43.

SUR LES RÉCIPROCANTS PURS IRRÉDUCTIBLES DU QUATRIÈME ORDRE.

[*Comptes Rendus*, CII. (1886), pp. 152, 153.]

DANS une Note précédente *, nous avons voulu donner le système de réciprocants irréductibles par rapport aux lettres a, b, c, d, e.

Malheureusement une erreur de calcul s'est glissée dans la détermination de la forme numérotée (5) [p. 248, above], et conséquemment la forme (6) qui, d'après notre méthode de calcul, dépend en partie de la forme (5) est aussi erronée. L'erreur est grave, car, en conséquence, un terme contenant b^3d se trouve dans cette dernière forme qui ne doit pas y paraître; cela empêcherait une combinaison ultérieure linéaire de cette forme avec le carré de la forme (4), qui donne naissance à une nouvelle forme irréductible.

Dans la forme (5) donnée, au lieu de $1585ab^2c^2$ on doit lire $1485ab^2c^2$, et, au lieu de $-18000b^4c$, on doit lire $-3600b^4c$. Ainsi corrigée, la forme, en divisant par 9, devient

$$45a^2d^2 - 450a^2bcd + 192a^2c^3 + 165ab^2c^2 + 400ab^3d - 400b^4c,$$

et, en combinant celle-ci linéairement avec le produit de (2) et (4), on obtient, en divisant par a, pour la forme (6),

$$240a^2ce - 400ab^2e - 315a^2d^2 + 1470abcd - 1008ac^3 - 35b^2c^2.$$

Sans aucun calcul arithmétique, on aurait dû prévoir que l'argument b^3d ne doit pas paraître là-dedans; car le terme qui contient $b^2\delta_d$ dans V, opérant sur b^3d, donne b^5, et évidemment aucune autre partie de V, opérant sur un terme quelconque de la forme commençant par a^2ce, ne peut donner ce même argument.

En combinant linéairement le produit de cette forme par la forme $ac-b^2$ avec le carré de (4) [p. 248, above], on obtient, en divisant par a, une nouvelle forme irréductible (7). C'est M. Hammond qui m'a averti de mon erreur de calcul et qui a calculé lui-même cette nouvelle forme dont il a vérifié l'exactitude par le moyen de l'équation différentielle partielle. On peut donc accepter avec pleine confiance pour (7) la forme

$$\begin{aligned}25a^3e^2 - 350a^2bde - 4970a^2c^2e + 17150ab^2ce + 6615a^2cd^2 \\ - 9800ab^2d^2 - 31360abc^2d + 21217ac^4 - 14000b^4e \\ + 49000b^3cd - 34055b^2c^3.\end{aligned}$$

Avec ces conventions le système complet de *Grundformen*, pour le système de lettres a, b, c, d, e, sera constitué par les formes (1), (2), (3), (4), (6), (7).

[* Above, p. 242.]

44.

SUR UNE EXTENSION DU THÉORÈME RELATIF AU NOMBRE D'INVARIANTS ASYZYGÉTIQUES D'UN TYPE DONNÉ À UNE CLASSE DE FORMES ANALOGUES.

[*Comptes Rendus*, CII. (1886), pp. 1430—1435.]

[Cf. p. 459, above.]

NOUS employons toujours aujourd'hui le mot *invariant* pour désigner les sous-invariants et les invariants (ainsi ordinairement nommés) sans distinction.

Le type d'un invariant est l'ensemble de trois éléments, le poids, le degré et l'étendue, que nous désignerons ordinairement par les lettres w, i, j, et nous nous servons de cet ensemble entre parenthèses $(w:i,\ j)$ pour signifier le nombre de manières de composer w avec i des chiffres $0, 1, 2, \ldots, j$ ou bien, ce qui revient au même, avec j des chiffres $0, 1, 2, \ldots, i$.

Il est quelquefois utile d'ajouter à ces trois éléments un autre dont il est fonction, à savoir l'*excès* qu'on prend égal à $ij - 2w$.

Quand on considère un invariant comme source d'un covariant, l'excès coïncide avec l'ordre dans les variables de ce dernier.

Le théorème connu, dont nous parlons dans le titre de cette Note, se divise en deux parties:

(1) Il n'existe aucun invariant dont l'excès du type soit négatif;

(2) Quand l'excès est positif, le nombre des invariants asyzygétiques du type $w:i,j$ est $(w:i,j)-(w-1:i,j)$ qu'on peut représenter par $\Delta(w:i,j)$.

Évidemment, ces résultats peuvent être étendus au cas des formes rationnelles et entières qui sont *anéanties* par l'opérateur

$$\lambda_1 a_0 \delta_{a_1} + \lambda_2 a_1 \delta_{a_2} + \ldots + \lambda_j a_{j-1} \delta_{a_j},$$

pourvu qu'aucun des λ ne soit nul; car alors, en remplaçant les a par des multiples numériques convenables, l'anéantisseur peut être changé dans la forme $a_0\delta_a + 2a_1\delta_{a_2} + \ldots + ja_{j-1}\delta_{a_j}$.

Quand tous les λ dans l'opérateur sont pris égaux à l'unité, on peut donner aux formes qu'il anéantit le nom de *binariants*.

De même, on peut considérer un anéantisseur

$$a_0\delta_{a_k} + a_1\delta_{a_{k+1}} + \ldots + a_{j-k}\delta_{a_j},$$

et donner aux formes qu'il anéantit le nom de *binariants de raison* k*; en particulier, quand $k=2$, on peut les nommer *transbinariants*. C'est sur les transbinariants pour lesquels l'étendue j est un nombre PAIR que nous allons démontrer un théorème analogue à celui que nous avons énoncé sur les binariants ordinaires.

Si nous considérons les binariants de raison k, voici comment on pourrait procéder pour trouver toutes les formes du type $(w:i, j)$:

On prendra la forme la plus générale de ce type qui contiendra $(w:i, j)$ constantes disponibles. On opérera sur elle avec l'anéantisseur $a_0\delta_{a_k} + \ldots$, ce qui donnera une forme du type $(w-k:i, j)$ dont les $(w-k:i, j)$ coefficients seront des fonctions linéaires de ceux de la forme primitive, et l'on égalera à zéro tous ces coefficients. Ainsi l'on pourrait être porté à croire que, pourvu que le nombre des coefficients de la forme primitive excède le nombre de coefficients de la dérivée, la différence de ces deux nombres doit être le nombre de binariants de raison k asyzygétiques. Mais tout ce qu'on peut légitimement conclure dans ce cas, c'est que ce dernier nombre ne peut pas être moindre que cette différence; car les équations dont on a parlé ne sont pas nécessairement indépendantes. Cette précaution n'est nullement surérogatoire; un seul exemple suffira à le démontrer. Prenons $k=2$ et cherchons le nombre des transbinariants du type $(6:2, 5)$.

On a

$(6:2, 5)=3$, car 6 peut être composé avec $5+1$, $4+2$, $3+3$,
$(4:2, 5)=3$, car 4 „ $4+0$, $3+1$, $2+2$.

Donc $$(6:2, 5)-(4:2, 5)=0.$$

Cependant le nombre des transbinariants du type donné n'est pas zéro, mais 1; car, évidemment, $2bf-d^2$ est anéanti par l'opérateur

$$a\delta_c + b\delta_d + c\delta_e + d\delta_f.$$

On voit donc que c'est un théorème bien réel et nullement nugatoire, qui énonce que, pour le cas où j est un nombre pair, le nombre des transbinariants du type $(w:i, j)$ est égal exactement à $(w:i, j)-(w-2:i, j)$ quand cette différence n'est pas négative. On peut ajouter que cette différence est négative seulement dans le cas où l'*excès* du type est négatif et qu'alors (comme on va le démontrer) il n'y a pas de binariants de ce type.

Si l'on a $$\Theta = a_0\delta_{a_2} + a_1\delta_{a_3} + \ldots + a_{2\eta-2}\delta_{a_{2\eta}},$$

on peut écrire $$\Theta = \theta_1 + \theta_2,$$

* Le théorème de Brioschi montre qu'un binariant de raison k est une fonction de

$$s_1, s_2, \ldots, s_{k-1}, s_{k+1}, \ldots, s_j, s_0$$

étant la somme des puissances $\theta^{\text{ièmes}}$ des racines de l'équation $a_0x^j + a_1x^{j-1} + \ldots + a_j = 0$.

en posant $$\theta_1 = a_0\delta_{a_2} + a_2\delta_{a_4} + a_4\delta_{a_6} + \ldots + a_{2\eta-2}\delta_{a_{2\eta}},$$
$$\theta_2 = a_1\delta_{a_3} + a_3\delta_{a_5} + \ldots\ldots\ldots + a_{2\eta-3}\delta_{a_{2\eta-1}}.$$

En faisant $$t = t_1 + t_2,$$

avec $$t_1 = 1 \,.\, \eta a_2\delta_{a_0} + 2(\eta-1)\,a_4\delta_{a_2} + 3(\eta-2)\,a_6\delta_{a_4} + \ldots + \eta \,.\, 1 \,.\, a_{2\eta}\delta_{a_{2\eta-2}},$$
$$t_2 = 1(\eta-1)\,a_3\delta_{a_1} + 2(\eta-2)\,a_5\delta_{a_3} + \ldots + (\eta-1)\,1 \,.\, a_{2\eta-1}\delta_{a_{2\eta-3}},$$

on trouvera

$$\theta_1 t_1 - t_1\theta_1 = \eta a_0\delta_{a_0} + (\eta-2)\,a_2\delta_{a_2} + \ldots - (\eta-2)\,a_{2\eta-2}\delta_{a_{2\eta-2}} - \eta a_{2\eta}\delta_{a_{2\eta}},$$
$$\theta_2 t_2 - t_2\theta_2 = (\eta-1)\,a_1\delta_{a_1} + (\eta-3)\,a_3\delta_{a_3} + \ldots - (\eta-1)\,a_{2\eta-1}\delta_{a_{2\eta-2}}.$$

Donc, si I est une fonction homogène et isobarique dans les lettres a du type $w; i, j$, on aura

$$(\Theta T - T\Theta)\,I = [\eta a_0\delta_{a_0} + (\eta-1)\,a_1\delta_{a_1} + \ldots - (\eta-1)\,a_{2\eta-1}\delta_{a_{2\eta-1}} - \eta a_{2\eta}\delta_{a_{2\eta}}]\,I$$
$$= (i\eta - j)\,I = \frac{2\eta i - 2w}{2}\,I;$$

car on remarquera que ni l'un ni l'autre θ n'agit sur l'un ou l'autre t, et que ni l'un ni l'autre t n'agit sur l'un ou l'autre θ.

Le coefficient de I, on le remarquera, est la moitié de l'*excès* au type $w:i, 2\eta$.

Il est bon d'observer qu'il n'est pas possible d'obtenir un résultat semblable dans le cas où j est impair, c'est-à-dire qu'on ne peut pas trouver, dans ce cas, une forme T telle que le résultat de l'opération $(\Theta T - T\Theta)$ sur une forme homogène et isobarique soit équivalent au produit de cette forme par une fonction quelconque de $w; i, j$.

Avec l'aide de la formule ci-dessus, suivant la même marche que nous avons prise pour les invariants dans le *Philosophical Magazine** (mars 1878), on parvient à des résultats tout à fait semblables.

En appelant ϵ la moitié de l'excès et en supposant que I est un transbinariant, on trouve

$$\epsilon I = \Theta T I$$

et, plus généralement, $$\mu T^{q-1} I = \Theta T^q I,$$

où $\mu = q(\epsilon - q + 1)$.

Or il est évident que, puisque l'effet de T est d'augmenter (par deux unités) le poids de la forme sur laquelle il agit sans en changer le degré ni l'étendue, et que le poids d'une forme homogène et isobarique ne peut pas excéder le produit du degré par l'étendue, en prenant q suffisamment grand, on aura

$$TI = 0,$$

et, à plus forte raison, $$\Theta T I = 0.$$

On trouvera donc successivement $T^{q-1}I = 0$, $T^{q-2}I = 0$, ..., $TI = 0$, $I = 0$, pourvu que le μ ne devienne pas nul dans le cours de cette déduction: ceci

[* Vol. III. of this Reprint, p. 117.]

ne peut pas arriver quand ϵ est négatif, car on trouvera que les valeurs de μ, dans ce cas, resteront toujours négatives.

Cela démontre qu'un transbinariant, dont le type a un excès négatif, ne peut pas être autre que zéro, c'est-à-dire n'a pas d'existence actuelle quand l'excès est non négatif; en désignant par $E(w:i,j)$ le nombre

$$(w:i,j)-(w-2:i,j),$$

et par $D(w:i,j)$ le nombre de transbinariants du type $(w:i,j)$, on prouve que $D(w:i,j)=E(w:i,j)$ de la manière suivante.

En remarquant que, pour w négatif, $E(w:i,j)=0$, on trouve immédiatement

$$\sum_{q=\infty}^{q=0} E(w-2q:i,j)=(w:i,j),$$

et, puisque chaque D est au moins égal au E correspondant, on a

$$\sum_{q=\infty}^{q=0} D(w-2q:i,j) \geqq (w:i,j).$$

Or on peut démontrer facilement que, si $ij-2w$ est non négatif, en appelant $I_{w:i,j}$ un transbinariant du type $(w:i,j)$, $\Theta^q T^q I_{w-2q:i,j}$ sera égal à un multiple numérique de $I_{w-2q:i,j}$ différent de zéro pour toutes les valeurs de q qu'on a besoin de considérer.

Or, dans l'ensemble des transbinariants asyzygétiques, dont le type est $w-2q:i,j$, on peut substituer à chacun, pour ainsi dire, son *image* $T^q I_{w-2q:i,j}$. Le nombre de ces images sera

$$\sum_{q=\infty}^{q=0} D(w-2q:i,j).$$

De plus, chaque image sera du même type $(w:i,j)$.

On démontre facilement qu'il ne peut pas exister entre ces images une relation linéaire; car, dans le cas contraire, en opérant sur l'équation qui les lie ensemble avec une puissance convenable de Θ, on tomberait sur une équation linéaire entre les transbinariants asyzygétiques eux-mêmes. Donc, évidemment, le nombre des images ne peut pas excéder la valeur de $(w:i,j)$. Donc

$$\sum_{q=\infty}^{q=0} D(w-2q;i,j)$$

n'est ni plus grand ni plus petit que $\sum_{q=\infty}^{q=0} E(w-2q;i,j)$; il lui est donc égal, et conséquemment, puisque aucun D ne peut être moins que le E qui lui correspond pour chaque valeur de q,

$$D(w-2q;i,j)=E(w-2q;i,j);$$

car si un D quelconque était plus grand que le E qui lui correspond, un autre D serait nécessairement plus petit, ce qui est inadmissible.

On aura donc $\qquad D(w:i,j)=E(w:i,j),$

pourvu que $ij-2w$ ne soit pas négatif. C.Q.F.D.

On démontre facilement les mêmes théorèmes pour des formes anéantissables par une *somme* d'opérateurs

$$a_0\delta_{a_2}+\ldots+a_{j-2}\delta_{a_j},$$
$$a_0'\delta_{a_2'}+\ldots+a'_{j'-2}\delta_{a'_{j'}},$$

....

En supposant que chaque j soit pair et en regardant $w:i,j:i',j',\ldots$ comme leur type, on parvient à cette conclusion qu'aucun transbinariant d'un tel type n'existe dans le cas où $ij+i'j'+\ldots-2w$ est négatif et que, quand cette quantité n'est pas négative, le nombre des transbinariants asyzygétiques est égal à $(w:i,j:i',j':\ldots)-(w-2:i,j:i',j';\ldots)$, où $(w:i,j:i',j':\ldots)$ désigne le nombre de manières de composer w avec i des chiffres 0, 1, 2, …, combinés avec i' des chiffres 0, 1, 2, …, j', etc.

Il est utile de remarquer que les formes et les syzygies fondamentales des intégrales de l'équation

$$(a_0\delta_{a_2}+a_1\delta_{a_3}+\ldots+a_{\eta-2}\delta_{a_{2\eta}})I=0$$

sont des mêmes types que les invariants et les syzygies fondamentales d'un système formé avec deux *quantics* d'ordres η et $\eta-1$ respectivement; ce qui donne un moyen facile de vérifier la formule que nous avons démontrée pour le nombre de transbinariants asyzygétiques d'un type donné. Il va sans dire que nous n'avons pas négligé de nous servir de cette méthode pour vérifier la justesse de nos conclusions.

45.

NOTE SUR LES INVARIANTS DIFFÉRENTIELS.

[*Comptes Rendus*, CII. (1886), pp. 31—34.]

En affirmant, dans notre Lettre à M. Hermite (dont un Extrait a paru dans les *Comptes rendus*), que les invariants différentiels de M. Halphen sont identiques avec nos réciprocants purs, nous sommes allé trop loin ; nous aurions dû dire qu'ils sont identiques avec la classe spéciale de ces derniers que nous avons nommés *réciprocants projectifs* ; en effet, en prenant pour *éléments*

$$\frac{1}{1.2}\frac{d^2y}{dx^2}, \quad \frac{1}{1.2.3}\frac{d^3y}{dx^3}, \quad \frac{1}{1.2.3.4}\frac{d^4y}{dx^4}, \quad \ldots,$$

regardés comme quantités algébriques, lesquelles on peut nommer (selon l'usage quand on parle de formes binaires) $a, b, c, d, \ldots$, un invariant différentiel possède la propriété vraiment étonnante d'être en même temps un réciprocant et un sous-invariant ordinaire.

En accommodant la valeur de V à cette notation nouvelle, il devient

$$4aa\delta_b + 5(ab+ba)\delta_c + 6(ac+bb+ca)\delta_d, \quad \ldots;$$

et, en posant $$a\delta_b + 2b\delta_c + 3c\delta_d + \ldots = \Omega,$$

un invariant différentiel I satisfait en même temps aux deux équations partielles différentielles

$$V.I=0, \quad \Omega.I=0.$$

Voici comment on peut établir le fait que $\Omega.I=0$.

En commençant avec les trois premiers invariants différentiels, c'est-à-dire a, $a^2d - 3abc + 2b^3$, et le Δ de M. Halphen (dans sa thèse immortelle), on sait que les deux premiers, et l'on vérifie sans trop de peine que le troisième sont tous les trois des sous-invariants.

De plus, on sait que, en commençant avec ces trois invariants que nous nommerons I_0, I_1, I_2, on peut former une suite indéfinie de formes protomorphiques

$$I_0, \quad I_1, \quad I_2, \quad I_3, \quad \ldots, \quad I_p, \quad \ldots,$$

dont tous les autres seront des fonctions rationnelles.

Pour obtenir cette suite, on n'a qu'à former une fonction J de $I_0, I_1, \ldots, I_p, \ldots$, dont le degré et le poids soient tous deux zéro ; en opérant alors sur J (considéré comme fonction des dérivées de y par rapport à x) avec δ_x, on obtient I_{p+1}.

Si donc on peut démontrer que $\Omega\delta_x J = \delta_x \Omega J$, il s'ensuivra que I_{p+1} sera un sous-invariant, pourvu que I_p en soit un, et le théorème en question sera démontré.

Or remarquons en premier lieu que, à cause de la valeur zéro du degré et du poids de J, la quantité

$$(\lambda a\delta_a + \mu b\delta_b + \nu c\delta_c + \ldots)J$$

sera nulle si $\lambda, \mu, \nu, \ldots$ forment une progression arithmétique quelconque ; et, en second lieu, que (par rapport à une fonction de dérivées de J par rapport à x), $\delta_x = 3b\delta_a + 4c\delta_b + 5d\delta_c + \ldots$ identiquement.

Conséquemment

$$(\Omega\delta_x - \delta_x\Omega)J = [(3a\delta_a + 8b\delta_b + 15c\delta_c + \ldots) - (3b\delta_b + 8c\delta_c + \ldots)]J$$
$$= (3a\delta_a + 5b\delta_b + 7c\delta_c + \ldots)J = 0,$$

ce qu'il fallait démontrer.

M. Halphen, à qui j'avais communiqué ce résultat, en a trouvé une tout autre démonstration qu'il m'autorise à communiquer à l'Académie. Elle possède sur la mienne l'avantage d'aller plus au fond de la question, en faisant voir que l'équation $\Omega.I=0$ équivaut à dire que, en se servant de x, y, z au lieu de $x, y, 1$, un invariant différentiel peut subir le changement entre eux de x et z. Or, puisque $V.I=0$ signifie qu'on peut imposer des substitutions linéaires quelconques sur x et y, il s'ensuit, en combinant les deux équations, que la même chose aura lieu quand x, y, z subissent tous les trois des substitutions linéaires quelconques. Voici la démonstration très élégante de M. Halphen :

"Si l'on fait le changement de variables

$$X = \frac{1}{x}, \quad Y = \frac{y}{x},$$

et qu'on écrive

$$\frac{dy}{dx} = y', \quad \frac{d^2y}{dx^2} = y'', \quad \ldots, \quad \frac{d^ny}{dx^n} = y^{(n)}, \quad \ldots,$$

on a $Y = +x^{-1}y,$

$$\frac{dY}{dX} = -x^{+1}\left(y' - \frac{1}{x}y\right),$$

$$\frac{d^2Y}{dX^2} = +x^3y'',$$

$$\frac{d^3Y}{dX^3} = -x^5\left(y''' + \frac{3}{x}y''\right),$$

$$\frac{d^4Y}{dX^4} = +x^7\left(y^{\text{IV}} + \frac{8}{x}y''' + \frac{12}{x^2}y''\right),$$

$$\frac{d^5Y}{dX^5} = -x^9\left(y^{\text{V}} + \frac{15}{x}y^{\text{IV}} + \frac{60}{x^2}y''' + \frac{60}{x^3}y''\right),$$

....,

$$\frac{d^nY}{dX^n} = (-1)^n x^{2n-1}\left[y^{(n)} + \frac{n(n-2)}{x}y^{(n-1)} + \frac{\alpha}{x^2}y^{(n-2)} + \frac{\beta}{x^3}y^{(n-3)} + \dots\right].$$

"Posant $\frac{d^nY}{dX^n} = n'A_n, \quad \frac{d^ny}{dx^n} = n'a_n, \quad \frac{1}{x} = \epsilon,$

on a $A_n = (-1)^n x^{2n-1}[a_n + (n-2)\epsilon a_{n-1} + \alpha'\epsilon^2 a_{n-2} + \dots].$

"Soit une fonction $f(A_0, A_1, \dots, A_n)$ dont tous les termes soient de poids et de degré constants p, δ ; en supposant ϵ infiniment petit, on aura

$$f(A_0, A_1, \dots, A_n) = (-1)^p x^{2p-\delta}\left\{f(a_0, a_1, \dots, a_n) + \epsilon\left[-a_0\frac{\partial f}{\partial a_1} + 2a_2\frac{\partial f}{\partial a_3} + 2a_3\frac{\partial f}{\partial a_4} + \dots + (n-2)a_{n-1}\frac{\partial f}{\partial a_n}\right]\right\}.$$

"Donc, pour que f soit invariant pour la substitution considérée, il faut qu'on ait

$$a_2\frac{\partial f}{\partial a_3} + 2a_3\frac{\partial f}{\partial a_4} + 3a_4\frac{\partial f}{\partial a_5} + \dots + (n-2)a_{n-1}\frac{\partial f}{\partial a_n} = a_0\frac{\partial f}{\partial a_1}.$$

"En particulier, si f ne contient pas a_1, ce qui est le cas des *réciprocants purs*, on aura

$$a_2\frac{\partial f}{\partial a_3} + 2a_3\frac{\partial f}{\partial a_4} + \dots + (n-2)a_{n-1}\frac{\partial f}{\partial a_n} = 0. \qquad \text{C.Q.F.D."}$$

Ainsi, l'on voit qu'un invariant différentiel est en même temps réciprocant et sous-invariant ; ce n'est nullement un mélange ou une combinaison de deux choses différentes, mais plutôt, pour ainsi dire, une personnalité seule et indivisible douée de deux natures tout à fait distinctes.

Afin de compléter la théorie, il faut démontrer la réciproque, c'est-à-dire que toute forme douée de ces deux natures est un réciprocant projectif. M. Halphen effectue cela en trouvant le développement complet de sa série et en faisant voir que, quand le coefficient de la première puissance de ϵ disparaît,

la même chose aura lieu pour tous les coefficients suivants. Voici notre méthode, à nous de l'effectuer.

Soit H une forme rationnelle et entière dont le terme principal (c'est-à-dire celui qui contient la plus haute puissance du terme le plus avancé) est Gh^i. On suppose que le théorème à démontrer est vrai jusqu'à la lettre g incluse, et que $VA = 0$, $\Omega H = 0$ sans que H soit projectif.

Alors évidemment $VG = 0$, $\Omega G = 0$ et G, par hypothèse, sera projectif. Soit H' une puissance d'un protomorphe pour laquelle le terme principal est $G'h^i$, alors, si $H_1 = G'H - GH'$, G, G', H' sont projectifs, mais H non projectif ; donc, H_1 (qui, comme H, est anéanti par V et par Ω) sera non projectif : de plus, dans H_1 le degré du terme principal en h est abaissé. De la même manière on peut construire H_2, H_3, ... jusqu'à ce qu'on parvienne à une forme* qui ne contient pas h, laquelle possédera les mêmes caractères que H, ce qui est impossible par hypothèse. Donc, si le théorème à démontrer est vrai pour un nombre quelconque donné de lettres, il sera vrai universellement : mais il est évidemment vrai pour la fonction a qui est le seul réciprocant à une lettre. Donc, si $VI = 0$ et $RI = 0$, I est un réciprocant projectif, c'est-à-dire un invariant différentiel. Ce qui était à démontrer.

* Cette forme sera, en effet, le résultant de H et de la première puissance du protomorphe. Nous avons jugé inutile de dire dans le texte que G', comme G, sera anéanti par V et par Ω et conséquemment, par hypothèse, sera lui aussi projectif.

46.

SUR L'ÉQUATION DIFFÉRENTIELLE D'UNE COURBE D'ORDRE QUELCONQUE.

[*Comptes Rendus*, CIII. (1886), pp. 408—411.]

[Also, above, p. 492.]

On peut obtenir une solution directe et universelle de ce problème: *Trouver l'équation différentielle d'une courbe de l'ordre n*, en représentant la fonction de l'équation (avec l'unité pour terme constant), soit U ou $(x, y, 1)^n$, sous la forme symbolique u^n, où $u = a + bx + y$. Alors, en mettant $\left(\frac{d}{dx}\right)^n y = y_r$, on aura

$$\frac{du}{dx} = b + y_1, \quad \frac{d^{i+1}u}{dx^{i+1}} = y_{i+1}\,^*.$$

Égalons à zéro les dérivées de u^n des degrés $n+1, n+2, \ldots, \frac{(n+1)(n+2)}{2}$; il en résultera $\frac{n^2+n}{2}$ équations entre lesquelles on peut éliminer le même nombre de coefficients, c'est-à-dire tous les coefficients en U, sauf ceux qui ne contiennent nulle puissance de y, lesquels ne paraîtraient pas dans les équations dont nous parlons.

Pour obtenir le déterminant qui correspond à ce système d'équations, remarquons que le théorème de Taylor donne immédiatement†

$$\frac{1}{\Pi r}\partial_x{}^r u^n = \text{co}_r\left(u + u'h + u''\frac{h^2}{1.2} + u'''\frac{h^3}{1.2.3} + \ldots\right)^n$$

$$= \text{co}_r\left((u+u'h)^n + n.(u+u'h)^{n-1}V + n.\frac{n-1}{2}(u+u'h)^{n-2}V^2 + \ldots\right),$$

où l'on peut prendre

$$V = y_2\frac{h^2}{1.2} + y_3\frac{h^3}{1.2.3} + y_4\frac{h^4}{1.2.3.4} + \ldots,$$

ce qui suffit à résoudre le problème.

* On remarquera qu'avec cette notation toute fonction entière de u et $\partial_x u$ représentera sans ambiguïté une quantité algébrique ordinaire, pourvu que l'on sache *a priori* qu'elle doit être linéaire dans les coefficients de u^n. C'est pourquoi dans le texte on est libre d'exprimer toute dérivée différentielle de U comme fonction de u et u'.

† Par co_r on sous-entend les mots "le coefficient de h^r dans."

Pour cela, on considère toutes les dérivées de U comme fonctions linéaires des termes qui paraissent dans le développement de $(u, u', 1)^{n-1}$*.

Alors, en représentant par $m.\mu$ le coefficient de h^m dans

$$\left(\frac{y^2}{1.2}h^2 + \frac{y^3}{1.2.3}h^3 + \ldots\right)^\mu,$$

on trouvera, sans calcul algébrique aucun, que la $q^{\text{ième}}$ ligne du déterminant cherché peut être prise sous la forme

$$(1+q).1\ (2+q).1\ (2+q).2\ (3+q).1\ (3+q).2\ (3+q).3\ \ldots$$
$$(n+q).1\ (n+q).2\ \ldots\ (n+q).n.$$

Par exemple, prenons le cas de $n = 4$; le déterminant

2.1	3.1	3.2	4.1	4.2	4.3	5.1	5.2	5.3	5.4
3.1	4.1	4.2	5.1	5.2	5.3	6.1	6.2	6.3	6.4
4.1	5.1	5.2	6.1	6.2	6.3	7.1	7.2	7.3	7.4
5.1	6.1	6.2	7.1	7.2	7.3	8.1	8.2	8.3	8.4
6.1	7.1	7.2	8.1	8.2	8.3	9.1	9.2	9.3	9.4
7.1	8.1	8.2	9.1	9.2	9.3	10.1	10.2	10.3	10.4
8.1	9.1	9.2	10.1	10.2	10.3	11.1	11.2	11.3	11.4
9.1	10.1	10.2	11.1	11.2	11.3	12.1	12.2	12.3	12.4
10.1	11.1	11.2	12.1	12.2	12.3	13.1	13.2	13.3	13.4
11.1	12.1	12.2	13.1	13.2	13.3	14.1	14.2	14.3	14.4

sera le premier membre de l'équation différentielle (disons le critérium différentiel) d'une courbe du quatrième degré.

Si l'on se borne aux termes contenus dans les six premières lignes et colonnes, on aura le critérium pour la cubique, et, en se bornant aux termes contenus dans les trois premières lignes et colonnes, celui pour la conique, ou plutôt ce critérium multiplié par 2.1, ce qui constitue un cas exceptionnel.

2.1 lui-même, c'est-à-dire $\frac{\partial_x{}^2 y}{2}$, est naturellement le critérium pour la ligne droite. On remarquera que 3.2, 4.3, 5.3, 5.4, 6.4, 7.4 sont des combinaisons pour ainsi dire fictives, qui ont pour valeur zéro†. De même, en général, il y aura toujours des termes nuls dans les $(n-1)$ premières lignes du critérium de la courbe de degré n; au-dessous de la $(n-1)^{\text{ième}}$ ligne, toutes les places seront remplies par des combinaisons qui correspondent à des non-zéros.

Quand $n = 3$, en substituant pour $\frac{y''}{1.2}, \frac{y'''}{1.2.3}, \frac{y^{\text{IV}}}{1.2.3.4}, \ldots$ les lettres $a, b, c, \ldots$, on retombe sur la formule trouvée pour la cubique par M. Samuel

* Ou plutôt les termes avec leurs coefficients numériques de $(u, u', 1)^n$, en omettant les $(n+1)$ termes du degré n.

† Évidemment $m.\mu$ est zéro quand $m < 2\mu$.

Roberts (voir *Mathematical Questions from the Educational Times*, t. X. p. 47)*, c'est-à-dire la même matrice que celle donnée par M. Roberts, mais avec ses colonnes autrement présentées.

On voit immédiatement que le degré du critérium pour une courbe du $n^{\text{ième}}$ ordre sera $\frac{n(n+1)(n+2)}{6}$ et, par un calcul facile, que son poids sera $\frac{(n-1)n(n+1)(n+2)}{8}+\frac{n(n+1)(n+2)}{3}$†. Ce dernier nombre suppose que le poids de $d_x^i y$ est compté comme i. Dans le calcul des réciprocants, on le compte toujours comme étant $i-2$ et, en faisant cette réduction, le poids devient tout simplement $\frac{(n-1)n(n+1)(n+2)}{8}$.

M. Halphen nous a appris que les formules qu'il a données dans son Mémoire intitulé : *Recherches des points d'une courbe algébrique plane*, etc. (*Journal de Mathématiques*, 3ᵉ série, t. II. pp. 373, 374 et 400 ; 1876) fournissent un moyen pour calculer le degré et le poids du critérium $n^{\text{ième}}$ et conduisent aux mêmes résultats que ceux donnés ci-dessus. Dans le cas de la conique, le déterminant, comme nous l'avons dit, se divise par y'', de sorte que son poids-degré s'abaisse et, au lieu d'être 3.4, devient 3.3 ; en effet, c'est la forme bien connue $a^2d-3abc+2b^3$, trouvée par Monge.

* Ce travail a été cité et reproduit dans le *Philosophical Magazine* de février 1886, par M. Muir, qui y construit pour ainsi dire le tableau du calcul dont M. Roberts avait déjà fait le procès-verbal.

† Car le degré sera la somme de n termes de la série $1+3+6+\ldots$, c'est-à-dire $\frac{n(n+1)(n+2)}{6}$, et le poids, moins deux fois le degré, la somme de n termes de la série

$$0+(2+1)+(5+4+3)+(9+8+7+6)+\ldots$$

ou bien de $\frac{n^2+n-2}{2}$ termes de la progression naturelle $1+2+3+4+5+\ldots$, c'est-à-dire

$$\frac{n^2+n-2}{2}\,\frac{n^2+n}{4}.$$

47.

SUR UNE EXTENSION D'UN THÉORÈME DE CLEBSCH RELATIF AUX COURBES DU QUATRIÈME DEGRÉ.

[*Comptes Rendus*, CII. (1886), pp. 1532—1534.]

En appliquant un terme quelconque du développement de

$$(\delta_x, \delta_y, \delta_z, \ldots)^\eta$$

au quantic $(x, y, z, \ldots)^{2\eta}$, on obtient autant de fonctions de degré η qu'il y a de termes dans chaque fonction. L'ensemble de leurs coefficients peut donc être regardé comme la matrice d'un déterminant auquel nous donnerons le même nom de *catalecticant*, dont on fait usage dans le cas des formes binaires.

On voit très aisément que la matrice catalectique, pour une puissance d'une fonction linéaire de variables, possède cette propriété que chaque déterminant mineur du second ordre qu'elle contient s'évanouit. Conséquemment, deux colonnes quelconques d'une telle matrice, associées à d'autres colonnes arbitraires, en nombre suffisant pour former une matrice carrée nouvelle, feront s'évanouir le déterminant de cette dernière.

Or la matrice catalectique d'une somme de puissances de fonctions linéaires des mêmes variables est la somme des matrices qui appartiennent à chacune prise séparément ; donc, comme conséquence immédiate de cette propriété dont nous avons parlé, si le nombre de ces matrices est moindre que l'ordre de chacune, le déterminant de leur somme s'évanouira, car il pourra être résolu dans une somme de déterminants dont chacun aura la valeur zéro*.

* S'il y a n matrices, chacune de l'ordre N (de sorte que N est le nombre des colonnes dans chaque matrice), on associera à volonté la première colonne d'une quelconque des n matrices avec la seconde, avec la troisième, etc. colonne, prises ou dans la même ou dans aucune autre matrice, en sorte que le nombre des nouvelles matrices partielles sera n^N. Il est évident que, N étant par hypothèse plus grand que n, deux colonnes *au moins* de chaque matrice ainsi formée appartiendront à une même matrice fondamentale, c'est-à-dire à la matrice catalectique d'une puissance d'une fonction linéaire des variables. Voilà la raison pour laquelle chacun des n^N déterminants partiels est égal à zéro.

(1) Prenons deux variables. Le catalecticant sera de l'ordre $\eta + 1$; on retrouve ainsi cette règle bien connue, et qui ne contient rien d'exceptionnel ni de paradoxal: pour qu'une forme binaire d'ordre 2η soit équivalente à la somme de η puissances de fonctions linéaires, il faut que le catalecticant de la forme soit nul.

(2) Prenons trois variables et faisons $\eta = 2$: l'ordre du déterminant catalectique de $(ax + by + cz)^4$ étant 6, le catalecticant de

$$\sum_{\theta=5}^{\theta=1} (a_\theta x + b_\theta y + c_\theta z)^4 = 0.$$

Cela donne le théorème de Clebsch, à savoir que le premier membre de l'équation d'une courbe du quatrième degré n'est pas, en général, exprimable en une somme de cinq puissances de fonctions linéaires des variables.

(3) Prenons cinq variables, en faisant encore $\eta = 2$. L'ordre du déterminant catalectique $(ax + by + cz + dt + eu)^4$ étant 15, le catalecticant de

$$\sum_{\theta=14}^{\theta=1} (a_\theta x + b_\theta y + c_\theta z + d_\theta t + e_\theta u)^4$$

s'évanouit.

Or $5 \times 14 = 70$, ce qui est justement le nombre $\frac{5 \,.\, 6 \,.\, 7 \,.\, 8}{1 \,.\, 2 \,.\, 3 \,.\, 4}$ des coefficients de $(x, y, z, t, u)^4$.

On arrive ainsi à cette conclusion nouvelle, et un peu paradoxale, que l'équation d'une hypersurface du quatrième degré, bien que contenant le même nombre de constantes que la somme de 14 puissances biquadratiques de fonctions linéaires des variables, ne peut pas en général être exprimée comme une telle somme; car, pour que cela fût possible, il faudrait que le catalecticant de l'hypersurface s'évanouît.

(4) Prenons encore $\eta = 2$, et considérons la somme de 9 puissances quatrièmes de fonctions linéaires de x, y, z, t. Le catalecticant de cette somme sera de l'ordre 10 et, conséquemment, zéro.

Donc le premier membre de l'équation d'une surface du quatrième degré qui ne contient que 35 constantes ne peut pas en général être mis sous la forme d'une somme de 9 puissances de fonctions linéaires des variables, quoique cette somme contienne 36 constantes disponibles.

Ce résultat pour les surfaces est, on le voit, un peu plus *paradoxal*, en apparence, que le théorème de Clebsch, sur les courbes du quatrième degré, quoiqu'en effet il n'y ait aucun paradoxe, ni dans l'un ni dans l'autre de ces théorèmes, pour ceux qui sont convaincus qu'on ne doit jamais se fier, sans contrôle, aux conclusions apparentes, fournies par la comparaison numérique de constantes.

48.

ON THE DIFFERENTIAL EQUATION TO A CURVE OF ANY ORDER.

[*Nature*, XXXIV. (1886), pp. 365, 366.]

To Mr Samuel Roberts (see Reprint of *Educational Times*, X. p. 47) is due the credit of having been the first to show that a direct method of elimination properly conducted leads to the differential equation for a cubic curve; but he has not attempted to obtain the general formula for a curve of any order. By aid of a very simple idea explained in a paper intended to appear in the *Comptes Rendus* of the Institute, I find* without calculation the general form of this equation. The left-hand member of it may be conveniently termed the differential *criterion* to the curve. One single matrix will then serve to express the criteria for all curves whose order does not exceed any prescribed number. For instance, suppose we wish to have the criteria for the orders 1, 2, 3, 4:—

Let $m \cdot \mu$ be used in general to denote the coefficient of h^m in

$$\left(\frac{1}{1 \,.\, 2} y'' h^2 + \frac{1}{1 \,.\, 2 \,.\, 3} y''' h^3 + \frac{1}{1 \,.\, 2 \,.\, 3 \,.\, 4} y'''' h^4 + \ldots\right)^\mu.$$

Write down the matrix—

2·1	3·1	3·2	4·1	4·2	4·3	5·1	5·2	5·3	5·4
3·1	4·1	4·2	5·1	5·2	5·3	6·1	6·2	6·3	6·4
4·1	5·1	5·2	6·1	6·2	6·3	7·1	7·2	7·3	7·4
5·1	6·1	6·2	7·1	7·2	7·3	8·1	8·2	8·3	8·4
6·1	7·1	7·2	8·1	8·2	8·3	9·1	9·2	9·3	9·4
7·1	8·1	8·2	9·1	9·2	9·3	10·1	10·2	10·3	10·4
8·1	9·1	9·2	10·1	10·2	10·3	11·1	11·2	11·3	11·4
9·1	10·1	10·2	11·1	11·2	11·3	12·1	12·2	12·3	12·4
10·1	11·1	11·2	12·1	12·2	12·3	13·1	13·2	13·3	13·4
11·1	12·1	12·2	13·1	13·2	13·3	14·1	14·2	14·3	14·4

[* Cf. pp. 492, 524 above.]

The determinant of the entire matrix, which is of the tenth order, is the criterion for a quartic curve. The determinant of the minor of the sixth order, comprised within the first six lines and columns, is the criterion for a cubic. The determinant of the third order, comprised within the first three lines and columns (subject to a remark about to be made) will furnish the criterion for a conic, and the apex of the matrix is the criterion for the straight line. By adding on five more lines and columns, according to an obvious law, the matrix may be extended so as to give the criterion for a quintic; then six more lines and columns a sextic, and so on as far as may be required.

The remark to be made concerning the determinant of the third order referred to is that it contains the irrelevant factor 2·1, that is, $\frac{y''}{2}$, so that the criterion for a conic (Monge's) is this determinant divested of such factor. It is *certain* that the next determinant is indecomposable, and is therefore the criterion for a cubic. There is no reason that I know of to suppose that any other determinant except that one which corresponds to the conic, is decomposable into factors. If this is made out, then, observing that the single term which is the criterion for the right line is indecomposable, we have another example of what may be called, in Babbage's words, a miraculous exception to a general law.

A well-known similar case of such miraculous exception I had occasion many years ago to notice in connection with the criteria for determining the number of real and imaginary roots in an algebraical equation. Such criteria may, with one single exception, be expressed by means of invariants. The case of exception is the biquadratic equation, for which it is impossible to assign an invariantive criterion that shall serve to distinguish between the cases of all the roots being real and all imaginary.

It is proper to notice that it follows, from the definition of the symbol $m \cdot \mu$, that its value is zero whenever m is less than 2μ. Thus, in the matrix written out above, the symbols 3·2, 4·3, 5·3, 5·4, 6·4, 7·4 may be replaced by zeros.

The above general result for a curve of any order is actually obtained by a far less expenditure of thought and labour than was employed by Monge, Halphen, and others to obtain it for the trifling case of a conic. I touch a secret spring, and the doors of the cabinet fly wide open*.

* Adopting the convention for degree and weight of a differential coefficient usual in the theory of reciprocants the deg : weight of the differential criterion of the nth order will be easily found to be

$$\frac{n \, . \, n+1 \, . \, n+2}{6} : \frac{n-1 \, . \, n \, . \, n+1 \, . \, n+2}{8}$$

except that for $n=2$ it is 3 : 3 instead of 4 : 3.

49.

ON THE SO-CALLED TSCHIRNHAUSEN TRANSFORMATION.

[*Crelle's Journal*, c. (1887), pp. 465—486.]

EXACTLY one hundred years ago, E. S. Bring (Dissertation, University of Lund, 1786. Meletemata quaedam mathematica circa transformationem aequationum algebraicarum) gave the method to which the name of Tschirnhausen by a common consent in error is now usually attached*. Sometimes but more rarely the method is attributed to Jerrard who came much later into the field. This is especially the case in England; Hamilton for instance in his "Report on Jerrard's method" published exactly 50 years ago in the

* The expression $$P_\theta - L_{n-1} Q_{n-1-\theta} + M_n R_{n-2-\theta}$$

where L, M are given entire functions in x of degrees $n-1$, n,

P, Q, R ,, disposable ,, ,, ,, ,, θ, $n-1-\theta$, $n-2-\theta$,

may be made identically zero by solving $2n-1-\theta$ homogeneous linear equations between the $2n-\theta$ disposable constants contained collectively in P, Q, R, and when this is done we have

$$\frac{P_\theta}{Q_{n-1-\theta}} \equiv L_{n-1} \, [\text{mod. } M_n].$$

Hence it follows that the Tschirnhausen substitution has a one-to-one correspondence with any fractional substitution containing the requisite number of disposable constants: so for instance in the case of a *quintic* the Bring substitution

$$lx^4 + mx^3 + nx^2 + px + q$$

is only *another name* for the general *quadratic substitution* $\frac{ax^2+bx+c}{dx^2+ex+f}$.

This change of form in the substitution, supposed to be generalised, is interesting for the reason that it completes the analogy between the Tschirnhausen method of simplifying an algebraical equation and Combescure's method of simplifying a linear differential equation. Sir James Cockle appears to have arrived at the same result as M. Combescure in a paper on Linear Differential Equations. (*Quarterly Journal of Mathematics*, Aug. 1864.)

This method involves two quadratures, the integration of a differential equation of the second order, and substitutions impressed simultaneously upon *the two variables*.

The quadratures and solution of an equation of the second order are, of course, analogous to the solution of two simple and one quadratic algebraical equation; the substitutions impressed on the two variables run parallel to the two integral substitutions to be performed upon the two variables of the algebraical equation put under the form of a quantic which are equivalent to a fractional substitution performed upon the single variable of a non-homogeneous form.

Reports of the British Association makes hardly any mention of any other author but Jerrard in connexion with the subject.

In the following memoir I propose to present Hamilton's process under what appears to me to be a clearer and more easily intelligible form, to extend his numerical results and to establish the principles of a more general method than that to which he has confined himself.

But previously to entering upon this part of my work I think it may be well to call attention to a circumstance connected with the so-called Tschirnhausen transformation, as bearing upon the character of the transformed equation to which it leads, which hitherto appears to have escaped observation, and which is of particular interest as regards the application of the method to the equation of the 5th degree when it is reduced to the form

$$y^5 + By + C = 0,$$

for I shall be able to show in that case that in general the coefficients which remain (notwithstanding the large element of indeterminateness of which the method admits) cannot be made real when more than one of the roots of the original equation is real; this remark will be found to apply whether the method be used under its original form or under the modified form employed so advantageously by Hermite.

In order to make out this proposition it will be useful to give a somewhat more extended statement of the Law of Inertia (Trägheitsgesetz) for quadratic forms than that originally presented by me in the memoir: "On a theory of the syzygetic relations of two rational integral functions comprising an application to the theory of Sturm's functions and that of the greatest algebraical common measure" (*Phil. Trans.* for 1853)*.

Let us suppose a quadratic function of $m+n$ letters, either independent or connected by linear relations which in the latter case reduce the number of independent quantities to $\mu+\nu$.

Let the function be supposed to be expressed

(1) by the sum of m positive and n negative squares,

(2) by the sum of μ positive and ν negative squares

of *real* linear functions of the variables.

Then I affirm the impossibility of either of the two inequalities

$$\mu > m; \quad \nu > n.$$

(1) I say that the conjunction of the inequalities $m > \mu$, $\nu > n$ is impossible.

For suppose the two expressions of the same quadratic function to be

$$a_1^2 + a_2^2 + \ldots + a_m^2 - b_1^2 - b_2^2 - \ldots - b_n^2$$

and

$$\alpha_1^2 + \alpha_2^2 + \ldots + \alpha_\mu^2 - \beta_1^2 - \beta_2^2 - \ldots - \beta_\nu^2.$$

[* Vol. I. of this Reprint, p. 511.]

Then

$$a_1^2 + a_2^2 + \ldots + a_m^2 + \beta_1^2 + \beta_2^2 + \ldots + \beta_\nu^2$$
$$= b_1^2 + b_2^2 + \ldots + b_n^2 + \alpha_1^2 + \alpha_2^2 + \ldots + \alpha_\mu^2.$$

By hypothesis

$$\mu + n < \mu + \nu,$$
$$\mu + n < m + \nu.$$

By virtue of the first inequality it must be possible to establish $\mu+n$ relations between the $\mu+\nu$ independent variables.

Consequently we may equate each square on the right-hand side of the equation to some distinct square on the other side, and then by virtue of the second inequality some squares will remain over on the left-hand side of the equation whose sum will be identically zero. Which is impossible. Hence the inequalities $m > \mu$, $\nu > n$ cannot exist simultaneously. In like manner it follows that $n > \nu$, $\mu > m$ cannot exist simultaneously.

Now the only suppositions of combined relations of greater and less that can connect m, n; μ, ν are the following:

$$m < \mu,\ n < \nu; \quad m < \mu,\ n = \nu; \quad m < \mu,\ n > \nu;$$
$$m = \mu,\ n < \nu; \quad m = \mu,\ n = \nu; \quad m = \mu,\ n > \nu;$$
$$m > \mu,\ n < \nu; \quad m > \mu,\ n = \nu; \quad m > \mu,\ n > \nu.$$

Of these 9 suppositions the 1st, 2nd, and 4th are excluded by the condition $m+n =$ or $> \mu+\nu$, and the 3rd and 7th by virtue of what has just been proved. Hence the only hypotheses admissible are the four contained in the negative statements:

μ not $> m$ and ν not $> n$. Q.E.D

Although the only application which I shall have to make of this Lemma is to the case where $m+n = \mu+\nu+1$, I have thought that it is of sufficient interest in itself and collaterally in the logical process of its proof to deserve setting out in full.

Suppose now that we have the equation $f(x) = (x, 1)^n = 0$ where all the coefficients in f are supposed to be real, and that we write in conformity with the ordinary so-called Tschirnhausen process:

$$y = u_1 x + u_2 x^2 + \ldots + u_{n-1} x^{n-1} - S,$$

where

$$nS = u_1 \Sigma x + u_2 \Sigma x^2 + \ldots + u_{n-1} \Sigma x^{n-1}$$

so that the transformed equation will be of the form:

$$y^n + B_2 y^{n-2} + B_3 y^{n-3} + \ldots + B_n = 0,$$

where B_i is a quantic of degree i in the letters $u_1, u_2, \ldots u_{n-1}$. Let us consider the projective character of the quadratic function B_2. This character is determined by the nature of the succession of algebraical signs in the sum of positive and negative squares to which B_2 regarded as a function of the $n-1$ letters u may be reduced by *real* linear transformations.

Since

$$y_1 + y_2 + \ldots + y_n = 0,$$
$$-2B_2 = -\Sigma 2 y_1 y_2 = \Sigma y^2,$$

so that it is the character of Σy^2 which determines the projective character of B_2. The number of real values of y is the same as of x. Hence if f has i pairs of imaginary roots, Σy^2 will be the sum of $n - i$ positive and i negative squares of real linear functions of $u_1, u_2, \ldots u_{n-1}$.

Consequently, by virtue of the lemma above proved, there is only one element of uncertainty as to the character of Σy^2, that is, it must we know *à priori*, when reduced to a sum of $n - 1$ positive and negative squares of linear functions of $u_1, u_2, \ldots u_{n-1}$, contain either i or $i - 1$ negative squares. This uncertainty may be removed by means of a second lemma, namely, that the discriminant of B_2 is a numerical multiplier of the discriminant of f.

When two of the roots of f are equal, two of the values of y become equal so that Σy^2 becomes reducible to a sum of $n - 2$ instead of a sum of $n - 1$ squares.

Hence the former contains the latter as a factor: moreover it is obvious from the form of each value of y that its discriminant regarded as a function of the n roots of f will be of the degree $2\{1 + 2 + \ldots + (n-1)\}$, that is, $n(n-1)$ which is the same as that of the squared product of the differences of the roots of f. Hence B_2 is a *numerical* multiplier of such squared product. To find the value of the multiplier, I observe that in general it follows from known algebraical principles that if F is a sum of the squares of n linear functions of $n - 1$ variables the discriminant of F may be found as follows. Form an oblong matrix with the coefficients of the several linear functions. The determinant represented by what Cauchy would have called the square of this matrix, but which is more correctly to be called the product of this matrix by its transverse, will be the discriminant in question, or which is the same thing this discriminant is the sum of the squares of all the *complete* minors that are contained in the oblong matrix.

In the case before us if we make $f = x^n - 1$* it will easily be seen that

* When $f = x^n - 1$ the value of S (the mean of the values of y) is obviously zero. Suppose now by way of illustration that $n = 5$, then calling the imaginary 5th roots of unity $\rho_1, \rho_2, \rho_3, \rho_4$, one of the complete minors referred to in the text will be the determinant of the matrix

$$\begin{matrix} \rho_1 & \rho_2 & \rho_3 & \rho_4 \\ \rho_1^2 & \rho_2^2 & \rho_3^2 & \rho_4^2 \\ \rho_1^3 & \rho_2^3 & \rho_3^3 & \rho_4^3 \\ \rho_1^4 & \rho_2^4 & \rho_3^4 & \rho_4^4, \end{matrix}$$

and when the columns of this matrix are divided respectively by $\rho_1^\theta, \rho_2^\theta, \rho_3^\theta, \rho_4^\theta$, $[\theta = 1, 2, 3, 4]$, which will leave the value of the determinant unaltered, the determinant of the matrix so modified will represent in succession each of the other 4 minors.

The value of the one above written, paying no attention to the algebraical sign, is by a well known theorem the product of the differences of $\rho_1, \rho_2, \rho_3, \rho_4$, that is, inasmuch as

$$(1 - \rho_1)(1 - \rho_2)(1 - \rho_3)(1 - \rho_4) = 5$$

the n minors in question, paying no regard to algebraical sign, become all equal, and each will be the product of the differences of the roots of $x^n - 1$ when the root 1 is excluded, or which is the same thing will be the product of the differences of all the roots (not excluding 1) divided by n.

Hence the sum of the n squared minors will be the nth part of the square of the products of the differences of the roots of $x^n - 1$. Consequently in general the discriminant of Σy^2 is the nth part of the product of the squares of the differences of the roots of the function f, and therefore by the process of reduction of $-\Sigma y^2$ to a sum of $n - 1$ squares it is the *positive* sign always which will undergo the diminution of a unit, the number of negative signs remaining unaltered.

Hence when there are no imaginary roots in f, $-B_2$ will have all its signs positive; but when there are i pairs of imaginary roots in f, i of the signs in $-B_2$ will be negative, and thus the character of B_2, or of the quadratic *contour* (that is, curve, surface, hypersurface, etc.) represented by $B_2 = 0$ is completely determined when the number of real and imaginary roots in f is given.

If we suppose $n = 5$ we see that according as the number of real roots in f is 5, 3, or 1, the signs of $-B_2$ regarded as a sum of positive and negative squares of real linear functions of 4 letters will be:

$$\begin{matrix} + & + & + & + \\ + & + & + & - \\ + & + & - & - \end{matrix}$$

In the first case the contour B_2 is completely imaginary, and it is not only not possible to apply the Bring-Tschirnhausen method so as to make simultaneously $B_2 = 0$, $B_3 = 0$ by real quantities u_1, u_2, u_3, u_4, but it is also the case that such values of u_1, u_2, u_3, u_4 do not exist. This indeed is evident *à priori*, from the fact that the equation

$$y^5 + B_4 y + B_5 = 0$$

must have at least two imaginary roots and therefore the equation in x would have at least two imaginary roots if the quantities u_1, u_2, u_3, u_4 were all real and unequal; whereas all the roots of that equation are supposed to be real.

In the second case the intersection of the contours B_2, B_3 may be real or imaginary: but even if it be real the method will not serve to determine any

it is the 5th part of the product of the differences of $1, \rho_1, \rho_2, \rho_3, \rho_4$, and consequently the sum of the squares of the 5 minors is 5 times the 25th part of the squared product of the differences of the 5 roots. Here $\frac{5}{25}$ represents the general numerical multiplier $\frac{n}{n^2}$, that is, $\frac{1}{n}$.

single point in such section, because no *real* right line can be drawn to B_2 at any point which shall lie on the surface.

In the 3rd case at each point of B_2 two real right lines can be drawn each of which will intersect B_3 in one real point at least, and accordingly there will be a duplex-infinity of systems of real values of the u's which will make $B_2=0$, $B_3=0$ capable of being found by solving only a quadratic and a cubic equation in succession, and any one of such systems will lead to an equation of the form

$$y^5+B_4y+B_5=0,$$

where B_4, B_5 (which it is hardly necessary to notice become respectively $\frac{1}{4}\Sigma y^4$, $-\frac{1}{5}\Sigma y^5$) will each be real.

The B_2 found by Hermite's method may be obtained from the B_2 above given by a real linear substitution impressed on the letters u_1, u_2, u_3, u_4, and consequently the same conclusions continue to apply, that is, the coefficient of y and the constant will not in general be real unless four of the roots of the equation in x are imaginary*.

I will now proceed to the principal object of this paper, namely, the elucidation and extension of the method, contained in Hamilton's report, for determining the least number of letters which must be contained in one or more equations in order that they may admit of being solved by means of equations whose degrees are subject to satisfy certain prescribed conditions.

Before proceeding to the Lemma upon which all that follows is based, it will be useful to give one or two definitions.

1. Let S be a system of homogeneous equations in an indefinite number of variables $x, y, \ldots$, and let $x=a$, $y=b$, ... satisfy all the equations. I call $a, b, \ldots$ a solution of S.

2. If $a, b, \ldots$ is a given solution of S, I call the equation obtained by operating upon any of those in S with $(a\partial_x+b\partial_y+\ldots)^q$ where q has any integer value whatever not excluding zero, an emanant of such equation in respect to the solution $a, b, \ldots$, and the new system S_1 which contains all the emanants of all the equations in S an emanant to S in respect to the given solution.

* Hamilton remarks (*Report of British Association*, 1836, p. 307) that "the coefficients of the new or transformed equation will *often* be imaginary even when the coefficients of the original equation are real." Apparently he was not aware that the criterion for determining when this is so, depends solely on the intrinsic character of the equation to be transformed.

It should have been noticed before that when two of the roots in the given quintic are equal the quadratic surface represented by the coefficient of y^3 in the transformed equation becomes a cone and the reasoning employed in the text falls to the ground. But inasmuch as in this case two of the values of y become equal, we know *à priori* that the equation in y must be reducible to a form with real coefficients, namely,

$$y^5-5y+4=0.$$

3. Let $a, b, \ldots, a_1, b_1, \ldots$, be any two solutions of a system of equations. I call

$$a+\lambda a_1,\quad b+\lambda b_1,\quad c+\lambda c_1,\quad \ldots$$

an alliance of the two solutions.

We may now state the following Sub-lemma. The alliance of any solution of a system S with a solution of S_1, its emanant in respect to the first named solution, is a solution of S_1.

For let $a, b, \ldots$ be the solution of S which gives rise to S_1. Then, calling

$$a\partial_x+b\partial_y+\ldots=E,$$

the general form of the equations in S_1 is $E^q\Phi(x, y, \ldots)$, and, supposing Φ to be of n dimensions,

$$E^q\Phi(x+\lambda a,\ y+\lambda b,\ \ldots)$$
$$=E^q\Phi+\lambda E^{q+1}\Phi+\tfrac{1}{2}\lambda^2E^{q+2}\Phi+\ldots+\lambda^{n-1}E^{q+n-1}\Phi.$$

Hence the effect of substituting $x+\lambda a$, $y+\lambda b$, ... for $x, y, \ldots$ in S_1 is merely to effect upon it a linear transformation, and consequently the alliance of the solution $a, b, c, \ldots$ with any solution of S_1 will be a solution of that system. If now we find a solution $a_2, b_2, \ldots$ of S_1 and form an emanant S_2 of it in respect to that solution, it will follow from the sub-lemma that an alliance of the solutions $a, b, \ldots$; $a_1, b_1, \ldots$; with any solution $a_2, b_2, \ldots$ of S_2, that is, the solution $a+\lambda a_1+\mu a_2$, $b+\lambda b_1+\mu b_2$, ... will be a solution of S_2, and so in general. This I call the Lemma.

An ordinary solution of a system of equations may be called a point solution, an alliance in which 1, 2, 3, ... parameters enter a line, a plane, a hyperplane, ... solution.

It will of course be observed that any solution of an emanant to a system is *à fortiori* a solution of the primitive which as observed forms a portion of its emanant.

If S be a system of $q_i, q_{i-1}, \ldots q_1$ equations of degrees $i, i-1, \ldots 1$ respectively in the variables, it is obvious the 1st emanant will consist of

$$q_i,\quad q_i+q_{i-1},\quad q_i+q_{i-1}+q_{i-2},\quad \ldots\quad q_i+q_{i-1}+\ldots+q_1$$

equations of the degrees $i, i-1, i-2, \ldots 1$ respectively, and more generally in the rth emanant the number of equations of the degree

i will be q_i

$i-1$ „ „ rq_i+q_{i-1}

$i-2$ „ „ $\frac{1}{2}r(r+1)q_i+(r-1)q_{i-1}+q_{i-2}$

....

....

1 „ „ $[r+i-1]_iq_i+[r+i-2]_{i-1}q_{i-1}+\ldots+q_1$,

where in general $[\theta]_i$ is used to denote $\dfrac{\theta(\theta-1)\ldots(\theta-i+1)}{1\,.\,2\ldots i}$.

The question now arises as to what must be the number of variables in a system S in order that its rth emanant S_r may admit of a general solution. If the total number of equations in S_r be called N, it might at first sight be supposed that the number of variables, or letters as I prefer to call them, in S must have $N+1$ as an inferior limit: but the case is not so—the least number of variables required will be r greater than this, that is, $N+r+1$.

Thus, for example, suppose we consider a first emanant S_1; then if $a_1, b_1, c_1, \ldots$ is a solution we know that $a_1+\lambda a,\ b_1+\lambda b,\ c_1+\lambda c, \ldots$ is also a solution whatever λ may be. Hence making $\lambda = -\frac{a_1}{a}$ and remembering that the equations are homogeneous we see that zero associated with any system of independent minors of the matrix

$$\begin{matrix} a & b & c & \ldots, \\ a_1 & b_1 & c_1 & \ldots \end{matrix}$$

will constitute a solution, as for instance 0; $ab_1 - ba_1$; $ac_1 - ca_1$; …*. Hence the number of independent quantities in S_1 will be 1 less than the number of letters in S.

* As an illustration suppose Φ is a quantic of degree n in $(n+2)$ letters representing what may be termed a *contour*, the analogue in general space of a curve in 2-dimensional or a surface in 3-dimensional space. If we take all the successive emanants of Φ in respect to a point upon it $a, b, c, \ldots$ the n resulting functions [Φ included] being functions of the $n+1$ minors to the matrix [$(n+2)$ places in length]

$$\begin{vmatrix} a & b & c & \ldots \\ x & y & z & \ldots \end{vmatrix}$$

the contours which they represent will intersect in a faisceau of right lines—showing that on a contour of the nth degree in $(n+1)$-dimensional space $1 \cdot 2 \cdot 3 \ldots n$ right lines lying in the contour will pass through every point thereof, a fact we are familiar with in the case of a quadric surface where $n=2$. We might with equal propriety and more convenience say that Πn straight lines may be drawn upon and at every point of an n-fold contour of the nth *order*.

As I have already referred in this footnote to right lines drawn on contours I venture upon a slight digression connected with this conception. If we have a cubic twofold contour (an ordinary cubic surface) expressed as a quantic in x, y, z, t, we see that on writing x, y as linear functions of z, t and substituting their values in Φ in order to make the result, a cubic function of z, t vanish, we have to satisfy 4 equations between the 4 coefficients of substitution, which at once shows that a finite number of right lines may be drawn upon such contour of which the number we see at once cannot exceed 3^4 and which we know aliunde is 3^3.

It would seem then that for a contour in n letters of the degree $2n-5$ (unless there is some lurking fallacy in the counting of the constants) we ought in like manner to be able, by expressing $n-2$ of the letters as linear functions of the two remaining ones, to make the result vanish by solving $2n-4$ non-homogeneous equations of the degree $2n-5$ between the like number of coefficients of substitution, and as if upon such a contour we must be able to draw a definite number of straight lines of which the number, supposing that there is no latent fallacy of constant-counting, would be not greater and in all probability less than $(2n-5)^{2n-4}$, in fact $(2n-5)^{2n-5}$.

Also it may be shown that, as by Bedetti's theorem we know that every twofold contour (an ordinary surface) is cut by its linear polar (its tangent plane) in respect to a point upon it, in a curve having a double point thereat, so a contour of the 3rd order will be cut by its linear and quadratic polars in respect to any point upon it in a curve having a sextuple point thereat, and so in general an n-fold contour will be cut by $n-1$ consecutive polars (starting from the tangential

Similarly for the system S_r; r zeros associated with any independent system of complete minors of the matrix

$$\begin{matrix} a & b & c & \ldots \\ a_1 & b_1 & c_1 & \ldots \\ a_2 & b_2 & c_2 & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_r & b_r & c_r & \ldots \end{matrix}$$

may be taken as the variables, and consequently it is $N-r-1$ and not $N-1$ which has for its inferior limit the number of equations in S_r. We may restore to the variables their independence by associating with the equations in S_r r additional perfectly arbitrary linear functions and there is sometimes a convenience in substituting in place of the rth emanant as it stands such emanant augmented by r arbitrary linear functions, which may be called the *completed* emanant.

For the purpose of greater clearness of exposition there will be an advantage in ignoring in the first instance all considerations based upon any other alliance except of the 1st order, that is, involving only one arbitrary parameter.

Suppose a system of equations S_1 consisting of a system S and one equation more Q. If we are in possession of a linear solution of S, that is, a solution

$$x = a_1 + \lambda a, \quad y = b_1 + \lambda b, \quad \ldots$$

by substituting these values in Q, λ may be found by solving an equation whose degree is that of Q, and thus a point (or ordinary) solution of S_1 will have been found.

Let us now consider the question of a linear solution of S containing $q_i, q_{i-1}, \ldots q_1$ equations of degree $i, i-1, \ldots 1$ respectively. This we shall call of the type $[q_i, q_{i-1}, \ldots q_1]$. Let

$a, b, \ldots$ be any point solution of S,

and $a_1, b_1, \ldots$ any point solution of ES,

homaloid as the first of them) in respect to any point upon it in a curve having thereat a point of multiplicity $1 \cdot 2 \cdot 3 \ldots n$.

It may be well here to notice that a uni-parametrical solution of $\Phi=0$ corresponds to drawing a straight line upon the contour represented by Φ, and in like manner a bi-parametric solution corresponds to drawing a plane upon the contour, a tri-parametric solution to drawing a hyperplane upon the contour, and so in general. This is why I call such solutions linear, planar, hyperplanar, etc.

So again in this connexion it may be remarked that upon a quadratic contour in trans-hyperspace 6 planes lying on the contour pass through every point and in like manner upon a quadratic contour in $2n$ letters, $1 \cdot 2 \cdot 3 \ldots n$ n-fold homaloids may be drawn upon the contour through every point thereon.

the *completed* emanant of S in regard to $a, b, \ldots$. This will be of the type

$$q_i;\ q_i+q_{i-1};\ \ldots;\ q_i+q_{i-1}+\ldots+q_2;\ q_i+q_{i-1}+\ldots+q_1+1.$$

The alliance of these two point solutions will be a linear solution of S. Again, the number of variables required for the point solution of ES need not exceed the number required for the linear solution of the system to which S is reduced by the abstraction of one equation of the degree i. Hence if we use $[p, q, r, \ldots \eta, \theta]$ to denote the number of variables sufficient for the solution of a system of p equations of degree i, q of degree $i-1$, etc. (i being the number of indices $p, q, \ldots \theta$) we obtain the formula of reduction

$$[p, q, r, \ldots \theta] = [p-1, p+q, p+q+r, \ldots p+q+r+\ldots+\theta]+1^*.$$

Continuing this process of reduction until the first index is reduced to zero a very easy calculation leads to the *formula of obliteration*

$$[p, q, r, s, \ldots \theta] = p + [q_1, r_1, s_1, \ldots \theta_1]\dagger,$$

where

$$q_1 = \frac{p(p+1)}{2} + q,$$

$$r_1 = \frac{p(p+1)(2p+1)}{6} + pq + r,$$

$$s_1 = \frac{p(p+1)(p+2)(3p+1)}{24} + \frac{p(p+1)}{2}q + pr + s,$$

....

$$\theta_1 = \frac{p(p+1)\ldots(p+i-1)(ip+1)}{1.2.3\ldots(i+1)} + \frac{p(p+1)\ldots(p+i-2)}{1.2.3\ldots(i-1)}q + \ldots + \theta.$$

In applying the process of reduction in the way indicated, the system S will have been replaced by two systems which we may call a diminished S and a diminished emanant of S, that is, S and ES each deprived of an equation (not necessarily the same in both) of degree i.

In like manner each of these will give rise to two systems, namely, a diminished self-system and a diminished emanant-system; but as the object is to obtain a formula of reduction for the number of letters required to obtain a linear solution of S, and as this number is greater for an emanant of any system than for the system itself, it was sufficient to follow the main-stream of deduction, in which the first alone is taken account of, in order to arrive at the required formula. In doing so, 2^p independent equations of the degree i will have been set apart each of which will have to be solved in its proper turn.

In the formula of *obliteration* the index in the first place has disappeared. Repeating the process we shall come to

$$p + q_1 + [r_2, s_2, \ldots \theta_2]$$

* In this and all subsequent formulae of reduction or obliteration the sign "=" is to be understood to mean "not greater than."

† Compare Hamilton, *Report of British Association*, 1836, p. 335, formula 244, and p. 345, formula 320.

where $r_2, s_2, \ldots \theta_2$ are derived from $q_1, r_1, s_1, \ldots \theta_1$ in the same way as $q_1, r_1, s_1, \ldots \theta_1$ from $p, q, r, s, \ldots \theta$ except that i will be replaced by $i-1$; and thus pursuing the same process we shall arrive at

$$[p + q_1 + r_2 + \ldots + \eta_{i-1} + \theta_i]$$

or say $[\sigma]$. The number of variables required for a solution involving one arbitrary parameter of σ homogeneous linear equations being $\sigma + 2$, this latter will be the number sufficient for S to admit of a linear solution without giving occasion to solve any equation of a degree exceeding i, and also without having occasion to solve any simultaneous system of equations other than linear ones.

Suppose a system of equations of the respective degrees $1, 2, 3, \ldots i$ and a single equation of the degree $i+1$.

The type of the former will be $1, 1, 1, \ldots 1$ to i places,
and of the latter $1, 0, 0, 0, \ldots 0$ „ $i+1$ „ .

By the rule which has been established the number of letters required for the linear solution of the latter will be *one* more than for the former.

Hence the determination of the Tschirnhausen question of finding what the degree of an equation must be in order that i consecutive terms following immediately after the first term in the transformed equation, conjoined with any more advanced term, may admit of a solution of minimum weight, contains a determination of the number of variables required to ensure the possibility of obtaining a linear solution by a system of equations of minimum weight of a single equation of degree $i+1$; for the latter number will be the former increased by a unit*. The first form of the question is the more simple in itself; but as the other is more immediately connected with the object in which the theory originated, I prefer to put it in the latter form.

We may apply the obliteration formula to the indefinite type and obtain the annexed Table.

Triangle of Obliteration.

1	1	1	1	1	1	1	......
	2	3	4	5	6	7	......
		6	15	29	49	76	......
			36	210	804	2449	......
				876	24570	401134	......
					408696	246382080	......
						83762796636	......
							......
							....
							...

* For example, to take away the 2nd, 3rd, and another term the degree required is 5: and to obtain a linear solution of a cubic the number of variables required is 6.

To take away the 2nd, 3rd, 4th, and another term, employing a solution of *the lowest weight*, 11 variables are required; in order to obtain a solution, of lowest weight, of a single function of the fourth degree, 12 variables are required, and so on.

The degree of the equation sufficient to allow

$$2, \ 3, \ 4, \ 5, \ 6, \ 7, \ \ldots$$

consecutive terms following the first to be removed by a solution of *minimum weight* of the auxiliary equations, will be the continued sum of

$$1, \ 2, \ 6, \ 36, \ 876, \ 408\,696, \ 83\,762\,796\,636, \ \ldots$$

each increased by 2, that is,

$$3, \ 5, \ 11, \ 47, \ 923, \ 409\,619, \ 83\,763\,206\,255, \ \ldots$$

These numbers up to 923 agree with those found by Hamilton (*Report*, p. 346), the two last have been calculated here probably for the first time.

It would be too arduous a task to seek to give a much further extension to the table inasmuch as each successive term in the series 1, 2, 6, 36, ... is a fraction converging to $\frac{1}{2}$ of the square of the preceding term. This becomes obvious from inspection of the series formed by dividing each number in the above series by the square of the one before it; we thus obtain the fractions:

$$\frac{4}{1}, \ \frac{6}{4}, \ \frac{36}{36}, \ \frac{876}{1296}, \ \frac{408696}{767376}, \ \frac{83762796636}{167032420416},$$

which are continually diminishing.

But if we call two successive and infinitely distant rows of the Triangle of Obliteration

$$\begin{matrix} a & b & \ldots \\ & B & \ldots, \end{matrix}$$

$$B = \frac{a^2 + a}{2} + b.$$

Hence $\frac{B}{a^2}$ converges to $\frac{1}{2} + \frac{b}{a^2}$ which is always greater than $\frac{1}{2}$. Moreover $\frac{b}{a^2}$, calculated for the successive values as far as the table extends, will be seen to be a continually decreasing fraction and assuming (what awaits exact proof) that it eventually vanishes, $\frac{B}{a^2}$ must converge to $\frac{1}{2}$.

The successive values of $\frac{b}{a^2}$ for the different rows are

$$\frac{3}{4}, \ \frac{15}{36}, \ \frac{210}{1296}, \ \frac{24570}{767376}, \ \frac{246382080}{167032420416}.$$

Inverting these fractions the values, to the nearest integer, become 1, 2, 6, 31, 678, so that there can be no doubt of the truth of the law that the asymptotic value of the square of each term divided by the square of its antecedent is $\frac{1}{2}$.

Moreover the numbers last found themselves obviously obey a parallel law to that of the original series which raises a presumption that it may be possible to obtain an exact expression for the general term in the original series or even in the Obliteration Table in its entirety. But be that as it may, as evidently the asymptotic law is equally true for the sums of the terms in the first diagonal as for the terms themselves, we arrive at the interesting fact that if $\Phi(i)$ is the minimum degree of an equation from which i consecutive terms immediately following the first can be removed, $2\Phi(i+1)$ converges to a ratio of equality with $\Phi(i)^2$ when i increases indefinitely.

The minimum number of letters thus found is we see a minimum, at all events in this sense that the *method employed* to obtain a solution is inapplicable if that number of letters be reduced. In the words of Jerrard as quoted by Hamilton (*Report*, pp. 326, 327) "to discover $m-1$ ratios of m disposable quantities,

$$a_1, \ a_2, \ \ldots \ a_m$$

which shall satisfy a given system of h_1 rational and integral and homogeneous equations of the first degree

$$A' = 0, \quad A'' = 0, \quad \ldots \quad A^{(h_1)} = 0,$$

h_2 such equations of the second degree

$$B' = 0, \quad B'' = 0, \quad \ldots \quad B^{(h_2)} = 0,$$

h_3 of the third degree

$$C' = 0, \quad C'' = 0, \quad \ldots \quad C^{(h_3)} = 0,$$

and so on, as far as h_t equations of the tth degree

$$T' = 0, \quad T'' = 0, \quad \ldots \quad T^{(h_t)} = 0$$

without being obliged, in any part of the process, to introduce any elevation of degree by elimination."

But this definition may be superseded by another in which only the intrinsic character of the result arrived at is in question, and not the particular method pursued to reach it.

Let us agree to consider all equations of the same degree to have the same weight and that this weight is infinitely greater than that of an equation of any lower degree. The weight of a system of equations to be regarded as the sum of the weights of the equations which it contains.

We may, extending but not altering the meaning previously attached to the word "solution," call the *ensemble* of the equations to be solved in order to obtain any solution of the given system a solution thereof. If now a system of equations is given in number and in the degree of each, and each equation is supposed to be the most general of its kind, but the number of variables in the system is left disposable, it is easy to see that the above process, when it is practicable, leads to a solution of the lowest weight, so that no increase in the number of letters will have any effect in diminishing the weight of the solution, whatever may be the process employed to obtain it. Thus the numbers given by the linear method are *minima* in regard to solutions of the *lowest weight*.

