + T(1,1,2)sT(s+1) - 4T(1,1,1,2)T(s), \\
\psi(1,2,1,1;s) &= -6T(5;z) - 3sT(4,s+1) - \frac{s(s+1)}{2}T(3,s+2) \\
&\quad + T(1,2)\frac{s(s+1)}{2}T(s+2) - 3T(1,1,2)sT(s+1) + 6T(1,1,1,2)T(s), \\
\psi(2,1,1,1;s) &= 4T(5;z) + sT(4,s+1) + T(2)\frac{s(s+1)(s+2)}{6}T(s+3) \\
&\quad - T(1,2)s(s+1)T(s+2) + 3T(1,1,2)sT(s+1) - 4T(1,1,1,2)T(s).
\end{aligned}$$

4.2.1 Some explicit forms of Arakawa-Kaneko zeta functions of level two

Theorem 4.8 and Theorem 4.9 can be written explicitly for some special arguments. In this section, we obtain some explicit forms of Theorem 4.8 and Theorem 4.9.

Let us consider the following lemma which will be needed in proving our main results under this section.

Lemma 4.10. *For integers $m \geq 0$ and $n > 0$, we get*

$$\int_0^z \log^m(t)\log^n\left(\frac{1-t}{1+t}\right)\frac{dt}{t} = (-1)^n n! \sum_{l=0}^{m} l! \binom{m}{l} (-1)^l (\log(z))^{m-l} \mathrm{A}(\{1\}_{n-1}, l+2; z). \tag{4.8}$$

In particular,

$$\int_0^1 \log^m(t)\log^n\left(\frac{1-t}{1+t}\right)\frac{dt}{t} = (-1)^{n+m} n!\, m!\, T(\{1\}_{n-1}, m+2). \tag{4.9}$$

Proof. From Lemma (4.6), we have

$$\begin{aligned}\int_0^z \log^m(t)\log^n\left(\frac{1-t}{1+t}\right)\frac{dt}{t} &= (-1)^n n! \int_0^z \frac{\log^m(t)\mathrm{A}(\{1\}_n; t)}{t} dt \\ &= (-1)^n n! \sum_{l=0}^{m} l! \binom{m}{l} (-1)^l (\log(z))^{m-l} \mathrm{A}(\{1\}_{n-1}, l+2; z).\end{aligned}$$

By setting $z \to 1$ in the above equation, we get

$$\begin{aligned}\int_0^1 \log^m(t)\log^n\left(\frac{1-t}{1+t}\right)\frac{dt}{t} &= (-1)^{n+m} n!\, m!\, \mathrm{A}(\{1\}_{n-1}, m+2; 1) \\ &= (-1)^{n+m} n!\, m!\, T(\{1\}_{n-1}, m+2).\end{aligned}$$

This completes the proof of the lemma. ■

We obtain the following identity as a level two generalization of Lemma 3.7 in [12].

Theorem 4.11. *For any positive integers j and r with $j \le r$,*

$$\begin{aligned}\mathrm{A}\left(\{1\}_{j-1}, 2, \{1\}_{r-j}; \frac{1-z}{1+z}\right) &= \sum_{i=0}^{r-j} (-1)^i \binom{i+j}{i} T(i+j+1) \mathrm{A}\left(\{1\}_{r-j-i}; \frac{1-z}{1+z}\right) \\ &\quad + (-1)^{r-j-1} \sum_{l=r-j}^{r} \binom{l}{r-j} \mathrm{A}\left(\{1\}_{r-l}; \frac{1-z}{1+z}\right) \mathrm{A}(l+1; z).\end{aligned}$$

Proof. Set $k_1 = \cdots = k_{j-1} = 1, k_j = 2, k_{j+1} = \cdots = k_r = 1$ and replacing z by $\frac{1-z}{1+z}$ in (4.4). For convenience, we let $E'_r(z) := E_r\left(\frac{1-z}{1+z}\right)$, where

$$E_r(z) := \left\{(t_1, \ldots, t_r) \mid \frac{1-z}{1+z} < t_1 < \cdots < t_r < 1\right\}.$$

