31.

LECTURES ON THE PRINCIPLES OF UNIVERSAL ALGEBRA.

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LECTURE I.

Preliminary Conceptions and Definitions.

Apotheosis of Algebraical Quantity.

A matrix of a quadrate form historically takes its rise in the notion of a linear substitution performed upon a system of variables or carriers; regarded apart from the determinant which it may be and at one time was almost exclusively used to represent, it becomes an empty *schema* of operation, but in conformity with Hegel's principle that the Negative is the course through which thought arrives at another and a fuller positive, only for a moment loses the attribute of quantity to emerge again as quantity, if it be allowed that that term is properly applied to whatever is the subject of functional operation, of a higher and unthought of kind, and so to say, in a glorified shape,—as an organism composed of discrete parts, but having an essential and undivisible unity as a whole of its own. *Naturam expellas furcâ, tamen usque recurret**. The conception of multiple quantity thus rises upon the field of vision.

At first undifferentiated from their content, matrices came to be regarded as susceptible of being multiplied together; the word multiplication, strictly applicable at that stage of evolution to the content alone, getting transferred by a fortunate confusion of language to the schema, and superseding, to some extent, the use of the more appropriate word composition applied to the reiteration of substitution in the Theory of Numbers. Thus there came into view a process of multiplication which the mind, almost at a glance, is able to recognize must be subject to the associative law of ordinary

* *Chassez le naturel, il revient au galop*, a familiar quotation which I thought was from Boileau, but my friend Prof. Rabillon informs me is from a comedy of Destouches (born in 1680, died 1754).

multiplication, although not so to the commutative law; but the full significance of this fact lay hidden until the subject-matter of such operations had dropped its provisional mantle, its aspect as a mere schema, and stood revealed as *bona-fide* multiple quantity subject to all the affections and lending itself to all the operations of ordinary numerical quantity. This revolution was effected by a forcible injection into the subject of the concept of addition, that is, by choosing to regard matrices as susceptible of being added to one another; a notion, as it seems to me, quite foreign to the idea of substitution, the *nidus* in which that of multiple quantity was laid, hatched and reared. This step was, as far as I know, first made by Cayley in his Memoir on Matrices, in the *Phil. Trans.* 1858, wherein he may be said to have laid the foundation-stone of the science of multiple quantity. That memoir indeed (it seems to me) may with truth be affirmed to have ushered in the reign of Algebra the 2nd; just as Algebra the 1st, in its character, not as mere art or mystery, but as a science and philosophy, took its rise in Harriot's *Artis Analyticae Praxis*, published in 1631, ten years after his death, and exactly 250 years before I gave the first course of lectures ever delivered on Multinomial Quantity, in 1881, at the Johns Hopkins University. Much as I owe in the way of fruitful suggestion to Cayley's immortal memoir, the idea of subjecting matrices to the additive process and of their consequent amenability to the laws of functional operation was not taken from it, but occurred to me independently before I had seen the memoir or was acquainted with its contents; and indeed forced itself upon my attention as a means of giving simplicity and generality to my formula for the powers or roots of matrices, published in the *Comptes Rendus* of the Institute for 1882 (Vol. XCIV. pp. 55, 396). My memoir on Tchebycheff's method concerning the totality of prime numbers within certain limits, was the indirect cause of turning my attention to the subject, as (through the systems of difference-equations therein employed to contract Tchebycheff's limits) I was led to the discovery of the properties of the latent roots of matrices, and had made considerable progress in developing the theory of matrices considered as quantities, when on writing to Prof. Cayley upon the subject he referred me to the memoir in question: all this only proves how far the discovery of the quantitative nature of matrices is removed from being artificial or factitious, but, on the contrary, was bound to be evolved, in the fulness of time, as a necessary sequel to previously acquired cognitions.

Already in Quaternions (which, as will presently be seen, are but the simplest order of matrices viewed under a particular aspect) the example had been given of Algebra released from the yoke of the commutative principle of multiplication—an emancipation somewhat akin to Lobatchewsky's of Geometry from Euclid's noted empirical axiom; and later on, the Peirces, father and son (but subsequently to 1858) had prefigured the universalization of Hamilton's theory, and had emitted an opinion to the effect that probably all systems of algebraical symbols subject to the associative law of multiplication would be eventually found to be identical with linear transformations of schemata susceptible of matricular representation.

That such must be the case it would be rash to assert; but it is very difficult to conceive how the contrary can be true, or where to seek, outside of the concept of substitution, for matter affording pabulum to the principle of free consociation of successive actions or operations.

Multiplication of Matrices.

A matrix written in the usual form may be regarded as made up of parallels of latitude and of longitude, so that to every term in one matrix corresponds a term of the same latitude and longitude in any other of the same order.

Every matrix possesses a principal axis, namely, the diagonal drawn from the intersection of the first two parallels to the intersection of the last two of latitude and longitude; and by a symmetrical matrix is always to be understood one in which the principal diagonal is the axis of symmetry. If there were ever occasion to consider a symmetrical matrix in which this coincidence does not exist, it might be called improperly symmetrical. This designation might and probably ought to be extended to matrices symmetrical, not merely in regard to the second visible diagonal, but to all the $(\omega - 1)$ rational diagonals of a matrix of the order ω, a rational diagonal being understood to mean any line straight or broken, drawn through ω elements, of which no two have the same latitude or longitude.

The composition of substitutions directly leads to the following rule for the multiplication of matrices. If $m, n,$ be matrices corresponding to substitutions in which m is the antecedent or passive, and n the consequent or active, their product may be denoted by mn (that is, m multiplied by n), and then any term in the product of the two matrices will be equal to its parallel of latitude taken in the antecedent or passive and multiplied by its parallel of longitude taken in the consequent or active matrix. Cauchy has taught us what is to be understood by the product of one rectangular array or matrix by another of the same length and breadth, and we have only to consider the case of rectangles degenerating each to a single line and column respectively, to understand what is meant by the product of the multiplication of the two parallels spoken of above. It may, however, be sometimes convenient to speak of the *disjunctive product* of two sets of the same number of elements, meaning by this the sum of the products of each element in the

one by the corresponding element in the other. Thus $(\lambda l)\, mn$ denoting the term in mn of latitude λ and longitude l, we have the equation

$$(\lambda l)\, mn = \lambda m \times ln,$$

where, of course, λm means the λth parallel of latitude, and ln the lth parallel of longitude, in m and n respectively. This notation may be extended so as to express the value of any minor determinant of mn; such minor may obviously be denoted by

$$\begin{matrix} \lambda_1 l_1, & \lambda_1 l_2, \ldots \lambda_1 l_i, \\ \lambda_2 l_1, & \lambda_2 l_2, \ldots \lambda_2 l_i, \\ \cdots\cdots & \cdots\cdots\cdots \\ \cdots\cdots & \cdots\cdots\cdots \\ \lambda_i l_1, & \lambda_i l_2, \ldots \lambda_i l_i, \end{matrix}$$

and its value will be the product of the two rectangles (in Cauchy's sense) formed respectively by the $\lambda_1, \lambda_2, \ldots \lambda_i$ parallels of latitude in m, and the $l_1, l_2, \ldots l_i$ parallels of longitude in n.

Any other definition of multiplication of matrices, such as the rule for multiplying lines by lines, or columns by columns, sins against good method, as being incompatible with the law of consociation, and ought to be inexorably banished from the text-books of the future. It is almost unnecessary to add that by a pth power of a matrix m is to be understood the result of multiplying p m's together; and by the qth root of m, a matrix which multiplied by itself q times produces m: hence we can attach a clear idea to any positive integral or fractional power. The complete extension of the ordinary theory of surds to multinomial quantity will appear a little further on. But it is well at this point to draw attention to the fact that at all events, if M, M' are positive integer powers of the same matrix m, the factors M, M' are convertible, that is, $MM' = M'M$, this commutative law being an immediate consequence (too obvious to insist upon) of the associative law of multiplication.

On Zero and Nullity.

The absolute zero for matrices of any order is the matrix all of whose elements are zero. It possesses so far as regards multiplication (and as will presently be evident as regards addition also) the distinguishing property of the ordinary zero, namely, that when entering into composition with any other matrix, either actively or passively, the product of such composition is itself over again; so that it may be said to absorb into itself any foreign matrix (of its own order) with which it is combined. This is the highest degree of nullity which any matrix can possess, and (regarded as an integer) will be called ω, the order of the matrix. On the other hand, if the matrix has finite content, its nullity will be regarded as zero. Between these two limits the nullity may have any integer value; thus, if its content, that is, its determinant, vanishes without any other special relation existing between its elements, the nullity will be called 1; if all the first minors vanish, 2; and, in general and more precisely, if all the minors of order $\omega - i + 1$ vanish, but the minors of order $\omega - i$ do not all vanish, the nullity will be said to be i: as an example, if the elements are not all zero, but every minor of the second order vanishes, the nullity is $\omega - 1$.

In general, a substitution impressed on a set of variables may be reversed, and the problem of reversal is perfectly determinate; but when the matrix—the schema of the substitution—is affected with any degree of nullity, such reversal becomes indeterminate. Hence the use of the word indeterminate employed by Cayley to characterize matrices affected with any degree of nullity, in which he has been followed by Clifford, who goes a step further in distinguishing the several degrees of indeterminateness from one another.

On Addition and Monomial Multiplication of Matrices.

The sum of two matrices of like order is the matrix of which each element is the sum of the elements of the same latitude and longitude as its own in the component matrices; thus, as stated by anticipation in what precedes, the addition of a zero matrix to any matrix of like order leaves the latter entirely unchanged.

Addition of matrices obviously will be subject to the same two associative and commutative laws as the addition of monomial quantities. This seems to me a sufficient ground for declining to accept *associative* as the distinguishing name of the algebra of multinomial quantity; for the emphasis thereby laid on association would seem to imply the entire absence of the commutative principle from the theory, whereas, although not having a place in multinomial multiplication, it flourishes in full vigour in the not less important, and, so to say, collateral process of multinomial addition. If k is any positive integer, the addition of the same matrix taken k times obviously leads to a matrix of which each element is k times the corresponding element of the given one; and if p times one matrix is q times another, the elements of the first are obviously $\frac{q}{p}$ into the corresponding ones of the other: hence, if k is any positive monomial quantity, k times a given matrix, by a legitimate use of language, should and will be taken to mean the matrix obtained by multiplying each element in the given one by k. And as the negative of a given matrix ought to mean the matrix which added to the given one should produce the zero-matrix previously defined, the meaning of multiplying a matrix by k may be extended, with the certainty of leading to no contradiction, to the case of any commensurable value of k positive or negative, and consequently, by the usual and

valid course of inference, to the case of k being any monomial symbol whatever, whether possessing arithmetical content or not.

On the Multinomial Unit and Scalar Matrix.

On subjecting a matrix of any order ω to a resolution similar to that by which one of the second order may be resolved into a scalar and a vector, it will be shown hereafter that the ω^2 components separate into a group of $\omega^2 - 1$ terms analogous to the vector and to a single term analogous to the scalar of a quaternion. This outstanding single term is of an invariable form, namely, its principal diagonal consists of elements having the same value, which may be called its parameter, and all the other elements are zeros.

A matrix of such form I shall call a scalar. When the parameter is unity it may be termed a multinomial unity and denoted by $\overset{\omega}{\Upsilon}$*, or in place of ω we may write ω dots over Υ, or for greater simplicity when desirable write simply Υ. Any scalar, by virtue of what precedes, is a mere monomial multiplier of some such Υ.

Let $k\Upsilon$ be any scalar of order ω. It will readily be seen, by applying the laws of multiplication and addition previously laid down, that

$$\phi(k\Upsilon) = \phi(k).\Upsilon, \text{ and that } k\Upsilon.m = m.k\Upsilon = km.$$

Thus a scalar possesses all the essential properties of a monomial quantity, and a multinomial unity of ordinary unity; in particular, the faculty of being absorbed in any other coordinate matrix with which it comes in contact. A scalar whose parameter vanishes of course becomes a zero-matrix.

The properties stated of a scalar $k\Upsilon$ serve to show that in all operations into which it enters the Υ may be dropped, and supplied or understood to be supplied at the end of the operations when needed to give homogeneity to an expression. Thus, for example,

$$(m + h\Upsilon)(m + k\Upsilon) = m^2 + (h + k)\Upsilon m + hk\Upsilon^2 = m^2 + (h + k)m + hk\Upsilon;$$

but this result may be obtained by the multiplication of $(m + h)(m + k)$, and supplying Υ (or imagining it to be supplied) to the final term in order to preserve the homogeneity of the form. In like manner, 0_ω or 0 with ω points over it may be used to denote the absolute zero of the order ω; but it will be more convenient to use the ordinary 0, having only recourse to the additional notation when thought necessary or desirable in order to make obvious the homogeneity of the terms in any equation or expression. Thus, for example, such an expression as $m^2 + 2bm + d = 0$, where m is a matrix, say of the 2nd

* Perhaps more advantageously by 1_ω. I shall hold myself at liberty in what follows to use whichever of these two notations may appear most convenient in any case as it arises.

order, and b and d monomials, set out in full would read $m^2 + 2bm + d\ddot{\Upsilon} = \ddot{0}$, meaning $m . m + 2\,bm + \begin{matrix} d & 0 \\ 0 & d \end{matrix} = \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}$.

On the Inverse and Negative Powers of a Matrix.

The inverse of a matrix, denoted by m^{-1}, means the matrix which multiplied by m on either side produces multinomial unity. It is a matter of demonstration that when a matrix is non-vacuous (that is, has a finite content or determinant appertaining to it), an inverse to it fulfilling this *double* condition can always be found, and that if the product of mn is unity, so also must be that of nm.

It is a well-known fact, proved in the ordinary theory of determinants, that if every element in the first of two matrices is the logarithmic differential derivative, in respect to its correspondent in the second, of the content of that second, so conversely, every element of the second is the logarithmic derivative, in respect to its correspondent in the first, of the content of the first.

But two such matrices multiplied together in either sense would not give for their product multinomial unity; to obtain this product either matrix must be multiplied indifferently into or by the *transverse* of the other (meaning by the transverse of a matrix, the new matrix obtained by rotating the original one through 180° about its principal diagonal). In other words, if m be a given matrix and n be obtained from it by substituting for each element the logarithmic derivatives of its content in respect to its opposite, then $mn = \overset{\omega}{\Upsilon}$ and $nm = \overset{\omega}{\Upsilon}$, where ω means (as will always be the case throughout these lectures) the order of the matrices concerned. The n which satisfies these two equations (and it cannot satisfy the one without satisfying the other) will be called the inverse of m and be denoted by m^{-1}.

For brevity and suggestiveness it will be advantageous to write in general 1 for $\overset{\omega}{\Upsilon}$ as we write 0 for 0_ω, so that $mn = 1$ will imply $nm = 1 = mn$ and $n = m^{-1}$.

We may define in general (as in monomial algebra) m^{-i} to mean the inverse of m^i, that is, $(m^i)^{-1}$. We shall then have $(m^{-1})^i = m^{-i}$, for $mn . mn = 1$ implies $m . mn . n = mn = 1$ or $m^2n^2 = 1$. Hence $n^2 = m^{-2}$, that is, $(m^{-1})^2 = m^{-2}$. Also since $m^2n^2 = 1$, $m^3n^3 = mn = 1$ or $n^3 = m^{-3}$, that is, $(m^{-1})^3 = m^{-3}$, and so in general for all positive integer values of i, $(m^{-1})^i = m^{-i}$. And, as in monomial algebra, it may now be proved and taken as proved that, for all real values of i and j, whether positive or negative, $m^i . m^j = m^{i+j}$, and the same relation may be assumed to continue when i, j become general quantities. The elements in the inverse to any matrix m all involving the reciprocal of the

determinant to m, if D be the content of m we may write $m^{-1} = \frac{1}{D}\mu$, where μ is a matrix all of whose elements are always finite. Hence we come to the important conclusion that for vacuous matrices inverses only exist in idea and are incapable of being realized so as to have an actual existence. In the sequel it will be shown that the inverse is only a single instance of an infinite class of matrices which exist ideally as functions of actual matrices, but are incapable of realization.

Suppose now that M, N are any two matrices such that $MN = 0$ or that $NM = 0$; multiplying each side of the equation by M^{-1} if such expression has an actual existence (that is, if M is non-vacuous), we obtain, from the known properties of zero, $N = 0$, but if M is vacuous no such conclusion can be drawn. So further if $m^i = 0$ (i being any positive integer), it will be seen under the third law of motion that m is necessarily vacuous. Hence from this equation it cannot be inferred that any lower power than the ith of m is necessarily zero.

On the Latent Roots and Different Degrees of Vacuity of Matrices.

If m be any matrix, the augmented matrix $m - \lambda\Upsilon$ or $m - \lambda . 1_\omega$ or $m - \lambda$ will be found simply by subtracting λ from each element in the principal diagonal of m. The content of this matrix or the same multiplied by -1 or any other constant, I term the latent function to m, which will be an algebraical function of the degree ω in λ (which may be termed the latent variable or carrier); and the ω roots of this function (that is, the ω values of the carrier which annihilate the latent function) I call the latent roots of the unaugmented matrix m. It is obvious from this definition that if λ_1 be any latent root of m, the content of $m - \lambda_1$ will vanish, that is, $m - \lambda_1$ will be vacuous, and conversely that if $m - \lambda_1$ is vacuous, λ_1 must be one of the latent roots to m. Thus if m is vacuous, one of the latent roots must be zero; if only one of them is zero I call m simply vacuous and say that its vacuity is 1: thus zero vacuity and simple vacuity mean the same thing as zero nullity and simple nullity respectively. More generally if any number i, but not $i+1$, of the latent roots of m are all of them zero, m will be said to have the vacuity i.

By a principal minor determinant to any matrix I mean any minor determinant whose matrix is divided by the principal diagonal into two triangles. It will then easily be seen that if s_i means in general the sum of the principal ith minors to m, and s_0 means the complete determinant, the assertion of m having the vacuity i is exactly coextensive with the assertion that

$$s_0 = 0, \quad s_1 = 0, \quad s_2 = 0, \ldots s_{i-1} = 0.$$

If the nullity of m is i, *every* qth minor of m is zero when $q < i$. Hence the vacuity cannot fall short of the nullity, but the converse is not true.

A matrix may not have any vacuity up to ω inclusive without the nullity being greater than 1. It will hereafter be shown, under the 2nd law of motion, that if $\lambda_1, \lambda_2, \ldots \lambda_\omega$ are the ω latent roots of m, then

$$(m-\lambda_1)(m-\lambda_2)\ldots(m-\lambda_\omega)=0 \text{ or say } M=0.$$

But it will be interesting even at this early stage to show that a theorem closely approaching this may be deduced from the distinction drawn between vacuous and non-vacuous matrices as regards their possession of real inverses.

I propose to prove instantaneously by this means that at all events $M^{\omega-1}=0$. It is obvious from any single instance of multiplication that mn and nm are not in general coincident. But if n could be expressed as a linear function of powers of m (including m^0 or 1_ω among such powers), mn and nm must be coincident. If now we take the ω^2 matrices

$$1, m, m^2, \ldots m^{\omega^2-1},$$

n at first blush one would say ought to be expressible as a linear function of these ω^2 quantities determinable by means of the solution of ω^2 linear equations, and can only escape being so expressible in consequence of the fact that these ω^2 powers of m are linearly related. Hence we must have an identical equation of the form

$$Am^{\omega^2-1}+Bm^{\omega^2-2}+Cm^{\omega^2-3}\ldots+Gm+H=0_\omega \text{ or say } Fm=0.$$

If now Fm were supposed to contain any factor other than

$$m-\lambda_1,\ m-\lambda_2, \ldots m-\lambda_\omega,$$

such factors being non-vacuous may be expelled from Fm; consequently the equation in question must be of the form

$$(m-\lambda_1)^{\alpha_1}(m-\lambda_2)^{\alpha_2}(m-\lambda_\omega)^{\alpha_\omega}=0,$$

and as the coefficients of the equation in m are necessarily rational we must have $\alpha_1=\alpha_2, \ldots, \alpha_\omega=\alpha$. Hence $\omega\alpha=\alpha_1+\alpha_2+\ldots\alpha_\omega<\omega^2$, and consequently $\alpha<\omega$.

Hence, at all events (since $M^{\omega-1-\theta}=0$ on multiplication by M^θ gives $M^{\omega-1}=0$),

$$\{(m-\lambda_1)(m-\lambda_2)\ldots(m-\lambda_\omega)\}^{\omega-1}=M^{\omega-1}=0. \quad \text{Q.E.D.}$$

LECTURE II.

On Reduction.

It follows from what has been already shown in Lecture I, when m is a matrix of the second order ($\omega-1$ being here unity) that $(m-\lambda_1)(m-\lambda_2)=0$.