We may however suppose another and more natural condition attached to the solution to be obtained; let n be the highest degree of any equation in a given general system proposed for solution; we know that it is impossible to avoid the solution of one or more equations of the nth degree. We may therefore propose to ourselves the problem of determining what is the least number of letters necessary in order that no equation in the solution shall be of a degree exceeding n. The minimum thus obtained will in general be inferior to the minimum required for obtaining a solution of the lowest weight, and to arrive at it in any particular case it becomes necessary to make use of the Lemma in its general form which introduces the notion of alliances above the first order. Hamilton has not touched upon this part of the subject except in a single case which it was impossible to overlook: namely, where he considers the problem of taking away four consecutive terms from the general equation of the tenth or any higher degree.

The process we have seen leads to the conclusion that as many letters are required as are needed for the solution of two quadratics and seven linear equations. The solution of one biquadratic equation in the application of the process being indispensable, he felt the absurdity (if I may use the word) of stickling at the introduction of one biquadratic more, the use of which has the effect of lowering the minimum from 11 to 10. See *Report of British Association*, 1836, p. 326.

The linear method however or theory of solutions of lowest weight enjoys this prerogative that the reduction formulae are of a purely algebraical kind, whereas when the other condition above referred to is introduced, questions of numerical equality and inequality have to be considered and the theory ceases to be strictly algebraical. In what follows therefore I shall confine myself to the only case of any particular interest, namely, that which arises from the original problem of removing any given number of consecutive terms (immediately following the first) from an algebraical equation.

We may accept as the general condition to be observed that the degree of no equation appearing in the solution of a system of equations shall exceed the highest degree which must perforce figure in such solution, that is, the highest degree in the system of equations to be solved. In the case then of n equations of the successive degrees 1, 2, 3, ... i the condition will be that no equation in the solution shall be of a higher degree than i.

Thus, for example, if we look back to the easy case of a quaternary succession of such terms to be removed, we find that the problem reduces itself to finding the number of letters required to obtain a line-solution of the system whose type is 1, 1, 1, and that again to finding the number of letters required to obtain a line-solution to its augmented emanant 2, 4, that is, a system of 2 quadratic and 4 linear solutions, that is, a point solution of the completed emanant to this system which will be of the type 2, 7. The condition imposed here is that no equation shall appear of a higher degree than a biquadratic. Consequently subject to this condition the number of letters required to solve a system of one linear, one quadratic, and one cubic equation, is that sufficient for the plane-solution of a system of 7 linear equations, that is, 10, which is less by 1 than the number required in order to obtain a solution of the same system which shall be of the lowest weight.

It might at first sight be supposed that in general the introduction of solutions involving 2 or more parameters would lead to a very considerable reduction of the numbers found in the obliteration table; this however is not the case, the reduction in the values obtained by this extended method bears in general a very small ratio to the number reduced. This is a consequence of the following rule:—

In passing from the point solution of a system to a solution of any kind with a reduced type, the reduction is effected by *segregating* a certain number i of the given equations and obtaining a solution of the remainder which shall contain i arbitrary parameters.

Now it will be found that the *literant* (by which I mean the number of letters sufficient for the solution) will never be diminished by any other kind of segregation than what may be termed an *external segregation**.

* Imagine the type of a set of equations to be represented by a broad ribbon, in which each group of equations of the same degree is represented by a band of a distinct colour occupying as many units of space as there are units in the group. The legitimate process of segregation will then consist in dividing the band into two, obeying the same conditions as the original one, and the rule of "external segregation" amounts to saying that this separation must be effected by a single straight cut so that no middle portion is to be cut out.

According to this (which is a perfectly natural) representation the rule of external segregation may in the language of logic be described as the rule of the *excluded middle*. Thus, for example, suppose we wish to find the smallest number of variables required for the solution of a system of equations of which the type is 1, 1, 1, 0 without solving an equation beyond the 8th degree. The number required may be made equal to (cf. p. 547)

$$.[1, 1, 0] \text{ or to } :[1, 0, 0].$$

But

$$.[1, 1, 0]=[1, 2, 3]=:[3],$$

and

$$:[1, 0, 0]=[1, 2, 5]=:[5].$$

Thus the simultaneous segregation of the equations of the 4th and 2nd degrees *contrary to the rule* not only raises the weight of the solution but also increases the number of variables required in the given system in order that the solution may be possible.

As a consequence of this rule it may easily be seen (in the problem of determining the

Let $f, g, \ldots k, l$ be the type of the system of equations segregated, this will have no effect in diminishing the literant unless $f, g, \ldots k$ are the initial numbers of the type of the given system, in such case I call the segregation *external*.

Thus in starting with a system of the type $1, 1, \ldots 1, 1$ the first act of segregation must consist in setting apart the equation of the highest degree and finding a line-solution of the system thus reduced. Suppose, to fix the ideas, that the highest degree is 6 and that we have arrived in the course of the deduction at a system of linear, quadratic, and cubic equations denoted by the type m, n, p.

So far as regards observance of the limit 6 for the highest degree in any substituted system, it would be permissible to segregate one cubic and one quadratic, but according to the rule of external segregation this will not be profitable (it will in general be quite the reverse unless $m = 1$) and so in general.

Let us now proceed to obtain the literant required for the point-solution of a sequence of i equations of all degrees from 1 to i subject to the condition that no auxiliary system shall contain an equation of degree higher than i for the values $i = 5, 6, 7, 8$ which is as far as the table of obliteration extends. The rule teaches that this is the same as the literant of a line-solution of a system of $i-1$ equations whose degrees extend from 1 to $i-1$.

It will be useful in what follows to obtain a general formula for the plane-literant of a system of i quadratics denoted by the type $i, 0$.

Let us signify by a symbol consisting of a type preceded by q points the literant to the form of solution containing q parameters of the system to which the type refers.

Then calling the plane-literant for $[i, 0]$ v_i, we have by virtue of the Lemma

$$v_i = :[i, 0] = [i-2, 2i+2] \quad = v_{i-2} + 2i + 2,$$
$$v_1 = :[1, 0] = .[1, 2] \qquad = .[4] = 6,$$
$$v_2 = :[2, 0] = .[2, 3] = .[1, 6] = [8] = 9.$$

Hence by integrating $v_i - v_{i-2} = 2i + 2$ we shall easily obtain:

$$v_{2q} \ = 2q^2 + 4q + 3,$$
$$v_{2q-1} = 2q^2 + 2q + 2.$$

In treating of the literant to $[1, 1, 1, 1, 1, 1, 1, 1]$ it will be convenient to find

minimum degree of the equation required for taking away i consecutive terms without any equation in the solution exceeding the ith degree) that the occasion can never arise in the act of segregation to take account of any other numerical equalities and inequalities than one or the other of the two following

$$q^i = \text{ or } < n, \quad q^i(q-1)^j = \text{ or } < n.$$

the value of $\vdots[i, 0, 0]$ the general expression of which rid of exponentials will give rise to 3 cases.

Not being desirous of encumbering this memoir with formulae, and as we shall only have occasion to consider a single case of these formulae, I adjourn the calculation until we know what the form is of i in regard to 3 in the case to be calculated, and shall obtain the value of $\vdots[i, 0, 0]$ for that case alone.

I will now consider in succession the *literants* denoted by

$$.[1, 1, 1, 1] \quad .[1, 1, 1, 1, 1] \quad .[1, 1, 1, 1, 1, 1] \quad .[1, 1, 1, 1, 1, 1, 1]$$

subject to the conditions of the solution containing no equation of a degree higher than the 5th, 6th, 7th, 8th respectively

$$.[1, 1, 1, 1] = .[2, 3, 5] = .[1, 5, 11] = .[6, 18]$$
$$= :[4, 25] = 25 + 2 . 2^2 + 4 . 2 + 3 = 44.$$

This is the literant for the solution of minimum highest degree and is 3 units less than 47, the literant for the solution of lowest weight.

It will be observed that $.[6, 18]$ has been expressed in the course of the deduction by $:[4, 25]$ instead of $.[5, 25]$. In fact $.[6, 18] = [6, 25]$ and this latter according as we segregate 1 or 2 of the quadratics is expressible by $.[5, 25]$ or by $:[4, 25]$.

The expression $.[6, 18]$ might have been obtained immediately from the triangle of obliteration

$$\begin{matrix} 1 & 1 & 1 & 1 & \ldots \\ & 2 & 3 & 4 & \ldots \\ & & 6 & 15 & \ldots \\ & & & . & \ldots \end{matrix}$$

by simply substituting $1 + 2 + 15$ for 18. (It is worth noticing that in the table of obliteration after the 2nd line every initial number in any line ends with 6 and after the 3rd line every second number in each line ends with 0.)

So in like manner observing that $1 + 2 + 6 + 210 = 219$, we have

$$.[1, 1, 1, 1, 1] = .[36, 219]$$

which must have been *led up to* from

$$[1, 36, 219].$$

Hence $\quad .[1, 1, 1, 1, 1] = .[1, 35, 182] = [1, 36, 219] = :[35, 219]$

$$= 219 + 2 . 18^2 + 2 . 18 + 2 = 905$$

which is 18 units less than the corresponding literant of lowest weight 923. Similarly observing that

$$1 + 2 + 6 + 36 + 24\,570 = 24\,615,$$

$$.[1, 1, 1, 1, 1, 1] = :[875, 24\,615] = 24\,615 + 2\,(438)^2 + 2\,(438) + 2 = 409\,181$$

which is 438 less than the corresponding literant of lowest weight 409 619.
In like manner calling

$$246\,382\,080 + 876 + 36 + 6 + 2 + 1 = 246\,383\,175 = s,$$

$$.[1, 1, 1, 1, 1, 1, 1] = s + :[408\,695, 0] = [408\,695, 0] + t = \vdots[408\,692, 0] + t$$

where $$t = s + 2 \times 408\,695 + 2 = 247\,200\,567.$$

Here $$408\,695 \equiv 2 \text{ [mod. 3]}.$$

But in general
$$\vdots[3q+2, 0] = \vdots[3q-1, 0] + 9q + 9$$
$$= \vdots[2, 0] + 9\{(q+1) + q + (q-1) + \ldots + 2\}$$
$$= \vdots[2, 0] + \frac{9(q^2+3q)}{2} = \frac{9q^2+27q+24}{2}\,*$$
$$= \frac{(3q+2)^2 + 5(3q+2)}{2} + 5.$$

Therefore
$$.[1, 1, 1, 1, 1, 1, 1] = t + 5 + (204\,346)(408\,697)$$
$$= 247\,200\,572 + 83\,515\,597\,162$$
$$= 83\,762\,797\,734.$$

This number is the minimum degree of equation which admits of 8 of its terms being removed without solving any equation above the 8th degree in the same sense as 5 is the minimum degree of equation from which 3 terms can be removed without solving an equation above the 3rd degree.

The Hamiltonian numbers corresponding to the solutions of lowest weight, have been found to be

3, 5, 11, 47, 923, 409 619, 83 763 206 255

the reduced numbers due to the introduction of planar and hyperplanar solutions

3, 5, 10, 44, 905, 409 181, 83 762 797 734,

the differences are 1, 3, 18, 438, 408 521.

The ratio of these last numbers to the numbers above them constituting a rapidly decreasing series, it is obvious that the "asymptotic law" will remain good for the second as well as for the first line of numbers: so that if $\phi(i)$ expresses the minimum degree of an equation from which i terms can be abstracted without solving an equation above the ith degree, $\frac{2\phi(i+1)}{\phi(i)^2}$ will continually decrease towards and finally (when i is infinite) coincide with unity.

I have already defined the weight of a solution. According to analogy (as, for example, in the case of a given symmetric function $\Sigma a^{\alpha}.b^{\beta}.c^{\gamma}\ldots$) the degree of the equation of highest degree in a solution may be termed its *order*.

* For $\vdots[2, 0] = [2, 9] = :[9] = 12$.

Thus then the two first series of numbers which have been given express the first of them the literant of the solution of lowest *weight*, the second the literant of the solution of lowest *order*. The numbers in the first series up to 923 and in the second series up to 10 appear in Hamilton's Report, all the others are here presented (it is believed) for the first time.

A solution is of course to be understood to mean a *non-simultaneous* but *not independent* system of equations from which a solution of a given system of equations may be derived. The equations in the solution-system form an arborescence or a ramification of consecutive systems, meaning thereby that the solution of any one of them depends upon a successive process of substitution of values of variables deduced from equations which precede it in such ramification. Some of the simpler of these arborescences I propose to *delineate* graphically in a subsequent communication.

Invited to participate in the centenary number of the leading Mathematical Journal in the world, it occurred to me that compatibly with my feeble means no more suitable contribution could be made than one which at the same time celebrates the centenary of the discovery due to the long and persistently ignored author of the method which it is the object of this memoir to elucidate and extend. I offer it (an aloe-flower of 100 years' growth) as a tardy Bessarabian "satisfaction to the Manes of" Bring.

50.

SUR UNE DÉCOUVERTE DE M. JAMES HAMMOND RELATIVE À UNE CERTAINE SÉRIE DE NOMBRES QUI FIGURENT DANS LA THÉORIE DE LA TRANSFORMATION TSCHIRNHAUSEN.

[*Comptes Rendus*, CIV. (1887), pp. 1228—1231.]

ON peut se proposer le problème suivant:

Étant donné un quantic, le faire disparaître en exprimant chaque variable comme une fonction linéaire et homogène de deux variables.

Si le nombre des variables dans le quantic est suffisamment grand, quel que soit son degré n, ce problème peut s'effectuer au moyen d'un système auxiliaire d'équations, tel que pour résoudre le système on n'aura jamais occasion de résoudre une équation d'un degré supérieur à n.

En nommant N le nombre minimum des variables nécessaire pour que cela soit possible, cette question se présente: *trouver la valeur de N pour une valeur donnée de n.*

Par exemple, pour $n = 2$, on voit bien que N est 4.

Pour $n = 3$, on peut démontrer que N est 6; pour $n = 4$, $N = 11$, etc.

Mais on peut imposer une condition plus rigoureuse sur le caractère du système auxiliaire d'équations qui aura l'effet d'augmenter la valeur minimum N. On peut exiger que le type du système auxiliaire d'équations sera *le plus simple possible* ou, comme je préfère le dire, sera d'*un poids minimum*. Le poids d'une équation dépend seulement de son degré i et peut être pris égal à ρ^i, où ρ est une constante indéfiniment grande. De plus, le poids d'un système d'équations peut être défini comme étant la somme des poids des équations individuelles qu'il contient.

On a ainsi un criterium exact pour déterminer lequel des deux systèmes a son poids inférieur à celui d'un autre; le terme *poids minimum* devient exempt de toute ambiguïté, et l'on comprend ce que veut dire le système d'équations le plus simple d'un nombre quelconque de tels systèmes.

Avec la première définition de N, ses valeurs successives seront

$$3,\quad 4,\quad 6,\quad 11,\quad 45,\quad 906,\quad 409182,\quad 83762797735,\quad \ldots.$$

En imposant la condition la plus rigoureuse, on obtient la série moins transcendante

$$3,\quad 4,\quad 6,\quad 12,\quad 48,\quad 924,\quad 409620,\quad 83763206256,\quad \ldots$$

que je nommerai $E_0, E_1, E_2, E_3, \ldots$.

En diminuant ces derniers chiffres de l'unité, on trouve la série de nombres

$$2,\quad 3,\quad 5,\quad 11,\quad 47,\quad 923,\quad 409619,\quad 83763206255,\quad \ldots,$$

dont les six premiers ont été calculés par Hamilton (voir *Report of 6th Meeting of British Association*, pp. 346—7, 1837).

Hamilton a, en effet, montré que le degré d'une équation algébrique, étant pris successivement égal à 2, 3, 5, 11, 47, ..., on peut, par la méthode dite *de Tschirnhausen*, la transformer dans une autre où 1, 2, 3, 4, 5, ... termes consécutifs, après le premier, manquent, sans avoir occasion de résoudre aucune équation au-dessus des degrés 1, 2, 3, 4, 5, ... respectivement.

J'ajoute que le système d'équations auxiliaires, auquel on parvient par la méthode qu'il emploie, sera *du type le plus simple possible*. Si, pour ôter i termes consécutifs, on voulait se borner à la seule condition de n'avoir pas à résoudre une équation au-dessus du degré i, alors, au lieu des nombres 2, 3, 5, 11, 47, ..., on aurait les nombres plus transcendants 2, 3, 5, 10, 44, ... C'est la série 2, 3, 5, 11, 47, ... que je nomme les *nombres de Hamilton*, et que je désigne par $H_0, H_1, H_2, H_3, H_4, \ldots$. Pour les obtenir (ou plutôt leurs différences) par la méthode de Hamilton, on a besoin de construire un triangle de chiffres (*voir* mon Mémoire dans le *Journal de Kronecker*, t. C. p. 477 [above, p. 541]).

Mon collaborateur, M. James Hammond, a trouvé un très beau théorème pour déduire les N immédiatement et successivement les uns des autres, sans introduire de nombres étrangers.

En se servant de $\beta_r(q)$ pour représenter $\dfrac{q(q-1)\ldots(q-r+1)}{1.2\ldots r}$, il a trouvé la formule vraiment remarquable

$$H_i = 2 + \beta_2(H_{i-1}) - \beta_3(H_{i-2}) + \beta_4(H_{i-3}) - \ldots.$$

A ce théorème, j'ajoute comme corollaire une formule qui se rapporte à la série de nombres E (qui ne sont autre chose que les nombres H, augmentés chacun de l'unité), qui est bonne pour toutes les valeurs de r supérieures à l'unité,

$$\beta_0(E_r) - \beta_1(E_{r-1}) + \beta_2(E_{r-2}) - \ldots + (-)^r \beta_r(E_0) = 0,$$

c'est-à-dire $$E_{r-1} = 1 + \beta_2(E_{r-2}) - \beta_3(E_{r-3}) + \ldots + (-)^r \beta_r(E_0).$$

Par exemple,

$$1-\frac{4}{1}+\frac{3.2}{1.2}=0,$$

$$1-\frac{6}{1}+\frac{4.3}{1.2}-\frac{3.2.1}{1.2.3}=0,$$

$$1-\frac{12}{1}+\frac{6.5}{1.2}-\frac{4.3.2}{1.2.3}=0,$$

$$1-\frac{48}{1}+\frac{12.11}{1.2}-\frac{6.5.4}{1.2.3}+\frac{4.3.2.1}{1.2.3.4}=0,$$

$$1-\frac{924}{1}+\frac{48.47}{1.2}-\frac{12.11.10}{1.2.3}+\frac{6.5.4.3}{1.2.3.4}=0.$$

C'est par la méthode de fonctions génératrices que M. Hammond a réussi à établir cette échelle de relation entre les nombres de Hamilton, lequel évidemment n'avait pas le moindre soupçon de l'existence d'une échelle pareille.

Si l'on prend les différences des nombres de Hamilton, on obtient la série 1, 2, 6, 36, 876, ..., qu'on peut nommer $h_1, h_2, h_3, h_4, h_5, \ldots$. On savait déjà par démonstration que $h_{i+1} \div h_i^2$ est plus grand que $\frac{1}{2}$ pour toute valeur finie de i et avec certitude morale que ce rapport devient $\frac{1}{2}$ quand i est infini. Avec la formule de M. Hammond, on peut donner une démonstration rigoureuse de ce dernier fait et en même temps établir ce nouveau théorème: *$H_{i+1} \div H_i^2$ est plus petit que $\frac{1}{2}$ pour toute valeur de i finie et plus grande que l'unité, et égal à $\frac{1}{2}$ quand i est infini.*

51.

ON HAMILTON'S NUMBERS.

By J. J. Sylvester and James Hammond.

[*Philosophical Transactions of the Royal Society of London*, CLXXVIII. (1887), pp. 285—312; CLXXIX. (1888), pp. 65—71.]

Introduction.

In the year 1786 Erland Samuel Bring, Professor at the University of Lund in Sweden, showed how by an extension of the method of Tschirnhausen it was possible to deprive the general algebraical equation of the 5th degree of three of its terms without solving an equation higher than the 3rd degree. By a well-understood, however singular, academical fiction, this discovery was ascribed by him to one of his own pupils, a certain Sven Gustaf Sommelius, and embodied in a thesis humbly submitted to himself for approval by that pupil, as a preliminary to his obtaining his degree of Doctor of Philosophy in the University*. The process for effecting this reduction seems to have been overlooked or forgotten, and was subsequently rediscovered many years later by Mr Jerrard. In a memoir contained in the *Report of the British Association*, for 1836, Sir William Hamilton showed that Mr Jerrard was mistaken in supposing that the method was adequate to taking away more than three terms of the equation of the 5th degree, but supplemented this somewhat unnecessary refutation of a result known *à priori* to be impossible, by an extremely valuable discussion of a question raised by Mr Jerrard as to the number of variables required in order that any system of equations of given degrees in those variables shall

* Bring's "Reduction of the Quintic Equation" was republished by the Rev. Robert Harley, F.R.S., in the *Quarterly Journal of Pure and Applied Mathematics*, vol. VI. 1864, p. 45. The full title of the Lund Thesis, as given by Mr Harley (see *Quart. Journ. of Math.*, pp. 44, 45) is as follows: "B. cum D. Meletemata quaedam mathematica circa transformationem aequationum algebraicarum, quae consent. Ampliss. Facult. Philos. in Regia Academia Carolina Praeside D. Erland Sam. Bring, Hist. Profess. Reg. & Ord. publico Eruditorum Examini modeste subjicit Sven Gustaf Sommelius, Stipendiarius Regius & Palmcrentzianus Lundensis. Die XIV Decemb., MDCCLXXXVI, L.H.Q.S.—Lundae, typis Berlingianis."

admit of being satisfied without solving any equation of a degree higher than the highest of the given degrees.

In the year 1886 the senior author of this memoir showed in a paper* in Kronecker's (better known as *Crelle's*) *Journal* that the trinomial equation of the 5th degree, upon which by Bring's method the general equation of that degree can be made to depend, has necessarily imaginary coefficients except in the case where four of the roots of the original equation are imaginary, and also pointed out a method of obtaining the absolute minimum degree M of an equation from which any given number of specified terms can be taken away subject to the condition of not having to solve any equation of a degree higher than M†. The numbers furnished by Hamilton's method, it is to be observed, are not minima unless a more stringent condition than this is substituted, namely, that the system of equations which have to be resolved in order to take away the proposed terms shall be the simplest possible, that is, of the lowest possible weight and not merely of the lowest order; in the memoir in *Crelle*, above referred to, the author has explained in what sense the words weight and order are here employed. He has given the name of Hamilton's Numbers to these relative minima (minima, that is, in regard to weight) for the case where the terms to be taken away from the equation occupy consecutive places in it, beginning with the second.

Mr James Hammond has quite recently discovered by the method of generating functions a very simple formula of reduction, or scale of relation, whereby any one of these numbers may be expressed in terms of those that precede it: his investigation will be found in the second section of this paper, and constitutes its most valuable portion. The principal results obtained by its senior author, consequential in great measure to Mr Hammond's remarkable and unexpected discovery, refer to the proof of a theorem left undemonstrated in the memoir in *Crelle* above referred to, and the establishment of certain other asymptotic laws to which Hamilton's Numbers and their differences are subject, by a mixed kind of reasoning, in the main apodictic, but in part also founded on observation‡. It thus

[* Above, p. 531.]

† For instance, an equation of not lower than the 905th degree may be transformed into another of that degree, in which the 2nd, 3rd, 4th, 5th, 6th, 7th, terms are all wanting, by means of the successive solution of a ramificatory system of equations, of no one of which the degree exceeds 6, whereas by the Jerrard-Hamiltonian method this transformation could not be effected for the general equation of degree lower than the 6th Hamiltonian Number, namely, 923. So for the analogous removal of 5 consecutive terms the inferior limit of degree of the equation to be transformed would be 47 by the one method, but 44 (the lowest possible) by the other. In the case of 4 consecutive terms Hamilton could not avoid being aware that 11, the 4th number which I have named after him, might be replaced by 10, as the lowest possible inferior limit of the equation to be transformed.

‡ In the 3rd section, communicated to the Society after the 1st and 2nd had gone to press, the empirical element is entirely eliminated, and the results reduced to apodictic certainty.

became necessary to calculate out the 10th Hamiltonian Number, which contains 43 places of figures. The highest number calculated by Hamilton (the 6th) was the number 923, which comes third in order after 5 (the Bring Number), 11 and 47 being the two intervening numbers. It is to be hoped that some one will be found willing to undertake the labour (considerable, but not overwhelming) of calculating some further numbers in the scale.

The theory has been "a plant of slow growth." The Lund Thesis, of December 1786 (a matter of a couple of pages), Hamilton's Report of 1836, with the tract of Mr Jerrard therein referred to, and the memoir in *Crelle*, of December 1886, constitute, as far as we are aware, the complete bibliography of the subject up to the present date.

§ 1. *On the Asymptotic Laws of the Numbers of Hamilton and their Differences.*

Consider the following Table:—

1	0	0	0	0	0	0	0 ......
	1	1	1	1	1	1	1 ......
		2	3	4	5	6	7 ......
			6	15	29	49	76 ......
				36	210	804	2449 ......
					876	24570	401134 ......
						408696	246382080 ......
							83762796636 ......
							......
							......

Any line of figures, say $p, q, r, s, t, \ldots \theta$, in the Table being given, to form the subsequent line $q_1, r_1, s_1, t_1, \ldots \theta_1$, we write

$$q_1 = \frac{p(p+1)}{1.2} + q,$$

$$r_1 = \frac{p(p+1)(2p+1)}{1.2.3} + pq + r,$$

$$s_1 = \frac{p(p+1)(p+2)(3p+1)}{1.2.3.4} + \frac{p(p+1)}{1.2} q + pr + s,$$

$$t_1 = \frac{p(p+1)(p+2)(p+3)(4p+1)}{1.2.3.4.5} + \frac{p(p+1)(p+2)}{1.2.3} q + \frac{p(p+1)}{1.2} r + ps + t,$$

....

$$\theta_1 = \frac{p(p+1)\ldots(p+i-1)(ip+1)}{1.2.3\ldots(i+1)} + \frac{p(p+1)\ldots(p+i-2)}{1.2.3\ldots(i-1)} q + \ldots + \theta.$$

If we call the nth term of the mth line $[m, n]$, the general law of deduction may be expressed by the formula

$$[m+1, n] = -B_{n+1}([m, 1]-1) + \sum_{i=0}^{i=n} [m, n+1-i]\, B_i [m, 1],$$

where $B_i k$ means the coefficient of z^i in $(1-z)^{-k}$.

The negative term $-B_{n+1}([m, 1]-1)$, it may be noticed, arises from decomposing the first term of $[m+1, n]$, as given by the original formulæ, into two parts, of which it is one. Thus, for example,

$$\frac{p(p+1)(p+2)(p+3)(4p+1)}{1.2.3.4.5}$$

is changed into

$$-\frac{(p-1)p(p+1)(p+2)(p+3)}{1.2.3.4.5} + \frac{p(p+1)(p+2)(p+3)}{1.2.3.4} p.$$

The numbers in the hypothenuse of this infinite triangle, namely,

1, 1, 2, 6, 36, 876, 408696, 83762796636, 3508125906207095591916,

6153473687096578758445014683368786661634996, ... ,

are what I call the Hamiltonian Differences, or Hypothenusal Numbers*; and their continued sums augmented by unity, namely,

2, 3, 5, 11, 47, 923, 409619, 83763206255, 3508125906290858798171,

6153473687096578758448522809275077520433167, ...,

are what I call the Hamiltonian Numbers. The two latter of these have been calculated by means of Mr Hammond's formula, presently to be mentioned, and the corresponding Hypothenusal Numbers deduced from them by simple subtraction. Their connection with the theory of the Tschirnhausen Transformation will be found fully explained in my memoir on the subject in Vol. C. of *Crelle*. My present object is to speak of the numbers as they stand, without reference to their origin or application†.

* The other numbers of the "triangle," whose properties it may be some day desirable to investigate, may be termed co-hypothenusal numbers of *order* measured by their horizontal distance from the hypothenuse—their vertical distance below the top line denoting their *rank*. In the sequel the development is given of the half of a hypothenusal number (of the first order) in a descending series of powers (with fractional indices) of the half of its antecedent, the coefficients in the principal part of such series being (not, as might have been the case, functions of the rank, but) absolute constants. These may be termed the hypothenusal constants. The values of the first four of them are shown to be 1, $\frac{4}{3}$, $\frac{11}{18}$, $\frac{16}{81}$.

† The reader will be disappointed who seeks in Hamilton's Report any systematic deduction of the numbers which I have called after his name. He treats therein the more general question of finding the number of letters sufficient for satisfying any system of equations of given degrees by means of a certain prescribed uniform process whereby the necessity is obviated of solving any equation of a higher degree than the highest one of the given equations, and among, and mixed up with, other examples considers systems of equations of degrees 1, 2, 3; 1, 2, 3, 4; 1, 2, 3, 4, 5; 1, 2, 3, 4, 5, 6; for which the minimum numbers of letters required to make such process possible (when the equations are homogeneous) are 5, 11, 47, 923, respectively. Accordingly he has no occasion to employ the infinitely developable Triangle which gives unity and cohesion to the problem which deals with an indefinite number of equations of all consecutive degrees from 1 upwards. This triangle, which plays an important part in the systematic treatment of the problem, first appears in my memoir on the subject in the 100th volume of *Crelle*.

The question arises as to whether it is possible to deduce the Hamiltonian Differences, or to deduce the Hamiltonian Numbers, directly in a continued chain from one another without the use of any intermediate numbers. Mr James Hammond has shown that it is possible, and has made the remarkable discovery that it is the Numbers of Hamilton, and not the Hypothenusal Numbers, which are subject to a very simple scale of relation. These being found, of course the Differences become known. This is contrary to what one would have expected. *À priori*, one would have anticipated that the determination of the Hypothenusal Numbers would have preceded that of their sums.

I leave Mr Hammond to give his own account of his mode of obtaining the wonderful formula of reduction, which, by a slight modification, I find, may be expressed as follows:—Using E_i to denote the $(i+1)$th Hamiltonian Number augmented by unity, so that $E_0=3$, $E_1=4$, $E_2=6$, $E_3=12$, $E_4=48, \ldots$;

It is proper also again to notice that what I call the Numbers of Hamilton (at all events those subsequent to the number 5) are not the smallest numbers requisite for fulfilling the condition above specified. Smaller numbers will serve to satisfy that condition taken alone; but when such smaller numbers are substituted for Hamilton's the resolving equations will be less simple, inasmuch as they will contain a greater number of equations of the higher degrees than when the larger Hamiltonian numbers are employed. This distinction will be found fully explained in the memoir cited, and the smallest numbers substitutable for Hamilton's are there actually determined for r equations of degrees extending from 1 to r for all values of r up to 8 inclusive.

I have added nothing (for there is nothing to be added) to the fundamental formula of Hamilton expressed by the equation

$$[\lambda, \mu, \nu, \ldots \pi] = 1 + [\lambda - 1, \lambda + \mu, \lambda + \mu + \nu, \ldots, \lambda + \mu + \nu + \ldots + \pi],$$

where, supposing the letters $\lambda, \mu, \nu, \ldots \pi$, to be i in number, $[\lambda, \mu, \nu, \ldots \pi]$ means the number of letters required in order that it may be possible to satisfy, according to the process employed by Hamilton (in conformity with a certain stipulation of Jerrard), a system of λ equations of degree i, μ equations of degree $i-1$, ν equations of degree $i-2$, ..., π equations of the degree 1, without solving any single equation of a degree higher than i. This formula, applied λ times successively, will have the effect of abolishing λ and causing $[\lambda, \mu, \nu, \ldots \pi]$ to depend on $[\mu', \nu', \ldots \pi']$, where $\mu', \nu', \ldots \pi'$ are connected with $\lambda, \mu, \nu, \ldots \pi$ by means of the formulæ given at the commencement of the present paper, but where instead of the letters $\lambda, \mu, \nu, \ldots$ I have used the letters $p, q, r, \ldots$.

It is presumable that the *reduced* Hamiltonian numbers would be found much less amenable to algebraical treatment than the Hamiltonian numbers proper; for numerical equalities and inequalities have to be taken account of, in determining them, which have no place in the determination of the latter numbers. Hamilton, as already stated, expressly alludes to the reduction of 11 to 10, but with that exception has avoided the general question of finding the *absolutely* lowest number of letters required in order that a system of equations (expressed in terms of those letters) of given degrees may admit of being satisfied without the necessity arising to solve any equation of a higher degree than the highest of the given ones.

and $\beta_i m$ to signify the coefficient of t^i in $(1+t)^m$; then, for any value of i greater than unity,

$$\beta_0 E_i - \beta_1 E_{i-1} + \beta_2 E_{i-2} - \beta_3 E_{i-3} + \ldots + (-)^i \beta_i E_0 = 0.$$

Or in other words, writing $\beta_0 E_i = 1$, $\beta_1 E_{i-1} = E_{i-1}$, and replacing $i-1$ by i,

$$E_i = 1 + \beta_2 E_{i-1} - \beta_3 E_{i-2} + \ldots + (-)^{i+1} \beta_{i+1} E_0$$

for all values of i greater than zero.

This is eminently a practical formula, as all the numerical calculations made use of to obtain any E are available for finding the E which follows. Dispensing with the symbol β, we may deduce all the values of E successively from those that go before by means of the *equivalence*

$$S = (1-t)^{E_0} + t(1-t)^{E_1} + t^2(1-t)^{E_2} + \ldots \equiv 1 - 2t,$$

which, by equating the powers of t on the two sides of the equivalence, gives

$$E_0 = 3,$$

$$E_1 = 1 + \frac{3 \,.\, 2}{1 \,.\, 2} = 4,$$

$$E_2 = 1 + \frac{4 \,.\, 3}{1 \,.\, 2} - \frac{3 \,.\, 2 \,.\, 1}{1 \,.\, 2 \,.\, 3} = 6,$$

$$E_3 = 1 + \frac{6 \,.\, 5}{1 \,.\, 2} - \frac{4 \,.\, 3 \,.\, 2}{1 \,.\, 2 \,.\, 3} + \frac{3 \,.\, 2 \,.\, 1 \,.\, 0}{1 \,.\, 2 \,.\, 3 \,.\, 4} = 12,$$

and so on.

I use the term equivalence and its symbol in order to convey the necessary caution that the relation indicated is not one of quantitative equality; for, although the series on the left-hand side of the symbol converges for all positive values of t less than 2, it is never equal to the expression on the right-hand side except when $t = 0$. Thus, for example, when t is unity the two terms of the equivalence are 0 and -1, and when $t = \frac{1}{2}$ they are

$$2^{-E_0} + 2^{-E_1-1} + 2^{-E_2-2} + \ldots \text{ and } 0, \text{ respectively;}$$

and for all values of t within the limits of convergence the value of the left-hand side is in excess of the value of the right-hand side of the equivalence by a finite quantity which decreases continuously as t decreases from 2 to 0, and which vanishes when $t = 0$*.

In a word, the *generating equation* is not an equation in the usual sense of the term. Conceiving each term of the series S to be expanded in ascending powers of t, and like powers of t to be placed in columns under and above

* Of the truth of the statement that the excess never changes sign, and continually decreases, I have scarcely a doubt, but it requires proof. Mr Hammond remarks that

$$(1-t)^{E_0} + t(1-t)^{E_1} + t^2(1-t)^{E_2} + \ldots + t^n(1-t)^{E_n} = (1-2t) + t^2(1-t)^{E_n - 2} F_n(t) - t^{n+1}(1-t)^{E_n - 1},$$

where $F_n(t)$ is positive for all positive values of t. Probably a proof of the point in question might be deduced from this expression, but I have not thought it necessary to investigate the matter.

each other, the double sum may be taken as a vertical sum of line-sums or as a horizontal sum of column-sums, and, although for licit values of t each sum has a finite value, the two finite values are not identical, just as a double definite integral may undergo a change of value when the order of its integrations is reversed *.

I have noticed [see above, p. 542] that the value of any Hamiltonian Difference divided by the square of the preceding one was always greater than $\frac{1}{2}$, and stated as morally certain, but "awaiting exact proof," that this ratio ultimately becomes $\frac{1}{2}$. By aid of Mr Hammond's formula for the numbers, I shall now be able to supply this proof, and at the same time to show that the ratio of a Hamiltonian *Number* to the square of its antecedent (which, of course, converges to the same asymptotic value $\frac{1}{2}$) is always *less* than that limit†.

We must in the first place prove that in the series

$$\beta_2 E_{i-1} - \beta_3 E_{i-2} + \beta_4 E_{i-3} - \beta_5 E_{i-4} + \ldots$$

the absolute value of each term is greater than that of the one which follows it.

In proving this, I shall avail myself of the property of the Hypothenusal Numbers disclosed in the process of forming the triangle given at the outset of the memoir, namely, that $E_i - E_{i-1}$ is greater than $(E_{i-1} - E_{i-2})^2/2$.

Let us suppose that the law to be established holds good for a certain value of i. For the sake of brevity, I denote $E_i, E_{i-1}, E_{i-2}, E_{i-3}, \ldots$ by $N, P, Q, R, \ldots$.

We have then

$$P - 1 = \frac{Q(Q-1)}{2} - \frac{R(R-1)(R-2)}{2 \,.\, 3} + \frac{S(S-1)(S-2)(S-3)}{2 \,.\, 3 \,.\, 4} - \ldots$$

$$N - 1 = \frac{P(P-1)}{2} - \frac{Q(Q-1)(Q-2)}{2 \,.\, 3} + \frac{R(R-1)(R-2)(R-3)}{2 \,.\, 3 \,.\, 4} - \frac{S(S-1)(S-2)(S-3)(S-4)}{2 \,.\, 3 \,.\, 4 \,.\, 5} + \ldots.$$

* Professor Cayley has brought under my notice a not altogether dissimilar, but perhaps less striking, phenomenon, pointed out by Cauchy, that, although the series

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ldots$$

is convergent, its square

$$u_0^2 + (2u_0u_1) + (2u_0u_2 + u_1^2) + \ldots,$$

that is,

$$1 - \sqrt{2} + \left(\frac{2}{\sqrt{3}} + \frac{1}{2}\right) - \ldots$$

is divergent.

† The fortunate circumstance of the two ratios in question being always respectively less and greater than the common asymptotic value of each of them enables us to find the value of the constant in the expression c^{2^x}, which is asymptotically equivalent to the half of the xth Hamiltonian or Hypothenusal Number by a method exactly analogous to that of exhaustions for finding the Archimedian constant correct to any required number of decimal places. See end of this section [p. 566, below].

If, then, the law to be proved is true for all the consecutive terms of the upper series it will obviously be true for the second series, *abstraction being made of its first term*, provided that no antecedent is less than its consequent in the series

$$\frac{Q-2}{3},\ \frac{R-3}{4},\ \frac{S-4}{5},\ \ldots,$$

which is true *à fortiori* if

$$\frac{Q}{3},\ \frac{R}{4},\ \frac{S}{5},\ \ldots$$

continually decrease, as is obviously the case, inasmuch as

$$Q,\quad R,\quad S,\ \ldots$$

form a descending series.

In order, then, to establish the necessary chain of induction, it only remains to show that

$$3P(P-1)-Q(Q-1)(Q-2)$$

is positive.

Now $\quad (P-Q)-\dfrac{(Q-R)^2}{2}$, and *à fortiori* $P-\dfrac{(Q-R)^2}{2}$,

is positive for a reason previously given.

And, if in the series 3, 4, 6, 12, 48, 924, ... we make exclusion of the first three terms, we have always

$$R = \text{ or } < \frac{Q}{4},$$

and consequently $\quad P > \dfrac{9Q^2}{32}$*.

And, since under the same condition $(P-1)/(Q-1) > 4$,

$3P(P-1)-Q^2(Q-1)$, and *à fortiori* $3P(P-1)-Q(Q-1)(Q-2)$,

is positive if $12P-Q^2$ is positive, which is the case, since $P > 9Q^2/32$.

Hence, since the theorem to be proved is true for the several series

$$(1)\quad \frac{4\,.\,3}{1\,.\,2}-\frac{3\,.\,2\,.\,1}{1\,.\,2\,.\,3},$$

$$(2)\quad \frac{6\,.\,5}{1\,.\,2}-\frac{4\,.\,3\,.\,2}{1\,.\,2\,.\,3},$$

$$(3)\quad \frac{12\,.\,11}{1\,.\,2}-\frac{6\,.\,5\,.\,4}{1\,.\,2\,.\,3}+\frac{4\,.\,3\,.\,2\,.\,1}{1\,.\,2\,.\,3\,.\,4},$$

$$(4)\quad \frac{48\,.\,47}{1\,.\,2}-\frac{12\,.\,11\,.\,10}{1\,.\,2\,.\,3}+\frac{6\,.\,5\,.\,4\,.\,3}{1\,.\,2\,.\,3\,.\,4},$$

* The proof that the ratio of each term of the series 4, 6, 12, 48, 924, ... to its antecedent continually increases is too easy and too tedious to be worth setting forth in the text.

it will be true universally; for in all the succeeding series the term we have called R will be higher than the term 6 in the scale 3, 4, 6, 12, 48, ...

Hence $\quad P-1 = \text{ or } < \tfrac{1}{2}(Q^2-Q)$.

For the initial values of Q, P, (namely, 3, 4)

$$P-1=\tfrac{1}{2}(Q^2-Q).$$

(When P represents any term beyond the first it is very easy to prove, but too tedious to set out the proof, that the sum of all the terms after the first in the series equated to $P-1$ will be less than -2; so that, except in the case stated, $P<\tfrac{1}{2}(Q^2-Q)$.)

For the series 12, 48, 924, ... we have seen that $P > 9Q^2/32$.

Hence, for the series 48, 924, ...,

$$Q > \frac{9R^2}{32} \text{ or } R < \sqrt{\frac{32Q}{9}}.$$

But $\quad P > \dfrac{Q^2-Q}{2}-\dfrac{R(R-1)(R-2)}{6}$

$$> \frac{Q^2-Q}{2}-\frac{R^3}{6}.$$

Hence $\quad P > \dfrac{Q^2-Q}{2}-\dfrac{64\sqrt{2}}{81}Q^{\frac{3}{2}}$, and $P < \dfrac{Q^2-Q}{2}$.

Hence, when P, Q, are at an infinite distance from the origin,

$$\frac{P}{Q^2}=\tfrac{1}{2}.$$

Hence, also, $\quad \dfrac{P-Q}{(Q-R)^2}$ ultimately $=\dfrac{P}{Q^2}=\tfrac{1}{2}$,

which proves the theorem left over for "exact proof" in the memoir referred to.

It is convenient to deal with the halves of the *sharpened** Numbers of Hamilton, which may be called the reduced Hamiltonian Numbers, and denoted by h with a subscript, or, when required, by p, q, r, ... (the halves of P, Q, R, ... respectively).

We have then

$$2p < \frac{4q^2-2q}{2},$$

or

$$p < q^2-\frac{q}{2},$$

$$p > q^2-\frac{q}{2}-\tfrac{128}{81}q^{\frac{3}{2}}.$$

* Numbers increased by unity may conveniently be denominated sharpened numbers, and numbers diminished by unity flattened numbers.

We may find a closer superior limit to p in terms of q as follows—

$$P-1 = \text{or} < \frac{Q^2-Q}{2} - \frac{R(R-1)(R-2)}{6} + \frac{S(S-1)(S-2)(S-3)}{24},$$

in which inequality it may be shown by inspection up to a certain point, and after that by demonstration, the tedium of writing out or reading which I spare my readers and myself, that P may be substituted for its flattened value $P-1$.

We have then $$P < \frac{Q^2-Q}{2} - \frac{R^3-3R^2}{6} + \frac{S^4}{24}.$$

Let us suppose that S, R, are not lower in the scale of the E's than 12, 48, respectively; so that P is not lower than E_6, which is 409620.

Then, as we have previously shown,

$$Q^2 < \tfrac{32}{9}P, \quad R^2 < \tfrac{32}{9}Q, \quad S^2 < \tfrac{32}{9}R.$$

Moreover, we have

$$P < \tfrac{1}{2}(Q^2-Q), \text{ whence it follows that } Q^2 > 2P+Q,$$

and, *à fortiori*, $$Q^2 > 2P.$$

Similarly $$R^2 > 2Q,$$

and $$S^2 > 2R.$$

Now $$P < \frac{Q^2-Q}{2} - \frac{R^3}{6} + \frac{R^2}{2} + \frac{S^4}{24}$$

$$< \frac{Q^2-Q}{2} - \tfrac{1}{6}(2Q)^{\frac{3}{2}} + \tfrac{1}{2}(\tfrac{32}{9}Q) + \tfrac{1}{24}(\tfrac{32}{9}R)^2$$

$$< \frac{Q^2}{2} - \frac{\sqrt{2}}{3}Q^{\frac{3}{2}} + \tfrac{23}{18}Q + \tfrac{1}{24}(\tfrac{32}{9})^3 Q,$$

that is, $$P < \tfrac{1}{2}Q^2 - \frac{\sqrt{2}}{3}Q^{\frac{3}{2}} + \tfrac{13781}{4374}Q.$$

This result, expressed in terms of the reduced numbers p, q, takes the form

$$p < q^2 - \tfrac{2}{3}q^{\frac{3}{2}} + \tfrac{13781}{4374}q,$$

and we have previously shown that

$$p > q^2 - \tfrac{128}{81}q^{\frac{3}{2}} - \frac{q}{2},$$

at all events when P is not lower in the scale than E_6.

The fraction $\frac{128}{81}$ arises from our having substituted for R^3 the inferior value $(\frac{32}{9}Q)^{\frac{3}{2}}$; but, the higher we advance P in the scale, the nearer R^2 approaches to $2Q$, and is ultimately in a ratio of equality with it. But, if we had written $(2Q)^{\frac{3}{2}}$ for R^3, the coefficient, which now stands at $-\frac{128}{81}$, would

have been $-\frac{2}{3}$. In like manner, as P and Q are travelled on in the scale, R^2 and S^4 become indefinitely near to $2Q$ and $(2R)^2$, that is, $8Q$, so that the coefficient of Q in the superior limit approximates indefinitely near to

$$-\tfrac{1}{2} + 1 + \tfrac{1}{3}, \text{ that is, } \tfrac{5}{6},$$

and the two limits of p which have been obtained become

$$q^2 - \tfrac{2}{3}q^{\frac{3}{2}} + (\tfrac{5}{6}+\epsilon)q,$$
$$q^2 - (\tfrac{2}{3}+\eta)q^{\frac{3}{2}} - \tfrac{1}{2}q,$$

where ultimately ϵ and η are infinitesimals*.

Hence it follows that the ultimate value of

$$(p-q^2) \div q^{\frac{3}{2}} \text{ is } -\tfrac{2}{3},$$

that is, $$\frac{2E_i - E^2_{i-1}}{E^{\frac{3}{2}}_{i-1}} = -\sqrt{\tfrac{8}{9}} \text{ when } i=\infty.$$

Let $\lambda, \mu, \nu, \ldots$ represent the halves of the Hypothenusal Numbers in the triangle given at the commencement of the paper, that is, the differences of the numbers which we have called $p, q, r, \ldots$.

Since $$p = q^2 - \tfrac{2}{3}q^{\frac{3}{2}} \text{ and } q = r^2 - \tfrac{2}{3}r^{\frac{3}{2}},$$

$$p-q = q^2 - \tfrac{2}{3}q^{\frac{3}{2}} - q, \text{ and } q-r = r^2 - \tfrac{2}{3}r^{\frac{3}{2}} - r.$$

Obviously, therefore, as a first approximation when λ, μ, are very advanced terms in the hypothenuse,

$$\lambda = \mu^2.$$

Let us write $$\lambda = \mu^2 + \kappa\mu^a$$

for a second approximation.

Then $$q^2 - \tfrac{2}{3}q^{\frac{3}{2}} - q = (r^2 - \tfrac{2}{3}r^{\frac{3}{2}} - r)^2 + \kappa(r^2 - \tfrac{2}{3}r^{\frac{3}{2}} - r)^a,$$

or, *neglecting* terms of lower dimensions than r^3,

$$(r^2 - \tfrac{2}{3}r^{\frac{3}{2}})^2 - \tfrac{2}{3}r^3\left(1 - \frac{1}{r^{\frac{1}{2}}} + \frac{1}{6r} - \cdots\right) = (r^2 - \tfrac{2}{3}r^{\frac{3}{2}} - r)^2 + \kappa r^{2a}.$$

Therefore $$-\tfrac{2}{3}r^3 = -2r^3 + \kappa r^{2a}.$$

Consequently $$a = \tfrac{3}{2} \text{ and } \kappa = \tfrac{4}{3}.$$

Thus, then, for the consecutive Hypothenusal Numbers λ, μ,

$$\lambda = \mu^2 + \tfrac{4}{3}\mu^{\frac{3}{2}} + \ldots.$$

Let $$\lambda = \mu^2 + \tfrac{4}{3}\mu^{\frac{3}{2}} + \theta\mu,$$

or say $$\eta_{x+1} = \eta_x^2 + \tfrac{4}{3}\eta_x^{\frac{3}{2}} + \rho_x\eta_x,$$

where η_x is the xth term in the series $\frac{1}{2}$, 1, 3, 18, ...

* As a matter of fact, it will be found that, as soon as q and p attain the values 6, 24, $q^2 - \frac{2}{3}q^{\frac{3}{2}}$ may be taken as a superior limit. It may be noticed also, to prevent a wrong inference being drawn from the above expressions, that, as will hereafter appear, η is an infinitesimal of the order $1/q^{\frac{1}{2}}$, when q is infinite.

The successive values of ρ_x and their differences are given in the annexed Table.

x	η_x	ρ_x	$\Delta\rho_x$
1	·5	·55719096	
2	1	·66666666	+ ·10947570
3	3	·69059893	+ ·02393227
4	18	·67647909	− ·01411984
5	438	·64334761	− ·03313148
6	204348	·61769722	− ·02565039
7	41881398318	·61139243	− ·00630479
8	1754062953103547795958	·61111171	− ·00028072

The decimal figures following those given in ρ_8, required for ulterior purposes, being 5795.

An examination of the column of differences for $x = 5, 6, 7, 8$, shows that the ratios of each to the rest go on decreasing somewhat faster than their squares: this makes it almost certain that $\rho_8 - \rho_9$ will be between the 400th and 500th part of ·000280, and that accordingly the value of ρ_9 will be ·6111111, &c. I believe it is beyond all moral doubt that the ultimate value of ρ is exactly $\frac{11}{18}$; and, indeed, it was the conviction I entertained of this being its true value, when I had calculated ρ_7, that led me to undertake the very considerable labour of ascertaining the 10th Hamiltonian Number in order to deduce from it the value of ρ_8. This being taken for granted*, we may proceed to ascertain a further term in the asymptotic value of η_{x+1} expressed as a function of η_x.

For, calling $\quad \rho_x - \frac{11}{18} = \delta_x$ and $\sqrt[4]{\eta_x} = q_x$,

we have

$$\delta_6 = \cdot 00658611,$$
$$\delta_7 = \cdot 00028132,$$
$$\delta_8 = \cdot 0000006047,$$

$$\left.\begin{aligned} q_6 &= 21, \\ q_7 &= 452, \\ q_8 &= 204649, \end{aligned}\right\} \text{neglecting decimals.}$$

Thus

$$(\delta q)_6 = \cdot 1383,$$
$$(\delta q)_7 = \cdot 1272,$$
$$(\delta q)_8 = \cdot 12375.$$

The value of $\quad (\delta q)_6 - (\delta q)_7$ being ·0111,

and of $\quad (\delta q)_7 - (\delta q)_8$ „ ·0035,

* It is reduced to *certainty* in the supplemental 3rd section.

we may feel tolerably certain, from the Law of Squares, that $(\delta q)_8 - (\delta q)_9$ will be somewhere in the neighbourhood of the tenth part of ·0035, and accordingly that $(\delta q)_9$ is about ·1234, so that the probable value of $(\delta q)_\infty$ is ·1234 ....

Thus we have found

$$\eta_{x+1} = \eta_x^2 + \tfrac{4}{3}\eta_x^{\frac{3}{2}} + \tfrac{11}{18}\eta_x + [\quad]\,\eta_x^{\frac{1}{2}} + \ldots,$$

the only moral doubt being as to the degree of closeness of propinquity of the coefficient of $\eta_x^{\frac{1}{2}}$ to the decimal ·1234 ...*.

For the benefit of those who may wish to carry on the work, I give the following numerical results which have been employed in the preceding arithmetical determinations:—

$$\frac{E_8(E_8-1)}{1\,.\,2} = 61534736871945297028957640011158846858717 06$$

$$\frac{E_7(E_7-1)(E_7-2)}{1\,.\,2\,.\,3} = 9795094444841421613760720063752 0$$

$$\frac{E_6(E_6-1)(E_6-2)(E_6-3)}{1\,.\,2\,.\,3\,.\,4} = 117302430235229583844 5$$

$$\frac{E_5(E_5-1)(E_5-2)(E_5-3)(E_5-4)}{1\,.\,2\,.\,3\,.\,4\,.\,5} = 5552272910184$$

$$\frac{E_4(E_4-1)(E_4-2)(E_4-3)(E_4-4)(E_4-5)}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6} = 12271512$$

$$\frac{E_3(E_3-1)(E_3-2)(E_3-3)(E_3-4)(E_3-5)(E_3-6)}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7} = 792$$

$$\eta_5 \div \eta_4 = 24{\cdot}33333333 \ldots$$
$$\eta_6 \div \eta_5 = 466{\cdot}54794520 \ldots$$
$$\eta_7 \div \eta_6 = 204951{\cdot}34925714 \ldots$$
$$\eta_8 \div \eta_7 = 41881671184{\cdot}54776412 \ldots$$
$$\eta_9 \div \eta_8 = 1754062953159389842293{\cdot}346657805 \ldots$$

$$\sqrt{\eta_4} = 4{\cdot}24264068 \ldots$$
$$\sqrt{\eta_5} = 20{\cdot}92844819 \ldots$$
$$\sqrt{\eta_6} = 452{\cdot}04866994 \ldots$$
$$\sqrt{\eta_7} = 204649{\cdot}45227877 \ldots$$
$$\sqrt{\eta_8} = 41881534751{\cdot}051659567667 \ldots$$

Finally, it is interesting to find the asymptotic values of h_x and η_x (the halves of the sharpened Hamiltonian and of the Hypothenusal Numbers), which are ultimately in a ratio of equality to each other, in terms of x.

* The exact value of the coefficient of $\eta_x^{\frac{1}{2}}$, left blank in the text, is proved in section 3 to be $\frac{10}{81}$, that is, the recurring decimal $\cdot\dot{1}23456790\dot{0}$.

Obviously each of these is ultimately in a ratio of equality with M^{2^x}, where M is a constant to be determined.

Let $$M = 10^{2^\alpha} \text{ and } u_x = 10^{2^{x+\alpha}}.$$

Then, for finite values of x, remembering that (in the preceding notation)

$$p < q^2 \text{ and } \lambda > \mu^2,$$

u_x must be intermediate between the corresponding terms of the two series

$$\eta = \tfrac{1}{2},\ 1,\ 3,\ 18,\ 438,\ 204348,\ 41881398318,\ \ldots,$$
$$h = 2,\ 3,\ 6,\ 24,\ 462,\ 204810,\ 41881603128,\ \ldots.$$

By means of this formula, writing for u_x corresponding values of η and h, and retaining so much of the two corresponding determinations of α as is common to both, we can find α precisely to any desired number of places of decimals, as shown in the following Table, in which 18 and 24 are taken as the terms of place zero in the respective series:

$u_x = 18$,	438,	204348,	41881398318,
$\alpha = \cdot 32$,	·401,	·4088,	·4089863 …
$u_x = 24$,	463,	204810,	41881603128,
$\alpha = \cdot 46$,	·413,	·4090,	·4089866 ….

Hence, if we now change the origin, taking $\frac{1}{2}$ and 2 as the zero terms, we have approximately

$$M^{2^{x+3}} = 10^{2^{x+\alpha}}$$

and $$8 \log M = 2^{\cdot 408986},$$

which gives $$M = 1\cdot 4654433\ \ldots^*.$$

As a verification, since $2^3 = 8$, $(1\cdot 46544)^8$ should lie between 18 and 24; and, as a matter of fact, a rough calculation gives

$$(1\cdot 46544)^2 = 2\cdot 1473\ \ldots,$$
$$(2\cdot 1473)^2 = 4\cdot 608\ \ldots,$$
$$(4\cdot 608)^2 = 21\cdot 234\ \ldots,$$

which is about midway between the two limits.—J. J. S.

§ 2. *Proof of the Formula for the Successive Determination of each in turn of Hamilton's Numbers from its Antecedents.*

Let
$$\begin{aligned} 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \ldots &= F_0(x), \\ 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + \ldots &= F_1(x), \\ 6x^2 + 15x^3 + 29x^4 + 49x^5 + 76x^6 + \ldots &= F_2(x), \\ 36x^3 + 210x^4 + 804x^5 + 2449x^6 + \ldots &= F_3(x), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \end{aligned}$$

* See Note 1, p. 578 [below].

where the coefficients of the various powers of x are the numbers set out in the triangular Table at the commencement of this paper.

If, in general, we write

$$F_n(x) = a_n x^n + b_n x^{n+1} + c_n x^{n+2} + d_n x^{n+3} + \ldots,$$

the coefficients of $F_{n+1}(x)$, expressed in terms of those of $F_n(x)$, are as follows:—

$$a_{n+1} = b_n + \frac{a_n(a_n+1)}{1\,.\,2}$$

$$b_{n+1} = c_n + a_n b_n + \frac{a_n(a_n+1)(2a_n+1)}{1\,.\,2\,.\,3}$$

$$c_{n+1} = d_n + a_n c_n + \frac{a_n(a_n+1)}{1\,.\,2} b_n + \frac{a_n(a_n+1)(a_n+2)(3a_n+1)}{1\,.\,2\,.\,3\,.\,4}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Now $$(1-x)^{-a_n} = 1 + a_n x + \frac{a_n(a_n+1)}{1\,.\,2} x^2 + \frac{a_n(a_n+1)(a_n+2)}{1\,.\,2\,.\,3} x^3 + \ldots,$$

when multiplied by

$$F_n(x) = a_n x^n + b_n x^{n+1} + c_n x^{n+2} + d_n x^{n+3} + \ldots,$$

gives
$$\begin{aligned} (1-x)^{-a_n} F_n(x) = a_n x^n + b_n x^{n+1} + c_n x^{n+2} + d_n x^{n+3} + \ldots \\ + a_n^2 x^{n+1} + a_n b_n x^{n+2} + a_n c_n x^{n+3} + \ldots \\ + \frac{a_n^2(a_n+1)}{1\,.\,2} x^{n+2} + \frac{a_n(a_n+1)}{1\,.\,2} b_n x^{n+3} + \ldots \\ + \frac{a_n^2(a_n+1)(a_n+2)}{1\,.\,2\,.\,3} x^{n+3} + \ldots \\ + \ldots. \end{aligned}$$

Comparing this with

$$\begin{aligned} F_{n+1}(x) = b_n x^{n+1} + c_n x^{n+2} + d_n x^{n+3} + \ldots \\ + \frac{a_n(a_n+1)}{1\,.\,2} x^{n+1} + a_n b_n x^{n+2} + a_n c_n x^{n+3} + \ldots \\ + \frac{a_n(a_n+1)(2a_n+1)}{1\,.\,2\,.\,3} x^{n+2} + \frac{a_n(a_n+1)}{1\,.\,2} b_n x^{n+3} + \ldots \\ + \frac{a_n(a_n+1)(a_n+2)(3a_n+1)}{1\,.\,2\,.\,3\,.\,4} x^{n+3} + \ldots \\ + \ldots, \end{aligned}$$

we see that the difference of the two expressions is

$$a_n x^n + \frac{(a_n-1)a_n}{1\,.\,2} x^{n+1} + \frac{(a_n-1)a_n(a_n+1)}{1\,.\,2\,.\,3} x^{n+2} + \frac{(a_n-1)a_n(a_n+1)(a_n+2)}{1\,.\,2\,.\,3\,.\,4} x^{n+3} + \ldots,$$

which is equal to $$x^{n-1}(1-x)^{-(a_n-1)} - x^{n-1}(1-x).$$

Thus $F_{n+1}(x) = (1-x)^{-a_n} F_n(x) - x^{n-1}(1-x)^{-a_n+1} + x^{n-1}(1-x)$*.

Multiplying this equation by $(1-x)^{s_{n+1}}$, where

$$s_{n+1} = a_0 + a_1 + a_2 + \ldots + a_{n-1} + a_n,$$

we obtain

$$(1-x)^{s_{n+1}} F_{n+1}(x) = (1-x)^{s_n} F_n(x) + x^{n-1}(1-x)^{s_{n+1}+1} - x^{n-1}(1-x)^{s_n+1},$$

which gives, when we write successively $n-1$, $n-2$, $n-3$, ... 0 in the place of n,

$$(1-x)^{s_n} F_n(x) = (1-x)^{s_{n-1}} F_{n-1}(x) + x^{n-2}(1-x)^{s_n+1} - x^{n-2}(1-x)^{s_{n-1}+1};$$

$$(1-x)^{s_{n-1}} F_{n-1}(x) = (1-x)^{s_{n-2}} F_{n-2}(x) + x^{n-3}(1-x)^{s_{n-1}+1} - x^{n-3}(1-x)^{s_{n-2}+1};$$

....

$$(1-x)^{s_1} F_1(x) = (1-x)^{s_0} F_0(x) + x^{-1}(1-x)^{s_1+1} - x^{-1}(1-x)^{s_0+1}.$$

Hence, by addition of these n equations, we find

$$(1-x)^{s_n} F_n(x) = (1-x)^{s_0} F_0(x) + x^{n-2}(1-x)^{s_n+1} - x^{-1}(1-x)^{s_0+1}$$
$$+ x^{n-3}(1-x)^{s_{n-1}+2} + x^{n-4}(1-x)^{s_{n-2}+2} + \ldots + x^{-1}(1-x)^{s_1+2},$$

where it has been assumed that it is possible to assign to s_0 (previously undefined) such a value as will make the last of the above n equations, namely,

$$(1-x)^{s_1} F_1(x) = (1-x)^{s_0} F_0(x) + x^{-1}(1-x)^{s_1+1} - x^{-1}(1-x)^{s_0+1},$$

identically true. That this can be done is obvious; for, if in that equation we write for $F_1(x)$, $F_0(x)$, and s_1 their values, namely,

$$F_1(x) = (1-x)^{-2} - 1, \quad F_0(x) = (1-x)^{-1}, \text{ and } s_1 = a_0 = 1,$$

then, on making $s_0 = 0$, the equation becomes

$$(1-x)^{-1} - (1-x) = (1-x)^{-1} + x^{-1}(1-x)(1-x-1).$$

Thus the general value of $F_n(x)$ is given by the equation

$$(1-x)^{s_n} F_n(x) = (1-x)^{-1} + x^{n-2}(1-x)^{s_n+1} - x^{-1}(1-x)$$
$$+ x^{n-3}(1-x)^{s_{n-1}+2} + x^{n-4}(1-x)^{s_{n-2}+2} + \ldots + x^{-1}(1-x)^{s_1+2},$$

which is equivalent to

$$(1-x)^{s_n} F_n(x) - (1-x)^{-1} + x^{-1}(1-x) - x^{n-1}(1-x)^{s_n+1}$$
$$= x^{n-2}(1-x)^{s_n+2} + x^{n-3}(1-x)^{s_{n-1}+2} + x^{n-4}(1-x)^{s_{n-2}+2} + \ldots + x^{-1}(1-x)^{s_1+2},$$

where, a_0, a_1, a_2, a_3, ... being the Hypothenusal Numbers 1, 2, 6, 36, ... we have

$$s_1 = a_0 = 1,$$
$$s_2 = a_0 + a_1 = 3,$$
$$s_3 = a_0 + a_1 + a_2 = 9,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots;$$

that is, the successive values of $s_n + 2$ are the Hamiltonian Numbers 3, 5, 11, 47 ....

* See Note 2, p. 578 [below].

Now $F_n(x) = a_n x^n + \ldots$, so that the coefficient of x^n in $(1-x)^{s_n} F_n(x)$ is the same as the coefficient of x^n in $F_n(x)$, namely, a_n. Consequently, equating coefficients of x^n on each side of the equation just obtained, we find

$$a_n - 1 + (s_n + 1) = \frac{(s_n+2)(s_n+1)}{1.2} - \frac{(s_{n-1}+2)(s_{n-1}+1)s_{n-1}}{1.2.3}$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ (-)^{n+1} \frac{(s_1+2)(s_1+1)\ldots(s_1+2-n)}{1.2.3\ldots(n+1)}.$$

Remembering that $\qquad a_n + s_n = s_{n+1},$

if we call the Hamiltonian Number $s_n + 2$, H_n, the above relation may be written thus:

$$H_{n+1} - 2 = \frac{H_n(H_n - 1)}{1.2} - \frac{H_{n-1}(H_{n-1}-1)(H_{n-1}-2)}{1.2.3}$$
$$+ \frac{H_{n-2}(H_{n-2}-1)(H_{n-2}-2)(H_{n-2}-3)}{1.2.3.4}$$
$$- \ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ (-)^{n+1} \frac{H_1(H_1-1)(H_1-2)\ldots(H_1-n)}{1.2.3\ldots(n+1)}.$$

To obtain Professor Sylvester's modification of this formula given in the preceding portion of this memoir, we multiply the equation from which it was obtained by $1-x$ before proceeding to equate coefficients. Thus we have to equate coefficients of x^n on both sides of

$$(1-x)^{s_n+1} F_n(x) - 1 + x^{-1}(1-x)^2 - x^{n-1}(1-x)^{s_n+2}$$
$$= x^{n-2}(1-x)^{s_n+3} + x^{n-3}(1-x)^{s_{n-1}+3} + x^{n-4}(1-x)^{s_{n-2}+3} + \ldots + x^{-1}(1-x)^{s_1+3}.$$

Or, writing $\qquad s_n + 3 = E_n,$

we equate coefficients on both sides of

$$(1-x)^{E_n-2} F_n(x) - 1 + x^{-1}(1-x)^2 - x^{n-1}(1-x)^{E_n-1}$$
$$= x^{n-2}(1-x)^{E_n} + x^{n-3}(1-x)^{E_{n-1}} + x^{n-4}(1-x)^{E_{n-2}} + \ldots + x^{-1}(1-x)^{E_1}.$$

This equation is easily transformed into

$$(1-x)^{E_0} + x(1-x)^{E_1} + x^2(1-x)^{E_2} + \ldots + x^n(1-x)^{E_n}$$
$$= 1 - 2x + x^2(1-x)^{E_n-2} F_n(x) - x^{n+1}(1-x)^{E_n-1},$$

from which, as Professor Sylvester has pointed out in this memoir, by equating coefficients of all powers of x from 0 to n, we can obtain the successive values of E_n.

The general formula

$$1 - E_{n-1} + \frac{E_{n-2}(E_{n-2}-1)}{1.2} - \ldots + (-)^n \frac{E_0(E_0-1)\ldots(E_0-n+1)}{1.2\ldots n} = 0$$

arises from equating the coefficients of x^n.—J. H.*

* See Note 3, p. 578 [below].

§ 3. *Sequel to the Asymptotic Theory contained in* § 1.

The relation $$p=q^2-\tfrac{2}{3}q^{\frac{3}{2}},\ \text{etc.}$$
previously obtained supplies only the two first terms of the remarkable asymptotic development

$$\frac{q^2-p}{q}=\tfrac{2}{3}\left(q^{\frac{1}{2}}+q^{\frac{1}{4}}+q^{\frac{1}{8}}+\ldots+q^{(\frac{1}{2})^i}\right)+\Xi q,$$

where i is any assigned integer and Ξ is of a lower order of magnitude than the lowest power of q in the series which precedes it. This may be easily established as follows:—

By the scale of relation proved in the preceding section we have

$$p=q^2-\tfrac{2}{3}r^3+\frac{s^4}{3}+\ldots$$
$$=q^2-\tfrac{2}{3}r^3+\text{terms whose maximum order is that of } r^2.$$

Let, now, $$p=q^2-\tfrac{2}{3}q^{\frac{3}{2}}-\tfrac{2}{3}hq^{\alpha}-\tfrac{2}{3}kq^{\beta}-\tfrac{2}{3}lq^{\gamma}\ldots;$$
therefore $$q=r^2-\tfrac{2}{3}r^{\frac{3}{2}}-\tfrac{2}{3}hr^{\alpha}-\tfrac{2}{3}kr^{\beta}-\tfrac{2}{3}lr^{\gamma}\ldots$$
and $$p=q^2-\tfrac{2}{3}r^3\left(1-r^{-\frac{1}{2}}-hr^{\alpha-2}-kr^{\beta-2}-lr^{\gamma-2}\ldots\right)+\ldots$$
$$-\tfrac{2}{3}hr^{2\alpha}-\tfrac{2}{3}kr^{2\beta}-\tfrac{2}{3}lr^{2\gamma}\ldots$$
$$=q^2-\tfrac{2}{3}r^3+\tfrac{2}{3}\left(r^{\frac{5}{2}}+hr^{\alpha+1}+kr^{\beta+1}+lr^{\gamma+1}+\ldots\right)+\ldots$$
$$-\tfrac{2}{3}hr^{2\alpha}-\tfrac{2}{3}kr^{2\beta}-\tfrac{2}{3}lr^{2\gamma}-\ldots\ldots$$

Therefore $$h=1,\quad k=1,\quad l=1,\quad m=1,\ \ldots$$
$$2\alpha=\tfrac{5}{2},\quad 2\beta=1+\alpha,\quad 2\gamma=1+\beta,\quad 2\delta=1+\gamma,\ \ldots$$
that is, $$\alpha=\tfrac{5}{4},\quad \beta=\tfrac{9}{8},\quad \gamma=\tfrac{17}{16},\quad \delta=\tfrac{33}{32},\ \ldots$$
and thus $$p=q^2-\tfrac{2}{3}q\left(q^{\frac{1}{2}}+q^{\frac{1}{4}}+q^{\frac{1}{8}}+q^{\frac{1}{16}}+\ldots+q^{(\frac{1}{2})^i}\right)+\Xi q,$$
as was to be shown*.

* This theorem may be rigorously demonstrated, and reduced to a more precise analytical form, as follows:—

For the sake of brevity, we may call $-p/q+q$ the relative deficiency of p, and denote it by Δ.

First it may be noticed that, if in the equation
$$F(q)=\Sigma_0^\infty\left(q^{2^{-s}}-q^{-2^{-s}}\right)$$
we write $\log q=k$,
$$F(q)=2\left(k+\frac{k^3}{1.2.3.7}+\frac{k^5}{1.2.3.4.5.31}+\frac{k^7}{1.2.3.4.5.6.7.127}+\ldots\right),$$
which is always convergent.

Moreover, the value of $F(q)$ may be calculated for any given value of q within close limits. For, if we call U the right-hand branch of the series in q, beginning with $z-z^{-1}$, the terms of U will easily be seen to lie between those of two geometrical series of which $z-z^{-1}$ is the first term, and of one of which $\frac{1}{2}$, and of the other $(z^{\frac{1}{2}}+z^{-\frac{1}{2}})^{-1}$, is the common ratio.

Hence U is intermediate between $2(z^2-1)/z$ and $(z^2-1)(z+1)/z(z-z^{\frac{1}{2}}+1)$.

It is interesting to notice that the formula *apparently* remains arithmetically true for *finite* values of p and q, provided that q is not less than

---

The difference between these limits, it may be parenthetically observed, is
$$(z-z^{-1})\frac{(z^{\frac{1}{4}}-z^{-\frac{1}{4}})^2}{z^{\frac{1}{2}}-1+z^{-\frac{1}{2}}},$$
which, when z is nearly unity (the limit to which $q^{(\frac{1}{2})^x}$ converges), is nearly equal to $\frac{1}{16}(z-z^{-1})^3$; that is, if $z=1+\tau$, the difference between the limits (for τ small) is very near to $\tau^3/2$.

Now on p. 575 (*post*) it is shown that $\sqrt{q}-r+s-\frac{2}{3}\sqrt{r}=\epsilon$, and that, when the rank of q is taken indefinitely great, ϵ converges to $\frac{1}{3}$. Hence ϵ always lies between finite limits.

For, in general, x being any one of a series of increasing numbers, and $\psi(x)$ a function of x which is always finite for finite values of x, but ultimately converges to c, by taking for x a value of L sufficiently great, we make the series of terms for $x>L$ intermediate between $c+\delta$ and $c-\delta$, where δ is any assigned positive quantity; and consequently, if μ, ν, are the greatest and least values of $\psi(x)$ when x does not exceed L, the greater of the two values, $c+\delta$, μ, and the lesser of the two, $c-\delta$, ν, will be superior and inferior limits to the value of $\psi(x)$ for all values of x.

Hence, writing
$$\sqrt{(p)}-q+r-\tfrac{2}{3}\sqrt{(q)}=\epsilon_1,$$
$$\sqrt{(q)}-r+s-\tfrac{2}{3}\sqrt{(r)}=\epsilon_2,$$
$$\sqrt{(r)}-s+t-\tfrac{2}{3}\sqrt{(s)}=\epsilon_3,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\sqrt{(6)}-3+2-\tfrac{2}{3}\sqrt{(3)}=\epsilon_{x-1},$$
we obtain, by summation,
$$\sqrt{(p)}-q+\tfrac{1}{3}\left\{\sqrt{(q)}+\sqrt{(r)}+\sqrt{(s)}+\ldots+\sqrt{(6)}\right\}=\Sigma\epsilon-2+\tfrac{2}{3}\sqrt{(3)},$$
and, consequently, $$\sqrt{(p)}-q+\tfrac{1}{3}\left\{\sqrt{(q)}+\sqrt{(r)}+\sqrt{(s)}+\ldots+\sqrt{(6)}\right\}=\rho x,$$
where ρ is always a finite quantity lying between determinable limits. But again (p. 573)—
$$p=\{q-\theta\sqrt{(q)}\}^2,$$
where θ (whose ultimate value is $\frac{1}{3}$) is always a proper fraction. Hence
$$\frac{q^2-p}{q}=2\{q-\sqrt{(p)}\}-\theta^2.$$
Hence, from what has been shown above,
$$\frac{q^2-p}{q}=\tfrac{2}{3}\left\{\sqrt{(q)}+\sqrt{(r)}+\sqrt{(s)}+\ldots+\sqrt{(6)}\right\}-2\rho' x.$$
In this equation we may write
$$\left.\begin{aligned}\sqrt{(r)}&=q^{\frac{1}{4}}+k_1\\ \sqrt{(s)}&=q^{\frac{1}{8}}+k_2\\ \sqrt{(t)}&=q^{\frac{1}{16}}+k_3\\ &\ldots\ldots\ldots\end{aligned}\right\}\ (x-3\text{ equations}),$$
where $k_1, k_2, k_3, \ldots$ are all of them finite (and, as a matter of fact, of no consequence for our immediate object, positive proper fractions). For, ultimately,
$$k_1=\sqrt{(r)}-q^{\frac{1}{4}}=\frac{r-q^{\frac{1}{2}}}{r^{\frac{1}{2}}+q^{\frac{1}{4}}}=\frac{\theta_1 r^{\frac{1}{2}}}{r^{\frac{1}{2}}+q^{\frac{1}{4}}}=\tfrac{1}{2}\theta_1=\tfrac{1}{6}\ \text{(see p. 574)},$$
and consequently the finiteness of each k is a direct inference from the general principle previously applied in the case of the ϵ's.

Applying this result to the equation previously given, it follows that
$$q^{\frac{1}{2}}+q^{\frac{1}{4}}+\ldots+q^{(\frac{1}{2})^{x-2}}=\tfrac{3}{2}\Delta-\upsilon x\ \text{(where } \upsilon \text{ is finite)}$$
$$=F(q)+\left(q^{-\frac{1}{2}}+q^{-\frac{1}{4}}+\ldots+q^{-(\frac{1}{2})^{x-2}}\right)-\left\{(z-z^{-1})+(z^{\frac{1}{2}}-z^{-\frac{1}{2}})+(z^{\frac{1}{4}}-z^{-\frac{1}{4}})+\ldots\right\},$$
where z lies between 1 and 2.