Then, we have

$$\begin{aligned}\mathrm{A}\left(\{1\}_{j-1}, 2, \{1\}_{r-j}; \frac{1-z}{1+z}\right) &= \int_{E'_r(z)} \log \frac{(1+t_{r+1-j})(1-t_{r-j})}{(1-t_{r+1-j})(1+t_{r-j})} \frac{dt_1}{t_1} \cdots \frac{dt_r}{t_r} \\ &= \int_{E'_r(z)} \log\left(\frac{1-t_{r-j}}{1+t_{r-j}}\right) \frac{dt_1}{t_1} \cdots \frac{dt_r}{t_r} - \int_{E'_r(z)} \log\left(\frac{1-t_{r+1-j}}{1+t_{r+1-j}}\right) \frac{dt_1}{t_1} \cdots \frac{dt_r}{t_r}.\end{aligned}$$

By using (4.2), we get

$$\begin{aligned}
&\mathrm{A}\left(\{1\}_{j-1}, 2, \{1\}_{r-j}; \frac{1-z}{1+z}\right)\\
&\quad= \int_z^1 \frac{1}{(r-j-1)!}\log^{r-j-1}\left(\frac{t_{r-j}}{z}\right)\cdot\frac{1}{j!}\log^{j}\left(\frac{1}{t_{r-j}}\right)\log\left(\frac{1-t_{r-j}}{1+t_{r-j}}\right)\frac{dt_{r-j}}{t_{r-j}}\\
&\qquad-\int_z^1 \frac{1}{(r-j)!}\log^{r-j}\left(\frac{t_{r-j+1}}{z}\right)\cdot\frac{1}{(j-1)!}\log^{j-1}\left(\frac{1}{t_{r-j+1}}\right)\log\left(\frac{1-t_{r-j+1}}{1+t_{r-j+1}}\right)\frac{dt_{r-j+1}}{t_{r-j+1}}.
\end{aligned}$$

Applying Binomial theorem, we get

$$\begin{aligned}
&\mathrm{A}\left(\{1\}_{j-1}, 2, \{1\}_{r-j}; \frac{1-z}{1+z}\right)\\
&\quad= \sum_{i=0}^{r-1-j}\binom{r-1-j}{i}\frac{(-1)^{r-1-i}}{(r-j-1)!\,j!}\log^{r-1-j-i}(z)\int_z^1 \log^{i+j}(t)\log\frac{(1-t)}{(1+t)}\frac{dt}{t}\\
&\qquad-\sum_{i=0}^{r-j}\binom{r-j}{i}\frac{(-1)^{r-1-i}}{(r-j)!\,(j-1)!}\log^{r-j-i}(z)\int_z^1 \log^{i+j-1}(t)\log\frac{(1-t)}{(1+t)}\frac{dt}{t}\\
&\quad= \sum_{i=0}^{r-j}\frac{(-1)^{r-i-1}}{(r-j-i)!}\frac{(i+j)}{i!\,j!}\log^{r-j-i}(z)\int_1^z \log^{i+j-1}(t)\log\left(\frac{1-t}{1+t}\right)\frac{dt}{t}.
\end{aligned} \tag{4.10}$$

Substitute (4.8) and (4.9) with $n=1$ into (4.10). Then, we get

$$\begin{aligned}
&\mathrm{A}\left(\{1\}_{j-1}, 2, \{1\}_{r-j}; \frac{1-z}{1+z}\right)\\
&\quad= \sum_{i=0}^{r-j}\frac{(-1)^{r+j}}{(r-j-i)!}\binom{i+j}{i}\log^{r-j-i}(z)T(i+j+1)\\
&\qquad\qquad+\sum_{i=0}^{r-j}\sum_{l=0}^{i+j-1}\frac{(-1)^{r-i+l}}{(r-l-1)!}\binom{i+j}{i}\binom{r-l-1}{r-j-i}\log^{r-l-1}(z)\mathrm{A}(l+2;z).
\end{aligned} \tag{4.11}$$