Understanding by m the matrix $\begin{matrix} t_1, & \tau_1, \\ t_2, & \tau_2, \end{matrix}$ the latent equation to m is

$$\begin{vmatrix} t_1-\lambda, & \tau_1 \\ t_2 \quad , & \tau_2-\lambda \end{vmatrix}=0,$$

that is, $$\lambda^2-(t_1+\tau_2)\lambda+(t_1\tau_2-t_2\tau_1)=0,$$

so that $$m^2-(t_1+\tau_2)m+(t_1\tau_2-t_2\tau_1)=0,$$

or, using the literation applied to the parametric triangle,

$$m^2-2bm+d=0; \qquad (1)$$

for since the content of $x+ym+zn$ is supposed to be

$$x^2+2bxy+2cxz+dy^2+2eyz+fz^2,$$

that of $-\lambda+m$ will be found by making $z=0$, $x=-\lambda$, $y=1$. The variation of equation (1) obtained by taking ϵn for the increment of m (remembering that the variation of m^2 is $(m+\epsilon n)(m+\epsilon n)-m^2$, that is, $\epsilon(mn+nm)$) gives rise to the identical equation

$$mn+nm-2bn-2cm+2e=0, \qquad (2)$$

and the variation of this again gives

$$n^2+n^2-2cn-2cn+2f=0,$$

or $n^2-2cn+f=0$, as of course will be obtained immediately from (1) by substituting n, c, f in place of m, b, d.

The parameters c, f, if n represents $\begin{matrix} u_1 & v_1 \\ u_2 & v_2 \end{matrix}$ are the sum of the principal diagonal elements and the content of u, just as b, d are such sum and content in respect to m.

The parameter e (the connective to d and f) or rather its double $2e$ is obviously the emanant of d in respect to the operator

$$u_1\delta_{t_1}+u_2\delta_{t_2}+v_1\delta_{\tau_1}+v_2\delta_{\tau_2},$$

or, if we please, of f in respect to the inverse operator

$$t_1\delta_{u_1}+t_2\delta_{u_2}+\tau_1\delta_{v_1}+\tau_2\delta_{v_2},$$

that is, $$t_1v_2+u_1\tau_2-t_2v_1-u_2\tau_1.$$

With the aid of the *catena* of equations in m, in m and n, and in n, any combination of functions of m and n may be reduced to the standard form

$$Amn+Bm+Cn+D.$$

For, in the first place,

$$\phi m=P(m^2-2bm+d)+rm+s=rm+s,$$

and similarly $$\psi n=\rho n+\sigma.$$

Hence the most general combination referred to is expressible as the product of alternating linear functions of m and n, and may therefore be reduced to a sum of terms of which each is a product of alternate powers of m and of n, each of which powers may again be reduced to the form of linear functions, and this process admits of being continually repeated.

Suppose then, at any stage of it, that the greatest number of occurrences of linear functions of m and n in the aggregate of terms is i; then at the next stage of the process the new aggregate will consist of monomial multipliers of one or more *simple* successions of m and n, and of terms in which the number of alternating linear functions never exceeds $i-1$; hence, eventually we must arrive at a stage when the aggregate will be reduced to a sum of monomial multipliers of simple successions of m and n, every such succession being of the form

$$(mn)^q \text{ or } m^{-1}(mn)^q \text{ or } (mn)^q n^{-1} \text{ or } m^{-1}(mn)^q n^{-1}.$$

But
$$\begin{aligned}(mn)^2 &= m\,.\,nm\,.\,n = -m(mn-2bn-2cm+2e)\,n\\ &= -m^2n^2+2bmn^2+2cm^2n-2emn\\ &= -(2bm-d)(2cn-f)+2bm(2cn-f)+2c(2bm-d)\,n-2emn\\ &= -(2e-4bc)\,mn-df.\end{aligned}$$

Hence
$$(mn)^2+2(e-2bc)\,mn+df=0.$$

Hence $(mn)^q = P\{(mn)^2+2(e-2bc)\,mn+df\}+Amn+B = Amn+B$,
where A and B are known functions of $(e-2bc)$ and f; and therefore

$$m^{-1}(mn)^q = An+Bm^{-1} = An-\frac{B}{d}\,m+\frac{2Bb}{d}.$$

Similarly
$$(mn)^q n^{-1} = Am-\frac{B}{f}\,n+\frac{2Bc}{f},$$

and
$$m^{-1}(mn)^q n^{-1} = A+B(mn)^{-1} = -\frac{B}{df}\,mn+\left(A-B\,\frac{2e-4bc}{df}\right).$$

And this being true (*mutatis mutandis*) for all values q, it follows that the function expressed by any succession of products of functions of m and n is reducible to the form of a linear expression in m, n, mn, in which the 4 monomial coefficients are known or determinable functions of the parameters to the corpus m, n.

The latent function to any such linear expression, say

$$Amn+Bm+Cn+D,$$

may be found in the same way as the latent function to mn has been found, namely, as follows:

$$\begin{aligned}(Amn+Bm+Cn+D)^2 &= A^2(mn)^2+AB(mnm+mmn)+AC(mnn+nmn)\\ &\quad+2ADmn+B^2m^2+BC(mn+nm)+C^2n^2+2BDm+2CDn+D^2\\ &= A^2(-2e+4bc)\,mn-A^2df+ABm(2bn+2cm-2e)\\ &\quad+AC(2bn+2cm-2e)\,n+2ADmn+B^2m^2+BC(2bn+2cm-2e)+C^2n^2\\ &\quad+2BDm+2CDn+D^2.\end{aligned}$$

Let
$$(Amn+Bm+Cn+D)^2-2P(Amn+Bm+Cn+D)+Q=0$$
be the identical equation to $Amn+Bm+Cn+D$.

The coefficient of mn in the development of the first term being

$$(4bc-2e)\,A^2+2bAB+2cAC+2AD,$$

and m^2, n^2 being reducible to linear functions of m, n respectively, it follows that

$$P = A(2bc-e)+Bb+Cc+D.$$

To find Q it is only needful to fasten the attention upon the constant terms in the before named development reduced to the standard form. These will be

$$-A^2df-2ABcd-2ACbf-B^2d-2BCe-C^2f+D^2, \text{ say } K,$$

and the constant part in $-2P(Amn+Bm+Cn+D)$ being $-2DP$, it follows that

$$\begin{aligned}Q &= 2AD(2bc-e)+2BDb+2CDc+D^2-K\\ &= A^2df+2ABcd+2ACbf+2AD(2bc-e)\\ &\quad+B^2d+2BCe+C^2f+2BDb+2CDc,\end{aligned}$$

and consequently the latent function $\Lambda^2-2P\Lambda+Q$, of which the algebraical roots are the latent roots of $Amn+Bm+Cn+D$, is completely determined. Thus, for example, if the latent function of $m+n$ is required, making $A=D=0$, $B=C=1$, its value will be seen to be $\Lambda^2-2(b+c)\Lambda+d+2e+f=0$, so that the roots will be $b+c\pm\sqrt{\{(b+c)^2-(d+2e+f)\}}$.

On Involution.

In general, if m and n be two given binary matrices, and p any third matrix, say

$$m = \begin{matrix} t_1 & t_2 \\ t_3 & t_4 \end{matrix}, \quad n = \begin{matrix} \tau_1 & \tau_2 \\ \tau_3 & \tau_4 \end{matrix}, \quad p = \begin{matrix} T_1 & T_2 \\ T_3 & T_4 \end{matrix},$$

p may be expressed as a linear function of $\ddot{\Upsilon}$, m, n, mn or of $\ddot{\Upsilon}$, m, n, nm. For in order that p may be expressible under the form $A+Bm+Cn+Dnm$, observing that

$$nm = \begin{matrix} t_1\tau_1+t_3\tau_2 & t_2\tau_1+t_4\tau_2 \\ t_1\tau_3+t_3\tau_4 & t_2\tau_3+t_4\tau_4 \end{matrix},$$

and that $\ddot{\Upsilon} = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$, it is only necessary to write

$$\begin{aligned}A+Bt_1+C\tau_1+D(t_1\tau_1+t_3\tau_2) &= T_1,\\ Bt_2+C\tau_2+D(t_2\tau_1+t_4\tau_2) &= T_2,\\ Bt_3+C\tau_3+D(t_1\tau_3+t_3\tau_4) &= T_3,\\ D+Bt_4+C\tau_4+D(t_2\tau_3+t_4\tau_4) &= T_4,\end{aligned}$$

and then A, B, C, D may be found by the solution of these four linear equations: and this solution must always be capable of being effected unless the determinant

$$\begin{vmatrix} 1, & t_1, & \tau_1, & t_1\tau_1+t_3\tau_2 \\ 0, & t_2, & \tau_2, & t_2\tau_1+t_4\tau_2 \\ 0, & t_3, & \tau_3, & t_1\tau_3+t_3\tau_4 \\ 1, & t_4, & \tau_4, & t_2\tau_3+t_4\tau_4 \end{vmatrix}$$

vanishes.

When this is the case the matrices m, n, in the order in which they are written, will be said to be in sinistral involution. In like manner, if $1, n, m, mn$ are linearly related, m, n may be said to be in dextral involution. But it is very easy to see from the identical equation (2) that in this case these two involutions are really identical, for, since $A + Bm + Cn + Dmn = 0$, by subtraction

$$A + Bm + Cn - Dnm + 2Dcm + 2Dbn - 2De = 0,$$

that is, $$(A - 2eD) + (B + 2cD)m + (C + 2bD)n - Dnm = 0.$$

The above determinant then will be called the involutant to m, n or n, m, indifferently, for it will be seen, and indeed may be shown, *à priori*, that its value remains absolutely unaltered (not merely to a numerical factor *près*, but in sign and in arithmetical magnitude as well) when the Latin and Greek letters, or which is the same thing, when the matrices m and n are interchanged.

On the Linearform or Summatory Representation of Matrices, and the Multiplication Table to which it gives rise.

This method by which a matrix is robbed as it were of its areal dimensions and represented as a linear sum, first came under my notice incidentally in a communication made some time in the course of the last two years to the Mathematical Society of the Johns Hopkins University, by Mr C. S. Peirce, who, I presume, had been long familiar with its use. Each element of a matrix in this method is regarded as composed of an ordinary quantity and a symbol denoting its place, just as 1883 may be read

$$1\theta + 8h + 8t + 3u,$$

where θ, h, t, u, mean thousands, hundreds, tens, units, or rather, the places occupied by thousands, hundreds, tens, units, respectively.

Take as an example matrices of the second order, as

$$\begin{matrix} \alpha & \beta \\ \gamma & \delta \end{matrix} \qquad \begin{matrix} a & b \\ c & d. \end{matrix}$$

These may be denoted respectively by

$$\alpha\lambda + \beta\mu + \gamma\nu + \delta\pi, \quad a\lambda + b\mu + c\nu + d\pi;$$

their product by

$$(a\alpha + c\beta)\lambda + (b\alpha + d\beta)\mu + (a\gamma + c\delta)\nu + (b\gamma + d\delta)\pi,$$

which therefore must be capable of being made identical with

$$\begin{aligned} & a\alpha\lambda^2 + a\beta\lambda\mu + a\gamma\lambda\nu + a\delta\lambda\pi \\ + & b\alpha\mu\lambda + b\beta\mu^2 + b\gamma\mu\nu + b\delta\mu\pi \\ + & c\alpha\nu\lambda + c\beta\nu\mu + c\gamma\nu^2 + c\delta\nu\pi \\ + & d\alpha\pi\lambda + d\beta\pi\mu + d\gamma\pi\nu + d\delta\pi^2, \end{aligned}$$

when a proper system of relations is established between the quadric combinations and the simple powers of λ.

The arguments of like coefficients in the two sums being equated together, there result the equations

$$\lambda^2 = \lambda, \quad \lambda\nu = \nu, \quad \mu\lambda = \mu, \quad \mu\nu = \pi,$$
$$\nu\mu = \lambda, \quad \nu\pi = \nu, \quad \pi\mu = \mu, \quad \pi^2 = \pi,$$

and again, the arguments to the 8 coefficients in the second sum which are not included among the coefficients of the first, being equated to zero, there result the equations

$$\lambda\mu = 0, \quad \lambda\pi = 0, \quad \mu^2 = 0, \quad \mu\pi = 0,$$
$$\nu\lambda = 0, \quad \nu^2 = 0, \quad \pi\lambda = 0, \quad \pi\nu = 0.$$

These 16 equalities may be brought under a single *coup d'œil* by the following multiplication table:

	λ	ν	μ	π
λ	λ	ν	0	0
ν	0	0	λ	ν
μ	μ	π	0	0
π	0	0	μ	π

In like manner it will be found that any matrix of the 3rd order as $\begin{matrix} a & b & c \\ d & e & f, \\ g & h & k \end{matrix}$

regarded as a quantity, may be expressed linearformly by the sum

$$a\lambda + b\mu + c\nu + d\pi + e\rho + f\sigma + g\tau + h\upsilon + k\phi,$$

where the topical symbols are subject to the multiplication table below written:

	λ	π	τ	μ	ρ	υ	ν	σ	ϕ
λ	λ	π	τ	0	0	0	0	0	0
π	0	0	0	λ	π	τ	0	0	0
τ	0	0	0	0	0	0	λ	π	τ
μ	μ	ρ	υ	0	0	0	0	0	0
ρ	0	0	0	μ	ρ	υ	0	0	0
υ	0	0	0	0	0	0	μ	ρ	υ
ν	ν	σ	ϕ	0	0	0	0	0	0
σ	0	0	0	ν	σ	ϕ	0	0	0
ϕ	0	0	0	0	0	0	ν	σ	ϕ

And, in like manner, matrices of any order ω may be expressed linearformly as the sum of ω^2 terms, each consisting of a monomial multiplier of a topical symbol, the entire ω^2 symbols being subject to a multiplication table containing ω^4 places, of which ω^3 will be occupied by the ω^2 simple symbols, each appearing ω times, and the remaining $\omega^4 - \omega^3$ places by the ordinary zero.

This conception applied to quadratic matrices might have served to establish the connection between them and Hamilton's quaternions, regarded as homogeneous functions of 1, i, j, k, themselves linear functions of the topical symbols λ, μ, ν, π; but the same result may be arrived at somewhat more simply by a method given in a *subsequent* lecture.

On the Corpus *formed by two Independent Matrices of the same order, and the Simple* Parameters *of such Corpus.*

By the latent function of a *corpus* (m, n) we may understand the content or any numerical multiplier of the content of (that is, the determinant to) the matrix $x + ym + zn$, where x, y, z are monomial carriers. This function will be a quantic of the order ω in x, y, z, and in the standard form the coefficient of x^ω may be supposed to be unity, so that it will contain $\frac{1}{2}(\omega^2 + 3\omega)$ coefficients, which may be termed the parameters of the corpus.

To fix the ideas, suppose $\omega = 3$ and let the latent function to

$$\begin{matrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'', \end{matrix} \qquad \begin{matrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{matrix}$$

be called F, where

$$F = x^3 + 3bx^2y + 3cx^2z + 3dxy^2 + 6exyz + 3fxz^2 + gy^3 + 3hy^2z + 3kyz^2 + lz^3.$$

Let m become $m + \epsilon n$, where ϵ is a monomial infinitesimal. Then the function to the corpus becomes the content of

$$x + y(m + \epsilon n) + zn, \text{ that is, } x + ym + (z + \epsilon y)n,$$

and consequently the variation of the function to (m, n) is $\epsilon y \delta_z F$. If then the *rate* of variation of any of the parameters, when n is the rate of variation of m, be denoted by prefixing to such parameter the symbol E, we shall find

$$Eb = c; \quad Ed = 2e; \quad Ee = f; \quad Eg = 3h; \quad Eh = 2k; \quad Ek = l;$$

and similarly, if $\exists$, preceding a parameter, be used to indicate its rate of variation corresponding to n's rate of variation being m, then

$$\exists c = b; \quad \exists f = 2e; \quad \exists e = d; \quad \exists l = 3k; \quad \exists k = 2h; \quad \exists h = g;$$

and the variations of c, f, l, as regards E, and of b, d, g, as regards $\exists$, are of course zero.

By forming the triangle of parameters

$$\begin{matrix} 1 \\ b \; c \\ d \; e \; f \\ g \; h \; k \; l \\ p \; q \; r \; s \; t \end{matrix}$$

the law of variations of the parameters of the function to (m, n) (expressed in the ordinary manner by a ternary quantic affected with the proper numerical multipliers) becomes evident, whatever may be the order of the corpus (that is, of the matrices m and n, of which it is constituted): thus, for example, when $\omega = 4$, in addition to the previous expressions we shall find

$$Ep = 4q, \quad Eq = 3r, \quad Er = 2s, \quad Es = t, \quad Et = 0,$$

$$\exists t = 4s, \quad \exists s = 3r, \quad \exists r = 2q, \quad \exists q = p, \quad \exists p = 0.$$

By means of the above relations, any identical equation, into which enters one or more matrices, admits of being varied, so as to give rise to an identical equation connecting one additional number of the same.

Scholium.—In what precedes it will have been observed that the matter under consideration has always regard to matrices, or, as we may say, quantities of a fixed order ω, combined exclusively with one another and with ordinary monomial quantities. Every such combination forms as it were a *clausum* or world of its own, lying completely outside and having no relations with any other. It is, however, possible, and even probable, that as the theory is further evolved, this barrier may be found to give way and the worlds of all the various orders of quantity be brought into relation and intercommunion with one another.

LECTURE III.

On Quantity of the Second Order.

The theory of matrices of the second order seems to me to deserve a special preliminary investigation on various grounds. First, as affording a facile and natural introduction to the general theory (as the study of Conic Sections is usually made to precede that of universal Geometry); secondly, because it presents certain very special features distinguishing it from all other kinds of quantity, such as the coincidence of the two involutants (reminding one of the single image in the case of ordinary refraction as contrasted with the double image seen through iceland spar), or, again, the rational relation between the products of matrices of the second order, in whatever order the factors are introduced in the performance of the multiplication; and thirdly, because the theory of this kind of quantity has already been extensively studied and developed under the name or aspect of Quaternions. Hence it may not be out of place to make the remark that, as it surely would not be logical to seek for the origin of the conception included in the symbol $\sqrt{(-1)}$ in geometrical considerations, however important its application to geometrical exegesis, so now that an independent algebraical foundation has been discovered for the introduction and use of the symbols employed in Hamilton's theory, it would (it seems to me) be exceedingly illogical and contrary to good method to build the pure theory of the same upon space conceptions; the more so, as it will hereafter be shown that quantities of every order admit of being represented in a mode strictly analogous to that in which quantity of the second order is represented by quaternions, namely, if the order is ω, by ω^2-ions, or as I shall in future say, by *Ions*, of which the geometrical interpretation, although there is little doubt that it exists, is not yet discovered, and it must, it is certain, draw upon the resources of inconceivable space before it can be effected.

32.

ON THE SOLUTION OF A CLASS OF EQUATIONS IN QUATERNIONS.

[*Philosophical Magazine*, XVII. (1884), pp. 392—397.]

THE general equation of the degree ω in Quaternions or Binary Matrices is obviously ω^4, but in certain cases some of these roots evaporate and go off to infinity. The only equation considered by Sir William Hamilton in his Lectures is the Quadratic Equation of a form which I call unilateral, because the quaternion coefficients in it are supposed all to lie on the same side of the unknown quantity. I propose here to show how Hamilton's equation, and indeed a unilateral one of any order, may be solved by a general algebraical method and the number of its roots determined.

It will be convenient to begin by setting out certain general equations relating to any two binary matrices m, n.

Writing the determinant of $x + ym + zn$ under the form

$$x^2 + 2bxy + 2cxz + dy^2 + 2eyz + fz^2$$

(b, c, d, e, f, thus constituting what I call the parameters of the *corpus* m, n), we have universally

$$m^2 - 2bm + d = 0, \quad n^2 - 2cn + f = 0, \quad d(m^{-1}n)^2 - 2e(m^{-1}n) + f = 0.$$

Moreover if m, n receive the scalar increments μ, ν; d, e, f become respectively

$$d - 2\mu b + \mu^2, \quad e - \mu c - \nu b + \mu\nu, \quad f - 2\nu c + \nu^2.$$

Let us begin with Hamilton's form, say

$$x^2 - 2px + q = 0,$$

and suppose

$$x^2 - 2Bx + D = 0,$$

where B, D are scalars to be determined.