The series of negative powers of q is obviously less than x, and the z-series, which follows it, is less than the finite quantity $2(z-1/z)$, that is, $<2(2-\frac{1}{2})$. Hence $\frac{3}{2}\Delta=F(q)+\Theta x$, where Θ is

24, when we replace each term in the formula by its integer portion, and in the series on the right stop at the term immediately preceding the first term for which

$$Eq^{(\frac{1}{2})^i} = 1.$$

Thus, when $p = 462$ and $q = 24$,

we have $$E\left(\frac{q^2-p}{q}\right) = E\left(\frac{576-462}{24}\right) = E\left(\frac{114}{24}\right) = 4,$$

and $$E\{\tfrac{2}{3}(Eq^{\frac{1}{2}} + Eq^{\frac{1}{4}})\} = E\{\tfrac{2}{3}(4+2)\} = 4.$$

So also, when $p = 41881603128$, $q = 204810$,

$$E\left(\frac{q^2-p}{q}\right) = 319,$$

and $E\{\frac{2}{3}(Eq^{\frac{1}{2}} + Eq^{\frac{1}{4}} + Eq^{\frac{1}{8}} + Eq^{\frac{1}{16}})\} = E\{\frac{2}{3}(452+21+4+2)\} = E(\frac{958}{3}) = 319.$

But, if we had included the term $Eq^{\frac{1}{32}}$, the result would have been

$$E\{\tfrac{2}{3}(452+21+4+2+1)\} = 320.$$

---

a number lying between fixed limits, and x, the rank of q, is of the same order of magnitude as $\log\log q$. This equation contains as a consequence the asymptotic theorem to be proved; for, using i to denote any positive integer,

$$\tfrac{3}{2}\Delta - \Sigma_1^i q^{(\frac{1}{2})^i} = F(q) - \Sigma_1^i q^{(\frac{1}{2})^i} - \Theta x = q^{(\frac{1}{2})^{i+1}} + \sum_{s=i+2}^{s=\infty}(q^{(\frac{1}{2})^s} - q^{-(\frac{1}{2})^s}) - \Sigma_0^i 1/q^{(\frac{1}{2})^{i+1}} - \Theta x.$$

Hence, remembering that x is of the same order of magnitude as $\log\log q$, and that

$$\sum_{s=i+2}^{s=\infty}(q^{(\frac{1}{2})^s} - q^{-(\frac{1}{2})^s}) < 2(q^{(\frac{1}{2})^{i+2}} - q^{-(\frac{1}{2})^{i+2}}),$$

which is of a lower order of magnitude than $q^{(\frac{1}{2})^{i+1}}$, it follows that $\frac{3}{2}\Delta - \Sigma_1^i q^{(\frac{1}{2})^i}$ for all values of i is ultimately in a ratio of equality with $q^{(\frac{1}{2})^{i+1}}$, which is the theorem to be proved.

We have thought it desirable to obtain the formula $\frac{3}{2}\Delta = Fq + \Theta x$ for its own sake, but, so far as regards the proof in question, that might be obtained more expeditiously from the expression given for $3\Delta/2 - vx$ without introducing the series Fq.

It is easy to ascertain the ultimate value to which Θ converges. In the first place, the series of fractions $1/q^{\frac{1}{2}} + 1/q^{\frac{1}{4}} + 1/q^{\frac{1}{8}} + \ldots$ to $x-2$ terms (where x is the rank of q) may be shown to be always finite, and consequently, when divided by x, converges to zero.

For we know that $(p-q) > (q-r)^2 > (r-s)^4 \ldots > (6-3)^{2^{x-2}}$. Hence the last term of the series $q^{\frac{1}{2}}, q^{\frac{1}{4}}, q^{\frac{1}{8}} \ldots$ (namely, $q^{(\frac{1}{2})^{x-2}}$) > 3. Hence the finite series $1/q^{\frac{1}{2}} + 1/q^{\frac{1}{4}} + 1/q^{\frac{1}{8}} + \ldots$ for a double *à fortiori* reason is less than the infinite geometrical series $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots < \frac{1}{2}$.

In fact, from § 1 (p. 566) it may easily be shown that the last term of the series

$$q^{\frac{1}{2}}, q^{\frac{1}{4}}, q^{\frac{1}{8}} \ldots > M^4 > (1{\cdot}465)^4 > 4{\cdot}608,$$

so that the sum is really less than $\frac{1}{3{\cdot}608}$.

Hence, retracing the steps by which Θ has been obtained, and observing that ρ' differs from ρ by a finite multiple of $1/x$, we have ultimately $\Theta = v = k - 3\rho' = k - 3\rho = k - 3\epsilon = \frac{1}{6} - \frac{3}{4} = -\frac{7}{12}$. If, then (using u_x to denote the half of the sharpened xth Hamiltonian number), we write $u_x - 1/u_x = v_x$, and understand by $G(t - 1/t)$ the infinite series

$$(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) + (t^{\frac{1}{4}} - t^{-\frac{1}{4}}) + (t^{\frac{1}{8}} - t^{-\frac{1}{8}}) + \ldots,$$

it is easily seen that the *principal* part of $\sqrt{(v_{x+1})}$, regarded as a function of v_x and x, is

$$v_x - \tfrac{1}{3}Gv_x + \tfrac{7}{12}x.$$

Again, when

$$p = 307673684354828937922426140463753876021658 4,$$
$$q = 175406295314542939 9086,$$

$$E\left(\frac{q^2-p}{q}\right) = 27921159919,$$

and $E\{\frac{2}{3}(Eq^{\frac{1}{2}} + Eq^{\frac{1}{4}} + Eq^{\frac{1}{8}} + Eq^{\frac{1}{16}} + Eq^{\frac{1}{32}} + Eq^{\frac{1}{64}})\}$

$= E\{\frac{2}{3}(41881534751 + 204649 + 452 + 21 + 4 + 2)\} = 27921159919$*.

We will now proceed to consider afresh the asymptotic development of any *Hypothenusal* Number $p-q$ in terms of its antecedent $q-r$, and to reduce to apodictic certainty results which in the first section were partly obtained by observation. It has already been shown in that section that

$$p > q^2 - \tfrac{128}{81}q^{\frac{3}{2}} - \frac{q}{2}$$

when p is not lower than 204810 in the scale 2, 3, 6, 24, 462, 204810, ..., that is, when q is not less than 462.

Hence $$p > q^2 - 2q^{\frac{3}{2}} + q + (\tfrac{34}{81}q^{\frac{3}{2}} - \tfrac{3}{2}q),$$

or, since $\frac{34}{81}q^{\frac{3}{2}} - \frac{3}{2}q$ is a positive quantity,

$$p > (q - \sqrt{q})^2,$$

at all events when $q =$ or > 462.

It will be found also on trial that this formula remains true for all the values of q inferior to 462.

Thus $$462 > (24 - \sqrt{24})^2,$$
$$24 > (6 - \sqrt{6})^2,$$
$$6 > (3 - \sqrt{3})^2,$$
$$3 > (2 - \sqrt{2})^2.$$

Hence, universally, $p > (q - \sqrt{q})^2$†.

But we know that $p < q^2$.

We may therefore write $p = (q - \theta\sqrt{q})^2$,

where θ is some quantity between 0 and 1.

Similarly, $$q = (r - \theta_1\sqrt{r})^2,$$
$$r = (s - \theta_2\sqrt{s})^2,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

where $\theta_1, \theta_2, \ldots$ are also positive fractions.

* The authors must be understood merely to affirm the *possibility* of the theorem being true, and to offer no opinion on the strength of the presumption raised that it is so.

† Had this inequality been true only for values of q sufficiently great, it would have been enough for the purposes of the text.

When p and q become infinite,

$$\frac{q^2-p}{q^{\frac{3}{2}}}=\tfrac{2}{3}=2\theta.$$

Hence the ultimate value of θ is $\frac{1}{3}$. Similarly, θ_1, θ_2, ... all of them converge to the value $\frac{1}{3}$.

This agrees with the result previously demonstrated (p. 563), and is the starting point of all that follows.

We know that letters $p, q, r, s, \ldots$, being used to denote the halves of the augmented Hamiltonian Numbers, they are connected by the scale of relation

$$p=\tfrac{1}{2}+\frac{q\,(2q-1)}{2}-\frac{r\,(2r-1)\,(2r-2)}{2\,.\,3}+S-T,$$

where $$S=\frac{s\,(2s-1)\,(2s-2)\,(2s-3)}{2\,.\,3\,.\,4}$$

and T stands for the remaining terms, involving

$$t,\ u,\ v,\ \ldots.$$

Considering $\qquad q,\ r,\ s,\ t,\ \ldots$

to be of the order $\qquad 1,\ \tfrac{1}{2},\ \tfrac{1}{4},\ \tfrac{1}{8},\ \ldots,$

we may reject the term $\frac{1}{2}$, which is of zero order, and write

$$p=q^2-\tfrac{2}{3}r^3;\quad -\frac{q}{2}+r^2-\frac{r}{3}+S-T.$$

Hence, rejecting terms of order less than $\frac{3}{2}$ (which have, however, to be retained in obtaining the subsequent approximations),

$$\left.\begin{matrix}(p-q)\\-(q-r)^2\end{matrix}\right\}=\begin{cases}q^2-\tfrac{2}{3}r^3; & -\tfrac{3}{2}q+r^2-\dfrac{r}{3}+S-T\\ -q^2+2qr; & -r^2\end{cases}$$
$$=(2qr-\tfrac{2}{3}r^3);$$

that is, $$(p-q)-(q-r)^2=\tfrac{4}{3}q^{\frac{3}{2}}$$

when q is infinite.

Again, writing for S its expanded value, namely,

$$\frac{s^4}{3}-s^3+\tfrac{11}{12}s^2-\frac{s}{4},$$

we have

$$\left.\begin{matrix}(p-q)\\-\ (q-r)^2\\-\tfrac{4}{3}(q-r)^{\frac{3}{2}}\end{matrix}\right\}=\begin{cases}2qr-\tfrac{2}{3}r^3-\tfrac{4}{3}q^{\frac{3}{2}} & \text{Order } \tfrac{3}{2};\\ +2q^{\frac{1}{2}}r-\tfrac{3}{2}q+\tfrac{1}{3}s^4 & \quad ,,\quad 1;\\ -\tfrac{1}{2}q^{-\frac{1}{2}}r^2-\dfrac{r}{3}-s^3+\tfrac{11}{12}s^2-\tfrac{1}{4}s-T & \quad ,,\quad <1,\end{cases}$$

rejecting the terms $q^{-\frac{3}{2}}r^3$, $q^{-\frac{5}{2}}r^4$, ... in the expansion of $(q-r)^{\frac{3}{2}}$ because the order of none of them is superior to zero.

We now write $$q=(r-\theta_1\sqrt{r})^2,$$

so that

$$\begin{aligned}2qr-\tfrac{2}{3}r^3-\tfrac{4}{3}q^{\frac{3}{2}}&=(2r^3-4\theta_1r^{\frac{5}{2}}+2\theta_1^2r^2)-\tfrac{2}{3}r^3-(\tfrac{4}{3}r^3-4\theta_1r^{\frac{5}{2}}+4\theta_1^2r^2-\tfrac{4}{3}\theta_1^3r^{\frac{3}{2}})\\&=-2\theta_1^2r^2+\tfrac{4}{3}\theta_1^3r^{\frac{3}{2}}.\end{aligned}$$

Hence

$$\left.\begin{matrix}(p-q)\\-\ (q-r)^2\\-\tfrac{4}{3}(q-r)^{\frac{3}{2}}\end{matrix}\right\}=\begin{cases}-2\theta_1^2r^2+2q^{\frac{1}{2}}r-\tfrac{3}{2}q+\tfrac{1}{3}s^4 & \text{Order } 1;\\ +\tfrac{4}{3}\theta_1^3r^{\frac{3}{2}}-s^3 & \quad ,,\quad \tfrac{3}{4};\\ -\tfrac{1}{2}q^{-\frac{1}{2}}r^2-\dfrac{r}{3}+\tfrac{11}{12}s^2-\tfrac{1}{4}s-T & \quad ,,\quad <\tfrac{3}{4}.\end{cases}$$

Since $$q=r^2=s^4\ \text{(ultimately)},$$

the terms of Order 1 (which are the only ones with which we have to do at present) are ultimately equal to

$$(-2\theta_1^2+2-\tfrac{3}{2}+\tfrac{1}{3})\,q;$$

or, giving θ_1 its ultimate value $\frac{1}{3}$, to $\frac{11}{18}q$, or to the same order of approximation to $\frac{11}{18}(q-r)$.

Hence, ultimately,

$$(p-q)=(q-r)^2+\tfrac{4}{3}(q-r)^{\frac{3}{2}}+\tfrac{11}{18}(q-r)\text{*}.$$

We use this result to obtain a closer approximation to $\sqrt{q}$ than $r-\theta_1\sqrt{r}$, and to find the relation between the general values of θ_1 and θ_2.

Thus, assuming $\quad \sqrt{(q-r)}=r-s+\tfrac{2}{3}\sqrt{(r-s)}+k,$

we have, ultimately,

$$\begin{aligned}q-r&=(r-s)^2+\tfrac{4}{3}(r-s)^{\frac{3}{2}}+(\tfrac{4}{9}+2k)(r-s)\\&=(r-s)^2+\tfrac{4}{3}(r-s)^{\frac{3}{2}}+\tfrac{11}{18}(r-s).\end{aligned}$$

Consequently, as r becomes indefinitely great, k converges to the value $\frac{1}{2}(\frac{11}{18}-\frac{4}{9})=\frac{1}{12}$.

Now $$\sqrt{(q-r)}=\sqrt{(q)}-\tfrac{1}{2}\frac{r}{\sqrt{q}}\ldots=\sqrt{(q)}-\tfrac{1}{2}\ \text{ultimately};$$

and similarly $$\sqrt{(r-s)}=\sqrt{(r)}-\tfrac{1}{2}\ \text{ultimately}.$$

Hence, ultimately,

$$\sqrt{(q)}=r-s+\tfrac{2}{3}\sqrt{(r)}+\tfrac{1}{12}+\tfrac{1}{2}-\tfrac{1}{3}=r-s+\tfrac{2}{3}\sqrt{(r)}+\tfrac{1}{4}.$$

We may therefore write

$$\sqrt{(q)}=r-s+\tfrac{2}{3}\sqrt{(r)}+\epsilon\ \text{(where ultimately } \epsilon=\tfrac{1}{4}\text{)}.$$

But $$\sqrt{(q)}=r-\theta_1\sqrt{r},$$

and therefore $$\theta_1\sqrt{(r)}=s-\tfrac{2}{3}\sqrt{(r)}-\epsilon.$$

* As previously obtained by observation in § 1 (p. 563). It will, of course, be understood that in the above and similar passages the sign = is to be interpreted to mean "is in a ratio of equality with."

Moreover $\sqrt{(r)} = s - \theta_2 \sqrt{s}$,

whence it follows that

$$\theta_1 \sqrt{(r)} = \tfrac{1}{3}s + \tfrac{2}{3}\theta_2 \sqrt{(s)} - \epsilon \text{ (where } \epsilon = \tfrac{1}{4} \text{ ultimately).}$$

Resuming the development of $(p-q)$ in terms of $(q-r)$, we have

$$\left.\begin{array}{r} (p-q) \\ -\quad (q-r)^2 \\ -\tfrac{4}{3}(q-r)^{\frac{3}{2}} \\ -\tfrac{11}{18}(q-r) \end{array}\right\} = \left\{\begin{array}{ll} -2\theta_1 r^2 + 2q^{\frac{1}{2}} r - \tfrac{10}{9}q + \dfrac{s^4}{3} & \text{Order } 1, \\ +\tfrac{4}{3}\theta_1^3 r^{\frac{3}{2}} - s^3 & \quad ,, \quad \tfrac{3}{4}, \\ & \\ -\tfrac{1}{2}q^{-\frac{1}{2}} r^2 + \tfrac{5}{18}r + \tfrac{11}{12}s^2 - \dfrac{s}{4} - T & \quad ,, \; < \tfrac{3}{4}. \end{array}\right.$$

The terms of order inferior to $\frac{3}{4}$ are of no value for present purposes, and are only retained for the benefit of those who may wish to carry on the work.

To reduce the terms of Order 1, we write, in succession,

$$q = (r - \theta_1 \sqrt{r})^2,$$
$$\theta_1 \sqrt{(r)} = \tfrac{1}{3}s + \tfrac{2}{3}\theta_2 \sqrt{(s)} - \epsilon,$$
$$r = (s - \theta_2 \sqrt{s})^2.$$

Thus $\dfrac{s^4}{3} - 2\theta_1^2 r^2 + 2q^{\frac{1}{2}} r - \tfrac{10}{9}q$

$$= \frac{s^4}{3} - 2\theta_1^2 r^2 + 2r^2 - \tfrac{10}{9}r^2 ;\; -2\theta_1 r^{\frac{3}{2}} + \tfrac{38}{9}\theta_1 r^{\frac{3}{2}} ;\; -\tfrac{10}{9}\theta_1^2 r$$

$$= \frac{s^4}{3} - \frac{r^2}{9} - 2\theta_1^2 r^2 ;\; +\tfrac{20}{9}\theta_1 r^{\frac{3}{2}} ;\; -\tfrac{10}{9}\theta_1^2 r$$

$$= \frac{s^4}{3} - \frac{r^2}{9} - 2r\left(\frac{s}{3} + \tfrac{2}{3}\theta_2 \sqrt{s}\right)^2 ;\; + 4\epsilon r\left(\frac{s}{3} + \tfrac{2}{3}\theta_2 \sqrt{s}\right) + \tfrac{20}{9}\theta_1 r^{\frac{3}{2}} ;\; -2\epsilon^2 r - \tfrac{10}{9}\theta_1^2 r$$

$$= \frac{s^4}{3} - \tfrac{1}{9}(s^4 - 4\theta_2 s^{\frac{7}{2}} + 6\theta_2^2 s^3 - 4\theta_2^3 s^{\frac{5}{2}} + \theta_2^4 s^2) - \tfrac{2}{9}s^2(s^2 + 4\theta_2 s^{\frac{3}{2}} + 4\theta_2^2 s)$$
$$+ \tfrac{4}{9}\theta_2 s^{\frac{3}{2}}(s^2 + 4\theta_2 s^{\frac{3}{2}} + 4\theta_2^2 s) - \tfrac{2}{9}\theta_2^2 s(s^2 + 4\theta_2 s^{\frac{3}{2}} + 4\theta_2^2 s) ;$$
$$+ \tfrac{4}{3}\epsilon rs + \tfrac{20}{9}\theta_1 r^{\frac{3}{2}} ;\; + \tfrac{8}{3}\epsilon\theta_2 r \sqrt{(s)} - 2\epsilon^2 r - \tfrac{10}{9}\theta_1^2 r$$
$$= \tfrac{4}{3}\epsilon rs + \tfrac{20}{9}\theta_1 r^{\frac{3}{2}} \qquad \text{Order } \tfrac{3}{4},$$
$$+ \tfrac{4}{3}\theta_2^3 s^{\frac{5}{2}} + \tfrac{8}{3}\epsilon\theta_2 r \sqrt{(s)} - \theta_2^4 s^2 - 2\epsilon^2 r - \tfrac{10}{9}\theta_1^2 r \qquad ,, \; < \tfrac{3}{4}.$$

Hence

$$\left.\begin{array}{r} (p-q) \\ -\quad (q-r)^2 \\ -\tfrac{4}{3}(q-r)^{\frac{3}{2}} \\ -\tfrac{11}{18}(q-r) \end{array}\right\} = \left\{\begin{array}{ll} \tfrac{4}{3}\theta_1^3 r^{\frac{3}{2}} - s^3 + \tfrac{4}{3}\epsilon rs + \tfrac{20}{9}\theta_1 r^{\frac{3}{2}} & \text{Order } \tfrac{3}{4}, \\ -T + \tfrac{4}{3}\theta_2^3 s^{\frac{5}{2}} + \tfrac{8}{3}\epsilon\theta_2 r s^{\frac{1}{2}} & \quad ,, \quad \tfrac{5}{8}, \\ -\tfrac{1}{2}q^{-\frac{1}{2}} r^2 + \tfrac{5}{18}r + \tfrac{11}{12}s^2 - \dfrac{s}{4} - \theta_2^4 s^2 - 2\epsilon^2 r - \tfrac{10}{9}\theta_1^2 r & \quad ,, \; < \tfrac{5}{8}. \end{array}\right.$$

Here the terms of Order $\frac{3}{4}$ are ultimately equal to

$$(\tfrac{4}{3}\theta_1^3 - 1 + \tfrac{4}{3}\epsilon + \tfrac{20}{9}\theta_1)\, q^{\frac{3}{4}},$$

which, when θ_1 and ϵ receive their ultimate values, $\frac{1}{3}$ and $\frac{1}{4}$, becomes

$$(\tfrac{4}{81} - 1 + \tfrac{1}{3} + \tfrac{20}{27})\, q^{\frac{3}{4}} = \tfrac{10}{81} q^{\frac{3}{4}}\text{*}.$$

From this it follows immediately that (rejecting terms of an order of magnitude inferior to that of $(q-r)^{\frac{3}{4}}$)

$$p - q = (q-r)^2 + \tfrac{4}{3}(q-r)^{\frac{3}{2}} + \tfrac{11}{18}(q-r) + \tfrac{10}{81}(q-r)^{\frac{3}{4}}.$$

The law of the indices in the complete development is easily deduced from the relation

$$p = \tfrac{1}{2} + \frac{q(2q-1)}{2} - \frac{r(2r-1)(2r-2)}{2\,.\,3} + \frac{s(2s-1)(2s-2)(2s-3)}{2\,.\,3\,.\,4} - \ldots.$$

The terms carrying the arguments

$$q^2,\ q,\ r^3,\ r^2,\ r,\ s^4,\ s^3,\ s^2,\ s,\ t^5, \ldots$$

furnish the indices $\quad 2,\ 1,\ \frac{3}{2},\ 1,\ \frac{1}{2},\ 1,\ \frac{3}{4},\ \frac{1}{2},\ \frac{1}{4},\ \frac{5}{8}, \ldots,$

which, arranged in order of magnitude, become

$$2,\ \tfrac{3}{2},\ 1,\ \tfrac{3}{4},\ \tfrac{5}{8},\ \tfrac{1}{2},\ \tfrac{3}{8},\ \tfrac{5}{16},\ \tfrac{1}{4}, \ldots.$$

Thus, calling $p-q$ and $q-r$ y and x respectively, the expansion for y in terms of x will be of the form

$$y = \Sigma A x^{\frac{2m+1}{2^n}},$$

where n has all values from 0 to ∞, and $2m+1$ does not exceed $n+2$, that is, m has all positive values from 0 to $n/2$ or $\frac{1}{2}(n+1)$, according as n is even or odd.

But, besides this expressed portion of the development of a Hypothenusal Number, say η_{x+1}, as a function of its antecedent, η_x, there will be another portion, consisting of terms with zero and negative indices of η_x having functions of x for their coefficients, which observation is incompetent to reveal, and with the nature of which we are at present unacquainted. The study of Hamilton's Numbers, far from being exhausted, has, in leaving our hands, little more than reached its first stage, and it is believed will furnish a plentiful aftermath to those who may feel hereafter inclined to pursue to the end the thorny path we have here contented ourselves with indicating, which lies so remote from the beaten track of research, and offers an example and suggestion of infinite series (as far as we are aware) wholly unlike any which have previously engaged the attention of mathematicians.

J. J. S. and J. H.

* Agreeing closely with what had been previously found by observation in § 1 (p. 563).

NOTE 1, p. 566.

It is easy to see that, if δM and $\delta\alpha$ are corresponding errors in the values of M and α respectively,

$$\delta M = (M \log_e M \log_e 2)\,\delta\alpha = (\cdot 38822\ \ldots)\,\delta\alpha$$

(since $M = 1\cdot 46544\ \ldots$, $\log_e M = \cdot 38220\ \ldots$, and $\log_e 2 = \cdot 69314\ \ldots$).

Hence, $\delta\alpha$ being intermediate between ·0000003 and ·0000006,

δM lies between ·000000116 and ·000000233.

The value of M (the base of the Hamiltonian Numbers) is thus found to be 1·465443 ..., correct to the last figure inclusive.—J. J. S.

NOTE 2, p. 568.

This equation may be obtained more simply from the *fundamental* formula of Hamilton (middle of above note). It follows from the law of derivation there given that, if we write ${}^1F_n = (1-x)^{-1}F_n - x^n$, and, in general, ${}^{j+1}F_n = (1-x)^{-1}\,{}^jF_n - x^n$, then $F_{n+1} = {}^{a_n}F_n$; and, consequently,

$$F_{n+1} - (1-x)^{-a_n}F_n = -x^n\{1 + (1-x)^{-1} + (1-x)^{-2} + \ldots + (1-x)^{-a_n+1}\}$$
$$= x^{n-1}\{(1-x) - (1-x)^{-a_n+1}\}.\text{—J. J. S.}$$

NOTE 3, p. 569.

It is curious to notice the sort of affinity which exists between a form of writing the scale of relation for Bernoulli's Numbers and that given at p. 569 for Hamilton's.

If we write

$G_0 = 1$, $G_1 = -1$, $G_2 = (-4)B_1$, $G_3 = 0$, $G_4 = (-4)^2B_2$, $G_5 = 0$, $G_6 = (-4)^3B_3$, ...

then, using β_κ in the same sense as at p. 558, we shall find the scale of relation between the B's (Bernoulli's Numbers) is given by the equation

$$\sum_{\kappa=0}^{\kappa=i}(-)^\kappa\beta_\kappa i\,.\,G_{i-\kappa} = 0, \textit{ provided } i \textit{ is odd.}$$

On striking out the i which intervenes between β_κ and $G_{i-\kappa}$, so as to make the former operate on the latter, the equation becomes that given at p. 569 for the E's, the sharpened numbers of Hamilton.—J. J. S.

§ 4. *Continuation, to an infinite number of terms, of the Asymptotic Development for Hypothenusal Numbers.*

"This was sometime a paradox, but now the time gives it proof."

Hamlet, Act III., Scene 1.

In the third section of this paper [above, p. 575] it was stated, on what is now seen to be insufficient evidence, that the asymptotic development of $p-q$, the half of any Hypothenusal Number, could be expressed as a series of powers of $q-r$, the half of its antecedent, in which the indices followed the sequence

$$2,\ \tfrac{3}{2},\ 1,\ \tfrac{3}{4},\ \tfrac{5}{8},\ \tfrac{1}{2},\ \ldots$$

It was there shown that, when quantities of an order of magnitude inferior to that of $(q-r)^{\frac{3}{4}}$ are neglected,

$$p-q = (q-r)^2 + \tfrac{4}{3}(q-r)^{\frac{3}{2}} + \tfrac{11}{18}(q-r) + \tfrac{10}{81}(q-r)^{\frac{3}{4}};$$

but, on attempting to carry this development further, it was found that, though the next term came out $\frac{88}{1215}(q-r)^{\frac{5}{8}}$, there was an infinite series of terms interposed between this one and $(q-r)^{\frac{1}{2}}$, namely, as proved in the present section, between $(q-r)^{\frac{3}{4}}$ and $(q-r)^{\frac{1}{2}}$ there lies an infinite series of terms whose indices are

$$\tfrac{5}{8},\ \tfrac{9}{16},\ \tfrac{17}{32},\ \tfrac{33}{64},\ \tfrac{65}{128},\ \ldots,$$

and whose coefficients form a geometrical series of which the first term is $\frac{88}{1215}$ and the common ratio $\frac{2}{3}$.

We shall assume the law of the indices (which, it may be remarked, is identical with that given in the introduction to this paper as originally printed in the *Proceedings**, but subsequently altered in the *Transactions*) and write

$$\begin{aligned} p-q = {} & (q-r)^2 + \tfrac{4}{3}(q-r)^{\frac{3}{2}} + \tfrac{11}{18}(q-r) + \tfrac{10}{81}(q-r)^{\frac{3}{4}} \\ & + \tfrac{2^3}{3^3}A(q-r)^{\frac{5}{8}} + \tfrac{2^4}{3^4}B(q-r)^{\frac{9}{16}} + \tfrac{2^5}{3^5}C(q-r)^{\frac{17}{32}} \\ & + \tfrac{2^6}{3^6}D(q-r)^{\frac{33}{64}} + \tfrac{2^7}{3^7}E(q-r)^{\frac{65}{128}} + \text{\&c., ad inf.} \\ & + \Theta\,\dagger. \end{aligned} \qquad (1)$$

The law of the coefficients will then be established by proving that

$$A = B = C = D = E = \ldots = \tfrac{11}{45}.$$

If there were any terms, of an order superior to that of $(q-r)^{\frac{1}{2}}$, whose indices did not obey the assumed law, any such term would make its presence felt in the course of the work; for, in the process we shall employ, the coefficient of each term has to be determined before that of any subsequent

[* See footnote, p. 584, below.]

† In the text above Θ represents some unknown function, the asymptotic value of whose ratio to $(q-r)^{\frac{1}{2}}$ is not infinite.

term can be found. It was in this way that the existence of terms between $(q-r)^{\frac{5}{8}}$ and $(q-r)^{\frac{1}{2}}$ was made manifest in the unsuccessful attempt to calculate the coefficient of $(q-r)^{\frac{1}{2}}$. It thus appears that the assumed law of the indices is the true one.

It will be remembered that $p, q, r, \ldots$, are the halves of the sharpened Hamiltonian Numbers $E_{n+1}, E_n, E_{n-1}, \ldots$, and that consequently the relation

$$E_{n+1} = 1 + \frac{E_n(E_n-1)}{1\,.\,2} - \frac{E_{n-1}(E_{n-1}-1)(E_{n-1}-2)}{1\,.\,2\,.\,3} + \ldots$$

may be written in the form

$$p = \tfrac{1}{2} + \frac{q(2q-1)}{2} - \frac{r(2r-1)(2r-2)}{2\,.\,3} + \frac{s(2s-1)(2s-2)(2s-3)}{2\,.\,3\,.\,4}$$
$$- \frac{t(2t-1)(2t-2)(2t-3)(2t-4)}{2\,.\,3\,.\,4\,.\,5} + \frac{u(2u-1)(2u-2)(2u-3)(2u-4)(2u-5)}{2\,.\,3\,.\,4\,.\,5\,.\,6}$$
$$- \ldots \qquad (2)$$

The comparison of this value of p with that given by (1) furnishes an equation which, after several reductions have been made, in which special attention must be paid to the order of the quantities under consideration, ultimately leads to the determination of the values $A, B, C, \ldots$, in succession.

Taking unity to represent the order of q, the orders of

$$p,\ q,\ r,\ s,\ t,\ u,\ v,\ w,\ \ldots$$

will be

$$2,\ 1,\ \tfrac{1}{2},\ \tfrac{1}{4},\ \tfrac{1}{8},\ \tfrac{1}{16},\ \tfrac{1}{32},\ \tfrac{1}{64},\ \ldots$$

Hence, after expanding each of the binomials on the right-hand side of (1) and arranging the terms in descending order, retaining only terms for which the order is superior to $\frac{1}{2}$, we shall find

Order	2	$p = q^2$
,,	$\frac{3}{2}$	$-2qr + \frac{4}{3}q^{\frac{3}{2}}$
,,	1	$+r^2 - 2q^{\frac{1}{2}}r + \frac{29}{18}q$
,,	$\frac{3}{4}$	$+\frac{10}{81}q^{\frac{3}{4}}$
,,	$\frac{5}{8}$	$+\frac{2^3}{3^3}A\,q^{\frac{5}{8}}$
,,	$\frac{9}{16}$	$+\frac{2^4}{3^4}B\,q^{\frac{9}{16}}$
,,	$\frac{17}{32}$	$+\frac{2^5}{3^5}C\,q^{\frac{17}{32}}$
,,	$\frac{33}{64}$	$+\frac{2^6}{3^6}D\,q^{\frac{33}{64}}$
,,	$\frac{65}{128}$	$+\frac{2^7}{3^7}E\,q^{\frac{65}{128}} + \ldots$ (3)

Again, retaining only those terms of (2) whose order is superior to $\frac{1}{2}$, we have

$$p = q^2;\ -\tfrac{2}{3}r^3;\ -\tfrac{1}{2}q + r^2 + \tfrac{1}{3}s^4;\ -s^3;\ -\tfrac{2}{15}t^5 \qquad (4)$$

Order $\quad 2;\quad \frac{3}{2};\quad 1;\quad \frac{3}{4};\quad \frac{5}{8}$.

From (3) and (4) we obtain by subtraction

Order	$\frac{3}{2}$	$0 = \frac{2}{3}r^3 - 2qr + \frac{4}{3}q^{\frac{3}{2}}$
,,	1	$-\frac{1}{3}s^4 - 2q^{\frac{1}{2}}r + \frac{10}{9}q$
,,	$\frac{3}{4}$	$+s^3 + \frac{10}{81}q^{\frac{3}{4}}$
,,	$\frac{5}{8}$	$+\frac{2}{15}t^5 + \frac{2^3}{3^3}A\,q^{\frac{5}{8}}$
,,	$\frac{9}{16}$	$+\frac{2^4}{3^4}B\,q^{\frac{9}{16}}$
,,	$\frac{17}{32}$	$+\frac{2^5}{3^5}C\,q^{\frac{17}{32}}$
,,	$\frac{33}{64}$	$+\frac{2^6}{3^6}D\,q^{\frac{33}{64}}$
,,	$\frac{65}{128}$	$+\frac{2^7}{3^7}E\,q^{\frac{65}{128}} + \ldots$ (5)

Changing $p, q, r, \ldots$ into $q, r, s, \ldots$ respectively, equation (4) becomes

$$q = r^2 - \tfrac{2}{3}s^3 - \tfrac{1}{2}r + s^2 + \tfrac{1}{3}t^4 - t^3 - \tfrac{2}{15}u^5,$$

so that, if we assume $q = r^2(1-\alpha)$, the order of α will be the same as that of $r^{-2}s^3$, namely, $-\frac{2}{2} + \frac{3}{4} = -\frac{1}{4}$.

Hence, if we substitute $r^2(1-\alpha)$ for q in (5), neglecting in the result quantities of the order $\frac{1}{2}$, we shall find

$$\tfrac{2}{3}r^3 - 2qr + \tfrac{4}{3}q^{\frac{3}{2}} - \tfrac{1}{3}s^4 - 2q^{\frac{1}{2}}r + \tfrac{10}{9}q$$
$$= \tfrac{2}{3}r^3 - 2r^3(1-\alpha) + \tfrac{4}{3}r^3(1 - \tfrac{3}{2}\alpha + \tfrac{3}{8}\alpha^2 + \tfrac{1}{16}\alpha^3)$$
$$-\tfrac{1}{3}s^4 - 2r^2(1-\tfrac{1}{2}\alpha) + \tfrac{10}{9}r^2(1-\alpha)$$
$$= \tfrac{1}{2}r^3\alpha^2 + \tfrac{1}{12}r^3\alpha^3 - \tfrac{1}{3}s^4 + \tfrac{1}{9}r^2 - \tfrac{10}{9}r^2\alpha;$$

while at the same time, since the order of $r^{\frac{3}{2}}\alpha$ does not exceed $\frac{1}{2}$, we have

$$q^{\frac{3}{4}} = r^{\frac{3}{2}}(1-\alpha)^{\frac{3}{4}} = r^{\frac{3}{2}},$$

and in like manner $\quad q^{\frac{5}{8}} = r^{\frac{5}{4}},\ q^{\frac{9}{16}} = r^{\frac{9}{8}}$, and so on.

Thus equation (5) becomes

Order	1	$0 = \frac{1}{2}r^3\alpha^2 - \frac{1}{3}s^4 + \frac{1}{9}r^2$
,,	$\frac{3}{4}$	$+\frac{1}{12}r^3\alpha^3 - \frac{10}{9}r^2\alpha + s^3 + \frac{10}{81}r^{\frac{3}{2}}$
,,	$\frac{5}{8}$	$+\frac{2}{15}t^5 + \frac{2^3}{3^3}A\,r^{\frac{5}{4}}$
,,	$\frac{9}{16}$	$+\frac{2^4}{3^4}B\,r^{\frac{9}{8}}$
,,	$\frac{17}{32}$	$+\frac{2^5}{3^5}C\,r^{\frac{17}{16}}$
,,	$\frac{33}{64}$	$+\frac{2^6}{3^6}D\,r^{\frac{33}{32}}$
,,	$\frac{65}{128}$	$+\frac{2^7}{3^7}E\,r^{\frac{65}{64}} + \ldots$ (6)

where $\quad \alpha = \frac{2}{3}r^{-2}s^3;\ +\frac{1}{2}r^{-1} - r^{-2}s^2 - \frac{1}{3}r^{-2}t^4;\ +r^{-2}t^3;\ +\frac{2}{15}r^{-2}u^5,$

order $\quad -\frac{1}{4};\quad -\frac{1}{2};\quad -\frac{5}{8};\quad -\frac{11}{16}$.

Let $\qquad \alpha = \frac{2}{3}r^{-2}s^3(1+\alpha')$

then $\qquad \alpha' = \frac{3}{2}s^{-3}(\frac{1}{2}r - s^2 - \frac{1}{3}t^4 + t^3 + \frac{2}{15}u^5)$

where terms as far as, but not beyond, $-\frac{7}{16}$ (which is the order of $s^{-3}u^5$) have been retained.

Now p consists of terms whose orders are $2, \frac{3}{2}, 1, \frac{3}{4}, \frac{5}{8}, \frac{1}{2}, \ldots$

q " " " $1, \frac{3}{4}, \frac{1}{2}, \frac{3}{8}, \frac{5}{16}, \frac{1}{4}, \ldots$

α " " " $-\frac{1}{4}, -\frac{1}{2}, -\frac{5}{8}, -\frac{11}{16}, -\frac{3}{4}, \ldots$

α' " " " $-\frac{1}{4}, -\frac{3}{8}, -\frac{7}{16}, -\frac{1}{2}, \ldots.$

Thus the order of α' is $-\frac{1}{4}$, and in the above expression all terms of α' superior to $-\frac{1}{2}$ have been retained, and consequently (rejecting the square of α' whose order is $-\frac{1}{2}$) in the first line of (6) we may write

$$\begin{aligned}\tfrac{1}{2} r^3\alpha^2 &= \tfrac{2}{9} r^{-1}s^6(1+2\alpha') \\ &= \tfrac{2}{9} r^{-1}s^6 + \tfrac{2}{3} r^{-1}s^3\left(\tfrac{1}{2} r - s^2 - \tfrac{1}{3} t^4 + t^3 + \tfrac{2}{15} u^5\right) \\ &= \tfrac{2}{9} r^{-1}s^6 + \tfrac{1}{3} s^3 - \tfrac{2}{3} r^{-1}s^5 - \tfrac{2}{9} r^{-1}s^3t^4 + \tfrac{2}{3} r^{-1}s^3t^3 + \tfrac{4}{45} r^{-1}s^3u^5.\end{aligned}$$

In the second line of (6) we may reject the whole of α', since its order is $-\frac{1}{4}$, and write

$$\begin{aligned}&\tfrac{1}{12} r^3\alpha^3 - \tfrac{10}{9} r^2\alpha + s^3 \\ &= \tfrac{2}{81} r^{-3}s^9 + \tfrac{7}{27} s^3.\end{aligned}$$

After substituting their values for the terms in (6) which contain α, and at the same time dividing throughout by $\frac{2}{3}$, we shall obtain

$$\begin{array}{lll}\text{Order} & 1 & 0 = \tfrac{1}{3} r^{-1}s^6 - \tfrac{1}{2} s^4 + \tfrac{1}{6} r^2 \\ \quad " & \tfrac{3}{4} & \quad + \tfrac{1}{27} r^{-3}s^9 - r^{-1}s^5 - \tfrac{1}{3} r^{-1}s^3t^4 + \tfrac{8}{9} s^3 + \tfrac{5}{27} r^{\frac{3}{2}} \\ \quad " & \tfrac{5}{8} & \quad + r^{-1}s^3t^3 + \tfrac{1}{5} t^5 + \tfrac{2^2}{3^2} A r^{\frac{5}{4}} \\ \quad " & \tfrac{9}{16} & \qquad + \tfrac{2}{15} r^{-1}s^3u^5 + \tfrac{2^3}{3^3} B r^{\frac{9}{8}} \\ \quad " & \tfrac{17}{32} & \qquad\qquad + \tfrac{2^4}{3^4} C r^{\frac{17}{16}} \\ \quad " & \tfrac{33}{64} & \qquad\qquad + \tfrac{2^5}{3^5} D r^{\frac{33}{32}} \\ \quad " & \tfrac{65}{128} & \qquad\qquad + \tfrac{2^6}{3^6} E r^{\frac{65}{64}} + \ldots \end{array} \qquad (7)$$

We now write

$$r = s^2(1-\beta) \text{ and } \beta = \tfrac{2}{3} s^{-2} t^3 (1+\beta')$$

where, observing that the values of β and β' can be immediately deduced from those of α and α' by changing $r, s, t, \ldots$ into $s, t, u, \ldots$, it is evident that β and β' are both of the order $-\frac{1}{8}$; for α and α' are both of the order $-\frac{1}{4}$. Thus (neglecting quantities whose order is equal to, or less than, $\frac{1}{2}$) we have

$$\begin{aligned}&\tfrac{1}{3} r^{-1}s^6 - \tfrac{1}{2} s^4 + \tfrac{1}{6} r^2 \\ &= \tfrac{1}{3} s^4(1+\beta+\beta^2+\beta^3) - \tfrac{1}{2} s^4 + \tfrac{1}{6} s^4(1-2\beta+\beta^2) = \tfrac{1}{2} s^4\beta^2 + \tfrac{1}{3} s^4\beta^3 \\ &= \tfrac{2}{9} t^6(1+2\beta') + \tfrac{8}{81} s^{-2} t^9 \\ &= \tfrac{2}{9} t^6 + \tfrac{2}{3} t^3\left(\tfrac{1}{2} s - t^2 - \tfrac{1}{3} u^4 + u^3 + \tfrac{2}{15} v^5\right) + \tfrac{8}{81} s^{-2} t^9 \\ &= \tfrac{2}{9} t^6;\ + \tfrac{1}{3} st^3 - \tfrac{2}{3} t^5 - \tfrac{2}{9} t^3u^4 + \tfrac{8}{81} s^{-2}t^9;\ + \tfrac{2}{3} t^3u^3;\ + \tfrac{4}{45} t^3v^5.\end{aligned}$$

Order $\frac{3}{4}$; $\frac{5}{8}$; $\frac{9}{16}$; $\frac{17}{32}$.

$$\begin{aligned}&\tfrac{1}{27} r^{-3}s^9 - r^{-1}s^5 - \tfrac{1}{3} r^{-1}s^3t^4 + \tfrac{8}{9} s^3 + \tfrac{5}{27} r^{\frac{3}{2}} \\ &= \tfrac{1}{27} s^3(1+3\beta) - s^3(1+\beta) - \tfrac{1}{3} st^4(1+\beta) + \tfrac{8}{9} s^3 + \tfrac{5}{27} s^3\left(1-\tfrac{3}{2}\beta\right) \\ &= \tfrac{1}{9} s^3 - \tfrac{7}{6} s^3\beta - \tfrac{1}{3} st^4(1+\beta) \\ &= \tfrac{1}{9} s^3 - \tfrac{1}{3} st^4;\ - \tfrac{7}{9} st^3 - \tfrac{2}{9} s^{-1}t^7.\end{aligned}$$

Order $\frac{3}{4}$; $\frac{5}{8}$.

$$r^{-1}s^3t^3 + \tfrac{1}{5} t^5 + \tfrac{2^2}{3^2} A r^{\frac{5}{4}} = st^3 + \tfrac{1}{5} t^5 + \tfrac{2^2}{3^2} A s^{\frac{5}{2}},$$
$$\tfrac{2}{15} r^{-1}s^3u^5 + \tfrac{2^3}{3^3} B r^{\frac{9}{8}} = \tfrac{2}{15} su^5 + \tfrac{2^3}{3^3} B s^{\frac{9}{4}},$$

and so on.

Hence (7) becomes

$$\begin{array}{lll}\text{Order} & \tfrac{3}{4} & 0 = \tfrac{2}{9} t^6 - \tfrac{1}{3} st^4 + \tfrac{1}{9} s^3 \\ \quad " & \tfrac{5}{8} & \quad + \tfrac{8}{81} s^{-2}t^9 - \tfrac{7}{15} t^5 - \tfrac{2}{9} t^3u^4 + \tfrac{5}{9} st^3 - \tfrac{2}{9} s^{-1}t^7 + \tfrac{2^2}{3^2} A s^{\frac{5}{2}} \\ \quad " & \tfrac{9}{16} & \quad + \tfrac{2}{3} t^3u^3 + \tfrac{2}{15} su^5 + \tfrac{2^3}{3^3} B s^{\frac{9}{4}} \\ \quad " & \tfrac{17}{32} & \qquad + \tfrac{4}{45} t^3v^5 + \tfrac{2^4}{3^4} C s^{\frac{17}{8}} \\ \quad " & \tfrac{33}{64} & \qquad\qquad + \tfrac{2^5}{3^5} D s^{\frac{33}{16}} \\ \quad " & \tfrac{65}{128} & \qquad\qquad + \tfrac{2^6}{3^6} E s^{\frac{65}{32}} + \ldots. \end{array} \qquad (8)$$

Dividing this throughout by $\frac{2}{3}s$, and then writing

$$s = t^2(1-\gamma) \text{ and } \gamma = \tfrac{2}{3} t^{-2} u^3 (1+\gamma'),$$

we obtain in exactly the same manner as before, merely altering the letters in the previous work,

$$\begin{aligned}&\tfrac{1}{3} s^{-1}t^6 - \tfrac{1}{2} t^4 + \tfrac{1}{6} s^2 \\ &= \tfrac{2}{9} u^6;\ + \tfrac{1}{3} tu^3 - \tfrac{2}{3} u^5 - \tfrac{2}{9} u^3v^4 + \tfrac{8}{81} t^{-2}u^9;\ + \tfrac{2}{3} u^3v^3;\ + \tfrac{4}{45} u^3w^5.\end{aligned}$$

Order $\frac{3}{8}$; $\frac{5}{16}$; $\frac{9}{32}$; $\frac{17}{64}$

where quantities of the order $\frac{1}{4}$, or less, are now neglected.

Similarly $\frac{4}{27} s^{-3}t^9 - \frac{7}{10} s^{-1}t^5 - \frac{1}{3} s^{-1}t^3u^4 + \frac{5}{6} t^3 - \frac{1}{3} s^{-2}t^7 + \frac{2}{3} A s^{\frac{3}{2}}$

$$\begin{aligned}&= \tfrac{4}{27} t^3(1+3\gamma) - \tfrac{7}{10} t^3(1+\gamma) - \tfrac{1}{3} tu^4(1+\gamma) + \tfrac{5}{6} t^3 - \tfrac{1}{3} t^3(1+2\gamma) + \tfrac{2}{3} A t^3\left(1-\tfrac{3}{2}\gamma\right) \\ &= \left(\tfrac{2}{3} A - \tfrac{7}{135}\right) t^3 - \tfrac{1}{3} tu^4 - \left(A + \tfrac{83}{90}\right) t^3\gamma - \tfrac{1}{3} tu^4\gamma \\ &= \left(\tfrac{2}{3} A - \tfrac{7}{135}\right) t^3 - \tfrac{1}{3} tu^4;\ - \left(\tfrac{2}{3} A + \tfrac{83}{135}\right) tu^3 - \tfrac{2}{9} t^{-1}u^7.\end{aligned}$$

Order $\frac{3}{8}$; $\frac{5}{16}$.

$$s^{-1}t^3u^3 + \tfrac{1}{5} u^5 + \tfrac{2^2}{3^2} B s^{\frac{5}{4}} = tu^3 + \tfrac{1}{5} u^5 + \tfrac{2^2}{3^2} B t^{\frac{5}{2}}$$
$$\tfrac{2}{15} s^{-1}t^3v^5 + \tfrac{2^3}{3^3} C s^{\frac{9}{8}} = \tfrac{2}{15} tv^5 + \tfrac{2^3}{3^3} C t^{\frac{9}{4}},$$

and so on.

Thus (8) becomes

$$\begin{array}{lll}\text{Order} & \tfrac{3}{8} & 0 = \tfrac{2}{9} u^6 - \tfrac{1}{3} tu^4 + \left(\tfrac{2}{3} A - \tfrac{7}{135}\right) t^3 \\ \quad " & \tfrac{5}{16} & \quad + \tfrac{8}{81} t^{-2}u^9 - \tfrac{7}{15} u^5 - \tfrac{2}{9} u^3v^4 + \left(\tfrac{97}{135} - \tfrac{2}{3} A\right) tu^3 - \tfrac{2}{9} t^{-1}u^7 + \tfrac{2^2}{3^2} B t^{\frac{5}{2}} \\ \quad " & \tfrac{9}{32} & \quad + \tfrac{2}{3} u^3v^3 + \tfrac{2}{15} tv^5 + \tfrac{2^3}{3^3} C t^{\frac{9}{4}} \\ \quad " & \tfrac{17}{64} & \qquad + \tfrac{4}{45} u^3w^5 + \tfrac{2^4}{3^4} D t^{\frac{17}{8}} \\ \quad " & \tfrac{33}{128} & \qquad\qquad + \tfrac{2^5}{3^5} E t^{\frac{33}{16}} + \ldots. \end{array}$$

Now the terms of the highest order in this equation must vanish when we write $t=u^2$, and therefore $\frac{2}{9}-\frac{1}{3}+\frac{2}{3}A-\frac{7}{135}=0$, which gives $A=\frac{11}{45}$. Substituting this value for A, we find

$$
\begin{array}{lll}
\text{Order} & \frac{3}{8} & 0=\frac{2}{9}u^6-\frac{1}{3}tu^4+\frac{1}{9}t^3 \\
\;\;" & \frac{5}{16} & +\frac{8}{81}t^{-2}u^9-\frac{7}{15}u^5-\frac{2}{9}u^3v^4+\frac{5}{9}tu^3-\frac{2}{9}t^{-1}u^7+\frac{2^2}{3^2}Bt^{\frac{5}{2}} \\
\;\;" & \frac{9}{32} & +\frac{2}{3}u^3v^3+\frac{2}{15}tv^5+\frac{2^3}{3^3}Ct^{\frac{9}{4}} \\
\;\;" & \frac{17}{64} & +\frac{4}{45}u^3w^5+\frac{2^4}{3^4}Dt^{\frac{17}{8}} \\
\;\;" & \frac{33}{128} & +\frac{2^5}{3^5}Et^{\frac{33}{16}}+\ldots,
\end{array}
$$

which is a mere repetition of equation (8), with all the letters moved forward one place. Hence it is evident that, if we treat this equation as we treated (8), we shall find $B=\frac{11}{45}$, arriving, at the same time, at another equation which will be merely a repetition of (8), with all its letters moved forward two places; and this process can be continued as long as we please.

Thus we arrive at the result—

$$A=B=C=D=E=\ldots=\tfrac{11}{45},$$

and the asymptotic development for Hypothenusal Numbers

$$p-q=(q-r)^2+\tfrac{4}{3}(q-r)^{\frac{3}{2}}+\tfrac{11}{18}(q-r)+\tfrac{10}{81}(q-r)^{\frac{1}{2}}$$
$$+\tfrac{11}{45}(q-r)^{\frac{1}{2}}\left\{\tfrac{2^3}{3^3}(q-r)^{\frac{1}{8}}+\tfrac{2^4}{3^4}(q-r)^{\frac{1}{16}}+\tfrac{2^5}{3^5}(q-r)^{\frac{1}{32}}+\ldots\right\}$$

is established.

Comparing this with the corresponding formula for Hamiltonian Numbers,

$$p=q^2-\tfrac{2}{3}q\,(q^{\frac{1}{2}}+q^{\frac{1}{4}}+q^{\frac{1}{8}}+q^{\frac{1}{16}}+\ldots+q^{(\frac{1}{2})^k})+\Xi q,$$

given at the beginning of the third section [p. 570], it will be noticed that each of the two developments begins with an irregular portion consisting respectively of four and one terms, followed by a regular series. In the one case the regular portion is $\frac{11}{45}(q-r)^{\frac{1}{2}}$, multiplied by a series whose general term is $\frac{2^n}{3^n}(q-r)^{(\frac{1}{2})^n}$; in the other it consists of a series of terms of the form $q^{(\frac{1}{2})^n}$ multiplied by $-\frac{2}{3}q$.

[To p. 579, footnote *. The reference is to *Proceedings of the Royal Society*, Vol. 42 (1887), pp. 470, 471, where is printed an Abstract identical with the Introduction to this paper (pp. 553–555 above) save for the insertion after the word "scale" (p. 555 above) of the words "in order to establish or disprove conclusively the presumptive law of the asymptotic branch of the series connecting any two consecutive semi-differences η_x, η_{x+1} of the Hamiltonian Numbers, viz.: —

$$\eta_{x+1}-\eta_x{}^2=\eta_x{}^{\frac{1}{2}}\sum_{r=0}^{\infty}c_r\eta_x{}^{(\frac{1}{2})^r}."$$

There is also a paper, *Proceedings of the Royal Society*, Vol. 44 (1888), pp. 99—101, containing what is here given on p. 579 and the first half of p. 580.]

52.

SUR LES NOMBRES DITS DE HAMILTON.

[*Compte Rendu de l'Assoc. Française (Toulouse)*, 1887, pp. 164—168.]

Considérons ce tableau formé en bas par un procédé qui à peu près s'explique de soi-même :

1	0	0	0	0	0	0	....
	1	1	1	1	1	1	....
		2	3	4	5	6	....
		1	5	9	14	20	....
			6	15	29	49	...
			5	21	50	89	...
			4	26	76	175	...
			3	30	106	231	...
			2	33	139	420	...
			1	35	174	594	...
				36	210	804	...
				.	.	.	...
				.	.	.	...

Ce tableau peut être étendu indéfiniment.

On voit qu'il se divise en étages et que les nombres initiaux des premières lignes de ces étages sont :

1, 1, 2, 6, 36.

En les additionnant et en ajoutant l'unité aux sommes, on obtient les nombres 2, 3, 5, 11, 47 ....

Ces nombres sont ce que j'appelle les nombres de Hamilton qui a trouvé les nombres 11, 47, et encore le nombre qui vient après 47, c'est-à-dire 923, dans un rapport qu'il a publié dans les *Reports of the British Association* 1836, sur la méthode de Jerrard pour réduire les équations du cinquième degré, méthode qui remonte, en effet, à Bring, professeur à Lund, qui l'a publié dans un opuscule en 1786 qui restait inconnu ou oublié. De même qu'on peut ôter 3 termes d'une équation dont le degré est au moins 5 sans résoudre aucune équation d'un degré supérieur à 3, de même aussi on peut ôter 4 termes d'une équation dont le degré est au moins 11 sans résoudre des équations d'un degré supérieur à 4; 5 termes d'une équation dont le degré est au moins 47 sans résoudre des équations d'un degré supérieur à 5 et ainsi de suite.

Mais il est nécessaire d'avertir ici que la même chose aura lieu pour des équations de degrés moindres, en général, que ceux fournis par les nombres de Hamilton. En effet, au lieu de 11, 47, 923 ... on peut substituer 10, 44, 905 ...: mais le système d'équations résolvantes deviendra plus compliqué quand on fait cette diminution du degré minimum. Ainsi, par exemple, il est bien vrai que pour ôter 4 termes à une équation du degré 10, le système d'équations à résoudre ne contiendra nulle équation d'un degré supérieur à 4: mais il y aura 3 équations de ce degré à résoudre tandis que quand l'équation donnée est du degré 11 ou plus haut que 11, on n'aura à résoudre (en combinaison bien entendu avec des équations cubiques quadratiques et linéaires) qu'une seule équation biquadratique au lieu de trois: et ainsi en général.

Pour trouver les nombres de Hamilton, mon coadjuteur, M. Hammond a trouvé une échelle de relation d'une simplicité merveilleuse.

On peut former avec les lignes successives du tableau les fonctions

$$\begin{array}{rll}
1+0x+0x^2+0x^3+0x^4\ldots & \text{disons} & F_0 \text{ (qui en effet est l'unité).} \\
x+x^2+x^3+x^4\ldots & ,, & F_1 \\
2x^2+3x^3+4x^4\ldots & ,, & F_2 \\
x^2+5x^3+9x^4\ldots & ,, & {}^1F_2 \\
6x^3+15x^4\ldots & ,, & {}^2F_2=F_3 \\
5x^3+21x^4\ldots & ,, & {}^1F_3 \\
4x^3+26x^4\ldots & ,, & {}^2F_3 \\
3x^3+30x^4\ldots & ,, & {}^3F_3 \\
2x^3+35x^4\ldots & ,, & {}^4F_3 \\
x^3+35x^5\ldots & ,, & {}^5F_3 \\
36x^5\ldots & ,, & {}^6F_3=F_4
\end{array}$$

et ainsi de suite.

Donnons à 1, 1, 2, 6, 36 ... les noms a_0, a_1, a_2, a_3, a_4 ... alors il est facile à voir qu'en général ${}^nF_n=F_{n+1}$; mais aussi on voit que

$${}^{i+1}F_n=(1-x)^{-1}\,{}^iF_n-x^n.$$

Donc

$$F_{n+1}-(1-x)^{-a_n}.F_n=-x^n\{1+(1-x)^{-1}+(1-x)^{-2}+\ldots+(1-x)^{-a_n+1}\}$$
$$=x^{n-1}\{(1-x)-(1-x)^{-a_n+1}\}.$$

Faisons $a_0+a_1+a_2+\ldots+a_n=S_{n+1}$ alors en multipliant l'équation par $(1-x)^{S_{n+1}}$, on obtient:

$$(1-x)^{S_{n+1}}.F_{n+1}-(1-x)^{S_n}.F_n=x^n(1-x)^{S_{n+1}+1}-x^n(1-x)^{S_{n+1}}.$$

Cette équation qui existe pour toutes les valeurs S_n jusqu'à S_1 exclusif reste vraie comme identité même pour S_0 si on met $S_0=0$. Alors en donnant à n toutes les valeurs depuis $n-1$ jusqu'à 0 inclusivement et en faisant la sommation des équations ainsi formées, on obtient facilement:

$$(1-x)^{S_n}F_n-1+x^{-1}(1-x)-x^{n-1}(1-x)^{S_n+1}$$
$$=x^{n-2}(1-x)^{S_n+2}+x^{n-3}(1-x)^{S_{n-1}+2}+x^{n-n}(1-x)^{S_{n-n}+2}+\ldots.$$

Si dans cette équation on compare les coefficients de x^n en se rappelant que le coefficient de x^n en F_n est a_n, c'est-à-dire $S_{n+1}-S_n$, et que S_n+1 est le nombre n^{me} de M. Hamilton, de sorte que S_n+2 que je nommerai E_n est ce nombre augmenté de l'unité, on trouve:

$$E_{n+1}=1+E_n.\frac{E_n-1}{2}-\frac{E_{n-1}(E_{n-1}-1)(E_{n-1}-2)}{2.3}+\ldots$$

formule de relation entre les nombres de Hamilton qu'on peut écrire sous la forme symétrique

$$1-(E_n)_1+(E_{n-1})_2-(E_{n-2})_3\ldots=0.$$

En augmentant les nombres de Hamilton de l'unité, on obtient pour E les valeurs successives

3, 4, 6, 12, 48, 924

qu'on trouve très facilement par la formule de la relation donnée.

Ainsi par exemple:

$$\begin{array}{lllllll}
\frac{3.2}{2} & & & & = & 4-1= & 3 \\
\frac{4.3}{2} & -\frac{3.2.1}{1.2.3} & & & = & 6-1= & 5 \\
\frac{6.5}{2} & -\frac{4.3.2}{1.2.3} & & & = & 12-1= & 11 \\
\frac{12.11}{2} & -\frac{6.5.4}{1.2.3} & +\frac{4.3.2.1}{1.2.3.4} & & = & 48-1= & 47 \\
\frac{48.47}{2} & -\frac{12.11.10}{1.2.3} & +\frac{6.5.4.3}{1.2.3.4} & & = & 924-1= & 923 \\
\frac{924.923}{2} & -\frac{48.67.44}{2.3} & +\frac{12.11.10.9}{1.2.3.4} & -\frac{6.5.4.3.2}{1.2.3.4.5} & = & 409620-1= & 409619.
\end{array}$$

Les nombres de Hamilton ainsi calculés sont:

2, 3, 5, 11, 47, 923, 409619, 83763206255 ...

où comme première approximation asymptotique on peut remarquer que si u_x est le nombre de rang x, $u_{x+1}\div u_x^2$ devient de plus en plus près de δ mais toujours moindre que l'unité quand x croît indéfiniment.

Telle est la formule bien remarquable trouvée par M. Hammond, dont j'ai un peu simplifié et abrégé la démonstration.

Un travail sur les nombres de Hamilton, fait par M. Hammond et moi-même va prochainement paraître dans les *Philosophical Transactions* [above, p. 553].

53.

NOTE ON A PROPOSED ADDITION TO THE VOCABULARY OF ORDINARY ARITHMETIC*.

[*Nature*, XXXVII. (1888), pp. 152, 153.]

THE total number of distinct primes which divide a given number I call its Manifoldness or Multiplicity.

A number whose Manifoldness is n I call an n-fold number. It may also be called an n-ary number, and for $n = 1, 2, 3, 4, \ldots$ a unitary (or primary), a binary, a ternary, a quaternary, ... number. Its prime divisors I call the *elements* of a number; the highest powers of these elements which divide a number its *components*; the degrees of these powers its *indices*; so that the indices of a number are the totality of the indices of its several components. Thus, we may say, a prime is a one-fold number whose index is unity.

So, too, we may say that all the components but one of an odd perfect number must have even indices, and that the excepted one must have its base and index each of them congruous to 1 to modulus 4.

Again, a remarkable theorem of Euler, contained in a memoir relating to the Divisors of Numbers (*Opuscula Minora*, II. p. 514), may be expressed by saying that *every even perfect number is a two-fold number, one of whose components is a prime, and such that when augmented by unity it becomes a power of* 2, *and double the other component*†.

* Perhaps I may without immodesty lay claim to the appellation of the Mathematical Adam, as I believe that I have given more names (passed into general circulation) to the creatures of the mathematical reason than all the other mathematicians of the age combined.

† It may be well to recall that a perfect number is one which is the half of the sum of its divisors. The converse of the theorem in the text, namely that $2^n(2^{n+1}-1)$, when $2^{n+1}-1$ is a prime, is a perfect number, is enunciated and proved by Euclid in the 36th (the last) proposition of the 9th Book of the "Elements," the second factor being expressed by him as the sum of a geometric series whose first term is unity and the common ratio 2. In Isaac Barrow's English translation, published in 1660, the enunciation is as follows: "If from a unite be taken how many numbers soever 1, A, B, C, D, in double proportion continually, until the whole

Euler's function $\phi(n)$, which means the number of numbers not exceeding n and prime to it, I call the *totient* of n; and in the new nomenclature we may enunciate that *the totient of a number is equal to the product of that number multiplied by the several excesses of unity above the reciprocals of its elements.* The numbers prime to a number and less than it, I call its *totitives*.

Thus we may express Wilson's generalized theorem by saying that any number is contained as a factor in the product of its totitives increased by unity if it is the number 4, or a prime, or the double of a prime, and diminished by unity in every other case.

I am in the habit of representing the totient of n by the symbol τn, τ (taken from the initial of the word it denotes) being a less hackneyed letter than Euler's ϕ, which has no claim to preference over any other letter of the Greek alphabet, but rather the reverse.

It is easy to prove that the half of any perfect number must exceed in magnitude its totient.

Hence, since $\frac{3}{2}\cdot\frac{5}{4}$ is less than 2, it follows that no odd two-fold perfect number exists.

added together E be a prime number; and if this whole E multiplying the last produce a number F, that which is produced F shall be a perfect number."

The direct theorem that every even perfect number is of the above form could probably only have been proved with extreme difficulty, if at all, by the resources of Greek Arithmetic. Euler's proof is not very easy to follow in his own words, but is substantially as follows:

Suppose P (an even perfect number) $= 2^n A$. Then, using in general $\int X$ to denote the sum of the divisors of X,

$$2 = \frac{\int P}{P} = \frac{\int 2^n \cdot \int A}{2^n A} = \frac{2^{n+1}-1}{2^n} \cdot \frac{\int A}{A}.$$

Hence
$$\frac{\int A}{A} = \frac{2^{n+1}}{2^{n+1}-1}, \text{ say} = \frac{Q+1}{Q}.$$

Hence $A = \mu Q$, and $\int A = 1 + \mu + Q + \mu Q + \ldots$ (if μ be supposed > 1). Hence unless $\mu = 1$ and at the same time Q is a prime

$$\int A > \mu (Q+1),$$

that is $\frac{\int A}{A}$ is greater than itself.

Hence an even number P cannot be a perfect number if it is not of the form $2^n(2^{n+1}-1)$, where $2^{n+1}-1$ is a prime, which of course implies that $n+1$ must itself be a prime.

It is remarkable that Euler makes no reference to Euclid in proving his own theorem. It must always stand to the credit of the Greek geometers that they succeeded in discovering a class of perfect numbers which in all probability are the only numbers which are perfect. Reference is made to *so-called* perfect numbers in Plato's "Republic," H, 546 B, and also by Aristotle, Probl. I E 3 and "Metaph." A 5. Mr Margoliouth has pointed out to me that Muhamad Al-Sharastani, in his *Book of Religious and Philosophical Sects*, Careton, 1856, p. 267 of the Arabic text, assigns reasons for regarding all the numbers up to 10 inclusive as perfect numbers, which he attributes to Pythagoras, but which are purely fanciful and entitled to no more serious consideration than the late Dr Cummings's ingenious speculations on the number of the Beast. My particular attention was called to perfect numbers by a letter from Mr Christie, dated from "Carlton, Selby," containing some inquiries relative to the subject.

Similarly, the fact of $\frac{3}{2} \cdot \frac{7}{6} \cdot \frac{11}{10}$ being less than 2 is sufficient to show that 3, 5 must be the two least elements of any three-fold perfect number; furthermore, $\frac{3}{2} \cdot \frac{5}{4} \cdot \frac{17}{16}$ being less than 2, shows that 11 or 13 must be the third element of any such number if it exists*—each of which hypotheses admits of an easy disproof. But to disprove the existence of a four-fold perfect number by my actual method makes a somewhat long and intricate, but still highly interesting, investigation of a multitude of special cases. I hope, *numine favente*, sooner or later to discover a general principle which may serve as a key to a universal proof of the non-existence of any other than the Euclidean perfect numbers, for a prolonged meditation on the subject has satisfied me that the existence of any one such—its escape, so to say, from the complex web of conditions which hem it in on all sides—would be little short of a miracle. Thus then there seems every reason to believe that Euclid's perfect numbers are the only perfect numbers which exist!

In the higher theory of congruences (see Serret's *Cours d'Algèbre Supérieure*) there is frequent occasion to speak of "a number n which does not contain any prime factor other than those which are contained in another number M."

In the new nomenclature n would be defined as *a number whose elements are all of them elements of M*.

As τN is used to denote the totient of N, so we may use μN to denote its multiplicity, and then a well-known theorem in congruences may be expressed as follows.

The number of solutions of the congruence

$$x^2 - 1 \equiv 0 \pmod{P}$$

is $2^{\mu P}$ if P is odd,

$2^{\mu P - 1}$ if P is the double of an odd number,

$2^{\mu P}$ if P is the quadruple of an odd number,

and $2^{\mu P + 1}$ in every other case.

In the memoir above referred to, Euler says that no one has demonstrated whether or not any odd perfect numbers exist. I have found a method for determining what (if any) odd perfect numbers exist of any specified order of manifoldness. Thus, for example, I have proved that there exist no perfect odd numbers of the 1st, 2nd, 3rd, or 4th orders of manifold-

* 3, 5, 7 can never co-exist as elements in *any* perfect number as shown by the fact that $\frac{1+3+3^2}{9} \cdot \frac{1+5}{5} \cdot \frac{1+7+49}{49}$, that is $\frac{26}{15}\left(1+\frac{1}{7}+\frac{1}{49}\right)$, is *greater* than 2. Thus we see that no perfect number can be a multiple of 105. So again the fact that $\frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{18}$ is *less* than 2 is sufficient to prove that any odd perfect number of multiplicity less than 7 must be divisible by 3.

ness, or in other words, no odd primary, binary, ternary, or quaternary number can be a perfect number. Had any such existed, my method must infallibly have dragged each of them to light*.

In connection with the theory of perfect numbers I have found it useful to denote $p^i - 1$ when p and i are left general as the Fermatian function, and when p and i have specific values as the ith Fermatian of p. In such case p may be called the base, and i the index of the Fermatian.

Then we may express Fermat's theorem by saying [cf. p. 625 below] that *either the Fermatian itself whose index is one unit below a given prime or else its base must be divisible by that prime*†.

It is also convenient to speak of a Fermatian divided by the excess of its base above unity as a Reduced Fermatian and of that excess itself as the Reducing Factor.

The spirit of my actual method of disproving the existence of odd perfect numbers consists in showing that an n-fold perfect number must have more than n elements, which is absurd. The chief instruments of the investigation are the two inequalities to which the elements of any perfect number must be subject and the properties of the prime divisors of a Reduced Fermatian with an odd prime index.

* I have, since the above was in print, extended the proof to quinary numbers, and anticipate no difficulty in doing so for numbers of higher degrees of multiplicity, so that it is to be hoped that the way is now paved towards obtaining a general proof of this *palmary* theorem.

† So too we may state the important theorem that if *an element of a Fermatian is its index the component which has that index for its base must be its square.*

54.

ON CERTAIN INEQUALITIES RELATING TO PRIME NUMBERS.

[*Nature*, XXXVIII. (1888), pp. 259—262.]

I SHALL begin with a method of proving that the number of prime numbers is infinite, which is not new, but which it is worth while to recall as an introduction to a similar method, by series, which will subsequently be employed in order to prove that the number of primes of the form $4n+3$, as also of the form $6n+5$, is infinite.

It is obvious that the reciprocal of the product

$$\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\left(1-\frac{1}{p_3}\right)\ldots\left(1-\frac{1}{p_{N.p}}\right)$$

(where p_i means the ith in the natural succession of primes, and $p_{N.p}$ means the highest prime number not exceeding N)* will be equal to

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots+\frac{1}{N}+R,$$

and therefore greater than $\log N$ (R consisting exclusively of positive terms).

Hence $$\left(1+\frac{1}{p_1}\right)\left(1+\frac{1}{p_2}\right)\ldots\left(1+\frac{1}{p_{N.p}}\right)>M\log N,$$

where $$M=\left(1-\frac{1}{p_1^2}\right)\left(1-\frac{1}{p_2^2}\right)\ldots\left(1-\frac{1}{p_{N.p}^2}\right),$$

and is therefore greater than $\frac{2}{\pi}$.

Hence the number of terms in the product must increase indefinitely with N.

By taking the logarithms of both sides we obtain the inequality

$$S_1-\frac{1}{2}S_2+\frac{1}{3}S_3-\frac{1}{4}S_4+\ldots>\log\log N+\log M,$$

* $N.p$ itself of course denotes in the above notation the number of primes (p) not exceeding N.

where in general S_i means the sum of inverse ith powers of all the primes not exceeding N; and accordingly is finite, except when $i=1$, for any value of N. We have therefore

$$S_1>\log\log N+\text{Const.}$$

The actual value of S_1 is observed to differ only by a limited quantity from the second logarithm of N, but I am not aware whether this has ever been strictly proved.

Legendre has found that for large values of N

$$\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\ldots\left(1-\frac{1}{p_{N.p}}\right)=\frac{1\cdot104}{\log N}.$$

Consequently

$$\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\ldots\left(1-\frac{1}{p_{N.p}}\right)=\frac{\cdot552}{\log N}.$$

This would show that the value of our R bears a finite ratio to $\log N$; calling it $\theta\log N$ we obtain, according to Legendre's formula,

$$\frac{1}{1+\theta}=\cdot552,\text{ which gives }\theta=\cdot811,$$

so that the nebulous matter, so to say, in the expansion of the reciprocal of the product of the differences between unity and the reciprocals of all the primes not exceeding a given number, stands in the relation of about 4 to 5 to the condensed portion consisting of the reciprocals of the natural numbers.

I will now proceed to establish similar inequalities relating to prime numbers of the respective forms $4n+3$ and $6n+5$.

Beginning with the case $4n+3$, I shall use q_j to signify the jth in the natural succession of primes of the form $4n+3$, and $q_{N.q}$ to signify the highest q not exceeding N, $N.q$ itself signifying the number of q's not exceeding N.

Let us first, without any reference to convergence, consider the product obtained by the usual mode of multiplication of the infinite series

$$S=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\ldots\textit{ ad inf.}$$

by the product

$$\frac{1}{1-\frac{1}{2}}\cdot\frac{1+\frac{1}{q_1}}{1-\frac{1}{q_1}}\cdot\frac{1+\frac{1}{q_2}}{1-\frac{1}{q_2}}\cdot\frac{1+\frac{1}{q_3}}{1-\frac{1}{q_3}}\ldots\textit{ ad inf.}$$

It is clear that the effect of the multiplication of S by the numerator of the above product will be to deprive the series S of all its negative terms. Then the effect of dividing by the denominator of the product, with the exception of the factor $1-\frac{1}{2}$, will be to restore all the obliterated terms, but with the sign + instead of −. Lastly, the effect of multiplying by the reciprocal of $(1-\frac{1}{2})$ will be to supply the even numbers that were wanting in the denominators of the terms of S, and we shall thus get the indefinite series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots \textit{ ad inf.}$$

Call now

$$Q_N=\frac{1}{1-\frac{1}{2}}\cdot\frac{1+\frac{1}{q_1}}{1-\frac{1}{q_1}}\cdot\frac{1+\frac{1}{q_2}}{1-\frac{1}{q_2}}\cdots\frac{1+\frac{1}{q_{N.q}}}{1-\frac{1}{q_{N.q}}}.$$

Q_N, which is finite when N is finite, may be expanded into an infinite aggregate of positive terms, found by multiplying together the series

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots$$

$$1+\frac{2}{q_1}+\frac{2}{q_1^2}+\frac{2}{q_1^3}+\dots$$

$$1+\frac{2}{q_2}+\frac{2}{q_2^2}+\frac{2}{q_2^3}+\dots$$

....

....

$$1+\frac{2}{q_{N.q}}+\frac{2}{q_{N.q}^2}+\frac{2}{q_{N.q}^3}+\dots.$$

Let $$S_N=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\dots\pm\frac{1}{N},$$

then from what has been said it is obvious that we may write

$$Q_NS_N=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{N}+V-R,$$

where V and R may be constructed according to the following rule: Let the denominator of any term in the aggregate Q_N be called t, and let θ be the smallest odd number which, multiplied by t, makes $t\theta$ greater than N; then if θ is of the form $4n+1$ it will contribute to V a portion represented by the product of the term by some portion of the series S_N of the form

$$\frac{1}{\theta}-\frac{1}{\theta+2}+\frac{1}{\theta+4}-\dots$$

and if θ is of the form $4n+3$ it will contribute to $-R$ a portion equal to the term multiplied by a series of the form

$$-\frac{1}{\theta}+\frac{1}{\theta+2}-\frac{1}{\theta+4}+\dots.$$

Hence R is made up of the sum of products of portions of the aggregate Q_N multiplied respectively by the series

$$\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\frac{1}{9}+\frac{1}{11}-\frac{1}{13}+\dots$$

$$\frac{1}{7}-\frac{1}{9}+\frac{1}{11}-\frac{1}{13}+\dots$$

$$\frac{1}{11}-\frac{1}{13}+\dots$$

of which the greatest is obviously the first, whose value is $1-S_N$.

Consequently R must be less than the total aggregate Q_N multiplied by $1-S_N$.

Therefore

$$Q_NS_N+Q_N(1-S_N)>1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{N}>\log N,$$

that is, $$Q_N>\log N,$$

from which it follows that when N increases indefinitely the number of factors in Q_N also increases indefinitely, and there must therefore be an infinite number of primes of the form $4n+3$.