By substituting Lemma 4.6 into the above equation, we get

$$\begin{aligned}
&\mathrm{A}\left(\{1\}_{j-1}, 2, \{1\}_{r-j}; \frac{1-z}{1+z}\right)\\
&\quad= \sum_{i=0}^{r-j}(-1)^{i}\binom{i+j}{i}T(i+j+1)\mathrm{A}\left(\{1\}_{r-j-i}; \frac{1-z}{1+z}\right)\\
&\qquad+\sum_{i=0}^{r-j}\sum_{l=0}^{i+j-1}(-1)^{i+1}\binom{i+j}{i}\binom{r-l-1}{r-j-i}\mathrm{A}\left(\{1\}_{r-l-1}; \frac{1-z}{1+z}\right)\mathrm{A}(l+2;z).
\end{aligned} \tag{4.12}$$

In order to obtain the desired formula, let us simplify the last term as follows.

$$\begin{aligned}
&\sum_{i=0}^{r-j}\sum_{l=0}^{i+j-1}(-1)^{i+1}\binom{i+j}{i}\binom{r-l-1}{r-j-i}\mathrm{A}\left(\{1\}_{r-l-1};\frac{1-z}{1+z}\right)\mathrm{A}(l+2;z)\\
&\qquad=\sum_{i=0}^{r-j}\sum_{l=1}^{i+j}(-1)^{i+1}\binom{i+j}{i}\binom{r-l}{r-j-i}\mathrm{A}\left(\{1\}_{r-l};\frac{1-z}{1+z}\right)\mathrm{A}(l+1;z)\\
&\qquad=\sum_{l=1}^{r}\sum_{i=0}^{r-j}(-1)^{i+1}\binom{i+j}{i}\binom{r-l}{r-j-i}\mathrm{A}\left(\{1\}_{r-l};\frac{1-z}{1+z}\right)\mathrm{A}(l+1;z).
\end{aligned}$$

Here,

$$\binom{r-l}{r-j-i}=\binom{r-l}{i+j-1}.$$

By using the binomial identity 176 in [17], we get

$$\sum_{i=0}^{r-j}(-1)^{i+1}\binom{i+j}{i}\binom{r-l}{i+j-l}=\begin{cases}0 & (j<r-l),\\ (-1)^{r-j-1}\binom{l}{r-j} & (r-l\le j\le r).\end{cases}$$

Substituting this into the above equation, we get

$$\begin{aligned}
&\sum_{i=0}^{r-j}\sum_{l=0}^{i+j-1}(-1)^{i+1}\binom{i+j}{i}\binom{r-l-1}{r-j-i}\mathrm{A}\left(\{1\}_{r-l-1};\frac{1-z}{1+z}\right)\mathrm{A}(l+2;z)\\
&\qquad=\sum_{l=r-j}^{r}(-1)^{r-j-1}\binom{l}{r-j}\mathrm{A}\left(\{1\}_{r-j};\frac{1-z}{1+z}\right)\mathrm{A}(l+1;z).
\end{aligned}$$

By substituting this in equation 4.12, we get the desired results. ■

Accordingly, we can obtain the following theorem for ψ which can be regarded as an explicit form of Theorem 4.9 at some special arguments.

Theorem 4.12. *For positive integers j, r and $Re(s) > 1$ with $j \le r$,*

$$\begin{aligned}
&\psi(\{1\}_{j-1},2,\{1\}_{r-j};s)\\
&\quad=\sum_{i=0}^{r-j}(-1)^{i}\binom{i+j}{i}\binom{s+r-i-j-1}{r-i-j}T(i+j+1)T(s+r-i-j)\\
&\qquad\quad+(-1)^{r-j-1}\sum_{l=r-j}^{r}\binom{l}{r-j}\binom{s+r-l-1}{r-l}T(l+1,s+r-l).
\end{aligned}$$