Let b, c, d, e, f be the five known parameters of the *corpus* p, q. Then, since

$$(p - B)^{-1}(q - D) = 2x,$$

we shall have [cf. p. 188 above]

$$4(d-2bB+B^2)x^2-4(e-bD-cB+BD)x+f-2cD+D^2=0.$$

Hence, writing $\quad B-b=u,\quad D-c=v,$

$$d-b^2=\alpha,\quad e-bc=\beta,\quad f-c^2=\gamma,$$

we have $\quad u^2+\alpha=\lambda,\quad uv+\beta=2\lambda(u+b),\quad v^2+\gamma=4\lambda(v+c).$

From the last two equations, eliminating v, there results

$$(2\lambda u-2b\lambda-\beta)^2-4\lambda(2\lambda u-2b\lambda-\beta)u+(\gamma-4c\lambda)u^2=0.$$

Hence substituting $\lambda-\alpha$ for u^2,

$$(4\lambda^2+4c\lambda-\gamma)(\lambda-\alpha)-(2b\lambda-\beta)^2=0.$$

We have thus six values of u, namely

$$\pm\sqrt{(\lambda-\alpha)}$$

(where λ has three values), to which correspond six values of v, namely

$$2\lambda\pm\frac{2\lambda b-\beta}{\sqrt{(\lambda-\alpha)}};$$

and, finally, $\quad 2x=(p-u-b)^{-1}(q-v+c)$

$$=\{(p-b)^2-u^2\}^{-1}(p-b+u)(q-c-v),$$

or

$$x=\frac{pq-(c+v)p-(b-u)q+(b-u)(c+v)}{2(b^2-d-u^2)};$$

which equation gives six values for x, and shows that ten have evaporated.

It is easy to account *à priori* for the solution depending only upon a cubic in u^2.

For $x^2-2px+q=0$ is the same as $y^2-2yp+q=0$, where $y=-x+2p$. But obviously, from the nature of the process for determining them, B and C are independent of the *side* of the unknown on which the first coefficient lies. Hence the actual B will be associated with B', B' being what B becomes when x becomes $-x+2p$, which is obviously $-B+2b$.

Hence with any value of $B-b$, which is u, is associated a corresponding $B-b$, which is $-u$.

I will now proceed to apply a similar or the same method to the trinomial cubic equation in quaternions (or binary quantity) $x^3+px-q=0$, with a view to ascertain the number of its roots.

Retaining the same notation as before, and still supposing

$$x^2-2Bx+D=0,$$

we obtain $\quad x^3+(D-4B^2)x+2BD=0,$

and

$$x=\frac{:q+2BD}{p+4B^2-D}\,^*.$$

Hence $\quad \{(4B^2-D)^2-2b(4B^2-D)+d\}x^2$

$$-2\{2(4B^2-D)BD-c(4B^2-D)-2bBD+e\}x$$

$$+4B^2D^2-2cBD+f=0.$$

Hence we may write

$$(4B^2-D)^2-2b(4B^2-D)+d=\lambda,$$

$$2(4B^2-D)BD-c(4B^2-D)-2bBD+e=\lambda B,$$

$$4B^2D^2-2cBD+f=\lambda D;$$

from which equations B and D are to be determined. Eliminating λ between the first and second and between the first and third of these equations, we obtain two equations, of which the arguments are

$$D^3;\quad B^2D^2,\ D^2;\quad B^4D,\ B^2D,\ BD,\ D;\quad 1$$

for the one,

$$BD^2;\quad B^3D,\ BD,\ D;\quad B^5,\ B^3,\ B^2,\ B;\quad 1$$

for the other.

Eliminating D by the Dialytic method between these two equations, we shall have (using points to signify unexpressed coefficients) the following three linear equations in D^2, D, 1, namely:

$$\cdot BD^2+(\cdot B^3+\&c.)D+(\cdot B^5+\&c.)=0,$$

$$\cdot B^3D^2+(\cdot B^5+\&c.)D+(\cdot B^7+\&c.)=0,$$

$$\cdot B^5D^2+(\cdot B^7+\&c.)D+(\cdot B^9+\&c.)=0.$$

Hence in the final equation B rises to the 15th power; and by combining any two of the above equations, D is given linearly in terms of B; and, finally, x is known from the equation

$$x=\frac{:(p+D-4B^2-2b)(q+2BD)}{-(4B^2-D)^2-2(4B^2-D)+d},$$

and has 15 values.

A like process may be extended to a unilateral equation (of the Jerrardian form) of any degree, say $x^m+qx+r=0$.

Introducing the auxiliary equation with scalar coefficients as before, namely

$$x^2-2Bx+D=0,$$

x may be expressed as a function of q, r, B, D; and the term containing the

* I use $\frac{:L}{M}$ and $\frac{L:}{M}$ to signify $M^{-1}L$ and LM^{-1} respectively.

highest power of B in the equation for determining B (of which D is a one-valued function), when $\omega = 4$, will be found to be the determinant

$$\begin{matrix} \cdot B & \cdot B^3 & \cdot B^5 & \cdot B^7 \\ \cdot B^3 & \cdot B^5 & \cdot B^7 & \cdot B^9 \\ \cdot B^5 & \cdot B^7 & \cdot B^9 & \cdot B^{11} \\ \cdot B^7 & \cdot B^9 & \cdot B^{11} & \cdot B^{13}* \end{matrix}$$

and a similar determinant will fix the degree of B in the resolving equation for any value of ω. Hence the number of solutions of the unilateral equation in quaternions of the Jerrardian form of the degree ω is $\omega(2\omega - 1)$ or $2\omega^2 - \omega$, and the evaporation will accordingly be $\omega^4 - 2\omega^2 + \omega$, or

$$(\omega^2 - \omega)(\omega^2 + \omega - 1).$$

Moreover the same method with a slight addition will serve to determine the roots of the general unilateral equation in quaternions, the number of which will be a cubic function of ω, as I propose to show and to give its precise value in some future communication, either in this Journal, or at all events in the memoir on Universal Algebra now in the course of publication, under the form of lectures, in the *American Journal of Mathematics*†.

I very much question whether the old method of Hamilton, as taught by its most consummate masters, Tait in this country, or the late Prof. Benjamin Peirce in America, would be found sufficiently plastic to deal effectually with an analytical investigation in quaternions of this degree of complexity, so as to lead to the formula for the number of solutions of the unilateral equation of the Jerrardian form above given.

I invite my much esteemed and most capable former colleague and former pupil, Dr Story, of the Johns Hopkins, and Prof. Stringham, of the University of California, who carry on the traditions of the Harvard School, to put the power of the old method as compared with the new to this practical test.

Postscript.—If $$x^3 - 3px^2 + 3qx - r = 0,$$

(where p, q, r are perfectly general matrices of the second order which satisfy the general equations

$$q^2 - 2bq + d = 0, \quad qr + rq - 2bq - 2b_1q + 2\epsilon = 0, \quad r^2 - 2b_1q + d_1 = 0,$$
$$pq + qp - 2bp - 2\beta q + 2e = 0, \quad p^2 - 2\beta p + \delta = 0,$$
$$pr + rp - 2b_1p - 2\beta r + 2e_1 = 0),$$

* It may readily be seen that the highest term in the equation for finding B is identical with the resultant of

$$D^4 - 24B^2D^3 + 80B^4D^2 \text{ and } 4BD^3 - 40B^3D^2 + 64B^5D - 64B^7,$$

that is, will be $2^{18}.3.7.19B^{28}$; and that the last term (at all events to the sign *près*) will be $b^4\delta^3$, which is of $4.3 + 2.2.4$ (that is of 28) dimensions in x, and is therefore codimensional (as it ought to be) with B^{28}.

† It is given in the Postscript below.

and if we write $$x^2 - 2Bx + D = 0,$$

then $$px = \frac{:r + 3Dp - BD}{3q - 3Bp + B^2 - D};$$

and I find by perfectly easy and straightforward work that B, D may be determined by means of the following equations:

$$\frac{(B^2 - D)^2}{9} + 2(b - \beta B)\frac{B^2 - D}{3} + (d - 2eB + 4\delta B^2) = 9\lambda,$$
$$\frac{B^3D - BD^2}{3} + (b_1 + 3\beta D)\frac{B^2 - D}{3} + (\epsilon - e_1B + 3eD - 6\delta BD) = 3B\lambda,$$
$$B^2D^2 - 2(b_1 + 3\beta D)BD + d_1 + 6De_1 + 9\delta D^2 = D\lambda.$$

The order (by which I mean the number of solutions of this system of equations) is readily seen to be the same as that of

$$\cdot D^3 + \cdot B^2D + \cdot B^4D = 0$$
$$\cdot BD^2 + \cdot B^3D + \cdot B^5 = 0;$$

that is, is the same as the degree in B of $B^3(B^3)^2 \cdot R$, where R is the resultant of

$$\cdot D^2 + \cdot B^3 + \cdot B^4 \text{ and } \cdot D^2 + \cdot B^2D + \cdot B^4.$$

Hence* the number of solutions is $3 + 10 + 8$, that is, is 21.

Practically, therefore, we have now sufficient data to determine the number of solutions of a unilateral equation in quaternions of any order ω; for it is morally certain that such number is a rational function of ω; and as it cannot but be of a lower order than ω^4, we have only to determine a cubic function of ω whose values for $\omega = 0, 1, 2, 3$ are $0, 1, 6, 21$, which is easily found to be $\omega^3 - \omega^2 + \omega$; so that the evaporation is $\omega^4 - \omega^3 + \omega^2 - \omega$, that is

$$(\omega^2 + 1)(\omega^2 - \omega).$$

Practically also we can solve (subject to hardly needful verification) the number of roots of a unilateral equation of the special form

$$x^\omega + q_\theta x^\theta + q_{\theta-1}x^{\theta-1} + \ldots + q_0 = 0.$$

For when $\theta = \omega$, we know the number is ω^2; and when $\theta = 1$, the number is $\omega^3 + \omega^2 - \omega$; consequently if the second differences of the function of (ω, θ) which expresses the number of roots are constant, the value of this function when $\theta = \omega - 1$ is $\omega^3 - \omega^2 + \omega$, which we have found to be the actual number; and consequently, if the second differences are not constant, they must be sometimes positive and sometimes negative, which is in the highest degree improbable. Hence in all probability it will be found that the required number of solutions in the form supposed is $(1 + \theta)\omega^2 - \theta\omega$.

I need hardly add that the nine quantities $2b, 2b_1, 2\beta$; $2e, 2e_1, 2\epsilon$; d, δ, d_1, which occur in the discussion above given of the general unilateral cubic, or, say, rather the ten quantities obtained by adding on to these *unity*, are the

[* See footnote † p. 197 above.]

ten coefficients of the determinant to the binary matrix $(x+py+qz+rt)$, which of course there is not the slightest difficulty in expressing in terms of scalar and vector affections of p, q, r and their combinations, if any one chooses to regard them as given in quaternion form.

Scholium. In what precedes it is very requisite to notice that only *general* cases are considered; and that there are multitudinous others which escape the direct application of this method, and do not conform to the rule which assigns the number of solutions. Thus, for example, the equation $x^2+px=0$, besides the solutions $x=0$, $x=-p$, will have two others which will require the method of the text to be modified in order to determine. Or take the most elementary case of all, the simple equation $px=q$. If p is not vacuous (that is, if its determinant when regarded as a matrix, or its modulus when regarded as a quaternion, is finite), there is the one solution $x=p^{-1}q$. But if p is vacuous, then, unless q is also vacuous, the equation is insoluble. If $q=0$, there will be two solutions; one of them $x=0$, the other $x=$ conjugate of p in quaternion terminology; or

$$x=\begin{matrix}-d; & b\\ c; & -a\end{matrix},\ \text{when}\ p=\begin{matrix}a; & b\\ c; & d\end{matrix}$$

in the language of matrices. If, p still remaining vacuous, q is vacuous but not zero, a further condition must be satisfied, namely, if

$$p=\begin{matrix}a; & b\\ c; & d\end{matrix}\ \text{and}\ q=\begin{matrix}\alpha; & \beta\\ \gamma; & \delta\end{matrix},$$

the condition is $\qquad a\delta+\alpha d-b\gamma-c\beta=0;$

or if $\qquad p=a+bi+cj+dk$ and $q=\alpha+\beta i+\gamma j+\delta k,$

the condition is $\qquad a\alpha+b\beta+c\gamma+d\delta=0.$

When this condition (besides that of q being vacuous) is satisfied, the equation $px=q$ is soluble, and $p^{-1}q$ becomes finite but indeterminate, containing two arbitrary constants*.

* So in general if p, q be two simply vacuous matrices of any order, the condition that the equation $px=q$ may be soluble, or, in other words, that $p^{-1}q$ (a combination of an ideal with a vacuous matrix) may be non-ideal, may be shown to be that the determinant to the matrix $\lambda p+\mu q$ (where λ, μ are scalar quantities) shall vanish identically—which (p being supposed already to be vacuous) involves just as many additional conditions as there are units in the order of the matrix.

33.

ON HAMILTON'S QUADRATIC EQUATION AND THE GENERAL UNILATERAL EQUATION IN MATRICES.

[*Philosophical Magazine*, XVIII. (1884), pp. 454—458.]

In the *Philosophical Magazine* of May last I gave a purely algebraical method of solving Hamilton's equation in Quaternions, but did not carry out the calculations to the full extent that I have since found is desirable. The completed solution presents some such very beautiful features, that I think no apology will be required for occupying a short space of the *Magazine* with a succinct account of it.

Hamilton was led to this equation as a means of calculating a continued fraction in quaternions, and there is every reason for believing that the Gaussian theory of Quadratic Forms in the theory of numbers may be extended to quaternions or binary matrices, in which case the properties of the equation with which I am about to deal will form an essential part of such extended theory*. Let us take a form slightly more general than that before considered, namely, the form

$$px^2+qx+r=0,$$

with the understanding that the determinant of p (if we are dealing with matrices), or its tensor if with quaternions, differs from zero. Let us construct the ternary quadratic

$$au^2+2buv+2cuw+dv^2+2evw+fw^2,$$

defined as the determinant of $up+vq+wr$, on the one supposition, or by means of the equations

$$a=Tp^2,\quad d=Tq^2,\quad f=Tr^2,\quad b=SpSq-SVpVq,$$
$$c=SpSr-SVpVr,\quad e=SqSr-SVqVr,$$

on the other supposition.

* I have found, and stated, I believe, in the form of a question in the *Educational Times* some years ago, that any fraction whose terms are real integer quaternions may be expressed as a finite continued fraction, the greatest-common-measure process being applicable to its two terms, provided *both* their Moduli are not odd multiples of an odd power of 2, which can always be guarded against by a previous preparation of the fraction.

On referring to the article of May [p. 226 above], it will be seen that the solution of the equation may be made to depend on the roots of a cubic equation in the quantity therein called λ. When fully worked out, this equation will be found to take the remarkable form $e^{\lambda\Omega}. I = 0$, where I is the invariant of the ternary quadratic above written, and $\Omega = 2a\delta_c - a\delta_d$. It may also be shown that

$$x = -\frac{(p+b-u)(q-c-v)}{2\lambda},$$

where u is a two-valued function of λ, and v a linear function of u.

I shall suppose that I, the final term in the equation in λ, differs from zero: the solution of the given equation in x will then be what may be termed *regular*, and will consist of three pairs of actual and determinate roots. When $I = 0$, the solution ceases to be regular; some of the roots may disappear from the sphere of actuality, or may remain actual but become indeterminate, or these two states of things may coexist. The first coefficient of the equation in λ is a, the determinant of p (or its squared tensor), which also must not be zero, as in that case one root at least of λ would be infinite. Let us suppose, then, that neither a nor I vanishes. The very interesting question presents itself as to what kind of equalities can arise among the *three* pairs of roots, and what are the conditions of such arising.

This equation admits of an extremely interesting and succinct answer as follows:—Let m represent $\frac{c+2d}{3}$; the equalities between the roots of the given equation in x will be completely governed, and are definable by the equalities existing between those of the biquadratic binary form

$$(a, b, m, e, f)(X, Y)^4 \text{*}.$$

* If the equation is regarded as one in quaternions, the determining biquadratic is the modulus of $x^2 + xp + q$; from which it follows immediately that, if p, q are *real* quaternions, all the four roots, say α, β, γ, δ, are imaginary. It may be shown that the roots of Hamilton's determining cubic are

$$d - \frac{(\alpha+\beta)(\gamma+\delta)}{4}, \quad d - \frac{(\alpha+\gamma)(\beta+\delta)}{4}, \quad d - \frac{(\alpha+\delta)(\beta+\gamma)}{4},$$

and these therefore are (as shown also by Hamilton) all of them real. The biquadratic serves to determine the points in which the variable conic associated to the equation $px^2 + qx + r$ (that is, the determinant to $xp + yq + zr$) is intersected by the absolute conic $xz - y^2$. Each root of the given equation corresponds to a side of the complete quadrilateral formed by the four points of intersection of these two conics; and thus we see that there are five cases to consider when the variable conic is a conic proper, according as it intersects or touches the fixed conic (which can happen in four different ways); and seven other cases where the conic degenerates into two intersecting or two coincident lines (in which cases the solution becomes irregular); namely, the intersecting lines may cut or touch in one or two points the fixed one, and may cut or touch the conic at their point of intersection, which gives five cases; and the coincident lines may cut or touch the fixed conic, which gives two more. Hence there are in all twelve principal cases to consider in Hamilton's form of the Quadratic Equation in Quaternions; or rather thirteen, for the case of the variable and fixed conics coinciding must not be lost sight of.

If the biquadratic has two equal roots, the given quadratic will have two pairs of equal roots.

If the biquadratic has two pairs of equal roots, the given quadratic will have four equal roots.

If the biquadratic has three equal roots, the quadratic will have three pairs of equal roots.

If the biquadratic has all its roots equal, the quadratic will have all its roots equal.

In the first case two of the three pairs of roots of the given quadratic coincide, or merge into a single pair.

In the second case, not only two pairs merge into one pair, but the two roots of that pair coincide with one another.

In the third case the three pairs merge into a single pair.

In the fourth case the two members of that single pair coincide with one another.

So long as the equation in x remains regular, no kind of equalities can exist between the roots other than those above specified.

For instance, let us consider the possibility of two values of x, and no more, becoming equal. First, let us inquire what is the condition to be satisfied in order that the scalar parts of two roots which belong to the same pair shall become equal. It may be shown that the sufficient and necessary condition that this may take place is that the irreducible sub-invariant of degree 3 and weight 6 (that is, the first coefficient of the irreducible skew-covariant of the associated biquadratic form $[a, b, m, e, f]$) shall vanish.

If, now, the *vectors* as well as the *scalars* of the two roots are to be equal, it may be shown that the *second* as well as the first coefficient of the skew-covariant must vanish. But this cannot happen without the discriminant vanishing*; for it may easily be seen that the discriminant of a binary biquadratic with its sign changed is equal to sixteen times the product of the first and last coefficients, less the product of the second and penultimate coefficients of its irreducible skew-covariant. Hence when two roots belonging to the same pair of the given quadratic coincide, two values of λ become equal, and therefore all four roots belonging to two pairs merge into one.

Again, it is not possible for two roots belonging to two pairs corresponding to two different values of λ to coincide; for in such case the expression

* The first two coefficients of the skew-covariant vanishing implies the existence of two pairs of equal roots and *vice versâ*. This is on the supposition made that a, the first coefficient of the given quartic, is not zero.

given for x shows that $pq, p, q, 1$ would be connected by a linear equation. But when this happens (as has been shown by me elsewhere), the invariant of the associated ternary quartic vanishes and the equation ceases to be regular. Thus, then, it appears that it is impossible for a single relation of equality (*and no more*) to exist between the roots of the given equation when its form is regular. So, again, it may be shown that it is impossible for four, and no more, relations of equality to exist between the roots.

It need hardly be added, that the equation $px^2 + qx + r = 0$ ceases to be regular when q or r vanishes.

The reader may satisfy himself as to the truth of what has been alleged as to the relation of the discriminant of a binary biquadratic to the coefficients of its skew-covariant by simple verification of the identity

$$16(a^2d - 3abc + 2b^3)(e^2b - 3edc + 2d^3)$$
$$- (a^2e + 2abd - 9c^2a + 6b^2c)(e^2a + 2edb - 9ec^2 + 6d^2c)$$
$$= 27(ace + 2bcd - c^3 - b^2e - ad^2)^2 - (ae - 4bd + 3c^2)^3.$$

The biquadratic equation in X, Y is what the determinant of $\lambda p + \mu q + \nu r$ becomes when X^2, XY, Y^2 are substituted therein for λ, μ, ν; so that we may say that $(a, b, m, e, f)(x, 1)^4$ is the determinant of $px^2 + qx + r$, when x is regarded as an ordinary quantity. Let ϕx be any quadratic factor of this biquadratic function in x: I have found that $\phi x = 0$ will be the *identical* equation to one of the roots of the given equation $fx = 0$, where

$$fx = px^2 + qx + r.$$

Between the two equations $fx = 0$, $\phi x = 0$, x^2 may be eliminated and x found in terms of known quantities: ϕx will have six different values, which will give the six roots of $fx = 0$. It is far from improbable that a similar solution applies to a unilateral equation $fx = 0$ of any degree n in matrices of any order ω.