Denoting by M_N the quantity

$$\left(1-\frac{1}{q_1^2}\right)\left(1-\frac{1}{q_2^2}\right)\dots\left(1-\frac{1}{q_{N.q}^2}\right)$$

we obtain the inequality

$$\left(1+\frac{1}{q_1}\right)^2\left(1+\frac{1}{q_2}\right)^2\dots\left(1+\frac{1}{q_{N.q}}\right)^2>\frac{1}{2}M_N\log N,$$

and taking the logarithms of both sides

$$\Sigma_1-\frac{1}{2}\Sigma_2+\frac{1}{3}\Sigma_3-\dots>\frac{1}{2}\log\log N+\frac{1}{2}\log M_N-\frac{1}{2}\log 2,$$

where in general Σ_i denotes the sum of the ith powers of the reciprocals of all prime numbers of the form $4n+3$ not surpassing N.

Hence it follows that $\Sigma_1>\frac{1}{2}\log\log N$.

If we could determine the ultimate ratio of the sum of those terms of Q_N whose denominators are greater than N to the total aggregate, and should find that μ, the limiting value of this ratio, is not unity, then the method employed to find an inferior limit would enable us also to find a superior limit to Q_N; for we should have $V<\mu Q_N$ added to the sum of portions of what remains of the aggregate when μQ_N is taken from it multiplied respectively by the several series

$$\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\dots \textit{ ad inf.}$$
$$\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\dots \textit{ ad inf.}$$
$$\frac{1}{13}-\frac{1}{15}+\dots \textit{ ad inf.}$$

the total value of the sum of which products would evidently be less than

$$(1-\mu)\left(S-1+\frac{1}{3}\right)Q_N.$$

Hence the total value of V would be less than

$$\mu Q_N S+(1-\mu)\,Q_N\left(S-\frac{2}{3}\right),$$

that is, less than $$Q_N S-\frac{2}{3}(1-\mu)\,Q_N,$$

and consequently we should have

$$\frac{2}{3}(1-\mu)\,Q_N<\log N,$$

that is $$Q_N<\frac{3}{2(1-\mu)}\log N.$$

From which we may draw the important conclusion that if μ is less than 1, that is, if when N is infinite the portion of the aggregate $S_N Q_N$ comprising the terms whose denominators exceed N does not become infinitely greater than the remaining portion, the sum of the reciprocals of all the prime numbers of the form $4n+3$ not exceeding N would differ by a limited quantity from half the second logarithm of N.

A precisely similar treatment may be applied to prime numbers of the form $6n+5$. We begin with making

$$S_N=1-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}+\frac{1}{13}-\frac{1}{17}+\frac{1}{19}-\dots.$$

We write

$$Q_N=\frac{1}{1-\frac{1}{2}}\cdot\frac{1}{1-\frac{1}{3}}\cdot\frac{1+\frac{1}{r_1}}{1-\frac{1}{r_1}}\cdot\frac{1+\frac{1}{r_2}}{1-\frac{1}{r_2}}\cdots\frac{1+\frac{1}{r_{N.r}}}{1-\frac{1}{r_{N.r}}}.$$

We make $$Q_N S_N=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{N}+V-R.$$

We prove as before that $$R<(1-S)\,Q_N,$$

and thus obtain $$Q_N>\log N,$$

and then putting $$M_N=\left(1-\frac{1}{r_1^2}\right)\left(1-\frac{1}{r_2^2}\right)\dots\left(1-\frac{1}{r_{N.r}^2}\right),$$

and finally noticing that $$\frac{1}{1-\frac{1}{2}}\cdot\frac{1}{1-\frac{1}{3}}=3,$$

we obtain $$\left(1+\frac{1}{r_1}\right)^2\left(1+\frac{1}{r_2}\right)^2\dots\left(1+\frac{1}{r_{N.r}}\right)^2>\frac{1}{3}M_N\log N.$$

Taking the logarithms of both sides of the equation, we find

$$\Theta_1-\frac{1}{2}\Theta_2+\frac{1}{3}\Theta_3-\dots>\frac{1}{2}\log\log N+\frac{1}{2}\log M_N-\frac{1}{2}\log 3,$$

where Θ_i means the sum of ith powers of the reciprocals of all the prime numbers, not exceeding N, of the form $6n+5$.

Either from this equation or from the one from which it is derived it at once follows that the number of primes of the form $6n+5$ is greater than any assignable limit.

Parallel to what has been shown in the preceding case, if it could be ascertained that the sum of the terms of the aggregate Q_N whose denominators do not exceed N bears a ratio which becomes indefinitely small to the total aggregate, it would follow by strict demonstration that the sum of the reciprocals of the primes of the form $6n+5$ inferior to N would always differ by a limited quantity from the half of the second logarithm of N.

It is perhaps worthy of remark that the infinitude of primes of the forms $4n+3$ and $6n+5$ may be regarded as a simple rider to Euclid's proof (Book IX., Prop. 20) of the infinitude of the number of primes in general.

The point of this is somewhat blunted in the way in which it is presented in our ordinary text-books on arithmetic and algebra.

What Euclid gives is something more than this*: his statement is, "There are more prime numbers than any proposed multitude (πλῆθος) of prime numbers"; which he establishes by giving a formula for finding at least one more than any proposed number. He does not say, as our text-book writers do, "if possible let $A, B, \dots C$ be *all* the prime numbers," &c., but simply that if $A, B, \dots C$ are *any* proposed prime numbers, one or more additional ones may be found by adding unity to their product which will either itself be a prime number, or contain at least one additional prime; which is all that can correctly be said, inasmuch as the augmented product may be the power of a prime.

* Whereas the English elementary book writers content themselves with showing that to suppose the number of primes finite involves an absurdity, Euclid shows how from any given prime or primes to generate an infinite succession of primes.

Thus from one prime number arbitrarily chosen, a progression may be instituted in which one new prime number at least is gained at each step, and so an indefinite number may be found by Euclid's formula: for example, 17 gives birth to 2 and 3; 2, 3, 17 to 103; 2, 3, 17, 103 to 7, 19, 79; and so on.

We may vary Euclid's mode of generation and avoid the transcendental process of decomposing a number into its prime factors by using the more general formula, $a, b, \ldots c+1$, where $a, b, \ldots c$, are any numbers relatively prime to each other; for this formula will obviously be a prime number or contain one or more distinct factors relatively prime to $a, b, \ldots c$.

The effect of this process will be to generate a continued series of numbers all of which remain prime to each other: if we form the progression

$$a,\ a+1,\ a^2+a+1,\ a(a+1)(a^2+a+1)+1, \ldots$$

and call these successive numbers

$$u_1, u_2, u_3, u_4, \ldots$$

we shall obviously have $\quad u_{x+1} = u_x^2 - u_x + 1.$

It follows at once from Euclid's point of view that no primes contained in any term up to u_x can appear in u_{x+1}, so that all the terms must be relatively prime to each other. The same consequence follows *a posteriori* from the scale of relation above given; for, as I had occasion to observe in the *Comptes Rendus* for April 1888 [see p. 620, below], if dealing only with rational integer polynomials,

$$\phi(x) = (x-a) f(x) + a,$$

then, whatever value, c, we give to x, no two forms $\phi^i(c)$, $\phi^j(c)$ can have any common measure not contained in a: in this case $\phi(x) = (x-1)x+1$; so that $\phi^i(c)$ and $\phi^j(c)$ must be relative primes for all values of i and j*.

It is worthy of remark that all the primes, other than 3, implicitly obtained by this process will be of the form $6i+1$.

Euclid's own process, or the modified and less transcendental one, may be applied in like manner to obtain a continual succession of primes of the form $4n+3$ and $6n+5$.

As regards the former, we may use the formula

$$2 . a . b \ldots c + 1$$

(where $a, b, \ldots c$ are any "proposed" primes of the form $4n+3$), which will necessarily be of the form $4n+3$, and must therefore contain *one* factor at least of that form.

* Another theorem of a similar kind is that, whatever integer polynomial $\phi(x)$ may be, if i, j have for their greatest common measure k, then $\phi^k[\phi(0)]$ will be the greatest common measure of $\phi^i[\phi(0)]$, $\phi^j[\phi(0)]$.

As regards the latter, we may employ the formula

$$3 . a . b \ldots c + 2$$

(where $a, b, \ldots c$ are each of the form $6n+5$), which will necessarily itself be, and therefore contain *one* factor at least, of that form.

The scale of relation in the first of these cases will be, as before,

$$u_{x+1} = u_x^2 - u_x + 1;$$

so that each term in the progression, abstracting 3, will be of the form $4i+3$ and $6j+1$ conjointly, and consequently of the form $12n+7$; as for example,

$$3,\ 7,\ 43,\ 1807, \ldots.$$

In the latter case the scale of relation is

$$u_{x+1} = u_x^2 - 2u_x + 2,$$

which is of the form $(u_x - 2)u_x + 2$. It is obvious that in each progression at each step one new prime will be generated, and thus the number of ascertained primes of the given form go on indefinitely increasing, as also might be deduced *a posteriori* by aid of the general formula above referred to from the scale of relation applicable to each. Each term in the second case (the term 3, if it appears, excepted) will be simultaneously of the form $6i-1$ and $4j+1$, and consequently of the form $12n+5$, as in the example 5, 17, 257, 65537, ...

The same simple considerations *cease* to apply to the genesis of primes of the forms $4n+1$, $6n+1$. We may indeed apply to them the formulae

$$(2 . a . b \ldots c)^2 + 1 \quad \text{and} \quad 3(a . b \ldots c)^2 + 1$$

respectively, but then we have to draw upon the theory of quadratic forms in order to learn that their divisors are of the form $4n+1$ and $6n+1$ respectively.

Of course the difference in their favour is that in their case *all* the divisors locked up in the successive terms of the two progressions respectively are of the prescribed form; whereas in the other two progressions, whose theory admits of so much simpler treatment, we can only be assured of the presence of *one* such factor in each of the several terms.

Euler has given the values of two infinite products, without any evidence of their truth except such as according to the lax method of dealing with series without regard to the laws of convergence prevalent in his day, and still held in honour in Cambridge down to the times of Peacock, De Morgan, and Herschel inclusive (and this long after Abel had justly denounced the use of divergent series as a crime against reason), was erroneously supposed to amount to a proof, from which the same consequences may be derived as shown in the foregoing pages, and something more besides*. These two theorems are

$$(1) \qquad \frac{3}{3+1}\cdot\frac{5}{5-1}\cdot\frac{7}{7+1}\cdot\frac{11}{11+1}\cdot\frac{13}{13-1}\cdots=\frac{\pi}{4}$$

(where, corresponding to the primes 3, 7, 11, ... of the form $4n+3$, the factors of the product on the left are

$$\frac{3}{3+1},\ \frac{7}{7+1},\ \frac{11}{11+1},\ \cdots$$

all of them with the sign + in the denominator; while the fractions corresponding to primes of the form $4n+1$ have the − sign in their denominators).

$$(2) \qquad \frac{5}{5+1}\cdot\frac{7}{7-1}\cdot\frac{11}{11+1}\cdot\frac{13}{13-1}\cdot\frac{17}{17+1}\cdots=\frac{\pi}{2}\sqrt{3}$$

where, as in the previous product, the sign in the denominator of each fraction depends on the form of the prime to which it corresponds (being + for primes of the form $6n-1$, and − for primes of the form $6n+1$).

Dr J. P. Gram (*Mémoires de l'Académie Royale de Copenhague*, 6me série, Vol. II. p. 191) refers to a paper by Mertens ("Ein Beitrag zur analytischen Zahlentheorie," *Borchardt's Journal*, Bd 78), as one in which the truth of the first of the two theorems is demonstrated—"fuldstœndigt Bevis af Mertens" are Gram's words†.

* It follows from the first of these theorems that with the understanding that no denominator is to exceed n (an indefinitely great number),

$$\left(1+\frac{1}{3}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1+\frac{1}{19}\right)\cdots$$

bears a finite ratio to

$$\left(1+\frac{1}{5}\right)\left(1+\frac{1}{13}\right)\left(1+\frac{1}{17}\right)\cdots$$

so that as their *product* is known to be infinite, each of these two partial products must be separately infinite; in like manner from Euler's second theorem a similar conclusion may be inferred in regard to each of the two products

$$\left(1+\frac{1}{5}\right)\left(1+\frac{1}{11}\right)\left(1+\frac{1}{17}\right)\left(1+\frac{1}{23}\right)\left(1+\frac{1}{29}\right)\left(1+\frac{1}{41}\right)\cdots$$

and

$$\left(1+\frac{1}{7}\right)\left(1+\frac{1}{13}\right)\left(1+\frac{1}{19}\right)\left(1+\frac{1}{31}\right)\cdots.$$

† It always seems to me absurd to speak of a complete proof, or of a theorem being rigorously demonstrated. An incomplete proof is no proof, and a mathematical truth not rigorously demonstrated is not demonstrated at all. I do not mean to deny that there are mathematical truths, morally certain, which defy and will probably to the end of time continue to defy proof, as, for example, that every indecomposable integer polynomial function must represent an infinitude of primes. I have sometimes thought that the profound mystery which envelops our conceptions relative to prime numbers depends upon the limitation of our faculties in regard to time, which like space may be in its essence poly-dimensional, and that this and such sort of truths would become self-evident to a being whose mode of perception is according to *superficially* as distinguished from our own limitation to *linearly* extended time.

Assuming this to be the case, we shall easily find when N is indefinitely great, so that S_N becomes $\frac{\pi}{4}$,

$$Q_N S_N = \frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{N}\right)},$$

which, according to Legendre's empirical law (Legendre, *Théorie des Nombres*, 3rd edition, Vol. II. p. 67, Art. 397), is equal to $\frac{2\log N}{K}$, where $K=1{\cdot}104$; and as we have written $Q_N S_N = \log N + (V-R)$, we may deduce, upon the above assumptions,

$$V-R=\left(\frac{2}{K}-1\right)\log N = 0{\cdot}811\ldots\log N.$$

R, we know, is demonstrably less than $\left(1-\frac{\pi}{4}\right)\log N$, consequently V must be less than $(0{\cdot}812+0{\cdot}215)\log N$, that is, less than $1{\cdot}027\log N$, and *a fortiori* the portion of the omnipositive aggregate Q_N, which consists of terms whose denominators exceed N, when N is indefinitely great, cannot be less than $\frac{4}{\pi}\left(1-\frac{\pi}{4}\right)\log N$, that is, $0{\cdot}273\log N$.

Before concluding, let me add a word on Legendre's empirical formula for the value of

$$\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\cdots\left(1-\frac{1}{p_{N.p}}\right),$$

referred to in the early part of this article.

If N is any odd number, the condition of its being a prime number is that when divided by any odd prime less than its own square root, it shall not leave a remainder zero. Now if N (an unknown odd number) is divided by p, its remainder is equally likely to be 0, 1, 2, 3, ... or $(p-1)$. Hence the chance that it is not divisible by p is $\left(1-\frac{1}{p}\right)$, and, if we were at liberty to regard the like thing happening or not for any two values of p within the stated limit as independent events, the expectation of N being a prime number would be represented by

$$\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\cdots\left(1-\frac{1}{p_{N^{\frac{1}{2}}.p}}\right),$$

which, according to the formula referred to, for infinitely large values of N is equal to $\frac{1{\cdot}104}{\log N^{\frac{1}{2}}}$. It is rather more convenient to regard N as entirely unknown instead of being given as odd, on which supposition the chance of its being a prime would be $\frac{1{\cdot}104}{2\log N^{\frac{1}{2}}}$ or $\frac{1{\cdot}104}{\log N}$.

Hence for very large values of N the sum of the logarithms of all the primes inferior to N might be expected to be something like $(1\cdot104)\,N$. This does not contravene Tchebycheff's formula (Serret, *Cours d'Algèbre Supérieure*, 4me ed., Vol. II. p. 233), which gives for the limits of this sum AN and BN, where $A = 0\cdot921292$, and $B = \frac{6A}{5} = 1\cdot10555$; but does contravene the narrower limits given by my advance upon Tchebycheff's method [see Vol. III. of this Reprint, p. 530], according to which for A, B, we may write A_1, B_1, where

$$A_1 = 0\cdot921423, \quad B_1 = 1\cdot076577\,^*.$$

That the method of probabilities *may* sometimes be successfully applied to questions concerning prime numbers I have shown reason for believing in the two tables published by me [above, p. 101] in the *Philosophical Magazine* for 1883†.

* Namely $A_1 = \frac{51072}{50999}A$, and $B_1 = \frac{59595}{50999}A$, the values of which are incorrectly stated in the memoir. Strange to say, Dr Gram, in his prize essay, previously quoted, on the number of prime numbers under a given limit, has omitted all reference to this paper in his bibliographical summary of the subject, which is only to be accounted for by its having escaped his notice; a narrowing of the asymptotic limits assigned to the sum of the logarithms of the prime numbers series being the most notable fact in the history of the subject since the publication of Tchebycheff's memoir. Subjectively, this paper has a peculiar claim upon the regard of its author, for it was his meditation upon the two simultaneous difference-equations which occur in it that formed the starting-point, or incunabulum, of that new and boundless world of thought to which he has given the name of Universal Algebra. But, apart from this, that the superior limit given by Tchebycheff as 1·1055 should be brought down by a more stringent solution of his own inequalities to only 1·076577—in other words, that the excess above the probable mean value (unity) should be reduced to little more than $\frac{2}{3}$rds of its original amount—is in itself a surprising fact. Perhaps the numerous (or innumerable) misprints and arithmetical miscalculations which disfigure the paper may help to account for the singular neglect which it has experienced. It will be noticed that the mean of the limits of Tchebycheff is 1·01342, the mean of the new limits being 0·99900. The excess in the one case above and the defect in the other below the probable true mean are respectively 0·01342 and 0·00100.

† A principle precisely similar to that employed above if applied to determining the number of reduced proper fractions whose denominators do not exceed a given number n, leads to a correct result. The expectation of two numbers being prime to each other will be the product of the expectations of their not being each divisible by any the same prime number. But the probability of one of them being divisible by i is $\frac{1}{i}$, and therefore of two of them being not each divisible by i is $\frac{1}{i^2}$. Hence the probability of their having no common factor is

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{49}\right)\left(1-\frac{1}{121}\right)\dots \textit{ad inf.}, \text{ that is, is } \frac{6}{\pi^2}.$$

If, then, we take two sets of numbers, each limited to n, the probable number of relatively prime combinations of each of one set with each of the other should be $\frac{6n^2}{\pi^2}$, and the number of reduced proper fractions whose denominators do not exceed n should be the half of this or $\frac{3n^2}{\pi^2}$. I believe M. Césaro has claimed the prior publication of this mode of reasoning, to which he is heartily welcome. The number of these fractions is the same thing as the sum of the *totients* of all

numbers not exceeding n. In the *Philosophical Magazine* for 1883 (Vol. xv. p. 251), a table of these sums of totients has been published by me for all values of n not exceeding 500, and [above, p. 101] in the same year (Vol. xvi. p. 231) the table was extended to values of n not exceeding 1000. In every case without any exception the estimated value of this totient sum is found to be intermediate between

$$\frac{3n^2}{\pi^2} \text{ and } \frac{3(n+1)^2}{\pi^2}.$$

Calling the totient sum to n, $T(n)$, I stated the exact equation

$$T(n) + T\left(\frac{n}{2}\right) + T\left(\frac{n}{3}\right) + T\left(\frac{n}{4}\right) + \dots = \frac{n^2+n}{2}.$$

from which it is capable of proof, without making any assumption as to the form of Tn, that its asymptotic value is $\frac{3n^2}{\pi^2}$. The functional equation itself is merely an integration (so to say) of the well-known theorem that any number is equal to the sum of the totients of its several divisors. The introduction to these tables will be found very suggestive, and besides contains an interesting bibliography of the subject of Farey series (*suites de Farey*), comprising, among other writers upon it, the names of Cauchy, Glaisher, and Sir G. Airy, the last-named as author of a paper on toothed wheels, published, I believe, in the "Selected Papers" of the Institute of Mechanical Engineers. The last word on the subject, as far as I am aware, forms one of the *interludes*, or rather the *postscript*, to my "Constructive Theory of Partitions," published in the *American Journal of Mathematics* [above, p. 55].

55.

SUR LES NOMBRES PARFAITS.

[*Comptes Rendus*, CVI. (1888), pp. 403—405.]*

Existe-t-il des nombres parfaits impairs? C'est une question qui reste indécise.

Dans un article intéressant de M. Servais, paru dans le journal *Mathesis* en octobre 1887, on trouve cette proposition qu'*un nombre parfait* (*s'il y en a*) *qui ne contient que trois facteurs premiers distincts est nécessairement divisible par* 3 *et* 5. Je vais démontrer ici qu'un tel nombre n'existe pas, au moyen d'un genre de raisonnement qui m'a fourni aussi une démonstration de ce théorème qu'*il n'existe pas de nombre parfait qui contienne moins de six facteurs premiers distincts.*

On voit facilement que la somme de la série géométrique

$$1 + c + c^2 + \ldots + c^i,$$

où c est impair, sera elle-même paire quand i est impair; de plus, quand i est pair, cette somme sera toujours paire, mais impairement paire seulement dans le cas où $c \equiv i \equiv 1 \pmod 4$.

Donc, si un nombre parfait impair est de la forme $p^i q^j r^k \ldots$, ($p, q, r, \ldots$ étant des nombres premiers distincts), tous les indices $i, j, k, \ldots$ doivent être pairs à l'exception d'un seul, soit i, lequel, de même que sa base p, sera congru à 1 par rapport au module 4; car on doit avoir

$$\int p^i \int q^j \int r^k \ldots = 2 p^i q^j r^k \ldots,$$

$\int x^i$ représentant $1 + x + \ldots + x^i$, c'est-à-dire $\dfrac{x^{i+1} - 1}{x - 1}$.

Ainsi, on voit qu'un nombre parfait impair (si un tel nombre existe) sera de la forme

$$M^2 (4q+1)^{4k+1},$$

$4q+1$ étant un nombre premier qui ne divise pas M.

[* See also below, p. 615.]

Comme corollaire, on peut déduire qu'aucun nombre parfait impair ne peut être divisible par 105; en effet, soit un tel nombre

$$3^{2i} 5^x 7^{2k} \ldots,$$

on aura $$\frac{\int 3^{2i} \int 5^x \int 7^{2k}}{3^{2i} 5^x 7^{2k}} \geqq \left(1 + \frac{1}{3} + \frac{1}{3^2}\right)\left(1 + \frac{1}{5}\right)\left(1 + \frac{1}{7} + \frac{1}{7^2}\right);$$

c'est-à-dire $\geqq \dfrac{2 \,.\, 13 \,.\, 19}{5 \,.\, 49}$, c'est-à-dire $\dfrac{494}{245}$; qui est plus grand que 2.

Remarquons qu'en général, si $p^i q^j r^k \ldots$ est un nombre parfait, il fait que $\dfrac{p^{i+1}}{p^i (p-1)} \dfrac{q^{j+1}}{q^j (q-1)} \ldots$, c'est-à-dire $\dfrac{p}{p-1} \dfrac{q}{q-1} \dfrac{r}{r-1} \ldots$, soit plus grand que 2.

Ainsi, à moins que le plus petit des éléments $p, q, r, \ldots$ ne soit plus grand que 3, on doit avoir

$$\frac{5}{4} \frac{7}{6} \frac{11}{10} \frac{13}{12} \frac{17}{16} \frac{19}{18} \ldots > 2;$$

mais en ne dépassant pas 19, ce produit est moindre que 1,94963. Conséquemment le nombre des éléments, dans ce cas, doit être 7, au moins. Puisque $1{,}95 \times \left(1 + \frac{1}{40}\right) < 2$, on voit immédiatement que, si un nombre parfait à 7 éléments parmi lesquels 3 ne figurent pas existe, le septième élément ne pourrait pas dépasser 37.

Passons au cas de 3 éléments 3, q, r d'un nombre parfait impair. Puisque $\dfrac{3}{2} \dfrac{7}{6} \dfrac{11}{10} = \dfrac{231}{120} < 2$, on voit que $3^i 7^j 11^k$, et à plus forte raison $3^i p^j q^k$, où p, q sont des nombres quelconques autres que 3 ou 5, ne peut être un nombre parfait.

Supposons donc que 3, 5, q sont les éléments d'un nombre parfait: puisque $\dfrac{3}{2} \dfrac{5}{4} \dfrac{17}{16} = \dfrac{255}{128} < 2$, on voit que q ne peut être ni 17, ni un nombre quelconque plus grand que 17. Donc $q = 11$ ou $q = 13$; car nous avons vu que 3, 5, 7 ne peuvent jamais se trouver réunis comme éléments d'un nombre parfait quelconque.

(1) Soient 3, 5, 13 les éléments. L'indice de 13 ne peut pas être impair, car alors le nombre $\int 13^{2i+1} = \dfrac{13^{2i+2} - 1}{13 - 1}$ contiendrait le facteur 7, et 7 devrait être un des éléments. Il s'ensuit que $(3^{2i+1} - 1)(13^{2j+1} - 1)$ devrait contenir 5; mais, par rapport au module 5, une puissance impaire quelconque de 3 ou 13 est congrue à 3 ou à 2. Donc la combinaison 3, 5, 13 est inadmissible.

(2) Soient 3, 5, 11 les éléments.

L'indice de 5 doit être de la forme $4j+1$; mais, si $j>0$,

$$\int 5^{4j+1} = \frac{5^{4j+2}-1}{5-1}$$

contiendra les trois nombres impairs premiers entre eux*

$$\frac{5^{2j+1}-1}{5-1}, \quad \frac{5^{2j+1}+1}{5+1}, \quad \frac{5+1}{2}.$$

Conséquemment, il y aura au moins trois autres éléments en plus de 5, ce qui est inadmissible : donc le nombre sera de la forme $3^{2i}5\ 11^{2k}$.

Donc $(1+5)(11^{2k+1}-1)$ doit contenir 9, ce qui est impossible ; car $11^{2k+1} \equiv 2 \pmod 3$.

Ainsi, on voit qu'un nombre impair avec 3 éléments seulement ne peut exister.

Quant aux nombres parfaits pairs, Euclide a démontré que $2^n \int 2^n$, c'est-à-dire $2^n(2^{n+1}-1)$, est un nombre parfait pourvu que $2^{n+1}-1$ soit un nombre premier. Mais on doit à Euler la seule preuve que je connaisse de la proposition réciproque qu'*il n'existe pas de nombres pairs parfaits autres que ceux d'Euclide.*

[* See below, p. 615.]

56.

SUR UNE CLASSE SPÉCIALE DES DIVISEURS DE LA SOMME D'UNE SÉRIE GÉOMÉTRIQUE.

[*Comptes Rendus*, CVI. (1888), pp. 446—450.]

En l'honneur du grand et surprenant Fermat, dont j'ai vu avec une émotion indicible gravés sur le buste au musée de Toulouse les mots qui lui étaient adressés par Blaise Pascal : "Au plus grand homme de l'Europe," je me propose de nommer la fonction fondamentale de la haute Arithmétique $\Theta^M - 1$ le *fermatien* à la *base* Θ et à l'*indice* M.

De plus, je nommerai la fonction $\dfrac{\Theta^M - 1}{\Theta - 1}$, qui n'est autre chose que la somme d'une série géométrique dont la raison est un entier, le *fermatien réduit.* M (bien entendu) est un entier positif quelconque, mais Θ un entier positif ou *négatif.*

Les nombres premiers qui divisent un nombre quelconque, je les nomme ses *éléments.*

On sait, d'après Euler, que tout diviseur d'un fermatien sera de la forme $x\mu + 1$, où μ est M ou bien un diviseur quelconque de M. Parmi ces diviseurs, il y a une classe toute spéciale qui correspond aux cas de $\mu = 1$ et de $\mu = -1$. Le caractère spécial de ces diviseurs du fermatien, c'est qu'ils doivent nécessairement être (comme on verra immédiatement) en même temps diviseurs de son indice. Je remarque préalablement que, $\Theta^{p^a} - 1$ (où p est un nombre premier) étant, par rapport au module p, congru à $\Theta - 1$, afin que ce fermatien contienne p, il faut que $\Theta - 1$ le contienne.

(1) Soit $M = p$ un nombre premier *impair* : je dis que le fermatien réduit $\dfrac{\Theta^p - 1}{\Theta - 1}$ contiendra p, mais non pas p^2. Car, en mettant $\Theta = kp + 1$, on voit que le fermatien réduit $\dfrac{\Theta^p - 1}{\Theta - 1}$, envisagé comme la somme d'une série géométrique, sera congru par rapport au module p^2 à $p + k\dfrac{p^2 - p}{2}p$, c'est-à-dire à p.

(2) Soit M la puissance d'un nombre premier impair p^a. En supposant toujours que $\Theta - 1$ contient p, $\Theta^x - 1$ le contiendra.

Conséquemment, puisque $\dfrac{\Theta^{p^a}-1}{\Theta-1} = \dfrac{\Theta^{p^a}-1}{\Theta^{p^{a-1}}-1}\,\dfrac{\Theta^{p^{a-1}}-1}{\Theta^{p^{a-2}}-1}\cdots\dfrac{\Theta^{p}-1}{\Theta-1}$, il suit comme conséquence de ce qui précède que $\dfrac{\Theta^{p^a}-1}{\Theta-1}$ sera divisible par p^a, mais non pas par p^{a+1}.

(3) Soit $M = Np^a$, où N est premier à p; on a

$$\frac{\Theta^{Np^a}-1}{\Theta-1} = \frac{\Theta^{Np^a}-1}{\Theta^{N}-1}\,\frac{\Theta^{N}-1}{\Theta-1};$$

le premier facteur peut être envisagé comme fonction de Θ^N et par le cas précédent sera divisible par p^a, mais non pas par p^{a+1}. Le second facteur, envisagé comme la somme d'une série géométrique, sera congru à N par rapport à p (quel que soit N pair ou impair) et conséquemment ne contiendra pas p. Donc $\dfrac{\Theta^{Np^a}-1}{\Theta-1}$ sera divisible par p^a, mais non par p^{a+1}.

Ainsi, si p est un élément quelconque impair de $\Theta - 1$ et p^a la plus haute puissance de p contenu dans M, le fermatien réduit $\dfrac{\Theta^{M}-1}{\Theta-1}$ contiendra p^a, mais ne contiendra pas p^{a+1} et, comme conséquence particulière, ne contiendra nul élément de $\Theta - 1$ qui n'est pas un diviseur de M.

On peut aussi supposer que $\Theta - 1$ contient chaque élément de M, et l'on obtient le théorème suivant:

Un fermatien réduit à indice impair, dont le dénominateur est divisible par chaque élément de son indice, sera lui-même divisible par cet indice, et de plus le quotient qui résulte de la division de l'une de ces quantités par l'autre sera premier relatif à l'indice.

C'est dans les recherches sur la possibilité de l'existence de nombres parfaits autres que ceux d'Euclide que se rencontre cette théorie des fermatiens réduits qui y joue un rôle indispensable. Comme exemple de son utilité, je vais faire voir qu'un nombre de la forme $3N \pm 1$ à 7 éléments ne peut pas être un nombre parfait.

Remarquons que, si g est un des nombres gaussiens 3, 5, 17, 257,..., c'est-à-dire un nombre premier de la forme $2^n + 1$, g ne peut pas diviser un fermatien réduit à indice impair s'il ne divise pas le dénominateur; car, afin que cela eût lieu, $g - 1$ par le théorème déjà cité d'Euler devrait contenir un facteur impair.

Donc un tel fermatien réduit sera de la forme $\dfrac{(gx+1)^{\mu g}-1}{(gx+1)-1}$.

Or nous avons vu, dans la Note précédente [p. 604, above], qu'un nombre $3N \pm 1$ à 6 éléments ne peut pas être un nombre parfait, et que, si un tel nombre à 7 éléments est un nombre parfait, le plus grand d'entre eux ne peut pas excéder 37.

Il est facile de voir que ce nombre doit contenir 5, parce que

$$\frac{7}{6}\cdot\frac{11}{10}\cdot\frac{13}{12}\cdot\frac{17}{16}\cdot\frac{19}{18}\cdot\frac{23}{22}\cdot\frac{29}{28} < 2;$$

en effet, ce produit est moindre que 1,69.

Soit donc, s'il est possible, $3N \pm 1$ un nombre parfait à 7 éléments.

Les nombres premiers de la forme $4x + 1$ pas plus grands que 37 sont 13, 17, 29, 37. Mais 17 ne peut pas être l'élément exceptionnel de $3N \pm 1$ parce que la somme des diviseurs du component* qui répond à 17 sera la somme d'un nombre pair de termes de la série $1 + 17 + 17^2 + 17^3 + \ldots$, laquelle nécessairement contient 3. La même chose est évidemment vraie pour un nombre quelconque, comme $2q$, qui est de la forme $12x + 5$.

Donc le component exceptionnel aura pour élément ou 13 ou 37; mais ni $13^2 - 1$ ni $37^2 - 1$ ne contient 5. Il faut donc que la somme des diviseurs du component ou à l'élément 11 ou sinon à l'élément 31 soit respectivement de la forme $\dfrac{11^{5\mu}-1}{11-1}$ ou $\dfrac{31^{5\nu}-1}{31-1}$, car 11 et 31 sont les seuls nombres pas plus grands que 37 de la forme $5x + 1$. Conséquemment tous les diviseurs d'une au moins des deux quantités $\dfrac{11^5-1}{11-1}$ ou $\dfrac{31^5-1}{31-1}$ seront compris parmi les éléments de $3N \pm 1$.

Selon notre théorème, les diviseurs ni de l'un ni de l'autre de ces deux fonctions ne peuvent contenir 5 et conséquemment par le théorème d'Euler seront de la forme $10x + 1$.

Or, puisque 11 n'est pas un résidu quadratique de 31, $11^5 - 1$ ne peut pas contenir 31; donc les diviseurs de $\dfrac{11^5-1}{11-1}$ sont compris parmi les nombres 41, 61, 71, 101, ...

$\dfrac{31^5-1}{31-1}$ contiendra 11, mais ne peut pas être une puissance de 11, car au module 11^2

$$4^5(31^5 - 1) \equiv 3^5 - 4^5 \equiv 1 - 4^3 \equiv -1023,$$

c'est-à-dire

$$-11 . 93,$$

de sorte que $31^5 - 1$ n'est pas divisible même par 11^2.

Donc les diviseurs de $\dfrac{31^5-1}{31-1}$ sont aussi compris parmi les nombres 41, 61, 71, 101, ...

* La plus haute puissance d'un élément d'un nombre qu'il contient se nomme un *component* de ce nombre.

Conséquemment il y aura au moins un élément du nombre parfait $3N \pm 1$ qui n'est pas moindre que 41; cette conclusion est contradictoire à l'existence de la limite supérieure 37 à la grandeur des éléments. Donc on peut affirmer en toute sûreté qu'un nombre non divisible par 3 qui contient moins que 8 facteurs premiers distincts ne peut pas être un nombre parfait.

Il y a une méthode un peu plus expéditive pour parvenir au résultat dernièrement acquis; mais, tout de même, supprimer la première méthode serait un procédé mal avisé, puisque son principe est applicable à d'autres cas où celui dont je vais faire usage se trouverait en défaut; par exemple en combinant les deux méthodes, c'est-à-dire en tenant compte en même temps des conséquences de la présence de 17 quand il figure comme élément, et de la présence de l'élément 5 dans le cas où 17 manque, je crois avoir démontré qu'un entier $3N \pm 1$ à 8 éléments ne peut pas être un nombre parfait.

Remarquons que, puisque le produit suivant, à 7 termes, où 17 manque dans le numérateur, $\frac{5}{4}\frac{7}{6}\frac{11}{10}\frac{13}{12}\frac{19}{18}\frac{23}{22}\frac{29}{28}$, est moindre que 1,988, un nombre parfait à 7 éléments non divisible par 3 ne peut pas exister sans l'élément 17. Supposons qu'un tel nombre existe. Soit η un de ses éléments (autre que 17). La somme des diviseurs du *component* qui y correspond sera de la forme $\frac{\eta^{2i+2}-1}{\eta-1}$ si η est un élément ordinaire, et de la forme $\frac{(\eta^2)^{2j+1}-1}{\eta^2-1}(\eta+1)$ si η est l'élément exceptionnel.

Dans l'un et dans l'autre cas, cette somme ne peut contenir 17 que sous la condition que η^2-1 soit divisible par 17.

Donc, puisque le produit des sommes des diviseurs des components d'un nombre parfait doit contenir tous ses éléments, il existe au moins un élément η tel que η^2-1 contient 17, c'est-à-dire il y a un élément qui est un nombre premier compris dans l'une ou l'autre des formules $17x+1$, $17x-1$; mais le plus petit nombre premier contenu dans ces formules est 67*. Ainsi, puisque

$$\frac{5}{4}\frac{7}{6}\frac{11}{10}\frac{13}{12}\frac{17}{16}\frac{19}{18}\frac{67}{66} < (1{,}95)\left(1+\frac{1}{66}\right) < 1{,}98,$$

l'existence d'un nombre parfait $3N \pm 1$ à 7 éléments est impossible.

* On pourrait facilement prouver (s'il était nécessaire pour les besoins de la démonstration du théorème) que η doit être un nombre premier de la forme $17x+1$ ou un nombre premier en même temps de la forme $17x-1$ et $12y+1$, c'est-à-dire de la forme $204x+169$, et ainsi il y aurait au moins un élément plus grand que 103.

57.

SUR L'IMPOSSIBILITÉ DE L'EXISTENCE D'UN NOMBRE PARFAIT IMPAIR QUI NE CONTIENT PAS AU MOINS 5 DIVISEURS PREMIERS DISTINCTS.

[*Comptes Rendus*, CVI. (1888), pp. 522—526.]

Nous avons vu, dans une Note précédente, qu'un nombre parfait impair avec moins de 7 facteurs doit être divisible par 3, et aussi que nul nombre parfait ne peut être divisible par 105. Ajoutons que, puisque

$$\frac{3}{2}\cdot\frac{11}{10}\cdot\frac{13}{12}\cdot\frac{17}{16}=\frac{151\frac{15}{16}}{80}<2$$

et que, en changeant 11, 13, 17 pour d'autres éléments, on ne peut diminuer ce produit qu'en empiétant sur les chiffres 5 ou 7, il s'ensuit que l'élément 3 doit être associé ou avec 7 ou avec 5 dans un nombre parfait à quatre éléments, s'il y en a.

Supposons donc qu'un tel nombre N existe.

(1) Soient 3 et 7 deux de ses éléments. Le troisième élément en ordre de grandeur ne peut pas excéder 13; car

$$\frac{3}{2}\cdot\frac{7}{6}\cdot\frac{17}{16}\cdot\frac{19}{18}=\frac{119}{64}\left(1+\frac{1}{18}\right)<\frac{126}{64}<2.$$

(α) Soit 11 le troisième élément; puisque

$$\frac{3}{2}\cdot\frac{7}{6}\cdot\frac{11}{10}\cdot\frac{29}{28}=\frac{77}{40}\left(1+\frac{1}{28}\right)<2,$$

on voit que le quatrième élément ne peut être qu'un des nombres 13, 17, 19, 23.

Mais, parmi les éléments, un au moins doit être de la forme $4x+1$.

De plus, nous avons vu dans une Note précédente que *nul* nombre parfait ne peut contenir l'élément 17 sans contenir en même temps un élément pas plus petit que 67. Donc les quatre éléments seront 3, 7, 11, 13.

Le diviseur-somme* à 7 ne peut pas contenir le facteur algébrique 7^9-1, car alors $\frac{1}{3}\cdot\frac{7^3-1}{7-1}$, $\frac{1}{3}\cdot\frac{7^9-1}{7^3-1}$ seront diviseurs de cette somme premiers entre eux, à 3 et à 7, et en plus ne contenant pas 13 parce que 13 n'est ni une fonction unilinéaire† de q ni diviseur de 7^3-1. Ainsi sur cette supposition il y aurait au moins cinq éléments distincts. Donc le diviseur-somme à 7 ne peut pas contenir 9, mais le component à 3 contient nécessairement 3^2; conséquemment, puisque le diviseur-somme à 11 (élément ordinaire et non pas de la forme $3x+1$) ne peut pas contenir 3, le diviseur-somme à 13 contiendra un facteur algébrique de la forme $\frac{13^3-1}{13-1}$ qui est égal à $169+13+1$. Donc 61 sera un élément en plus de 3, 7, 11, 13 qui est contraire à l'hypothèse.

1. (β) Soit 13 le troisième élément.

Puisque $\frac{3}{2}\cdot\frac{7}{6}\cdot\frac{13}{12}\cdot\frac{23}{22}\doteq\frac{91}{48}\left(1+\frac{1}{22}\right)<2$, le quatrième élément sera nécessairement moins que 23, et le système des éléments sera 3, 7, 13, 19, car 17 est exclus.

Les diviseurs-sommes, ni à 13 ni à 19, ne peuvent pas contenir 3; parce qu'ils contiendraient nécessairement les facteurs $\frac{13^3-1}{13-1}$ et $\frac{19^3-1}{19-1}$, et ainsi $\frac{1+13+13^2}{3}$, c'est-à-dire 61, et $\frac{1+19+19^2}{3}$, c'est-à-dire 127.

Donc le diviseur-somme à 7 doit contenir algébriquement les facteurs $\frac{1}{3}\cdot\frac{7^9-1}{7^3-1}$, $\frac{1}{3}\cdot\frac{7^3-1}{7-1}$; ce dernier est égal à 19; le premier sera nécessairement premier à 3, 7, 19 et, pour la raison déjà donnée, à 13.

Il est donc démontré que 7 ne peut pas être un élément de N.

(2) Supposons que 3 et 5 sont deux de ses éléments.

2. A. Soit 5 l'élément exceptionnel.

2. A (α). Si l'indice à l'élément 3 est 2, alors, puisque $1+3+3^2=13$, on aura les éléments 3, 5, 13; donc le diviseur-somme à 13 doit contenir 3, et, conséquemment, contiendra algébriquement le facteur $\frac{13^2+13+1}{3}$, c'est-à-dire 61.

Ainsi on aura les éléments 3, 5, 13, 61.

Mais $\frac{1+3+3^2}{9}\cdot\frac{1+5}{5}\cdot\frac{13}{12}\cdot\frac{61}{60}<2$, ce qui est inadmissible.

* Si p est un élément et p^i un *component* d'un nombre N, on nomme p^i le *component* à p, et $\frac{p^{i+1}-1}{p-1}$ le *diviseur-somme* à p.

† Il est très commode, dans ce genre de recherches, de se servir de la phrase "fonction unilinéaire de x" pour signifier $kx+1$.

2. A (β). On peut donc supposer l'indice du component à 3 au moins 4.

Soient 3, 5, p les trois éléments; l'indice du diviseur-somme à p ne peut pas être 9, car alors on aurait en plus de 3, 5, p deux autres éléments au moins premiers entre eux et à 3, 5, p.

Soit q le quatrième élément; la même chose sera vraie du diviseur-somme à q.

Donc le produit des diviseurs-sommes à 3, 5, p, q ne peut pas contenir une plus haute puissance de 3 que 3^3; mais elle doit contenir au moins 3^4.

Ainsi l'hypothèse que 5 est l'élément exceptionnel est inadmissible.

2. B. Passons à l'hypothèse que 5 est un élément ordinaire.

Remarquons que $\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{31}{30}\cdot\frac{37}{36}<1{,}992<2$.

Conséquemment, il y aura au moins un élément, disons p, qui n'excède pas 29: je dis que p ne peut pas être contenu dans le diviseur-somme de 5; car, si cela avait lieu, l'indice de cette somme serait nécessairement un diviseur impair de l'excès au-dessus de l'unité de quelque nombre premier inférieur à 31, c'est-à-dire 3, 5, 7, 9 ou 11, dont les quatre derniers correspondent respectivement aux nombres premiers 11, 29, 19 et 23.

Il ne peut pas être 3, car $\frac{5^3-1}{5-1}=31$; ni 5, car $\frac{5^5-1}{5-1}=11\,.\,71$ (et l'on aurait une combinaison d'éléments 3, 5, 11, 71; laquelle est inadmissible, parce que 5 est, par hypothèse, non exceptionnel, et les autres éléments sont de la forme $4x+3$).

Il ne peut pas être 7, car on trouve facilement que 5^7-1 ne contient pas 29 ni 9; car, quoiqu'il soit vrai que (5 étant résidu quadratique de 19) 5^9-1 contient 19, il contient en même temps 5^3-1, et l'on aurait la combinaison 3, 5, 19, 31, qui est défendue par la même raison que l'est 3, 5, 11, 71.

Reste seulement 11, mais $5^{11}-1$ ne peut pas contenir 23, parce que 5 n'est pas résidu quadratique de 23.

Ainsi l'élément 5 ne peut pas *engendrer* (au moyen du diviseur-somme qui lui répond) un élément qui n'est pas en dehors de la limite 29.

Le diviseur-somme à un tel élément (s'il est 11 et seulement dans ce cas-là) peut contenir 5, mais non pas 5^2; car, s'il contenait 5^2, on aurait au moins deux diviseurs de cette somme premiers entre eux et à 3, 5, 11.

Remarquons que le component à l'élément exceptionnel ne peut pas être une puissance (à exponent $4j+1$) d'un nombre; car, si $j>0$, $q^{4j+2}-1$ contiendrait nécessairement deux facteurs premiers distincts en addition à 3, 5 et p; donc $j=0$; ainsi l'on voit que $q+1$ doit contenir au moins les puissances de 3 et 5 contenues en $3^2\,.\,5^2$, qui ne sont pas contenues dans le diviseur-somme de l'autre élément indéterminé, lequel on montre facilement ne pouvoir contenir que 3 ou 5 et non pas 3^2, $3 . 5$, ou 5^2; car, sur la première ou la dernière de ces trois hypothèses, le nombre des éléments serait plus grand que 4, et sur l'hypothèse qui reste plus grand même que 5. Donc l'élément exceptionnel augmenté par l'unité sera de la forme ou $2k . 3^2 . 5 - 1$ ou $2k . 3 . 5^2 - 1$: conséquemment sa valeur doit excéder 89; cela prouve que le p dont nous avons parlé n'est pas l'élément exceptionnel.

Soit q cet élément, on aura

$$q = 30\lambda - 1.$$

Or le diviseur-somme à 5 ne contient ni 3 ni p.

On aura donc forcément

$$\frac{5^x - 1}{5 - 1} = q = 30\lambda - 1,$$

c'est-à-dire

$$5^x - 120\lambda + 3 = 0,$$

ce qui est impossible.

Cela démontre que l'hypothèse 2. B est inadmissible, et finalement le résultat est acquis qu'il n'existe pas de nombres parfaits impairs qui soient divisibles par moins de 5 facteurs premiers; car ce théorème, pour les cas d'une multiplicité 3, 2, 1, a déjà été démontré.

Ajoutons quelques mots sur les nombres parfaits à cinq éléments.

Ici, puisque

$$\frac{3}{2} \cdot \frac{11}{10} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{23}{22} < 1{,}986,$$

mais

$$\frac{3}{2} \cdot \frac{11}{10} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{18} > 2{,}004.$$

On voit qu'un nombre parfait à cinq éléments, où 5 et 7 manquent, ne peut avoir pour ces éléments que les chiffres 3, 11, 13, 17, 19.

Mais 17 (un nombre cyclotomique de Gauss) ne peut pas exister sans un élément satellite de la forme $17k \pm 1$. Donc un nombre parfait à cinq éléments, s'il existe, aura nécessairement ou les éléments 3, 5 ou les éléments 3, 7.

J'ai réussi à démontrer l'impossibilité de l'une et de l'autre de ces hypothèses; mais la preuve est trop longue pour être insérée ici.

58.

SUR LES NOMBRES PARFAITS.

[*Comptes Rendus*, CVI. (1888), pp. 641, 642; *Mathesis*, VIII. (1888), pp. 57—61.]

DANS la démonstration de l'impossibilité qu'un nombre à 3 éléments soit un nombre parfait, qui a paru dans les *Comptes rendus* du 6 février dernier, il y a une petite omission que M. Mansion a eu la bonté de me signaler. Il est dit [p. 606, above], que les nombres $\frac{5^{2j+1} - 1}{5 - 1}$, $\frac{5^{2j+1} + 1}{5 + 1}$, $\frac{5 + 1}{2}$ sont premiers entre eux.

Cela n'est pas vrai si $2j + 1$ contient 3, mais, dans ce cas-là, $5^{2j+1} + 1$ contiendra $5^3 + 1$ qui contient 7 : conséquemment, on aura les quatre éléments 3, 5, 7, 11. Donc la démonstration reste bonne.

M. Sylvester vient de publier [p. 604, above], dans les *Comptes Rendus de l'Académie des Sciences de Paris* (séance du 6 février 1888, t. CVI. pp. 403—405), une importante contribution à l'étude des nombres parfaits, à l'occasion de remarques de notre collaborateur M. Servais (*Mathesis*, t. VII. pp. 228—230).

Nous sommes heureux de reproduire ici les considérations développées par l'illustre géomètre anglais, comme complément des articles publiés à ce sujet dans *Mathesis* (t. VI. pp. 100—101, 145—148, 178, 248—250, et t. VII. pp. 228—230, 245—246).

La notation $c \equiv i \equiv 1 \pmod 4$ est équivalente à la notation plus explicite:

$$c = i + \mathfrak{M}\,4 = 1 + \mathfrak{M}\,4$$

et se prononce : *c est congru à i et à* 1, *suivant le module* 4.

Nous ajoutons quelques notes à l'article un peu bref de M. Sylvester pour en faciliter l'intelligence*.

P. MANSION.

Existe-t-il des nombres parfaits impairs? C'est une question qui reste indécise.

* Dans les nos. des *C. R.* du 13 et du 20 février, M. Sylvester a publié de nouvelles recherches sur les nombres parfaits dont nous ne pouvons, faute d'espace, que signaler plus bas, les conclusions en note. Il s'est aussi occupé des nombres parfaits dans les nos. de *Nature*, du 15 et du 22 décembre 1887, et dans l'*Educational Times* du 1er mars 1888.

Dans un article intéressant de M. Servais, paru dans le journal *Mathesis*, en octobre 1887, on trouve cette proposition qu'*un nombre parfait impair (s'il y en a) qui ne contient que trois facteurs premiers distincts est nécessairement divisible par* 3 *et* 5. Je vais démontrer ici qu'un tel nombre n'existe pas, au moyen d'un genre de raisonnement qui m'a fourni aussi une démonstration de ce théorème qu'*il n'existe pas de nombre parfait impair qui contienne moins de six facteurs premiers distincts.*

On voit facilement que la somme de la série géométrique

$$1 + c + c^2 + \ldots + c^i$$

où c est impair, sera elle-même paire quand i est impair; de plus, quand i est pair, cette somme sera toujours impaire, mais impairement paire seulement dans le cas où $c \equiv i \equiv 1 \pmod 4$.

Donc, si un nombre parfait impair est de la forme $p^i q^j r^k \ldots$ ($p, q, r, \ldots$ étant des nombres premiers distincts), tous les indices $i, j, k, \ldots$ doivent être pairs à l'exception d'un seul, soit i, lequel, de même que sa base p, sera congru à 1 par rapport au module 4; car on doit avoir

$$\int p^i \int q^j \int r^k \ldots = 2 p^i q^j r^k \ldots,$$

$\int x^i$ représentant $1 + x + \ldots + x^i$, c'est-à-dire $\dfrac{x^{i+1} - 1}{x - 1}$.

Ainsi, on voit qu'un nombre parfait impair (si un tel nombre existe) sera de la forme $M^2 (4q+1)^{4k+1}$, $4q + 1$ étant un nombre premier qui ne divise pas M*.

Comme corollaire, on peut déduire qu'aucun nombre parfait impair ne peut être divisible par 105. En effet, soit un tel nombre $3^{2i} 5^{2j} 7^{2k} \ldots$; on aura

$$\frac{\int 3^{2i} \int 5^{2j} \int 7^{2k}}{3^{2i} 5^{2j} 7^{2k}} \geqq \left(1 + \frac{1}{3} + \frac{1}{3^2}\right)\left(1 + \frac{1}{5}\right)\left(1 + \frac{1}{7} + \frac{1}{7^2}\right);$$

c'est-à-dire $\geqq \dfrac{2 \,.\, 13 \,.\, 19}{5 \,.\, 49}$; c'est-à-dire $\dfrac{494}{245}$, qui est plus grand que 2.

Remarquons qu'en général, si $p^i q^j r^k \ldots$ est un nombre parfait, il faut que

$$\frac{p^{i+1}}{p^i (p-1)} \frac{q^{j+1}}{q^j (q-1)} \ldots \text{ c'est-à-dire } \frac{p}{p-1} \frac{q}{q-1} \frac{r}{r-1} \ldots$$

soit plus grand que 2†.

* Théorème démontré aussi, en 1886, par M. Stern, dans *Mathesis*, t. VI. pp. 248—250, mais que l'on trouve également au no. 109, du chapitre III, de l'opuscule d'Euler: *Tractatus de numerorum doctrina*, publié dans les *Commentationes arithmeticae collectae* (voir t. II. pp. 514—515).

Il en résulte que, si 3, 7, ou 11, etc. entrent comme facteur dans un nombre parfait impair, ils y entrent avec un exposant pair, car ils sont de la forme $(4p+3)$.

† Voir, par exemple, l'article de M. Servais, p. 230. D'après la définition des nombres parfaits, on a

$$\frac{\int p^i \,.\, \int q^j \,.\, \int r^k}{p^i q^j r^k} = 2,$$

Ainsi, à moins que le plus petit des éléments $p, q, r \ldots$ ne soit pas plus grand que 3, on doit avoir

$$\frac{5}{4} \frac{7}{6} \frac{11}{10} \frac{13}{12} \frac{17}{16} \frac{19}{18} \ldots > 2;$$

mais en ne dépassant pas 19, ce produit est moindre que 1,94963. Conséquemment le nombre des éléments, dans ce cas, doit être 7 au moins. Puisque

$$1{,}95 \times \left(1 + \frac{1}{40}\right) < 2,$$

on voit immédiatement que, si un nombre parfait à 7 éléments parmi lesquels 3 ne figure pas, existe, le septième élément ne pourrait pas dépasser 37*.

Passons au cas de 3 éléments 3, q, r d'un nombre parfait impair. Puisque

$$\frac{3}{2} \frac{7}{6} \frac{11}{10} = \frac{231}{120} < 2,$$

on voit que $3^i . 7^j . 11^k$, et à plus forte raison $3^i p^j q^k$, où p, q sont des nombres quelconques autres que 3 ou 5, ne peut être un nombre parfait.

Supposons donc que 3, 5, q sont les éléments d'un nombre parfait; puisque

$$\frac{3}{2} \frac{5}{4} \frac{17}{16} = \frac{255}{128} < 2,$$

on voit que q ne peut être ni 17, ni un nombre quelconque plus grand que 17. Donc $q = 11$ ou $q = 13$; car nous avons vu que 3, 5, 7 ne peuvent jamais se trouver réunis comme éléments d'un nombre parfait quelconque.

(1) Soient 3, 5, 13 les éléments. L'indice de 13 ne peut pas être impair, car alors le nombre

$$\int 13^{2i+1} = \frac{13^{2i+2} - 1}{13 - 1}$$

ou encore

$$\frac{p^{i+1} - 1}{p^i (p-1)} \cdot \frac{q^{j+1} - 1}{q^j (q-1)} \cdot \frac{r^{k+1} - 1}{r^k (r-1)} = 2.$$

On déduit aisément de là (1) que $(q^{j+1} - 1)(r^{k+1} - 1)$ doit être divisible par p. (2) En supprimant (-1) dans les numérateurs,

$$\frac{p}{p-1} \cdot \frac{q}{q-1} \cdot \frac{r}{r-1} > 2.$$

* Dans les *C. R.* du 13 février, M. Sylvester a prouvé qu'il ne peut y avoir de nombre parfait premier avec 3, ayant même 7 ou 8 éléments. Il se sert pour arriver à ce résultat de propriétés (déduites du théorème de Fermat) des expressions $\theta^n - 1$, $[(\theta^n - 1) : (\theta - 1)]$; il nomme ces expressions *fermatien* de base θ et d'*indice n*, et *fermatien réduit* en l'honneur du grand géomètre de Toulouse. Il rappelle, à ce propos, les mots adressés à celui-ce par Pascal : "Au plus grand homme de l'Europe," mots gravés sur le buste de Fermat au musée de Toulouse. La citation exacte de Pascal est : "Quoique vous soyez celui *de toute l'Europe* que je tiens pour *le plus grand géomètre*, etc." (Lettre du 10 août 1660).

contiendrait le facteur 7, et 7 devrait être un des éléments*. Il s'ensuit que $(3^{2i+1}-1)(13^{2j+1}-1)$ devrait contenir 5; mais, par rapport au module 5, une puissance impaire quelconque de 3 ou 13 est congrue à 3 ou à 2. Donc la combinaison 3, 5, 13 est inadmissible.

(2) Soient 3, 5, 11† les éléments. L'indice de 5 doit être de la forme $4j+1$; mais, si $j>0$,

$$\int 5^{4j+1} = \frac{5^{4j+2}-1}{5-1}$$

contiendra les trois nombres impairs premiers entre eux‡

$$\frac{5^{2j+1}-1}{5-1}, \quad \frac{5^{2j+1}+1}{5+1}, \quad \frac{5+1}{2}$$

[pourvu que $2j+1$ ne soit pas divisible par 3; dans ce cas, $5^{2j+1}+1$ contiendrait $5^3+1=18 \,.\, 7$, de sorte que 7 serait un élément].

Conséquemment, il y aura au moins trois autres éléments en plus de 5, ce qui est inadmissible; donc le nombre sera de la forme $3^{2i} \,.\, 5 \,.\, 11^{2k}$.

Donc $(1+5)(11^{2k+1}-1)$ doit contenir 9, ce qui est impossible; car $11^{2k+1} \equiv 2 \pmod 3$.

Ainsi, on voit qu'un nombre parfait impair avec 3 éléments seulement ne peut exister§.

Quant aux nombres parfaits pairs, Euclide a démontré que $2^n \int 2^n$, c'est-à-dire $2^n(2^{n+1}-1)$ est un nombre parfait pourvu que $2^{n+1}-1$ soit un nombre premier. Mais on doit à Euler la seule preuve‖ que je connaisse de la proposition réciproque qu'*il n'existe pas de nombres pairs parfaits autres que ceux d'Euclide.*

Note. On peut encore établir le (2) comme il suit. Le nombre $\frac{5^{2j+1}-1}{5-1}$ introduit dans le premier membre de l'égalité hypothétique

$$\frac{3^{2i+1}-1}{3-1} \,.\, \frac{5^{4j+2}-1}{5-1} \,.\, \frac{11^{2k+1}-1}{11-1} = 2 \,.\, 3^{2i} \,.\, 5^{4j+1} \,.\, 11^{2k}$$

* $13^{2i+2}-1$ est divisible par $13^2-1=168=7\times 24$.

† 3 et 11 ont des exposants pairs (voir la première note, p. [616]).

‡ Les nombres $5^{2j+1}-1$, $5^{2j+1}+1$ n'ont d'autre diviseur commun que leur différence 2; ensuite on a

$$5^{2j+1} = \mathfrak{M}\, 3 + 2,$$

donc $5^{2j+1}-1$ et $\frac{1}{4}(5^{2j+1}-1)$ ne sont pas divisibles par $\frac{5+1}{2}=3$. Mais $\frac{5^{2j+1}+1}{5+1}$ n'est pas toujours premier avec 3; en effet, $5^{2j+1}+1$ est un multiple de 9 plus 6, 0 ou 3 suivant que $2j+1$ est de la forme $3p+1$, $3p$, ou $3p+2$. Les lignes entre crochets manquent dans les *C. R.*; elles nous ont été obligeamment communiquées par l'auteur, pour compléter la démonstration, dans le cas où 5^{2j+1} est divisible par 9. (Voir aussi la note à la suite de l'article.)

§ Dans les *C. R.* du 20 février, M. Sylvester démontre qu'il n'y a pas nombre parfait impair avec quatre éléments et annonce qu'il a prouvé qu'il n'en existe pas même avec cinq éléments.

‖ *Commentationes arithm. coll.*, p. 514, no. 107 (cité par M. Sylvester, *Nature*, 15 déc. 1887, p. 152). Voir une autre démonstration due à M. Lucas, dans *Mathesis*, t. VI. pp. 146—147.

au moins un facteur différent de 3, 5, 11. En effet, ce nombre $\frac{5^{2j+1}-1}{5-1}$ s'il n'est pas divisible par 11, introduit un autre facteur que 3, 5, 11, puisqu'il est premier avec 3, 5, 11. D'autre part, s'il est divisible par 11, il est aussi divisible par 71; car on a

$$\begin{array}{ll}
5-1=4, & 5^{5(2p+1)-4}-1=\mathfrak{M}\,11+4, \\
5^3-1=\mathfrak{M}\,11+3, & 5^{5(2p+1)-2}-1=\mathfrak{M}\,11+3, \\
5^5-1=4\,.\,11\,.\,71, & 5^{5(2p+1)}-1=\mathfrak{M}\,(5^5-1)=\mathfrak{M}\,71, \\
5^7-1=\mathfrak{M}\,11+2, & 5^{5(2p+1)+2}-1=\mathfrak{M}\,11+2, \\
5^9-1=\mathfrak{M}\,11+8, & 5^{5(2p+1)+4}-1=\mathfrak{M}\,11+8.
\end{array}$$

P. Mansion.

59.

PREUVE ÉLÉMENTAIRE DU THÉORÈME DE DIRICHLET SUR LES PROGRESSIONS ARITHMÉTIQUES DANS LES CAS OÙ LA *RAISON* EST 8 OU 12.

[*Comptes Rendus*, CVI. (1888), pp. 1278—1281, 1385—1386.]

LE principe (ou pour ainsi dire le moment intellectuel) dont nous nous servons est le suivant:

Pour démontrer que le nombre de nombres premiers d'une forme donnée est infini, cherchons à construire une progression infinie d'entiers relativement premiers entre eux, et dont chacun contiendra un nombre premier (au moins) de la forme donnée.

Dans ce qui suit, f signifie une forme fonctionnelle rationnelle entière et ne contenant que des coefficients rationnels.

LEMME I.—*Si* $u_{x+1} = fu_x$ *et si* $ff0 = f0$, *alors*, r *et* s *étant deux entiers quelconques, le plus grand diviseur commun à* u_r *et* u_s *sera un diviseur de* $f0$.

Car évidemment $u_{r+s} \equiv ff \dots f0$ (c'est-à-dire $f^s 0$) [mod u_r]. Mais $f^s 0$, par hypothèse, $= f0$.

Conséquemment, tout diviseur de u_r et u_s sera un diviseur de $f0$.

LEMME II.—*Si* $u_{x+1} = fu_x$ *et si, de plus*, $u_1 = f0$, *le plus grand diviseur commun de* u_r *et* u_s *sera* u_t, *où* t *est le plus grand diviseur commun de* r *et* s.

(1) On aura évidemment

$$u_{s+t} \equiv u_t \pmod{u_s}.$$

Conséquemment u_t sera un diviseur de $u_{2t}, u_{3t}, \dots, u_{mt}$ quel que soit m.

(2) Écrivons un *schéma* pareil à celui qui s'applique à la recherche du plus grand diviseur de r et s, c'est-à-dire

$$r - hs = v, \quad s - kv = w, \quad \dots, \quad z - ly = t, \quad y - mt = 0;$$

alors, en vertu de ce qui précède, u_t sera un diviseur de u_r et u_s, et tout diviseur de u_r et de u_s sera un diviseur de u_t.

Donc, si t est le plus grand diviseur commun à r et s, u_t sera le plus grand diviseur commun à u_r et u_s, ce qui était à démontrer. Il s'ensuit que, si r est premier relativement à s, u_r et u_s auront u_1 pour leur plus grand diviseur commun.

Je vais faire l'application de ce principe: (A) aux progressions arithmétiques à la raison 8, (B) à la raison 12.

A. 1. *Cas de* $8x + 3$.—Écrivons

$$u_1 = 1, \quad u_2 = 2u_1^2 + 1 = 3, \quad u_3 = 2u_2^2 + 1 = 19, \quad \dots.$$

On démontre facilement que tout u est de la forme $8m + 3$, et l'on sait que les facteurs premiers de tout u sont de la forme $8n + 1$ ou $8n + 3$.

Conséquemment, tout u contiendra au moins un facteur de la forme $8m + 3$, et tout terme de la progression infinie

$$u_3, \quad u_5, \quad u_7, \quad u_{11}, \quad u_{13}, \quad \dots$$

contiendra un facteur premier de la forme voulue.

De plus, en vertu du second lemme, tous ces facteurs seront distincts l'un de l'autre; car sinon u_r et u_s, où r est premier à s, auraient un facteur commun autre que u_1.

On pourrait prendre une série plus générale en écrivant u_1 égal à un produit d'un nombre quelconque de nombres premiers dont aucun n'est de la forme $8m + 3$, tellement combinés que $u_1 \equiv 1$ [mod 8]; le résultat restera acquis que chaque terme de la progression des u contiendra un facteur premier de la forme $8x + 3$, et que tous ces facteurs seront distincts entre eux.

A. 2. *Cas de* $8x + 7$.—Écrivons

$$u_1 = 1, \quad u_2 = 2(u_1 + 1)^2 - 1 = 7, \quad u_3 = 2(u_2 + 1)^2 - 1 = 127, \quad \dots.$$

Tout $u \equiv 7$ [mod 8]: chaque diviseur premier de tout u sera de la forme $8m + 1$ ou $8m + 7$. Donc il entrera dans chaque terme de la progression

$$u_2, \quad u_3, \quad u_5, \quad u_7, \quad \dots$$

un facteur de la forme $8x + 7$, et de plus, en vertu du second lemme (puisque $f0 = 1$), tous ces facteurs seront distincts.

A. 3. *Cas de* $8x + 1$.—Écrivons

$$u_1 = 1, \quad u_2 = u_1^4 + 1 = 2, \quad u_3 = u_2^4 + 1 = 17, \quad \dots.$$

Tous les facteurs de chaque u, à l'exception de 2, seront de la forme $8x + 1$, et, en vertu du second lemme $u_3, u_5, u_7, u_{11}, u_{13}, u_{17}$, seront premiers entre eux.

A. 4. *Cas de* $8x + 5$.—Écrivons

$$u_1 = 1, \quad u_2 = u_1^2 + 1 = 2, \quad u_3 = u_2^2 + 1 = 5,$$
$$u_4 = u_3^2 + 1 = 26, \quad u_5 = u_4^2 + 1 = 677, \quad \dots.$$

Chaque u_{2i+1} sera de la forme $8m + 5$, et chaque diviseur premier sera ou de la forme $8n + 1$ ou $8n + 5$, de sorte qu'il s'en trouvera un au moins de la forme $8x + 5$. Donc par le second lemme la progression

$$u_3, \quad u_5, \quad u_7, \quad u_{11}, \quad u_{13}, \quad \dots$$

contiendra un nombre infini de nombres premiers distincts de cette forme.

B. 1. *Cas de* $12x+5$.—On démontre facilement par induction que chaque terme de rang pair de la progression précédente au delà du second sera de la forme $2(24n+13)$, et chaque terme de rang impair au delà du premier de la forme $24n+5$.

Les diviseurs premiers de chaque u seront de l'une ou l'autre des six formes $24x+1$, 5, 19, 17, 13, 21.

Supposons qu'il n'existe aucun facteur premier de la forme $24x+17$ ni de la forme $24x+5$. Alors les résidus des facteurs (par rapport à 12) appartiendront au groupe 1, 9, 13, 21. Mais on voit facilement que ce groupe est un groupe fermé : car toutes ces combinaisons binaires ne font que reproduire ces mêmes nombres.

Conséquemment, tout terme de rang impair contiendra nécessairement un facteur ou de la forme $24x+5$ ou de la forme $24x+17$, et ainsi, en vertu du second lemme, on voit que la progression déjà écrite contiendra un nombre infini de nombres premiers de la forme $12n+5$.

B. 2. *Cas de* $12x+7$.—Écrivons

$$u_1 = 7, \quad u_2 = u_1^2 - u_1 + 1 = 43, \quad u_3 = u_2^2 - u_2 + 1 = 1807, \quad \dots.$$

Les diviseurs premiers de chaque u seront de la forme $12n+1$ ou $12n+7$ et u lui-même de la forme $12m+7$. Donc, en vertu du premier lemme, la suite $u_1, u_2, u_3, u_4, \dots$ contiendra un nombre infini de nombres premiers de la forme $12x+7$*.

B. 3. *Cas de* $12x+11$.—Écrivons

$$u_1 = -1, \quad u_2 = 3u_1^2 - 1 = 2, \quad u_3 = 3u_2^2 - 1 = 11,$$
$$u_4 = 3u_3^2 - 1 = 362, \quad \dots.$$

Tous les u de rang impair seront de la forme $12m+11$, de sorte que leurs diviseurs premiers étant, ou de la forme $12x+1$ ou $12x+11$, il y aura un nombre infini de nombres premiers distincts contenus dans les termes de la progression

$$u_3, \quad u_5, \quad u_7, \quad u_{11}, \quad \dots.$$

B. 4. *Cas de* $12x+1$.—Écrivons

$$u_1 = \theta^4 - \theta^2 + 1, \quad u_2 = u_1^4 - u_1^2 + 1, \quad u_3 = u_2^4 - u_2^2 + 1, \quad \dots.$$

Chaque u, selon la loi cyclotomique, ne contiendra que des facteurs de la forme $12x+1$ et, en vertu du premier lemme, $u_1, u_2, u_3, u_4, u_5, \dots$ seront tous

* Par un procédé analogue à celui que nous avons appliqué à la progression dont nous nous sommes servis dans les cas A. 4 et B. 1 ; on peut démontrer avec l'aide de la progression 7, 43, 1807, ..., donnée plus haut, que le nombre de nombres premiers dans la double progression arithmétique à raison 30,

$$7, 13, 37, 43, 67, 73, \dots,$$

contient un nombre infini de nombres premiers : à plus forte raison cette conclusion s'applique à la double progression à raison 5

$$2, 3, 7, 8, 12, 13, \dots.$$

premiers entre eux : donc cette progression contiendra un nombre infini de facteurs de la forme $12x+1$.

L'application du principe général énoncé au commencement n'est nullement astreinte aux progressions de la forme $\phi\theta$, $\phi\phi\theta$, $\phi\phi\phi\theta$, ... C'est ce que j'ai montré au Congrès scientifique d'Oran.

Au Congrès scientifique d'Oran nous avons indiqué :

(1) Une démonstration instantanée du théorème de Dirichlet pour le cas $Ax+1$, quel que soit A, en nous servant des fonctions cyclotomiques de l'espèce ordinaire en u, en prenant pour les indices successifs $A, 2A, 3A, \dots$ et en donnant à u une valeur quelconque. Ces fonctions cyclotomiques sont les facteurs irréductibles des fermatiens. Par exemple, en prenant 3 pour la base des fonctions cyclotomiques, et en ôtant de chaque cyclotome dont l'indice est une puissance de 2 le *facteur singulier* 2, on obtient la progression 2, 2, 13, 5, 121, 7, 1093, ..., dont tous les termes, en omettant le second, sont premiers entre eux, et où le terme à l'indice i (le second excepté) ne contient d'autres facteurs premiers que ceux de la forme $ix+1$. Conséquemment, en se bornant aux $i^{\text{ème}}$, $(2i)^{\text{ème}}$, $(3i)^{\text{ème}}$, $(4i)^{\text{ème}}$, ... termes, et en décomposant chacun de ces termes dans un produit de facteurs premiers distincts, la totalité de ces facteurs fournira un nombre infini de nombres premiers de la forme $ix+1$;

(2) Une démonstration beaucoup plus cachée pour le cas $Ax-1$, quand A est une puissance d'un nombre premier, au moyen des fonctions cyclotomiques qui se déduisent des fonctions dont nous avons parlé en les divisant par une puissance convenable de u, en exprimant le quotient comme fonction de $u+\frac{1}{u}$, disons v, et en attribuant à v une valeur constante dont la forme par rapport au module A ou bien à un multiple de A (capable de grandir indéfiniment) dépend de la forme du nombre premier dont A est une puissance, par rapport au module 8.

Plus récemment, nous avons étendu la même démonstration aux cas où A est une combinaison de puissances de 2, 3, 5, 7, de sorte qu'il nous paraît peu douteux que les propriétés cyclotomiques donnent le moyen de prouver le théorème de Dirichlet aussi bien pour le cas de $Ax-1$, comme pour le cas de $Ax+1$, quelle que soit la forme de A. Il nous semble donc qu'il y a quelque lieu d'espérer que le principe général (qu'on peut nommer constructif ou cosmothétique) peut servir à donner une démonstration pour le cas le plus général du théorème de Dirichlet. En addition à la méthode ici donnée et celle fournie par la théorie cyclotomique pour obtenir des progressions infinies de nombres relativement premiers entre eux, on peut se servir comme troisième méthode des *cumulants* (les numérateurs et dénominateurs de fractions continues) et sans doute d'une infinité d'autres espèces de fonctions. Toute la difficulté consiste à trouver la *forme* de progression convenable à chaque cas donné.

En ce qui regarde la théorie générale des diviseurs des fonctions cyclotomiques de toute espèce, nous renvoyons à notre article, intitulé: *Excursus A: On the divisors of cyclotomic functions* [Vol. III. of this Reprint, p. 317]; et en ce qui regarde la propriété des nombres cyclotomiques de la première et seconde espèce, privés de leur *facteur singulier*, d'être relativement premiers entre eux, à un article paru dans le journal *Nature* [see pp. 591, 625 of this Volume] du mois de mars de cette année*.

* Le cas de $12x+5$ (page [622] de la Note précédente) est mal expliqué. Afin de démontrer le théorème de Dirichlet pour ce cas il suffit de remarquer que chaque terme de rang impair (après le premier) dans la progression 1, 2, 5, 26, 677, ... est de la forme $12m+5$, et chacun de ses facteurs *premiers* de la forme $4x+1$, c'est-à-dire de la forme $12x+1$ ou $12x+5$; conséquemment il contiendra au moins un facteur premier de la forme $12x+5$.

60.

ON THE DIVISORS OF THE SUM OF A GEOMETRICAL SERIES WHOSE FIRST TERM IS UNITY AND COMMON RATIO ANY POSITIVE OR NEGATIVE INTEGER.

[*Nature*, XXXVII. (1888), pp. 417, 418.]

"Nein! Wir sind Dichter*."
—*Kronecker in Berlin.*

A REDUCED Fermatian†, $\frac{r^p-1}{r-1}$, is obviously only another name for the sum of a geometrical series whose first term is unity and common ratio an integer, r.

If p is a prime number, it is easily seen that the above reduced Fermatian will not be divisible by p, unless $r-1$ is so, in which case (unless p is 2) it will be divisible by p, but not by p^2.

This is the theorem which I meant to express [p. 591, above] in the footnote to the second column of this journal for December 15, 1887, p. 153, but by an oversight, committed in the act of committing the idea to paper, the expression there given to it is erroneous.

Following up this simple and almost self-evident theorem, I have been led to a theory of the divisors of a reduced Fermatian, and consequently of the Fermatian itself, which very far transcends in completeness the condition

* Such were the pregnant words recently uttered by the youngest of the splendid triumvirate of Berlin, when challenged to declare if he still held the opinion advanced in his early inaugural thesis (to the effect that mathematics consists exclusively in the setting out of self-evident truths, —in fact, amounts to no more than showing that two and two make four), and maintained unflinchingly by him in the face of the elegant raillery of the late M. Duhamel at a dinner in Paris, where his interrogator—the writer of these lines—was present. This doctoral thesis ought to be capable of being found in the archives of the University (I believe) of Breslau.