Proof. Let us consider Theorem 4.11.

$$\mathrm{A}\left(\{1\}_{j-1},2,\{1\}_{r-j};\frac{1-z}{1+z}\right)=\sum_{i=0}^{r-j}(-1)^i\binom{i+j}{i}T(i+j+1)\mathrm{A}\left(\{1\}_{r-j-i};\frac{1-z}{1+z}\right)$$
$$+(-1)^{r-j-1}\sum_{l=r-j}^{r}\binom{l}{r-j}\mathrm{A}\left(\{1\}_{r-l};\frac{1-z}{1+z}\right)\mathrm{A}(l+1;z). \quad (4.13)$$

Now we can see that the right side the the above equation is in the form of Theorem 4.8. We can write the each term of the equation (4.13) in the form of Theorem 4.9. This readily gives the desired result. ■

In order to prove the next result we consider the following shuffle product identity.

Lemma 4.13. *For the integers $m, n \geq 1$, we have*

$$\sum_{j=1}^{m}(-1)^j(y^{m-j}\sqcup y^jx^n)=-\sum_{\alpha_1+\cdots+\alpha_m=m+n,\forall\alpha_i\geq 1}yx^{\alpha_1}\cdots yx^{\alpha_{m-1}}yx^{\alpha_m}.$$

Proof. Consider the left hand-side of the above equation.

$$\begin{aligned}
&\sum_{j=1}^{m}(-1)^jy^{m-j}\sqcup y^jx^n\\
&\quad=(-1)^my^mx^n+\sum_{j=1}^{m-1}(-1)^j\left(y(y^{m-j-1}\sqcup y^jx^n)+y(y^{m-j}\sqcup y^{j-1}x^n)\right)\\
&\quad=(-1)^my^mx^n+\sum_{j=2}^{m}(-1)^{j-1}y(y^{m-j}\sqcup y^{j-1}x^n)+\sum_{j=1}^{m-1}(-1)^jy(y^{m-j}\sqcup y^{j-1}x^n)\\
&\quad=(-1)^my^mx^n+(-1)^{m-1}y^mx^n-y(y^{m-1}\sqcup x^n)\\
&\quad=-y(y^{m-1}\sqcup x^n).
\end{aligned} \quad (4.14)$$

By using the shuffle product formula. we obtain the desired result. ■

We obtain the following formula for $\mathrm{A}(\mathbf{k};z)$.

Theorem 4.14. *For any positive integers r and k,*

$$\mathrm{A}\left(\{1\}_{r-1},k;\frac{1-z}{1+z}\right)=\sum_{j=0}^{k-2}(-1)^{k-j}T(\{1\}_j,r+1)\mathrm{A}(\{1\}_{k-2-j};z)$$
$$+(-1)^{k-1}\sum_{\substack{a_1+\cdots+a_k=r\\ \forall a_j\geq 0}}\mathrm{A}\left(\{1\}_{a_k};\frac{1-z}{1+z}\right)\mathrm{A}(a_1+1,\cdots,a_{k-1}+1;z).$$

Proof. Set $k_1 = \cdots = k_{r-1} = 1, k_r = k$ in (4.4) and replacing z by $\frac{1-z}{1+z}$. For convenience, we let $E'_r(z) := E_r\left(\frac{1-z}{1+z}\right)$. Then, we get

$$\begin{aligned}
\mathrm{A}\left(\{1\}_{r-1}, k; \frac{1-z}{1+z}\right) &= \frac{1}{(k-1)!}\int_{E'_r(z)} \log^{k-1}\frac{(1-z)(1+t_1)}{(1+z)(1-t_1)}\frac{dt_1}{t_1}\cdots\frac{dt_r}{t_r} \\
&= \sum_{j=1}^{k-1}\frac{(-1)^j}{(k-1-j)!\,j!}\log^{k-1-j}\left(\frac{1-z}{1+z}\right)\int_{E'_r(z)}\log^j\left(\frac{1-t_1}{1+t_1}\right)\frac{dt_1}{t_1}\cdots\frac{dt_r}{t_r} \\
&\quad + \frac{1}{(k-1)!}\log^{k-1}\left(\frac{1-z}{1+z}\right)\int\limits_{E'_r(z)}\frac{dt_1}{t_1}\cdots\frac{dt_r}{t_r}.
\end{aligned}$$