Call Fx the determinant of fx when x is regarded as an ordinary quantity; then, if ϕx is an algebraical factor of the degree ω in x contained in Fx, it would seem to be in all probability true that $\phi x = 0$ is the identical equation to one of the roots of $fx = 0$; and, *vice versâ*, that the function identically zero of any such root is a factor of Fx. By combining the equations $fx = 0$, $\phi x = 0$, all the powers of x except the first may be eliminated, and thus every root of x determined. The solution of the given equation will depend upon the solution of an ordinary equation of the degree $n\omega$, and the number of roots will be the number of ways of combining $n\omega$ things ω and ω together. Thus, for a cubic equation in quaternions the number of roots would be $\frac{1}{2}6 . 5$, or 15. In the May number of this *Magazine* [p. 229 above] it was supposed to be shown to be 21; but it is quite conceivable that this determination may

be erroneous, especially as it was deduced from general considerations of the degrees of a certain system of equations without attention being paid to their particular form, which might very well be such as to occasion a fall in the *order* of the system. I am strongly inclined, with the new light I have gained on the subject, to believe that such must be the case, and that the true number of roots for a unilateral equation in quaternions of the degree n is $2n^2 - n$*; in which case the theorem above stated, and which may be viewed as a marvellous generalization of the already marvellous Hamilton-Cayley Theorem of the identical equation, will be undoubtedly true for all values of n and ω. But I can only assert positively at present that it is true for the case of $n = 1$ whatever ω may be, and for the case of $n = 2$, $\omega = 2$†.

* From the number 21 above referred to, now known to be erroneous, the general value was inferred to be $n^3 - n^2 + n$, whereas it is demonstrably $2n^2 - n$ only for the *general* unilateral equation of degree n in quaternions, as I proved it to be for the *Jerrardian* form of that equation.

† I have since obtained an easy proof of the truth of the conjectural theorem for all values of n and ω; see the *Comptes Rendus* of the Institute of France for October 20th last [p. 197 above].

34.

NOTE ON CAPTAIN MACMAHON'S TRANSFORMATION OF THE THEORY OF INVARIANTS.

[*Messenger of Mathematics*, XIII. (1884), pp. 163—165.]

THE whole question as is well known consists in finding the free forms of $\Omega^{-1}0$, where

$$\Omega = a_0\delta a_1 + 2a_1\delta a_2 + \ldots + ia_{i-1}\delta a_i;$$

but, as long ago noticed by me* in the *Am. Math. Journal*, $\Omega^{-1}0$ is only a deformation of $V^{-1}0$, where

$$V = a_0\delta a_1 - a_1\delta a_2 + \ldots \pm a_{i-1}\delta a_i,$$

$\Omega^{-1}0$ being deducible from $V^{-1}0$ by altering the dimensions of the a elements which it contains in known numerical proportions, so that $\Omega^{-1}0$ may be said to be $V^{-1}0$ subjected to a known *strain*†.

To fix the ideas let $i = 3$ and call the a's by the names a, b, c, d or, for greater simplicity, $1, b, c, d$.

Let
$$b = r + s + t,$$
$$c = rs + rt + st,$$
$$d = rst.$$

Then the matrix

$$\frac{D(b, c, d)}{D(r, s, t)} = \begin{matrix} 1 & 1 & 1\ , \\ s+t & t+r & r+s, \\ st & tr & rs\ , \end{matrix}$$

so that

$$\frac{D(r, s, t)}{D(b, c, d)} = \begin{matrix} \frac{r^2}{(r-s)(r-t)} & \frac{s^2}{(s-r)(s-t)} & \frac{t^2}{(t-r)(t-s)}, \\ \frac{r}{(r-s)(r-t)} & \frac{s}{(s-r)(s-t)} & \frac{t}{(t-r)(t-s)}, \\ \frac{1}{(r-s)(r-t)} & \frac{1}{(s-r)(s-t)} & \frac{1}{(t-r)(t-s)}. \end{matrix}$$

[* Vol. III. of this Reprint, p. 570.]

† In fact the numerical multipliers of the terms in Ω may be taken perfectly arbitrary without producing any effect upon the form $\Omega^{-1}0$ than what may be represented by a *strain*.

Consequently

$$V = \Sigma \frac{r^2 - (r+s+t)r + (rs+rt+st)}{(r-s)(r-t)}\delta_r = \Sigma \frac{st}{(r-s)(r-t)}\delta_r.$$

In like manner in general for $1, a_1, a_2, \ldots a_i$ we shall find, on writing

$$a_1 = r_1 + r_2 + \ldots + r_i,$$
$$a_2 = r_1r_2 + r_2r_3 + \ldots + r_{i-1}r_i,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$a_i = r_1r_2 \ldots r_i,$$

$$V = \delta a_1 - a_1\delta a_2 + \ldots \pm a_{i-1}\delta a_i = \Sigma \frac{r_2r_3 \ldots r_i}{(r_1 - r_2)(r_1 - r_3)\ldots(r_1 - r_i)}\delta r_1.$$

Hence
$$V^{-1}0 = F(s_1, s_2, \ldots s_i),$$
where, in general,
$$s_\omega = r_1^\omega + r_2^\omega \ldots + r_i^\omega;$$
and consequently the theory of invariants, which endoscopically treated in the ordinary way hinges upon symmetrical functions of the differences of a set of letters, is made to depend upon functions of the simple sums of powers commencing with the second power and ending with a power whose index is the order of any given finite quantic, but in the case of *perpetuants* taking in all the powers except the first.

It goes without saying that the same method applied to the *constrained* V will show that it is equal to $\Sigma\delta r_1$, so that V_0^{-1} is an arbitrary function of the differences of the r's corresponding to that hypothesis, as we know ought to be the case.

What has been established in the foregoing investigation is a principle of correspondence whose importance as a simplifying agent recalls Ivory's use of such principle in Attractions, namely, the remarkable algebraical law that any symmetrical function of the differences of a set of i quantities is a symmetrical function of the sums of the 2nd, 3rd, ..., ith powers of another equi-numerous set.

By virtue of this principle the numerical part of the Calculus of Invariants is capable of being entirely divorced from all question of algebraical content and a Zahl-Invariant theory comes into being, in its fundamental conception analogous to the Zahl-Geometrie of Schubert.

Further remarks on this subject will be found in the *Comptes Rendus de l'Institut* presumably for March 31 and April 7 of this year [p. 163 above].

35.

ON THE D'ALEMBERT-CARNOT GEOMETRICAL PARADOX AND ITS RESOLUTION.

[*Messenger of Mathematics*, XIV. (1885), pp. 92—96.]

I WILL presently state the simple geometrical problem which led D'Alembert to call into question the validity of the received Cartesian doctrine of positive and negative geometrical magnitudes, and which, according to Carnot, furnishes an unanswerable argument against it. See Mouchot, *La réforme Cartésienne*, pp. 74, 75.

Against this doctrine, presented in its crude form, the objections of these illustrious impugners of it are unquestionably well founded and unanswerable; but the inference to be drawn from this is not that no such or such-like doctrine reposing on an unassailable logical basis exists or is capable of being established (woe worth the day! when such a conclusion should be admitted), but that the doctrine as usually stated is incomplete and requires a supplement.

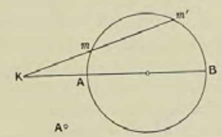
This has been anticipatively furnished by me many years ago in this very *Journal*, and in conjunction with the substitution of positive and negative indefinite rotation in lieu of Euclid's positive and limited angular magnitude, made the basis of a strictly logical deduction (which was before wanting) of the trigonometrical canon.

It consists in the notion of a line having, so to say, sides (returning upon itself at its two semi-points at infinity), or to put the matter in a more practical form, in regarding an Euclidean indefinite straight line as representing two distinct lines locally coincident, but running in contrary directions, and in referring the algebraical sign of any rectilinear segment to the concurrence or discordance of its flow (which is represented by the order in which its two extremities are named or written down) with that of the indefinite line, upon which it is supposed to be carried.

Thus, for example, AB taken on the upper side of a line or line-pair will be the negative of AB taken on the same side, but the same as BA taken on the under side.

I will now state the D'Alembert-Carnot problem. "Voici" says Carnot, "un exemple aussi simple que frappant, qui seul suffit pour renverser toute cette doctrine" of positive and negative magnitudes.

"D'un point K, pris hors d'un cercle donné, soit proposé de mener une droite Kmm', telle que la portion mm', interceptée dans le cercle, soit égale à une droite donnée.



"Du point K, et par le centre du cercle menons une droite KAB qui rencontre la circonférence en A et B. Supposons $KA = a$, $KB = b$, $mm' = c$, $Km = x$. On aura donc par les propriétés du cercle

$$ab = x(c + x) = cx + x^2$$

donc
$$x^2 + cx - ab = 0$$

ou
$$x = -\tfrac{1}{2}c \pm \sqrt{(\tfrac{1}{4}c^2 + ab)}.$$

x a deux valeurs: la première, qui est positive, satisfait sans difficulté à la question; mais que signifie la seconde, qui est négative? Il paraît qu'elle ne peut répondre qu'au point m', qui est le second de ceux où Km coupe la circonférence; et, en effet, si l'on cherche directement Km', en prenant cette droite pour l'inconnue x, on aura

$$x(x - c) = ab$$

ou
$$x = \tfrac{1}{2}c \pm \sqrt{(\tfrac{1}{4}c^2 + ab)}$$

dont la valeur positive est précisément la même que celle qui s'était présentée dans le premier cas avec le signe négatif. Donc, quoique les deux racines de l'équation

$$x = -\tfrac{1}{2}c \pm \sqrt{(\tfrac{1}{4}c^2 + ab)}$$

soient l'une positive et l'autre négative, elles doivent être prises toutes les deux dans le même sens par rapport au point fixe K. Ainsi, la règle qui veut que ces racines soient prises en sens opposés porte à faux. Si au contraire le point fixe K était pris sur le diamètre même AB et non sur le prolongement, on trouverait pour x deux valeurs positives et cependant elles devraient être prises en sens contraires l'une de l'autre. La règle est donc encore fausse pour ce cas.

"Si l'on dit que ce n'est pas ainsi qu'il faut entendre ce principe, que les racines positives et négatives doivent être prises en sens opposés, je demanderai comment il faut l'entendre? et j'en conclurai par là même qu'il faut une explication pour empêcher qu'il ne soit pris dans l'acceptation la plus naturelle. Il suit que ce principe est obscur et vague."

The answer has been already given to the question, "comment il faut entendre ce principe," and it will be seen in such a way as to remove all grounds for the charge of its being *any longer* "obscur et vague."

This is how the problem set out in full ought to be enunciated:

A complete line (that is, a line-pair or two-sided line) drawn from K cuts the circle in the points m, m'; mm' measured on either side of the line (and of course denoted quantitatively by the number of units of given length which it contains) is to be equal to c a given positive or negative number. Required the value of Am.

(1) Suppose K to be exterior to the circle as in the diagram above.

I distinguish the two sides of the complete line, as the under and upper line, and suppose the flow of the under one to make an acute Euclidean angle with the flow from K to the centre of the circle. In all cases

$$Km' = Km + mm',$$

and consequently the equation for finding x remains always $x^2 + cx = ab$, of which the two roots are $-\frac{1}{2}c + \sqrt{(\frac{1}{4}c^2 + ab)}$ and $-\frac{1}{2}c - \sqrt{(\frac{1}{4}c^2 + ab)}$.

Adhering to the letters of the diagram, if c is positive the two values of x will correspond to Am on the under line and Am' on the upper line of the line-pair. If, again, c is negative, the two values of x will correspond to Am on the upper and Am' on the under one.

(2) Suppose K to be within the circle.

It will still be true (paying attention to the signs) that $Km' = Km + mm'$ (that being a universal identity in algebraical geometry), but the algebraical values of KA, KB being contrary, we may regard KA as positive and equal to a, KB as negative and equal to $-b$, and shall have the equation

$$x^2 + cx = -ab,$$

of which the two roots are

$$-\tfrac{1}{2}c + \sqrt{(\tfrac{1}{4}c^2 - ab)}, \quad -\tfrac{1}{2}c - \sqrt{(\tfrac{1}{4}c^2 - ab)}.$$

Understand by the *two segments* Km and Km'.

We may suppose the indefinite line-pair mKm' to swing round K, its under-side in the position of coincidence with the diameter having the same flow as KA; then, if c is positive, until the swinging line revolving with the sun has described a right angle, the first root will be the *infrà*-diametral segment taken on the lower line (or side), and the second root the *suprà*-diametral segment taken on the upper line (or side) of the line-pair (or complete line); in the next quadrant of rotation the first root will be the *suprà*-diametral segment on the under and the second root the *infrà*-diametral segment on the upper side of the complete line. When c is negative a similar statement may be made if only the words *under* and *upper* are interchanged. In the critical position, when the swinger is at right angles to the diameter, the two roots become equal and undistinguishable; but throughout and subject to no exception, the complex of the two roots contains the complete solution of the problem, and the complete solution of the problem necessitates the retention of the complex of the two roots.

Thus, then, as in the preceding case, it has been shown that the Cartesian view of the equipollence of positive and negative roots (the latter Descartes influenced by hereditary prepossessions calls *radices falsae*) is made exact through the intermediation of the conception of sides to a line. D'Alembert and Carnot are entitled to the gratitude of Geometers and all lovers of truth for raising objections so perfectly well founded to the then, and even now, too prevalent interpretation of the meaning of the geometrical positive and negative, but the difficulty which they so justly appreciated and so clearly expressed is overcome and exists no longer.

P.S. I am informed that M. Laguerre has emitted the same view as that I have set forth relative to the sign to be given to geometrical distances, and made use of the same conception of the double or complete line-carrier.

My note on the subject appeared before my exodus across the Atlantic, probably nine or ten years ago. M. Laguerre's publication must have been many years posterior to this. The references to the reappearance of the theory on the other side of the Channel, obligingly furnished to me by M. Mannheim in Paris, have unfortunately got mislaid. I believe the communication containing it was made by M. Laguerre within the last three or four years, but it has already had time to find its way into some of the most esteemed French text-books. Being not only true but *the* truth, it must eventually find universal acceptance. It is not without interest (it seems to me) that we may regard a double or complete right line as a sort of embryonic embodiment of the idea of a Riemann Surface.

36.

SUR UNE NOUVELLE THÉORIE DE FORMES ALGÉBRIQUES*.

[*Comptes Rendus*, CI. (1885), pp. 1042—1046, 1110—1111, 1225—1229, 1461—1464.]

Si l'on imagine une fonction de dérivées différentielles (toutes d'un ordre supérieur à l'unité) de y par rapport à x, qui, sauf l'introduction d'un facteur multiple numérique, d'une puissance de $\frac{dy}{dx}$, ne change pas sa valeur quand on remplace x par y et y par x, il est évident qu'une telle fonction restera invariable (sauf l'introduction d'une constante comme facteur) quand pour x et y on substitue des fonctions linéaires quelconques, homogènes ou non homogènes de y et x. Ainsi une telle fonction conduira immédiatement à la connaissance d'un point singulier d'une courbe d'un degré quelconque. Le seul exemple d'une telle fonction, traité jusqu'à ce jour, est la simple fonction $\frac{d^2y}{dx^2}$ qui, par cette seule propriété, sans aucune autre considération, sert à démontrer l'existence d'une propriété projective de courbes dont la condition est $\frac{d^2y}{dx^2}=0$. Il nous paraît donc très utile de chercher un moyen de produire toutes les fonctions de cette espèce auxquelles nous donnerons le nom de *réciprocants purs* ou simplement *réciprocants*. On verra qu'il existe des réciprocants mixtes, c'est-à-dire contenant des puissances de $\frac{dy}{dx}$ (comme la forme bien connue de M. Schwarz, $\frac{dy}{dx}\frac{d^3y}{dx^3}-\frac{3}{2}\frac{d^2y}{dx^2}\frac{d^2y}{dx^2}$) qui possèdent la même faculté d'invariance par rapport à l'échange de y avec x, comme les réciprocants purs, mais qui évidemment ne peuvent pas indiquer l'existence de points singuliers dans les courbes.

Nous écrirons, au lieu de $\delta_x y$, $\delta_x^2 y$, $\delta_x^3 y$, $\delta_x^4 y$, ..., les lettres t, a, b, c, ..., et pour leurs réciproques $\delta_y x$, $\delta_y^2 x$, $\delta_y^3 x$, ..., τ, α, β, γ, ... On verra facilement que, pour que $F(t, a, b, c, \ldots)$ soit un réciprocant pur, F doit être d'un degré et d'un poids constant dans les lettres de chaque terme; de plus (pour un

[* See the lectures, below p. 303.]

réciprocant F d'une nature quelconque), on aura $F(a\ldots)/F(\alpha\ldots)=(-1)^\theta t^{2\lambda}$, où θ sera le plus petit nombre des lettres a, b, c, ... dans un terme quelconque de F, et λ sera la moyenne arithmétique entre le poids et trois fois le degré de F, en comptant le poids de t, a, b, c, ... comme étant -1, 0, 1, 2, ... Cela donne lieu à une remarque importante par rapport aux *réciprocants mixtes*: pour qu'on puisse additionner deux formes mixtes afin de former un nouveau réciprocant, il faut non seulement que le degré et le poids soient les mêmes pour tous les deux, mais aussi le *caractère* qui dépend de la valeur de θ et que l'on peut qualifier comme caractère pair ou impair selon la parité de θ. Ainsi, par exemple, $2tb-3a^2$ et a^2 sont tous deux réciprocants, mais $2tb$ ne le sera pas, parce que les *caractères* des deux données sont contraires. Il est facile de démontrer que, si R est un réciprocant quelconque,

$$(2tb-3a^2)\,\delta_a R+(2tc-4ab)\,\delta_b R+(2td-5ac)\,\delta_c R+\ldots$$

sera aussi un réciprocant de même caractère que R. Ainsi, en commençant avec le réciprocant a, on peut obtenir une suite infinie de réciprocants mixtes: ces réciprocants ainsi obtenus ne seront pas en général irréductibles; mais, sans les réduire, leur forme fait voir immédiatement que tout réciprocant, qu'il soit pur ou mixte, peut être exprimé comme une fonction rationnelle et aussi (si l'on regarde t comme unité) entière de combinaisons *légitimes** de ces quantités.

Pour obtenir tous les réciprocants purs de poids, degré et ordre (c'est-à-dire nombre de lettres) donnés, linéairement indépendants les uns des autres, on peut former une équation partielle différentielle, linéaire, où R est la variable dépendante, et a, b, c, ... les variables indépendantes; elle exprimera la condition nécessaire et suffisante pour que R soit un tel réciprocant et fournira un moyen sûr de résoudre le problème proposé. Voici la manière de démontrer ce théorème fondamental.

Si, dans l'équation

$$F(a, b, c, \ldots)=(-1)^\theta t^{2\lambda} F(\alpha, \beta, \gamma, \ldots),$$

on donne à y la variation ϵx, on voit que a, b, c, ..., et conséquemment F, restent invariables. Les variations de α, β, γ, ... sont faciles à déterminer, et la variation de t est donnée.

Ainsi, après quelques calculs faciles, en égalant à zéro, séparément, dans la variation de $t^{2\lambda}F(\alpha, \beta, \ldots)$, les termes qui contiennent t et ceux qui ne le contiennent pas, on arrive à deux équations dont l'une sera

$$\left(3a\frac{d}{da}+4b\frac{d}{db}+5c\frac{d}{dc}+\ldots\right)F(a, b, \ldots)=2\lambda F,$$

* Je nomme *légitime* une combinaison quelconque de réciprocants où l'on évite *d'additionner* ceux dont le poids, le degré, l'ordre et le *caractère* ne sont pas les mêmes pour tous.

qui exprime la valeur numérique de λ, comme fonction du poids et du degré de F; l'autre équation, en écrivant

$$V = 3a^2\delta_b + 10ab\delta_c + (15ac + 10b^2)\,\delta_d + (21ad + 35bc)\,\delta_e + (28ae + 56bd + 35c^2)\,\delta_f + \ldots,$$

sera

$$VR = 0.$$

Pour voir la loi des chiffres arithmétiques dans V, formons les suites des coefficients de $(1+x)^i$ en commençant avec $i = 4$; divisons chaque coefficient *central* en deux parties égales, et supprimons la dernière moitié des séries numériques ainsi formées; on obtiendra ainsi la Table :

1	4	3		
1	5	10		
1	6	15	10	
1	7	21	35	
1	8	28	56	35
.....	.....	.....	.....	.....

En négligeant les deux premières colonnes, on trouve les nombres qui paraissent dans la formule.

On démontre ainsi que $VR = 0$ est une condition nécessaire pour que R soit un réciprocant. Mais il faut aussi démontrer que cette condition est suffisante. Soit donc D la valeur de $F(a, b, \ldots) - t^{2\lambda}F(\alpha, \beta, \ldots)$, exprimée comme une fonction de $a, b, c, \ldots$ seulement. D sera donc une fonction de la même forme que $F(a, b, \ldots)$.