† The word Fermatian, formed in analogy with the words Hessian, Jacobian, Pfaffian, Bezoutiant, Cayleyan, is derived from the name of Fermat, to whom it owes its existence among recognized algebraical forms.

in which the subject was left by Euler (see Legendre's *Theory of Numbers*, 3rd edition, vol. I. chap. 2, § 5, pp. 223—27, of Maser's literal translation, Leipzig, 1886)*, and must, I think, in many particulars be here stated for the first time. This theory was called for to overcome certain difficulties which beset my phantom-chase in the chimerical region haunted by those doubtful or supposititious entities called odd perfect numbers. Whoever shall succeed in demonstrating their absolute non-existence will have solved a *problem of the ages* comparable in difficulty to that which previously to the labours of Hermite and Lindemann (whom I am wont to call the Vanquisher of PI, a prouder title in my eyes than if he had been the conqueror at Solferino or Sadowa) environed the subject of the quadrature of the circle. Lambert had proved that the Ludolphian† number could not be a fraction nor the square root of a fraction. Lindemann within the last few years, standing on the shoulders of Hermite, has succeeded in showing that it cannot be the root of any algebraical equation with rational coefficients (see Weierstrass' abridgment of Lindemann's method, *Sitzungsberichte der A. D. W. Berlin*, Dec. 3, 1885).

It had already been shown by M. Servais (*Mathesis*, Liège, October 1887), that no one-fold integer or two-fold odd integer could be a perfect number, of which the proof is extremely simple. The proof for three-fold and four-fold numbers will be seen in articles of mine in the course of publication in the *Comptes Rendus* [above, pp. 604—619], and I have been able also to extend the proof to five-fold numbers. I have also proved that no odd number not divisible by 3 containing less than eight elements can be a perfect number, and see my way to extending the proof to the case of nine elements.

How little had previously been done in this direction is obvious from the fact that, in the paper by M. Servais referred to, the non-existence of three-fold perfect numbers is still considered as problematical; for it contains a "Theorem" that if such form of perfect number exists it must be divisible by fifteen: the ascertained fact, as we must know, being that this hypothetical

* I find, not without surprise, that some of the theorems here produced, including the one contained in the corrected footnote, have been previously stated by myself in a portion of a paper "On certain Ternary Cubic Form Equations," entitled "Excursus A—On the divisors of Cyclotomic Functions" [Vol. III. of this Reprint, p. 317] the contents and almost the existence of which I had forgotten: but the mode of presentation of the theory is different, and I think clearer and more compact here than in the preceding paper; the concluding theorem (which is the important one for the theory of perfect numbers) and the propositions immediately leading up to it in this, are undoubtedly not contained in the previous paper.

I need hardly add that the term *cyclotomic* function is employed to designate the core or primitive factor of a Fermatian, because the resolution into factors of such function, whose index is a given number, is virtually the same problem as to divide a circle into that number of equal parts.

† So the Germans wisely name π, after Ludolph van Ceulen, best known to us by his second name, as the calculator of π up to thirty-six places of decimals.

theorem is the first step in the *reductio ad absurdum* proof of the non-existence of perfect numbers of this sort (see *Nature*, December 15, 1887, p. 153, written before I knew of M. Servais' paper, and recent numbers of the *Comptes Rendus*).

But after this digression it is time to return to the subject of the numerical divisors of a reduced Fermatian.

We know that it can be separated algebraically into as many irreducible functions as there are divisors in the index (unity not counting as a divisor, but a number being counted as a divisor of itself), so that if the components of the index be a^α, b^β, c^γ, ... the number of such functions augmented by unity is

$$(\alpha+1)(\beta+1)(\gamma+1)....$$

All but one of these algebraical divisors, with the exception of a single one, will also be a divisor of some other reduced Fermatian with a lower index: that one, the core so to say (or, as it is more commonly called, the irreducible primitive factor), I call a cyclotomic function of the base, or, taken absolutely, a cyclotome whose index is the index of the Fermatian in which it is contained.

It is obvious that the whole infinite number of such cyclotomes form a single infinite complex. Now it is of high importance in the inquiry into the existability of perfect numbers to ascertain under what circumstances the divisors of the same reduced Fermatian, that is, cyclotomes of different indices to the same base, can have any, and what, numerical factor in common. For this purpose I distinguish such divisors into superior or external and inferior or internal divisors, the former being greater, and the latter less, than the index.

As regards the superior divisors, the rule is that any one such cannot be other than a unilinear function of the index (I call $kx+1$ a unilinear function of x, and k the unilinear coefficient) and that a prime number which is a unilinear function of the index will be a divisor of the cyclotome when the base in regard to the index as modulus is congruous to a power of an integer whose exponent is equal to the unilinear coefficient.

As regards the inferior divisors, the case stands thus. If the index is a prime, or the power of a prime, such index will be itself a divisor. If the index is not a prime, or power of a prime, then the only possible internal divisor is the largest element contained in the index, and such element will not be a divisor unless it is a unilinear function of the product of the highest powers of all the other elements contained in the index.

It must be understood that such internal divisor in either case only appears in the first power; its square cannot be a divisor of the cyclotome.

It is easy to prove the important theorem that no two cyclotomes to the same base can have any the same *external* divisor*.

We thus arrive at a result of great importance for the investigation into the existence or otherwise of perfect odd numbers, which (it being borne in mind that in this theorem the divisors of a number include the number itself, but *not* unity) may be expressed as follows:

The sum of a geometrical series whose first term is unity and common ratio any positive or negative integer other than $+1$ *or* -1 *must contain at least as many distinct prime divisors as the number of its terms contains divisors of all kinds; except when the common ratio is* -2 *or* 2, *and the number of terms is* *even in the first case, and* 6 *or a multiple of* 6 *in the other, in which cases the number of prime divisors may be one less than in the general case**.

In the theory of odd perfect numbers, the fact that, in every geometrical series which has to be considered, the common ratio (which is an element of the supposed perfect number) is necessarily odd prevents the exceptional case from ever arising.

The establishment of these laws concerning the divisors and mutual relations of cyclotomes, so far as they are new, has taken its origin in the felt necessity of proving a purely negative and seemingly barren theorem, namely the non-existence of certain classes of those probably altogether imaginary entities called odd perfect numbers: the moral is obvious, that every genuine effort to arrive at a secure basis even of a negative proposition, whether the object of the pursuit is attained or not, and however unimportant such truth, if it were established, may appear in itself, is not to be regarded as a mere gymnastic effort of the intellect, but is almost certain to bring about the discovery of solid and positive knowledge that might otherwise have remained hidden†.

* The proof of this valuable theorem is extremely simple. It rests on the following principles:

(1) That any number which is a common measure to two cyclotomes to the same base must divide the Fermatian to that base whose index is their greatest common measure. This theorem needs only to be stated for the proof to become apparent.

(2) That any cyclotome is contained in the quotient of a Fermatian of the same index by another Fermatian whose index is an aliquot part of the former one. The truth of this will become apparent on considering the form of the linear factors of a cyclotome.

Suppose now that any prime number, k, is a common measure to two cyclotomes whose indices are PQ, PR respectively, where Q is prime to R, and whose common base is Θ. Then k must measure Θ^P-1 and also $\frac{\Theta^{PQ}-1}{\Theta^P-1}$; it will therefore measure Q, and similarly it will measure R; therefore $k=1$ [unless $Q=1$ or $R=1$; for suppose $Q=1$, then $\frac{\Theta^{PQ}-1}{\Theta^P-1}$ is unity, and no longer contains the *core* of $\Theta^{PQ}-1$]. Hence k being contained in R can only be an internal factor to one of the cyclotomes (namely, the one whose index is the greater of the two). (See footnote at end.)

The other theorem preceding this one in the text, and already given in the "Excursus," may be proved as follows:

Let k, any non-unilinear function of P, the index of a cyclotome χ, be a divisor thereto. Then, by Euler's law, there exists some number, μ, such that k divides $x^{\frac{P}{\mu}}-1$, but the cyclotome is contained algebraically in $\frac{x^P-1}{x^{\frac{P}{\mu}}-1}$; hence k must be contained in μ, and therefore in P. Also, k will be a divisor of $x^{\frac{P}{k}}-1$ and of $\frac{x^P-1}{x^{\frac{P}{k}}-1}$, which contain $x^{\frac{P}{\mu}}-1$ and χ respectively; consequently, if k is odd, k^2 will not be a divisor of $\frac{x^P-1}{x^{\frac{P}{k}}-1}$, and *a fortiori* not of χ. (A proof may easily be given applicable to the case of $k=2$.)

Again, let $P=Qk^i$, where Q does not contain k. Then, by Fermat's theorem, $x^{k^i}\equiv x$ [mod. k] and therefore k divides x^Q-1; but it is prime to Q. Hence, by what has been shown, k must be an external divisor of this function, and consequently a unilinear function of Q. Thus, it is seen that a cyclotome can have only one internal divisor, for this divisor, as has been shown, must be an element of the index, and a unilinear function of the product of the highest powers of all the other elements which are contained in the index.

For an extension of this law to "cyclotomes of the second order and conjugate species," see the "Excursus," where I find the words *extrinsic* and *intrinsic* are used instead of *external* and *internal*.

* A reduced Fermatian obviously may be resolved into as many cyclotomes, less one, as its index contains divisors (unity and the number itself as usual counting among the divisors). But, barring the internal divisors, all these cyclotomes to a given base have been proved to be prime to one another, and, consequently, there must be at least as many distinct prime divisors as there are cyclotomes, except in the very special case where the base and index are such that one at least of the cyclotomes becomes equal to its internal divisor or to unity. It may easily be shown that this case only happens when the base is -2 and the index any even number, or when the base is $+2$ and the index divisible by 6; and that in either of these cases there is only a single unit lost in the inferior limit to the number of the elements in the reduced Fermatian.

† Since receiving the revise, I have noticed that it is easy to prove that the algebraical resultant of two cyclotomes to the same base is unity, except when their indices are respectively of the forms $Q(kQ+1)^h$ and $Q(kQ+1)^i$, where $(kQ+1)$ is a prime number, and Q any number (unity *not* excluded), in which case the resultant is $kQ+1$. This theorem supplies the *raison raisonnée* of the proposition proved otherwise in the first part of the long footnote.

61.

NOTE ON CERTAIN DIFFERENCE EQUATIONS WHICH POSSESS AN UNIQUE INTEGRAL.

[*Messenger of Mathematics*, XVIII. (1888–9), pp. 113—122.]

FOR greater simplicity suppose in what follows that a difference equation is expressed in terms of the arguments

$$u_x,\ u_{x+1},\ \dots\ u_{x+i}.$$

I shall call u_{x+i} the highest and u_x the lowest argument respectively, or collectively the extreme or principal arguments, and the degrees in which they enter into the equation the upper and lower or extreme or principal degrees. It is these partial degrees rather than the total degree of the entire equation which determine the essential character of the solution.

If m is the upper degree and $u_0, u_1, \dots u_{i-1}$ be given it is obvious that for any value of x higher than $(i-1)$, u_x will have m^{x-i+1} values, and consequently in general there will be an infinite number of integrals whether complete or of a given order of deficiency (the deficiency being estimated by the number of relations connecting the initial values $u_0, u_1, \dots u_{i-1}$); but it may be, and is in some cases, possible to assign an integral which shall have m^{x-i+1} values, and in such case there can exist no other; such an integral may be called an unique or exhaustive one, and the equations which possess such integrals may be termed uni-solutional.

As the simplest example of such, suppose

$$u^m{}_{x+1} - u_x{}^n = 0,$$

where m and n are integers.

If we write $$u_x = \alpha^{\left(\frac{n}{m}\right)^x}$$

we have $$u_{x+1} = \left(\alpha^{\left(\frac{n}{m}\right)^x}\right)^{\frac{n}{m}}$$

or $$u^m{}_{x+1} = u_x{}^n.$$

Here $u_x = \alpha^{\left(\frac{n}{m}\right)^x}$ is the one and sole complete integral of the equation; for it possesses m^x values so that there can be no other integrals whatever.

Let us now seek to form difference uni-solutional equations of the 2nd order.

To this end let $u_x = C(\alpha^{2^x} - \beta^{2^x})$, where $\alpha\beta = 1$.

Then calling $\alpha^{2^x} = P$ and $\beta^{2^x} = Q$, $PQ = 1$,

$$u_x = C(P - Q),$$
$$u_{x+1} = C(P^2 - Q^2),$$
$$u_{x+2} = C(P^4 - Q^4).$$

Hence $$\frac{u_{x+1}}{u_x} = P + Q,\quad \frac{u_{x+2}}{u_{x+1}} = P^2 + Q^2 = (P+Q)^2 - 2,$$

and $$\frac{u_{x+2}}{u_{x+1}} = \left(\frac{u_{x+1}}{u_x}\right)^2 - 2.$$

Hence the equation

$$u_x{}^2 u_{x+2} - u^3{}_{x+1} + 2u_x{}^2 u_{x+1} = 0$$

has for its complete integral $u_x = C(\alpha^{2^x} - \alpha^{-2^x})$, and there can be no other because when u_0, u_1 are given u_x is absolutely determined.

But furthermore we may invert the above equation by interchanging u_x and u_{x+2}, which gives the equation

$$(u_x + 2u_{x+1})\, u^2{}_{x+2} - u^3{}_{x+1} = 0,$$

of which the solution will obviously be $u_x = C\left(P - \frac{1}{P}\right)$, where $P = \alpha^{(\frac{1}{2})^x}$.

Suppose u_0, u_1 to be given; then

$$C\left(\alpha - \frac{1}{\alpha}\right) = u_0,\quad C\left(\alpha^{\frac{1}{2}} - \frac{1}{\alpha^{\frac{1}{2}}}\right) = u_1,$$

and calling $$\frac{u_0}{u_1} = 2r,\quad \alpha^{\frac{1}{2}} + \frac{1}{\alpha^{\frac{1}{2}}} = 2r,\quad \alpha^{\frac{1}{2}} - \frac{1}{\alpha^{\frac{1}{2}}} = 2\sqrt{(r^2-1)},$$

$$C = \frac{u_0}{4r\sqrt{(r^2-1)}}.$$

Hence

$$u_x = \frac{u_0}{4r\sqrt{(r^2-1)}}\left[\{r + \sqrt{(r^2-1)}\}^{(\frac{1}{2})^{x-1}} - \{r - \sqrt{(r^2-1)}\}^{(\frac{1}{2})^{x-1}}\right],$$

has exactly 2^{x-1} values, for the change of $\sqrt{(r^2-1)}$ into $-\sqrt{(r^2-1)}$ changes simultaneously the signs of the numerator and denominator of this fraction. But by the general principle u_x ought to have 2^{x-1} values in terms of u_0, u_1. Hence the above integral is *exhaustive*.

Suppose now we were to write

$$u_x = C(\alpha^{2^x} + \beta^{2^x})\ \text{with}\ \alpha\beta = 1;$$

for brevity sake call $u_x = f$, $u_{x+1} = g$, $u_{x+2} = h$, then

$$C(P+Q) = f,$$
$$C(P^2+Q^2) = g,$$
$$C(P^4+Q^4) = h,$$
$$PQ = 1.$$

Hence
$$f^2 = Cg + 2C^2,$$
$$g^2 = Ch + 2C^2,$$
$$C = \frac{f^2-g^2}{g-h},$$
$$f^2 = \frac{f^2-g^2}{g-h} \cdot \frac{2f^2-g^2-gh}{g-h},$$

or
$$f^2g^2 - 2f^2gh + f^2h^2 = 2f^4 - 3f^2g^2 + g^4 - f^2gh + g^2h,$$
or
$$f^2h^2 - (g^2+f^2g)h - g^4 + 4f^2g^2 - 2f^4 = 0,$$
or
$$u_x^2u^2_{x+2} - (u^2_{x+1} + u_x^2u_{x+1})u_{x+2} - u^4_{x+1} + 4u_x^2u^2_{x+1} - 2u_x^4 = 0,$$

of which the correlative equation is

$$-2u^4_{x+2} + (4u^2_{x+1} - u_{x+1}u_x + u_x^2)u^2_{x+2} - u^2_{x+1}u_x - u^4_{x+1} = 0.$$

A complete solution of the former of these will therefore be

$$u_x = C(\alpha^{2^x} + \beta^{2^x}),$$

and of the latter
$$u_x = C(\alpha^{(\frac{1}{2})^x} + \beta^{(\frac{1}{2})^x}),$$

but neither of these will be an *exhaustive* solution, for in the one the most general value of u_x ought to be a 2^{x-1}-valued function and in the latter a 4^{x-1}-valued function, whereas the actual value is only one-valued in the one case and 2^{x-1}-valued in the other.

Suppose again we write

$$u_x = C(\alpha^{3^x} - \beta^{3^x}), \text{ where } \alpha\beta = 1, \text{ as before,}$$

say
$$u_x = C(P-Q), \text{ where } PQ = 1.$$

Then with the same notation as before

$$C(P-Q) = f,$$
$$C(P^3-Q^3) = g,$$
$$C(P^9-Q^9) = h,$$
$$\frac{g}{f} - 1 = P^2 + Q^2, \quad \frac{h}{g} - 1 = P^6 + Q^6,$$
$$\frac{h}{g} - 1 = \left(\frac{g}{f} - 1\right)^3 - 3\left(\frac{g}{f} - 1\right),$$

or
$$\frac{h}{g} = \left(\frac{g}{f}\right)^3 - 3\left(\frac{g}{f}\right)^2 + 3,$$
$$f^3h - 3f^3g = g^4 - 3g^3f,$$
$$\frac{h-3g}{g-3f} = \frac{g^3}{f^3}.$$

Whence it follows that the integrals of

$$\frac{u_{x+2} - 3u_{x+1}}{u_{x+1} - 3u_x} - \frac{u^3_{x+1}}{u_x^3} = 0,$$

and of
$$\frac{u^3_{x+2}}{u^3_{x+1}} - \frac{3u_{x+2} - u_{x+1}}{3u_{x+1} - u_x} = 0,$$

are respectively
$$u_x = C(\alpha^{3^x} - \alpha^{-3^x}),$$

and
$$u_x = C(\alpha^{(\frac{1}{3})^x} - \alpha^{-(\frac{1}{3})^x}),$$

with the understanding that $\alpha^{-\frac{1}{3}} \cdot \alpha^{\frac{1}{3}} = 1$.

These integrals are evidently *exhaustive*.

By writing $\sqrt{(-1)}\,\alpha$, $-\sqrt{(-1)}\,\alpha^{-1}$ for α, α^{-1} respectively, f, g, h become increased in the ratio of $\sqrt{(-1)}$, $-\sqrt{(-1)}$, $\sqrt{(-1)}$, respectively.

Hence the equations

$$\frac{u_{x+2} + 3u_{x+1}}{u_{x+1} + 3u_x} - \frac{u^3_{x+1}}{u_x^3} = 0,$$

and
$$\frac{u^3_{x+2}}{u^3_{x+1}} - \frac{3u_{x+2} + u_{x+1}}{3u_{x+1} + u_x} = 0,$$

have for their solutions

$$u_x = C(\alpha^{3^x} + \alpha^{-3^x}) \text{ and } u_x = C(\alpha^{(\frac{1}{3})^x} + \alpha^{-(\frac{1}{3})^x}).$$

Hitherto we have been dealing with *homogeneous* uni-solutional equations. It is easy, however, to form non-homogeneous ones by an obvious process. For, if we write

$$u_x = a_1^{m^x} + a_2^{m^x} + \ldots + a_i^{m^x} \ (m \text{ being an integer}),$$

by eliminating between

$$f_0 = \Sigma a, \ f_1 = \Sigma a^m, \ f_2 = \Sigma a^{m^2}, \ \ldots f_i = \Sigma a^{m^i},$$

we shall obtain a relation between the f's of the first degree in f_i and of the degree m^i in f_0, corresponding to which there will be a difference equation of the ith order in which the upper extreme degree is unity and the lower one m^i, of which the integral will be the value of u_x above written, and by interchanging $u_x, u_{x+1}, \ldots u_{x+i}$ respectively with $u_{x+i}, u_{x+i-1}, \ldots u_x$, another in which the lower degree is unity and the upper one m^i, of which the integral will be

$$u_x = a_1^{\left(\frac{1}{m}\right)^x} + a_2^{\left(\frac{1}{m}\right)^x} + \ldots + a_i^{\left(\frac{1}{m}\right)^x},$$

each of which equations will evidently be uni-solutional.

Or, again, if instead of the a's being independent we make their product equal to unity we shall obtain uni-solutional equations of the $(i-1)$th instead of the ith order.

Thus, for example, let

$$u_x = a^{2^x} + b^{2^x} + c^{2^x} \text{ with the condition } abc = 1.$$

Then writing $u_x = f$, $u_{x+1} = g$, $u_{x+2} = h$,

$$f = A + B + C,\ g = A^2 + B^2 + C^2,\ h = A^4 + B^4 + C^4,$$

$$f^2 - g = 2(AB + AC + BC),$$

$$2(g^2 - h) = 4(A^2B^2 + A^2C^2 + B^2C^2)$$

$$= (f^2 - g)^2 - 8f.$$

Hence we obtain the uni-solutional equations

$$2u_{x+2} - u^2_{x+1} - 2u_{x+1}u_x^2 + u_x^4 - 8u_x = 0,$$

$$u^4_{x+2} - 2u_{x+1}u^2_{x+2} - 8u_{x+2} - u^2_{x+1} + 2u_x = 0,$$

of which the integrals are known and are exhaustive.

We may in a similar manner obtain uni-solutional *simultaneous* difference equations.

Thus let $u_x = C(\alpha^{3^x} - \beta^{3^x}),\ v_x = C'(\alpha^{3^x} + \beta^{3^x}),$

and call u_x, u_{x+1}, u_{x+2} as before $f, g, h,$

and v_x, v_{x+1}, v_{x+2} $l, m, n.$

Then
$$\frac{g}{f} = P^2 + PQ + Q^2,\quad \frac{m}{l} = P^2 - PQ + Q^2,$$

$$\frac{h}{g} = P^6 + P^3Q^3 + Q^6,\ \frac{n}{m} = P^6 - P^3Q^3 + Q^6.$$

Hence
$$\frac{h}{g} - \frac{n}{m} = \tfrac{1}{4}\left(\frac{g}{f} - \frac{m}{l}\right)^3,$$

$$\begin{aligned}\frac{h}{g} + \frac{n}{m} &= 2(P^6 + Q^6)\\ &= 2(P^2 + Q^2)(P^4 - P^2Q^2 + Q^4)\\ &= 2(P^2 + Q^2)\{(P^2 + Q^2)^2 - 3P^2Q^2\}\\ &= \tfrac{1}{4}\left(\frac{g}{f} + \frac{m}{l}\right)\left\{\left(\frac{g}{f} + \frac{m}{l}\right)^2 - 3\left(\frac{g}{f} - \frac{m}{l}\right)^2\right\}\\ &= -\tfrac{1}{4}\left(\frac{g}{f} + \frac{m}{l}\right)\left(2\frac{g^2}{f^2} - 8\frac{g}{f}\cdot\frac{m}{l} + 2\frac{m^2}{l^2}\right);\end{aligned}$$

Hence
$$\frac{h}{g} = \tfrac{1}{8}\left(-\frac{g^3}{f^3} + 3\frac{g^2}{f^2}\cdot\frac{m}{l} + 9\frac{g}{f}\cdot\frac{m^2}{l^2} - 3\frac{m^3}{l^3}\right),$$

$$\frac{n}{m} = \tfrac{1}{8}\left(-3\frac{g^3}{f^3} + 9\frac{g^2}{f^2}\cdot\frac{m}{l} + 3\frac{g}{f}\cdot\frac{m^2}{l^2} - \frac{m^3}{l^3}\right).$$

Obviously, when $u_0, u_1; v_0, v_1$ are given, each u_x and v_x deduced from the above system of equations has only one value, so that their exhaustive integrals will be

$$u_x = C(\alpha^{3^x} - \beta^{3^x}),\ v_x = C'(\alpha^{3^x} + \beta^{3^x}).$$

The related system found by interchanging f with h and l with n will be

$$\frac{f}{g} = \tfrac{1}{8}\left(-\frac{g^3}{h^3} + 3\frac{g^2}{h^2}\cdot\frac{m}{n} + 9\frac{g}{h}\cdot\frac{m^2}{n^2} - 3\frac{m^3}{n^3}\right),$$

$$\frac{l}{m} = \tfrac{1}{8}\left(-3\frac{g^3}{h^3} + 9\frac{g^2}{h^2}\cdot\frac{m}{n} + 3\frac{g}{h}\cdot\frac{m^2}{n^2} - \frac{m^3}{n^3}\right).$$

When $f, g; l, m$ are given the system $\frac{1}{h}, \frac{1}{n}$ may be found by solving an equation of the 9th degree. Hence, when $u_0, u_1; v_0, v_1$ are given, u_2, v_2 will have 9; u_3, v_3, 81, and in general u_x, v_x will have $3^{2(x-1)}$ values which will correspond to the $3^{x-1}\,.\,3^{x-1}$ values of u_x, v_x.

The apparent number of values of each of these is $(3^x)^2$, which, however, must be reducible to $3^{x-1}\,.\,3^{x-1}$ when expressed in terms of the two initial values of u and of v, similarly to what was noticed at the outset on the reduction of the apparent multiplicity 2^x to a multiplicity 2^{x-1}.

In fact, we write

$$u_x = C\left(\alpha^{(\frac{1}{3})^x} - \beta^{(\frac{1}{3})^x}\right),\quad v_x = C'\left(\alpha^{(\frac{1}{3})^x} + \beta^{(\frac{1}{3})^x}\right),$$

$$u_0 = C(\alpha - \beta),\ u_1 = C(\alpha^{\frac{1}{3}} - \beta^{\frac{1}{3}});\ v_0 = C'(\alpha + \beta),\ v_1 = C'(\alpha^{\frac{1}{3}} + \beta^{\frac{1}{3}}),$$

$$\alpha^{\frac{2}{3}} + \alpha^{\frac{1}{3}}\beta^{\frac{1}{3}} + \beta^{\frac{2}{3}} = \frac{u_0}{u_1},\quad \alpha^{\frac{2}{3}} - \alpha^{\frac{1}{3}}\beta^{\frac{1}{3}} + \beta^{\frac{2}{3}} = \frac{v_0}{v_1},$$

$$\alpha^{\frac{1}{3}}\beta^{\frac{1}{3}} = \tfrac{1}{2}\left(\frac{u_0}{u_1} - \frac{v_0}{v_1}\right),$$

$$\alpha^{\frac{2}{3}} + \beta^{\frac{2}{3}} = \tfrac{1}{2}\left(\frac{u_0}{u_1} + \frac{v_0}{v_1}\right),$$

$$\alpha^{\frac{1}{3}} - \beta^{\frac{1}{3}} = \sqrt{\left\{\tfrac{1}{2}\left(\frac{3v_0}{v_1} - \frac{u_0}{u_1}\right)\right\}},$$

$$\alpha^{\frac{1}{3}} + \beta^{\frac{1}{3}} = \sqrt{\left\{\tfrac{1}{2}\left(\frac{3u_0}{u_1} - \frac{v_0}{v_1}\right)\right\}},$$

$$C = \frac{u_1}{\sqrt{\left\{\tfrac{1}{2}\left(\frac{3v_0}{v_1} - \frac{u_0}{u_1}\right)\right\}}},\quad C' = \frac{v_1}{\sqrt{\left\{\tfrac{1}{2}\left(\frac{3u_0}{u_1} - \frac{v_0}{v_1}\right)\right\}}},$$

$$\alpha^{\frac{1}{3}} = \sqrt{\left\{\tfrac{1}{8}\left(\frac{3u_0}{u_1} - \frac{v_0}{v_1}\right)\right\}} + \sqrt{\left\{\tfrac{1}{8}\left(\frac{3v_0}{v_1} - \frac{u_0}{u_1}\right)\right\}},$$

$$\beta^{\frac{1}{3}} = \sqrt{\left\{\tfrac{1}{8}\left(\frac{3u_0}{u_1} - \frac{v_0}{v_1}\right)\right\}} - \sqrt{\left\{\tfrac{1}{8}\left(\frac{3v_0}{v_1} - \frac{u_0}{u_1}\right)\right\}},$$

and thus for the final values of u_x and v_x, we find

$$u_x = \frac{u_1}{\sqrt{\left\{\frac{1}{2}\left(\frac{3v_0}{v_1} - \frac{u_0}{u_1}\right)\right\}}} \times \left\{\left[\sqrt{\left\{\frac{1}{8}\left(\frac{3u_0}{u_1} - \frac{v_0}{v_1}\right)\right\}} + \sqrt{\left\{\frac{1}{8}\left(\frac{3v_0}{v_1} - \frac{u_0}{u_1}\right)\right\}}\right]^{(\frac{1}{3})^{x-1}} - \left[\sqrt{\left\{\frac{1}{8}\left(\frac{3u_0}{u_1} - \frac{v_0}{v_1}\right)\right\}} - \sqrt{\left\{\frac{1}{8}\left(\frac{3v_0}{v_1} - \frac{u_0}{u_1}\right)\right\}}\right]^{(\frac{1}{3})^{x-1}}\right\},$$

$$v_x = \frac{v_1}{\sqrt{\left\{\frac{1}{2}\left(\frac{3u_0}{u_1} - \frac{v_0}{v_1}\right)\right\}}} \times \left\{\left[\sqrt{\left\{\frac{1}{8}\left(\frac{3u_0}{u_1} - \frac{v_0}{v_1}\right)\right\}} + \sqrt{\left\{\frac{1}{8}\left(\frac{3v_0}{v_1} - \frac{u_0}{u_1}\right)\right\}}\right]^{(\frac{1}{3})^{x-1}} + \left[\sqrt{\left\{\frac{1}{8}\left(\frac{3u_0}{u_1} - \frac{v_0}{v_1}\right)\right\}} - \sqrt{\left\{\frac{1}{8}\left(\frac{3v_0}{v_1} - \frac{u_0}{u_1}\right)\right\}}\right]^{(\frac{1}{3})^{x-1}}\right\},$$

each of which is unaffected by a change in the signs of the square roots, so that u_x and v_x are seen to be 3^{x-1}-valued functions, and (u_x, v_x) a 9^{x-1}-valued system, as should be the case for an exhaustive solution of the last written difference equations.

Let us tentatively go a step further in the same direction and suppose that we are given

$$u_x = C(\alpha^{5^x} - \beta^{5^x}), \quad v_x = C'(\alpha^{5^x} + \beta^{5^x}),$$

and use $f, g, h; l, m, n$ in the same way as before, and furthermore, write

$$\tfrac{1}{2}\left(\frac{g}{f} + \frac{m}{l}\right) = L, \quad \tfrac{1}{2}\left(\frac{h}{g} + \frac{n}{m}\right) = N,$$

$$\tfrac{1}{2}\left(\frac{g}{f} - \frac{m}{l}\right) = M, \quad \tfrac{1}{2}\left(\frac{h}{g} - \frac{n}{m}\right) = P,$$

we shall find

$$L = A^4 + A^2B^2 + B^4, \quad N = A^{20} + A^{10}B^{10} + B^{20},$$

$$M = AB(A^2 + B^2), \quad P = A^5B^5(A^{10} + B^{10}),$$

(where $A = \alpha^{5^x}$ and $B = \beta^{5^x}$).

Let $\qquad A^2 + B^2 = \lambda, \quad AB = \mu.$

Then $\qquad L = \lambda^2 - \mu^2, \quad M = \lambda\mu,$

and it will be seen that

$$N = (\lambda^5 - 5\lambda^3\mu^2 + 5\lambda\mu^4)^2 - \mu^{10},$$

$$P = \lambda^5\mu^5 - 5\lambda\mu^7(\lambda^2 - \mu^2).$$

For $\qquad A^6 + B^6 = \lambda^3 - 3\lambda\mu^2,$

and consequently

$$\lambda^5 = A^{10} + B^{10} + 5A^2B^2(A^6 + B^6) + 10A^4B^4(A^2 + B^2)$$

$$= A^{10} + B^{10} + 5\mu^2(\lambda^3 - 3\lambda\mu^2) + 10\lambda\mu^4,$$

that is $\qquad A^{10} + B^{10} = \lambda^5 - 5\lambda^3\mu^2 + 5\lambda\mu^4.$

The above values of N and P (remembering that $AB = \mu$) are found by substituting the expression just obtained for $A^{10} + B^{10}$ in

$$N = A^{20} + A^{10}B^{10} + B^{20},$$

$$P = A^5B^5(A^{10} + B^{10}).$$

From $\qquad P = \lambda^5\mu^5 - 5\lambda\mu^7(\lambda^2 - \mu^2),$

(remembering that $\lambda\mu = M$, $\lambda^2 - \mu^2 = L$), we obtain

$$P = M^5 - 5LM\mu^6.$$

Hence

$$\left.\begin{aligned} \frac{M^5 - P}{5LM^4} &= \frac{\mu^3}{\lambda^3} \\ \frac{L}{M} &= \frac{\lambda}{\mu} - \frac{\mu}{\lambda} \end{aligned}\right\}.$$

From these equations we obtain by elimination

$$\left(\frac{P}{M}\right)^2 + (2M^4 + 15L^2M^2 + 5L^4)\frac{P}{M} + M^4(M^4 - 10L^2M^2 + 5L^4) = 0. \qquad (1)$$

Similarly by an elimination into the details of which it is unnecessary to enter we obtain

$$3LMP + (L^2 + M^2)N = L(L^2 - 2M^2)(L^2 - M^2)^2, \qquad (2)$$

which gives a *linear* relation between N and P.

Equations (1) and (2) form a non-uni-solutional system of which (as also of its inverse) we are in possession of one complete integral, and I have some grounds for suspecting that it may be possible to obtain from this a second (so-called *indirect*) integral, but am unable for the present to pursue the subject further.

The preceding investigation originated in my attention happening to be called to Vieta's well known theorem for approximating to the *Archimedean* constant (π) by means of an indefinite product of cosines of continually bisected angles. The implied connection of ideas will become apparent when one considers that any one of such cosines may be expressed as a sum of two binary exponentials with $\frac{1}{2}$ for the first index, and that thus Vieta's theorem (although presumably obtained by him as a very simple consequence of the method of exhaustions) in its essence depends on the integrability of a uni-solutional difference equation of the 2nd order of the form treated of at the outset of this paper.

62.

SUR LA RÉDUCTION BIORTHOGONALE D'UNE FORME LINÉO-LINÉAIRE À SA FORME CANONIQUE.

[*Comptes Rendus*, CVIII. (1889), pp. 651—653.]

Soit F une fonction linéo-linéaire des deux séries de lettres

$$x_1, x_2, \ldots, x_n; \quad \xi_1, \xi_2, \ldots, \xi_n;$$

alors F contiendra n^2 termes. En assujettissant les x et les ξ respectivement à deux substitutions orthogonales indépendantes, on introduit dans la transformée n^2-n quantités arbitraires, de sorte que, en leur donnant des valeurs convenables, on doit pouvoir faire disparaître ce nombre de termes en ne conservant que les n paires dont les arguments seront (par exemple)

$$x_1\xi_1, x_2\xi_2, \ldots, x_n\xi_n.$$

On peut nommer les multiples de ces arguments les *multiplicateurs canoniques*; je vais donner la règle pour les déterminer, et en même temps pour trouver les deux substitutions orthogonales simultanées qui amènent la forme canonique. La marche à suivre sera parfaitement analogue à celle qui s'applique à la réduction d'une forme quadrique à n lettres à sa forme canonique au moyen d'une seule substitution orthogonale; mais on remarquera, *a priori*, une distinction essentielle entre les deux questions. Pour le cas d'une seule quadrique, les multiplicateurs canoniques sont absolument déterminés; mais, pour le cas actuel, il est évident que chacun de ces multiplicateurs peut changer son signe, de sorte que ce sont les carrés de ces multiplicateurs qui doivent se présenter dans le résultat.

Il sera utile de rappeler quelques faits élémentaires sur les matrices. Le carré d'une matrice est la matrice qui se produit par la multiplication des lignes par les colonnes; il sera une matrice non symétrique dont les *racines latentes* seront les carrés des racines latentes d'une matrice donnée. Au contraire, le produit d'une matrice par son transverse donnera (selon l'ordre de la multiplication) lieu à deux matrices symétriques qu'on obtient par la multiplication des lignes par des lignes ou bien par celle des colonnes par

les colonnes; ces matrices seront distinctes, mais posséderont les mêmes racines latentes, c'est-à-dire en affectant tous les termes dans la diagonale de symétrie de l'un ou de l'autre avec la même addition, soit $-\lambda$, le déterminant d'une matrice ainsi affectée sera le même pour l'un comme pour l'autre*.

En différentiant F par rapport aux x et aux ξ, on obtient deux matrices, dont l'une sera la transverse de l'autre, que je nommerai les matrices déterminatives. Avec l'aide de ces matrices on obtient une solution complète du problème voulu.

(1) Pour déterminer les multiplicateurs canoniques:

Je dis que les racines latentes de leur produit seront les carrés des multiplicateurs canoniques.

Il peut arriver qu'un de ces multiplicateurs soit zéro; alors le dernier terme de l'équation aux racines latentes, qui n'est autre chose que le carré du déterminant d'une matrice déterminative, s'évanouit; et l'on voit que le cas de la disparition d'un des n termes dans la réduite canonique est indiqué par l'évanouissement du déterminant de la matrice déterminative.

(2) Pour trouver les deux substitutions orthogonales canoniques:

Prenons une des deux matrices symétriques affectées de $-\lambda$ dans chaque terme de sa diagonale; en supprimant une quelconque de ses lignes, les n premiers mineurs de la matrice diminuée qui restent divisés chacun par la racine carrée de la somme de leurs carrés (fonctions de λ), en donnant à λ successivement les valeurs des n racines latentes, fourniront les n^2 termes d'une des substitutions orthogonales, et de même on obtient l'autre substitution orthogonale en agissant semblablement *pas à pas* sur l'autre matrice affectée: ainsi le problème de la réduction voulue est complètement résolu.

Prenons, par exemple,

$$F = 8x\xi - x\eta - 4y\xi + 7y\eta.$$

* Toutes ces racines latentes seront non seulement réelles (comme elles doivent l'être à cause de la forme symétrique de la matrice), mais aussi positives; car, en substituant λ à $-\lambda$, les coefficients de l'équation latente (en commençant avec le dernier) sont, respectivement, le carré du déterminant complet, la somme des carrés des premiers mineurs, des seconds mineurs, etc., de la matrice déterminative (le premier coefficient étant l'unité et le second la somme des carrés des coefficients de la forme bilinéaire). Chacune de ces sommes sera un invariant biorthogonal, et le déterminant de la matrice déterminative lui-même sera un invariant gauche de la forme bilinéaire.

Ajoutons que les deux matrices qui sont les carrés cauchiens de cette matrice, envisagées comme discriminants, fourniront deux quadriques (dont chacune contiendra un seul des deux systèmes donnés de lettres) qui seront des covariants orthogonaux simultanés de la fonction bilinéaire donnée.

(1) Pour trouver les multiplicateurs canoniques:

On prend la matrice déterminative dans ses deux formes

$$\begin{matrix} 8; & -1 \\ -4; & 7 \end{matrix}, \qquad \begin{matrix} 8; & -4 \\ -1; & 7 \end{matrix},$$

dont les produits affectés seront

$$\begin{matrix} 65-\lambda; & -39 \\ -39\;; & 65-\lambda \end{matrix}, \qquad \begin{matrix} 80-\lambda; & -36 \\ -36\;; & 50-\lambda \end{matrix}.$$

Ainsi, en se servant de l'un ou de l'autre, on obtient

$$\lambda^2 - 130\lambda + 2704 = 0,$$

dont les racines sont 26 et 104, de sorte que $\sqrt{26}$ et $2\sqrt{26}$ seront les multiplicateurs canoniques.

(2) Pour trouver les substitutions, on assigne ses deux valeurs à

$$39 : 65 - \lambda, \quad \text{c'est-à-dire} \quad 39 : 39 \text{ et } 39 : -39$$

et à

$$36 : 80 - \lambda, \quad \text{c'est-à-dire} \quad 36 : 54 \text{ et } 36 : -24.$$

Ainsi l'on aura, pour les deux matrices de substitution,

$$\begin{matrix} \frac{1}{\sqrt{2}}; & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}}; & \frac{1}{\sqrt{2}} \end{matrix} \quad \text{et} \quad \begin{matrix} \frac{2}{\sqrt{13}}; & \frac{3}{\sqrt{13}} \\ -\frac{3}{\sqrt{13}}; & \frac{2}{\sqrt{13}} \end{matrix},$$

et, en effet, on vérifie facilement que

$$\sqrt{26}\left(\frac{x}{\sqrt{2}}+\frac{y}{\sqrt{2}}\right)\left(\frac{2\xi}{\sqrt{13}}+\frac{3\eta}{\sqrt{13}}\right)+2\sqrt{26}\left(-\frac{x}{\sqrt{2}}+\frac{y}{\sqrt{2}}\right)\left(-\frac{3\xi}{\sqrt{13}}+\frac{2\eta}{\sqrt{13}}\right)$$
$$= 8x\xi - x\eta - 4y\xi + 7y\eta.$$

Si l'on donne les deux matrices symétriques ayant les mêmes racines latentes qui doivent représenter respectivement les deux produits *cauchiens* d'une matrice de l'ordre n par elle-même, on verra facilement que le problème de trouver cette dernière matrice a été virtuellement résolu plus haut, et que, comme le problème de trouver la véritable racine carrée d'une seule matrice générale donnée, il admet 2^n solutions.

63.

SUR LA CORRESPONDANCE COMPLÈTE ENTRE LES FRACTIONS CONTINUES QUI EXPRIMENT LES DEUX RACINES D'UNE ÉQUATION QUADRATIQUE DONT LES COEFFICIENTS SONT DES NOMBRES RATIONNELS.

[*Comptes Rendus*, CVIII. (1889), pp. 1037—1041.]

SI $u_i = \lambda_i u_{i-1} + u_{i-2}$ et $u_{-1} = 0$, $u_0 = 1$, on peut appeler u_i un cumulant dont la succession $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_i$ est le type; désignons-le par t.

Alors on peut représenter

Par $`t$ la succession............ $\lambda_2, \lambda_3, \ldots, \lambda_i$
Par t' " ....$\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{i-1}$
Par $`t'$ " .... $\lambda_2, \lambda_3, \ldots, \lambda_{i-1}$.

De plus, on peut représenter par θt la réunion du type θ suivi par le type t; par $\theta 0 t$ ce que devient θt quand on intercale un zéro entre la succession θ et la succession t; par $\theta(0t)^i$ la succession θ suivie par la succession $0t$ répétée i fois; et par $t(0\theta)^i\tau$ ce que devient $t\tau$ quand on intercale 0θ i fois entre le t et le τ.

T étant un type quelconque, on peut désigner par $[T]$ le cumulant dont T est le type.

Ainsi, si les éléments en T sont regardés comme les quotients partiels d'une fraction continue, et que, suivant la notation de l'immortel Lejeune-Dirichlet, on représente par (T) la dernière convergente à cette fraction, on aura

$$(T) = [T] \div [`T].$$

Désignons par $\underline{\theta}$ ce que devient θ quand on renverse l'ordre, et par $\overline{\theta}$ ce qu'il devient quand on change le signe de chacun de ses éléments. Posons

$$T_i = \underline{\theta} t (0t)^i \overline{\theta};$$

j'ai trouvé et démontré le lemme suivant*:

* Pour établir cette proposition, on n'a besoin que de se servir des *deux* identités suivantes. Si $T = t\theta$,

$$[T] = [t][\theta] + [t'][`\theta].$$

Les rapports des trois quantités $[T_i] : [{}^{\backprime}T_i] - [T_i'] : [{}^{\backprime}T_i']$ *sont indépendants de* $\underline{i}$*; c'est-à-dire sont les mêmes que les rapports de*

$$[\underline{\theta}t\bar{\theta}] : [{}^{\backprime}\underline{\theta}t\bar{\theta}] - [\underline{\theta}t\bar{\theta}'] : [{}^{\backprime}\underline{\theta}t\bar{\theta}'].$$

Avec l'aide de ce théorème et de l'équation qui exprime une propriété bien connue des convergentes successives de fractions continues, savoir

$$[T][{}^{\backprime}T'] - [{}^{\backprime}T][T'] = \pm 1,$$

on établit facilement le théorème suivant :

On peut écrire et d'une seule manière les deux racines d'une équation quadratique simultanément sous les formes

$$(\theta t(0t)^{\infty}), \quad -(\bar{\theta}\underline{t}(0\underline{t})^{\infty}),$$

où tous les éléments de θ*, sauf le dernier (qui peut être zéro), et tous les éléments de* t *sont positifs.*

Comme un simple corollaire de ce théorème de correspondance, en appliquant à la seconde forme la méthode donnée par Dirichlet pour *régulariser* une succession de quotients partiels dont quelques-uns au commencement sont négatifs, on voit que les périodes des deux fractions convergentes contiendront les mêmes éléments, mais en ordre inverse.

Un exemple fera mieux comprendre la portée du théorème.

Prenons l'équation

$$23x^2 - 68x + 50 = 0,$$

dont les racines sont

$$\frac{34 + \sqrt{6}}{23}, \quad \frac{34 - \sqrt{6}}{23}.$$

On trouve, pour le développement de ces deux quantités, les fractions périodiques en fractions continues

$$(1, 2, 1, 2 ; 4, 2 ; 4, 2 ; 4, 2 ; \ldots)$$

et

$$(1, 1, 1, 2 ; 2, 4 ; 2, 4 ; 2, 4 ; \ldots)$$

respectivement.

Si $T = t\theta\tau$, $\quad [T] = [t][\theta][\tau] + [t'][{}^{\backprime}\theta][\tau] + [t][\theta'][{}^{\backprime}\tau] + [t'][{}^{\backprime}\theta'][{}^{\backprime}\tau].$

On peut cependant ajouter que, de même, si $T = t\theta\tau\omega$,

$$[T] = [t][\theta][\tau][\omega] + [t'][{}^{\backprime}\theta][\tau][\omega] + [t][\theta'][{}^{\backprime}\tau][\omega] + [t][\theta][\tau'][{}^{\backprime}\omega]$$
$$+ [t'][{}^{\backprime}\theta'][{}^{\backprime}\tau][\omega] + [t'][{}^{\backprime}\theta][\tau'][{}^{\backprime}\omega] + [t][\theta'][{}^{\backprime}\tau'][{}^{\backprime}\omega] + [t'][{}^{\backprime}\theta'][{}^{\backprime}\tau'][{}^{\backprime}\omega],$$

où l'on remarquera que les trois premiers produits de la deuxième ligne sont composés de deux (le premier et le dernier) de formes analogues, et d'un troisième d'une forme différente, et ainsi, en général, si le nombre des types partiels $t, \theta, \tau, \ldots$ est $\underline{i}$, on aura 2^{i-1} produits de cumulants partiels et de leurs dérivés simples et doubles ; car il y aura $(i-1)$ intervalles entre les i types sur lesquels on doit faire tomber dans chaque manière possible $1, 2, 3, \ldots (i-1)$ paires d'accents. Quand les types partiels deviennent monomiaux, les termes avec les accents doubles dans la somme des produits deviennent zéros, et l'on retrouve la règle connue pour exprimer un cumulant comme somme des produits des agrégats de ses éléments, en élisant ou en traitant comme unités des paires et combinaisons de paires d'éléments consécutifs.

Or, en écrivant

$$\theta = 1, 2, \quad t = 1, 2, 3,$$

on aura

$$(\theta t(0t)^{\infty}) = (1, 2, 1, 2, 3, 0, 1, 2, 3, 0, 1, 2, 3, 0, 1, 2, \ldots)$$
$$= (1, 2, 1, 2 ; \quad 4, 2 ; \quad 4, 2 ; \quad 4, 2 ; \ldots),$$

ce qui répond à la première racine.

On aura aussi

$$(\bar{\theta}\underline{t}(0\underline{t})^{\infty}) = (-1, -2, 3, 2, 1, 0, 3, 2, 1, 0, 3, \ldots)$$
$$= (-1, -2, 3 ; 2, \quad 4 ; 2, \quad 4 ; \ldots),$$

laquelle convergente, *régularisée* selon les règles de Dirichlet*, peut être remplacée par

$$(-2, 1, 0, 1, 2 ; 2, 4 ; 2, 4 ; \ldots),$$

c'est-à-dire

$$(-2, 2, 2 ; 2, 4 ; 2, 4 ; \ldots),$$

ce qui, selon les mêmes règles, équivaut à

$$-(1, 1, 1, 2 ; 2, 4 ; 2, 4 ; \ldots),$$

laquelle est la valeur prise négativement de la seconde racine.

Terminons par l'exemple très simple

$$x^2 - 10x - 1 = 0,$$

dont les deux racines sont $5 + \sqrt{26}$, $5 - \sqrt{26}$, qui équivalent aux fractions continues

$$(10, 10, 10, \ldots), \quad -(0, 10, 10, 10, \ldots).$$

Faisons

$$\theta = 9, 0, \quad t = 1, 9.$$

Alors $(\theta t(0t)^{\infty})$ devient

$$(9, 0 ; 1, 9 ; 0, 1, 9 ; 0, 1, 9 ; \ldots),$$

c'est-à-dire

$$(10 ; 10 ; 10 ; \ldots),$$

la première racine ; et $(\bar{\theta}\underline{t}(0\underline{t})^{\infty})$ devient

$$(-9, 0 ; 9, 1 ; 0, 9, 1 ; 0, 9, 1 ; \ldots),$$

ce qui équivaut à

$$(0, 10, 10, \ldots),$$

laquelle est la valeur prise négativement de la seconde racine.

On comprendra que dans les formules pour une racine et la négative de l'autre, rien n'empêche que le θ disparaisse et qu'ainsi les formules deviennent

$$(t(0t)^{\infty}), \quad (\underline{t}(0\underline{t})^{\infty})$$

respectivement.

* *Vorlesungen über Zahlentheorie*, § 80 ; 1871.

Dans le cas où les deux racines sont égales, mais de signes contraires, non seulement le $\underline{\theta}$ disparaît, mais aussi le $\underline{t}$ devient symétrique : ainsi l'on retrouve la forme applicable à l'équation $Ax^2 - \beta = 0$, pour lequel cas la racine positive peut être mise sous la forme

$$(abc, \ldots, cba, 0, abc, \ldots, cba, 0, abc),$$

c'est-à-dire

$$(a_j bc, \ldots, cb, 2a_j bc, cb, 2a_j).$$

On peut encore simplifier un peu les expressions pour x et x' (où x et x' sont les racines de la même équation quadratique) en écrivant

$$x = (\theta\,(t, 0)^\infty), \quad x' = -(\bar{\theta}\,(\underline{t}, 0)^\infty),$$

formule vraiment surprenante par sa simplicité et sa symétrie.

64.

SUR LA REPRÉSENTATION DES FRACTIONS CONTINUES QUI EXPRIMENT LES DEUX RACINES D'UNE ÉQUATION QUADRATIQUE.

[*Comptes Rendus*, CVIII. (1889), pp. 1084—1086.]

NOUS avons donné dans une Note précédente [p. 644, above], pour les deux racines x et x' d'une équation quadratique à coefficients entiers, les formules jumelles

$$x = (t\,(\tau 0)^\infty), \quad -x' = (\bar{t}\,(\underline{\tau} 0)^\infty).$$

Mais ces formules admettent encore une simplification importante au moyen des considérations suivantes.

Un type peut être nommé *omni-positif* ou *omni-négatif* quand tous ses éléments sont positifs pour un des cas et tous négatifs pour l'autre : il sera nommé *homonyme* quand il est *ou* omni-positif *ou* omni-négatif sans spécifier lequel des deux il est.

Le zéro sera regardé comme un *nombre* (non pas neutre, mais) *amphibolique*, c'est-à-dire qui est en même temps positif et négatif, de sorte qu'un type omni-positif ou omni-négatif ne cesse pas d'être homonyme en y ajoutant ou y entremêlant un ou plusieurs zéros.

De plus, on remarquera que $(\bar{T}) = -(T)$.

Alors la théorie, atteignant son dernier terme de simplicité et de généralité, donne lieu à l'énoncé suivant :

En supposant que t est un type homonyme quelconque et τ un autre, et que x, x' sont les deux racines d'une équation quadratique à coefficients entiers, on aura toujours

$$x = (t\tau^\infty), \quad x' = (t0\,(\bar{\tau})^\infty)$$

avec la faculté à t de disparaître.

Ainsi, par exemple, en supposant que t disparaisse et que τ devienne monomial et égal à a, si

$$x = (a, a, a, \ldots, \textit{ad infinitum}),$$

on aura

$$x' = (0, -a, -a, \ldots, \textit{ad infinitum}),$$

c'est-à-dire $$x' = -(0, a, a, a, \ldots, \textit{ad infinitum});$$

de sorte que $$x' = -\frac{1}{x}\,*.$$

On remarquera que les types $t\tau^{\infty}$, $t0\underline{\bar{\tau}}^{\infty}$ sont mutuellement inverses l'un de l'autre, car $(t00\underline{\underline{\bar{\tau}}}^{\infty}) = (t\tau^{\infty})$.

Nous nous sommes déjà servi† dans nos conférences, tenues à King's College London en 1859, sur la détermination du nombre de solutions en nombres entiers d'un système d'équations numériques‡, avec grand avantage de cette idée d'une série de quantités omni-positive, omni-négative ou homonyme et de la conception du caractère du zéro comme appartenant aux deux catégories des quantités positives et négatives à la fois.

Dans une Note à suivre, nous nous proposons de faire connaître la connexion§ remarquable qui subsiste entre les racines de l'équation

$$ax^2 + 2bx + c = 0$$

et les développements en fractions continues des fractions ordinaires $\dfrac{p \pm bq}{aq}$, où p, q sont les nombres de Pell qui appartiennent au déterminant $b^2 - ac$, et, si nous ne nous sommes pas trompé, nous espérons fonder là-dessus une règle pour l'extraction simultanée des deux racines de l'équation au moyen de ces deux développements.

* Et, en général, quand $x = -\dfrac{1}{x'}$, on aura

$$x = ((\theta)^{\infty}),$$

où θ est un type symétrique, ce qui est le théorème de Gallois (*Journal de Liouville*, t. II. p. 385).

De même, si $x = ((\theta 0)^{\infty})$ (θ étant symétrique) et ainsi $\underline{\theta} = \theta$, on aura

$$-x' = ((00\theta 0)^{\infty}) = ((\theta 0)^{\infty}) = x,$$

de sorte que $((\theta 0)^{\infty})$ est la forme générale de la fraction continue qui exprime la racine carrée d'une quantité rationnelle quelconque.

[† See Vol. II. of this Reprint, p. 122.]

‡ Inédites jusqu'à ce jour, mais qui doivent paraître prochainement dans l'*American Journal of Mathematics*. C'est dans nos recherches sur ce sujet que nous avons rencontré et discuté la théorie géométrique de dispositions de points dans un plan et dans l'espace que notre éminent Confrère M. Halphen a retrouvée indépendamment depuis et à laquelle il a donné le nom de *théorie d'aspects*. C'est en réduisant la détermination du nombre de solutions en nombres entiers d'un système de 3 équations à dépendre d'un agrégat de pareilles déterminations pour des systèmes de 2 équations que cette théorie s'est forcément mise en évidence pour les points dans un plan. De même, en faisant dépendre le problème pour un système de 4 de celui de systèmes de 3 équations, on est amené à une théorie semblable pour l'espace; bien entendu, l'œil regardé comme un seul point dans la théorie pour le plan devient linéaire, ou, ce qui revient à la même chose, un système de deux points, pour l'espace.

§ Pour l'établir, nous nous servons encore de notre théorème de l'immutabilité des rapports de $[`T'] : [T'] - [`T] : [T]$ quand $T = t\tau\,(0\tau)^i\,\underline{\bar{t}}$ pour toute valeur positive et entière de i.

65.

SUR LA VALEUR D'UNE FRACTION CONTINUE FINIE ET PUREMENT PÉRIODIQUE.

[*Comptes Rendus*, CVIII. (1889), pp. 1195—1198.]

On sait que la valeur de la fraction purement périodique infinie (t^{∞}), où t est un type (c'est-à-dire une succession) d'éléments quelconques, est la racine positive de l'équation

$$[`t]x^2 - ([t] - [`t'])x - [t'] = 0. \qquad (1)$$

Cela conduit naturellement à la question de trouver la valeur de la fraction continue analogue périodique mais finie (t^n).

Avec l'aide de notre formule donnée dans une Note précédente, qui sert à exprimer un cumulant à un type composé de i types partiels comme une somme de 2^{i-1} produits des i cumulants partiels et leurs dérivées simples et doubles, on peut résoudre cette question sans aucune difficulté.

On a $$(t^n) = \frac{[t^n]}{[`t^n]} = \frac{[t^n]}{[`tt^{n-1}]}.$$

Soient $$[t^n] = u_n, \quad [`tt^{n-1}] = `tv_n,$$

on trouve que v_n sera une fonction entière et l'on établit, au moyen de la formule citée, entre u_n et v_n les équations aux différences

$$u_n - au_{n-1} - Bu_{n-2} = cBv_{n-2}, \quad v_{n-1} - cv_{n-2} = u_{n-2},$$

où $$a = [t], \quad B = [`t][t'], \quad c = [`t'].$$

Donc $$Bv_{n-1} = u_n - au_{n-1},$$

$$av_n + (B - ac)\,v_{n-1} = u_n = v_{n+1} - cv_n,$$

$$v_{n+1} - (a + c)\,v_n + (-)^{\mu-1}\,v_{n-1} = 0$$

[car $B - ac = (-)^{\mu-1}$, μ étant le nombre d'éléments en t].

Conséquemment, par un principe bien connu, v_n et u_n seront les coefficients de k^n dans le développement d'une fraction de la forme

$$\frac{A + Bk}{1 - (a + c)\,k - \epsilon k^2},$$

où $\epsilon = (-)^n$, A et B étant convenablement déterminés pour l'un et pour l'autre cas.

Or

$$u_0 = 1, \quad u_1 = a,$$
$$v_0 = 0, \quad v_1 = 1.$$

Donc u_n est le coefficient de k^n en $\frac{1 - ck}{1 - (a+c)k - \epsilon k^2}$ et v_n le coefficient de k^n en $\frac{k}{1 - (a+c)k - \epsilon k^2}$, de sorte que, si l'on écrit

$$\Phi_n(x) = x^n + (n-1)\epsilon x^{n-2} + \frac{(n-2)(n-3)}{2}\epsilon^2 x^{n-4} + \ldots$$

jusqu'au premier terme qui devient zéro, on aura

$$v_n = \Phi_{n-1}(a+c)$$

et

$$u_n = \Phi_n(a+c) - c\Phi_{n-1}(a+c).$$

Ainsi l'on voit que

$$(t^n) = \frac{(\Phi_n - [{}^{\backprime}t']\Phi_{n-1})(a+c)}{[{}^{\backprime}t]\Phi_{n-1}(a+c)}.$$

On peut aussi exprimer u_n et v_n au moyen des racines de l'équation

$$m^2 - ([t] + [{}^{\backprime}t'])m - \epsilon = 0,$$

dont on remarquera que le déterminant $\frac{1}{4}([t] + [{}^{\backprime}t'])^2 + \epsilon$ est le même que celui de l'équation (1), puisque

$$\tfrac{1}{4}([t] - [{}^{\backprime}t'])^2 + [{}^{\backprime}t][t'] = \tfrac{1}{4}([t] + [{}^{\backprime}t'])^2 + \epsilon;$$

car, en supposant que ρ et σ sont les deux racines, on aura

$$\frac{u_n}{v_n} = \frac{A\rho^n - B\sigma^n}{\rho^n - \sigma^n},$$

où A, B sont des quantités connues; et, en supposant que $\rho^2 => \sigma^2$, on aura $\frac{u_\infty}{v_\infty} = A$ et $(t^\infty) = \frac{A}{{}^{\backprime}t}$, laquelle valeur on identifiera facilement avec la racine positive de l'équation

$$[{}^{\backprime}t]x^2 - ([t] - [{}^{\backprime}t'])x - [t'] = 0.$$

Si l'on suppose que les éléments de t sont m en nombre et tous identiques avec l'unité, on aura

$$[{}^{\backprime}t] = [1^{m-1}], \quad [{}^{\backprime}t^n] = [1^{mn-1}],$$

et l'on obtient la formule peut-être nouvelle

$$\frac{\Phi_{mn-1}(1)}{\Phi_{m-1}(1)} = \Phi_{n-1}(\Psi_m),$$

où $\Psi_m = \Phi_m(1) + \Phi_{m-2}(1)$.

Si l'on suppose que m est impair, ϵ sera positif et Ψ_m prendra la forme

$$1 + m + m\frac{m-3}{2} + m\frac{(m-4)(m-5)}{2.3} + \ldots,$$

en s'arrêtant au premier terme qui devient zéro.

Cette formule donne naissance à un corollaire intéressant. Supposons que la somme de deux termes séparés par un seul dans la série *phyllotactique* 1, 2, 3, 5, 8, 13, 21, ... est un nombre premier p. Soit m, $m-2$ l'ordre de ces deux termes; alors je dis que le quotient du nombre de l'ordre $mi - 1$ par celui de l'ordre $m-1$ (nombre toujours entier) par rapport au module p sera congru à l'unité si i est impair et à zéro si i est pair; de plus, dans ce dernier cas où $i = 2j$, le quotient de ce quotient divisé par p sera congru à $(-)^j(j+1)$ par rapport au même module p.

On pourrait tirer sans doute d'autres théorèmes analogues, mais apparemment moins simples, au moyen de l'équation

$$[t^n] = \Phi_n[t] - [t']\Phi_{n-1}[t].$$

C'est une chose qu'on n'avait nul droit (*a priori*) d'attendre que le quotient $[{}^{\backprime}t^n] \div [{}^{\backprime}t]$, au lieu d'être une fonction rationnelle et entière de quatre quantités $[t]$, $[{}^{\backprime}t]$, $[t']$, $[{}^{\backprime}t']$ ou (ce qui est équivalent) rationnelle et fractionnelle de $[t]$, $[{}^{\backprime}t]$, $[t']$, est en effet une fonction rationnelle et entière d'une seule quantité, savoir de $[t] + [{}^{\backprime}t']$, c'est-à-dire est un nombre *phyllotactique* affecté ou paramétrique, nom qu'on peut convenablement donner à la valeur de $[x^n]$, où x est monomial et entier, $[1^n]$ prenant alors le nom de *nombre phyllotactique simple* ou *unitaire*.

66.

A NEW PROOF THAT A GENERAL QUADRIC MAY BE REDUCED TO ITS CANONICAL FORM (THAT IS, A LINEAR FUNCTION OF SQUARES) BY MEANS OF A REAL ORTHOGONAL SUBSTITUTION.

[*Messenger of Mathematics*, XIX. (1890), pp. 1—5.]

ALL the proofs that I am acquainted with (and their name is legion) of the possibility of depriving a quadric, in three or more variables, of its mixed terms by a real orthogonal transformation are made to depend on the theorem that the "*latent roots*" of any symmetrical matrix are all real.

By the latent roots is understood the roots of the determinant expressed by tacking on a variable $-\lambda$ to each term in the diagonal of symmetry to such matrix.

I shall show that the same conclusion may be established *à priori* by purely algebraical ratiocination and without constructing any equation, by the method of cumulative variation. The proof I employ is inductive: that is, if the theorem is true for two or any number of variables I prove that it will be true for one more.

To illustrate the method let us begin with two variables. Consider the form $ax^2 + 2hxy + by^2$.

If in any such form $b = a$, then by an obvious orthogonal transformation, namely, writing $\frac{x+y}{\sqrt{(2)}}$ and $\frac{x-y}{\sqrt{(2)}}$ for x and y, the form becomes

$$a(x^2+y^2) + h(x^2-y^2),$$

or

$$(a+h)x^2 + (a-h)y^2.$$

Now in general on imposing on x, y any orthogonal infinitesimal substitution, so that

$$x \text{ becomes } x + \epsilon y,$$
$$y \quad ,, \quad y - \epsilon x,$$

h in the new form becomes $h + (a-b)\epsilon$, or say $\delta h = (a-b)\epsilon$, and

$$\tfrac{1}{2}\delta(h^2) = (a-b)h\epsilon;$$

the variations of a and b need not be set forth.

Let an infinite succession of such transformations be instituted; then either a and b become equal and the orthogonal substitution above referred to reduces the quadric to its canonical form, in which case this one combined with the preceding infinite series of such substitutions may be compounded into a single substitution, or else by giving ϵ the sign of $(b-a)$ the variation of h^2 may at each step be made negative so that h^2 continually decreases, unless h vanishes. If h does not vanish it must have a minimum value, and this minimum value may be diminished, which involves a contradiction: hence, in the infinite series of substitutions supposed, either a and b become equal or h vanishes, and in either case the quadric is reduced or reducible to its canonical form.

Let us now take the case of three variables x, y, z.

Obviously, by the preceding case, we may make the term involving xy disappear and commence with the initial form

$$ax^2 + by^2 + 2fxz + 2gyz + cz^2.$$

If f or g become zero the quadric may be canonified by virtue of the preceding case.

Again, if $b = a$, by imposing on x, y the orthogonal substitution

$$\frac{g}{\sqrt{(f^2+g^2)}}x + \frac{f}{\sqrt{(f^2+g^2)}}y$$
$$-\frac{f}{\sqrt{(f^2+g^2)}}x + \frac{g}{\sqrt{(f^2+g^2)}}y,$$

the term involving xz will disappear and the final result is the same as if f were zero.

Let us now introduce the infinitesimal orthogonal substitution which changes

$$x \text{ into } \quad x + \epsilon y + \eta z,$$
$$y \quad ,, \quad -\epsilon x + y + \theta z,$$
$$z \quad ,, \quad -\eta x - \theta y + z,$$

where ϵ, η, θ are supposed to be of the same order of magnitude so that only first powers of them have to be considered.

Then

$$\delta f = (a-c)\eta - g\epsilon,$$
$$\delta g = (b-c)\theta + f\epsilon,$$

also the coefficient of $2xy$ becomes $(a-b)\epsilon - f\theta - g\eta$.

Now whatever η, θ may be, we may determine ϵ in terms of η, θ so that this may be made to vanish, and the initial form of the quadric will be maintained, provided that b is not equal to a.

Hence instituting an infinite series of these infinitesimal substitutions, provided we do not reach a stage where a and b become equal, we may maintain the original form keeping η, θ arbitrary, and shall have

$$\tfrac{1}{2}\delta(f^2+g^2)=(a-c)f\eta+(b-c)g\theta.$$

Suppose a and b to be unequal; therefore $(a-c)$, $(b-c)$ do not vanish simultaneously, and consequently we may make $\delta(f^2+g^2)$ negative unless at least one of the two quantities f, g vanishes.

If neither of them vanishes f^2+g^2 may be made continually to decrease and will have a minimum other than zero, which involves a contradiction.

Hence the infinite series of infinitesimal orthogonal substitutions may be so conducted that either $a-b$ or one at least of the letters f, g shall become zero; and then two additional orthogonal substitutions at most will serve to reduce the Quadric immediately to its canonical form.

I shall go one step further to the case of four variables x, y, z, t and then the course of the induction will become manifest. We may, by virtue of what has been shown, take as our quadric

$$ax^2+by^2+cz^2+2fxt+2gyt+2hzt+dt^2.$$

Here, if any one of the mixed terms disappears, the quadric is immediately reducible by the preceding case, and if any two of the grouped pure coefficients a, b, c become equal (as for instance a, b), then by an orthogonal transformation one of the mixed terms (f or g in the case supposed) may be got rid of; so that this supposition merges in the preceding one.

Impose on x, y, z, t an infinitesimal orthogonal substitution, writing

$$\begin{array}{rl} x+\epsilon y+\theta z+\lambda t & \text{for } x,\\ -\epsilon x+y+\eta z+\mu t & \text{,, } y,\\ -\theta x-\eta y+z+\nu t & \text{,, } z,\\ -\lambda x-\mu y-\nu z+t & \text{,, } t.\end{array}$$

Then
$$\delta f=(a-d)\lambda-g\epsilon-h\theta,$$
$$\delta g=(b-d)\mu+f\epsilon-h\eta,$$
$$\delta h=(c-d)\nu+f\theta+g\eta.$$

Also the coefficients of $2xy$, $2xz$, $2yz$ respectively become

$$(a-b)\epsilon-f\mu-g\lambda,$$
$$(a-c)\theta-f\nu-h\lambda,$$
$$(b-c)\eta-g\nu-h\mu.$$

Suppose that no two of the grouped pure coefficients a, b, c are equal; then ϵ, θ, η can be, and are to be, expressed in terms of λ, μ, ν so as to make these three expressions vanish; that being done the initial form of the Quadric is maintained throughout the series of substitutions and we may write

$$f\delta f+g\delta g+h\delta h=(a-d)f\lambda+(b-d)g\mu+(c-d)h\nu.$$

Of the three quantities λ, μ, ν it is sufficient for the purpose of the argument to retain any two as λ, μ and to suppose $\nu=0$.

Then, since we suppose that a and b are not equal,

$$(a-d)f\lambda+(b-d)g\mu$$

(where λ, μ are arbitrary) can always be made negative unless f, g are none of them zero; so that if a and b never become equal *nor* f or g vanish $f^2+g^2+h^2$ cannot have any minimum value other than zero, which involves a contradiction; hence in the course of the series of infinitesimal transformations either a and b must become equal, or f or g or both of them vanish. If f and g vanish simultaneously or even if one only of them vanish, then *one* succeeding substitution, and if a and b become equal *two* succeeding substitutions, will effect the reduction to the canonical form. This proves the theorem for four variables.

The method is obviously extendible to any number of variables; in the case just considered it is seen that in the infinitesimal orthogonal matrix of substitution for the *exceptional* line or column (that which relates to the *excepted* variable the t) it is not necessary to employ more than two arbitrary infinitesimals and a like remark applies to the general case, so that if there are n variables, whilst $\frac{1}{2}(n^2-n)$ is the number of infinitesimals that would appear in the complete matrix, $\frac{1}{2}(n^2-3n+6)$, that is $\frac{1}{2}[(n-1)(n-2)]+2$, are sufficient for the purpose of the demonstration.

Thus then without recourse to any theorem of Equations it is proved that any Quadric may be reduced by a real orthogonal substitution to its canonical form*.

* I have applied the same method to prove that by two real independent orthogonal substitutions operated on

$$x_1, x_2, \ldots x_n;\ y_1, y_2, \ldots y_n$$

the general lineo-linear Quantic in the x's and y's (with real coefficients) may be reduced to the canonical form $\Sigma x_i y_i$, and have sent for insertion in the *Comptes Rendus* of the Institute a Note in which I give the rule for effecting this reduction [above, p. 638].

It may be sufficient here to mention that if U is the given lineo-linear Quantic, its n canonical multipliers are the square roots of the n canonical multipliers of the Quadric $\Sigma\left(\frac{dU}{dy}\right)^2$, or if we please of $\Sigma\left(\frac{dU}{dx}\right)^2$, which it may easily be shown à *posteriori* are necessarily omni-positive; and I need hardly add that although these two Quadrics are different, their canonical multipliers are the same.

67.

ON THE REDUCTION OF A BILINEAR QUANTIC OF THE n^{TH} ORDER TO THE FORM OF A SUM OF n PRODUCTS BY A DOUBLE ORTHOGONAL SUBSTITUTION*.

[*Messenger of Mathematics*, XIX. (1890), pp. 42—46.]

A HOMOGENEOUS lineo-linear function in two sets of variables

$$x, y, \ldots z;\ u, v, \ldots w$$

will contain n^2 terms: two independent orthogonal substitutions performed on the two sets will introduce twice $\frac{1}{2}n(n-1)$ disposable constants, and by a suitable choice of these, n^2-n terms of the transformed function may be made to vanish so as to leave a sum of products of the new $x, y, \ldots z$ paired with the new $u, v, \ldots w$: it will of course be found in general impossible to obliterate *any* arbitrarily chosen (n^2-n) terms in the transformed function; since if in the n remaining products one letter of one set were combined with more than one of the other set, this would (by means of a further superimposed orthogonal substitution) be equivalent to taking away more than (n^2-n) terms by means of only (n^2-n) disposable constants. It is very easy to effect the transformation indicated by a method very analogous to that of reducing a quadric in n variables by an orthogonal substitution to its canonical form, and to show *à posteriori* that the substitutions are always real in this case as in the other, when the original coefficients are real; but it will, I think (although not necessary), be found interesting and instructive to prove *à priori* the latter assertion by a similar method to that applied to Quadrics in the last number of the *Messenger*. I will begin then with this proof, reserving the complete solution of the problem to the end of the article. The leading idea in this as in the preceding article is to regard a finite orthogonal substitution as the product of an infinite number of infinitesimal ones.

For $$axu + \alpha xv + \beta yu + byv.$$

Let $x, y;\ u, v$ become $x+\epsilon y,\ -\epsilon x+y;\ u+\lambda v,\ -\lambda u+v$ respectively, then

$$\delta\alpha = a\lambda - b\epsilon, \quad \delta\beta = a\epsilon - b\lambda,$$
$$\alpha\delta\alpha + \beta\delta\beta = (a\alpha - b\beta)\lambda + (a\beta - b\alpha)\epsilon.$$

[* Cf. p. 638 above.]

Hence $\alpha^2+\beta^2$ may be made to decrease unless $a=0, b=0$, or $\alpha=0, \beta=0$, or $\frac{a}{b}=\frac{\alpha}{\beta}=\pm 1$, in which case since

$$(a+\alpha)(x+y)(u+v) + (a-\alpha)(x-y)(u-v) = 2a(xu+yv) + 2\alpha(xv+yu),$$
$$(a-\alpha)(x+y)(u-v) + (a+\alpha)(x-y)(u+v) = 2a(xu-yv) + 2\alpha(xv-yu),$$

the form is immediately canonizable.

Hence in the infinite succession of infinitesimal orthogonal substitutions (equivalent to a single one) either a and b or α and β must vanish simultaneously, on which supposition the form is canonical or else it is reducible to the canonical form by a second finite orthogonal substitution.

Let us now proceed to the case of a *ternary* bilinear form in $x, y, z;\ u, v, w$.

I suppose by the previous case the form to be deprived of two terms, and that we have to deal with the form

$$axu + byv + fxw + guz + hyw + kvz + czw.$$

Lemma. If $f=0,\ g=0$, or $h=0,\ k=0$ the above form is reducible by the previous case. Also if $a^2=b^2$ and $f=0,\ h=0$, or $g=0,\ k=0$, or $a^2=b^2$ and $\left(\frac{f}{h}\right)^2 = \left(\frac{g}{k}\right)^2$ the form is reducible to the previous case by a single additional finite orthogonal transformation.

For the sake of brevity I leave the proof to my readers.

Introducing now two infinitesimal orthogonal substitutions with parameters $\epsilon, \eta, \theta;\ \lambda, \mu, \nu$*, we obtain the variations

$$\delta f = a\mu - h\epsilon - c\eta, \quad \delta h = b\nu + f\epsilon - c\theta,$$
$$\delta g = a\eta - k\lambda - c\mu, \quad \delta k = b\theta + g\lambda - c\nu,$$

also in order to keep the coefficients of xv, yu at *null*, we must have

$$a\lambda - b\epsilon - f\nu - k\eta = 0,$$
$$-b\lambda + a\epsilon - g\theta - h\mu = 0.$$

From the previous equations we obtain

$$f\delta f + g\delta g + h\delta h + k\delta k = (af-cg)\mu + (bh-ck)\nu + (ag-cf)\eta + (bk-ch)\theta.$$

(1) Suppose a^2-b^2 *not* zero; then μ, ν, η, θ will be independent and their coefficients cannot all become zero unless $f^2=g^2$ and $h^2=k^2$, or else $f=0$ and $g=0$, or $h=0$ and $k=0$, on either of which suppositions the form becomes canonizable by virtue of the Lemma.

(2) Let $a^2=b^2$. Then we must have

$$f\nu + k\eta \pm (g\theta + h\mu) = 0,$$

which I shall satisfy by making $f\nu \pm g\theta = 0,\ k\eta \pm h\mu = 0$.

* The positive values of the parameters in each system are supposed to belong to the upper, and the negative values to the lower half of each orthogonal matrix.