By using (4.2), we get

$$\begin{aligned}
\mathrm{A}\left(\{1\}_{r-1}, k; \frac{1-z}{1+z}\right) &= \sum_{j=1}^{k-1}\frac{(-1)^{r-1+j}}{(k-1-j)!\,j!\,(r-1)!}\log^{k-1-j}\left(\frac{1-z}{1+z}\right)\int_z^1\log^{r-1}(t)\log^j\left(\frac{1-t}{1+t}\right)\frac{dt}{t} \\
&\quad + \frac{(-1)^r}{(k-1)!\,r!}\log^{k-1}\left(\frac{1-z}{1+z}\right)\log^r(z). \qquad (4.15)
\end{aligned}$$

Substitute (4.8) and (4.9) into above equation. Then, we get

$$\begin{aligned}
&\mathrm{A}\left(\{1\}_{r-1}, k; \frac{1-z}{1+z}\right) \\
&= \sum_{j=1}^{k-1}\frac{1}{(k-1-j)!}\log^{k-1-j}\left(\frac{1-z}{1+z}\right)T(\{1\}_{j-1}, r+1) \\
&\quad + \sum_{j=1}^{k-1}\sum_{l=0}^{r-1}\frac{(-1)^{r-l}}{(k-1-j)!\,(r-1-l)!}\log^{k-1-j}\left(\frac{1-z}{1+z}\right)\log^{r-l-1}(z)\mathrm{A}(\{1\}_{j-1}, l+2; z) \\
&\quad + \frac{(-1)^r}{(k-1)!\,r!}\log^{k-1}\left(\frac{1-z}{1+z}\right)\log^r(z). \qquad (4.16)
\end{aligned}$$

By substituting Lemma 4.6 into the above equation, we get

$$\begin{aligned}
&\mathrm{A}\left(\{1\}_{r-1}, k; \frac{1-z}{1+z}\right) \\
&= (-1)^{k-1}\mathrm{A}(\{1\}_{k-1}; z)\mathrm{A}\left(\{1\}_r; \frac{1-z}{1+z}\right) + \sum_{j=1}^{k-1}(-1)^{k-1-j}\mathrm{A}(\{1\}_{k-1-j}; z)T(\{1\}_{j-1}, r+1) \\
&\qquad + \sum_{j=1}^{k-1}\sum_{i=0}^{r-1}(-1)^{k-j}\mathrm{A}(\{1\}_{k-1-j}; z)\mathrm{A}\left(\{1\}_{r-i-1}; \frac{1-z}{1+z}\right)\mathrm{A}(\{1\}_{j-1}, i+2; z).
\end{aligned}$$

This can be written as

$$
\begin{aligned}
&\mathrm{A}\left(\{1\}_{r-1}, k; \frac{1-z}{1+z}\right) \\
&= \sum_{j=1}^{k-1} (-1)^{k-1-j} T(\{1\}_{j-1}, r+1) \mathrm{A}(\{1\}_{k-1-j}; z) \\
&\qquad + (-1)^k \sum_{i=0}^{r-1} \mathrm{A}\left(\{1\}_i; \frac{1-z}{1+z}\right) \left(\sum_{j=1}^{k-1} (-1)^j \mathrm{A}(\{1\}_{k-1-j}; z) \mathrm{A}(\{1\}_{j-1}, r+1-i; z) \right) \\
&\qquad\quad + (-1)^{k-1} \mathrm{A}\left(\{1\}_r; \frac{1-z}{1+z}\right) \mathrm{A}(\{1\}_{k-1}; z).
\end{aligned} \tag{4.17}
$$