On suppose que

$$\Delta D = 0\,;$$

c'est-à-dire que la variation de D produite par la substitution de $x + \epsilon y$ à x est égale à zéro, en vertu de l'équation $VR = 0$.

Donnons à y une variation arbitraire $y + \eta u$; alors, si D devient D', la variation de D' sera nulle, quand on substitue, pour x, $x + \epsilon y + \epsilon\eta u$, et, conséquemment, quand on substitue $x + 2y$ pour x; on aura donc

$$\Delta D' = 0,$$

et, en prenant la différence des variations de D et D', on obtient

$$\Delta\left(u'\frac{d}{da}R + u''\frac{d}{db}R + u'''\frac{d}{dc}R + \ldots\right) = 0.$$

Donc, à cause de la forme arbitraire de u, il faut que

$$\Delta\frac{d}{da}D = 0, \quad \Delta\frac{d}{db}D = 0, \quad \ldots;$$

et, en raisonnant sur $\frac{d}{da}D$, $\frac{d}{db}D, \ldots$ comme on a raisonné sur D, on voit que le Δ de chacune des dérivées secondes différentielles de D sera zéro; en

poursuivant le même calcul, on trouve évidemment que le Δ d'une dérivée de D d'un ordre quelconque par rapport à $a, b, c, \ldots$ sera nul.

Donc D est nul; car, dans le cas contraire, s'il contient un terme quelconque, dont les lettres peuvent être distinctes ou identiques, en isolant une seule de ces lettres et prenant la dérivée de D par rapport à toutes les autres lettres, on aura le Δ de la lettre isolée, c'est-à-dire de $\delta_x y, \delta_x^2 y, \ldots$, zéro quand on substitue $x + \epsilon y$ pour x, ce qui est absurde. Ainsi l'on voit que, quand $\Delta D = 0$, c'est-à-dire quand $VR = 0$, $D = 0$, ce qui était à démontrer.

Soient ω, i, j le poids, le degré et l'ordre d'un réciprocant quelconque : de même que pour les sous-invariants, le nombre de formes linéairement indépendantes s'exprime par $(\omega;\, i, j) - (\omega - 1;\, i, j)$, où, en général, $(\omega;\, i, j)$ signifie le nombre de partitions de ω en i parties dont nulle n'excède j; ainsi l'on voit que, en vertu de l'équation $VR = 0$, on aura, pour le nombre des réciprocants linéairement indépendants, la formule

$$(\omega;\, i, j) - (\omega - 1;\, i + 1, j).$$

Mon long exil en Amérique expliquera, je l'espère, comment j'ai pu ignorer l'identité des invariants différentiels de M. Halphen avec les formes que j'ai nommées *réciprocants purs*. Les travaux vraiment remarquables de M. Halphen n'ont pas besoin de mes éloges et auront été couronnés par l'admiration de tous les géomètres dignes de ce nom.

Je crois cependant qu'il y a assez de différence entre le but et la marche de mes recherches sur ce terrain et ceux de M. Halphen pour justifier l'insertion dans les *Comptes rendus* de ma discussion de la théorie regardée comme une théorie de formes algébriques. Si je ne me trompe pas, M. Halphen, s'il l'a découverte, n'a fait nul usage de l'équation partielle différentielle que j'ai donnée et qui sert à établir le parallélisme merveilleux entre les invariants différentiels et les semi-invariants ordinaires.

De plus, il n'a pas eu occasion de faire allusion aux formes que j'appelle *réciprocants mixtes orthogonaux*, qui ne sont point compris dans la définition des *invariants différentiels*, et qui sont essentiels pour expliquer les singularités quasi-métriques des courbes.

Nous rappelons que par le mot *réciprocant* (sans qualification) il a été convenu de sous-entendre une forme de cette espèce qui ne contient pas t $\left(\text{c'est-à-dire } \frac{dy}{dx}\right)$ et nous avons trouvé que le nombre de ces réciprocants linéairement indépendants, du degré i, de l'étendue j (c'est-à-dire contenant $j + 1$ lettres distinctes) et du poids ω, s'exprime par la formule

$$(\omega;\, i, j) - (\omega - 1;\, i + 1, j),$$

où en général $(l;\, m, n)$ signifie le nombre de partitions de l en m ou un plus petit nombre que m de parties dont aucune n'excède n en grandeur ; de sorte que $(l;\ m,\ n)$, quand m est plus grand que l, signifie la même chose que $(l;\ l,\ n)$, car tous les deux sont équivalents à $(l;\ \infty,\ n)$. Conséquemment

$$(i;\ i,\ j)-(i-1;\ i+1,\ j)=(i;\ i,\ j)-(i-1;\ i,\ j),$$

lequel sera toujours positif quand i et j sont tous les deux plus grands que l'unité ; et, puisque a, qui est du degré 1, est un réciprocant, il s'ensuit que, pour un degré quelconque donné, il existe toujours des réciprocants (car on peut faire $\omega=i$), mais en nombre fini, car, en faisant croître ω, $(\omega-1;\ i+1,\ \infty)$, au delà d'une certaine valeur de ω, deviendra nécessairement plus grand que $(\omega;\ i,\ \infty)$. On peut exprimer par $(l:m)$ ce que devient $(l;\ m,\ n)$ quand $n=\infty$, et alors $(\omega:i)-(\omega-1:i+1)$ exprimera le nombre de réciprocants linéairement indépendants du poids ω et du degré i sans autre limitation. Ainsi on trouvera que du degré 1 il n'existe qu'un seul réciprocant du poids 0 ; pour le degré 2, un seul du poids 2 ; pour le degré 3, deux qui seront respectivement du poids 3 et du poids 4 ; etc.

On trouvera qu'étant donné j il existe toujours, sauf pour le cas où $j=1$, un réciprocant qui contient toutes les $j+1$ lettres et qui de plus contiendra un terme qui est un produit de la dernière lettre par une puissance de a. Ces formes, qu'on peut nommer les *protomorphes*, sont les analogues des formes $a,\ ac-b^2,\ a^2d+\ldots,\ ae+\ldots$, qu'on connaît dans la théorie des sous-invariants. Dans le cas des réciprocants, ces protomorphes seront $a,\ ac,\ \ldots,\ a^2d,\ \ldots,\ a^2e,\ \ldots,\ a^3f,\ \ldots,\ a^3g,\ \ldots$, etc.

Évidemment une fonction rationnelle *quelconque* des lettres peut, au moyen de substitutions successives, être exprimée comme une fonction rationnelle des protomorphes et de b divisée par une puissance de a. Soit donc R un réciprocant quelconque ; on aura

$$a^2R+P+Qb+\ldots+Jb^i=0,$$

où $P,\ Q,\ \ldots,\ J$ sont eux-mêmes des réciprocants. En opérant i fois sur cette équation avec notre opérateur V, on voit qu'on obtient $a^{2i}J=0$; donc J est nul, et l'on voit ainsi que tous les termes $Q,\ \ldots,\ J$ disparaissent et que R (en faisant $a=1$) devient une fonction rationnelle et entière des protomorphes. Nous allons appliquer ce principe fondamental, commun aux deux théories des sous-invariants et des réciprocants, pour obtenir les formes irréductibles (les *Grundformen*) des réciprocants pour les ordres 2, 3, 4.

Faisons $j=2,\ i=2,\ \omega=2$ et supposons que le réciprocant R soit $\lambda ac+\mu b^2$; on obtient

$$VR=(3a^2\delta_b+10ab\delta_c)\,R=(6\mu+10\lambda)\,a^2b=0.$$

Donc $-\lambda:\mu::3:5$ et nous obtenons le réciprocant $3ac-5b^2$*.

* Il est bon de remarquer que $3ac-5b^2=0$, c'est-à-dire

$$3\frac{d^2y}{dx^2}\frac{d^4y}{dx^4}-5\left(\frac{d^3y}{dx^3}\right)^2=0,$$

indique que le point $(x,\ y)$, quand cette équation est satisfaite par telles coordonnées d'une courbe quelconque, est un point supra-parabolique, c'est-à-dire où une parabole passe par 5 au lieu de 4 points consécutifs seulement.

Passons au cas $j=3,\ i=3,\ \omega=3$, et posons

$$R=\lambda a^2d+\mu abc+\nu b^3.$$

On aura

$$VR=(3a^2\delta_b+10ab\delta_c+15ac+10b^2\delta_d)\,R$$
$$=(3\mu+15\lambda)\,a^2c+(9\nu+10\mu+10\lambda)\,a^2b^2=0.$$

On aura donc

$$\mu=-5\lambda,\quad 9\nu=40\lambda,$$

de sorte qu'on peut écrire

$$R=9a^2d-45abc+40b^3.$$

On reconnaîtra immédiatement que $R=0$ est l'équation différentielle donnée par Monge et retrouvée par M. Halphen à une conique et que

$$9\,(\delta_x^2y)^2\,(\delta_x^5y)-45\delta_x^2y\,\delta_x^3y\,\delta_x^4y+40\,(\delta_x^3y)^3=0$$

exprime la condition que le point $(x,\ y)$ d'une courbe quelconque sera un point d'inflexion du second ordre, c'est-à-dire un point où une conique passe par six points consécutifs. Le nombre de ces points peut être trouvé en fonction linéaire de n, ordre d'une courbe donnée, en opérant sur cette équation une transformation analogue à celle au moyen de laquelle on passe du système $y=0,\ \dfrac{d^2y}{dx^2}=0$ au système équivalent, mais épuré, $\phi=0$; $H\phi=0$*.

Passons au cas où $j=4,\ i=3,\ \omega=4$, et écrivons

$$R=\lambda a^2e+\mu abd+\nu ac^2+\pi b^2c.$$

On aura

$$V=3a^2\delta_b+10ab\delta_c+(15ac+10b^2)\,\delta_d+(21ad+35bc)\,\delta_e,$$

et, en posant $RV=0$, on obtient, en égalant séparément à zéro les coefficients de $a^3d,\ a^2bc,\ ab^3$, les équations

$$21\lambda+3\mu=0,\quad 35\lambda+15\mu+20\nu+6\pi=0,\quad 10\mu+10\pi=0.$$

* Pour le cas d'une cubique, le nombre de ces points d'inflexion du second ordre est vingt-sept ; on démontre facilement que ce sont les intersections de la courbe avec son covariant du *degré-ordre* 12 . 9.

On voit immédiatement, au moyen de notre théorie connue de *résidus géométriques*, que ces vingt-sept points sont les points de la cubique où elle est rencontrée par les neuf faisceaux des tangentes qu'on peut mener des neuf points d'inflexion ordinaire. Car un quelconque de ces points doit être tel que sa dérivée à l'indice 5 sera coïncidente avec le point lui-même. On aura donc 1, 1=1, 5, c'est-à-dire 2=4, ce qui veut dire que le tangentiel du point est un point d'inflexion ; ce qui était à démontrer.

Soit dit, par parenthèse, que la même théorie de résiduation enseigne que le point fixe Q, où une cubique donnée sera coupée par une autre cubique quelconque qui a en commun avec la première 8 points consécutifs à un point donné P, sera le troisième tangentiel de P et peut être nommé son *satellite* ; quand le satellite coïncide avec son primaire, en se servant pour le moment de la forme canonique pour exprimer la cubique donnée, et en nommant $x,\ y,\ z$ les coordonnées du primaire, celles du satellite seront (d'après notre théorie exposée dans l'*American Journal of Mathematics*) $x,\ y,\ z$ multipliés respectivement par des fonctions rationnelles de $x^3,\ y^3,\ z^3$, chacune du degré 21. [Vol. III. of this Reprint, p. 339.]

C'est un fait depuis longtemps connu que les points primaires qui coïncident avec leurs satellites (en ne tenant pas compte des neuf inflexions) sont en nombre 72.

et ainsi on peut écrire

$$R = 5a^2e - 35abd + 7ac^2 + 35b^2c.$$

Voici donc le système de protomorphes pour tous les ordres jusqu'au quatrième inclusivement:

$$a, \qquad (1)$$

$$3ac - 5b^2, \qquad (2)$$

$$9a^2d - 45abc + 40b^3, \qquad (3)$$

$$5a^2e - 35abd + 7ac^2 + 35b^2c. \qquad (4)$$

En combinant le cube du deuxième avec le carré du troisième, et en divisant par a, on obtient la forme (analogue au discriminant) de la cubique, mais d'un degré plus élevé,

$$\left.\begin{array}{l} 405a^3d^2 - 4050a^2bcd + 1728a^2c^3 \\ \quad + 1585ab^2c^2 + 3600ab^3d - 18000b^4c\,^*. \end{array}\right\} \qquad (5)$$

En combinant le produit de (2) et de (4), linéairement, avec (5), on obtient

$$\left.\begin{array}{l} 4800a^2ce - 8000ab^2e - 2835a^2d^2 - 5376ac^3 \\ \quad - 5250abcd + 30800b^3d + 11305b^2c^2. \end{array}\right\} \qquad (6)$$

Si l'on se borne aux lettres a, b, c, d, les formes (1), (2), (3), (5) formeront un système complet de *Grundformen*: si on laisse entrer la nouvelle lettre e, (5) n'est plus irréductible, et le système complet de *Grundformen* est constitué par les formes (1), (2), (3), (4), (6).

Tout cela se passe précisément comme avec les sous-invariants avec les mêmes lettres: les poids des formes sont les mêmes pour les deux systèmes, et la seule différence essentielle entre les deux consiste en ce fait, que les trois dernières formes subissent chacune une élévation d'une unité de degré en passant du système des sous-invariants à celui des réciprocants.

Il est nécessaire d'ajouter quelques mots sur les réciprocants mixtes, qui se distinguent en deux espèces, homogènes et hétérogènes. Comme exemple des premiers, on a la dérivée Schwarzienne $2tb - 3a^2$, laquelle, égalée à zéro, ne donne aucune espèce de singularité, mais signifie seulement qu'au point (x, y) on peut mener une conique qui passera par cinq points consécutifs, en ayant ses deux asymptotes parallèles aux axes, ou bien la forme $tc - 5ab$. Comme exemple de l'autre classe, on a la forme connue $(1 + t^2)b - 3ta^2$, dont l'évanouissement (pourvu que x, y soient des coordonnées *rectangulaires*) signifie que le point (x, y) est un point de courbure maximum ou minimum.

* Cette fonction, égalée à zéro, exprime que x, y sont les coordonnées d'un point par où l'on peut faire passer une parabole *cubique* ayant 5 points consécutifs communs à la courbe dont x, y sont les coordonnées.

Nous avons remarqué, par parenthèse, que l'équation

$$(1 + t^2)b - 3ta^2 = 0$$

indique l'existence d'une singularité au point dont les coordonnées sont les x, y sous-entendus dans t, a, b de l'équation.

Mais, pour que cela soit vrai, il faut introduire la restriction que x, y sont des coordonnées *rectangulaires*.

On peut donner le nom de *réciprocant orthogonal* à tout réciprocant mixte qui jouit de la propriété de rester invariable (sauf l'introduction d'une puissance de t) quand on opère sur x et y une transformation linéaire orthogonale. Cela étant convenu, on peut démontrer facilement que le coefficient différentiel par rapport à t d'un réciprocant est lui-même un réciprocant ou pur ou mixte. La proposition réciproque est aussi vraie, de sorte qu'on a le beau théorème suivant:

Si R et $\frac{dR}{dt}$ sont tous les deux réciprocants, alors R est un réciprocant orthogonal.

Par exemple, le réciprocant que nous avons cité plus haut a pour coefficient différentiel par rapport à t la Schwarzienne $2tb - 3a^2$; donc c'est un réciprocant orthogonal; et, en effet, il exprime qu'au point (x, y), où l'équation $2tb - 3a^2 = 0$ est satisfaite, on peut appliquer un cercle qui aura un contact du troisième ordre avec la courbe dont x et y sont les coordonnées; au contraire, la Schwarzienne elle-même ne correspond pas à une singularité quelconque, car sa dérivée par rapport à t, c'est-à-dire $2b$, n'est pas un réciprocant.

De même nous avons trouvé qu'en intégrant le réciprocant $2tc - 10ab$ par rapport à t, entre les limites t et $-c - 15a^3$, la forme résultante

$$(t^2 + 1)c - 10abt + 15a^3$$

sera un réciprocant et conséquemment un réciprocant orthogonal, de sorte que l'équation

$$(1 + t^2)c - 10abt + 15a^3 = 0$$

sera la condition d'une singularité de la courbe $f(y, x) = 0$ qui se rapporte aux points circulaires à l'infini*. Peut-être trouvera-t-on que l'intégrale, par rapport à t, d'un réciprocant mixte quelconque, prise entre des limites convenables, conduira nécessairement à un réciprocant orthogonal. Les singularités d'une courbe peuvent être partagées en trois classes: celles de la première classe seront projectives et peuvent être définies indifféremment au moyen de covariants de formes ternaires ou par des réciprocants purs;

* M. James Hammond, dont on connaît les belles et importantes découvertes dans la théorie invariantive des formes binaires, a trouvé l'intégrale de cette équation, que nous avons donnée dans un discours inaugural, prononcé devant l'Université d'Oxford, lequel va être publié dans le journal anglais *Nature*. [p. 278 below.]

celles de la deuxième classe seront non projectives, mais n'auront affaire qu'avec la ligne à l'infini ; les singularités de cette classe seront exprimables au moyen de réciprocants purs, mais non pas au moyen de covariants de formes ternaires. Restent celles de la troisième classe qui non seulement ne sont pas projectives, mais sont quasi métriques en caractère, c'est-à-dire ont des rapports avec les points circulaires à l'infini ; les singularités de cette classe sont signalées par l'évanouissement de réciprocants orthogonaux. Les réciprocants mixtes, qui ne sont ni purs ni orthogonaux, comme celui, par exemple, de M. Schwarz, ne répondront à aucune de ces trois espèces de singularités ; mais, quoique ne servant pas à représenter une propriété invariable d'une courbe, ils serviront souvent, peut-être toujours, comme bases des réciprocants orthogonaux, c'est-à-dire qu'ils seront les coefficients différentiels par rapport à t de ces derniers.

L'échelle des *protomorphes*, aussi bien dans la théorie des réciprocants purs que dans celle des sous-invariants, joue un rôle si capital, en ce qui concerne la détermination des formes irréductibles, qu'il nous semble indispensable de donner une démonstration rigoureuse de son existence dans l'une et l'autre théorie.

1° Quant aux sous-invariants, soit j l'ordre (c'est-à-dire $j+1$ le nombre des lettres que l'on considère). Si j est pair, on connaît les formes invariantives $ac+\ldots$, $ae+\ldots$, $ag+\ldots$, et l'on peut passer au cas où j est impair. Dans ce cas, le nombre de sous-invariants du poids j et du degré 3 sera

$$(j;\,3,\,j)-(j-1;\,3,\,j).$$

Mais il faut démontrer qu'il existe une forme de ce type, dans laquelle le coefficient du produit de a^2 et de la dernière lettre n'est pas nul.

Or je dis que le nombre des formes du type supposé, qui ne contiennent pas cette lettre, sera

$$(j;\,3,\,j-1)-(j-1;\,3,\,j-1).$$

Mais $$(j-1;\,3,\,j)=(j-1;\,3,\,j-1)$$

et, évidemment, $$(j;\,3,\,j)-(j;\,3,\,j-1)=1\,;$$

car les partitions dont le nombre est $(j;\,3,\,j)$ contiendront toutes les partitions dont le nombre est $(j;\,3,\,j-1)$ et en plus la partition constituée par j combiné avec des zéros.

Conséquemment il existe un sous-invariant dont un terme sera le produit de a^2 par la dernière des lettres que l'on considère.

2° Quant aux réciprocants purs de l'ordre j, nous avons déjà démontré qu'on peut satisfaire à l'inégalité

$$(j;\,x,\,j)-(j-1;\,x+1,\,j)>0$$

en donnant à x une certaine valeur pas plus grande que $j-1$; et, pour démontrer qu'il y aura un réciprocant pur qui contient actuellement un terme

a^{x-1} multiplié par la dernière lettre, on pourrait faire précisément le même raisonnement que nous avons fait ci-dessus pour le cas précédent, et, puisque

$$(j;\,x,\,j)-(j-1;\,x+1,\,j)$$

excède de l'unité la valeur de $(j;\,x,\,j-1)-(j;\,x+1,\,j-1)$, on conclura avec certitude l'existence d'un protomorphe pour l'ordre j.