Hence

$$\Sigma f\delta f = \{(af - cg)k \mp (ag - cf)h\}\rho + \{(ah - ck)g \mp (ak - ch)f\}\tau,$$

ρ, τ being two arbitrary infinitesimals.

Therefore $\Sigma f\delta f$ may be made negative unless the multipliers of ρ and τ are both zero, in which case by addition or subtraction we obtain $fk = gh$; consequently two out of the four variables f, g, h, k are zero, or else $\frac{f}{h} = \frac{g}{k}$, and on either of these suppositions the transformed function may be canonized by virtue of what has been proved in the case of two biliteral sets, or may by a finite orthogonal substitution be brought to a form so canonizable.

Hence it is clear that either f, g, h, k may all be made to vanish, or else we must pass through a form known to be canonizable. This is the proof for a bilinear function of triliteral sets, which may be easily extended to a bilinear function of n-literal sets.

I will now give the method for effecting the reduction which is thus proved to be always capable of being effected by real substitutions.

Let $\Sigma a_{r,s} x_r y_s$ be the given bilinear function B.

Then $\Sigma \left(\frac{dB}{dy_s}\right)^2$, which is an orthogonal invariant of B quâ the y's, is a Quadratic function of the x's, which will have an orthogonal substitute quâ the x's of the form $\Sigma[\lambda_r x_r^2]$.

If then B is reducible by a double orthogonal substitution to the form $\Sigma[\theta_r x_r y_s]$, we must have $\Sigma[\theta_r x_r]^2$ orthogonally equivalent to $\Sigma[\lambda_r x_r^2]$, and this can only be the case when the θ's are respectively (in any order) the squares of the λ's.

The θ's I call the Canonical Multipliers to B.

This gives rise to the following rule:

Form the Matrix $[m]$.

$$\begin{matrix} a_{1,1}, & a_{2,1}, \dots a_{n,1}, \\ a_{1,2}, & a_{2,2}, \dots a_{n,2}, \\ \dots\dots & \dots\dots\dots\dots, \\ a_{1,n}, & a_{2,n}, \dots a_{n,n}. \end{matrix}$$

From this derive a Matrix $[M]$, a *false* square of $[m]$, obtained by multiplying each line in it by *all* the lines (according to Cauchy's rule, in fact, for the multiplication of *Determinants*). Then the latent roots of $[M]$ are the squares of the Canonical Multipliers to B.

But if instead of $\Sigma \left(\frac{dB}{dy_s}\right)^2$ we take $\Sigma \left(\frac{dB}{dx_r}\right)^2$ and deal with it in like manner, we shall obtain a matrix $[n]$, such that $[m]$ and $[n]$ are transverse

to each other, the lines and columns of the one being the columns and lines of the other: the Cauchian Square of $[n]$ will give rise to a matrix $[N]$ different from $[M]$ but having the same latent roots: in fact the coefficients of the equation to the latent roots alike of $[m]$ and of $[n]$ with the signs in the alternate places changed will be unity, the sum of the squares of all the terms in $[m]$ or $[n]$, the sum of the squares of the minors of the 2nd, 3rd, ... orders in $[m]$ or $[n]$; and finally the last coefficient will be the square of the determinant to $[m]$ or $[n]$: so that we shall obtain as we ought the same set of canonical multipliers whichever matrix $[M]$ or $[N]$ we employ; but in order to obtain the substitutions which must be impressed on the x set and the y set to arrive at the Canonical form in which only n products appear we shall want both $[M]$ and $[N]$. Let me, however, pause for a moment to call attention to the interesting fact that the sum of the squares of the coefficients in B by virtue of being a coefficient of the latent function to $[M]$ or $[N]$ is necessarily a bi-orthogonal invariant to B; so, too, all the other coefficients in this function are such invariants: and among them the last, which is the square of the determinant to $[m]$ or $[n]$. Thus then this determinant (which may be termed the discriminant) is an invariant alike for the two theories; namely the better known one in which the x set and the y set are subjected to the same general substitution, and the one here considered where these sets are subjected to two independent orthogonal substitutions.

In either theory the vanishing of the discriminant is the signal of the Canonical form becoming short of one term.

It is also proper to notice that the latent roots of $[M]$ or $[N]$, which by virtue of $[M]$ and $[N]$ being symmetrical matrices are necessarily real, are for these particular forms of $[M]$ and $[N]$ *positive* as well as real since the coefficients with the alternate signs changed are all positive, being the sums of squares of real numbers.

To complete the solution it remains to find the two canonizing orthogonal matrices, but these are known by the ordinary theory for quadrics: thus the x substitution will be that which canonizes $[M]$ and the y substitution that which canonizes $[N]$.

Conversely, if $[M]$ and $[N]$ are supposed given, we shall know the linear functions of the x's which substituted for $x_1, x_2, \dots x_n$ and the linear function of the y's which substituted for $y_1, y_2, \dots y_n$, such that $\Sigma\lambda_1^{\frac{1}{2}} x_1 y_1$ shall be identical with B, the λ's being the latent roots common to $[M]$ and $[N]$. There will be 2^n systems of values represented by $\lambda_1^{\frac{1}{2}}, \lambda_2^{\frac{1}{2}}, \dots \lambda_n^{\frac{1}{2}}$: thus then 2^n matrices transverse to one another can be found such that their *false* squares shall be respectively identical with any two given symmetrical matrices having the same latent roots, and we are thus enabled indirectly, through the theory of bi-orthogonal canonization, to obtain the solution of a problem which intrinsically has or seems to have nothing to do with orthogonal or other transformation.

It is worthy of observation that this problem of finding the so-to-say *false* square root common to two given symmetrical matrices having the same latent equation, admits of precisely the same number (2^n) solutions as the problem of finding the true square root of one general matrix. For if $[M]$ be any given matrix of order n and $[1]$ represents the unit matrix of that order, namely the matrix all of whose terms are zeros except those in the principal diagonal which are units, we know by virtue of a general theorem that calling $\lambda_1, \lambda_2, \ldots \lambda_n$ its n latent roots, each true square root of $[M]$ is represented by

$$\Sigma \lambda_1^{\frac{1}{2}} \frac{([M]-\lambda_2[1])([M]-\lambda_3[1]) \ldots ([M]-\lambda_n[1])}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3) \ldots (\lambda_1-\lambda_n)}.$$

68.

ON AN ARITHMETICAL THEOREM IN PERIODIC CONTINUED FRACTIONS.

[*Messenger of Mathematics*, XIX. (1890), pp. 63—67.]

THE well-known form of continued fraction for the square root of N, an integer, is

$$(a\,;\ b, c, d, \ldots, d, c, b, 2a\,;\ b, c, d, \ldots, d, c, b, 2a\,;\ \text{indefinitely continued})$$

which, if we denote the type $a, b, c, d, \ldots, d, c, b, a$ by t, may be written under the more convenient form

$$(t, 0, t, 0, t, 0, \ldots \textit{ad inf.}).$$

If now we use $[t]$ to signify the cumulant of which t is the type, and $[`t]$, $[t']$, $[`t']$ respectively, the cumulants of the types got by cutting off a from either end and from both ends of t, it is easily shown that *whatever* numbers $a, b, c, \ldots$ represent, the value of the continued fraction $\{(t, 0)^\infty\}$ is $\sqrt{\frac{[t]}{[`t']}}$, so that if $\{(t, 0)^\infty\}$ represents the square root of an integer, $[t]$ must be divisible by $[`t']$.

At first sight one would imagine that it would be a difficult matter to give a rule for determining whether such condition is fulfilled or not by any assigned value of the symmetrical type t, but Mr C. E. Bickmore, of New College, Oxford, has noticed that the case is quite otherwise, for that if we put t under the form a, τ, a, then, in order that $\{(a, \tau, a, 0)^\infty\}$ may satisfy the requirement of being the square root of an integer, the sufficient and necessary condition is the equivalence

$$2a \equiv (-)^\mu [\tau'][`\tau'] \,(\text{mod. } [\tau]),$$

where μ is the number of elements in τ.

Consequently τ may be taken quite arbitrarily, and then an infinite number of values be assigned to a, except in the case where $[\tau]$ is even, and at the same time $[\tau']$ and $[`\tau']$ are each of them odd.

The proof in my notation is as follows:

Since $t = a, \tau, a$, we have ${}^{\backprime}t' = \tau$, and consequently $\frac{[t]}{[{}^{\backprime}t']}$ will be an integer if

$$[a, \tau, a] \equiv 0 \text{ (mod. } [\tau]).$$

Expanding and remembering that $[{}^{\backprime}\tau] = [\tau']$ (the type τ being symmetrical), we obtain

$$a^2 [\tau] + 2a [\tau'] + [{}^{\backprime}\tau'] \equiv 0 \text{ (mod. } [\tau]).$$

Hence
$$2a [\tau'] + [{}^{\backprime}\tau'] \equiv 0 \text{ (mod. } [\tau]), \qquad (1)$$

and
$$2a [\tau']^2 + [\tau'][{}^{\backprime}\tau'] \equiv 0 \text{ (mod. } [\tau]). \qquad (2)$$

But
$$[\tau']^2 - [\tau][{}^{\backprime}\tau'] = (-1)^{\mu+1},$$

so that
$$[\tau']^2 \equiv (-1)^{\mu+1} \text{ (mod. } [\tau]),$$

and therefore (2) becomes

$$2a \equiv (-)^{\mu} [\tau'][{}^{\backprime}\tau'] \text{ (mod. } [\tau]), \qquad (3)$$

which is thus shown to be a necessary condition.

It is also a sufficient condition, for multiplying (3) by $[\tau']$ we have

$$2a [\tau'] \equiv (-)^{\mu} [\tau']^2 [{}^{\backprime}\tau'] \text{ (mod. } [\tau]),$$

or, since
$$[\tau']^2 \equiv (-)^{\mu+1} \text{ (mod. } [\tau]),$$
$$2a [\tau'] \equiv -[{}^{\backprime}\tau'] \text{ (mod. } [\tau]),$$

which is the same as (1).

Suppose now that ${}^{\backprime}\tau'$ is given and that we wish to ascertain if a can be found of such a value that the congruence (3) shall be soluble. This will obviously be the case if $[\tau]$ is odd. It will also be the case if $[\tau]$ is even, provided $[{}^{\backprime}\tau']$ is also even, and only in that case; for, when $[\tau]$ is even, then by virtue of the equation

$$[{}^{\backprime}\tau'][\tau] - [\tau']^2 = \pm 1,$$

$[\tau']$ must be odd.

We have, therefore, to find under what circumstances $[{}^{\backprime}\tau']$ will be odd and $[\tau]$ even; in all other cases but these the congruence (3) will be soluble, and then the most general value of a will be any term in an arithmetical series of which the common difference is $[\tau]$, unless $[\tau]$ and $[{}^{\backprime}\tau']$ are both of them even, in which case the common difference will be $\frac{1}{2}[\tau]$.

I proceed now to give a rule for determining the possible and impossible cases of the solution of (3), to explain the grounds of which the following statement will suffice.

(1) The value of a cumulant is not affected by striking out any even number of consecutive zeros from its type.

(2) The *parity* (that is the character *quâ* the modulus 2) of any cumulant will not be affected if we strike out three consecutive odd terms, whether

they occur in the middle or at either extremity. For if t, τ be any two types, the cumulant

$$[t, 1, 1, 1, \tau] = 3[t][\tau] + 2[t'][\tau] + 2[t][{}^{\backprime}\tau] + [t'][{}^{\backprime}\tau]$$
$$\equiv [t][\tau] + [t'][{}^{\backprime}\tau] \text{ (mod. 2)},$$

that is
$$\equiv [t, \tau] \text{ (mod. 2)}.$$

Also
$$[1, 1, 1, t] = [t, 1, 1, 1] = 3[t] + 2[{}^{\backprime}t] \equiv [t] \text{ (mod. 2)}.$$

(3) The value of any cumulant in the type of which 1, 0, 1 occurs anywhere is the same as if 2 is substituted for 1, 0, 1; and therefore its parity is not affected if the units on each side of the 0 are omitted.

In what precedes in Nos. (1), (2), (3) the result, to modulus 2, is obviously unaffected if for 0 we write any even and for 1 any odd number.

In order then to determine the parity of $[{}^{\backprime}\tau']$ and of $[\tau]$ we may proceed as follows:

Let τ be any assigned symmetrical type, ${}^{\backprime}\tau'$ will then represent the type divested of its two equal terminals.

Rules—(1) for each even number in ${}^{\backprime}\tau'$ write 0, and for each odd number, 1;

(2) elide any even number of consecutive *zeros*, and any number divisible by 3 of consecutive *units*;

(3) elide any pair of units lying on each side of a zero;

(4) repeat these processes as often as possible;

then, I say, eventually we must arrive at one or other of the six following irreducible types, namely

( ); 0; 1; 1, 1; 0, 1, 0; 0, 1, 1, 0*,

where ( ) means absolute vacuity; accordingly ${}^{\backprime}\tau'$ may be said to be affected with one or the other of these six *characters*.

If now the reduced form of ${}^{\backprime}\tau'$ is 0; 1, 1; 0, 1, 0, $[{}^{\backprime}\tau']$ is even, and the congruence (3) will be soluble. In the other three cases $[{}^{\backprime}\tau']$ is odd, but $[\tau]$ will also be odd unless its terminal elements are odd in the case where the reduced form of ${}^{\backprime}\tau'$ is ( ), and even for the reduced forms 1, and 0, 1, 1, 0.

In the following exhaustive table the second column indicates the evenness or oddness of the terminals of τ denoted by e and u respectively.

The third and fourth columns indicate the evenness or oddness (denoted as above) of $[{}^{\backprime}\tau']$ and $[\tau]$, along with the character of ${}^{\backprime}\tau'$ in the third column. In the fifth column the answer is given as to the determining congruence

* Except for the symmetrical form of τ there would be two additional (virtually undistinguishable) reduced forms 0, 1 and 1, 0.

being soluble or insoluble, denoted by s and i respectively; and the last column shows whether the common difference of the arithmetical series of the values of either terminal, in the case of solubility, is equal to the modulus $[\tau]$ or its moiety.

Cases	Terminals	'τ'		$[\tau]$	Sol. or Insol.	C. D.
1	e	( )	u	u	s	$[\tau]$
2	u	( )	u	e	i	
3	e	1	u	e	i	
4	u	1	u	u	s	$[\tau]$
5	e	0, 1, 1, 0	u	e	i	
6	u	0, 1, 1, 0	u	u	s	$[\tau]$
7	e	0	e	e	s	$\frac{1}{2}[\tau]$
8	u	0	e	e	s	$\frac{1}{2}[\tau]$
9	e	1, 1	e	u	s	$[\tau]$
10	u	1, 1	e	u	s	$[\tau]$
11	e	0, 1, 0	e	u	s	$[\tau]$
12	u	0, 1, 0	e	u	s	$[\tau]$

The following examples are given to prevent the possibility of misapprehension in the application of the Algorithm.

(α) Let

$$\tau = 1, 9, 1, 1, 1, 2, 1, 7, 4, 2, 2, 2, 4, 7, 1, 2, 1, 1, 1, 9, 1.$$

Then

$$\begin{aligned} `\tau' &\equiv 1, 1, 1, 1, 0, 1, 1, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1 \\ &\equiv 0, \quad 1, \qquad 0, 1, \quad 0 \\ &\equiv 0, \qquad 0, \quad 0 \\ &\equiv 0. \end{aligned}$$

This corresponds to case (8), which is a soluble one, and accordingly we have from Degen's Table

$$\{(15, \tau, 15, 0)^{\infty}\} = \sqrt{(251)},$$

15 being the first term of an arithmetical series whose common difference is $\frac{1}{2}[\tau]$.

(β) Let $\tau = 2, 3, 1, 2, 4, 1, 6, 6, 1, 4, 2, 1, 3, 2.$

Then

$$\begin{aligned} `\tau' &\equiv 1, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 1 \\ &\equiv 1, 1, \quad 1, \quad 1, \quad 1, 1 \\ &\equiv \quad (\ ). \end{aligned}$$

This corresponds to the soluble case (1), and accordingly we find from Degen's Table $\{(10, \tau, 10, 0)^{\infty}\} = \sqrt{(109)}$; 10 being the first term of an arithmetical series whose common difference is $[\tau]$.

69.

ON A FUNICULAR SOLUTION OF BUFFON'S "PROBLEM OF THE NEEDLE" IN ITS MOST GENERAL FORM.

[*Acta Mathematica*, XIV. (1890–1), pp. 185—205.]

"...quaintly made of cords."
(*Two Gentlemen of Verona*, Act III. Sc. 1.)

THE founder of the theory of Local Probability appears to have been Buffon (better known as a Naturalist, but who began his career as a Mathematician). Among a few other questions of a similar kind, which he proposed in his *Essai d'Arithmétique Morale*, the one which has obtained the greatest notoriety is the celebrated one which goes by the name of the *Problème de l'Aiguille*, the purport of which is as follows.

On an area of indefinite extent (say a planked floor) a number of parallel straight lines are ruled at equal distances, upon which a needle, not long enough to cross more than one of the parallels at the same time, is thrown down: the probability is required of its falling in such a position as to be intersected by one of the parallels.

An easier question of the same kind, which Buffon treats before the other, is when a circle is used instead of the needle. This latter question he solves by simple geometrical considerations too obvious to need recapitulation; to obtain a solution of the former he, and after him Laplace, had recourse to a process of integration.

In a question given in the late Mr Todhunter's *Integral Calculus* (1st edition, 1857, p. 268) the solution of the problem is correctly stated for an ellipse, whose major axis is less than the distance between two consecutive parallels, instead of for a circle or straight line: this important step in the development of the theory is, I am informed, currently attributed to the late Mr Leslie Ellis, of the University of Cambridge.

In the year 1860, Lamé proposed to give a course of lectures on the subject at the Sorbonne, and, apparently without knowledge of the result contained in Todhunter's treatise, reproduced the solution for the ellipse and for any equilateral polygon. In the same year M. Emile Barbier, whose lamented decease occurred in the course of the present year and who had attended Lamé's lectures, discovered and published in *Liouville's Journal* for that year a universal solution for an undivided plane contour of any form whatever.

The subsequent history I am not able to trace further than to state that in Czuber's *Geometrische Wahrscheinlichkeiten* (Leipzig, 1884) Barbier's solution is extended to the case of any two rigidly connected convex figures (in a plane)*. I propose to give here the finishing stroke to the theory as regards plane figures by extending it to any number of them, rigidly connected and of any forms, in the same plane. It is always to be understood, in what precedes as in what follows, that the greatest diameter of the figure, or system of figures, is less than the distance between two consecutive parallels.

Barbier's principle (see Czuber, pp. 117, 125) leads at once to the conclusion that the probability of any figure (subject to the restriction above stated) intersecting the system of parallels is to certainty as the length of a cord stretched round the figure is to the circumference of a circle touched by two adjoining parallels†. This circumference (with a view to simplicity of expression) we shall adopt as the unit of length in all subsequent formulae.

By the disjunctive probability of a set of figures I shall understand the probability of *one or more* of them intersecting one of the parallels: by the conjunctive probability of the same, the probability of *all* of them intersecting one of the parallels.

I start from Barbier's theorem that for a single figure the probability of intersection is measured by the length of a stretched string passing round it: this, it should be observed, is universally true whether the contour be curvilinear or rectilinear or mixtilinear, composed of a single line straight or curved or of any number of such—a theorem almost unexampled for its generality. The disjunctive probability for any number of figures $A, B, C, \ldots, H$ I shall for the present denote by $A:B:C:\ldots:H$, the conjunctive by $A.B.C\ldots H$.

Let there be $n+1$ figures given, let p_i be the sum of the conjunctive and ϖ_i of the disjunctive probabilities for these figures taken i and i together; so that ϖ_1 and p_1 are identical, and ϖ_{n+1}, p_{n+1} are monomial quantities. Then by a universal theorem of *logic* we have the reciprocal formulae

$$\varpi_{n+1} = \sum_{i=1}^{i=n+1} (-)^{i+1} p_i, \qquad (1)$$

$$p_{n+1} = \sum_{i=1}^{i=n+1} (-)^{i+1} \varpi_i. \qquad (2)$$

* See *Postscriptum*, p. [679, below].

† The case of a straight line (the original question of the *needle*) may be made to fall under this rule: for the line, as Barbier has observed, may be regarded as an indefinitely narrow ellipse or other oval.

Let us now suppose that we have obtained expressions for the disjunctive and conjunctive probabilities of any number not exceeding n figures of any kind: we may extend these to the case of $n+1$ figures as follows.

(1) When the $n+1$ figures are so situated that it is impossible for all of them to be cut by the same straight line, we have $p_{n+1}=0$ so that ϖ_{n+1} can be found immediately in terms of $p_1, p_2, \ldots, p_n$ by using formula (1), or in terms of $\varpi_1, \varpi_2, \ldots, \varpi_n$ by using (2); that is ϖ_{n+1} can be found in terms of known quantities; for by hypothesis all the terms of p_i or of ϖ_i are known when i is any number not exceeding n.

(2) When all the $n+1$ figures are capable of being cut by the same straight line, let XY be some straight line which cuts them all and call the figures taken in the order in which they are cut by XY

$$A_1, A_2, A_3, \ldots, A_{n+1}\text{*}.$$

Let a stretched string be made to wind round these $n+1$ contours passing alternately from one side of XY to the other, as in Fig. 1, and crossing itself

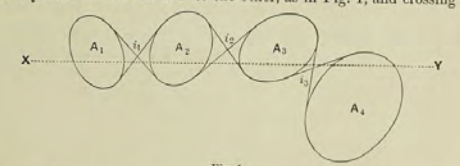

Fig. 1.

in the n points $i_1, i_2, \ldots, i_n$ lying between A_1, A_2; A_2, A_3; ... A_n, A_{n+1} respectively. Let us call the figures enclosed by the successive $n+1$ *loops* of the winding string

$$B_1, B_2, B_3, \ldots, B_{n+1}.$$

It is obvious that any straight line which cuts all these loops will cut all the given figures, and *vice versâ*.

Hence $$A_1.A_2.A_3\ldots A_{n+1} = B_1.B_2.B_3\ldots B_{n+1}.$$

Let P_i, Π_i represent what p_i, ϖ_i become when for the figures A we substitute the loops B, so that

$$\Pi_{n+1} = \sum_{i=1}^{i=n+1} (-)^{i+1} P_i,$$

$$P_{n+1} = \sum_{i=1}^{i=n+1} (-)^{i+1} \Pi_i,$$

and $$P_{n+1} = p_{n+1}.$$

* It may be well to draw at once attention to the fact that different systems of straight lines do not necessarily cut the figures $A_1, A_2, A_3, \ldots$ in the same order; as, for example, if three circles touch, or so nearly touch one another that each blocks the channel between the other two, straight lines may be drawn whose intersections with *any* one of the three shall be intermediate to their intersections with the other two.

Π_{n+1} is known by Barbier's rule, because the loops taken together form a single figure, in fact

$$\Pi_{n+1} = L,$$

where L is the length of the uncrossed string stretched round the system of figures B, which is no other than that stretched round the given figures A. Also, by hypothesis, Π_i is known for all values of i not exceeding n. We therefore know p_{n+1} which is the same as P_{n+1}. Hence ϖ_{n+1} is known from (1): thus then p_{n+1} and ϖ_{n+1} are both known, so that when the conjunctive and disjunctive probabilities are known in general for n figures they become known for $n+1$ figures; but when $n = 1$, p_1 and ϖ_1 are equal to one another and to the length of a given stretched string. Hence, by the usual process of induction, we may conclude that the conjunctive and disjunctive probabilities for any number of figures can always be expressed as a linear function with positive and negative integer coefficients, or in a word as a Diophantine linear function, of a finite number of lengths of certain stretched strings.

When there are only *two* figures A_1, A_2 we pass a stretched string between them crossing itself in i (see Fig. 2): then using $(A_1 \times A_2)$ to

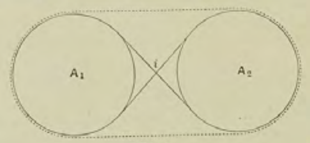

Fig. 2.

denote the length of this string, and (A_1A_2) to denote the length of the uncrossed string (indicated by dots in the figure) stretched round A_1, A_2 we have

$$\Pi_2 = (A_1 \times A_2) - P_2$$

and

$$\varpi_2 = (A_1) + (A_2) - p_2$$

(where (A_1), (A_2) denote the lengths of the separate bands round A_1, A_2 respectively).

But

$$\Pi_2 = (A_1A_2),$$

and consequently

$$p_2 = P_2 = (A_1 \times A_2) - (A_1A_2),$$

$$\varpi_2 = (A_1) + (A_2) + (A_1A_2) - (A_1 \times A_2).$$

We will now proceed to consider in detail the application of the inductive method to the case of *three* figures for which, since each of these may be replaced by a convex band passing round it, we may if we please for greater

graphical simplicity substitute three convexes (that is contours which any secant must intersect in exactly two points). Many cases requiring separate discussion will arise, but one important consequence, rising to the dignity of a principle, which holds good whatever may be the number of figures, governs them all; namely that the final result for either probability is a linear homogeneous function of lengths of stretched bands drawn in various ways round the given figures and depending for their course on the forms and disposition of these figures exclusively, *wholly uninfluenced* by the presence of any points external to them. Lines drawn from the pointed ends, or apices, of the loops enclosing them do it is true make their appearance in the computations but, either coalesce into portions of the bands referred to, or else, entering in pairs with opposite algebraical signs, disappear from the final result. As a consequence, if for the sake of illustration we suppose the figures to be any closed *curves* without singular points, the probability, disjunctive or conjunctive, to be ascertained is a function exclusively of the complete system of lengths of double tangents that can be drawn between the curves and of the arcs into which they are severally divided by their points of contact with those tangents.

We have for all the cases of three figures

$$\varpi_3 = p_1 - p_2 + p_3$$

where

$$p_1 = (A_1) + (A_2) + (A_3)$$

and $p_2 = (A_2 \times A_3) - (A_2A_3) + (A_3 \times A_1) - (A_3A_1) + (A_1 \times A_2) - (A_1A_2)$.

Thus

$$\varpi_3 - p_3 = (A_1) + (A_2) + (A_3) + (A_2A_3) + (A_3A_1) + (A_1A_2) - (A_2 \times A_3) - (A_3 \times A_1) - (A_1 \times A_2). \quad (3)$$

Similarly

$$\Pi_3 - P_3 = (B_1) + (B_2) + (B_3) + (B_2B_3) + (B_3B_1) + (B_1B_2) - (B_2 \times B_3) - (B_3 \times B_1) - (B_1 \times B_2),$$

where B_1, B_2, B_3 are the loops of the string which passes round the figures A_1, A_2, A_3 and crosses itself at i and j, as shown in Fig. 3. But $P_3 = p_3$,

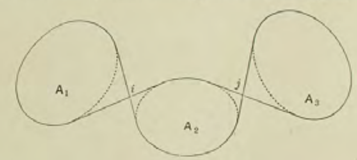

Fig. 3.

and Π_3 is the length of an uncrossed band stretched round the entire system of figures A_1, A_2, A_3 (which will be expressed in symbols by writing

$$\Pi_3 = (A_1A_2A_3).)$$

Hence $p_3 = (A_1 A_2 A_3) + (B_2 \times B_3) + (B_3 \times B_1)$
$+ (B_1 \times B_2) - (B_1) - (B_2) - (B_3) - (B_2 B_3) - (B_3 B_1) - (B_1 B_2)$.

Moreover $(B_1 \times B_2) = (B_1) + (B_2)$

and $(B_2 \times B_3) = (B_2) + (B_3)$,

because B_1, B_2 and B_2, B_3 are pairs of consecutive loops. And whenever the three given figures are capable of being cut by a straight line in the order A_1, A_2, A_3 (that is except in the case $p_3 = 0$, which is separately considered)

$$(B_3 B_1) = (A_3 A_1),$$

because both the crossing points, i and j, of the looped string necessarily fall inside the uncrossed band round A_1, A_3. Thus the value of p_3 is given by the equation

$$p_3 = (A_1 A_2 A_3) - (A_1 A_3) + (B_3 \times B_1) + (B_2) - (B_2 B_3) - (B_1 B_2) \qquad (4)$$

which, for immediate purposes, we shall find convenient to write under the form

$$p_3 = (A_1 A_2 A_3) - (A_1 A_3) + (B_2 \times B_3) - (B_2 B_3) + (B_3 \times B_1) - (B_1 B_2) - (B_3). \qquad (5)$$

We shall apply the formula to the two classes which between them comprise all the cases of three figures, namely

Class A. One of the figures, which we call A_2, lies either wholly or partially inside the crossed band round the other two.

Class B. Each figure lies entirely outside the crossed band round the other two.

In Class A we recognize three species, namely

Aa. The figure A_2 does not cut either of the crossed strings ab, cd of the band looped round A_1, A_3 (Fig. 4), but lies wholly in the same loop as one of them, which we call A_1.

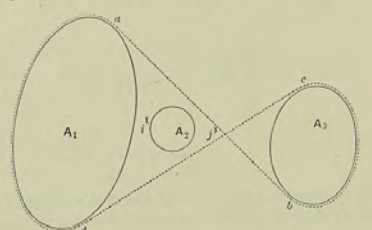

Fig. 4.

Ab. The figure A_2 cuts one, but not both, of the crossed strings ab, cd (Fig. 5), and part of it lies in the same loop as A_1.

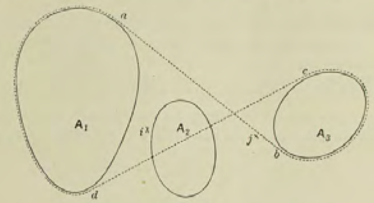

Fig. 5.

Ac. The figure A_2 cuts both the crossed strings ab, cd (Figs. 6 and 7) and part of it lies in the same loop as A_1.

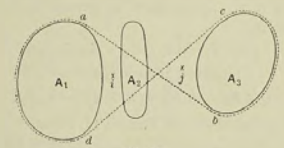

Fig. 6.

To avoid complicating these figures (4, 5, 6, 7) the band (looped round A_1, A_2, A_3 as shown in Fig. 3) which crosses itself at i, j is not given, but the position of each crossing point is marked by a small cross. It should be

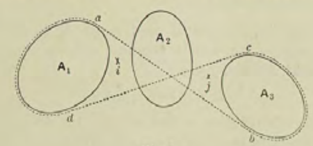

Fig. 7.

observed that in Fig. 5 (species Ab) j lies outside the crossed band round A_1, A_3; in Fig. 4 (species Aa) i and j lie in the same loop, and in Figs. 6, 7 (species Ac) i and j lie in opposite loops of the crossed band round A_1, A_3.

The discussion of species Aa is very simple; for it is clear that the conjunctive probability is

$$p_3 = (A_2 \times A_3) - (A_2 A_3)$$

since it is obviously impossible for a straight line to cut A_2 and A_3 without cutting A_1. Substituting this value for p_3 in formula (3) we obtain the disjunctive probability

$$\varpi_3 = (A_1) + (A_2) + (A_3) + (A_1 A_2) + (A_1 A_3) - (A_1 \times A_2) - (A_1 \times A_3).$$

The remaining two species belonging to class A may be discussed simultaneously; for we have in all the cases (see Fig. 8), using e, f to denote the

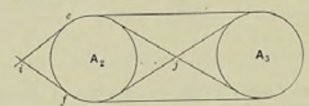

Fig. 8.

points of contact with the figure A_2 of the strings which cross at the point i (between A_1 and A_2),

$$(B_2 \times B_3) = (A_2 \times A_3) + fi + ie - ef,$$
$$(B_2 B_3) = (A_2 A_3) + fi + ie - ef,$$

so that $$(B_2 \times B_3) - (B_2 B_3) = (A_2 \times A_3) - (A_2 A_3).$$

Hence, for all the species of class A, formula (5) becomes

$$p_3 = (A_1 A_2 A_3) - (A_1 A_3) + (A_2 \times A_3) - (A_2 A_3) + (B_1 \times B_3) - (B_1 B_2) - (B_3).$$

In reducing the last three terms of this expression to a form which involves the lengths of bands round the A's, a slight difference arises between species Ab (in which, see Fig. 5, the point j and the figure A_1 are on the same side of the string ab) and species Ac (in which j and A_1 are on opposite sides of the string ab, see Figs. 6 and 7).

Thus, for species Ac, the crossed band round B_1, B_3 will not encounter either of the points i, j, but will be identical with the crossed band ($abcda$, Figs. 6 and 7) round A_1, A_3; that is

$$(B_3 \times B_1) = (A_3 \times A_1).$$

Moreover, a moment's reflexion will show that the uncrossed band round B_1, B_2 will combine with the loop B_3 so as to form a single band: in fact we have

$$(B_1 B_2) + (B_3) = D,$$

where D is the crossed band round A_1, A_3 with the loop which contains A_1 distended until it also contains A_2.

But in species Ab (see Fig. 9), let the points of contact with A_3 of the strings which cross at j (between A_2, A_3) be g, h; and let a string jk, in contact with A_1 at k, be stretched from j to the figure A_1: then

$$(B_1 \times B_3) = (A_1 \times A_3) + gj + jk + ka - ab - bg,$$

and $$(B_1 B_2) + (B_3) = D + gj + jk + ka - ab - bg,$$

where D is the band ($abgchjlmna$), derived from the crossed band ($abgcdna$) round A_1, A_3 by distending the loop which contains A_1 until it also contains A_2.

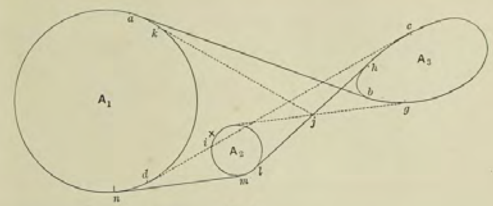

Fig. 9.

Hence $$(B_1 \times B_3) - (B_1 B_2) - (B_3) = (A_1 \times A_3) - D,$$

and the general formula for the conjunctive probability (for class A) becomes

$$p_3 = (A_1 A_2 A_3) + (A_1 \times A_3) + (A_2 \times A_3) - (A_1 A_3) - (A_2 A_3) - D. \qquad (6)$$

Combining this with formula (3), which belongs to all cases of three figures, we obtain

$$\varpi_3 = (A_1) + (A_2) + (A_3) + (A_1 A_2) + (A_1 A_2 A_3) - (A_1 \times A_2) - D.$$

The species Aa, Ab, Ac are distinguishable from one another by the difference in shape of the band D belonging to each. Thus in Aa the band

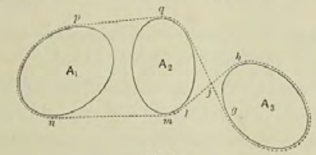

Fig. 10.

D is not distended at all, but is simply $(A_1 \times A_3)$; in Ab the loop containing A_1 is distended on one side only; and in Ac is distended on both sides (see Figs. 10 and 11). This difference in shape will be denoted by writing D_1

for D in the general formula when the species is Ab, and D_2 for D when the species is Ac.

The dotted bands (*pqjghjlmnp*) of Fig. 10, and (*abhlmna*) of Fig. 11 are what the dotted bands of Fig. 7 (species Ac) and Fig. 5 (species Ab) become, when the former is doubly and the latter singly distended.

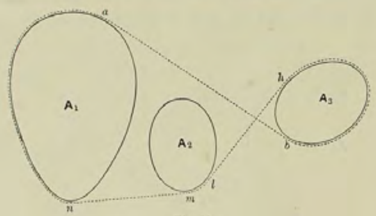

Fig. 11.

Varieties of the species in class A (namely one variety for Aa, two for Ab, and three for Ac, making 6 cases in all) occur when we consider the situation of the figure A_2 with respect to the uncrossed band round A_1, A_3. In all cases where A_2 lies wholly inside this band we have $(A_1A_2A_3)=(A_1A_3)$, so that in all such cases the general formula (6), which gives the conjunctive probability, becomes

$$p_3=(A_1 \times A_3)+(A_2 \times A_3)-(A_2A_3)-D.$$

Aa. We have $$D=(A_1 \times A_3)$$
so that $$p_3=(A_2 \times A_3)-(A_2A_3)$$
(the same as the result previously obtained from *à priori* considerations).

Ab. 1. The figure A_2 lies wholly within the uncrossed band round A_1, A_3
$$p_3=(A_1 \times A_3)+(A_2 \times A_3)-(A_2A_3)-D_1.$$

Ab. 2. The figure A_2 cuts the uncrossed band round A_1, A_3
$$p_3=(A_1A_2A_3)+(A_1 \times A_3)+(A_2 \times A_3)-(A_1A_3)-(A_2A_3)-D_1.$$

Ac. 1. The figure A_2 lies wholly within the uncrossed band round A_1, A_3.

Ac. 2. The figure A_2 cuts only one string of the uncrossed band round A_1, A_3. In these two cases the formulae which give p_3 are the same as in the corresponding varieties of Ab, except that D_2 takes the place of D_1.

Ac. 3. The figure A_2 cuts both strings of the uncrossed band round A_1, A_3. In this case the formula for the conjunctive probability
$$p_3=(A_1A_2A_3)+(A_1 \times A_3)+(A_2 \times A_3)-(A_1A_3)-(A_2A_3)-D_2$$
becomes greatly simplified; for (see Fig. 12)
$$D_2-(A_1A_2A_3)=rsjgu+vljht-rt-vu=(A_2 \times A_3)-(A_2A_3)$$
so that $$p_3=(A_1 \times A_3)-(A_1A_3),$$
which is evidently true, since every straight line which cuts both A_1 and A_3 must also (in this case) cut A_2.

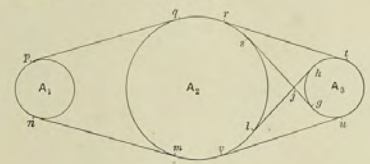

Fig. 12.

We have now enumerated all the six cases of Class A, and given in each case the formula for the conjunctive probability (from which, by means of formula (3), the disjunctive probability may be determined immediately). We proceed to the discussion of Class B.

In Class B (that is in the class where each figure lies entirely outside the crossed band round the other two) we recognize four species, and in one of them two varieties, making five cases in all. The enumeration is as follows.

Ba. There is one definite order of succession in which the three figures can be cut by a system of straight lines. There are two varieties of this species, namely

Ba. 1. The middle figure (A_2, see Fig. 13) lies wholly inside the uncrossed band round the other two. The small crosses in this figure, as in others, indicate the positions of the points i, j where the string looped round A_1, A_2, A_3 (see Fig. 3) crosses itself.

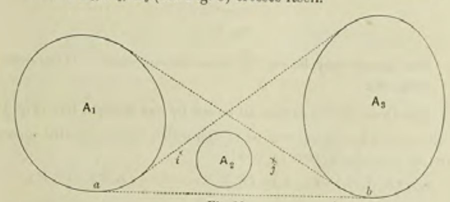

Fig. 13.

Ba. 2. The middle figure cuts the uncrossed band round the other two as shown in Fig. 14. In this, as in the preceding case, both i and j lie outside the crossed, but inside the uncrossed, band round A_1, A_3*.

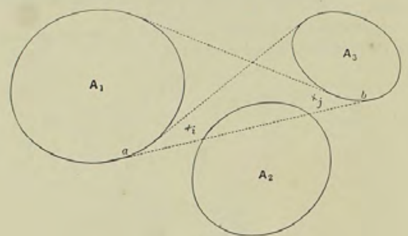

Fig. 14.

Bb. The figures may be cut in two different orders by two distinct systems of straight lines (see Fig. 15). One system of straight lines cuts the figures in the order A_1, A_2, A_3; the other system cuts them in the order A_3, A_1, A_2.

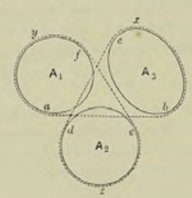

Fig. 15.

Bc. The figures may be cut by three distinct systems of straight lines (Fig. 16).

Bd. The three figures cannot all be cut by any straight line (Fig. 17).

In all cases with the exception of Bd, which will be treated separately, we have (see formula (4) *ante* [p. 668])

$$p_3 = (A_1 A_2 A_3) - (A_1 A_3) + (B_1 \times B_3) + (B_2) - (B_2 B_3) - (B_1 B_2).$$

* This circumstance enables us to discuss Ba. 1 and Ba. 2 simultaneously.

In Ba (see Fig. 18) we have

$$(B_2 B_3) = (A_2 A_3) + hi + ik - kc - cd - dh,$$
$$(B_1 B_2) = (A_1 A_2) + mj + jn - nf - fe - em,$$
$$(B_1 \times B_3) = (B_1) + (B_3) + ik - kc - cr - rj + jn - nf - fp - pi.$$

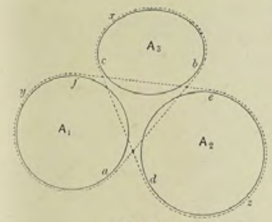

Fig. 16.

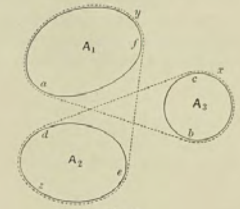

Fig. 17.

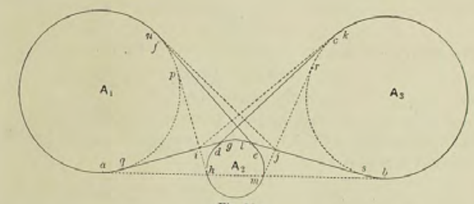

Fig. 18.

Substituting these values in the general expression for p_3, we obtain

$$p_3 = (A_1 A_2 A_3) - (A_2 A_3) - (A_3 A_1) - (A_1 A_2)$$
$$+ (B_1) + (B_2) + (B_3) - mr - rc + cd + dh - hp - pf + fe + em$$

where the term $-mr$ comes from $-mj - rj$, and the term $-hp$ comes from $-hi - pi$; the other terms involving the points i, j or the points of contact k, n of tangents drawn from them to the original figures disappear in pairs. The terms

$$(B_1) + (B_2) + (B_3) - mr - rc + cd + dh - hp - pf + fe + em$$

will be seen to coalesce into a single band (whose course is marked in Fig. 18 by the continuous line *aqigljsbkcdhmefna*, all other lines in the figure being dotted). This band we shall call Δ_1.

Fig. 18 is drawn for the case Ba. 2, but the investigation of case Ba. 1 is precisely the same as that of Ba. 2. In both cases we find

$$p_3 = (A_1A_2A_3) - (A_2A_3) - (A_3A_1) - (A_1A_2) + \Delta_1$$

for the conjunctive probability, and consequently

$$\varpi_3 = (A_1) + (A_2) + (A_3) + (A_1A_2A_3) - (A_2 \times A_3) - (A_3 \times A_1) - (A_1 \times A_2) + \Delta_1$$

gives the disjunctive probability in both cases.

The band Δ_1 for the case Ba. 1 is shown by the continuous line of Fig. 19, that is Δ_1 is the band *atqglsvbxcdwuefya*: its course is precisely the same as that of the Δ_1 for the case Ba. 2.

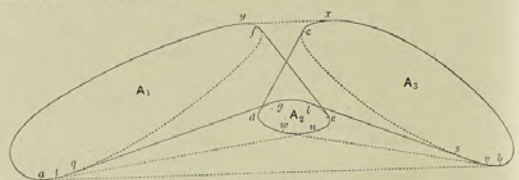

Fig. 19.

The difference between the two cases is this: in Ba. 1 we have

$$(A_1A_2A_3) = (A_1A_3)$$

so that
$$p_3 = \Delta_1 - (A_1A_2) - (A_2A_3)^*,$$

whereas in Ba. 2 (and in all the cases to be subsequently considered) the terms $(A_2A_3) + (A_3A_1) + (A_1A_2) - (A_1A_2A_3)$ coalesce into a single band which we shall call Δ, so that

$$p_3 = \Delta_1 - \Delta.$$

The course of the band Δ is marked by the letters *abkcdhmefna* in Fig. 18. The band Δ_1 may be derived from Δ by supposing its rectilinear portion *ab* to be pressed inwards by the figure A_2 so as to occupy the position *aqglsb*.

The investigation of the case Bb proceeds on exactly the same lines as that of Ba. 2; we start from the same general formula and, by performing precisely similar work, obtain the result

$$p_3 = \Delta_2 - \Delta,$$

where (see Fig. 15) Δ is the band *abxcdzefya* whose course is indicated by dots, and Δ_2 is the band derived from Δ by supposing *two* of its rectilinear portions *ab*, *cd* to be pressed inwards by the figures A_1 and A_2.

* By an easy rearrangement of the bands the value of p_3 for this case may be expressed as the difference of the two bands, *atuelgdwvbxya* and *atqgleuwdglsvbxya* (see Fig. 19), derived from the uncrossed band *abxya* round A_1, A_3 by *twisting* its rectilinear portion *ab* right round A_2 in opposite directions.

In the case Bc (Fig. 16) the work is simplified by observing that each of the figures A_1, A_2, A_3 blocks the channel between the other two (that is, no straight line can pass between any two of them without cutting the third). Hence every straight line which cuts the uncrossed band round all the figures must cut one or more of them; that is

$$\varpi_3 = (A_1A_2A_3)$$

and consequently formula (3) gives

$$p_3 = (A_1A_2A_3) - (A_2A_3) - (A_3A_1) - (A_1A_2)$$
$$+ (A_2 \times A_3) + (A_3 \times A_1) + (A_1 \times A_2) - (A_1) - (A_2) - (A_3).$$

Now it is easily seen that

$$(A_2A_3) + (A_3A_1) + (A_1A_2) - (A_1A_2A_3) = \Delta$$

and
$$(A_2 \times A_3) + (A_3 \times A_1) + (A_1 \times A_2) - (A_1) - (A_2) - (A_3) = \Delta_3$$

where Δ is the band *abxcdzefya* (shown by the dotted line in Fig. 16) and Δ_3 is what Δ becomes when its rectilinear portions *ab*, *cd*, *ef* are pressed inwards by the figures A_1, A_2, A_3.

Thus
$$p_3 = \Delta_3 - \Delta.$$

The sole remaining case of three figures is Bd (Fig. 17), the case in which no straight line can possibly cut all three figures. In it we have obviously

$$p_3 = 0,$$

and therefore

$$\varpi_3 = (A_1) + (A_2) + (A_3) + (A_2A_3) + (A_3A_1) + (A_1A_2)$$
$$- (A_2 \times A_3) - (A_3 \times A_1) - (A_1 \times A_2).$$

This case forms no exception to the general rule for finding the conjunctive probability in cases belonging to class B.

We have
$$\Delta = abxcdzefya$$

(that is, Δ is the dotted band of Fig. 17), and since this band is not pressed inwards by any of the figures the conjunctive probability according to the rule would be $\Delta - \Delta = 0$, which is right.

Having thus pointed out the general method of procedure, and illustrated it by treating in detail the case of three figures, it does not seem desirable to pursue the subject further in this direction for the present; but, before concluding, it may be worth while to notice that, in the general case of n limited right lines, the probabilities with which we have to do become Diophantine linear functions of the sides of the complete $2n$-gonal figure of which the n pairs of extremities of the lines are the angles. There will be a group of such linear functions depending on the mutual disposition of the n lines, but the number of formulae in any such group will be much greater than in the case of n general figures: for, when we pass from these to indefinitely narrow ovals, the portion of a definite band (appearing in any formula), partially surrounding any one of such ovals, may, according to the mutual disposition of their major axes, have in common with it an infinitesimal arc in some cases, in others an arc (to an infinitesimal *près*) equal to a circumference, and again in others to a semicircumference of the oval; which latter is ultimately the same as the length of the line whose double the complete circumference represents.

By way of illustration let us consider the question of two needles or limited straight lines rigidly connected. Neglecting the limiting cases, where one of the lines terminates in the other, there will remain three hypotheses:

A. The lines intersect.

B. The lines tend to intersect in a point external to each of them.

C. One of the lines tends towards a point lying within the other.

Let p_2 denote the chance of both the needles AB, CD being cut by one of the parallels, ϖ_2 the chance of one or other of them being cut: then we have the general formulae applicable to all cases

$$\varpi_2 = 2AB + 2CD - p_2,$$

p_2 = difference between the crossed and uncrossed bands round AB, CD.

A. When the lines intersect

$$\varpi_2 = AD + DB + BC + CA,$$
$$p_2 = 2AB + 2CD - AD - DB - BC - CA.$$

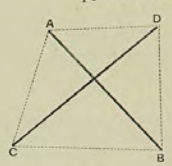

Fig. 20.

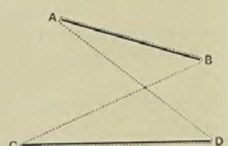

Fig. 21.

B. When the lines tend to intersect in a point external to each of them

$$p_2 = (AB + BC + CD + DA) - (AB + BD + DC + CA)$$
$$= BC - CA + AD - DB^*,$$
$$\varpi_2 = 2AB + 2CD - BC + CA - AD + DB.$$

* Imagine a string passing from B to C, from C to A, from A to D, and from D to B. This string cannot be kept tight unless fastened by pins at A, B, C, D. Inserting the necessary pins and tightening the string, we agree to consider the consecutive portions of the string as alternately positive and negative.

On these suppositions p_2 is the algebraical length of the band $BCADB$ stretched round the pins. The method of representation by means of pinned bands may be extended to the case of two (or any number of) general figures.

C. When one of the lines tends towards a point lying within the other

$$p_2 = (2AB + BC + CD + DB) - (AC + CD + DA)$$
$$= 2AB + BC - CA - AD + DB,$$
$$\varpi_2 = 2CD - BC + CA + AD - DB.$$

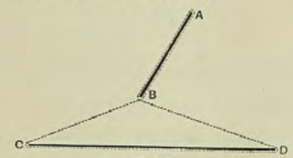

Fig. 22.

The complexity of cases for three right lines is such as would require a separate study even to obtain a perfect enumeration of them; consequently I shall leave it to others to pursue the subject further whether as regards principles or details. I will only add that the ascertainment of the general law that the formulae contain no other arguments than lengths of tight endless bands variously drawn round the given contours appears to me a distinct step achieved in the prosecution of this extensive theory, and one that is far from being obvious *à priori*. Buffon's problem of the needle, it will be seen, has now expanded into a problem of n needles rigidly connected, which may be treated as a corollary to that of n *entirely separate* general contours, the mode of solution of which, it is believed, has been sufficiently indicated in the investigations which form the subject of this memoir.

Postscriptum. Since the above was set up in print my attention has been called to the fact that the extension of Barbier's theorem referred to on p. [664] is due to Prof. Crofton and is given by him in his celebrated paper on the *Theory of Local Probability* contained in the *Philosophical Transactions* for 1868. Strange to say, no reference to this, so far as I can find, is made in Czuber's treatise. It is the more singular that I should have overlooked the fact inasmuch as it was an outcome of conversations with myself, when Prof. Crofton was serving under me in the Royal Military Academy at Woolwich, that he was put upon the track of investigations in local probability in which he has since earned for himself so great and well merited celebrity. It may be added that Prof. Crofton seems to have written in entire ignorance of Barbier's discovery as he makes no allusion to it in his paper.

It is indeed a romantic incident in mathematical history that Buffon's problem of the needle should have led up (as is undoubtedly the case) to Crofton's new and striking theorems in the integral calculus reproduced in Bertrand's *Calcul intégral*.

70.

SUR LE RAPPORT DE LA CIRCONFÉRENCE AU DIAMÈTRE.

[*Comptes Rendus*, CXI. (1890), pp. 778—780.]

[See p. 682, below ; footnote.]

En étudiant la preuve de Lambert, du théorème que π ne peut pas être la racine carrée d'un nombre entier, je crois avoir trouvé le moyen d'en faire l'extension au théorème de Lindemann, c'est-à-dire que π ne peut pas être la racine d'une équation rationnelle. Par exemple, supposons que π soit une racine de l'équation

$$Ax^2 + Bx + C = 0,$$

ou en mettant $Ax = \rho$, que $A\pi$ soit une racine de

$$\rho^2 + B\rho + AC = 0 ;$$

prenons un nombre entier K, tel que $K(B - A\pi)$ soit de la forme

$$2m\pi + (1-\theta)\frac{\pi}{2},$$

θ étant < 1 ; en mettant $K\rho = R$, nous aurons l'équation

$$R^2 + DR + E = 0, \qquad (1)$$

dont $KA\pi$ sera une racine et l'autre une quantité dont la tangente sera positive, η.

Considérons la fraction continue

$$S = 3 - \frac{R^2}{5 -}\,\frac{R^2}{7 -}\cdots,$$

en mettant $R = KA\pi$, on aura

$$S = 0 ;$$

en mettant $R = \eta$, on aura

$$S' = \eta.$$

Or, prenons un nombre ν tel que $2\nu > R^2$ et considérons les deux fractions continues

$$S_\nu = \frac{R^2}{2\nu+1 -}\,\frac{R^2}{2\nu+3 -}\,\frac{R^2}{2\nu+5}\cdots,$$

$$S'_\nu = \frac{R'^2}{2\nu+1 -}\,\frac{R'^2}{2\nu+3 -}\,\frac{R'^2}{2\nu+5}\cdots,$$

R, R' étant les deux racines de l'équation quadratique (1)

$$S_\nu = \frac{B}{A}, \qquad S_{\nu+1} = \frac{C}{B}, \qquad S_{\nu+2} = \frac{D}{C}, \qquad \ldots,$$

$A, B, C, D, \ldots$ étant des fonctions linéaires avec des coefficients entiers de R, et l'on aura

$$S'_\nu = \frac{B' - B'_1\eta}{A' - A'_1\eta}, \qquad S'_{\nu+1} = \frac{C' - C'_1\eta}{B' - B'_1\eta}, \qquad \ldots,$$

A', B', C' étant les mêmes fonctions de R' que le sont A, B, C de R.

Or, on peut démontrer que $A', B', C', \ldots$ seront des nombres positifs, et $\frac{A'}{A'_1}, \frac{B'}{B'_1}, \frac{C'}{C'_1} \ldots$ chacune $> \eta$.

De plus, toutes les fractions $\frac{B' - B'_1\eta}{A' - A'_1\eta}$ seront des quantités positives et moindres que l'unité.

Mais $\frac{B'}{A'} - \frac{B' - B'_1\eta}{A' - A'_1\eta} = \frac{R'^2\eta}{A'^2\left(1 - \frac{A'_1}{A'}\eta\right)}$, dont le dénominateur sera nécessairement positif.

Donc la quantité positive $\frac{B'}{A'}$ égale une fraction positive diminuée d'une autre fraction positive.

Donc $\frac{B'}{A'}$, et les quantités semblables, $\frac{C'}{B'}, \frac{D'}{C'}, \ldots$, seront toutes des fractions positives et moindres que l'unité.

Donc $\frac{BB'}{AA'}, \frac{CC'}{BB'}, \frac{DD'}{CC'}, \ldots$ seront des fractions possédant ce même caractère.

Mais tous ces *produits* AA', BB', CC' seront des *nombres entiers*, ce qui est impossible.

Je crois pouvoir faire une démonstration tout à fait semblable pour établir que π ne peut pas être la racine d'une équation d'un degré quelconque dont toutes les racines sont réelles. Pour le cas d'équations avec des racines imaginaires, il y aura quelque chose de plus à faire pour achever la démonstration ; mais j'ai lieu de croire qu'avec l'aide de la théorie des modules de quantités imaginaires il n'y aura pas de grosses difficultés à vaincre. Enfin j'ajoute que deux quantités réelles ou imaginaires, dont l'une est la tangente ou le logarithme népérien de l'autre, ne peuvent être toutes les deux fonctions algébriques des racines de la même équation irréductible, à coefficients entiers.

71.

PREUVE QUE π NE PEUT PAS ÊTRE RACINE D'UNE ÉQUATION ALGÉBRIQUE À COEFFICIENTS ENTIERS*.

[*Comptes Rendus*, CXI. (1890), pp. 866—871.]

LEMME. *Soit*

$$J = \cfrac{\epsilon m}{n + \cfrac{\epsilon' m'}{n' + \cfrac{\epsilon'' m''}{n'' + \ldots}}},$$

où $\epsilon^2 = \epsilon'^2 = \epsilon''^2 = \ldots = 1$; $n, n', n'', \ldots$ *sont des nombres réels positifs et plus grands que l'unité;* $m, m', m'', \ldots$, *des nombres réels ou complexes, et où chaque quotient partiel est assujetti à la condition que* $n - 1$ *est plus grand que le module de* m.

Alors je dis que le module de J sera moindre que l'unité.

Supposons que ces conditions soient satisfaites par $\frac{m}{n}, \frac{m_{,}}{n_{,}}$.

Soit $$m = \alpha + i\beta.$$

Par hypothèse $$n - 1 > \sqrt{(\alpha^2 + \beta^2)}.$$

Servons-nous de $M(x)$ pour signifier le module de x, alors

$$M\left(\frac{m}{n}\right) = \frac{M(m)}{n} < \frac{n-1}{n} < 1,$$

de sorte que, si $\frac{m}{n} = \alpha_{,} + i\beta_{,}$, $\alpha_{,}^2 + \beta_{,}^2 < 1$ et, à plus forte raison, $\alpha_{,}^2 < 1$,

$$M\left(\cfrac{m_{,}}{n_{,} + \cfrac{m}{n}}\right) = \frac{M(m_{,})}{M(n_{,} + \alpha_{,} + i\beta_{,})} = \frac{M(m_{,})}{\sqrt{\{(n_{,} + \alpha_{,})^2 + \beta_{,}^2\}}} < \frac{M(m_{,})}{n_{,} - 1},$$

* Cette Note doit être substituée à la Note de l'auteur qui a été insérée, par suite d'un malentendu, dans les *Comptes rendus* du 24 novembre dernier. La Note précédente, qui ne traitait que le cas le plus restreint du théorème du texte, est affectée d'inexactitudes qui la rendent de nulle valeur.

car $(n_{,} + \alpha_{,})^2$, quand $\alpha_{,}$ est compris entre les limites $1, -1$, est plus grand que $(n_{,} - 1)^2$.

Donc, par hypothèse,

$$M\left(\cfrac{m_{,}}{n_{,} + \cfrac{m}{n}}\right) < 1,$$

et, évidemment, par le même raisonnement, on trouve successivement

$$M\left(\frac{m}{n}\right), \quad M\left(\cfrac{m_{,}}{n_{,} + \cfrac{m}{n}}\right), \quad M\left(\cfrac{m_{,,}}{n_{,,} + \cfrac{m_{,}}{n_{,} + \cfrac{m}{n}}}\right), \quad \ldots$$

ou, ce qui revient à la même chose, toutes les quantités

$$M\left(\frac{\epsilon m}{n}\right), \quad M\left(\cfrac{\epsilon_{,} m_{,}}{n_{,} + \cfrac{\epsilon m}{n}}\right), \quad M\left(\cfrac{\epsilon_{,,} m_{,,}}{n_{,,} + \cfrac{\epsilon_{,} m_{,}}{n_{,} + \cfrac{\epsilon m}{n}}}\right), \quad \ldots$$

seront moindres que l'unité*.

Nous allons démontrer, à l'aide de ce lemme, que, si θ est une racine d'une équation *irréductible* à coefficients entiers, tang θ ne peut pas être rationnel ou même une fonction rationnelle à coefficients rationnels de θ.

Supposons que $A\theta^n + B\theta^{n-1} + \ldots + L = 0$ et que tang θ soit une fonction rationnelle de θ. On peut supposer que $A = 1$, car, si nous écrivons $\theta' = A\theta$†, alors l'équation pour θ' peut s'exprimer semblablement à celle pour θ, mais avec le premier coefficient égal à l'unité. De plus, si l'on peut démontrer que tang θ' ne peut pas être une fonction rationnelle de θ', alors, puisque $\theta' = A\theta$, et conséquemment tang θ', est une fonction rationnelle de tang θ, il s'ensuivra que, si tang θ est une fonction rationnelle de θ, tang θ' sera une fonction rationnelle de θ', ce qui est contraire à la supposition faite‡.

* Ce lemme peut être envisagé comme une application de la proposition 8, III d'Euclide. En prenant O le centre d'un cercle à rayon unité et N un point extérieur à ce cercle, Euclide y enseigne que le segment de ON, compris entre N et le contour convexe, sera moindre que toute autre ligne droite menée de N au cercle: à plus forte raison il sera moindre que la distance de N à un point quelconque d'un cercle intérieur au premier. Voir la Note au bas de la page [685, below] pour une addition qu'on doit faire à ce lemme.

† Voir le scolie pour le cas plus général où les coefficients de l'équation en θ sont des nombres complexes [p. 686, below].

‡ L'illustre Legendre aurait, il me semble, dû faire une transformation analogue dans sa présentation célèbre de la preuve de Lambert de son théorème (Note IV, *Éléments de Géométrie*). Pour avoir négligé cette précaution, la succession infinie de quantités toujours décroissantes qu'il trouve par le moyen du lemme de Lambert ne forme pas nécessairement une succession de nombres entiers, mais de tels nombres divisés par des puissances toujours croissantes de A, le dénominateur de θ, supposé rationnel, exprimé comme fraction vulgaire réduite, ce qui n'est nullement impossible.

Donc, nous pouvons supposer que l'équation en θ soit de la forme

$$\theta^n + B\theta^{n-1} + \ldots + L = 0.$$

Évidemment on peut aussi supposer que l'équation en θ soit irréductible.

Écrivons $\theta \operatorname{tang} \theta = \tau(\theta)$, de sorte que

$$\tau(\theta) = \cfrac{\theta^2}{1 - \cfrac{\theta^2}{3 - \cfrac{\theta^2}{5 - \ddots}}},$$

on trouvera

$$\cfrac{\theta^2}{3 - \cfrac{\theta^2}{5 - \ddots}} = \frac{\tau(\theta) - \theta^2}{\tau(\theta)},$$

$$\cfrac{\theta^2}{5 - \cfrac{\theta^2}{7 - \ddots}} = \frac{\tau(\theta)(3 - \theta^2) - 3\theta^2}{\tau(\theta) - \theta^2},$$

$$\cfrac{\theta^2}{7 - \cfrac{\theta^2}{9 - \ddots}} = \frac{\tau(\theta)(15 - 6\theta^2) - 15\theta^2 + \theta^4}{\tau(\theta)(3 - \theta^2) - 3\theta^2},$$

et, en nommant

$$\cfrac{\theta^2}{2r + 1 - \cfrac{\theta^2}{2r + 3 - \ddots}} = \Theta_r(\theta),$$

$$\Theta_r(\theta) = \frac{A_{r+1}(\theta)\,\tau(\theta) - B_{r+1}(\theta)}{A_r(\theta)\,\tau(\theta) - B_r(\theta)},$$

$$\Theta_{r+1}(\theta) = \frac{A_{r+2}(\theta)\,\tau(\theta) - B_{r+2}(\theta)}{A_{r+1}(\theta)\,\tau(\theta) - B_{r+1}(\theta)}.$$

....

....

Soit $\Theta_{r,i}(\theta)$ ce que devient $\Theta_r(\theta)$ quand on substitue θ_i pour θ dans la valeur de $\tau(\theta)$. Si, pour une certaine racine θ_i de l'équation supposée en θ, $\tau_{r,i}(\theta) = \tau_r(\theta_i)$, alors $\tau_{r,i}(\theta)$ en vertu du lemme aura un module moindre que l'unité ; sinon, ce module deviendra éventuellement et restera, pour une certaine valeur r, et pour toute valeur supérieure, au-dessous d'une certaine limite, parce que dans ce cas $\Theta_{r,i}(\theta)$ différera et continuera à différer par une quantité aussi petite qu'on veut de $\frac{A_{r+1}(\theta_i)}{A_r(\theta_i)}$ (dont le module a une limite supérieure dépendant de la grandeur de θ_i) quand r est pris suffisamment grand. Cela sera développé au long dans une Communication ultérieure.

Supposons que N soit le plus grand des modules carrés des n racines,

$\theta_1, \theta_2, \theta_3, \ldots, \theta_n$ les n racines de l'équation proposée en θ. Prenons $2r > N$; alors, en vertu du lemme* et à cause du principe énoncé plus haut, on aura éventuellement (en prenant $2r - N$ suffisamment grand) le produit des modules de $\Theta_r(\theta_1), \Theta_r(\theta_2), \ldots, \Theta_r(\theta_n)$ moindre que l'unité pour une certaine valeur de r et toute valeur de r supérieure à celle-ci.

Or, remarquons que, à cause de la valeur l'*unité* du coefficient de θ^n dans l'équation en θ, tous les $A(\theta)$ et les $B(\theta)$ seront des fonctions linéaires et *entières* de $\theta, \theta^2, \ldots, \theta^{n-1}$, car si $\mu > n - 1$, θ^μ devient une fonction linéaire et entière de $\theta, \theta^2, \ldots, \theta^{n-1}$.

Ainsi, en supposant que k soit un nombre tel qui rende $k\tau(\theta)$ une fonction linéaire entière de $\theta, \theta^2, \ldots, \theta^{n-1}$, pour toute valeur de r,

$$k[A_r(\theta)\,\tau(\theta) - B_r(\theta)]$$

sera une fonction rationnelle et entière de θ ; or, en vertu de ce qui a été dit, le produit des modules de

$$\Theta_\mu(\theta_1), \quad \Theta_\mu(\theta_2), \quad \ldots, \quad \Theta_\mu(\theta_n)$$

sera moindre que l'unité quand μ est plus grand que le nombre que nous avons nommé r. Mais le produit des modules de n quantités est le module de leur produit ; donc

$$k^n\,\Pi\,[A_r(\theta)\,\tau(\theta) - B_r(\theta)],$$
$$k^n\,\Pi\,[A_{r+1}(\theta)\,\tau(\theta) - B_{r+1}(\theta)],$$
$$k^n\,\Pi\,[A_{r+2}(\theta)\,\tau(\theta) - B_{r+2}(\theta)],$$

....

....

formeront une succession infinie de nombres entiers décroissants, ce qui est impossible†.

Ainsi $\tau(\theta)$ et conséquemment $\operatorname{tang} \theta$ ne peut pas être une fonction rationnelle de θ quand θ est racine d'une équation à coefficients entiers.

Si nous supposons que $\operatorname{tang} \theta$ soit une quantité rationnelle pure et simple, cela ne fait nul changement dans notre raisonnement ; ainsi, puisque $\operatorname{tang} \pi$ $\left(\text{ou bien si l'on veut } \operatorname{tang} \frac{\pi}{4}\right)$ est rationnel, π ne peut pas être la racine d'une équation algébrique à coefficients entiers.

Je démontre par un procédé à peu près pareil à ce qui précède, la proposition inverse, c'est-à-dire que, si $\operatorname{tang} \theta$ est racine d'une équation algébrique, alors θ ne peut pas être une fonction rationnelle à coefficients rationnels de $\operatorname{tang} \theta$. Or, dans cette théorie, il n'y a nulle distinction entre les quantités réelles et complexes, de sorte que $\sqrt{(-1)}$ compte comme quantité entière. Donc $\operatorname{tang} \sqrt{(-1)}$, et conséquemment e, base des logarithmes népériens (qui

* On doit sous-entendre par le *lemme* la proposition ainsi nommée au commencement de cette Note, mais avec l'addition essentielle, facilement prouvée, que quand les n croissent continuellement et les m restent constants, alors, en commençant avec un r suffisamment grand, le module de J deviendra une quantité aussi petite que l'on veut.

† Voir le scolie [p. 686, below] pour le cas plus général où l'équation en θ a des coefficients complexes.

en est une fonction algébrique) ne peut pas être racine d'une équation algébrique à coefficients entiers. En réunissant les deux procédés applicables à ces deux cas, on parvient à démontrer un théorème plus général, à savoir :

Si une fonction trigonométrique quelconque et son amplitude sont liées ensemble par une équation algébrique à coefficients entiers, ni l'une ni l'autre ne peut satisfaire à une équation algébrique à coefficients entiers, et comme cas particulier compris dans ce théorème, une fonction trigonométrique et son amplitude ne peuvent pas être l'une une racine d'une équation algébrique à coefficients entiers et l'autre aussi une racine d'une telle équation.*

Il y a un théorème un peu plus général, au moins en apparence, qu'on peut démontrer par un raisonnement tout à fait semblable.

Nommons une quantité qui est racine d'une équation algébrique irréductible à coefficients entiers, simples ou complexes, *quantité équationnelle*, et les racines de la même équation algébrique irréductible à coefficients entiers, *quantités équationnelles associées*; de plus, nommons une quantité qui est racine d'une équation dont les coefficients sont fonctions rationnelles d'un nombre quelconque d'autres quantités données *fonction équationnelle* de ces quantités; alors on peut affirmer qu'une fonction trigonométrique et son amplitude ne peuvent pas être, toutes les deux, fonctions équationnelles d'un même système de quantités équationnelles associées. Cette proposition donne lieu de soupçonner qu'au moyen de formules propres aux fonctions elliptiques on pourrait démontrer qu'une fonction elliptique, son amplitude et son paramètre ne peuvent pas être, tous les trois, fonctions équationnelles d'un même système de quantités équationnelles associées.

Scolie. On ne doit nullement exclure le cas où θ serait proposé comme racine d'une équation à coefficients entiers, mais complexes.

Dans ce cas, si le coefficient du premier terme en cette équation est $\alpha + i\beta$, alors afin de pouvoir réduire l'équation à sa forme canonique où ce coefficient est l'unité, sans que le tangent du nouveau θ cesse d'être fonction rationnelle de tang θ, il faut écrire $\theta' = (\alpha^2 + \beta^2)\,\theta$.

On remarquera aussi que les produits [p. 685, above]

$$k^n\, \Pi\, [A_r(\theta)\, \tau(\theta) - B_r(\theta)], \qquad k^n\, \Pi\, [A_{r+1}(\theta)\, \tau(\theta) - B_{r+1}(\theta)], \qquad \ldots,$$

au lieu d'être entiers et réels, deviendront quantités complexes, mais entières, dont les *modules* vont à l'infini en décroissant; de sorte que la démonstration donnée, pour le cas où les coefficients de l'équation en θ sont des nombres ordinaires, reste bonne pour le cas général.

* Ainsi on peut affirmer qu'une fonction trigonométrique et son amplitude, ou bien un nombre et son logarithme, ne peuvent pas être tous les deux racines de deux équations algébriques quelconques à coefficients entiers. Par exemple, cos (cos $\lambda\pi$) ne peut pas être un nombre algébrique de Kronecker, quand λ est rationnel, car son amplitude cos $\lambda\pi$ est un tel nombre. De même $e^{\sqrt[i]{\lambda}+\sqrt[j]{\mu}+\sqrt[k]{\nu}+\cdots}$ ne peut pas être racine d'une équation algébrique à coefficients entiers.

72.

ON ARITHMETICAL SERIES.

[*Messenger of Mathematics*, XXI. (1892), pp. 1—19, 87—120.]

THE first part of this article relates to the prime numbers (or so to say latent primes) contained as factors of the terms of given arithmetical series; the second part will deal with the actual or, say, visible primes included among such terms. Both investigations repose alike upon certain elementary theorems concerning the "index-sums" (relative to any given prime) of arithmetical series, whether simple and continuous as in the case of series ordinarily so called or compound and interstitial as such before named series become when subjected to certain periodic and uniform interruptions.

PART I.

§ 1. *Preliminary Notions.*

Consider any given sequence

$$m+1,\ m+2,\ m+3,\ \ldots,\ m+n,$$

in relation to any given prime number q.

Let r be the sum of the indices of the highest powers of q which are contained in the several terms of the natural sequence

$$1,\ 2,\ 3,\ \ldots,\ n,$$

s the sum of the indices of the highest powers of q contained in the given sequence.

Then it is almost immediately obvious that $s =$ or $> r$, that is $s > r - 1$.

For the index-sum of the natural sequence will be represented by

$$r = E\left(\frac{n}{q}\right) + E\left(\frac{n}{q^2}\right) + E\left(\frac{n}{q^3}\right) + \ldots,$$

and the index-sum of the given sequence by

$$s = E\left(\frac{m+n}{q}\right) + E\left(\frac{m+n}{q^2}\right) + E\left(\frac{m+n}{q^3}\right) + \ldots$$
$$- E\left(\frac{m}{q}\right) - E\left(\frac{m}{q^2}\right) - E\left(\frac{m}{q^3}\right) - \ldots$$

and this is at least equal to

$$E\left(\frac{n}{q}\right) + E\left(\frac{n}{q^2}\right) + E\left(\frac{n}{q^3}\right) + \ldots,$$

that is $s =$ or $> r$.

But there is another and more important theorem, less immediately obvious, and more germane to the subject-matter of the following section, which I proceed to explain.

Suppose $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_n$ to be the several exponents of the highest powers of q which are contained in

$$x+1,\ x+2,\ x+3,\ \ldots,\ x+n,$$

and let σ be one of these n exponents which is not less than any other of them.

Call any term in the sequence

$$x+1,\ x+2,\ x+3,\ \ldots,\ x+n$$

which contains q^σ, say P, a principal q-term.

On one side of P the terms are less, on the other greater than P; in lieu of any term substitute the difference between it and P, then I say that the q-index of such altered term will be the same as when it was unaltered.

For let the principal term, or the chosen principal term if there are more than one, be λq^σ, and let μq^ρ be any other term.

If $\rho < \sigma$, $\lambda q^\sigma \sim \mu q^\rho$ will obviously have ρ for its q-index; also if $\rho = \sigma$ the same will be true, that is supposing $\mu q^\rho - \lambda q^\rho$ to be positive, ρ will be its q-index: for if we write $\lambda = aq + b$ and $\mu = cq + d$, where $b < q$ and $d < q$, a and c must be equal, since otherwise between λq^ρ and μq^ρ there would be a term $(a+1)q \cdot q^\rho$ containing a higher power of q than the principal term: hence $\mu - \lambda = d - b$ and does not contain q. In like manner if $\lambda q^\rho - \mu q^\rho$ is positive, ρ is its q-index for the same reason as before.

Hence the index-sum, *quâ* any prime q, of the two sequences

$$m+1,\ m+2,\ \ldots,\ P-1;\quad P+1,\ P+2,\ \ldots,\ m+n-1,\ m+n$$

is the same as the sum of the index-sums of

$$1,\ 2,\ 3,\ \ldots,\ P-m-1,$$
$$1,\ 2,\ \ldots,\ m+n-P.$$

Call the sum of these two index-sums s', then

$$s' = E\left(\frac{P-m-1}{q}\right) + E\left(\frac{P-m-1}{q^2}\right) + E\left(\frac{P-m-1}{q^3}\right) + \ldots$$
$$+ E\left(\frac{m+n-P}{q}\right) + E\left(\frac{m+n-P}{q^2}\right) + E\left(\frac{m+n-P}{q^3}\right) + \ldots$$

and this is

$$= \text{or} < E\left(\frac{n-1}{q}\right) + E\left(\frac{n-1}{q^2}\right) + E\left(\frac{n-1}{q^3}\right) + \ldots$$
$$= \text{or} < E\left(\frac{n}{q}\right) + E\left(\frac{n}{q^2}\right) + E\left(\frac{n}{q^3}\right) + \ldots$$
$$= \text{or} < r.$$

Hence $s' =$ or $< r$. But the original index-sum of the sequence is diminished by σ on account of P being omitted.

Hence $s - \sigma$ or $s' =$ or $< r$.

Thus we have $s > r - 1$, $s - \sigma < r + 1$.

But this is not all: we may for certain relative values of m, n, and q (without regard to the situation of the principal term) establish the inequality $s - \sigma < r$.

I premise the obviously true statement that if $f + g < h$, then

$$f + E\left(\frac{f}{q}\right) + E\left(\frac{f}{q^2}\right) + \ldots + g + E\left(\frac{g}{q}\right) + E\left(\frac{g}{q^2}\right) + \ldots$$
$$< h + E\left(\frac{h}{q}\right) + E\left(\frac{h}{q^2}\right) + \ldots.$$

Let now h be the number of terms in the natural sequence from 1 to n which contain q.