We know that $\mathrm{A}(\mathbf{k}; z)$ satisfies the shuffle relation. Setting $m = k-1$ and $n = r-j$ in Lemma 4.13, we can write the inner summation of the second term of equation (4.17) as bellow.

$$
\sum_{j=1}^{k-1} (-1)^j \mathrm{A}(\{1\}_{k-1-j}; z) \mathrm{A}(\{1\}_{j-1}, r+1-i; z) = - \sum_{\substack{|\mathbf{k}'| = k-1+r-i, \forall k_i \geq 1 \\ d(\mathbf{k}') = k-1}} \mathrm{A}(\mathbf{k}'; z).
$$

By substituting this into equation 4.17, we get

$$
\begin{aligned}
\mathrm{A}\left(\{1\}_{r-1}, k; \frac{1-z}{1+z}\right) &= \sum_{j=0}^{k-2} (-1)^{k-j} T(\{1\}_j, r+1) \mathrm{A}(\{1\}_{k-2-j}; z) \\
&\quad + (-1)^{k-1} \sum_{i=0}^{r-1} \mathrm{A}\left(\{1\}_i; \frac{1-z}{1+z}\right) \sum_{\substack{|\mathbf{k}'| = k-1+r-i, \forall k_i \geq 1 \\ d(\mathbf{k}') = k-1}} \mathrm{A}(\mathbf{k}'; z) \\
&\quad + (-1)^{k-1} \mathrm{A}\left(\{1\}_r; \frac{1-z}{1+z}\right) \mathrm{A}(\{1\}_{k-1}; z) \\
&= \sum_{j=0}^{k-2} (-1)^{k-j} T(\{1\}_j, r+1) \mathrm{A}(\{1\}_{k-2-j}; z) \\
&\quad + (-1)^{k-1} \sum_{i=0}^{r} \mathrm{A}\left(\{1\}_i; \frac{1-z}{1+z}\right) \sum_{\substack{|\mathbf{k}'| = k-1+r-i, \forall k_i \geq 1 \\ d(\mathbf{k}') = k-1}} \mathrm{A}(\mathbf{k}'; z).
\end{aligned} \tag{4.18}
$$

We can write the second term of the above equation as

$$
(-1)^{k-1} \sum_{\substack{a_1 + \cdots + a_k = r \\ \forall a_j \geq 0}} \mathrm{A}\left(\{1\}_{a_k}; \frac{1-z}{1+z}\right) \mathrm{A}(a_1 + 1, \cdots, a_{k-1} + 1; z). \tag{4.19}
$$

From this we can obtain the desired result. ■

Kaneko and Tsumura obtained a formula for $\psi(\{1\}_{r-1},k;s)$ in [12, Theorem 5.3] which is the level two analogue of [2, Theorem 8]. We provide a proof by using Theorems 4.14, 4.8 and 4.3.

Theorem 4.15 ([12] Theorem 5.3)**.** *For $r,k\geq 1$, we have*

$$\psi(\{1\}_{r-1},k;s)=(-1)^{k-1}\sum_{\substack{a_1+\cdots+a_k=r\\ \forall a_j\geq 0}}\binom{s+a_k-1}{a_k}T(a_1+1,\ldots,a_{k-1}+1,a_k+s)$$
$$+\sum_{j=0}^{k-2}(-1)^jT(\{1\}_{r-1},k-j)T(\{1\}_{j-1},s).$$

Proof. Consider Theorem 4.14.

$$\mathrm{A}\left(\{1\}_{r-1},k;\frac{1-z}{1+z}\right)=\sum_{j=0}^{k-2}(-1)^{k-j}T(\{1\}_j,r+1)\mathrm{A}(\{1\}_{k-2-j};z)$$
$$+(-1)^{k-1}\sum_{\substack{a_1+\cdots+a_k=r\\ \forall a_j\geq 0}}\mathrm{A}\left(\{1\}_{a_k};\frac{1-z}{1+z}\right)\mathrm{A}(a_1+1,\ldots,a_{k-1}+1;z). \quad (4.20)$$

We can see that the right side the the above equation is in the form of Theorem 4.8. Then, we can write the each term of equation (4.20) in the form of Theorem 4.9. By using the duality relation of T, we can obtain the desired result. ■

We obtain the level two analogue of [2, Theorem 9] which gives the result on special values at positive integers..