On peut, en général, trouver plusieurs valeurs de x qui rendent $(j;\,x,\,j)-(j-1;\,x+1,\,j)$ positif ; parmi ces valeurs, il est commode d'adopter, comme *protomorphe* par excellence, une quelconque de celles pour lesquelles la valeur de x qui satisfait à cette inégalité est un minimum. Quand la lettre la plus avancée est inférieure à h, il n'y en a qu'un seul qui réponde à cette définition. Ainsi, par exemple, si $j=5$, l'inégalité

$$(5:x)-(4:x+1)>1$$

donne pour x la valeur minimum $x=4$ et, avec l'aide de l'anéantisseur

$$3a^2\delta_b+10ab\,\delta_c+(15ac+10b^2)\,\delta_d+(21ad+35bc)\,\delta_e+(28ae+56bd+35c^2)\,\delta_f,$$

on obtient le protomorphe

$$45a^3f-420a^2be-42a^2cd+1120ab^2d-315abc^2-1120b^3c.$$

Cela servira pour conduire à la connaissance de tous les réciprocants purs de l'ordre 5, dont le nombre sera au moins égal à celui des *Grundformen* du quantic binaire.

Dans une Communication qui suivra celle-ci, nous nous proposons de donner la théorie des réciprocants doubles ou multiples dont ceux de l'espèce pure sont précisément analogues aux invariants ou sous-invariants de systèmes de formes binaires.

La théorie des doubles réciprocants purs comprend nécessairement, comme cas particulier, l'étude des formes qui déterminent la position des tangentes communes à deux courbes et les points bitangentiels d'une seule.

Dans la remarque que nous avons faite, dans la première Note, sur le même sujet que la Note actuelle, à propos des réciprocants mixtes de la forme

$$[(2tb-3a^2)\,\delta_a+(2tc-4ab)\,\delta_b+(2td-5ac)\,\delta_c+\ldots]^i\,a,$$

nous avons affirmé que tout réciprocant pur ou mixte peut être exprimé en fonction rationnelle et, de plus (quand on fait t égal à l'unité), entière de réciprocants de cette famille ; nous n'avons pas limité, comme nous aurions dû le faire, cette affirmation au cas de réciprocants homogènes : la proposition a besoin d'une certaine modification si on veut la rendre applicable au cas de réciprocants non homogènes ; mais nous ne croyons pas nécessaire d'y insister en ce moment. Seulement, il est bon de remarquer que l'existence d'une équation partielle différentielle linéaire, que nous avons trouvée pour les réciprocants *purs*, suffit à établir immédiatement que ces réciprocants seront nécessairement, et sans exception aucune, ou homogènes ou séparables en parties homogènes, dont chacune sera elle-même un réciprocant.

37.

NOTE ON SCHWARZIAN DERIVATIVES.

[*Messenger of Mathematics*, XV. (1886), pp. 74—76.]

READING with great pleasure and profit Mr Forsyth's masterly treatise on Differential Equations (in my opinion the best written mathematical book extant in the English language), it occurred to me to find an easy proof of the fundamental and striking identity concerning Schwarzian derivatives, from which all others are immediate consequences, namely $(y, x) - (z, x) = \left(\frac{dz}{dx}\right)^2 (y, z)$, where one of which is, it may be observed, that (y, x) like y'' has the property of remaining a factor of what it becomes when x and y are interchanged; a persistent factor, so to say, of its altered self. I will return to this point subsequently, my present concern is to give a natural proof of the above striking identity; to do this, it will be sufficient to show that (considering y, z, x, the two former as fixed, and the last as a variable function of a common variable) $\dfrac{(y, x) - (z, x)}{\left(\frac{dz}{dx}\right)^2}$ does not vary when x becomes $x + \epsilon\phi(x)$ where ϵ may be regarded as infinitesimal*. For then this must remain true by successive accumulation when x becomes any function whatever of itself, and accordingly making $x = z$ we obtain (y, z) as the value of the invariable quotient as was to be shown. Call† $\epsilon\delta\phi x = \theta$, then using dashes to denote differentiation *quâ* x, and a parenthesis to signify the augmented value of the derivatives, we obtain

$$(y') = y' - \theta y',$$
$$(y'') = y'' - 2\theta y'' - \theta' y',$$
$$(y''') = y''' - 3\theta y''' - 3\theta' y'' - \theta'' y'.$$

* It is easy to see *à priori* that if the theorem is true, it can only be so in virtue of (y, x) when x receives an infinitesimal, becoming of the form

$$(1 - 2\theta)(y, x) + \lambda\theta'',$$

as is subsequently shown to be the case in the text.

[† Cf. p. 306 below.]

Hence
$$(y'y''') = y'y''' - 4\theta y'y''' - 3\theta' y'y'' - \theta'' y'^2,$$
$$\tfrac{3}{2}(y''^2) = \tfrac{3}{2}y''^2 - 6\theta y''^2 - 3\theta' y'y'',$$
$$(y')^2 = y'^2 - 2\theta y'^2,$$
$$((y, x)) = (1 + 2\theta)\{(y, x) - 4\theta(y, x)\} - \theta''$$
$$= (1 - 2\theta)(y, x) - \theta'',$$

and
$$((y, x) - (z, x)) = (1 - 2\theta)[(y, x) - (z, x)].$$

Hence
$$\left(\frac{(y, x) - (z, x)}{\left(\frac{dz}{dx}\right)^2}\right) = \frac{(y, x) - (z, x)}{\left(\frac{dz}{dx}\right)^2},$$

that is, the right-hand expression does not change, when y, z remaining fixed forms of function, x passes from one form of function of the independent variable to another; as was to be shown.

From what precedes, it appears that if y, z, x be regarded as functions of t, then $\{(y, x) - (z, x)\}\left(\frac{dx}{dt}\right)^2$ is a constant function in the sense that it remains unaltered, whatever function x may be of t, or which is the same thing if y and z functions of x when expressed as functions of x' (any function of x) are written y', z', then $(y', x') - (z', x')$ is identical with $(y, x) - (z, x)$, save as to a *factor* which depends only on the form of the *substitution* of x' for x. Hence to all intents and purposes, any function of the differences of the Schwarzian derivatives of any system of functions of the same variable, in respect thereto, is (in a sense comprising, but infinitely transcending the sense in which that word is used in Algebra) a *covariant* of the system.

ADDENDUM.—Let us for the moment call functions of x, y which either remain unaltered or only change their sign when x and y are interchanged self-reciprocating functions.

The first case of the kind is $\dfrac{y''}{y'^{\frac{3}{2}}}$, the next is $\dfrac{y'y''' - y''^2}{y'^2}$, and obviously a very general one of this sort will be the function

$$\left(\frac{1}{y'^{\frac{1}{2}}}\frac{d}{dx}\right)\log y'.$$

For greater simplicity, let us call the *numerator* of any such function when expanded and brought to the lowest possible common denominator, a *reciprocant*, the highest index of differentiation which such reciprocant contains its *order*, and the number of factors in each term its *degree*. Then in any reciprocant so formed the degree is always just one unit less than the order: but as a matter of fact the function so obtained is in general not irreducible, so that its degree may be depressed, and it becomes a question of much interest to form the scale of degrees of reciprocants of this sort. For the orders 2, 3, 4, 5, 6 the degrees in question are respectively 1, 2, 2, 3, 3. Calling the successive derivatives of y, a, b, c, d, ..., they will be found to be

$$a,$$
$$b,$$
$$2ac - 3b^2,$$
$$ad - 5bc,$$
$$2a^2e - 15acd - 10ad^2 + 35b^2c,$$
$$2a^2f - 21abc - 35acd + 60ab^2d + 110bc^2,$$

where each form is obtained by operating upon the preceding one with the operator $a(b\delta c + c\delta d + d\delta e + \dots) - \lambda b$ (λ meaning half the weight + the degree of the operand), combining the result of this operation in each *alternate* case with a *legitimate* combination of those that precede, and in that case dividing out by a. I have proved that in this way can be obtained an infinite progression of reciprocants, of which the leading terms (substituting numbers for letters), will be alternately of the forms $1^i.(2i+1)$ and $1^i.(2i+2)$. Every other reciprocant can be formed algebraically from these primordial forms, as every seminvariant can be obtained from the primordial forms a, $ac - b^2$, $a^2d - 3abc + 2b^3$, ... The two theories run in parallel courses, but their relationship is that which naturalists call homoplazy as distinguished from homogeny; I propose to give further developments of this new algebraical theory in a subsequent Note.

38.

ON RECIPROCANTS.

[*Messenger of Mathematics*, XV. (1886), pp. 88—92.]

In a note on Invariant Derivatives in the September number of the *Messenger* I have given a definition and examples of reciprocants.

If in any of the forms at the end of the postscript to the note we restore to a, b, c, ... their values $\delta_x y$, $\delta_x^2 y$, $\delta_x^3 y$, ... any such function divided by a certain power of $\delta_x y$ will change its sign, but otherwise remain unaltered when x and y are interchanged. The index of that power is the degree added to half the weight and will be called the index of the reciprocant. Any product of i of such reciprocants will be a reciprocant of the same kind or contrary kind to those in the table (subsequent to a) according as i is odd or even. In the latter case the interchange of x and y will leave the function absolutely unaltered. Reciprocants which cause a change of sign will be said to be of an odd, those which cause no change of sign of an even character. Any linear function of reciprocants of the same weight, degree, and *character* will be itself a reciprocant of that character, but reciprocants of opposite characters cannot be combined to form a new reciprocant: those of an odd character may be regarded as analogous to skew, those of an even character to non-skew seminvariants; the rule against combining forms of opposite characters becomes superfluous in the case of seminvariants, because those that offer themselves for combination as having the same weight and degree must of necessity be of like character. Any reciprocant being given there is a simple *ex post facto* rule for assigning its character without any knowledge of the mode of its genesis, namely its character is odd or even according as the smallest number of letters other than a in any of its terms is odd or even. Thus the *character* of a reciprocant whose leading term is a^2e, or ab^2e, or $abce$ is odd; that of one whose *leading* term is abe or abf is even, as is also that of the remarkable reciprocant $bd - 5c^2$ in which no power of a appears.

A further important distinction between the two theories* is that there are two linear reciprocants a and b but only one linear seminvariant. As an illustration of the combinatorial *law* of *like character* it will be seen that if we operate upon $2ac - 3b^2$ with the operator

$$a(b\delta a + c\delta b) - 3b,$$

* That is of reciprocants and invariants.

we obtain a new reciprocant

$$2ad - 10bc + 9b^3,$$

of which the character is the same as that of b^3, namely both are odd; we may therefore add $-9b^3$ to the latter expression, and then dividing out by $2a$ there results the reciprocant $ad - 5bc$, but we cannot combine $2ac - 3b^2$ with b^3 because these two reciprocants are of opposite characters.

Again, remembering that a is of an even and b of an odd character, the three reciprocants

$$-\tfrac{45}{4}b^4, \quad 5\,(ac - \tfrac{3}{2}b^2)^2, \quad 3ab\,(ad - 5bc)$$

are all of an even order, hence we may add them together and divide the sum by a^2, which gives the new reciprocant $3bd - 5c^2$ a form not containing the first letter a.

No seminvariant exists, nor, except the one just given $bd - 5c^2$, have I been able to discover any other reciprocant in which the first letter does not make its appearance†.

The infinite progression of odd reciprocants with the leading terms

$$ac, \quad ad, \quad a.a.e, \quad a.a.f, \quad a.a.a.g, \quad a.a.a.h, \quad \ldots$$

will easily be seen to exist by virtue of the general theorem that any reciprocant of degree, extent, and weight (say briefly of *dew* i, j, w) gives birth to two others of the same character as its own, one of *dew* $i+1, j+1, w+2$, the other of *dew* $i+1, j+2, w+3$.

For let $$\tfrac{1}{2}w + i = \lambda,$$

then denoting the operator

$$b\delta_a + c\delta_b + \ldots \text{ by } \Omega,$$

and the result of the action of Ω upon itself $(\Omega *)^2$, which is in fact $\Omega^2 + \Omega_2$ (Ω_2 meaning $c\delta_a + d\delta_b + \ldots$); $(a\Omega - \lambda b)\,R$ will obviously be a reciprocant of *dew* $i+1, j+1, w+2$, and will give rise to a second reciprocant

$$\{a\Omega - (\lambda + \tfrac{3}{2})\,b\}\,(a\Omega - \lambda b)\,R,$$

which is $$a^2\,(\Omega *)^2 - (2\lambda + \tfrac{1}{2})\,ab\Omega R - \lambda acR + (\lambda^2 + \tfrac{3}{2}\lambda)\,b^2R;$$

the last term of this being a reciprocant of the same character as the entire expression may be omitted, and dividing out the residue by a we obtain the second new reciprocant

$$\{a\,(\Omega *)^2 - (2\lambda + \tfrac{1}{2})\,b - \lambda c\}\,R,$$

which will be of *dew* $i+1, j+2, w+3$, as was to be shown.

It is easy to see that every reciprocant must be a rational integral function of the forms above stated commencing with $a, b, 2ac - 3b^2$ (whose *dew*'s are alternately of the form $i, 2i-1, 3i-2$; $i, i-2, 3i-1$) divided by some power of a. For if any reciprocant contains only the letters $a, b, \ldots$

† Since the above went to press I have made the capital discovery that there are an infinite number of such reciprocants, and that all those of a given weight, extent and degree may be obtained by aid of a certain quadratico-linear partial differential equation.

h, k, l, it may be expressed as a rational integral function of the protomorph in which l first appears and of the letters $a, b, \ldots k$ divided by a power of a, and consequently the reciprocant may be so expressed, and continually repeating this process of substitution it follows that the reciprocant will be a rational integer of the protomorphs exclusively divided by a power of a*: this of course will necessarily be found only to contain combinations of like character; we already know the converse that the sum of all combinations of like character of the protomorphs is a reciprocant†. If any homogeneous reciprocant consists of portions of unlike degree (although of the same index) it is obvious that each portion must be itself a reciprocant, for if $P, P', P'' \ldots$ be such portions, $P + P' + P'' \ldots$ must be identical with $\Pi + \Pi' + \Pi'' + \ldots$ when $\Pi, \Pi', \Pi'' \ldots$ are the same functions of $\alpha, \beta, \gamma \ldots$ (that is, $\delta_y x, \delta_y{}^2 x, \delta_y{}^3 x \ldots$) that $P, P', P'' \ldots$ are of $a, b, c \ldots$. If then we make

$$P - a^{2\lambda}\Pi = \Delta, \quad P' - a^{2\lambda}\Pi' = \Delta' \ldots,$$

we have $\Delta + \Delta' + \Delta'' + \ldots$ identically zero.

But $P, P' \ldots$ being of the same index but different degrees must be of different weights, and consequently $\Delta, \Delta', \ldots$ are of different weights. Hence we must have $\Delta = 0, \Delta' = 0$, &c., as was to be shown.

It follows from this that every reciprocating function whatever may be obtained by an algebraical combination of the protomorphs, and consequently by an algebraical combination of the forms

$$\left(\frac{1}{y'^{\frac{1}{2}}}\delta_x\right)^i \log y'.$$

* The proof that every seminvariant is a rational integral function of the protomorphs is very similar: any proposed seminvariant is by the method employed in the text shown to be at worst a function of the protomorphs and of b; but the terms involving any power of b must disappear because no identical equation can connect seminvariants with a non-seminvariant b. In the text we see in like manner that any given reciprocant may be reduced to the form $H + K$, where H and K are protomorphic combinations of opposite character, so that one of them will disappear.

† Another general mode of generating a class of reciprocants would be to express any function of $a, b, c, \ldots$ say $\phi(a, b, c, \ldots)$ under the form $\psi(\alpha, \beta, \gamma, \ldots)$. The product $\phi(a, b, c, \ldots)\,\psi(a, b, c, \ldots)$, or its numerator, will then obviously be a reciprocant. To take a simple example,

$$c = \frac{d^3y}{dx^3} = -\frac{\dfrac{dx}{dy}\cdot\dfrac{d^3x}{dy^3} - 3\left(\dfrac{d^2x}{dy^2}\right)^2}{\left(\dfrac{dx}{dy}\right)^5} = -\alpha\gamma + 3\beta^2 \div \alpha^5.$$

Hence, by the rule laid down, $c\,(ac - 3b^2)$, that is, $ac^2 - 3b^2c$ ought to be a reciprocant, which is right, for it is equal to $(2ac - 3b^2)^2 - 9b^4$ divided by a multiple of a. The law that the factors of seminvariants must be seminvariants cannot be extended to the theory of reciprocants. In this case the factors may some or none of them be reciprocants, and the others on reciprocation exchange forms monocyclically or polycyclically with one another. *I add the remark that this is not true of pure reciprocants, that is, those in which* $\frac{dy}{dx}$ *does not appear. Every factor of a pure reciprocant must be itself a reciprocant.*

and that we should gain nothing in generality by operating with successive operators of the form

$$\left(\frac{1}{y'^{\frac{1}{2}}}\delta_x\phi_1\cdot\right),\quad \left(\frac{1}{y'^{\frac{1}{2}}}\delta_x\phi_2\cdot\right),\ \ldots$$

where $\phi_1, \phi_2, \ldots$ are arbitrary functions of $y' \pm \frac{1}{y'}$ instead of with the simple operator $\frac{1}{y'^{\frac{1}{2}}}\delta_x$ continually repeated.

The results of using the more general operators would only amount to algebraical combinations of the results obtained from the simple forms

$$\left(\frac{1}{y'^{\frac{1}{2}}}\delta_x\right)^i \log y',$$

where i may take all values from zero to infinity*.

As in the case of seminvariants so also reciprocants would in extent contain only a finite number of ground-forms; but furthermore for reciprocants limited in degree the number of ground-forms will also be finite. Whether reciprocants which are irreducible for a given extent ever cease to be so and become reducible when the order is increased, as is the case with seminvariants, remains to be seen†.

In order to facilitate the verification of the results obtained and to be obtained it may be well to express the successive derivatives of x in regard to y in terms of those of y in regard to x, that is, of $\alpha, \beta, \gamma, \ldots$ in terms of $a, b, c, \ldots$ as shown in the following short table.

$$\begin{array}{ll}
 & \div \\
a = \alpha & \alpha^2, \\
b = -\beta & \alpha^3, \\
c = -\alpha\gamma + 3\beta^2 & \alpha^5, \\
d = -\alpha^2\delta + 10\alpha\beta\gamma - 15\beta^3 & \alpha^7, \\
e = -\alpha^3\epsilon + 15\alpha^2\beta\delta + 10\alpha^2\gamma^2 - 105\alpha\beta^2\gamma + 105\beta^4 & \alpha^9, \\
f = -\alpha^4\zeta + 21\alpha^3\beta\epsilon + 35\alpha^3\gamma\delta - 210\alpha^2\beta^2\delta - 280\alpha^2\beta\gamma^2 + 1260\alpha\beta^3\gamma - 945\beta^5 & \alpha^{11}, \\
\left.\begin{array}{l} g = -\alpha^5\eta + 28\alpha^4\beta\zeta + 56\alpha^4\gamma\epsilon + 35\alpha^4\delta^2 - 378\alpha^3\beta^2\epsilon - 1260\alpha^3\beta\gamma\delta + 3150\alpha^2\beta^3\delta \\ \qquad - 280\alpha^3\gamma^3 + 6300\alpha^2\beta^2\gamma^2 - 17325\alpha\beta^4\gamma + 10395\beta^6 \end{array}\right\} & \alpha^{13},
\end{array}$$

where $a, b, c, d, e, f, \ldots$ represent the successive derivatives of y with respect to x; and $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \ldots$ of x with respect to y.

In any subsequent paper on reciprocants in this Journal, I shall make the absolutely necessary transliteration referred to in a preceding footnote, replacing the present letters $a, b, c, d, \ldots$ by the letters $t, a, b, c, \ldots$ or possibly, for reasons which carry great weight, by the expressions

$$t,\quad 2a,\quad 2.3b,\quad 2.3.4c, \ldots$$

* This is not true of homogeneous reciprocants.

† I have since found that this *is* true for reciprocants, as for seminvariants.

39.

NOTE ON CERTAIN ELEMENTARY GEOMETRICAL NOTIONS AND DETERMINATIONS.

[*Proceedings of the London Mathematical Society*, XVI. (1885), pp. 201—215.]

A CURVE, as every one knows, may be regarded as a *locus* of points or as an *assembly* of directions, every point being common to two consecutive directions of the assembly, and every direction to two consecutive points of the locus; the locus is called the *envelop* of the assembly (that is part of the accepted language of geometry), and, conversely, the assembly may be called the *environment* of the locus. So we may regard a surface as an assembly of tangent planes or as a locus of points standing to each other in the relation of envelop and environment, and extend these definitions to space of any number of dimensions.

By a *plasm*, waiting a better word, we may understand a figure analogous to a point-pair in a line, a triangle in a plane, a pyramid in space, etc.; and an n-gonal plasm or n-gon will signify a plasm having n vertices and n faces themselves $(n-1)$-gons.

It is easy and desirable to find the general value of the content of a regular n-gon, say *abcde*, all whose edges we may call unity.