Then in the given sequence the number will be

$$h + E\left(\frac{m+n}{q}\right) - E\left(\frac{m}{q}\right) - E\left(\frac{n}{q}\right),\ \text{say } h + e,$$

and the sum of the number of terms divisible by q in the partial sequences on each side of P will be $h + e - 1$, where $e = 1$ or 0; let the respective numbers be f, g. Then $f + g = h - 1 + e$, where $e = 0$ or 1, and, using the same notation as before,

$$s - \sigma = f + E\left(\frac{f}{q}\right) + E\left(\frac{f}{q^2}\right) + \ldots$$
$$+ g + E\left(\frac{g}{q}\right) + E\left(\frac{g}{q^2}\right) + \ldots,$$

and

$$r = h + E\left(\frac{h}{q}\right) + E\left(\frac{h}{q^2}\right) + \ldots.$$

Hence if $\qquad e = 0, \quad s - \sigma < r,$

if $\qquad e = 1, \quad s - \sigma < r + 1,$

the former inequality subsisting whenever

$$E\left(\frac{m+n}{q}\right) - E\left(\frac{m}{q}\right) - E\left(\frac{n}{q}\right) = 0.$$

If for example $m = n$, then $s - \sigma < r$ when

$$E\left(\frac{2n}{q}\right) - 2E\left(\frac{n}{q}\right) = 0,$$

which it is easily seen happens whenever $E\left(\frac{2n}{q}\right)$ is an even number.

§ 2. *Proof that* $(m+1)(m+2)\dots(m+n)$ *when* $m>n-1$ *contains a prime not contained in* $1.2.3\dots n$*.

The universal condition independent of the relation between m, n, q, above found, namely, $s-\sigma=$ or $<r$ will be found sufficient to establish the theorem which constitutes the object of this section and which is as follows:—

"If the first term of a sequence is greater than the number of terms in it, then one term at least must be a prime or a multiple of a prime greater than that number."

When the first term exceeds by unity the number of terms, the sequence takes the form $m+1$, $m+2$, ..., $2m-1$, and since no term in this sequence can be a multiple of m, the theorem for such case is tantamount to affirming that one term at least is a prime number which is in accord with and an easy inference from the well-known "postulate of Bertrand," that between m and $2m-2$ there must always be included some prime number when $m>\frac{7}{2}$.

Suppose if possible that $m+1, m+2, \dots, m+n$ contains no other primes than such as are not greater than n, and which therefore divide some of the numbers from 1 to n.

Let q be any such prime, and P_q a principal term of the sequence

$$m+1,\ m+2,\ \dots,\ m+n, \quad \text{quâ } q.$$

Then, by virtue of the proposition above established,

$$\frac{(m+1)(m+2)\dots(m+n)}{P_q}$$

will contain no higher power of q than does $1.2.3\dots n$, and consequently if P be the least common multiple of the principal terms in respect to the several primes, say ν in number (unity not being reckoned one of them), none greater than n, we may infer that

$$\frac{(m+1)(m+2)\dots(m+n)}{P}$$

will be wholly contained in, and therefore not greater than $1.2.3\dots n$, if the sequence $m+1, m+2, \dots, m+n$ contains no prime or multiple of a prime greater than n. To fix the ideas let us agree to consider that term in the sequence which contains the highest power of q, and is the greatest of all that do the same (if there be more than one), *the* principal q-term. The least common multiple cannot be greater than the product of the principal terms which are *distinct* from each other, and since even if they are all distinct, their number cannot exceed ν (the number of primes other than

* It will readily be seen that, if this theorem is true, for n any prime, it will be so à *fortiori* when n is a composite number.

unity less than $n+1$), it follows that P cannot be greater than the product of the *highest* ν terms in the given sequence. Hence we may infer that unless

$$(m+1)(m+2)\dots(m+n-\nu)$$

is less than $1.2.3\dots n$, some prime greater than n must divide one term at least of the sequence

$$m+1,\ m+2,\ \dots,\ m+n.$$

We might go further and say that unless $1.2.3\dots n$ is greater than

$$(m+1)(m+2)\dots(m+n-\nu)\,D,$$

where

$$D=\Pi q^{1+E\left(\frac{m}{q}\right)+E\left(\frac{n}{q}\right)-E\left(\frac{m+n}{q}\right)},$$

(q being made successively each of the ν primes between 2 and n inclusive and Π being used in the ordinary sense of indicating products), this same conclusion must obtain.

Conversely the theorem is true when either of these inequalities is denied. The denial of the first of them, which is sufficient for the object in view, is implied in the inequality

$$(m+1)(m+2)\dots(m+n-\nu)>1.2.3\dots n,$$

which, since ν depends only on n, may be written under the form

$$F(m,n)>1.2.3\dots n.$$

This will be referred to hereafter, in this section, as the *fundamental inequality**.

Since $F(m,n)$ increases with m, the theorem if true for m must be true for any greater value of m, when n remains constant.

From this it will be seen at once that the theorem must be true when m has any value exceeding n^2 and $n>7$.

For when $n=8$ the number of primes in the range from 1 to 8 is 4 and is equal to $\frac{1}{2}n$: but as n increases the number of new primes being less than the number of odd numbers must be less than $\frac{1}{2}n$.

Hence if $n>7$ and $m>n^2$,

$$F(m,n)>m^{n-\nu}>(n^2)^{\frac{1}{2}n}>n^n>1.2.3\dots n.$$

This result enables us to prove that the theorem is true when

$$13<n<3000.$$

The theorem it will be borne in mind is true if some prime number occurs in the sequence $m+1, m+2, \dots, m+n$, or in other words if the above sequence does not consist exclusively of composite numbers. But

* The subsistence of the fundamental inequality for any given value of n implies for that value of n the truth of the theorem to be established: but the converse does not necessarily hold. The theorem may be true when the fundamental inequality is *not* satisfied.

Dr Glaisher has found* that the highest sequence of composite numbers within the first 9000000 contains only 153 terms, namely, the sequence 4652354 to 4652506 (both inclusive). Hence if the theorem is not true when $n < 3000$, in which case $n^2 + n < 9000000$, we must have $n =$ or < 153, and there ought to be a sequence of n composite numbers in which the first term is less than $(153)^2$ which is 23409. But the longest sequence of composite numbers under 23409 is that which extends from 19610 to 19660 containing 51 terms, the square of 51 is 2601 and the longest sequence under this number is that which extends from 1328 to 1360 comprising 33 terms. The square of 33 is 1089, the longest sequence below which is from 888 to 906 comprising 19 terms: the square of 19 is 361, the longest sequence below which stretches from 114 to 126 comprising 13 terms. Hence the theorem is true for all values of n not greater than 3000 and not less than 13.

It is easy to show that the theorem is true for all values of n not greater than 13.

(1) Suppose $n = 13$, which gives $\nu = 6$.

The theorem must be true when m is taken so great that

$$(m+1)(m+2)(m+3)(m+4)(m+5)(m+6)(m+7) > 1.2.3.4.5.6.7.8.9.10.11.12.13,$$

which is easily seen to be satisfied when $m =$ or > 100.

But there is no sequence of 13 composite numbers till we come to the sequence 114 to 126, so that when $m < 100$ the theorem must be true as well as when $m =$ or > 100.

(2) Suppose $n = 11$, for which value of n, $\nu = 5$.

The theorem is true if

$$(m+1)(m+2)(m+3)(m+4)(m+5)(m+6) > 1.2.3.4.5.6.7.8.9.10.11,$$

which is obviously satisfied as before when $m = 100$, but there is no sequence of 11 which precedes the sequence before named from 114 to 126. Hence the theorem is true generally for $n = 11$.

When $n = 7$, $\nu = 4$ and the theorem is true for all values of m which make

$$(m+1)(m+2)(m+3) > 1.2.3.4.5.6.7, \text{ that is, } > 5040,$$

which is obviously the case if $m =$ or > 20, but there is no sequence of 7 composite numbers till we come to 89 to 97. Hence the theorem is proved for $n = 7$.

When $n = 5$, $\nu = 3$ and the condition of the theorem is satisfied if

$$(m+1)(m+2) > 2.3.4.5, \text{ that is, } > 120,$$

* See table at the end of this section.

as is the case if $m =$ or > 10, but the first composite sequence of 5 terms is 24 to 28. In like manner when $n = 3$, $\nu = 2$ and the theorem is true when $m + 1 =$ or $> 1.2.3$, that is, $m =$ or > 5, but 8, 9, 10 is the first composite sequence of 3 terms. Similarly, when $n = 2$, $\nu = 1$ and the condition $m + 1 =$ or > 2 is necessarily satisfied since $m =$ or $> n$ by hypothesis.

Finally, the theorem is obviously true when $n = 1$, because $m + 1$, whatever m may be, contains a factor greater than 1.

Being true for the prime numbers not exceeding 13, the slightest consideration will serve to prove that, as previously remarked in a footnote, it must be true *à fortiori* for all the composite numbers between them. Hence the theorem is verified for all values of n not greater than 3000, and it only remains to establish it for values of n exceeding that limit.

To prove it for this case we must begin with finding a superior limit to ν, when $n > 3000$, under the convenient form of a multiple of $\frac{n}{\log n}$.

If we multiply together the first 9 prime numbers from 2 to 23 and divide their product by that of the natural numbers up to 9 increased in the ratio of 1 to 2^9, the quotient will be found to exceed unity; and since the following primes are all more than twice the corresponding natural numbers, if we denote by $p_1, p_2, p_3, \ldots$, the prime numbers 2, 3, 5, ..., we must have

$$p_1 . p_2 . p_3 \ldots p_\nu > 2^\nu (1.2.3 \ldots \nu),$$

(provided that $\nu > 22$, as is the case if $n =$ or > 89),

or
$$\log(1.2.3 \ldots \nu) + (\log 2)\nu < \log(p_1 . p_2 . p_3 \ldots p_\nu).$$

But by Stirling's theorem (Serret, *Cours d'Alg. Sup.*, ed. 4, vol. II. p. 226),

$$\nu \log \nu - \nu - \tfrac{1}{2}\log \nu + \tfrac{1}{2}\log 2\pi < \log(1.2.3 \ldots \nu),$$

and by Tchebycheff's theorem (Serret, vol. II. p. 236)*,

$$\log(p_1 . p_2 . p_3 \ldots p_\nu) < n',$$

where $n' = \frac{6}{5}An + \frac{5}{4\log 6}(\log n)^2 + \frac{5}{2}\log n + 2$, and $A = \cdot 921292 \ldots$.

Hence $(\log \nu)(\nu - \frac{1}{2}) - (1 - \log 2)(\nu - \frac{1}{2}) + (\frac{1}{2}\log 2\pi - \frac{1}{2}\log \frac{1}{2}e) < n'$,

and *à fortiori* $\quad \log(\nu - \frac{1}{2})(\nu - \frac{1}{2}) - (\log \frac{1}{2}e)(\nu - \frac{1}{2}) < n'$,

or
$$\frac{2}{e}(\nu - \tfrac{1}{2})\log\left\{\frac{2}{e}(\nu - \tfrac{1}{2})\right\} < \frac{2}{e}n'.$$

Hence, if we write $\quad \mu \log \mu = \frac{2}{e}n' = n_1$

we shall have $\quad \nu - \frac{1}{2} < \frac{1}{2}e\mu.$

* For greater simplicity I have left out the term $-An^{\frac{1}{2}}$, and thereby *increased* the superior limit.

But $$\mu = \frac{n_1}{\log \mu},$$
and therefore
$$\log \mu = \log n_1 - \log\log \mu = \log n_1 - \log(\log n_1 - \log\log \mu) > \log n_1 - \log\log n_1.$$

Hence $$\mu < \frac{n_1}{\log n_1 - \log\log n_1}$$
$$< \frac{2}{e}\frac{n'}{\log n' - \log\log n' + \log\frac{2}{e}}$$

and $$\nu < \tfrac{1}{2} + \frac{n'}{\log n' - \log\log n' - (1-\log 2)}\,*.$$

Hence, observing that $\frac{1}{u}$, $\frac{\log u}{u}$, $\frac{(\log u)^2}{u}$, $\frac{\log\log u}{\log u}$ *all* decrease as the denominators increase (provided as regards the second of these fractions that $u > e$, as regards the third that $u > e^2$, and as regards the fourth that $u > e^e$), we may find a superior limit to ν in the case before us, where $n > 3000$, by writing in the numerator of $\nu - \frac{1}{2}$,
$$\frac{(\log 3000)^2}{3000}n,\quad \frac{\log 3000}{3000}n,\quad \frac{2}{3000}n,$$
for $(\log n)^2$, $\log n$, 2,

and in its denominator, first, $\log n - \log\log n$ for $\log n' - \log\log n'$, and then
$$\frac{\log\log 3000}{\log 3000}\log n \quad\text{and}\quad \frac{1-\log 2}{\log 3000}\log n,$$
for $\log\log n$ and $1-\log 2$ respectively.

Making the calculations it will be found that we shall get
$$\nu - \tfrac{1}{2} < 1{\cdot}606\,\frac{n}{\log n}.$$

With the aid of this limit it will now be easy to prove the truth of the theorem when $n =$ or > 3000.

Let us suppose $n =$ or > 3000.

(1) Suppose $m < 2n$, then $m + n > \frac{3}{2}m$ and the theorem will be proved for this case, if it can be shown that in the range of numbers from m to $\frac{3}{2}m$, there is at least one prime number when $m =$ or > 3000.

* From this it will be seen that the asymptotic ratio of ν to $\frac{n}{\log n}$ is less than the asymptotic ratio which any superior limit to the sum of the logarithms of the primes not exceeding n bears to n: this perhaps is a new result, at all events it is not to be found in Serret nor indeed is it wanted for Tchebycheff's proof of the famous postulate which Serret has so lucidly expounded. The correlative theorem that the asymptotic ratio of ν to $\frac{n}{\log n}$ is always *greater* than the asymptotic ratio which any *inferior* limit to the sum aforesaid bears to n is of course an obvious and familiar fact.

This will be the case (Serret, vol. II. p. 239), if (on that supposition) $\frac{5}{6}\cdot\frac{3}{2}n - n$, that is, if
$$\frac{n}{4} > 2\sqrt{(\tfrac{3}{2}n)} + \frac{25(\log\frac{3}{2}n)^2}{16A\log 6} + \frac{125}{24A}(\log\tfrac{3}{2}n) + \frac{25}{6A},$$
where $A = {\cdot}92129202\ldots$.

But when $n = 3000$, it will be found that the terms on the second side of the inequality are respectively less than
$$134{\cdot}1641,\quad 66{\cdot}9773,\quad 47{\cdot}5546,\quad 4{\cdot}5227,$$
whose sum is less than 750.

Hence, the inequality is satisfied, and accordingly the theorem is true when $m < 2n$ and n is equal to or *greater* than 3000; for when n satisfies that condition the derivative in respect to n of the right-hand side of the above inequality will be always less than $\frac{1}{4}$.

(2) Suppose $m =$ or $> 2n$, then it is only necessary to prove that
$$\log(2n+1)(2n+2)\ldots(3n-\nu) > \log(1\,.\,2\,.\,3\ldots n),$$
or, what is the same thing, that
$$\log\{1\,.\,2\,.\,3\,.\,4\ldots(3n-\nu)\} > \log(1\,.\,2\,.\,3\ldots n) + \log(1\,.\,2\,.\,3\ldots 2n),$$
ν being the number of primes not greater than n, and n being at least 3000.

Call the two sides of the inequality P and Q. Then (Serret, vol. II. p. 226)
$$P > \log\sqrt{(2\pi)} + (3n-\nu)\log(3n-\nu) - (3n-\nu) - \tfrac{1}{2}\log(3n-\nu)$$
$$> \log\sqrt{(2\pi)} + (3n-\nu)\log 3n + (3n-\nu)\log\left(1-\frac{\nu}{3n}\right) - 3n + \nu - \tfrac{1}{2}\log 3n$$
$$> \log\sqrt{(2\pi)} + 3(\log n)n + (3\log 3 - 3)n - (\log n)\nu$$
$$+ (1-\log 3)\nu - \tfrac{1}{2}\log 3 - \tfrac{1}{2}\log n - \nu,$$
for $$-(3n-\nu)\log\left(1-\frac{\nu}{3n}\right) = \nu\left\{1 - \tfrac{1}{2}\left(\frac{\nu}{3n}\right) - \tfrac{1}{6}\left(\frac{\nu}{3n}\right)^2 - \tfrac{1}{12}\left(\frac{\nu}{3n}\right)^3 - \ldots\right\} < \nu.$$

On the other hand,
$$Q < \log\sqrt{(2\pi)} + n\log n - n + \tfrac{1}{2}\log n + \tfrac{1}{12}$$
$$+ \log\sqrt{(2\pi)} + 2n\log 2n - 2n + \tfrac{1}{2}\log 2n + \tfrac{1}{12}$$
$$< \{2\log\sqrt{(2\pi)} + \tfrac{1}{2}\log 2 + \tfrac{1}{6}\} + 3(\log n)n + (2\log 2 - 3)n + \log n.$$

Hence
$$P - Q > (3\log 3 - 2\log 2)n - (\log n)\nu - \tfrac{3}{2}\log n - (\log 3)\nu - \{\tfrac{1}{2}\log(12\pi) + \tfrac{1}{6}\}$$
$$> (3\log 3 - 2\log 2)n - \log n\,(\nu - \tfrac{1}{2})$$
$$- 2\log n - \log 3\,(\nu - \tfrac{1}{2}) - \{\tfrac{1}{2}\log(36\pi) + \tfrac{1}{6}\}$$
where $\nu - \frac{1}{2} < 1{\cdot}606\,\frac{n}{\log n}$.

But $$3 \log 3 - 2 \log 2 = 1{\cdot}9095415 > 1{\cdot}909.$$

Hence*

$$P - Q > ({\cdot}303)\, n - (1{\cdot}606 \log 3) \frac{n}{\log n} - 2 \log n - \{\tfrac{1}{2} \log (36\pi) + \tfrac{1}{6}\},$$

say $P - Q > f(n) > 0$ when $n = 3000$.

Also the derivative with respect to n of $(\log n) f(n)$ being

$$({\cdot}303)(1 + \log n) - 1{\cdot}606 \log 3 - \frac{4 \log n}{n} - \frac{\frac{1}{2} \log (36\pi) + \frac{1}{6}}{n},$$

$P - Q$ will increase as n increases and will remain positive for all values of n superior to 3000.

Hence the theorem is true, whatever m may be, when $n =$ or > 3000, and since it has been proved previously for the case of $n < 3000$, it is true universally.

I subjoin the valuable table, kindly communicated to me by Dr Glaisher, referred to in the text above.

Table of Increasing Sequences of Composite Numbers interposed between Consecutive Primes included in the first nine million numbers.

Limits to sequence	Number of terms
7 to 11	3
23 „ 29	5
89 „ 97	7
113 „ 127	13
523 „ 541	17
887 „ 907	19
1129 „ 1151	21
1327 „ 1361	33
9551 „ 9587	35
15683 „ 15727	43
19609 „ 19661	51
31397 „ 31469	71
155921 „ 156007	85
373261 „ 373373	111
492113 „ 492227	113
1349533 „ 1349651	117
1357201 „ 1357333	131
2010733 „ 2010881	147
4652353 „ 4652507	153

* It will now be seen why I take separately the two cases of m greater and m less than $2n$. If we were to take *simpliciter* $m =$ or $> n$ and were to attempt to prove

$$\log \{1 \,.\, 2 \,.\, 3 \ldots (2n - \nu)\} > 2 \log (1 \,.\, 2 \,.\, 3 \ldots n)$$

the inferior limit to the difference between these two quantities would then have for its principal term, not $(3 \log 3 - 2 \log 2 - 1{\cdot}606)\, n$ but $(2 \log 2 - 1{\cdot}606)\, n$, which would be *negative*.

Of course there is no special reason except of convenience (in dealing with an integer instead of a fraction) for making $2n$ the dividing point between the two suppositions separately considered in the text; κn where κ as far as regards the second inequality does not fall short of some certain limit, would have served as well: this inferior limit to κ would be some quantity a little greater (how much exactly would have to be found by trial) than the quantity θ which makes $\theta \log \theta - (\theta - 1) \log (\theta - 1)$ equal to the coefficient of $\frac{n}{\log n}$ in the superior limit to ν. As regards the first inequality κ would have to be a quantity somewhat less (how much less to be found by trial) than the quantity η which makes $\frac{\eta + 1}{\eta} = \frac{6}{5}$, that is, $\eta = 5$. This is on the supposition made throughout of using Tchebycheff's own limits, but if we use the more general, but less compact, limits indicated in my paper in vol. IV. of the *American Journal of Mathematics*‡, any fraction not less than $\frac{6}{5}$ and not so great as [illegible] would take the place of $\frac{6}{5}$, and the extreme value of η would be [illegible], which is a trifle under 6. By a judicious choice of the value given to κ, a value of n could be found considerably less than 3000, which would satisfy both inequalities, and this in the absence of Dr Glaisher's table would have been a matter of some practical importance, but is of next to none when we have that table to draw upon. How low down in the scale of number, n may be taken, without interruption of the existence of the fundamental inequality for the minimum value of n in the case treated of in this section, it has not been necessary for the purpose in hand to ascertain. That it holds good for all values of n above a certain limit follows from the fact that $2 \log 2$ is greater than the coefficient of the leading term in the superior functional limit to the sum of the logarithms of the primes not greater than n.

The table is to be understood as follows. The lowest sequence of as many as 3 consecutive composite numbers is that included between 7 and 11: the lowest of as many as 5 is that included between 23 and 29, of as many as 7 that included between 89 and 97; between 13 and 17 there is a break—this indicates that the lowest sequence of as many as 15, or as many as 17 first occurs in the sequence of 17 interposed between 523, 541. Similarly the break between 21 and 33 indicates that the lowest sequence containing 23 or 25 or 27 or 29 or 31 or 33 terms first occurs in the sequence of 33 composite numbers interposed between the primes 1327, 1361.

It is also necessary to add that in the first nine million numbers there is no succession of *more* than 153 consecutive composite numbers.

§ 3. *Relating to irreducible arithmetical series in general**.

Let P be a principal term quâ q in any *irreducible* arithmetical series whose common difference is i, N any other term greater or less than P, and D their difference. If q is not prime to i, no term in the series will be divisible by q.

Just as in the case of a natural sequence when there is only one principal term in the series it may be shown that the index of D quâ q will be the same as that of N; when there is more than one principal term it appears by the same reasoning as before that the index of N cannot be greater than that of D: (it will not now necessarily be equal unless q is greater than the common difference i).

The index-sum quâ q is zero when q has a common measure with i, and we may therefore consider only the case where q is relatively prime to i;

* An irreducible arithmetical series is one whose terms are prime to their common difference.

[‡ Vol. III. of this Reprint, p. 530.]

on this supposition, by virtue of what has been stated above, the index-sum quâ q of the series whose first term is $m+i$, and number of terms n, will be equal to or less than

$$E\left(\frac{P-m-i}{iq}\right)+E\left(\frac{P-m-i}{iq^2}\right)+E\left(\frac{P-m-i}{iq^3}\right)+\dots$$
$$+E\left(\frac{m+ni-P}{iq}\right)+E\left(\frac{m+ni-P}{iq^2}\right)+E\left(\frac{m+ni-P}{iq^3}\right)+\dots;$$

and therefore *à fortiori*

$$< \text{or} = E\left(\frac{(n-1)i}{iq}\right)+E\left(\frac{(n-1)i}{iq^2}\right)+E\left(\frac{(n-1)i}{iq^3}\right)+\dots$$
$$< \text{or} = E\left(\frac{n}{q}\right)+E\left(\frac{n}{q^2}\right)+E\left(\frac{n}{q^3}\right)+\dots,$$

that is, not greater than the index-sum of 2, 3, ..., n quâ q.

Consequently, by the same reasoning as that employed in the last section, the theorem now to be proved, namely, that if m (prime to i) = or $> n$, then $(m+i)(m+2i)\dots(m+ni)$ must contain some one or more prime numbers greater than n, must be true whenever

$$(m+i)(m+2i)(m+3i)\dots\{m+(n-\nu_1)i\} > 1.2.3\dots n \qquad (\Theta)^*$$

where ν_1 is the number of prime numbers not exceeding n, and not contained in i, and *à fortiori* when for ν_1, we substitute, as for the present we shall do, ν the *entire* number of primes not greater than n. This I term the *fundamental inequality* for the general case now under consideration.

Suppose n = or > 3000. The logarithm of the first side of the fundamental inequality when we write ν for ν_1 is obviously greater than the ith part of the logarithm of

$$(m+1)(m+2)\dots(m+i)(m+i+1)\dots\{m+(n-\nu)i\};$$

and the inequality (subject to certain suppositions) to be established will be satisfied, if on the same suppositions,

$$\frac{1}{i}\log[1.2.3\dots\{m+(n-\nu)i\}] > \log(1.2.3\dots n)+\frac{1}{i}\log(1.2.3\dots m).$$

Suppose $m=n$, and make

$$\log[1.2.3\dots\{(i+1)n-i\nu\}]=T,$$
$$(i+1)\log(1.2.3\dots n)=U,$$
$$F(n,i)=T-U.$$

* If it had been necessary the condition in the text might have been stated in the more stringent form that *some aliquot part* of the factorial of n (namely, this factorial divested of all powers of prime numbers contained in i) would have to be greater than

$$(m+i)(m+2i)\dots\{m+(n-\nu_1)i\}$$

if the theorem were not true for any specified values of m, n, i.

It will be noticed that when i is relatively prime to n, ν_1 is less than ν so that $n-\nu_1 > n-\nu$: some use will be made of the formula in the text when dealing with certain small values of n and $m-n$ towards the end of the section.

Then $T > \log(2\pi)+\{(i+1)n-i\nu\}\log\{(i+1)n-i\nu\}$
$$-\{(i+1)n-i\nu\}-\tfrac{1}{2}\log\{(i+1)n-i\nu\},$$
$$U < (i+1)\log\sqrt{(2\pi)}+(i+1)n\log n-(i+1)n+\tfrac{1}{2}(i+1)\log n+\tfrac{1}{12}(i+1).$$

Hence $F(n,i) > -i\log\sqrt{(2\pi)}+\{(i+1)n-i\nu\}\log\{(i+1)n\}$
$$+\{(i+1)n-i\nu\}\log\left\{1-\frac{i\nu}{(i+1)n}\right\}$$
$$+i\nu-(i+1)n\log n-\tfrac{1}{2}\log\{(i+1)n-i\nu\}-\tfrac{1}{2}(i+1)\log n-\tfrac{1}{12}(i+1)$$
$$>\{(i+1)\log(i+1)\}n-i\log\{(i+1)n\}\nu-\tfrac{1}{2}\log\{(i+1)n\}-\tfrac{1}{2}(i+1)\log n$$
$$-\tfrac{1}{2}i\log(2\pi)-\tfrac{1}{12}(i+1),$$

that is $>\{(i+1)\log(i+1)\}n-i\log\{(i+1)n\}\nu-\tfrac{1}{2}(i+2)\log n-\tfrac{1}{2}\log(i+1)$
$$-\tfrac{1}{2}i\log(2\pi)-\tfrac{1}{12}(i+1) \qquad (\mathrm{H}),$$

so that when $n > 3000$ and consequently $\nu < \tfrac{1}{2}+(1{\cdot}606)\dfrac{n}{\log n}$, the inequality to be established will be true *à fortiori* if

$$F(n,i) > \left\{(i+1)\log(i+1)-(1{\cdot}606)\,i\left[1+\frac{\log(i+1)}{\log n}\right]\right\}n-(i+1)\log n$$
$$-[\tfrac{1}{2}(i+1)\log(i+1)+\tfrac{1}{2}\{i\log(2\pi)\}+\tfrac{1}{12}(i+1)].$$

When $i=1$ or 2 or 3 the coefficient of n is negative; consequently the limit to ν before found is no longer applicable to bring out the desired result.

The case of $i=1$ has been already disposed of; that of $i=2$ may be disposed of, as I shall show, in a similar manner; when $i=3$, I shall raise the limit n from 3000 to 8100 of which the logarithm is so near to 9 that it may, for the purpose of the proof in hand, be regarded as equal to 9 without introducing any error in the inequality to be established, as the error involved will only affect the result in a figure beyond the 4th or 5th place of decimals, whereas the inequality in question depends on figures in the first decimal place. When this is done the theorem will be in effect demonstrated for the case of $i=3$ and $n > 8100$. For all values of n not greater than 8100 I shall be able to show that the fundamental inequality (Θ) is satisfied by employing the actual value of ν_1 or ν instead of a limiting value of the latter.

Thus the fundamental inequality will be shown to subsist for all values of n when $i=3$ and $m=n$, and *à fortiori* therefore for all values of m and i *not less* than n and 3 respectively.

Case of $i=2$.

Suppose n = or > 3000, and take separately the cases $m <$ or $= 2n$, $m > 2n$.

(1) Let m be not greater than $2n$ so that $m+2n$ is greater than $2m-1$.

By hypothesis m must be odd, and by Bertrand's Postulate

$$m+2,\ m+3,\ m+4,\ \ldots,\ 2m,$$

and therefore

$$m+2,\ m+4,\ m+6,\ \ldots,\ (2m-1)$$

(seeing that the interpolated terms are all even) must contain a prime, and thus the first case is disposed of.

(2) Since the fundamental inequality has been shown to be satisfied when $n>3000$, $m>2n$, $i=1$, it will *à fortiori* be so when $n>3000$, $m>2n$, $i=2$.

Hence the theorem is established for $i=2$ when $n>3000$. Finally as regards values of n inferior to 3000, the reasoning employed for the case of $i=1$ applies *à fortiori* to the case of $i=2$.

To see this let us recall the first step of the reasoning applicable to the supposition of $i=1$.

Because in the first nine million numbers there is no sequence of 3000 composite numbers, from Dr Glaisher's Table of Sequences (taken in conjunction with the fact that when $m>n^2$, the theorem has been proved to be true whatever n may be), we were able to infer that it must be true when n does not exceed 153: in the present case, if the theorem were not true when $3000>n>153$, there would be a sequence of 153 composite odd numbers and therefore of over 305 composite consecutive numbers in the first 9000000 numbers, whereas there are not more than 153, and so we may proceed step by step till we arrive at the conclusion that the theorem must be true when $n>13$; and when $n=13, 11, 7, 5, 3, 2, 1$ a like method of disproof (but briefer) will apply as for the case of $i=1$.

Case of $i =$ *or* >3.

Let $n=$ or >8100. Then we may without ultimate error write

$$\nu-\tfrac{1}{2}<\frac{1{\cdot}1056+\dfrac{5}{4\log 6}\dfrac{81}{8100}+\tfrac{5}{2}\dfrac{9}{8100}+\dfrac{2}{8100}}{1-\dfrac{\log 9}{9}-\dfrac{1-\log 2}{9}}\,\frac{n}{\log n}<1{\cdot}546\,\frac{n}{\log n},$$

and accordingly

$$F(n,3)>\left\{4\log 4-(3\times 1{\cdot}546)\left(1+\frac{\log 4}{9}\right)\right\}n$$
$$-4\log n-\left(2\log 4+\tfrac{3}{2}\log 2\pi+\tfrac{1}{3}\right)$$

and

$$F(8100,3)>(5{\cdot}545-5{\cdot}352)(8100)-36-5{\cdot}863>0.$$

Hence the Fundamental Inequality is satisfied when $n=$ or >8100.

To prove that it is satisfied for values inferior to 8100, observe that by virtue of the formula (H) it will be so, *ex abundantiâ*, for all values of n not

less than $'n$ and not greater than n', provided that, calling n'_ν the number of primes not exceeding n',

$$(5{\cdot}545)\,'n-3\log(4n')\,n'_\nu-\tfrac{5}{2}\log n'-C>0,$$

where

$$C=\tfrac{1}{3}+\log 2+\tfrac{3}{2}\log(2\pi)=3{\cdot}783.$$

On trial it will be found that the above inequality is satisfied when we successively substitute for $'n$, n', and for n'_ν (found from any Table for the enumeration of primes) the values given in the annexed table:

n'	n'_ν	$'n$
8100	1018	5725
5724	753	4096
4095	564	2967
2966	427	2172
2171	326	1604
1603	252	1200
1199	196	903
902	154	687
686	124	535
534	99	415
414	80	325
324	66	260
259	55	210
209	46	171
170	39	141
140	34	111
110	29	99
98	25	84
83	23	76
75	21	68
67	19	62
61	18	57
56	16	50
49	15	46
45	14	42
41	13	39
38	12	36
35	11	32
31	11	31
30	10	30
29	10	29

The fundamental theorem is therefore established when $i>2$ for all values of n down to 29 inclusive.

It remains to consider the case where n is any prime number less than 29.

Calling μ the difference between n and the number of primes (exclusive of 1) not greater than n, to

$$n=2,\ 3,\ 11,\ 17,\ 23$$

will correspond

$$\mu=1,\ 1,\ \ 6,\ 10,\ 14$$

and for each combination of these corresponding numbers it will be found that

$$1.2.3\ldots n = \text{or} < (n+3)(n+6)\ldots(n+3\mu).$$

Hence the theorem is proved for these values of n, whatever n may be, when $i = \text{or} > 3$. To

$$n = 13, \quad n = 19$$

corresponds

$$\mu = 7, \quad \mu = 11,$$

and for these combinations of n and μ it will be found that

$$1.2.3\ldots n < (n+4)(n+7)\ldots(n+1+3\mu),$$

so that the theorem is true for

$$n = 13, \ 19,$$

except in the case where

$$m = 13, \ 19.$$

That it is true in these excepted cases follows from inspection of the series,

$$16, 19, 22, 25, \&c.,$$
$$22, 25, 28, 31, \&c.,$$

where $19 > 13$, $31 > 19$: or it might be proved, but more cumbrously, by the same method as that applied below to the only two values of n remaining to be considered, namely

$$n = 5, \quad n = 7,$$

for which we have respectively

$$\mu = 2, \quad \mu = 3.$$

If $n = 5$ and i has no common measure with $2.3.4.5$, i must be not less than 7, but $1.2.3.4.5 < 12.19$.

On the other hand, if i has a common measure with $2.3.4.5$, then what we have called ν_1, in formula (Θ), is less than ν, so that $n - \nu_1 > 2$, but

$$1.2.3.4.5 < 8.11.14.$$

These two inequalities combined serve to prove that, whatever i may be, the inequality (Θ) is satisfied, and the theorem is consequently proved for $n = 5$.

So again, when $n = 7$, if i has no common measure with $2.3.4.5.6.7$ it must be 11 at least. In that case the inequality $2.3.4.5.6.7 < 18.29.40$, and in the contrary case the inequality $2.3.4.5.6.7 < 10.13.16.19$ serves to prove the theorem.

When $n = 1$ the truth of the theorem is obvious: hence combining the results obtained in this and the preceding section, it will be seen we have proved that whatever n and whatever i may be, provided that m is relatively prime to i and not less than n, the product

$$(m+i)(m+2i)\ldots(m+ni)$$

must contain some prime number by which $2.3\ldots n$ is not divisible, and the wearisome proof is thus brought to a close. It will not surprise the author of it, if his work should sooner or later be superseded by one of a less piece-meal character—but he has sought in vain for any more compendious proof. He has not thought it necessary to produce the figures or refer in detail to the calculations giving the numerical results inserted in various places in the text: had he done so the number of pages, already exceeding what he had any previous idea of, would probably have been more than doubled*.

PART II†.

Explicit Primes.

In this part I shall consider the asymptotic limits to the number of primes of certain *irreducible* linear forms $mz + r$ comprised between a number x and a given fractional multiple thereof kx, the method of investigation being such that the asymptotic limits determined will be unaffected by the value of r, and will be the same for all values of m which

* The author was wandering in an endless maze in his attempts at a general proof of his theorem, until in an auspicious hour when taking a walk on the Banbury road (which leads out of Oxford) the Law of Ademption flashed upon his brain: meaning thereby the law (the nerve, so to say, of the preceding investigation) that *if all the terms of a natural arithmetical series be increased by the same quantity so as to form a second such series, no prime number can enter in a higher power as a factor of the product of the terms of this latter series, when a suitable term has been* taken away *from it, than the highest power in which it enters as a factor into the product of the terms of the original series.*

In Part II. I shall be able to apply the same method to demonstrate a theorem showing that it is always possible to split up an infinite arithmetical series, if not in the general case, at least for certain values of the common difference, into an infinite number of successive finite and determinable segments such that one or more primes shall be found in each such segment: a theorem which is, so to say, Dirichlet's theorem on arithmetical progressions cut up into slices.

The whole matter is thus made to rest on an elementary fundamental *equality* (Tchebycheff's) which, with the aid of an application of Stirling's theorem, leads (as the former has so admirably shown) *inter alia* to a superior limit to the sum of the logarithms of the primes not exceeding a given number, from which as has been seen in § 2, a superior limit may be deduced to the number of such primes. With the aid of this last limit together with an elementary fundamental *inequality* and a *renewed* application of Stirling's theorem, all my results are made to flow. Thus a theorem of pure form is brought to depend on considerations of greater and less, or as we may express it, Quality is made to stoop its neck to the levelling yoke of Quantity.

Long and vain were my previous efforts to make the desired results hinge upon the properties of transposed Eratosthenes' scales: now we may hope to reverse the process and compel these scales to reveal the secret of their laws under the new light shed upon them by the successful application of the Quantitative method.

† I ought to have stated that the theorem contained in section 2 of Part I. originally appeared in the form of a question (No. 10951) in the *Educational Times* for April of this year.

have the same totient. The simplest case, and the foundation of all that follows, is that in which $k=0$ and $m=2$: this will form the subject of the ensuing chapter which may be regarded as a supplement to Tchebycheff's celebrated memoir of 1850*, and as superseding my article thereon in vol. IV. of the *Amer. Math. Journ.* [Vol. III. of this Reprint, p. 530].

CHAPTER I.

ON THE ASYMPTOTIC LIMITS TO THE NUMBER OF PRIMES INFERIOR TO A GIVEN NUMBER.

§ 1. *Crude determination of the asymptotic limits.*

Call the sum of the logarithms of primes not exceeding x (any real positive quantity) the prime-number-logarithmic sum, or more briefly the prime-log-sum to x, and the sum of such sums to x and all its positive integer roots the prime-log-sum-sum, which in Serret is called $\psi(x)$.

Then it follows from elementary arithmetical principles that the sum of this sum-sum to x and all its aliquot parts, that is

$$\psi(x)+\psi\left(\frac{x}{2}\right)+\psi\left(\frac{x}{3}\right)+\ldots,$$

which we may call the natural series of sum-sums and denote by $T(x)$, is identical with the logarithm of the factorial of the highest integer not exceeding x, and accordingly from Stirling's theorem may be shown to have for its asymptotic limit $x\log x-x$, the superior and inferior limits being this quantity with a residue which, as well for the one as for the other, is a known linear function of $\log x$. Serret, vol. II. p. 226.

If now we take two sets of positive integers,

$$p, p', p'', \ldots;\ q, q', q'', \ldots,$$

together forming what may be termed a *harmonic scheme*, meaning thereby that the sum of the reciprocals of the numbers in the two sets is the same, and extend the T series over x divided by the respective numbers in each set and take the difference between the two sums thus obtained, there will result a new series of the form

$$\sum_{n=1}^{n=\infty} f(n)\,\psi\left(\frac{x}{n}\right),$$

of which the asymptotic limit will be x multiplied by

$$\Sigma\frac{\log p}{p}-\Sigma\frac{\log q}{q},$$

and the value of $f(n)$ will be

$$\Sigma\frac{n}{p:}-\Sigma\frac{n}{q:},$$

* Published in the *St Petersburg Transactions* for 1854.

where, in general, $\frac{n}{t:}$ means 1 or 0 according as n does or does not contain t, or in other words the "denumerant" of the equation $ty=n$.

I shall call the p's and q's the *stigmata* of the scheme:

$$\Sigma\frac{\log p}{p}-\Sigma\frac{\log q}{q}$$

the stigmatic multiplier, and the new series in $\psi(x)$ a stigmatic series of sum-sums (obtained, it will be noticed, by a four-fold process of summation—namely, two infinite and two finite summations).

It is possible, in general (as will hereafter appear), to deduce from the asymptotic value of a stigmatic series of sum-sums, superior and inferior asymptotic limits to the sum-sum itself. The *asymptotic* limits to the simple sum will then be the same as those last named (Serret, vol. II. p. 236, formulae (8) and (9)*) and will be multiples of x: dividing these respectively by $\log x$, we obtain superior and inferior asymptotic limits to the number of primes not exceeding x (*Messenger*, May 1891, p. 9, footnote [above, p. 694]).

It is obviously simplest always to take unity as one of the stigmata; those employed by Tchebycheff are 1, 30; 2, 3, 5; this *scheme* as I term it leads to the relation

$$\psi\left(\frac{x}{1}\right)-\psi\left(\frac{x}{6}\right)+\psi\left(\frac{x}{7}\right)-\psi\left(\frac{x}{10}\right)+\psi\left(\frac{x}{11}\right)-\psi\left(\frac{x}{12}\right)$$
$$+\psi\left(\frac{x}{13}\right)-\psi\left(\frac{x}{15}\right)+\psi\left(\frac{x}{17}\right)-\psi\left(\frac{x}{18}\right)+\psi\left(\frac{x}{19}\right)-\psi\left(\frac{x}{20}\right)$$
$$+\psi\left(\frac{x}{23}\right)-\psi\left(\frac{x}{24}\right)+\psi\left(\frac{x}{29}\right)-\psi\left(\frac{x}{30}\right)$$
$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$=\left(\tfrac{1}{2}\log 2+\tfrac{1}{3}\log 3+\tfrac{1}{5}\log 5-\tfrac{1}{30}\log 30\right)x+\dagger,$$

the series extending to infinity but consisting of repetitions (with a difference) of the above period, obtained by adding for the second period 30, for the third period 60, for the fourth period 90, and so on, to each denominator in the period set out. We may call this a period of 30 terms in which the coefficients are $+1$, 0, or -1. So, in general, whatever the stigmata may be, the stigmatic series will consist of periods of terms in each of which the total number of terms will be the least common multiple of the stigmata.

* The fourth edition, 1879, of Serret's *Cours d'Algèbre Supérieure* is referred to here and throughout the paper.

† The + is used to denote that a quantity is omitted of inferior *order* of magnitude to x. The strict interpretation of the "relation" is that the sum of the stigmatic series less the stigmatic multiplier into x is intermediate to two known linear functions of $\log x$.

Thus, for example, the schemes 1; 2, 2 and 1, 6; 2, 3, 3 would give rise to the relations

$$\psi(x)-\psi\left(\frac{x}{2}\right)+\psi\left(\frac{x}{3}\right)-\psi\left(\frac{x}{4}\right)+\psi\left(\frac{x}{5}\right)-\psi\left(\frac{x}{6}\right)+\dots$$
$$=(\tfrac{1}{2}\log 2+\tfrac{1}{2}\log 2)\,x+=(\log 2)\,x+\dots,$$
$$\psi(x)-\psi\left(\frac{x}{3}\right)+\psi\left(\frac{x}{5}\right)-\psi\left(\frac{x}{6}\right)+\psi\left(\frac{x}{7}\right)-\psi\left(\frac{x}{9}\right)+\psi\left(\frac{x}{11}\right)-\psi\left(\frac{x}{12}\right)+\dots$$
$$=(\tfrac{1}{2}\log 2+\tfrac{2}{3}\log 3-\tfrac{1}{6}\log 6)\,x+=(\tfrac{1}{3}\log 2+\tfrac{1}{2}\log 3)\,x+\dots$$

of which the periods are 2 and 6 respectively.

The three schemes above given, whose keys, so to say, are 2, 3, 5 respectively (these being the highest prime numbers contained in the stigmata), possess the property that their effective coefficients are alternately plus and minus 1, and, in consequence thereof, we may *immediately* deduce from them asymptotic limits superior and inferior to the logarithmic sum-sum $\psi(x)$.

Thus, calling the stigmatic multipliers in the three cases

$$St_2,\ St_3,\ St_5,$$

we obtain as limits to the coefficient of x in $\psi(x)$,

	St_2	and	$2St_2$	from the	first,
	St_3	,,	$\frac{3}{2}St_3$	,, ,,	second,
and	St_5	,,	$\frac{6}{5}St_5$	,, ,,	third scheme.

(Compare Serret, pp. 233, 234, where the A is the present St_5.)

The three pairs of limits will thus be

$$\cdot 6931472 : 1\cdot 3862944,$$
$$\cdot 7803552 : 1\cdot 1705328,$$
$$\cdot 9212920 : 1\cdot 1055504,$$

which are in regular order of closer and closer propinquity to unity on each side of it*.

The question then arises can no further schemes be discovered which will enable us to bring the asymptotic coefficients still nearer to this empirical limit†?

* Mr Hammond has noticed that the harmonic scheme 1, 12; 2, 3, 4 will also give rise to a stigmatic series in which the effective terms are alternately positive and negative units, namely,

$$\psi(x)-\psi\left(\frac{x}{4}\right)+\psi\left(\frac{x}{5}\right)-\psi\left(\frac{x}{6}\right)+\psi\left(\frac{x}{7}\right)-\psi\left(\frac{x}{8}\right)+\psi\left(\frac{x}{11}\right)-\psi\left(\frac{x}{12}\right)+\psi\left(\frac{x}{13}\right)-\psi\left(\frac{x}{16}\right)+\dots,$$

the stigmatic multiplier corresponding to which, say St_{12}, is $\cdot 8522758\dots$, and therefore will furnish the asymptotic coefficients St_{12} and $\frac{4}{3}St_{12}$, that is, $\cdot 8522758\dots$ and $1\cdot 1363687\dots$.

† The true asymptotic limit to the number of primes below x being according to Legendre's empirical rule $\frac{x}{\log x}$, the asymptotic value of $\psi(x)$ should presumably be x.

It would, I believe, be perfectly futile to seek for stigmatic schemes, involving higher prime numbers than 5, that should give rise to stigmatic series of sum-sums in which the successive coefficients should be alternately positive and negative unity, as in the above instances, but this although a sufficient is not a necessary condition in order that limits to a sum-sum may be capable of being extracted from the known limits to the sum of a series of such sum-sums.

This will be most easily explained by actually exhibiting a new scheme which is effective to the end in view, and showing why it is so.

Such a scheme is 1, 6, 70; 2, 3, 5, 7, 210, which, it will be observed, satisfies the necessary *harmonic* condition: for we have

$$1+\tfrac{1}{6}+\tfrac{1}{70}=\frac{210+35+3}{210}=\tfrac{248}{210},$$
$$\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7}+\tfrac{1}{210}=\frac{105+70+42+30+1}{210}=\tfrac{248}{210}.$$

The *stigmatic multiplier* is here

$$\tfrac{1}{2}\log 2+\tfrac{1}{3}\log 3+\tfrac{1}{5}\log 5+\tfrac{1}{7}\log 7+\tfrac{1}{210}\log 210-\tfrac{1}{6}\log 6-\tfrac{1}{70}\log 70=\cdot 9787955,$$

which I shall call D.

The stigmatic series arranged in sets in two different ways then becomes as a first arrangement

$$\psi(x)-\psi\left(\frac{x}{10}\right);$$
$$+\psi\left(\frac{x}{11}\right)+\psi\left(\frac{x}{13}\right)-\psi\left(\frac{x}{14}\right)-\psi\left(\frac{x}{15}\right);\ +\psi\left(\frac{x}{17}\right)+\psi\left(\frac{x}{19}\right)$$
$$-\psi\left(\frac{x}{20}\right)-\psi\left(\frac{x}{21}\right);\ +\psi\left(\frac{x}{23}\right)-\psi\left(\frac{x}{28}\right);\ +\psi\left(\frac{x}{29}\right)-\psi\left(\frac{x}{30}\right);$$
$$+\psi\left(\frac{x}{31}\right)-\psi\left(\frac{x}{35}\right);\ +\psi\left(\frac{x}{37}\right)-\psi\left(\frac{x}{40}\right);\ +\psi\left(\frac{x}{41}\right)-\psi\left(\frac{x}{42}\right);$$
$$+\psi\left(\frac{x}{43}\right)-\psi\left(\frac{x}{45}\right);\ +\psi\left(\frac{x}{47}\right)-\psi\left(\frac{x}{50}\right);\ +\psi\left(\frac{x}{53}\right)-\psi\left(\frac{x}{56}\right);$$
$$+\psi\left(\frac{x}{59}\right)-\psi\left(\frac{x}{60}\right);\ +\psi\left(\frac{x}{61}\right)-\psi\left(\frac{x}{63}\right);\ +\psi\left(\frac{x}{67}\right)-\psi\left(\frac{x}{70}\right);$$
$$+\psi\left(\frac{x}{71}\right)+\psi\left(\frac{x}{73}\right)-\psi\left(\frac{x}{75}\right)+\psi\left(\frac{x}{79}\right)-\psi\left(\frac{x}{80}\right)+\psi\left(\frac{x}{83}\right)$$
$$-\psi\left(\frac{x}{84}\right)+\psi\left(\frac{x}{89}\right)-\psi\left(\frac{x}{90}\right)+\psi\left(\frac{x}{97}\right)-\psi\left(\frac{x}{98}\right)-\psi\left(\frac{x}{100}\right);$$
$$+\psi\left(\frac{x}{101}\right)+\psi\left(\frac{x}{103}\right)-\psi\left(\frac{x}{105}\right)-\psi\left(\frac{x}{105}\right);\ +\psi\left(\frac{x}{107}\right)$$

$$+\psi\left(\frac{x}{109}\right)-\psi\left(\frac{x}{110}\right)-\psi\left(\frac{x}{112}\right);\ +\psi\left(\frac{x}{113}\right)-\psi\left(\frac{x}{120}\right);$$

$$+\psi\left(\frac{x}{121}\right)-\psi\left(\frac{x}{126}\right);\ +\psi\left(\frac{x}{127}\right)-\psi\left(\frac{x}{130}\right);\ +\psi\left(\frac{x}{131}\right)$$

$$-\psi\left(\frac{x}{135}\right);\ +\psi\left(\frac{x}{137}\right)+\psi\left(\frac{x}{139}\right)-\psi\left(\frac{x}{140}\right)+\psi\left(\frac{x}{143}\right)$$

$$-\psi\left(\frac{x}{147}\right)+\psi\left(\frac{x}{149}\right)-\psi\left(\frac{x}{150}\right)+\psi\left(\frac{x}{151}\right)-\psi\left(\frac{x}{154}\right)$$

$$+\psi\left(\frac{x}{157}\right)-\psi\left(\frac{x}{160}\right)+\psi\left(\frac{x}{163}\right)-\psi\left(\frac{x}{165}\right)+\psi\left(\frac{x}{167}\right)$$

$$-\psi\left(\frac{x}{168}\right)+\psi\left(\frac{x}{169}\right)-\psi\left(\frac{x}{170}\right)+\psi\left(\frac{x}{173}\right)-\psi\left(\frac{x}{175}\right)$$

$$+\psi\left(\frac{x}{179}\right)-\psi\left(\frac{x}{180}\right)+\psi\left(\frac{x}{181}\right)-\psi\left(\frac{x}{182}\right)+\psi\left(\frac{x}{187}\right)$$

$$-\psi\left(\frac{x}{189}\right)-\psi\left(\frac{x}{190}\right);\ +\psi\left(\frac{x}{191}\right)+\psi\left(\frac{x}{193}\right)-\psi\left(\frac{x}{195}\right)$$

$$-\psi\left(\frac{x}{196}\right);\ +\psi\left(\frac{x}{197}\right)+\psi\left(\frac{x}{199}\right)-\psi\left(\frac{x}{200}\right)+\psi\left(\frac{x}{209}\right)$$

$$-\psi\left(\frac{x}{210}\right)-\psi\left(\frac{x}{210}\right);\ +\psi\left(\frac{x}{211}\right)-\psi\left(\frac{x}{220}\right);$$

....

....

the correlative arrangement being

$$\psi(x)-\psi\left(\frac{x}{10}\right)+\psi\left(\frac{x}{11}\right)+\psi\left(\frac{x}{13}\right):$$

$$-\psi\left(\frac{x}{14}\right)-\psi\left(\frac{x}{15}\right)+\psi\left(\frac{x}{17}\right)+\psi\left(\frac{x}{19}\right);\ -\psi\left(\frac{x}{20}\right)-\psi\left(\frac{x}{21}\right)$$

$$+\psi\left(\frac{x}{23}\right)-\psi\left(\frac{x}{28}\right)+\psi\left(\frac{x}{29}\right)-\psi\left(\frac{x}{30}\right)+\psi\left(\frac{x}{31}\right)-\psi\left(\frac{x}{35}\right)$$

$$+\psi\left(\frac{x}{37}\right)-\psi\left(\frac{x}{40}\right)+\psi\left(\frac{x}{41}\right)-\psi\left(\frac{x}{42}\right)+\psi\left(\frac{x}{43}\right)-\psi\left(\frac{x}{45}\right)$$

$$+\psi\left(\frac{x}{47}\right)-\psi\left(\frac{x}{50}\right)+\psi\left(\frac{x}{53}\right)-\psi\left(\frac{x}{56}\right)+\psi\left(\frac{x}{59}\right)-\psi\left(\frac{x}{60}\right)$$

$$+\psi\left(\frac{x}{61}\right)-\psi\left(\frac{x}{63}\right)+\psi\left(\frac{x}{67}\right)-\psi\left(\frac{x}{70}\right)+\psi\left(\frac{x}{71}\right)+\psi\left(\frac{x}{73}\right);$$

$$-\psi\left(\frac{x}{75}\right)+\psi\left(\frac{x}{79}\right);\ -\psi\left(\frac{x}{80}\right)+\psi\left(\frac{x}{83}\right);\ -\psi\left(\frac{x}{84}\right)+\psi\left(\frac{x}{89}\right);$$

$$-\psi\left(\frac{x}{90}\right)+\psi\left(\frac{x}{97}\right);\ -\psi\left(\frac{x}{98}\right)-\psi\left(\frac{x}{100}\right)+\psi\left(\frac{x}{101}\right)$$

$$+\psi\left(\frac{x}{103}\right);\ -\psi\left(\frac{x}{105}\right)-\psi\left(\frac{x}{105}\right)+\psi\left(\frac{x}{107}\right)+\psi\left(\frac{x}{109}\right);$$

$$-\psi\left(\frac{x}{110}\right)-\psi\left(\frac{x}{112}\right)+\psi\left(\frac{x}{113}\right)-\psi\left(\frac{x}{120}\right)+\psi\left(\frac{x}{121}\right)$$

$$-\psi\left(\frac{x}{126}\right)+\psi\left(\frac{x}{127}\right)-\psi\left(\frac{x}{130}\right)+\psi\left(\frac{x}{131}\right)-\psi\left(\frac{x}{135}\right)$$

$$+\psi\left(\frac{x}{137}\right)+\psi\left(\frac{x}{139}\right);\ -\psi\left(\frac{x}{140}\right)+\psi\left(\frac{x}{143}\right);\ -\psi\left(\frac{x}{147}\right)$$

$$+\psi\left(\frac{x}{149}\right);\ -\psi\left(\frac{x}{150}\right)+\psi\left(\frac{x}{151}\right);\ -\psi\left(\frac{x}{154}\right)+\psi\left(\frac{x}{157}\right);$$

$$-\psi\left(\frac{x}{160}\right)+\psi\left(\frac{x}{163}\right);\ -\psi\left(\frac{x}{165}\right)+\psi\left(\frac{x}{167}\right);\ -\psi\left(\frac{x}{168}\right)$$

$$+\psi\left(\frac{x}{169}\right);\ -\psi\left(\frac{x}{170}\right)+\psi\left(\frac{x}{173}\right);\ -\psi\left(\frac{x}{175}\right)+\psi\left(\frac{x}{179}\right);$$

$$-\psi\left(\frac{x}{180}\right)+\psi\left(\frac{x}{181}\right);\ -\psi\left(\frac{x}{182}\right)+\psi\left(\frac{x}{187}\right);\ -\psi\left(\frac{x}{189}\right)$$

$$-\psi\left(\frac{x}{190}\right)+\psi\left(\frac{x}{191}\right)+\psi\left(\frac{x}{193}\right);\ -\psi\left(\frac{x}{195}\right)-\psi\left(\frac{x}{196}\right)$$

$$+\psi\left(\frac{x}{197}\right)+\psi\left(\frac{x}{199}\right);\ -\psi\left(\frac{x}{200}\right)+\psi\left(\frac{x}{209}\right);\ -\psi\left(\frac{x}{210}\right)$$

$$-\psi\left(\frac{x}{210}\right)+\psi\left(\frac{x}{211}\right)-\psi\left(\frac{x}{220}\right)+\psi\left(\frac{x}{221}\right)+\psi\left(\frac{x}{223}\right):$$

....

....*.

The terms in each arrangement, it will be seen, are separated by marks of punctuation into groups: omitting the first group in either of them, which may be called the outstanding group, in each of the others the sum of the coefficients is zero.

Moreover, the sum of the coefficients from the beginning of each group is always homonymous in sign, that is, will be non-negative in the first and non-positive in the second arrangement: the consequence of this is that all the terms of such groups may be resolved into pairs, whose sum will be necessarily positive in the one and negative in the other.

Thus, for example, in the first arrangement the last but one of the groups may be resolved into the pairs

$$\psi\left(\frac{x}{197}\right)-\psi\left(\frac{x}{200}\right);\ \psi\left(\frac{x}{199}\right)-\psi\left(\frac{x}{210}\right);\ \psi\left(\frac{x}{209}\right)-\psi\left(\frac{x}{210}\right).$$

* Each of these arrangements is to be regarded as made up of the outstanding group and an infinite succession of periodic groups. In the text we have set out the outstanding group and the first period, the other periods will be formed from this one by adding to each denominator in it successive multiples of 210.

each of which is equal to zero or a positive quantity. So the eighth group of the second arrangement is resoluble into the pairs

$$-\psi\left(\frac{x}{98}\right)+\psi\left(\frac{x}{101}\right);\ -\psi\left(\frac{x}{100}\right)+\psi\left(\frac{x}{103}\right),$$

each of which is zero or a negative quantity.

It may be as well to notice in this place that the sum of the coefficients, reckoning from the first term of the outstanding group to the term whose denominator is n, is

$$\sum_{t=0}^{t=n}\Sigma\left(\frac{t}{p:}-\frac{t}{q:}\right),$$

which by virtue of the obvious identity,

$$\sum_{t=0}^{t=n}\left(\frac{t}{i:}\right)=E\left(\frac{n}{i}\right),$$

is equal to $$\Sigma\left\{E\left(\frac{n}{p}\right)-E\left(\frac{n}{q}\right)\right\}.$$

This formula supplies an easy and valuable test for ascertaining the correctness of the determination of the coefficients up to any given term in the series.

These observations may be extended to any harmonic scheme whatever: for it will be observed that

$$\Sigma\left\{E\left(\frac{n}{p}\right)-E\left(\frac{n}{q}\right)\right\}$$

is a periodic quantity, and therefore possesses both a maximum and a minimum; whence it is easy to see that, by taking the outstanding group of terms sufficiently extensive, all the remaining terms in either kind of arrangement may be separated into groups similar to those above set out; namely, such that the *complete* sum of the coefficients in each group from its first to its end term is zero and up to any intermediate term is *homonymous*, that is, always positive in one and always negative in the other arrangement*.

* For example, from the harmonic scheme 1, 15; 2, 3, 5, 30, we may derive a stigmatic series under the two forms of arrangement

$$\psi(x)-\psi\left(\frac{x}{6}\right):+\psi\left(\frac{x}{7}\right)-\psi\left(\frac{x}{10}\right);+\psi\left(\frac{x}{11}\right)-\psi\left(\frac{x}{12}\right):+\psi\left(\frac{x}{13}\right)+\psi\left(\frac{x}{17}\right)-\psi\left(\frac{x}{18}\right)$$
$$+\psi\left(\frac{x}{19}\right)-\psi\left(\frac{x}{20}\right)+\psi\left(\frac{x}{23}\right)-\psi\left(\frac{x}{24}\right)+\psi\left(\frac{x}{29}\right)-\psi\left(\frac{x}{30}\right)-\psi\left(\frac{x}{30}\right);+\psi\left(\frac{x}{31}\right)-\psi\left(\frac{x}{36}\right):\ \&c.,$$
$$\psi(x)-\psi\left(\frac{x}{6}\right)+\psi\left(\frac{x}{7}\right)-\psi\left(\frac{x}{10}\right)+\psi\left(\frac{x}{11}\right)-\psi\left(\frac{x}{12}\right)+\psi\left(\frac{x}{13}\right)+\psi\left(\frac{x}{17}\right):-\psi\left(\frac{x}{18}\right)$$
$$+\psi\left(\frac{x}{19}\right);-\psi\left(\frac{x}{20}\right)+\psi\left(\frac{x}{23}\right);-\psi\left(\frac{x}{24}\right)+\psi\left(\frac{x}{29}\right);-\psi\left(\frac{x}{30}\right)-\psi\left(\frac{x}{30}\right)+\psi\left(\frac{x}{31}\right)$$
$$-\psi\left(\frac{x}{36}\right)+\psi\left(\frac{x}{37}\right)-\psi\left(\frac{x}{40}\right)+\psi\left(\frac{x}{41}\right)-\psi\left(\frac{x}{42}\right)+\psi\left(\frac{x}{43}\right)+\psi\left(\frac{x}{47}\right):\ \&c.$$

In the above arrangements the groups are separated by semicolons and the period is marked out by the colons. In this instance it will be observed that minimum and maximum values of

The consequence of this is that the outstanding group in the first arrangement will always be less, and in the second arrangement always greater, than a function of which the principal, or, as we may call it, the asymptotic term, is the product of x by the stigmatic multiplier, say (St), the complete function being in each case of the form $(St)x$ associated with a known linear function of $\log x$. (Compare Serret, vol. II. p. 232.)

The importance of this observation will become apparent in a subsequent section.

In the case before us (that is, for the scheme in the key of 7) confining our attention to the principal term of either limit, the first arrangement leads immediately (Serret, p. 234) to the superior asymptotic limit $\frac{10}{9}Dx$.

As regards the inferior limit, we have

$$\psi(x)+\psi\left(\frac{x}{13}\right)>Dx,$$

$$\psi(x)>Dx-\tfrac{1}{13}\cdot\tfrac{10}{9}Dx>\tfrac{107}{117}Dx^{*}.$$

Substituting for D its value ·9787955, we obtain the asymptotic limits 1·0873505 and ·8951370.

The corresponding values got from the Tchebycheffian scheme (1, 30; 2, 3, 5) being 1·1055504 and ·9212920, which are the $\frac{6}{5}A$ and A of Serret.

We know *aliunde* that the true asymptotic values are each of them presumably unity. The superior value above obtained by the new scheme is thus seen to be better, and the inferior value worse than those given by Tchebycheff's scheme. But these values correspond to what may be termed the *crude* determination of the limits which the schemes are capable of affording. The contraction of these asymptotic limits by a method of continual successive approximation will form the subject of the following section†.

$E(n)+E\left(\frac{n}{15}\right)-E\left(\frac{n}{2}\right)-E\left(\frac{n}{3}\right)-E\left(\frac{n}{5}\right)-E\left(\frac{n}{30}\right)$ are 0 and 2, and accordingly in the first arrangement the outstanding group has to be continued until the sum of the coefficients of the terms which it contains is 0, and in the second until such sum is 2.

Writing $$Q=\tfrac{1}{2}\log 2+\tfrac{1}{3}\log 3+\tfrac{1}{5}\log 5+\tfrac{1}{30}\log 30-\tfrac{1}{15}\log 15=\cdot 96750\ldots,$$

we may deduce from the above, the asymptotic coefficients $\frac{6}{5}Q$ and $Q-\frac{1}{17}\cdot\frac{6}{5}Q$; that is, 1·1610 ... and ·8992....

* Compare the determination of the limits for the harmonic scheme 1; 2, 3, 6 (*American Journal of Mathematics*, vol. IV. pp. 243, 244 [Vol. III. of this Reprint, p. 542]).

† By the method about to be explained, it should be noticed, we may not merely improve upon the results obtained by the *crude* method from certain harmonic schemes (which form a very restricted class) but may also obtain limits to $\psi(x)\div x$ from harmonic schemes which without its aid would be absolutely sterile (see p. [715]).

§ 2. *On a method of obtaining continually contracting asymptotic limits to* $\dfrac{\psi(x)}{x}$.

To fix the ideas let us consider the scheme (1, 30; 2, 3, 5) which leads to the stigmatic series

$$(1)-(6)+(7)-(10)+(11)-(12)+(13)-(15)+(17)-(18)+(19)$$
$$-(20)+(23)-(24)+(29)-(30)+(31)\ldots,$$

in which for brevity (n) is used to denote $\psi\left(\dfrac{x}{n}\right)$.

The sum of this series is, we know, intermediate between

$$Dx+R(\log x) \quad\text{and}\quad D_1x+R_1(\log x),$$

where $\qquad D=\cdot 9212920\ldots,\quad D_1=1\cdot 1055504\ldots=\tfrac{6}{5}D,$

and R, R_1 signify two known quantities which for uniformity may both be regarded as quadratic functions of $\log x$ (in the first of which the coefficient of $(\log x)^2$ is zero). (Serret, pp. 233, 235.)

Omitting every pair of consecutive terms $-(m)+(\mu)$ in which $\dfrac{\mu}{m}<\tfrac{6}{5}$, and using $[\psi(x)]$ to signify the asymptotic value of $\psi(x)$, we find

$$[\psi(x)]>Dx+\left[\psi\left(\frac{x}{24}\right)\right]-\left[\psi\left(\frac{x}{29}\right)\right]>Dx+D\frac{x}{24}-D_1\frac{x}{29},$$

say $\qquad >D'x.$

Similarly, omitting every consecutive pair of terms $(m)-(\mu)$ in which $\dfrac{\mu}{m}<\tfrac{6}{5}$, we find

$$[\psi(x)]<Dx+D_1\frac{x}{6}-D\frac{x}{7}+D_1\frac{x}{10},$$

say $\qquad <D_1'x.$

If instead of $[\psi(x)]$ we had deduced limits to $\psi(x)$ in the manner indicated above, we should have found

$$\psi(x)>D'x+R'(\log x),\quad \psi(x)<D_1'x+R_1'(\log x);$$

the added terms being each of them quadratic functions of $\log x$.

Repeating this process we shall obtain

$$[\psi(x)]>D''x,\quad [\psi(x)]<D_1''x,$$

where $\qquad D''=D+\tfrac{1}{24}D'-\tfrac{1}{29}D_1',\quad D_1''=D+\tfrac{1}{6}D_1'-\tfrac{1}{7}D'+\tfrac{1}{10}D_1'.$

Similarly we may write

$$[\psi(x)]>D'''x,\quad [\psi(x)]<D_1'''x,$$

where $\qquad D'''=D+\tfrac{1}{24}D''-\tfrac{1}{29}D_1'',\quad D_1'''=D+\tfrac{1}{6}D_1''-\tfrac{1}{7}D''+\tfrac{1}{10}D_1'',$

and so on.

If then we write for $D,\ D',\ D'',\ \ldots,\ v_0,\ v_1,\ v_2,\ \ldots,$
and for $\qquad D_1,\ D_1',\ D_1'',\ \ldots,\ u_0,\ u_1,\ u_2,\ \ldots,$
we shall find in general

$$[\psi(x)]>v_ix,\quad [\psi(x)]<u_ix;$$

where
$$v_{i+1}=D+\frac{v_i}{24}-\frac{u_i}{29},$$
$$u_{i+1}=D+(\tfrac{1}{6}+\tfrac{1}{10})u_i-\tfrac{1}{7}v_i;$$

the complete statement of the inequalities being

$$\psi(x)>v_ix+R^{(i)}(\log x),\quad \psi(x)<u_ix+R_1^{(i)}(\log x),$$

where it is to be noticed that the supplemental terms always remain *quadratic* functions of $\log x$.

(The result thus obtained differs in this particular from that stated by me in the *Amer. Math. Jour.* (vol. IV. p. 241)*; the process therein employed giving as supplemental terms rational integral functions of continually rising degrees of $\log x$. I am indebted to Mr Hammond for drawing my attention to this simple but important circumstance which had strangely escaped my attention previously.) To integrate the equations in u, v we have only to write

$$v_i=V_i+F,\qquad u_i=U_i+E,$$
$$F(1-\tfrac{1}{24})+\tfrac{1}{29}E=D,\quad V_i=C_1\rho_1^i+C_2\rho_2^i,$$
$$\tfrac{1}{7}F+(1-\tfrac{1}{6}-\tfrac{1}{10})E=D,\quad U_i=K_1\rho_1^i+K_2\rho_2^i;$$

and to take for ρ_1, ρ_2 the two roots of the equation

$$\begin{vmatrix}\rho-\tfrac{1}{24}, & \tfrac{1}{29}\\ \tfrac{1}{7}, & \rho-\tfrac{1}{6}-\tfrac{1}{10}\end{vmatrix}=\rho^2-(\tfrac{1}{6}+\tfrac{1}{10}+\tfrac{1}{24})\rho+\tfrac{1}{24}(\tfrac{1}{6}+\tfrac{1}{10})-\tfrac{1}{203}=0,$$

that is $\qquad \rho^2-\tfrac{37}{120}\rho+\tfrac{113}{18270}=0.$

The roots of this equation being each less than 1, on making $i=\infty$ we obtain $v_\infty=F$, $u_\infty=E$, where E, F are deduced from the two algebraic equations

$$\tfrac{23}{24}F+\tfrac{1}{29}E=D,$$
$$\tfrac{1}{7}F+\tfrac{11}{15}E=D.$$

This gives

$$\frac{E}{F}=(\tfrac{23}{24}-\tfrac{1}{7})\div(\tfrac{11}{15}-\tfrac{1}{29})=\frac{137\times 145}{304\times 56}=\frac{19865}{17024}=q$$

(compare *Amer. Math. Jour.*, vol. IV. p. 242),

$$E=\tfrac{59595}{50999}D=1\cdot 0765779\ldots,$$
$$F=\tfrac{51072}{50999}D=\ \cdot 9226107\ldots;$$

whence we may infer that $\psi(x)$ may be made intermediate between two

[* See Vol. III. of this Reprint, p. 539.]

known functions $u_i x + r(\log x)$, $v_i x + s(\log x)$, where u_i, v_i may be brought *indefinitely near* to the numbers

$$1{\cdot}0765779\ldots, \quad {\cdot}9226107\ldots;$$

and the supplemental terms are quadratic functions of $\log x$ depending upon the value of i that may be employed. We may, therefore (subject to an obvious interpretation), treat E and F as asymptotic limits to $\frac{\psi(x)}{x}$.*

If we examine the ratio of the denominators m, μ of any pair of consecutive terms throughout the entire infinite series, whether of the form $(m) - (\mu)$ or $-(m) + (\mu)$, we shall find that $\frac{\mu}{m}$ is always less than q (namely $1{\cdot}16688\ldots$), except in the case of the pairs that have been retained in forming the equations between E and F, from which we may infer that if any of the discarded pairs had been retained we should have obtained values of E and F respectively greater and less than those above set forth.

If, on the other hand, q had turned out to be so much less than $\frac{6}{5}$ as to cause $\frac{\mu}{m}$ in any rejected pair to be greater than q, in such case in order to obtain a value of E the least, and of F the greatest, capable of being extracted from the given scheme, it would have been necessary to take account of every such pair and perform the calculations afresh, thereby obtaining a new value of q (say q') less than the former one; we should then have had to continue the process of examining the rejected pairs and reinstating those (if any) whose denominators furnished a ratio $\frac{\mu}{m}$ greater than q', thereby obtaining a still smaller value q''. Repeating these operations *toties quoties* we should at last arrive at a value of q superior to every ratio $\frac{\mu}{m}$ throughout the entire stigmatic series; the corresponding values of the asymptotic limits would then be the best capable of being deduced from the given scheme.

Per contra had we retained at the start any of the discarded pairs of terms, we should have found for q a value greater than the value of $\frac{\mu}{m}$ in some of the terms retained, which would be a sure indication that the retention of those terms had led to a greater value of q than was necessary; those pairs would then have to be omitted; the q calculated from the reformed equations would be diminished by so doing and the resulting values of E, F

* For the complete analytical determination of the limits to $\psi(x)$ see § 3 of this chapter.

By making i sufficiently great u_i, v_i may be brought indefinitely near to E, F: furthermore, when the superior and inferior limits of $\psi(x) \div x$ are expressed as functions of x and i of the form mentioned in the text, these limits may, by taking x sufficiently great, be brought indefinitely near to u_i, v_i, and therefore to E, F, which I therefore speak of throughout as asymptotic limits to $\psi(x) \div x$. But more strictly the optimistic limits actually arrived at are E' as little as we please greater than E, and F' as little as we please less than F.

would be the best attainable, provided that care was taken at the outset that no rejected pair gave a larger value to $\frac{\mu}{m}$ than any pair that had been retained.

In the case we have considered initial asymptotic limits (namely D and D_1) to $\frac{\psi(x)}{x}$ were obtained from the scheme itself, but it will not always be possible to do this when we are dealing with any harmonic scheme.

Thus, for example, from the fact that the minor arrangement of the stigmatic series corresponding to the scheme [1, 6, 10, 210, 231, 1155; 2, 3, 5, 7, 11, 105] has (1) + (13) for its outstanding group [see p. 718], we may deduce that $\psi(x) + \psi\left(\frac{x}{13}\right)$ has Nx for its inferior asymptotic limit, but are unable from this arrangement to obtain an initial inferior asymptotic limit to $\psi(x)$ itself, and still less shall we be able to obtain an initial superior asymptotic limit to $\psi(x)$ from the major arrangement of the same stigmatic series. It is therefore important to notice that the final asymptotic limits arrived at by the method explained in this section, depend only on the stigmatic multiplier and the coefficients of the stigmatic series, being quite independent of the *initial* values employed, so that in the general case we may start from any given asymptotic limits to $\frac{\psi(x)}{x}$, *however obtained*, without thereby producing any effect in the final result. The limits $u_0 = 2\log 2$ and $v_0 = \log 2$ obtained from the scheme [1; 2, 2] will do as well as any others for our initial asymptotic limits to $\frac{\psi(x)}{x}$, and we may, by substituting these limits in the retained portion of the stigmatic series, arrive at new limits u_1, v_1 which in their turn will give rise to fresh limits u_2, v_2 and so on. We shall in this way obtain a pair of difference equations (connecting u_{i+1}, v_{i+1} with u_i, v_i) which will be of the same form as those previously given [p. 713], and it is to be noticed that in the solutions of these equations, namely

$$u_i = C\rho^i + C_1\rho_1{}^i + E, \quad v_i = K\rho^i + K_1\rho_1{}^i + F,$$

only the values of C, C_1, K, K_1 will depend on the initial values of u, v; so that, provided the roots of the quadratic in ρ (which are always real) are each less than unity, we may, by taking i sufficiently great, make u_i and v_i approach as near as we please to E and F respectively; that is as near as we please to two quantities whose values depend solely on the stigmatic series employed.

The positive and negative divergences from unity of the E and F previously found are respectively

$${\cdot}0765779\ldots, \quad {\cdot}0773893\ldots;$$

these divergences as found by Tchebycheff being

$${\cdot}1055504\ldots, \quad {\cdot}0787080\ldots,$$

which is already an important gain; but by varying the scheme we shall obtain still better results.

Let us apply the method of indefinite successive approximation to the scheme in the key of 7 treated of in the preceding section, namely [1, 6, 70; 2, 3, 5, 7, 210], for which the stigmatic multiplier (the D of p. [707]), namely

$$\tfrac{1}{2}\log 2+\tfrac{1}{3}\log 3+\tfrac{1}{5}\log 5+\tfrac{1}{7}\log 7+\tfrac{1}{210}\log 210-\tfrac{1}{6}\log 6-\tfrac{1}{70}\log 70$$

is ·9787955 ....

Preliminary calculations having served to satisfy me that the asymptotic ratio $\frac{E}{F}$ (the q) for this system was not likely to differ much from 1·10, which may be called the *regulator*, I form the corresponding equations for E and F by retaining only those pairs $(m)-(\mu)$ in the stigmatic series for which $\frac{\mu}{m}$ is greater than 1·10.

As previously explained no *error* can result whatever regulator we employ; the worst that can happen will be that the result will not be the best attainable from the scheme, and such imperfection can be ascertained by means of the method previously explained; the result, if the best possible, will prove itself to be so, and, if not the best, will indicate whether the regulator (or criterion of retention) has been taken too small or too great.