Theorem 4.16. *Let $m,r\geq 1$, $k\geq 2$ be integers. Then, we have*

$$\psi(\{1\}_{r-1},k;m+1)+(-1)^k\psi(\{1\}_{m-1},k;r+1)=\sum_{j=0}^{k-2}(-1)^jT(\{1\}_{r-1},k-j)T(\{1\}_{m-1},2+j). \tag{4.21}$$

Proof. By the definition

$$\psi(\{1\}_{r-1},k;m+1)=\frac{1}{m!}\int_0^\infty\frac{t^m}{\sinh(t)}\mathrm{A}(\{1\}_{r-1},k;\tanh t/2)dt.$$

Then by using the identity

$$\mathrm{A}(\{1\}_m;\tanh t/2)=\frac{t^m}{m!},$$

we have

$$
\begin{aligned}
&\psi(\{1\}_{r-1},k;m+1)\\
&=\int_0^\infty \mathrm{A}(\{1\}_{r-1},k;\tanh t/2)\mathrm{A}(\{1\}_m;\tanh t/2)\frac{1}{\sinh(t)}dt\\
&=\int_0^\infty \mathrm{A}(\{1\}_{r-1},k;\tanh t/2)\frac{d}{dt}\mathrm{A}(\{1\}_{m-1},2;\tanh t/2)dt\\
&=T(\{1\}_{r-1},k)T(\{1\}_{m-1},2)-\int_0^\infty \frac{d}{dt}\mathrm{A}(\{1\}_{r-1},k;\tanh t/2)\mathrm{A}(\{1\}_{m-1},2;\tanh t/2)dt\\
&=T(\{1\}_{r-1},k)T(\{1\}_{m-1},2)-\int_0^\infty \mathrm{A}(\{1\}_{r-1},k-1;\tanh t/2)\frac{d}{dt}\mathrm{A}(\{1\}_{m-1},3;\tanh t/2)dt\\
&=T(\{1\}_{r-1},k)T(\{1\}_{m-1},2)-T(\{1\}_{r-1},k-1)T(\{1\}_{m-1},3)\\
&\qquad\qquad+\int_0^\infty \mathrm{A}(\{1\}_{r-1},k-2;\tanh t/2)\frac{d}{dt}\mathrm{A}(\{1\}_{m-1},4;\tanh t/2)dt\\
&=\cdots\\
&=\sum_{j=0}^{k-2}(-1)^jT(\{1\}_{r-1},k-j)T(\{1\}_{m-1},2+j)\\
&\qquad\qquad+(-1)^{k+1}\int_0^\infty \mathrm{A}(\{1\}_r;\tanh t/2)\mathrm{A}(\{1\}_{m-1},k;\tanh t/2)\frac{dt}{\sinh t}\\
&=\sum_{j=0}^{k-2}(-1)^jT(\{1\}_{r-1},k-j)T(\{1\}_{m-1},2+j)+(-1)^{k+1}\psi(\{1\}_{m-1},k;r+1).
\end{aligned}
$$

From this we can obtain the desired result. ■

Kaneko and Tsumura obtained the following formula for ψ at positive integers. It can be considered as a special case of Theorem 4.3.

Corollary 4.17 ([12] Theorem 5.5)**.** *Let $r,k\geq 1$ and $m\geq 0$ be integers. Then we have*

$$
\psi(\{1\}_{r-1},k;m+1)=\sum_{\substack{a_1+\cdots+a_k=m\\ \forall a_j\geq 0}}\binom{a_k+r}{r}T(a_1+1,\ldots,a_{k-1}+1,a_k+r+1).
$$

We give a proof to the above theorem by using Theorem 4.15 and Theorem 4.16.