If $$b\beta = \tfrac{1}{2}ab,\quad c\gamma = \tfrac{2}{3}c\beta,\quad d\delta = \tfrac{3}{4}d\gamma \ldots,$$

it is easily seen by an elementary process of integration that $\beta, \gamma, \delta \ldots$ are the centres of figure to the successive plasms ab, abc, $abcd$, ..., and, making

$$ba = p_1,\quad c\beta = p_2,\quad d\gamma = p_3 \ldots,$$

each term in $p_1, p_2, p_3 \ldots$ will be perpendicular to the one which precedes it, so that, if V_n is the content of the plasm,

$$(1, 2, 3 \ldots n)^2\, V_n = p_1 p_2 \ldots p_n.$$

Moreover, we shall have

$$p_n^2 = 1 - \left(\frac{n-1}{n}\right)^2 p^2_{n-1},$$

of which the general integral is

$$p_n^2 = \frac{n+1}{2 \,.\, n} + C(-)^n \frac{1}{n^2};$$

in the present case, since $p_1 = 1$, $C = 0$, so that

$$V_n^2 = \frac{n+1}{(1 \,.\, 2 \ldots n)^2\, 2^n}.$$

If a, b, c be the angles of a fixed triangle, and A, B, C are proportional to the distances of a variable line from a, b, c, respectively, we may denote the line by $A : B : C$; as regards a variable point, it will presently be seen to be advantageous to denote its proportional coordinates, not, as is rather more usually done, by equimultiples of its distances from the three sides, but as equimultiples of these distances multiplied by the sides of the triangle from which they are measured*; so that, calling these coordinates a, b, c, the image† of the line at infinity becomes $a + b + c$.

Consider now the universal mixed concomitant (which it will be convenient to call a *mutuant*) $Aa + Bb + Cc$ (where a, b, c, A, B, C are used in lieu of the more usual letters $x, y, z, \xi, \eta, \zeta$); it will readily be seen that, when a, b, c vary, and A, B, C are fixed, the mutuant images the line $A : B : C$, and that, when A, B, C vary and a, b, c are fixed, the mutuant images the *radiant* point $a : b : c$; that is to say, $Aa + Bb + Cc = 0$ is true for every point in the point-containing line $A : B : C$ in the one case, and to every line through the *radiant point* $a : b : c$ in the other.

Supposing, then, that the two kinds of coordinates are chosen in this manner, we see (what would not be the case if the simple distances were taken) that a form F and its "polar-reciprocal" ϕ image the self-same curve referred to the self-same fundamental triangle.

These consequences would moreover continue to subsist if, calling the distances of a line from the vertices P, Q, R, and of a point from the sides p, q, r, we took $\Lambda P : \mathrm{M}Q : \mathrm{N}R$, $\lambda p : \mu q : \nu r$ for the two sets of coordinates, provided only that $\lambda \Lambda F = \mu \mathrm{M} G = \nu \mathrm{N} H$; F, G, H being the distances of the sides from the vertices of the fundamental triangle, in which case the line at infinity would no longer be imaged by $a + b + c$. I shall, however, adhere in what follows to the convention above laid down. I need hardly add that in like manner, in space taking $A : B : C : D$ (the distances of a plane from the

* Or rather divided by the distances of these sides from the opposite angles of the fundamental triangle, whose coordinates thus become 1, 0, 0, 0, 1, 0, 0, 0, 1.

† If $F = 0$ is the equation to any *locus* or *assembly*, I call F the *image*, and such locus or assembly the *object*; to a given image responds in general an absolutely definite object, but, when the object is given, the image is only determined to a *constant* factor *près*.

vertices of a fundamental pyramid) as the coordinate-representation of such plane, and $a : b : c : d$ (the contents of the volumes which any variable point makes with the respective faces) as the coordinate-representation of such point, the mutuant $aA + bB + cC + dD$ will be the image of the radiant point $a : b : c : d$ when the capital letters are the variables, and of the plane $A : B : C : D$ when the small letters are the variables, meaning of course that $Aa + Bb + Cc + Dd = 0$ will be true of every point in the plane $A : B : C : D$ and of every plane through the point $a : b : c : d$, and, as before, F and ϕ polar-reciprocals to each other will image the self-same surface (referred to the self-same fundamental pyramid) viewed as a locus or envelop on the one hand, as an assembly or environment on the other.

If a, b, c, d be used to signify the actual as distinguished from the proportional coordinates of a point, a linear function of these is constant, whereas it is a quadratic function of $A, B, C, D \ldots$, when used to signify the actual distances of a variable line, plane, &c., from the vertices of the fundamental plasm which is constant; and it is the principal object of this note to determine the form of this quadratic function, which, as Prof. Cayley was the first to show, may be expressed by the determinant to a matrix standing in close relation to the well-known "invertebrate symmetrical matrix," the determinant to which represents a numerical multiple of any plasm in terms of its edges, as, for example:

$$\begin{vmatrix} . & ab & ac & ad & 1 \\ ba & . & bc & bd & 1 \\ ca & cb & . & cd & 1 \\ da & db & dc & . & 1 \\ 1 & 1 & 1 & 1 & . \end{vmatrix}$$

where $ab, ac, bc \ldots$ are used for brevity to signify the measure of absolute distance between a, b, a, c, $b, c \ldots$, that is, stand for what in ordinary notation would be denoted by $(ab)^2, (ac)^2, (bc)^2, \ldots$. This may be quoted as the mutual-distance matrix; its determinant, besides representing a numerical multiplier of the squared content of the pyramid when equated to zero, expresses the conditions of the four points a, b, c, d lying in a plane, the former property being a consequence immediately deducible by *strict* algebraical reasoning from the latter.

That this determinant does image the condition of the plasm to which the points $a, b, c, d \ldots$ are the vertices, losing one dimension of space, may be shown in a somewhat striking manner as follows. If for a moment we use x, y, z, the distances of any point in the plane of abc from bc, ca, ab as coordinates, the equation to a circle circumscribed about abc will be of the form $fyz + gzx + hxy$, and, calling the sides of the triangle a, b, c respectively,

$ax+by+cz$ is constant. Hence, substituting for z its value in terms of x and y, the image of the circle may be put under a form in which fb and ga will be the coefficients of y^2 and x^2 respectively; but, since x and y are *proportional* to the Cartesian coordinates y and x respectively, the coefficients of x^2 and y^2 must be equal. Hence $f:g:h::a:b:c$, and if now ax, by, cz, instead of x, y, z, be used as the coordinates of the variable point, the image to the circumscribing circle becomes $\Sigma \frac{ayz}{bc}$, or if we please Σa^2yz, that is, $\Sigma bcyz$, where bc stands as convened for $(bc)^2$.

Hence, if a, b, c, d be the vertices of a pyramid, $\Sigma abyz$ will be the image of the circumscribing sphere, for when any coordinate t is made zero the image becomes that of a circle; and so universally for a plasm of any number of dimensions.

Consider the case of a circle, and suppose that

$$\begin{vmatrix} . & ab & ac & 1 \\ ba & . & bc & 1 \\ ca & cb & . & 1 \\ 1 & 1 & 1 & . \end{vmatrix}$$

vanishes; this means that the line $x+y+z$ touches the circle

$$abxy+bcyz+cazx.$$

But, if $x+y+z$ images the line at infinity, it must *cut* this (as it cuts any other circle) in two distinct points, namely, the so-called circular points at infinity. Hence $x+y+z$ must, when the above determinant vanishes, cease to be the line at infinity, which can only come to pass by the triangle abc losing a dimension of space, and a, b, c coming into a straight line, in which case $x+y+z=0$, instead of being true of a particular line, is true of every point in the plane.

Just in like manner, if

$$\begin{vmatrix} . & ab & ac & ad & 1 \\ ba & . & bc & bd & 1 \\ ca & cb & . & cd & 1 \\ da & db & dc & . & 1 \\ 1 & 1 & 1 & 1 & . \end{vmatrix}$$

vanishes, unless $x+y+z+t$ ceases to image the plane at infinity, this plane would touch the sphere $\Sigma abxy$, that is, would cut it in a pair of straight lines, whereas it intersects it in a circle. Consequently the plasm $abcd$ must, as before, lose one dimension, and so in general. The content of a plasm vanishes when the mutual-distance determinant does so, and the latter as well as the former may be expressed rationally in terms of ordinary Cartesian coordinates; but the expression for the content (being linear in each set of coordinates) is obviously indecomposable, and must therefore be a numerical multiple of some power of the mutual-distance determinant; a comparison of dimensions shows at once that this power is the square root.

As regards the numerical multiplier, when the plasm has all its edges equal to unity (say a triangle, for example), the mutual-distance determinant becomes

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix},$$

which is easily transformable into

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & \bar{1} & 0 & 0 \\ 1 & 0 & \bar{1} & 0 \\ 1 & 0 & 0 & \bar{1} \end{vmatrix},$$

of which the value is -3; and so in general for a regular plasm with $(n+1)$ vertices; that is, in space of n dimensions the mutual-distance determinant, say D_n, becomes $(-)^{n+1}(n+1)$, whereas the $(\text{volume})^2$, say V_n^2, has been shown to be $\frac{n+1}{2^n(1.2\ldots n)^2}$.

Hence, universally,

$$D_n=(-)^{n+1}2^n(1.2\ldots n)^2V_n^2.$$

It may be here noticed that, if p be the perpendicular from any vertex on an opposite face of the plasm whose content is V_{n-1}, we shall have

$$V_{n-1}p=nV_n.$$

Consequently, $$D_{n-1}p^2=(-)^n2^{n-1}\{1.2\ldots(n-1)\}^2V^2_{n-1}p^2$$
$$=(-)^n2^{n-1}(1.2\ldots n)^2V_n^2=-\tfrac{1}{2}D_n.$$

I now pass on to the leading motive of this note, namely, the determination of the connection between the coordinates A, B, C ... drawn from a, b, c ....

It is clear *à priori* that the form of the condition will be in all cases that a homogeneous quadratic function of the distances must be constant. Thus, for example, when there are four points, if A, B, C be assumed, we may describe three spheres with these quantities as radii, and the fourth point will be determined by means of one of the pairs of tangent planes drawn to them, the particular pair depending on the relative signs attributed to

A, B, C. Hence, if $F(A, B, C, D) = \infty$ be the general equation, each of the quantities must enter in the second and no higher degree; moreover, since by transporting the plane from which the distances are measured parallel to itself, A, B, C, D will be all increased by the same quantity, F must express a function of their differences, and consequently, since any two distances may be interchanged, F can contain no terms of the first order in the variables, so that $F = 0$ must amount to the predication of a homogeneous quadratic function of the distances being constant.

Thus, for example, in the case of three points, we have the well-known equation

$$\Sigma (ab)(A - C)(B - C) = \tfrac{1}{4}(abc)^2.$$

Suppose now that A, B, C are taken in proportions consistent with making

$$\Sigma (ab)^2 (A - C)(B - C) = 0.$$

Let $\Sigma (ab)^2 (A - C)(B - C) = P \,.\, Q$, where P, Q are two linear functions of A, B, C; then P, Q image two radiant points, each of which will have the property that any of its rays is at an infinite distance from a, b, c, or at all events, if it should pass through one of them, from the other two, and it is easy to anticipate that these two points must be the circular points at infinity. That such is the fact is obvious, because (using Cartesian co-ordinates) the perpendicular distance from any point upon $x \pm \sqrt{-1} \,.\, y$ contains zero in its denominator; so that the two points of the absolute may be regarded as the centres of two points of rays, all of them infinitely distant from the finite region.

But these two points are the intersections of the circumscribing circle with the line at infinity, and consequently their collective equation will be found by taking the resultant of $\Sigma ab\,xy$, Σx, ΣAx, which is well known to be the determinant of the quadratic function bordered by the coefficients of the two linear ones. Hence the constant quadratic function in A, B, C, namely, $\Sigma ab (A - B)(A - C)$, ought to be a numerical multiple of the determinant

$$\begin{vmatrix} . & A & B & C & . \\ A & . & ab & ac & 1 \\ B & ba & . & bc & 1 \\ C & ca & cb & . & 1 \\ . & 1 & 1 & 1 & . \end{vmatrix},$$

as is the case, the value of this determinant being

$$-2\Sigma ab (A - C)(B - C).$$

The same thing may be shown in a more elementary manner as follows. Combining

$$x + y + z = 0, \quad abxy + bcyz + cazx = 0,$$

we have
$$acx^2 + (bc + ca - ab)\,xy + bcy^2 = 0,$$
at each point of the absolute. And, taking $x_1y_1z_1$, $x_2y_2z_2$ as the coordinates at these two points, it follows that

$$x_1x_2 : y_1y_2 : z_1z_2 : x_1y_2 + x_2y_1 : y_1z_2 + y_2z_1 : z_1x_2 + z_2x_1$$
$$:: bc : ca : ab : -bc - ca + ab : -ca - ab + bc : -ab - bc + ca.$$

And, as the two points will be imaged by

$$x_1A + y_1B + z_1C, \quad x_2A + y_2B + z_2C,$$

respectively, it follows that their collective image will be

$$\Sigma \{bcA^2 + (bc - ab - ac)\,BC\},$$

which is easily seen to be identical with

$$\Sigma bc (A - B)(A - C).$$

The universal algebraical theorem upon which the first method of proof depends is the well-known one that, if Q is a quadratic function and $L_1, L_2, \ldots L_i$ i linear functions of j variables, and if Q' (where j is not less than $i + 1$) is what Q becomes when i of its variables are expressed in terms of the rest, then the necessary and sufficient condition of the discriminant of *every* such Q' vanishing is that the determinant to Q bordered by the coefficients of the i linear functions shall vanish. When j is equal to $i + 1$, the theorem shows that the resultant of the quadratic and its i attendant linear functions will be the bordered determinant in question. In the above example we had $j = 3$, $i = 2$.

Let us now proceed to apply a similar principle to the case of four points a, b, c, d in space.

If we take the case $x^2 + y^2 + z^2 + t^2 = 0$, any tangent plane to it at x', y', z', t' will be

$$x'x + y'y + z'z + t't,$$

and, as
$$x'^2 + y'^2 + z'^2 + t'^2 = 0,$$

it follows that every tangent plane will be at infinite distance from any point external to it; and, as this is true wherever the centre of the cone be placed, and all the cones so obtained have the "circle at infinity" in common, —it follows that every tangent plane to the circle at infinity is infinitely distant from any external point in the finite region,—the infinitely-infinite system of planes thus obtained one may regard, if one pleases, as consisting of sheaves of planes whose axes form the environment to the circle at infinity, and will be the correlative to the infinitely-infinite system of points in the plane at infinity, which are infinitely distant from all external planes in the finite region. We see, then, that the coordinates to each such plane must satisfy the condition that, on making $\Sigma x = 0$ and $\Sigma Ax = 0$, and expressing any two of the variables x, y, z, t in terms of the two others, the discriminant of the form then assumed by $\Sigma abxy$ must vanish, and consequently, as before, the mutual-distance determinant to the points a, b, c, d, bordered with a row and column of units and a row and column consisting of the letters A, B, C, D, will represent to a numerical factor *près* the constant quadratic function of distances, that is, this function will be

$$\begin{vmatrix} . & A & B & C & D & . \\ A & . & ab & ac & ad & 1 \\ B & ba & . & bc & bd & 1 \\ C & ca & cb & . & cd & 1 \\ D & da & db & dc & . & 1 \\ . & 1 & 1 & 1 & 1 & . \end{vmatrix},$$

and obviously a similar algebraical conclusion will continue to apply, whatever may be the number of points n in a space of $n-1$ dimensions.

As regards the value of the constant, in any case, that may be obtained by taking a face of the plasm as the *term* (line, plane, etc.) from which the distances $A, B, C \ldots$, are measured; that is, we may make $B=0$, $C=0$, $D=0 \ldots$, provided we make A equal to the perpendicular from a on the opposite face. The value of the bordered determinant then becomes the *negative* of the squared perpendicular from a on $bcd \ldots$ multiplied by the mutual-distance determinant to $bcd \ldots$; that is, by virtue of what has previously been shown, will be half of the mutual-distance determinant of $abcd \ldots$.

Hence the complete relation between A, B, C, D may be exhibited by making

$$\begin{vmatrix} -\tfrac{1}{2} & A & B & C & D & . \\ A & . & ab & ac & ad & 1 \\ B & ba & . & bc & bd & 1 \\ C & ca & cb & . & cd & 1 \\ D & da & db & dc & . & 1 \\ . & 1 & 1 & 1 & 1 & . \end{vmatrix} = 0,$$

and similarly for any number of points.

Professor Cayley has obtained the same result by a more direct but not more instructive process, as follows. Taking, by way of example, three points, $A+k$, $B+k$, $C+k$, (where k is infinite,) may be regarded as the distances of a, b, c from a fourth point at an infinite distance, and accordingly we may write

$$\begin{vmatrix} . & ab & ac & (A+k)^2 & 1 \\ ba & . & bc & (B+k)^2 & 1 \\ ca & cb & . & (C+k)^2 & 1 \\ (A+k)^2 & (B+k)^2 & (C+k)^2 & . & 1 \\ 1 & 1 & 1 & 1 & . \end{vmatrix} = 0.$$

For the *gnomon* bordering the square formed by the small letters and dots, we may substitute

$$\begin{vmatrix} . & . & . & 2kA+A^2 & 1 \\ . & . & . & 2kB+B^2 & 1 \\ . & . & . & 2kC+C^2 & 1 \\ 2kA+A^2 & 2kB+B^2 & 2kC+C^2 & -2k^2 & 1 \\ 1 & 1 & 1 & 1 & . \end{vmatrix},$$

without altering the value of the determinant, which therefore, remembering that k is infinite, is in a ratio of equality to $(2k)^2$ multiplied into the determinant

$$\begin{vmatrix} . & ab & ac & A & 1 \\ ba & . & bc & B & 1 \\ ca & cb & . & C & 1 \\ A & B & C & -\tfrac{1}{2} & . \\ 1 & 1 & 1 & . & . \end{vmatrix}.$$

This last determinant therefore must vanish, agreeing with what has been shown above by a more purely geometrical method*. I will now proceed to develop this determinant deprived of its constant term, expressing it as a function of the differences of the capital letters.

It is obvious that it may be expressed as a sum of terms of which each variable part will be of one or the other of these three forms

$$(A-B)^2, \quad (A-B)(A-C), \quad (A-B)(C-D);$$

and accordingly we may distribute the totality of the terms of the constant function of difference into three families depending on the form of the variable argument.

In general, if we consider any *invertebrate* symmetrical determinant expressed by the *umbral* notation

$$\begin{vmatrix} aa & ab & ac \ldots al \\ ba & bb & bc \ldots bl \\ \ldots & \ldots & \ldots \quad \ldots \\ la & lb & lc \ldots ll \end{vmatrix},$$

* As a corollary, we may infer, from the vanishing of this determinant, that, using the notation previously employed,

$$\frac{D_n}{V_n^2} = -\tfrac{1}{2}n^2 \frac{D_{n-1}}{V^2_{n-1}},$$

and consequently that $$D_n = -(2)^n (1 \,.\, 2 \ldots n)^2 V_n^2,$$

and that thus the content of a regular plasm with unit edges and $(n+1)$ vertices is

$$\frac{n+1}{2^n (1 \,.\, 2 \ldots n)^2}, \text{ namely, } \frac{3}{16}, \frac{1}{72}, \frac{5}{9 \,.\, 2^{10}} \cdots$$

for triangle, pyramid, plu-pyramid, etc.

where $aa = bb = cc = ll \ldots = 0$ and $pq = qp$, we have this simple rule of proceeding:

Divide the letters $a \ldots l$ in every possible manner into cyclical sets, each set containing at least two letters.

Any cycle $a_1 a_2 \ldots a_i$ is to be interpreted as meaning

$$a_1 a_2 \, . \, a_2 a_3 \ldots a_{i-1} a_i \, . \, a_i a_1,$$

which, by virtue of the supposed condition $ab = ba$, will be the same in whichever direction the cycle is read, the effect of the inversion of the cycle being merely to give the same product over again, written under the form $a_1 a_i \, . \, a_2 a_1 \ldots a_1 a_{i-1}$.

The cycle of two letters $a_1 a_2$ must be interpreted to mean $(a_1 a_2)^2$. If now $C_1 C_2 \ldots C_i$ are cycles of two letters each, and $\chi_1 \chi_2 \ldots \chi_j$ cycles of three or more letters, the total value of the determinant will be

$$\Sigma (-)^{n+i+j} \, 2^j \, C_1 C_2 \ldots C_i \chi_1 \chi_2 \ldots \chi_j.$$

If, the principal diagonal terms remaining zero, the other terms were general, then the expression of the value of the determinant, calling the cycles $C_1 C_2 \ldots C_\nu$, and making no distinction between the case of their being binary or super-binary, would be $\Sigma (-)^{n+\nu} C_1 C_2 \ldots C_\nu$; only it would have to be understood that each cycle of two letters, as (ab), would mean $(ab)^2$, but a cycle of three or more letters, as (abc), would mean $ab \, . \, bc \, . \, ca + ac \, . \, cb \, . \, ba$.