Let us examine separately the two arrangements set out in the previous section, the first being employed to obtain by successive approximations the superior, and the second the inferior, limit.

Consider 1° the periodic part of the first arrangement: in the group $(11)+(13)-(14)-(15)$, the pair $(13)-(14)$ being rejected, $(11)-(15)$ remains. Similarly, in the following group $(19)-(20)$ being rejected, $(17)-(21)$ remains; in the third and fourth groups $(23)-(28)$ and $(31)-(35)$ are to be retained. In the following group, all the consecutive pairs from (73) to (98) both inclusive are to be rejected, leaving $(71)-(100)$ available. (The corresponding pair to this in the next period, namely $(281)-(310)$, gives $\frac{310}{281}$, which is less than the assumed regulator.) All the groups in the first period, following $-(100)$, will have to be rejected until we come to the group beginning with (137), which leads to the available pair $(137)-(190)$: in the next period all the ratios will be too small with the exception of $(347)-(400)$ which must be retained, but the term corresponding to this in the third period, namely $(557)-(610)$, will have to be neglected.

Hence, in approximating to the superior limit, we may write

$$u_{i+1}=M+\left(\tfrac{1}{10}+\tfrac{1}{15}+\tfrac{1}{21}+\tfrac{1}{28}+\tfrac{1}{35}+\tfrac{1}{100}+\tfrac{1}{190}+\tfrac{1}{400}\right)u_i$$
$$-\left(\tfrac{1}{11}+\tfrac{1}{17}+\tfrac{1}{23}+\tfrac{1}{31}+\tfrac{1}{71}+\tfrac{1}{137}+\tfrac{1}{347}\right)v_i.$$

2°. In the second arrangement, the first group in the periodic part being $-(14)-(15)+(17)+(19)$, and $\frac{17}{15}$ (and *à fortiori* $\frac{19}{14}$) exceeding the regulator, all these terms are to be preserved.

In addition to these, we shall find in the first period the available couples $-(20)+(73)$ and $-(110)+(139)$, and in the second period $-(230)+(283)$; no other couples will be available, and accordingly, we shall have

$$v_{i+1}=M+\left(\tfrac{1}{10}+\tfrac{1}{14}+\tfrac{1}{15}+\tfrac{1}{20}+\tfrac{1}{110}+\tfrac{1}{230}\right)v_i$$
$$-\left(\tfrac{1}{11}+\tfrac{1}{13}+\tfrac{1}{17}+\tfrac{1}{19}+\tfrac{1}{73}+\tfrac{1}{139}+\tfrac{1}{283}\right)u_i.$$

If then we write a, b for the coefficients of $u_i, -v_i$ in the first, and c, d for the coefficients of $v_i, -u_i$ in the second of the above equations, and make $u_i=U_i+E$, $v_i=V_i+F$, we shall obtain

$$u_i=C\rho^i+C_1\rho_1{}^i+E,$$
$$v_i=K\rho^i+K_1\rho_1{}^i+F,$$

where ρ, ρ_1 are the roots of the equation

$$\begin{vmatrix}\rho-a, & b\\ d, & \rho-c\end{vmatrix}=0,$$

that is
$$\rho^2-(a+c)\rho+(ac-bd)=0,$$

and E, F are subject to the equations

$$(1-a)E+bF=M,$$
$$dE+(1-c)F=M,$$

which give

$$E=\frac{1-b-c}{(1-a)(1-c)-bd}M,\quad F=\frac{1-a-d}{(1-a)(1-c)-bd}M.$$

On performing the calculations, we shall find

$$a=\cdot29633\ldots,\quad b=\cdot24973\ldots,$$
$$c=\cdot30153\ldots,\quad d=\cdot30371\ldots,$$
$$1-b-c=\cdot44873\ldots,\quad 1-a-d=\cdot39995\ldots,$$
$$ac=\cdot08935\ldots,\quad bd=\cdot07584\ldots,$$
$$a+c=\cdot59786\ldots,\quad (1-a)(1-c)-bd=\cdot41563\ldots,$$

ρ, ρ_1 will therefore be the roots of

$$\rho^2-\cdot59786\rho+\cdot01350=0,$$

which are each less than unity.

Also
$$E=1\cdot0567265\ldots,\quad F=\cdot9418543\ldots,$$
$$q=\frac{1-b-c}{1-a-d}=1\cdot12196\ldots.$$

This last number being *greater* than the assumed regulator 1·10, and *less* than any of the retained ratios "$\left[\frac{\mu}{m}\right]$," it follows that no better limits than E, F can be extracted from the scheme [1, 6, 70; 2, 3, 5, 7, 210]; or (as we may phrase it) E, F are the optimistic asymptotic limits to that scheme.

Obviously, there is no reason to suppose that these are the closest asymptotic limits that can be obtained from the infinite choice of schemes at our disposal: on the contrary, there is every reason to suppose that these limits may by schemes in higher and higher keys be brought to coincide as nearly as may be desired to each other and to unity.

We shall presently obtain by aid of a new scheme a better result than the E, F of the preceding investigation. But first it should be observed that instead of forming the difference equations in u, v from the two arrangements, say the major and minor, of one and the same stigmatic series (the former meaning the one used to find the superior and the latter the inferior asymptotic limit), we may take these two arrangements, if we please, from two distinct series corresponding to two different schemes.

I have had calculated, from beginning to end, the value of the coefficient of each term in the stigmatic series of sum-sums corresponding to the first natural period, containing 2310 terms of the scheme (1, 6, 10, 210, 231, 1155; 2, 3, 5, 7, 11, 105), the stigmatic multiplier to which, namely

$$\tfrac{1}{2}\log 2 + \tfrac{1}{3}\log 3 + \tfrac{1}{5}\log 5 + \tfrac{1}{7}\log 7 + \tfrac{1}{11}\log 11 + \tfrac{1}{105}\log 105$$
$$- \tfrac{1}{6}\log 6 - \tfrac{1}{10}\log 10 - \tfrac{1}{210}\log 210 - \tfrac{1}{231}\log 231 - \tfrac{1}{1155}\log 1155,$$

is ·9909532 ... (say N).

This stigmatic series, though too long for printing at full in the restricted space of this Journal, is given later on in a condensed tabular form (see Table A, p. 721). I will proceed to describe its essential features and the use made of it to bring the asymptotic limits closer together. The maximum and minimum sums of its coefficients are 2 and -2: the first terms being $(1) + (13) - (14) - (15)$, the maximum is first reached at the second term; so that the outstanding group in the minor arrangement will be $(1) + (13)$. But the minimum sum, -2, is not reached before the term whose argument is (616). The outstanding group in the major arrangement will therefore contain a very great number of terms, and there might be some trouble in handling the groups, so as to secure the greatest possible advantage. For this reason, I have thought it sufficient for the present to combine the major arrangement of the scheme [1, 6, 70; 2, 3, 5, 7, 210] with the minor one of the scheme [1, 6, 10, 210, 231, 1155; 2, 3, 5, 7, 11, 105].

Maintaining the regulator still at the same value as before, namely 1·10, the major arrangement will remain unaltered from what it was in the preceding case. In the minor arrangement there will be found to exist the

following 17 available pairs, all of which, except the last, belong to the first period (the last one belonging to the second period), namely

(14) – (19), (15) – (17), (21) – (31), (33) – (41), (44) – (53), (63) – (73), (84) – (97), (105) – (241), (110) – (131), (195) – (223), (315) – (481), (525) – (703), (735) – (943), (945) – (1231), (1484) – (1693), (1694) – (2323), (4004) – (4633).

We may accordingly write

$$u_{i+1} = M + a u_i - b v_i,$$
$$v_{i+1} = N + \gamma v_i - \delta u_i,$$

where

$$a = \frac{1}{10} + \frac{1}{15} + \frac{1}{21} + \frac{1}{28} + \frac{1}{35} + \frac{1}{100} + \frac{1}{190} + \frac{1}{400},$$

$$b = \frac{1}{11} + \frac{1}{17} + \frac{1}{23} + \frac{1}{31} + \frac{1}{71} + \frac{1}{137} + \frac{1}{347},$$

$$\gamma = \frac{1}{14} + \frac{1}{15} + \frac{1}{21} + \frac{1}{33} + \frac{1}{44} + \frac{1}{63} + \frac{1}{84} + \frac{1}{105} + \frac{1}{110}$$
$$+ \frac{1}{195} + \frac{1}{315} + \frac{1}{525} + \frac{1}{735} + \frac{1}{945} + \frac{1}{1484} + \frac{1}{1694} + \frac{1}{4004},$$

$$\delta = \frac{1}{13} + \frac{1}{17} + \frac{1}{19} + \frac{1}{31} + \frac{1}{41} + \frac{1}{53} + \frac{1}{73} + \frac{1}{97} + \frac{1}{131} + \frac{1}{223}$$
$$+ \frac{1}{241} + \frac{1}{481} + \frac{1}{703} + \frac{1}{943} + \frac{1}{1231} + \frac{1}{1693} + \frac{1}{2323} + \frac{1}{4633},$$

from which, writing

$$(1-a)E + bF = M,$$
$$\delta E + (1-\gamma)F = N,$$

we shall find

$$u_i = C\rho^i + C_1\rho_1{}^i + E,$$
$$v_i = K\rho^i + K_1\rho_1{}^i + F,$$

where ρ, ρ_1 are the roots of

$$\begin{vmatrix} \rho - a, & b \\ \delta\ , & \rho - \gamma \end{vmatrix} = 0,$$

that is

$$\rho^2 - (a+\gamma)\rho + a\gamma - b\delta = 0.$$

The values of a, b; γ, δ are respectively

·2963346 ..., ·2497346 ...; ·2992774 ..., ·3107808 ...,

from which we see that ρ, ρ_1 being each less than unity the values of u_∞, v_∞ will be E, F, where

$$E = \frac{(1-\gamma)M - bN}{(1-a)(1-\gamma) - b\delta},$$

$$F = \frac{(1-a)N - \delta M}{(1-a)(1-\gamma) - b\delta},$$

and on performing the calculation it will be found that

$$E = 1{\cdot}0551851\ldots,\quad F = {\cdot}9461974.$$

Also $$q = \frac{E}{F} = 1{\cdot}11518\ldots,$$

which being greater than the assumed regulator, but less than any of the retained ratios $\frac{\mu}{m}$, the results thus obtained are *optimistic*, that is no better can be found without having recourse to some other harmonic scheme.

The advance made upon the determination of the asymptotic limits beyond what was known previously is already remarkable. Tchebycheff's asymptotic numbers stood at

$$1{\cdot}1055504\ldots,$$
$${\cdot}9212920\ldots,$$

corresponding to a divergence from unity

·1055504 ... in excess,

and ·0787080 ... in defect;

by the combined effect of scheme variation and successive substitution we have succeeded in reducing these divergences to

·0551851 ... in excess,

and ·0538026 ... in defect;

in which it will be noticed that the divergence for the superior limit is only a little more than half the original one.

The mean of the two limits, it will be seen, is now less than

$$1{\cdot}0007.$$

The annexed table, in which for brevity $\bar{c}$ is written for $-c$, gives in a condensed form the stigmatic series to the scheme [1, 6, 10, 210, 231, 1155; 2, 3, 5, 7, 11, 105].

The coefficients, for all the terms $\psi\left(\frac{x}{m}\right)$ from $m = 1$ to $m = 1155$ (the half modulus), are written down in regular batches of 10. The coefficients for the ensuing terms up to 2309 can be got from these by the formula $c_{1155+t} = c_{1155-t}$, the term following will have the coefficient zero; the rest of the infinite series is then known from the formula $c_{t+2310} = c_t$.

TABLE A.

The coefficients of the first 1155 *terms of the stigmatic series to* [1, 6, 10, 210, 231, 1155; 2, 3, 5, 7, 11, 105]*.

1000000000	0011̅1̅01010	1̅1̅100001̅10	101̅01̅01000
11̅11̅1̅01000	00101̅1̅0010	101̅001̅1001̅	10101̅01̅010
0011̅0001̅10	00000011̅1̅0	10103̅01011̅	01̅10000000
000001̅1000	11̅001̅01011̅	0000001̅010	1002̅001000
00102̅011̅10	00101̅1̅0010	11̅0000001̅0	10101̅1̅11̅10
0000000001̅	1000000001̅	1011̅1̅01010	1̅0100001̅10
11̅001̅01000	11̅001̅01000	0011̅01̅0010	101̅01̅01001̅
10101̅1̅0010	0011̅001̅010	00000012̅00	10103̅01000
01̅10000001̅	100001̅1000	00001̅01011̅	01̅10001̅010
101̅1̅001000	0011̅1̅011̅10	00102̅00010	11̅00011̅01̅0
10101̅1̅0010	00000001̅11̅	100000001̅0	1011̅1̅01011̅
1̅0100001̅10	00001̅01000	11̅101̅01000	000001̅0010
101̅1̅001001̅	10102̅00010	0011̅01̅0010	00000001̅00
10103̅011̅10	01̅10000001̅0	100001̅1001̅	10001̅01011̅
1̅010001̅010	11̅01̅001000	00001̅011̅10	0011̅1̅00010
11̅001̅0101̅0	10101̅2̅1010	0000001̅011̅	10000001̅00
1011̅1̅01000	1̅0100001̅11̅	10001̅01000	01̅101̅01000
01̅1001̅0010	101̅0001001̅	1011̅1̅00010	0011̅1̅00010
000001̅11̅00	10103̅00010	01̅100001̅00	100001̅101̅0
10001̅01012̅	0010001̅010	0001̅001000	01̅101̅011̅10
00001̅00010	11̅01̅00101̅0	10102̅1̅1010	000001̅001̅
1000001̅000	1011̅1̅011̅10	1̅0100001̅00	10001̅01001̅
11̅101̅01000	1̅01001̅0010	11̅1̅0001001̅	10001̅00010
0011̅000010	00001̅011̅00	10103̅1̅1010	01̅10001̅000
100001̅11̅00	10001̅01001̅	0010001̅011̅	1001̅001000
1̅0101̅011̅10	01̅101̅00010	11̅1̅00010 1̅0	1011̅1̅1̅1010
00001̅00011̅	100001̅0000	1011̅1̅00010	1̅0100002̅10
10001̅0101̅0	11̅101̅01001̅	001001̅0010	001̅0001001̅
11̅101̅00010	0001̅000010	0001̅0011̅00	10102̅†.

* This table is to be read off in lines. The first three lines set out in full (omitting the *null* terms) will mean

$$\psi(x)+\psi\left(\frac{x}{13}\right)-\psi\left(\frac{x}{14}\right)-\psi\left(\frac{x}{15}\right)+\psi\left(\frac{x}{17}\right)+\psi\left(\frac{x}{19}\right)-\psi\left(\frac{x}{21}\right)-\psi\left(\frac{x}{22}\right)$$
$$+\psi\left(\frac{x}{23}\right)-\psi\left(\frac{x}{28}\right)+\psi\left(\frac{x}{29}\right)+\psi\left(\frac{x}{31}\right)-\psi\left(\frac{x}{33}\right)-\psi\left(\frac{x}{35}\right)+\psi\left(\frac{x}{37}\right)$$
$$+\psi\left(\frac{x}{41}\right)-\psi\left(\frac{x}{42}\right)+\psi\left(\frac{x}{43}\right)-\psi\left(\frac{x}{44}\right)-\psi\left(\frac{x}{45}\right)+\psi\left(\frac{x}{47}\right)+\psi\left(\frac{x}{53}\right)$$
$$-\psi\left(\frac{x}{55}\right)-\psi\left(\frac{x}{56}\right)+\psi\left(\frac{x}{59}\right)+\psi\left(\frac{x}{61}\right)-\psi\left(\frac{x}{63}\right)-\psi\left(\frac{x}{66}\right)+\psi\left(\frac{x}{67}\right)$$
$$-\psi\left(\frac{x}{70}\right)+\psi\left(\frac{x}{71}\right)+\psi\left(\frac{x}{73}\right)-\psi\left(\frac{x}{75}\right)-\psi\left(\frac{x}{77}\right)+\psi\left(\frac{x}{79}\right)$$
$$+\psi\left(\frac{x}{83}\right)-\psi\left(\frac{x}{85}\right)-\psi\left(\frac{x}{88}\right)+\psi\left(\frac{x}{89}\right)+\psi\left(\frac{x}{97}\right)-\psi\left(\frac{x}{98}\right)-\psi\left(\frac{x}{99}\right)+\psi\left(\frac{x}{101}\right)$$
$$+\psi\left(\frac{x}{103}\right)-3\psi\left(\frac{x}{105}\right)+\psi\left(\frac{x}{107}\right)+\psi\left(\frac{x}{109}\right)-\psi\left(\frac{x}{110}\right)-\psi\left(\frac{x}{112}\right)+\psi\left(\frac{x}{113}\right).$$

† By actual summation it will be found as stated above [p. 718] that the sum reckoned from the beginning of the positive and negative integers in this table always lies between 2 and −2 (both inclusive).

If we confine our attention exclusively to the outstanding group of the Major Arrangement, which extends to the 616th term inclusive, *without taking advantage* of any of the other groups, we shall find, on making $E = 1{\cdot}0551851$, $F = {\cdot}9461974$, and N (the stigmatic multiplier) $= {\cdot}9909532$,

$$\frac{[\psi(x)]}{x} < N + \left(\frac{1}{15}+\frac{1}{22}+\frac{1}{28}+\frac{1}{35}+\frac{1}{45}+\frac{1}{56}+\frac{1}{66}+\frac{1}{77}+\frac{1}{88}+\frac{1}{99} + \frac{1}{105}+\frac{1}{126}+\frac{1}{525}+\frac{1}{616}\right)E$$
$$-\left(\frac{1}{17}+\frac{1}{23}+\frac{1}{29}+\frac{1}{37}+\frac{1}{47}+\frac{1}{59}+\frac{1}{71}+\frac{1}{79}+\frac{1}{89}+\frac{1}{113}+\frac{1}{227}\right)F$$
$$< 1{\cdot}0542390\ \ldots$$ which is inferior in value to E.

This is enough to assure us that a better result than the one last found would be obtained by using the above scheme to furnish the major as well as the minor arrangement, instead of combining it, as we have done, with the scheme [1, 6, 70; 2, 3, 5, 7, 210].

Mr Hammond has been good enough to work out for me in the annexed scholium the *complete* approximation to the limits to $\psi(x)$ given by the original scheme of Tchebycheff [1, 30; 2, 3, 5]: this approximation preserves precisely the same form as that obtained by the crude method, and, although it lies a little out of the track which I had marked out for myself in this paper, will, I think, besides being possibly valuable for future purposes in a more or less remote future, serve as an example to clear up any obscurity that may have pervaded the previous exposition of the purely asymptotic portion of these limits*.

§ 3. *Scholium. Containing an example of the complete ith approximation to the limits to the prime-log-sum-sum to x.*

Using S to denote the stigmatic series

$$\psi(x) - \psi\left(\frac{x}{6}\right) + \psi\left(\frac{x}{7}\right) - \psi\left(\frac{x}{10}\right) + \ldots,$$

we have the inequalities

$$\left.\begin{aligned} S &> Ax - \tfrac{5}{2}\log x - 1 \\ S &< Ax + \tfrac{5}{2}\log x \end{aligned}\right\} \text{(Serret, p. 233)},$$

which, as explained in the preceding section, may be replaced by

$$\psi(x) > Ax - \tfrac{5}{2}\log x - 1 + \psi\left(\frac{x}{24}\right) - \psi\left(\frac{x}{29}\right) \qquad (1),$$

$$\psi(x) < Ax + \tfrac{5}{2}\log x + \psi\left(\frac{x}{6}\right) - \psi\left(\frac{x}{7}\right) + \psi\left(\frac{x}{10}\right) \qquad (2).$$

* In the paragraph [last but one] of p. [709] in the preceding number, a theorem (too simple to require a formal proof) is tacitly assumed which virtually amounts to saying:

If an equal number of black and white beads be strung upon a wire, in such a way that on telling them all, from left to right, more white than black ones are never told off, then the whole number of beads, as they stand, may be sorted into pairs, in each of which a black bead lies to the left of a white one.

If now we assume

$$\psi(x) > p_i Ax + q_i(\log x)^2 + r_i(\log x) + s_i \qquad (3),$$
$$\psi(x) < t_i Ax + u_i(\log x)^2 + v_i(\log x) + w_i \qquad (4),$$

we obtain, by combining these inequalities with (1),

$$\begin{aligned}\psi(x) > Ax &\qquad - \tfrac{5}{2}\log x \qquad - 1 \\ &+ \tfrac{1}{24}p_i Ax + q_i(\log x - \log 24)^2 + r_i(\log x - \log 24) + s_i \\ &- \tfrac{1}{29}t_i Ax - u_i(\log x - \log 29)^2 - v_i(\log x - \log 29) - w_i.\end{aligned}$$

Say $\qquad \psi(x) > p_{i+1}Ax + q_{i+1}(\log x)^2 + r_{i+1}(\log x) + s_{i+1}$,

where

$$\begin{aligned} p_{i+1} &= \tfrac{1}{24}p_i - \tfrac{1}{29}t_i + 1, \\ q_{i+1} &= q_i - u_i, \\ r_{i+1} &= r_i - v_i + 2u_i\log 29 - 2q_i\log 24 - \tfrac{5}{2}, \\ s_{i+i} &= s_i - w_i + q_i(\log 24)^2 - u_i(\log 29)^2 - r_i\log 24 + v_i\log 29 - 1. \end{aligned}$$

Similarly, combining (3) and (4) with (2), we find

$$\begin{aligned}\psi(x) < Ax &\qquad + \tfrac{5}{2}\log x \\ &+ \tfrac{1}{6}t_i Ax + u_i(\log x - \log 6)^2 + v_i(\log x - \log 6) + w_i \\ &- \tfrac{1}{7}p_i Ax - q_i(\log x - \log 7)^2 - r_i(\log x - \log 7) - s_i \\ &+ \tfrac{1}{10}t_i Ax + u_i(\log x - \log 10)^2 + v_i(\log x - \log 10) + w_i.\end{aligned}$$

Say $\qquad \psi(x) < t_{i+1}Ax + u_{i+1}(\log x)^2 + v_{i+1}(\log x) + w_{i+1}$,

where

$$\begin{aligned} t_{i+1} &= \tfrac{4}{15}t_i - \tfrac{1}{7}p_i + 1, \\ u_{i+1} &= 2u_i - q_i, \\ v_{i+1} &= 2v_i - r_i + 2q_i\log 7 - 2u_i\log 60 + \tfrac{5}{2}, \\ w_{i+1} &= 2w_i - s_i - q_i(\log 7)^2 + u_i\{(\log 6)^2 + (\log 10)^2\} + r_i\log 7 - v_i\log 60. \end{aligned}$$

These, together with the four given above, constitute a set of eight difference equations for the determination of $p_i, q_i, r_i, s_i, t_i, u_i, v_i, w_i$. Their initial values are furnished by the inequalities

$$\left.\begin{aligned} \psi(x) &> Ax - \tfrac{5}{2}\log x - 1 \\ \psi(x) &< \tfrac{6}{5}Ax + \frac{5}{4\log 6}(\log x)^2 + \tfrac{5}{4}\log x + 1 \end{aligned}\right\} \text{(Serret, p. 236)},$$

which give $\qquad p_0 = 1,\ q_0 = 0,\ r_0 = -\tfrac{5}{2},\ s_0 = -1,$

$$t_0 = \tfrac{6}{5},\ u_0 = \frac{5}{4\log 6},\ v_0 = \tfrac{5}{4},\ w_0 = 1.$$

The values of p_i, t_i will be found to be

$$p_i = \frac{1}{50999}\left\{51072 - 36\tfrac{1}{2}(\rho^i + \rho_1^i) - 47\tfrac{211}{2320}\left(\frac{\rho^i - \rho_1^i}{\rho - \rho_1}\right)\right\},$$

$$t_i = \frac{1}{50999}\left\{59595 + 801\tfrac{2}{15}(\rho^i + \rho_1^i) + 190\tfrac{2397}{2800}\left(\frac{\rho^i - \rho_1^i}{\rho - \rho_1}\right)\right\},$$

where ρ, ρ_1 are the roots of the equation

$$(\rho - \tfrac{4}{15})(\rho - \tfrac{1}{24}) = \frac{1}{203},$$

and it is easy to verify that these values (which agree with the general ones, involving arbitrary constants, obtained in the preceding section) satisfy the initial conditions

$$p_0 = 1,\ \ p_1 = \tfrac{1}{24}p_0 - \tfrac{1}{29}t_0 + 1 = 1\tfrac{1}{3480},$$
$$t_0 = \tfrac{6}{5},\ \ t_1 = \tfrac{4}{15}t_0 - \tfrac{1}{7}p_0 + 1 = 1\tfrac{31}{175}.$$

The values of q_i and u_i, obtained from the equations

$$q_{i+1} = q_i - u_i, \quad u_{i+1} = 2u_i - q_i,$$

with the initial conditions

$$q_0 = 0,\ \ u_0 = \frac{5}{4\log 6},$$

are

$$q_i = -\frac{5}{4\log 6}\left(\frac{\alpha^i - \alpha^{-i}}{\alpha - \alpha^{-1}}\right),$$
$$u_i = \frac{5}{8\log 6}\left(\alpha^i + \alpha^{-i} + \frac{\alpha^i - \alpha^{-i}}{\alpha - \alpha^{-1}}\right),$$

where α, α^{-1} are the roots of the equation

$$\alpha^2 - 3\alpha + 1 = 0.$$

The values of r_i, s_i, v_i, w_i are linear functions of q_i, u_i whose coefficients are linear functions of i in the case of r_i, v_i and quadratic functions of i in the case of s_i, w_i.

Thus we find, when the constants are properly determined,

$$r_i = -(2\log 6 + \lambda i)\,u_i + \{\kappa - \lambda - 2\log 29 + \log 6 - (\kappa + \lambda)\,i\}\,q_i,$$
$$v_i = \ \ (3\log 6 - \kappa i)\,u_i + (2\log 10 + \lambda - 2\kappa - \lambda i)\,q_i - \tfrac{5}{2},$$

where

$$\kappa = \tfrac{2}{5}\log\left(\frac{24^3 . 60^2}{7 . 29}\right), \quad \lambda = \tfrac{2}{5}\log\left(\frac{24^4 . 60}{7^2 . 29^3}\right).$$

The substitution of these values of r_i and v_i in the equations for determining s_i and w_i, will give a pair of equations of the form

$$s_{i+1} = s_i - w_i \ + (a + bi)\,q_i + (c + di)\,u_i - (1 + \tfrac{5}{2}\log 29),$$
$$w_{i+1} = 2w_i - s_i + (e + fi)\,q_i + (g + hi)\,u_i - \tfrac{5}{2}\log 60,$$

where a, b, c, d, e, f, g, h are known constants, and q_i, u_i are known linear functions of α^i, α^{-i}.

For example, the value of a is

$$(\log 24)^2 - (\kappa - \lambda - 2\log 29 + \log 6)\log 24 + (2\log 10 + \lambda - 2\kappa)\log 29.$$

From these equations we should obtain a result of the form

$$s_i = Q_1\alpha^i + R_1\alpha^{-i} + C_1,$$
$$w_i = Q_2\alpha^i + R_2\alpha^{-i} + C_2,$$

in which C_1, C_2 are constants and Q_1, Q_2, R_1, R_2 quadratic functions of i, but the complete determination of these would occupy too much space to be given here.

Sequel to Part II., Chapter I. § 2.

Since § 2 of this chapter was sent to press I have had asymptotic limits to $\psi(x) \div x$ computed by means of a scheme whose stigmata contain simply and in combination all the prime numbers up to 13 inclusive. The numerical results obtained on the one hand and on the other the process employed to determine *à priori* (so as to save the labour of working out the 30030 terms of a complete period) the minimum and maximum values (-1 and 4) of the sum of the coefficients of any number of consecutive terms (the first included) in the stigmatic series proper to the scheme, appear to me too noteworthy to be consigned to oblivion.

This calculation differs from those that precede it in the circumstance that it does not attempt to give the *optimistic* limits which the scheme will afford, notwithstanding which the limits actually obtained will be found to be each of them materially closer to unity than the optimistic limits furnished by any of the preceding schemes.

The scheme I adopt is [1, 6, 10, 14, 105; 2, 3, 5, 7, 11, 13, 385, 1001], which satisfies the necessary condition that the sums of the reciprocals of the numbers on the two sides of the semicolon are equal to one another.

The first thing to be done is to discover the maximum and minimum values of

$$S_n = E\left(\frac{n}{1}\right) + E\left(\frac{n}{6}\right) + E\left(\frac{n}{10}\right) + E\left(\frac{n}{14}\right) + E\left(\frac{n}{105}\right)$$
$$- E\left(\frac{n}{2}\right) - E\left(\frac{n}{3}\right) - E\left(\frac{n}{5}\right) - E\left(\frac{n}{7}\right) - E\left(\frac{n}{11}\right)$$
$$- E\left(\frac{n}{13}\right) - E\left(\frac{n}{385}\right) - E\left(\frac{n}{1001}\right).$$

On taking n equal to 66, it will be found that the value of S_n is -1: I shall proceed to show that this is the minimum, in other words that $-S_n$ cannot be so great as 2.

Denote the fractional part of any quantity x by $F(x)$: if $-S_n$ is not less than 2, then it may be shown that *à fortiori*

$$F\left(\frac{n}{6}\right) + F\left(\frac{n}{10}\right) + F\left(\frac{n}{14}\right) - F\left(\frac{n}{2}\right) - F\left(\frac{n}{3}\right) - F\left(\frac{n}{5}\right) - F\left(\frac{n}{7}\right) + F\left(\frac{n}{105}\right),$$

say $Q(n) + F\left(\frac{n}{105}\right)$ must not be less than 2, and therefore $Q(n)$ must be greater than 1: now it is not difficult to show that $Q(n)$ is only greater than unity when

$$n = 106 + 210\kappa \quad \text{or} \quad n = 136 + 210\kappa$$

(κ being a positive integer). But corresponding to these two values it will be found that

$$Q(106) + F\left(\frac{106}{105}\right) = \tfrac{1}{3} + \tfrac{2}{5} + \tfrac{3}{7} + \tfrac{1}{105},$$

$$Q(136) + F\left(\frac{136}{105}\right) = \tfrac{1}{3} + \tfrac{2}{5} + \tfrac{2}{7} + \tfrac{31}{105},$$

so that on either supposition $Q(n) + F\left(\frac{n}{105}\right)$ is less than 2.

Hence the minimum value of S_n is -1, and consequently, since the stigmatic excess is here $8 - 5$, the maximum value, as appears from the foot-note below, will be $8 - 5 + 1$, that is 4*. (By the stigmatic excess for any scheme I mean the number of stigmata in the right-hand less the number of those in the left-hand set. This excess is obviously equal to the coefficient, with its sign changed, of $\psi\left(\frac{x}{\mu}\right)$ in the stigmatic series, where μ is any common multiple of the stigmata.)

It will be found, on summing up the numbers in Table B, that S_n first attains the value 4 when $n = 1891$, and the value -1 when $n = 66$.

For the inferior limit the outstanding group consists of all the terms up to 1891 inclusive, and for the superior limit all the terms up to 66 inclusive. But in obtaining this limit advantage has been taken of the next three groups, which end with 78, 418, and 2068 respectively. Thus the extreme limit of the following table is 2068, instead of being 30030 (that is 2.3.5.7.11.13) which is the number of terms in a complete period. It contains the coefficients of the first 2068 terms of the stigmatic series for the scheme [1, 6, 10, 14, 105; 2, 3, 5, 7, 11, 13, 385, 1001] written down in horizontal order in regular batches of ten, as was done in Table A for the

* If we call c_n the coefficient of $\psi\left(\frac{x}{n}\right)$ and S_n the sum of such coefficients up to c_n inclusive (regarding c_0 and S_0 as zero), and take μ the least common multiple of the stigmata, we have, obviously,

$$S_\mu = 0, \quad c_n = c_{\mu-n}, \quad \text{and } (S_n + S_{\mu-1-n}) - (S_{n-1} + S_{\mu-n}) = c_n - c_{\mu-n} = 0.$$

Consequently, $\quad S_n + S_{\mu-1-n} = S_0 + S_{\mu-1} = -c_\mu = \eta$ (the stigmatic excess).

This is a valuable formula of verification, and moreover gives a rule for finding either the maximum or minimum coefficient-sum when the other sum is given; for if S_n has the maximum value, $S_{\mu-1-n} = \eta - S_n$; if this is not the minimum let S_n' be less than $\eta - S_n$, then $S'_{\mu-1-n}$ will be greater than S_n, contrary to hypothesis. Hence the minimum value of a coefficient-sum may be found by subtracting the maximum from the stigmatic excess and *vice versâ*.

(I may perhaps be allowed to add that this theorem suggests a generalization of itself, which I think it is safe to anticipate may be formally deduced from it, namely:

If $a_1, a_2, \ldots, a_n$; $\alpha_1, \alpha_2, \ldots, \alpha_\nu$ be any given positive quantities (integer or fractional, rational or irrational) such that $\Sigma a = \Sigma\alpha$, and if $-m$, M be the least and greatest values that $\Sigma E(ax) - \Sigma E(\alpha x)$ can assume when x is any positive quantity whatever, then $M - m = \nu - n$.)

scheme [1, 6, 10, 210, 231, 1155; 2, 3, 5, 7, 11, 105] with the unimportant difference that (for typographical convenience) negative coefficients are indicated by dots instead of by bars placed over them.

TABLE B.

The coefficients of the first 2068 *terms of the stigmatic series to* [1, 6, 10, 14, 105; 2, 3, 5, 7, 11, 13, 385, 1001].

1000000000	00001̇01010	1̇1̇1001̇0010	101̇01̇0101̇0
1011̇1̇01000	01̇101̇00010	101̇01̇1̇1000	10101̇01̇1̇10
00100001̇10	1̇000001̇01̇0	1011̇1̇01011̇	0010001̇000
000000100̇1̇	1̇1001̇01010	001̇0001̇010	1001̇01̇1000
00102̇01000	00101̇1̇0010	1̇10000001̇0	10102̇01110
00000001̇01	1000000001̇	00101̇01010	2̇011000010
1̇1001̇00000	10001̇01001̇	0011̇000010	102̇01̇01000
10101̇2̇0010	0010001̇000	0000001̇100	1̇1101̇01000
00101̇00001̇	1000001̇100	00001̇01010	1̇1̇10001̇010
101̇1̇001000	0011̇1̇00010	00103̇00011̇	100001̇101̇0
10001̇00010	000001̇01̇11	10000000 2̇0	10101̇01011̇
1̇1̇10000010	00002̇01000	1̇1101̇011̇00	0000000010
001̇1̇001000	1011̇2̇00010	001001̇1010	0000000001̇
10101̇011̇10	000000001̇0	100001̇1001̇	10001̇01000
1̇01̇0001̇010	12̇00001000	00002̇01010	0011̇1̇0011̇0
10001̇01̇01̇0	00101̇1̇1010	0001̇001̇011	1000001̇1̇00
101̇01̇01001̇	1̇010000011̇	101̇01̇01000	00101̇11000
01̇10000000	102̇0001000	1111̇100010	00102̇00010
000001̇1̇100	10101̇00010	1̇010000100	1001̇00101̇0
10001̇00012̇	0010001̇011̇	0000001000	01̇001̇01010
00001̇1̇0010	1001̇00102̇0	10102̇01010	01̇0001̇0011
10001̇01̇000	10101̇012̇10	1̇010000000	00001̇01001̇
1011̇1̇01000	1̇010001̇010	11̇10001001̇	10001̇00010
0001̇000010	00001̇1̇1000	10101̇1̇1000	0010001̇000
1̇10000 1̇100	10002̇01000	0010001̇1̇11̇	1000001000
3̇010101010	01̇11̇1̇00010	101̇000001̇0	1011̇1̇01011̇
00001̇00011	101̇001̇0000	10101̇1̇0010	1̇0100001̇00
10001̇01̇01̇0	11̇101̇01001̇	00101̇00010	001̇0001̇100
11̇101̇00010	1̇000000010	0002̇001000	10103̇00010
001001̇0001̇	1000000000	101̇01̇01110	001001̇1̇000
1000001̇01̇1̇	00101̇01010	1̇1̇101̇00010	11̇001̇01̇01̇0
10001̇011̇10	0001̇000011	00001̇00000	1011̇1̇1̇1010
1̇010002̇010	10001̇011̇01̇	10101̇01̇01̇0	0000000011̇
101̇001̇1000	00101̇00000	01̇10000010	01̇1̇0001000
1011̇2̇01010	00101̇001̇00	100001̇1000	00001̇00010
0011̇001̇1̇10	100000001̇0	00101̇01012̇	00101̇00010
00100010 1̇0	11̇101̇11010	001̇0000001	1001̇000000
11̇102̇01010	1̇01̇01̇1̇0010	10001̇001̇00	10101̇011̇00
1̇010000000	101̇1̇001002̇	10101̇01010	1̇0100000 11̇
01̇00001000	101̇01̇01010	0011̇01̇0000	10001̇01̇01̇0

$1000\dot{1}11010$	$0\dot{1}1000\dot{2}010$	$1000\dot{1}01\dot{1}00$	$0010\dot{1}01\dot{1}00$
$0010\dot{1}0001\dot{1}$	0000001010	$001\dot{1}101010$	$0\dot{1}0000\dot{1}011$
$10\dot{1}000000\dot{1}$	$101\dot{1}101010$	$\dot{1}000\dot{1}00010$	$1000\dot{1}\dot{2}1000$
$1010\dot{1}000\dot{1}0$	$0010000\dot{1}10$	$1\dot{1}\dot{1}0001010$	$1010\dot{2}0001\dot{1}$
$0010000\dot{1}10$	$\dot{1}00001000$	$0\dot{1}10\dot{1}01010$	$000\dot{1}000000$
$100\dot{1}000000$	$1000\dot{2}0101\dot{1}$	$00100\dot{1}1010$	$10\dot{1}0000000$
$0010\dot{1}\dot{1}1\dot{1}10$	$0010\dot{1}000\dot{1}0$	$10000010\dot{1}\dot{1}$	$1\dot{1}10\dot{1}01010$
$\dot{1}000\dot{1}00011$	$1\dot{1}00000\dot{1}00$	$1000\dot{1}01010$	$\dot{2}01\dot{1}000010$
$100\dot{1}\dot{3}01000$	$1010\dot{1}\dot{1}0000$	$001000\dot{1}01\dot{1}$	$10\dot{1}0001\dot{1}00$
$1000\dot{1}00000$	$00100\dot{1}001\dot{1}$	0000001010	0010101010
$\dot{3}010000000$	$10\dot{1}0\dot{1}01000$	$100\dot{1}10\dot{1}110$	$0010\dot{1}0\dot{1}010$
$0000\dot{1}1000$	$001\dot{1}100010$	$0010\dot{1}0\dot{1}1$	

In Tables I and II below, in addition to pairs of numbers $-(\eta)+(\eta+\theta)$, meaning

$$-\psi\left(\frac{x}{\eta}\right)+\psi\left(\frac{x}{\eta+\theta}\right),$$

and $+(\eta)-(\eta+\theta)$ meaning

$$+\psi\left(\frac{x}{\eta}\right)-\psi\left(\frac{x}{\eta+\theta}\right),$$

there will be found the unpaired numbers (15) and (66) in the one and (19), (229) and (1891) in the other; to understand how these are got, it should be observed that S_n (the sum of the first n numbers in Table B) first becomes 0 when $n=15$, first becomes -1 when $n=66$ and first becomes 2, 3, 4 when $n=19$, 229, 1891 respectively*.

TABLE I.

			+(15)
−	(17)+	(22)	
−	(19)+	(21)	
−	(23)+	(26)	
−	(29)+	(35)	
−	(41)+	(45)	
−	(47)+	(52)	
−	(59)+	(65)	+(66)
−	(67)+	(78)	
−	(79)+	(418)	
−	(107)+	(135)	
−	(210)+	(275)	
−	(289)+	(385)	
−	(419)+	(2068)	
−	(521)+	(585)	
−	(629)+	(795)	
−	(839)+	(936)	
−	(1049)+	(1144)	
−	(1717)+	(1925)	

TABLE II.

+	(15)−	(17)−	(19)
+	(21)−	(31)	
+	(26)−	(29)	
+	(33)−	(43)	
+	(44)−	(61)	
+	(63)−	(73)	
+	(65)−	(71)	
+	(75)−	(103)−	(229)
+	(242)−	(271)	
+	(285)−	(323)	
+	(385)−	(421)	
+	(385)−	(439)	
+	(440)−	(493)	
+	(494)−	(571)	
+	(770)−	(841)	
+	(1155)−	(1273)−	(1891)

* Call Σ the sum of the infinite series given by Table B: it may then easily be verified that

$$\{\psi(x)-\Sigma\}-\left\{\psi\left(\frac{x}{15}\right)+\psi\left(\frac{x}{66}\right)\right\}$$

The reasoning employed in dealing with previous schemes serves to show that superior and inferior asymptotic limits to $\psi(x)\div x$, which we shall call E_1, F_1 in order to distinguish them from the corresponding optimistic limits (E, F), may be found from the equations

$$\left.\begin{aligned} E_1 &= M+aE_1-bF_1 \\ F_1 &= M+cF_1-dE_1 \end{aligned}\right\},$$

where a is the sum of the reciprocals of the numbers occurring in Table I with the sign +
b " " " " " −
c " " " in Table II " +
d " " " " " −

and M is the stigmatic multiplier,

namely
$$a=\frac{1}{15}+\frac{1}{21}+\frac{1}{22}+\ldots+\frac{1}{2068}=\cdot 33352\ldots,$$
$$b=\frac{1}{17}+\frac{1}{19}+\frac{1}{23}+\ldots+\frac{1}{1717}=\cdot 30580\ldots,$$
$$c=\frac{1}{15}+\frac{1}{21}+\frac{1}{26}+\ldots+\frac{1}{1155}=\cdot 26966\ldots,$$
$$d=\frac{1}{17}+\frac{1}{19}+\frac{1}{29}+\ldots+\frac{1}{1891}=\cdot 27742\ldots\text{*},$$

may be resolved into term-pairs of the form

$$-\psi\left(\frac{x}{\eta}\right)+\psi\left(\frac{x}{\eta+\theta}\right)$$

that shall contain among them all those in Table I, and

$$\{\psi(x)-\Sigma\}+\left\{\psi\left(\frac{x}{19}\right)+\psi\left(\frac{x}{229}\right)+\psi\left(\frac{x}{1891}\right)\right\}$$

into term-pairs of the form $+\psi\left(\frac{x}{\eta}\right)-\psi\left(\frac{x}{\eta+\theta}\right)$ that shall contain among them all those in Table II above.

The maximum value of S_n is here 4: if it had been 2, then instead of 3 unpaired positive terms appended to $\{\psi(x)-\Sigma\}$ there would have been but 1. This is what happens for the scheme [1, 15; 2, 3, 5, 30] given in the footnote on p. [710]: and accordingly, we see that $\{\psi(x)-\Sigma\}+\psi\left(\frac{x}{17}\right)$, for that scheme, is resoluble into paired terms of the form

$$+\psi\left(\frac{x}{\eta}\right)-\psi\left(\frac{x}{\eta+\theta}\right).$$

So again, the minimum being 0 (instead of −1), there will be but 1 unpaired negative term to append to $\{\psi(x)-\Sigma\}$, and accordingly, we see that $\{\psi(x)-\Sigma\}-\psi\left(\frac{x}{6}\right)$ in that scheme is resoluble into term-pairs of the form $-\psi\left(\frac{x}{\eta}\right)+\psi\left(\frac{x}{\eta+\theta}\right)$.

* The above values of a, b, c, d give $a+c=\cdot 603\ldots$ and $ac-bd=\cdot 005\ldots$, and consequently the roots of the "characteristic" equation $\rho^2-(a+c)\rho+(ac-bd)=0$ satisfy the necessary condition of being each less than unity in absolute value.

and $$M=\frac{1}{2}\log 2+\frac{1}{3}\log 3+\frac{1}{5}\log 5+\frac{1}{7}\log 7$$
$$+\frac{1}{11}\log 11+\frac{1}{13}\log 13+\frac{1}{385}\log 385+\frac{1}{1001}\log 1001$$
$$-\frac{1}{6}\log 6-\frac{1}{10}\log 10-\frac{1}{14}\log 14-\frac{1}{105}\log 105=\cdot 98859\ldots.$$

Hence $$E_1=\frac{(1-c-b)M}{(1-a)(1-c)-bd}=1\cdot 04423\ldots,$$
$$F_1=\frac{(1-a-d)M}{(1-a)(1-c)-db}=\cdot 95695\ldots,$$
(so that the mean of E_1 and F_1 is less than ·0006), and $\frac{E_1}{F_1}=1\cdot 09120\ldots$ *.

Thus then (see footnote to p. [694]) by taking x sufficiently great, the number of primes not exceeding x, multiplied by $\log x$ and divided by x, may always be made to lie between the numbers

$$1\cdot 04423\ldots \quad \text{and} \quad \cdot 95695\ldots,$$

the divergences of which from unity are

·04423 ... and ·04304 ... (as against

Tchebycheff's ·10555 ... and ·07807 ...).

These divergences, there is little doubt, would become even more nearly equal than they are, if anyone should feel inclined to undertake the very laborious task of extracting the *optimistic* values (E, F) from the scheme employed.

In order to understand this necessarily abbreviated sketch of a method more easy to think out and apply than to find language to express, I must not conceal that a careful study of the several schemes given, and of the principles embodied in the calculations relating to them, is a *sine quâ non*. It may somewhat lighten the burden thrown upon the reader, if I add a few words concerning one or two points, perhaps inadequately explained in what precedes.

Let μ be the least common multiple of the stigmata of any given harmonic scheme and S_n the sum of the coefficients of

$$\psi(x),\quad \psi\left(\frac{x}{2}\right),\quad \psi\left(\frac{x}{3}\right),\ldots \psi\left(\frac{x}{n}\right)$$

* In Tables I and II above, the ratio $\frac{\eta+\theta}{\eta}$ is greater than 1·09120 ... for every pair of terms, except $-(1049)+(1144)$ in Table I. In the case of this pair, we have $\frac{1144}{1049}=1\cdot 0905\ldots$, which shows that the exclusion of it from that table would have led to asymptotic limits better (but very slightly so) than those arrived at in the text.

in the corresponding stigmatic series. Then from the formula of [p. 710] combined with the equation which connects the stigmata, it follows that

$$S_\mu=0,\quad S_{n+\mu}=S_n.$$

Hence an infinite number of values of n will give S_n its greatest value; the difference of these values will be of the form $k\mu-\mu'$ where μ' may, and in general will, besides zero have various other values less than μ, thus giving rise to the collections of terms called *groups* (see p. [709]) of which the period of μ terms will be composed. The same will be true when we substitute the word *least* for *greatest*.

If now i be taken *any* number such that S_i has its greatest value it may be shown that the sum of all the terms in the stigmatic series subsequent to the one containing $\psi\left(\frac{x}{i}\right)$ will be *negative* or zero, and similarly when S_i has its least value such sum will be *positive* or zero*; consequently when i is properly determined we can find immediately a superior limit in the one case and an inferior limit in the other, to the sum of the first i terms of the series.

I will conclude this portion of the subject with the remark that from the values of E_1 and F_1 it is easy to infer that if μ is equal to or less than $(\cdot 95695\ldots)k-(1\cdot 04423\ldots)$, and x exceeds a certain ascertainable number whose value depends on k and μ, then between x and kx there will be found more than $\mu\frac{x}{\log x}$ primes†.

* The reason of this is that the sum of all the terms beyond the ith may be separated into partial sums, each containing μ terms, which ultimately vanish. If now

$$\gamma_1(k\mu+i+1)+\gamma_2(k\mu+i+2)+\ldots+\gamma_\mu(k\mu+i+\mu)$$

be one of them, then $\gamma_1+\gamma_2+\ldots+\gamma_t$ will be zero when $t=\mu$, and will have a constant algebraical sign (or else be zero) when $t<\mu$; from which it follows (see footnote p. [722] where, be it observed, a coefficient $+\lambda$ or $-\lambda$ is supposed to be represented by a *sequence* of λ black or λ white beads) that each partial sum may be decomposed into an aggregate of quantities of the form $+(\eta)-(\eta+\theta)$ or $-(\eta)+(\eta+\theta)$ according as the first coefficient in each such sum is positive or negative, and will therefore, if not zero, have the same algebraical sign as that coefficient has, namely $-$ or $+$ according as S_i has its greatest or least value.

† In order that μ may be positive (which ensures the existence of *some* primes between x and kx, when x exceeds a certain limit) it is only necessary to take $k>1\cdot 09120\ldots$ (which differs very little from $\frac{12}{11}$), whereas if we limited ourselves to the results of the oft-quoted memoir of 1850 [see p. 704, above], we could not prove the existence of prime numbers between x and kx, for a given value of x, however great, unless k exceeds $\frac{6}{5}$.

73.

NOTE ON A NINE SCHOOLGIRLS PROBLEM.

[*Messenger of Mathematics*, XXII. (1893), pp. 159, 160.]

THIS is a parallel question to the well-known one of fifteen schoolgirls extended to the supposition of their walking for *one* week, three and three together, so that in any the same *day* no two, and at the end of the *week* no three, taking four walks a day, shall have walked more than once together.

Let us understand by the development of the array

$$\begin{array}{ccc} a, & b, & c, \\ d, & e, & f, \\ g, & h, & k, \end{array}$$

the four arrangements

$$\begin{array}{l} (abc,\ def,\ ghk), \\ (adg,\ beh,\ cfk), \\ (aek,\ bfg,\ cdh), \\ (afh,\ bdk,\ ceg), \end{array}$$

(corresponding, in fact, to the four sets of three lines through the nine inflexions of a cubic).

If we suppose the nine girls to walk out four times a day, the same two never being together more than once in the same day, and that at the week's end each has been associated with every pair of the remaining eight, the above will serve to represent one day's walks. To find the other six, I first form the three following pairs of subsidiary arrays, by circular motion performed successively on the three columns of the primitive array, namely

$$\begin{array}{ccccccc} g, & b, & c, & \quad & d, & b, & c, \\ a, & e, & f, & & g, & e, & f, \\ d, & h, & k, & & a, & h, & k, \\ & & & & & & \\ a, & h, & c, & & a, & e, & c, \\ d, & b, & f, & & d, & h, & f, \\ g, & e, & k, & & g, & b, & k, \\ & & & & & & \\ a, & b, & k, & & a, & b, & f, \\ d, & e, & c, & & d, & e, & k, \\ g, & h, & f, & & g, & h, & c. \end{array}$$

Then making any similarly placed line (I have taken the first) in each of the above six groups circulate in one direction as regards the three on the left, and in the opposite direction as regards the three on the right, we obtain six new arrays: these together with the original one furnish the following table:

$$\begin{array}{ccccccccccc} & & & & a, & b, & c, & & & & \\ & & & & d, & e, & f, & & & & \\ & & & & g, & h, & k, & & & & \\ c, & g, & b, & & & & & & b, & c, & d, \\ a, & e, & f, & & & & & & g, & e, & f, \\ d, & h, & k, & & & & & & a, & h, & k, \\ & & & & & & & & & & \\ c, & a, & h, & & & & & & e, & c, & a, \\ d, & b, & f, & & & & & & d, & h, & f, \\ g, & e, & k, & & & & & & g, & b, & k, \\ & & & & & & & & & & \\ k, & a, & b, & & & & & & b, & f, & a, \\ d, & e, & c, & & & & & & d, & e, & k, \\ g, & h, & f, & & & & & & g, & h, & c. \end{array}$$

When the seven arrays in the above table are developed according to the rule previously given, the triads thus arising will be found to be all distinct or, which is the same thing, will comprise among them the whole of the eighty-four ternary combinations of the nine symbols. We have therefore in this table a solution of the proposed problem.

Of course the general problem, when n is any odd multiple of 3, is to construct sets of $\frac{1}{2}(n-1)$ synthemes, each containing $\frac{1}{3}n$ triads with no element in common, and to distribute the whole number of triads into $(n-2)$ such sets.

This problem I solved very many years ago, but I believe have nowhere published, for the case where n is any power of 3, by a method of compound rhythmical displacement strictly analogous to (but of course more intricate than) the one here exhibited.

74.

ON THE GOLDBACH-EULER THEOREM REGARDING PRIME NUMBERS.

[*Nature*, LV. (1896–7), pp. 196, 197; 269.]

In the published correspondence of Euler there is a note from him to Goldbach, or, the other way, from Goldbach to Euler, in which a very wonderful theorem is stated which has never been proved by Euler or any one else, which I hope I may be able to do by an entirely original method that I have applied with perfect success to the problem of partitions and to the more general problem of denumeration, that is, to determine the number of solutions in positive integers of any number of linear equations with any number of variables. In applying this method I saw that the possibility of its success depended on the theorem named being true in a stricter sense than that used by its authors, of whom Euler verified but without proving the theorem by innumerable examples. As given by him, the theorem is this: *every even number* may be broken up in one or more ways into two primes.

My stricter theorem consists in adding the words "where, if $2n$ is the given number, one of the primes will be greater than $\frac{n}{2}$, and the other less than $\frac{3n}{2}$." This theorem I have verified by innumerable examples. Such primes as these may be called mid-primes, and the other integers between 1 and $2n-1$ extreme primes in regard to the range $1, 2, 3 \ldots, 2n-1$.

I have found that with the exception of the number 10, Euler's theorem is true for the resolution of $2n$ into two *extreme* primes; but this I do not propose to consider at present, my theorem being that every even number $2n$ may be resolved into the sum of two mid-primes of the range

$$(1, 2, 3 \ldots, 2n-1).$$

As, for example

$$\begin{array}{rlll}
4 = 2 + 2 & 6 = 3 + 3 & 8 = 3 + 5 & 10 = 3 + 7 \\
12 = 5 + 7 & 14 = 7 + 7 & 16 = 5 + 11 & \\
18 = 5 + 13 = & 7 + 11 & 20 = 7 + 13 & \\
40 = 11 + 29 = & 17 + 23 & 50 = 13 + 37 = 19 + 31 & \\
100 = 29 + 71 = & 41 + 59 & & \\
200 = 61 + 139 = & 73 + 127 = \&c. & & \\
500 = 127 + 373 = & 193 + 307 = \&c. & & \\
1000 = 257 + 743 = \&c. & & &
\end{array}$$

And so on.

My method of investigation is as follows. I prove that the number of ways of solving the equation $x + y = 2n$, where x and y are two mid-primes to the range $2n-1$, that is twice the number* of ways of breaking up $2n$ into two mid-primes + zero or unity, according as n is a composite or a prime number, is exactly equal to the coefficient of x^{2n} in the series

$$\left(\frac{1}{1-x^p} + \frac{1}{1-x^q} + \ldots + \frac{1}{1-x^l}\right)^2$$

where $p, q, \ldots, l$ are the mid-primes in question. This coefficient, we know *à priori*, is always a positive integer, and therefore if we can show that the coefficient in question is not zero, my theorem is proved, and as a consequence the narrower one of Goldbach and Euler. By means of my general method of expressing any rational algebraical fraction, say ϕx, as a residue, by taking the distinct roots of the denominator, say ρ, and writing the variable equal to $\rho\epsilon^t$, and taking the residue with changed sign of $\Sigma\rho^{-n}\epsilon^{-nt}\phi(\rho\epsilon^t)$, we can find the coefficient of x^n or (if we please to say so) of x^{2n} in the above square, and obtain a superior and an inferior limit to the same in terms of $p, q, \ldots, l$; and if, as I *expect* (or rather, I should say, *hope*) may be the case, these two limits do not include zero between them, the theorems (mine, and therefore *ex abundantia* Euler's) will be apodictically established.

The two limits in question will be algebraic functions of $p, q, \ldots, l$, whereas the *absolute* value of the coefficient included within these limits would require a knowledge of the residues of each of these numbers in respect to every other as a modulus, and of $2n$ in respect of each of them. In a word, the limits will be algebraical, but the quantity limited is an algebraical function of the mid-primes $p, q, r, \ldots, l$.

Postscript. The shortest way of stating my refinement on the Goldbach-Euler theorem is as follows:—"It is always possible to find two primes

* This number may be shown to be of the order $\frac{n}{(\log n)^2}$, and a very fair approximate value of it is $\frac{\mu^2}{n}$ where μ is the number of mid-primes corresponding to the frangible number $2n$.

differing by less than any given number whose sum is equal to twice that number."

Another more instructive and slightly more stringent statement of the new theorem is as follows. Any number n being given, it is possible to find two primes whose sum is $2n$, and whose difference is less than n, $n-1$, $n-2$, $n-3$, according as n divided by 4 leaves the remainders 1, 0, -1, -2 respectively.

Major MacMahon, to whom and to the Council of the Mathematical Society of London I owe my renewed interest in this subject, informs me that in a very old paper in the *Philosophical Magazine* I stated that I was in possession of "a subtle method, which I had communicated to Prof. Cayley," of finding the number of solutions in positive integers of any number of linear equations in any number of variables. This method (never printed) must have been in essence identical with that which within the last month I have discovered and shall, I hope, shortly publish.

I have verified the new law for all the even numbers from 2 to 1000, but will not encumber the pages of *Nature* with the details. The approximate formula hazarded for the number of resolutions of $2n$ into two primes, namely $\frac{\mu^2}{n}$, where μ is the number of mid-primes, does not always come near to the true value. I have reasons for thinking that when n is sufficiently great, $\frac{\mu^2}{2n}$ may possibly be an inferior limit. The generating function

$$\left[\Sigma \frac{1}{1-x^p}\right]^2$$

is subject to a singular correction when the partible number $2n$ is the double of a prime. In this case, since the development to be squared is

$$\mu + x^n + x^{2n} + \ldots + x^p + x^{2p} + \ldots + \&c.,$$

the coefficient of x^{2n} will contain 2μ, arising from the combination of 0 with $2n$, which is foreign to the question, and accordingly the result given by the generating function would be too great by 2μ.

This may be provided against by always rejecting the centre of the mid-range from the number of mid-primes. The formula will then in all cases give twice the number of ways of breaking up $2n$ into two unequal primes. Another method would be to take as the generating function not the square of the sum, but the product of the fractions $\frac{1}{1-x^p}$ (without casting out n when it is a prime), but this method would be inordinately more difficult to work with in computing series involving the roots of unity than the one

chosen, which is in itself a felicitous invention*. Whether the method turns out successful or not, it at the very least gives an analytical expression for the number of ways of conjoining the mid-primes to make up $2n$ without trial, which in itself is a somewhat surprising result. Having lost my preliminary calculations, it may be some little time before I shall be able to say whether the method does or does not contain a proof of the new theorem; but that this can be ascertained, there is no manner of doubt. This is the first serious attempt to deal with Euler's theorem, or to bring the question into line with the general theory of partitions.

It is proper to regard the range 1 to $2n-1$ as consisting of two complementary flank regions, two lateral mid-prime regions, and a region reduced to a single term in the middle, as for example,

1, 2, 3 : 4, 5 : 6 : 7, 8 : 9, 10, 11.

Or, again, 1, 2, 3 : 4, 5, 6 : 7 : 8, 9, 10 : 11, 12, 13.

And the question of $2n$ being resoluble into 2 primes breaks up into three, namely, whether $2n$ can be composed with two flank primes, two lateral mid-primes, or with the number in the central region repeated.

* For the generating function we may take any power greater than 2, instead of the square, and the coefficient of x^{2n} will then be the number of couples making up $2n$ multiplied by $(r^2-r)\mu^{r-1}$, which can be calculated by the same method as for the square, but is more difficult and must give rise to numerous theorems of great interest, arising from the multiform representation of the same quantity.

75.

ON THE NUMBER OF PROPER VULGAR FRACTIONS IN THEIR LOWEST TERMS THAT CAN BE FORMED WITH INTEGERS NOT GREATER THAN A GIVEN NUMBER.

[*Messenger of Mathematics*, XXVII. (1898), pp. 1—5.]

A SLIGHT reflexion will show that the number of such fractions $\left(\frac{1}{1}\text{ counting as one of them}\right)$ with the limit n is the sum of the totients of all the numbers from 1 to n.

Let us use Ej as usual to denote the integer part of j, τEj to denote the totient (or number of numbers not exceeding and prime) to Ej, and JEj to denote the sum of such totients for all numbers from 1 to j. Then we may establish the following exact equation given by the author of this article, but without proof and with some slight inaccuracy, in the *Phil. Mag.* for April, 1883 [p. 102, above]. The equation is

$$JEj + JE(\tfrac{1}{2}j) + JE(\tfrac{1}{3}j) + \text{etc.},$$

or, more shortly,

$$\sum_{1}^{\infty} JE\frac{j}{i} = \tfrac{1}{2}\{(Ej)^2 + (Ej)\}. \qquad (1)$$

The proof is as follows. Remarking that $E(j-1) = Ej - 1$, the right-hand side of equation (1), when j is reduced to $j-1$ obviously suffers a diminution equal to Ej.

On the left-hand side of the equation any term $JE\frac{j}{i}$ remains unaltered, when for j is written $(j-1)$, unless Ej is divisible by i, in which case the term undergoes a diminution JEj. Thus for example $J\frac{100}{11} - J\frac{99}{11} = 0$, but $J\frac{100}{5} - J\frac{99}{5} = J(20)$. And, as in the case supposed, $\frac{Ej}{i}$ is a factor of Ej, the total diminution, when $j-1$ replaces j, will be the sum of the totients

of the factors of Ej, which by a known theorem equals Ej. Hence equation (1) is satisfied for j if it is satisfied for $j-1$, and as it is true when $Ej = 1$ it is true for all values of j, as was to be proved. From equation (1) it follows that JEj is of the order $(Ej)^2$, and making

$$JEj = \tfrac{1}{2}\mu(Ej)^2 + \epsilon j,$$

where ϵj is zero when $j = \infty$, we obtain

$$\mu(1 + \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16} + \ldots) = 1,$$

or

$$\mu = \frac{6}{\pi^2}, \text{ or approximately } Jj = \frac{3j^2}{\pi^2}.$$

In the tables in the *Phil. Mag.* for April and September, 1883*, the value of Jj is computed up to $j = 1{,}000$ and compared with the mean value $\frac{3}{\pi^2}j^2$. From this table it appears that Jj is always intermediate between $\frac{3}{\pi^2}j^2$ and $\frac{3}{\pi^2}(j+1)^2$, and much nearer to their mean, which to an insignificant fraction *près* is the same as $\frac{3}{\pi^2}(j^2+j)$, than it is to either extreme. The first, at least, of these statements ought to be susceptible of proof.

As a matter of philosophical interest as embodying a principle applicable to other cases, I will show how I originally found the value $\frac{3}{\pi^2}j^2$ for the number of proper vulgar fractions in their lowest terms that can be formed by means of the first integers.

It is obvious that the probability of any unknown number being divisible by a prime number i is $\frac{1}{i}$, and of any two numbers, being each so divisible, is $\frac{1}{i^2}$, so that the probability of two unknown numbers being each *not* divisible either by 2, 3, 5, 7, n, or any other prime, will be

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)\left(1 - \frac{1}{7^2}\right)\left(1 - \frac{1}{n^2}\right)\ldots,$$

which we know is equal to the sum of the reciprocal of the squares of the natural numbers, that is, is equal to $\frac{6}{\pi^2}$. Hence the number of fractions in their lowest terms that can be got by combining each of j integers with each of i others, found *roughly* by adding together the probable expectation of any such combination consisting of two relative primes, will be $\frac{6}{\pi^2}j^2$, and the number of *proper* fractions in their lowest terms so capable of being formed will be the half of this or $\frac{3j^2}{\pi^2}$. It appears incidentally from this

[* p. 103, above.]

that the average or mean value of the totient of any number is $\frac{3}{\pi^2}$ into, or rather more than, $\frac{3}{10}$ths of that number.

In like manner, if we define a mid-prime to the number $2n$ to be one which is greater than $\frac{1}{2}n$ and less than $\frac{3}{2}n$, the *range* of numbers amongst which such primes are to be found will, to a unit *près*, be n. Let us call the number of such mid-primes μ. Then the probability of any number and its complement in respect to $2n$ being each of them primes will be $\frac{\mu^2}{n^2}$. If now we seek the number of solutions of the equation in prime numbers $x+y=2n$, which will be an even or an odd number, according as n is a composite number or a prime, we may suppose a row of n white balls and n black balls, each series being marked with all the numbers from 1 to n inclusive. It follows from what has been said that the sum of the expectation of x being inscribed on any one of the white balls being itself a prime, and its complement $2n-x$ upon one of the black balls being so likewise, will be $n \cdot \frac{\mu^2}{n^2}$, that is $\frac{\mu^2}{n}$,* and as the same will be true when x is a figure on a black ball and $2n-x$ on a white, the total value of the expectation of the equation in *primes* $x+y=2n$ being satisfied will be the double of this, or $\frac{2\mu^2}{n}$. I have had tables constructed for determining the number of solutions of this equation (x and y being primes) from $2n=2$ up to $2n=500$.

Call the number of solutions for any value of n, $\theta\frac{\mu^2}{n}$; on taking the average value of θ for all values of $2n$ on the 1st, 2nd, 3rd, 4th, 5th, centuries respectively, it will be found that

$$\begin{aligned}\tfrac{1}{2}\theta &= \cdot 96344\\ &= \cdot 99349\\ &= 1\cdot 00603\\ &= \cdot 98281\\ &= \cdot 99764,\end{aligned}$$

of which the sum is 4·94341 and the average is ·98868, agreeing with wonderful nearness to the rough estimate of the number of solutions being $\frac{2\mu^2}{n}$.

* μ is of the order of, and ultimately in a ratio of equality with, $\frac{n}{\log n}$, in the sense that, however small ϵ be taken, a limit $L\epsilon$ can be found such that for all values of n beyond it, μ will be limited on the two sides by $(1\pm\epsilon)\frac{n}{\log n}$; this follows demonstrably from a known theorem proved within the last few years, and as a consequence we see that the number of solutions in "mid-primes" of the equation $x+y=2n$ will necessarily be of the same order as $\frac{n}{(\log n)^2}$ and *presumably* in a ratio of equality with it in the sense explained above, but this, of course, awaits demonstration.

I ought not, however, to suppress the fact that, from another point of view, this number might be expected to eventuate as $\frac{\mu^2}{n}$ instead of $\frac{2\mu^2}{n}$.

In equation (1) we may write $F(j)$ for the sum of the totients of all the numbers not exceeding j, and it then takes the form

$$\phi j = \tfrac{1}{2}\{Ej + (Ej)^2\} = Fj + F(\tfrac{1}{2}j) + F(\tfrac{1}{3}j) + F(\tfrac{1}{4}j) + \text{etc.},$$

which, by the well-known formula of reversion (see *Phil. Mag.*, December, 1884*), gives

$$Fj = \phi j - \phi(\tfrac{1}{2}j) - \phi(\tfrac{1}{3}j) - \phi(\tfrac{1}{5}j) + \phi(\tfrac{1}{6}j) - \text{etc.}$$

Thus for example the number of terms in a Farey series with 17 as a limit should be equal to

$$\tfrac{1}{2}(17-8-5-3+2-2+1-1-1+1+1-1)$$
$$+\tfrac{1}{2}(289-64-25-9+4-4+1-1-1+1+1-1)$$

that is $\frac{1}{2}(1)+\frac{1}{2}(191)$ or 96, which is right†.

* I do not know whether the annexed important case of reversion has been noticed or not: i being greater than unity, let σ_i denote the sum of the *negative* ith powers of the prime numbers 2, 3, 5, 7, etc., and s_i the *logarithm* of the sum of the negative ith powers of the natural numbers 1, 2, 3, 4, etc. (which, when i is an even integer, is a known quantity), then it is easily shown that

$$s_i = \sigma_i + \tfrac{1}{2}\sigma_{2i} + \tfrac{1}{3}\sigma_{3i} + \tfrac{1}{4}\sigma_{4i} + \tfrac{1}{5}\sigma_{5i} + \text{etc.},$$

and therefore by reversion

$$\sigma_i = s_i - \tfrac{1}{2}s_{2i} - \tfrac{1}{3}s_{3i} - \tfrac{1}{5}s_{5i} + \tfrac{1}{6}s_{6i} - \tfrac{1}{7}s_{7i} + \tfrac{1}{10}s_{10i} + \text{etc.}$$

A very general case for reversion arises when $\phi i = \Sigma \frac{1}{n^r}\phi(n^s \cdot i)$. In this last application of the formula $r=1$, $s=1$; in the case considered in the text relating to Farey series $r=0$, $s=-1$.

† And so in general, since by a well-known theorem

$$Ej - E(\tfrac{1}{2}j) - E(\tfrac{1}{3}j) + E(\tfrac{1}{6}j) + \text{etc.}$$

is always equal to unity, so that

$$(Ej)^2 - 2JEj + 1 = E(\tfrac{1}{2}j)^2 + E(\tfrac{1}{3}j)^2 - E(\tfrac{1}{6}j)^2 + \text{etc.},$$

we have always

$$2JEj - 1 = (Ej)^2 - E(\tfrac{1}{2}j)^2 - E(\tfrac{1}{3}j)^2 + E(\tfrac{1}{6}j)^2 + \text{etc.}$$

a very convenient, and, I believe, new formula for calculating the number of fractions in their lowest terms where neither numerator nor denominator exceeds j.

To this E theorem there exists a pendant which may be called the H theorem, namely let Hx mean the nearest integer (when there is one) to x, but when x is midway between two integers Hx is to denote the first integer above x; let $p, q, r, \ldots$ be the primes not exceeding the integer n, and call

$$H_n = n - \Sigma H\frac{n}{p} + \Sigma H\frac{n}{pq} - \Sigma H\frac{n}{pqr} + \text{etc.};$$

then H_n will be the number of primes greater than n and less than $2n$, so that H_n is always greater than zero; and if $\epsilon(x)$ means unity or zero according as x is a prime or not, we shall always have

$$H_n - H_{n-1} = \epsilon(2n-1) - \epsilon(n).$$

I do not know whether this theorem has been previously noticed. It may be obtained by the Eratosthenes sieve process applied to the progression $n+1, n+2, n+3, \ldots, 2n$, replacing therein every prime number by unity.

If not already known, it may be worth while to register the two following additional theorems concerning $E_1 n$ and $H_1 n$, by which I mean what E_n and H_n become when the even prime 2 does not count among the primes p, q, r, which are less than n, namely

$$E_1 n = E\left(\frac{n}{2}\right) - \Sigma E\,\frac{n}{2p} + \Sigma E\,\frac{n}{2pq} + \text{etc.} = E\left(\frac{\log n}{\log 2}\right),$$

$$H_1 n = H\frac{n}{2} - \Sigma H\,\frac{n}{2p} + \Sigma H\,\frac{n}{2pq} + \text{etc.} = 1.$$

This paper was sent by Professor Sylvester to the editor on Feb. 12th, 1897, with a letter in which he wrote "I could subsequently send you the valuable table referred to in the text, giving the number of solutions of the equation $x + y = 2n$ in prime numbers for all values of n up to 500." In subsequent letters he made several slight additions to the paper. He corrected the proof sheets about the end of the month, and then added the first footnote and the last paragraph of the third note. His death took place on March 15th.

INDEX TO PROFESSOR SYLVESTER'S CONTRIBUTIONS TO "MATHEMATICAL QUESTIONS...FROM THE *EDUCATIONAL TIMES*."

In the following index, prepared by Mr C. C. Scott, of the Library of St John's College, Cambridge, the date is that of the volume of the "Mathematical Questions...from the *Educational Times*," the page is that of the volume, the number is that of the question, and the word 'Solution' indicates that a solution of a question is given by Professor Sylvester. It will be noticed that in some cases a question occurs several times, in different volumes (as indicated by the number of the question being the same).

Vol. VIII. (1868)		
p. xii	Q. 2285	
xiii	2337	
xiv	2391	
xv	2492	
18	2249	(Solution)
36	2371	(Solution)
43	1910	
59	2447	
61	2452	(Solution)
88	1805	
92	1849	
106	2473	(Solution)
Vol. IX. (1868)		
xiv	2531	
72	2411	(Solution)
Vol. X. (1868)		
35	2675	
53	2697	
74	2511	
112	2552	(Solution)
Vol. XI. (1869)		
18	2736	
21	2779	
38	2778	(Solution)
43	2636	
49	2823	
81	2877	(Solution)
Vol. XII. (1869)		
17	2802	
29	2864	
48	2930	
56	2950	
70	2296	
83	2977	
92	3003	
Vol. XIII. (1870)		
21	2975	
42	3041	
50	2845	(Solution)
94	3127	
Vol. XIV. (1871)		
68	3207	
Vol. XV. (1871)		
22	3279	
36	1950	
54	3330	
Vol. XV. (*cont.*)		
p. 57	Q. 3355	
99	3401	
Vol. XVI. (1872)		
—	—	
Vol. XVII. (1872)		
19	3230	(Solution)
33	3590	
Vol. XVIII. (1873)		
—	—	
Vol. XIX. (1873)		
17	3928	
25	3279	(Solution)
38	3979	
46	3876	
87	3959	
Vol. XX. (1874)		
17	4092	
Vol. XXI. (1874)		
57	4231	
58	4320	
111	4320	
Vol. XXII. (1875)		
31	4386	(Solution)
Vol. XXIII. (1875)		
43	4591	
48	4615	(Solution)
59	4637	
71	4660	
Vol. XXIV. (1876)		
40	4693	
57	4751	(Solution)
78	3067	(Solution)
Vol. XXV. (1876)		
27	4869	
68	4922	(Solution)
Vol. XXVI. (1876)		
32	4869	
83	5080	
Vol. XXVII. (1877)		
43	5131	
81	5208	
92	5131	
Vol. XXVIII. (1878)		
24	5327	(Solution)
106		

Vol. XXIX. (1878)		
p. 52	Q. 5563	
93	5563	
Vol. XXX. (1879)		
52	145	
81	573	
Vol. XXXI. (1879)		
36	5793	
87	5493	
Vol. XXXII. (1880)		
98	5521	
Vol. XXXIII. (1880)		
20	5713	
34	5762	
53	4817	(Solution)
63	5763	
79	1628	(Solution)
92	5762	
96	4031	(Solution)
97	5357	
115	1588	(Solution)
Vol. XXXIV. (1881)		
21	6243	(Solution)
34	5901	
34	6094	
46	6339	
55	5624	(Solution)
55	6188	
64	5926	
71	6034	
79	5983	
99	6218	
99	6008	
102	5452	
108	6154	(Solution)
110	5493	
Vol. XXXV. (1881)		
32	6532	
46	6405, 6531	
55	6563	
79	6596	
93	5080	
109	4994	
Vol. XXXVI. (1881)		
24	6469	
39	6373	(Solution)
97	6795, 6826	
Vol. XXXVII. (1882)		
p. 22	Q. 5052	
42	6919	
90	5688	
98	6978	
101	7002	(Solution)
Vol. XXXVIII. (1883)		
37	7073	
Vol. XXXIX. (1883)		
34	4971	
50	7249	
74	7277	(Solution)
85	7219	(Solution)
109	5820	
122	7322	
Vol. XL. (1884)		
21	7143	
25	5850	
32	7189	(Solution)
54	7403	(Solution)
77	7428	
112	7454	
Vol. XLI. (1884)		
21	7382	
53	7567	
58	7508	(Solution)
66	7351	
67	7377	
69	7536	
Vol. XLII. (1885)		
29	6218, 7454	
61	6118	
86	7705	
89	7769	(Solution)
101	7836	(Solution)
Vol. XLIII. (1885)		
21	4118	
38	7805	
51	4569	
53	4481	
85	4139	
93	8042, 8078	
105	7922	
110	4266	
Vol. XLIV. (1886)		
21	8115	

GENERAL INDEX

TO THE PAGES OF THE FOUR VOLUMES

CAMBRIDGE: PRINTED BY JOHN CLAY, M.A. AT THE UNIVERSITY PRESS.

4 vols
40/- net

MATH
PA
SYL
V
C
UNIV