Proof. To prove this, put $s=m+1$ in Theorem 4.15. Then, we have

$$
\begin{aligned}
\psi(\{1\}_{r-1},k;m+1)=(-1)^{k-1}&\sum_{\substack{a_1+\cdots+a_k=r\\ \forall a_j\geq 0}}\binom{a_k+m}{m}T(a_1+1,\ldots,a_{k-1}+1,a_k+m+1)\\
&+\sum_{j=0}^{k-2}(-1)^jT(\{1\}_{r-1},k-j)T(\{1\}_{j-1},m+1).
\end{aligned}
$$

Then compare this with Theorem 4.16 and the duality $T(\{1\}_j, m+1) = T(\{1\}_{m-1}, 2+j)$. Then, we obtain the desired formula for $m \geq 1$. By using the definition and the above mentioned duality we can directly obtain the desired formula for the case $m = 0$. ■

Bibliography

[1] T. Arakawa, T. Ibukiyama and M. Kaneko, *Bernoulli Numbers and Zeta Functions*, Springer, Tokyo, 2014.

[2] T. Arakawa and M. Kaneko, *Multiple zeta values, poly-Bernoulli numbers, and related zeta functions*, Nagoya Math. J. **153** (1999), 189–209.

[3] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, D. Reidel Publishing Co., Dordrecht (1974).

[4] D. Cvijovic, *Higher-order tangent and secant numbers*, Comp. and Math. with Appl. **62** (2011), 1879–1886.

[5] K. Ihara, M. Kaneko and D. Zagier, *Derivation and double shuffle relations for multiple zeta values*, Composition Math. **142** (2006), 307–338.

[6] K. Imatomi, M. Kaneko, E. Takeda, *Multi-poly-Bernoulli numbers and finite multiple zeta values*, J. Integer Seq. **17** (2014).

[7] M. Kaneko, *Poly-Bernoulli numbers*, J. Théor. Nombres Bordeaux **9** (1997), 199–206.

[8] M. Kaneko, *An introduction to classical and finite multiple zeta values*, Publications Mathématiques de Besançon, Algébre et théorie des nombres **19**, no. 1 (2019), 103–129.

[9] M. Kaneko, *Poly-Bernoulli numbers and related zeta functions, Algebraic and Analytic Aspects of Zeta Functions and L-functions*, MSJ Mem., Math. Soc. Japan, Tokyo, **21** (2010), 73–85.

[10] M. Kaneko, M. Pallewatta and H. Tsumura, *On poly-cosecant numbers*, J. Integer Seq. **23** (2020), Article 20.6.4.

[11] M. Kaneko and H. Tsumura, Multi-poly-Bernoulli numbers and related zeta functions, Nagoya Math. J. **232** (2018), 19–54.

[12] M. Kaneko and H. Tsumura, *Zeta functions connecting multiple zeta values and poly-Bernoulli numbers*, Avd. Stud. Pure Math. **84** (2020), 181–204.

[13] N. Kawasaki, Y. Ohno, *Combinatorial proofs of identities for special values of Arakawa-Kaneko multiple zeta functions*, Kyushu J. Math. **72** (2018), 215-222.

[14] L. J. Mordell, *On Mordell-Tornheim zeta values*, Proc. Amer. Math. Soc. **133** (2005), no.8, 2387–2393.

[15] M. Pallewatta, *On sum formulas for Mordell-Tornheim zeta values*, Rocky Mountain J. Math. **50** (2020), no. 1, 225–235.

[16] Y. Sasaki, *On generalized poly-Bernoulli numbers and related L-functions*, J. Number Theory **132** (2012), 156–170.

[17] M. Z. Spivey, *The Art of Proving Binomial Identities*, Chapman and Hall/ CRC, New York, 2019.

[18] L. Tornheim, *Harmonic double series*, Amer. J. Math. **72** (1950), 303–314.

[19] D. Zagier, *Zetafunktionen und quadratische Körper*, Springer, 1981.