This being premised, it is easy to deduce the following rule for the determination of the *three* different families of terms belonging to the constant determinant of distances, which, to avoid prolixity, must be left to the reader to verify.

FAMILY I.—Omitting any two letters, and forming all possible cyclical products with the remaining $(n-2)$ letters, if $C_1 C_2 \ldots C_\nu$ be any set thereof, and ν' the number of them containing more than two letters, the general term will be $\Sigma\Sigma (-)^{n+\nu} 2^{\nu'} C_1 \, . \, C_2 \ldots C_\nu (A-B)^2$, a, b being the two letters which do not occur in the cycles $C_1 C_2 \ldots C_\nu$.

FAMILY II.—Omitting any one letter, and forming with the remaining $n-1$ letters, in every possible way, *a chain* χ containing two or more letters, and cycles $C_1 C_2 \ldots C_\nu$, then, supposing the chain to be $bcd \ldots kl$, and understanding by (χ) the product $bc \, . \, cd \ldots kl$, the general term will be

$$\Sigma\Sigma (-)^{n+\nu} \, 2^{\nu'+1} C_1 C_2 \ldots C_\nu (\chi) (A-B)(A-L),$$

a being the letter which does not appear in the chain or any of the cycles, and ν' meaning as before the number of the cycles which contain at least three elements.

FAMILY III.—Form all the letters in every possible way into two *chains* (each containing two or more letters) χ, χ', and into cycles $C_1, C_2, \ldots C_\nu$;

then, supposing the initial and final letters of χ to be a, h, and of χ' to be k, l, the general term of this family will be

$$2\Sigma (-)^{n+\nu} \, 2^{\nu'+1} C_1 C_2 \ldots C_\nu (\chi)(\chi') \{(A-K)(H-L) + (A-L)(H-K)\}.$$

I subjoin in the following table the *types* of the coefficients of the several families for all the values of n from 2 up to 7; the vacant cycle ( ) of course means unity, and a cycle (ab) means $(ab)^2$; that is, the *fourth* power of the length ab.

Every cycle enclosed in a parenthesis of three or more letters, will be understood to be affected with a coefficient 2, and for greater brevity the variable part of each term is *left to be supplied.* A round parenthesis indicates a cycle, a square parenthesis a chain.

Number of Letters	Types	Name of Family
2	$(\ )$	1st
3	(bc)	2nd
4	$-(cd)$	1st
„	$2[bcd]$	2nd
„	$2[ab] \, . \, [cd]$	3rd
5	(cde)	1st
„	$-2[bcde] : 2(bc)[de]$	2nd
„	$-2[ab][cde]$	3rd
6	$-(cdef) : (cd)(ef)$	1st
„	$-2(bcd)[ef] : -2(bc)[def] : -2[bcdef]$	2nd
„	$-2(ab)[cd][ef] : [abc][def] : [ab][cdef]$	3rd
7	$(cdefg) - (cd)(efg)$	1st
„	$2(bcde)[fg] : 2(bcd)[efg] : -2(bc)(de)[fg] \; 2(bc)[defg] : -2[bcdefg]$	2nd
„	$2(abc)[de][fg] : 2(ab)[cd][efg] : -2[abc][defg] : -2[abcdefg]$	3rd

Thus, for example, the constant function of distances for three points in a plane is $2\Sigma bc (A-B)(A-C)$; for four points in space is

$$-\Sigma cd (A-B)^2 + 2\Sigma bc \, . \, cd (A-B)(A-D) + 2\Sigma ab \, . \, cd \{(A-C)(B-D) + (A-D)(B-C)\};$$

for five points in hyper-space is

$$2\Sigma (cd \, . \, de \, . \, ec)(A-B)^2 - 2\Sigma (bc \, . \, cd \, . \, de)(A-B)(A-E) + 2(bc)^2 (de)(A-D)(B-E)$$
$$- 2\Sigma ab \, . \, cd \, . \, de \, . \, ec \{(A-C)(B-E) + (A-E)(B-C)\}.$$

The part of the constant function of distances for seven points belonging to the 2nd family of terms will be

$$4\Sigma bc \, . \, cd \, . \, de \, . \, eb \, . \, fg (A-B)(A-E) + 4\Sigma bc \, . \, cd \, . \, db \, . \, ef \, . \, fg (A-E)(A-G)$$
$$- 2(bc)^2 (de)^2 fg (A-F)(A-G) + 2(bc)^2 (de \, . \, ef \, . \, fg)(A-D)(A-G)$$
$$- 2\, bc \, . \, cd \, . \, de \, . \, ef \, . \, fg (A-B)(A-G).$$

The number of types in each family for n points is easily expressible by a generating function.

Obviously in the 1st family this number is the number of ways of resolving n into parts none less than 2; that is, it is the coefficient of x^{n-2} in

$$\frac{1}{1-x^2\,.\,1-x^3\,.\,1-x^4\,\ldots}.$$

In the 2nd family, it is the sum of the number of ways of decomposing $n-3$, $n-4$, ... into parts none less than 2; that is, it is the coefficient of x^{n-3} in

$$\frac{1+x+x^2+\ldots}{(1-x^2)(1-x^3)\ldots},\text{ that is, in }\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots}.$$

In the 3rd family, if the number of ways of dividing r into two parts, neither of them less than 2, is called (r), and of dividing $(n-r)$ into any number of parts, none less than 2, is called $[n-r]$, the number of types is $\Sigma(r)\,[n-r]$; that is, it is the coefficient of x^{n-4} in

$$\frac{1+x+2x^2+2x^3+3x^4+3x^5+\ldots}{(1-x^2)(1-x^3)(1-x^4)\ldots},\text{ that is, in }\frac{1}{(1-x)(1-x^2)^2(1-x^3)(1-x^4)\ldots}.$$

Hence the total number of types in all three families combined will be the coefficient of x^{n-2} in

$$\frac{(1-x)(1-x^2)+x(1-x^2)+x^2}{1-x\,.\,1-x^2\,.\,1-x^3\,\ldots},\text{ that is, in }\frac{1}{1-x\,.\,(1-x^2)^2\,.\,1-x^3\,.\,1-x^4\,\ldots}.$$

Consequently, the indefinite partitions of 0, 1, 2, 3, 4, 5, 6, 7, ... being 1, 1, 2, 3, 5, 7, 11, 15, ..., the series for the type-number will be found by summing all the terms in the odd and even places successively. We thus obtain the series 1, 1, 3, 4, 8, 11, 19, 26, ... for the number of types in the constant-distance function for 2, 3, 4, 5, 6, 7, 8, 9, ... points respectively.

It may be worth while to exhibit the rule for the formation of the constant function of distances under a slightly different aspect.

As before, by the reading of any cycle, understand the product of its successive duads affected with the multiplier -1 or -2, according as the number of letters in the cycle is two or more than two.

By a *modified* reading of a cycle, understand what the reading becomes on substituting for any two duads pq, rs the product $(P-Q)(R-S)$, as for instance $(A-B)(C-D)$ in lieu of $ab\,.\,cd$, $(A-B)(B-C)$ in lieu of $ab\,.\,bc$, and (which can only happen in the case of a cycle of two letters), $(A-B)(B-A)$, that is, $-(A-B)^2$ in lieu of $ab\,.\,ba$.

Then, to find the constant function of distances to any given set of letters, we must begin with distributing the letters in every possible way into cycles containing between them two or more letters. Each such combination of cycles we may call a distribution.

In each distribution the cycle is to be taken (each in its turn), and the *sum* of its modified readings is to be multiplied by the product of the readings of the remaining cycles, if there are any. The sum of these sums (or the single sum, if there is but one cycle) is the portion of the quadratic function sought, due to the particular distribution dealt with; and the sum of these double sums, taken for each distribution in succession, is the total value of the function, and will be equal exactly to its representative determinant when the number of letters is odd, and to the same with its sign changed when that number is even.

As an example for five letters a, b, c, d, e, there will be ten distributions of the form $(ab)(cde)$, and twelve distributions of the form $(abcde)$.

From any one of the first ten distributions, as $(ab)(cde)$, by modifying first (ab) and then (cde), we obtain

(1) $2\,(cd\,.\,de\,.\,ec)(A-B)(B-A)$,

(2) $2\,(ab)^2\,\{ce\,(C-D)(D-E)+dc\,(D-E)(E-C)+ed\,(E-C)(C-D)\}$.

And from a distribution of the form $(abcde)$ we obtain, by operating on consecutive duads,

$$5\text{ terms of the form }-2\,\{cd\,.\,de\,.\,ea\,(A-B)(B-C)\},$$

and, by operating on non-consecutive duads,

$$5\text{ terms of the form }-2\,\{bc\,.\,de\,.\,ea\,(A-B)(C-D)\}\,^*.$$

The sum of all the sums of terms due to the twenty-two distributions is the constant function of distances for the five given letters.

In the case of six letters the distributions into cycles will be of four kinds, corresponding to the partitions 6; 4, 2; 3, 3; 2, 2, 2.

The first kind will contain two types of the 3rd family and one of the 2nd family; the second kind will contain one type of each of the three families, and the third and fourth kinds single types of the 2nd and 1st families respectively, thus giving eight distinct types of terms in all, as should be the case according to the rule.

* It will be observed that the distribution $(acbde)$ will give a term

$$-2\,\{cb\,.\,de\,.\,ea\,(A-C)(B-D)\},$$

in which the literal part $cb\,.\,de\,.\,ea$ is equal to the literal part $bc\,.\,de\,.\,ea$ in the term above expressed. This is how it comes to pass that the terms of the 3rd family may be grouped in pairs, as stated in the prior mode of arranging the result according to families instead of according to cycles.

40.

ON THE TRINOMIAL UNILATERAL QUADRATIC EQUATION IN MATRICES OF THE SECOND ORDER.

[*Quarterly Journal of Mathematics*, xx. (1885), pp. 305—312.]

In the May number [p. 225 above] of the present year of the *London and Edinburgh Philosophical Magazine* (disfigured by numerous errors or inaccuracies) I investigated the number of the solutions of an equation in quaternions or matrices of the second order, belonging to what I term the unilateral class, meaning one in which the coefficients of any actual power of the unknown quantity lie on the same side of it; this number for the Jerrardian Trinomial form I proved *strictly* is $2i^3 - i$ (i being the degree of the equation) and with evidence little short of moral certainty $i^3 - i^2 + i$* in the general case where none of the terms are wanting†.

But it must be well borne in mind that these numbers only apply when the coefficients are left general, and that for special relations between them some or all of the roots may become either ideal or indeterminate, or some the one and some the other. In all cases of equations in matrices one principal feature of the investigation is, or should be, to determine the equation of condition between the coefficients, in order that the solution may lose or retain its normal form; if we wish to avoid being compelled to enter upon a complicated consideration of exceptions piled upon exceptions, it is necessary to presuppose a certain criterion function to be other than zero; otherwise it is like the opening of Pandora's box, letting loose an almost incalculable train of vexatious inquiries scarcely worth the trouble they give to answer correctly.

* This article was written and sent to the press many months ago. I have since shown that the number of roots of a general unilateral equation of degree i in matrices of the order ω is the number of combinations of $i\omega$ things taken ω and ω together, and consequently for the case of quaternions is $2i^2 - i$ for the general and not merely for the Jerrardian form. See [above, pp. 197, 233. Also] *Nature*, Nov. 13, 1884.

† I made the assumption that the required number is an analytical function of ω.

Take as an instance the subject of monothetic equations. I have defined a monothetic equation to be one in which all the coefficients are functions of a single matrix, which may be called the base. In such an equation of the degree i and of the order ω in the matrices, we may suppose the unknown quantity to be a function of the base, and then the general formula for expressing a function of a matrix as a rational and integral function of the matrix with the aid of its latent roots, shows that i^ω and no more of such roots exist. But this in no manner precludes the possibility of the existence of other roots which are not functions of the base. Thus, for example, in the very simple case of the equation $x^2 + px = 0$, where x and p are quaternions or matrices of the second order, I have shown in the *Comptes Rendus* [pp. 174, 179 above] that besides the four determinate ones, all of which (0 included) may be regarded as functions of p, there are two other *indeterminate* ones, each one containing an arbitrary constant, and neither of them (to use quaternion language) coplanar with the base. Here there is a sort of reversion to the normal case of 3 pairs of roots to an unilateral quadratic, with the modification of two of them having become indeterminate. It becomes then of importance to fix accurately the condition of this normal state of things ceasing to exist. Being intent on the Denumeration theory of the roots in the general quadratic, I did not in the paper cited do this explicitly for the unilateral quadratic, although I gave there my own form of solution. Moreover, there are other features of much interest belonging to the question, which, for the same reason, I omitted to notice. These omissions and shortcomings it is the object of this present article to supply.

Starting with the form $x^2 - 2px + q = 0$, and for convenience of comparison with Hamilton's formulæ treating p, q indifferently as matrices or as quaternions, and forming the equation $x^2 - 2Bx + D = 0$, where B, D are scalars to be determined, so that $B = Sx$ and $D = Tx^2$, we shall have

$$2x = (p - B)^{-1}(q - D).$$

If now we understand by b, c, d, e, f

$$Sp,\ Sq,\ Tp^2,\ S(Vp\,Vq),\ Tq^2 \text{ respectively,}$$

by means of the general formula

$$T\pi^2 . (\pi^{-1}\chi)^2 - 2S(V\pi\,V\chi)(\pi^{-1}\chi) + T\chi^2 = 0^*,$$

[remembering that

$$T(p - B)^2 = d^2 - 2bB + B^2,$$
$$T(q - D)^2 = f^2 - 2cD + D^2,$$

* This formula, which I have not met with in Treatises on Quaternions, is a particular case only of the general Theorem in Matrices, that if

$$A\lambda^\omega + B\lambda^{\omega-1}\mu + \ldots + L\mu^\omega$$

is the determinant to $(\lambda L + \mu M)$, where L and M are two matrices of the order ω and λ and μ two ordinary quantities, then

$$A(L^{-1}M)^\omega - B(L^{-1}M)^{\omega-1} \ldots + (-)^\omega L = 0.$$

and $$S\{V(p-B)\,V(q-D)\} = e - bD - cB + BD],$$
we shall obtain [see p. 188 above]
$$4(d - 2bB + B^2)\,x^2 - 4(e - bD - cB + BD)\,x + (f - 2cD + D^2) = 0.$$

Hence, writing $$B - b = u, \quad D - c = v,$$
$$d - b^2 = \alpha, \quad e - bc = \beta, \quad f - c^2 = \gamma,$$
and comparing with each other the two quadratic equations in x, we may write
$$u^2 + \alpha = \lambda, \quad uv + \beta = 2\lambda(u + b), \quad v^2 + \gamma = 4\lambda(v + c).$$

Eliminating v from the latter two equations there results
$$-(2\lambda u + 2b\lambda - \beta)^2 + 4\lambda(2\lambda u + 2b\lambda - \beta)\,u - (\gamma - 4c\lambda)\,u^2 = 0,$$
and finally writing $\lambda - \alpha$ for u^2, we obtain
$$(4\lambda^2 + 4c\lambda - \gamma)(\lambda - \alpha) - (2b\lambda - \beta)^2 = 0.$$

There are thus 3 pairs of roots, for to each of the three values of λ correspond two values of u, namely
$$\pm(\lambda - d + b^2)^{\frac{1}{2}},$$
and to each value of λ and u one value of v, namely
$$2\lambda + \frac{2b\lambda + bc - e}{u}.$$

We have also $$x = \tfrac{1}{2}\{(p - b - u)^{-1}(q - D)\},$$
consequently, since $p^2 - 2bp + d = 0$,
$$x = \frac{(p - b + u)(q - c - v)}{2(b^2 - d - u^2)} = -\frac{(p - b + u)(q - c - v)}{2\lambda}.$$

Thus then we see that x can only cease to have 6 determinate values when $\lambda = 0$, and consequently the *Criterion of Normality* is the last term in the equation to λ.

This equation, written out at length, is
$$4\lambda^3 + 4(c - b^2 - \alpha)\lambda^2 + (-4c\alpha + 4b\beta - \gamma)\lambda + \alpha\gamma - \beta^2 = 0,$$
that is, $4\lambda^3 + 4(c - d)\lambda^2 + (-4cd + 4be - f + c^2)\lambda + (d - b^2)(f - c^2) - (e - bc)^2$.

Hence the Criterion in question is $(d - b^2)(f - c^2) - (e - bc)^2$ or $df - c^2d - b^2f - e^2 + 2bce$, which is the discriminant to the quadratic form
$$\lambda^2 + 2b\lambda\nu + 2c\mu\nu + d\mu^2 + 2e\mu\nu + f\nu^2;$$
this, as I have elsewhere shown, is the Criterion of the matrices p, q* being *in involution*†, that is, of a linear equation existing between the matrices $1, p, q, pq$; or if p, q are regarded as quaternions, it is the condition of the square of

* When p, q are regarded as matrices, then
$$p^2 - 2bp + d = 0, \quad q^2 - 2cq + f = 0, \quad \tfrac{1}{2}(pq + qp) - bq - cp + e = 0,$$
where $$\lambda^2 + 2b\lambda\nu + 2c\mu\nu + d\mu^2 + 2e\mu\nu + f\nu^2$$
is the determinant to $\lambda + \mu p + \nu q$.

[† Above p. 116.]

the sine of the angle between the vectors of p and q vanishing; a condition which of course does not imply the coincidence of the vectors unless accompanied by the futile limitation of such vectors being real.

It admits of easy demonstration by virtue of the foregoing that in the case of the more general equation
$$px^2 + qx + r = 0,$$
the Criterion of Normality will be the discriminant of the ternary quadratic, which is the determinant of
$$pu + qv + rw;$$
this seems to me a very remarkable and noteworthy theorem. When this Criterion does not vanish, the quadratic equation above written must have 3 pairs of determinate roots.

Why they go in pairs and can be found by solving only a cubic instead of a sextic is best seen *à priori* by reverting to the original form $x^2 - 2px + q = 0$.

It follows from the nature of the process for finding B and D that they will be the same for that equation as for the equation $y^2 - 2yp + q = 0$.

But on writing $x + y = 2p$ these two equations pass into one another.

Hence each value of B, say B_1, will be associated with another value, say B', where $B_1 + B' = 2b$*, that is to say, if u_1, namely $B - b$ is one value of u, then $b - B$, that is, $-u_1$ will be another value of u, so that the equation in u^2 ought to be (as it has been shown to be) a cubic.

It might for a moment be supposed that $\lambda = \alpha = d - b^2$ would lead to a breach of normality on account of the equation $v - 2\lambda = \dfrac{2b\lambda + bc - e}{u}$, where $u^2 = 0$.

This, however, is not the case. For the equation
$$v^2 + \gamma = 4\lambda(v + c)$$
becomes, when $\lambda = \alpha$,
$$v^2 - 4(d - b^2)\,v + f - c^2 - 4cd + 4b^2c = 0,$$
so that v remains *finite*; consequently $2b\lambda + bc - e$, that is, $2bd - 2b^3 + bc - e$, must vanish when $\lambda = d - b^2$, and $v - 2\lambda$ assumes the form $\frac{0}{0}$. Obviously then in this case, to the one value $u = 0$ will be associated the two values of v, say v_1 and v_2, given by the above quadratic, and to $\lambda = \alpha$ will still correspond two values of (u, v), namely $(0, v_1)$, $(0, v_2)$; where, ideally speaking, the two zeros may be regarded as the same infinitesimal affected with opposite signs.

* In quaternion phrase, if $x + y = 2p$, $Sx + Sy = 2Sp$.

It should be observed, in order to understand what follows in the text, that $b - B_1 = B' - b$, and that the values of B must obviously be the same in the equation $x^2 - 2px + q = 0$ as in the equation $x^2 - 2xp + q = 0$.

The equation in λ may be made to undergo a useful linear transformation. Let $\lambda = \mu + \alpha$, so that $\mu = u^2$.

Then

$$\mu\{4\mu^2 + (8\alpha + 4c)\mu + 4\alpha^2 + 4c\alpha - \gamma\} - (2b\mu + 2b\alpha - \beta)^2 = 0,$$

that is

$$4\mu^3 + \{4(c + 2d) - 12b^2\}\mu^2 + \{(c + 2d)^2 - 8(c + 2d)b^2 + 12b^4 + 4be - f\}\mu - \{b(c + 2d) - 2b^3 - e\}^2 = 0,$$

where it is noticeable that the number of parameters is reduced from 5 to 4, c and d only appearing together in the linear combination $c + 2d$. This is tantamount to the form obtained by Hamilton.

Let us make another linear transformation suggested by the preceding remark. Write $c + 2d = g$, and $\mu - b^2 = \gamma = \lambda - d$, the equation becomes

$$4\gamma^3 + 4g\gamma^2 + (g^2 + 4be - f)\gamma + 2beg - b^2f - e^2 = 0.$$

But obviously, notwithstanding this reduction of the parameters, λ itself is the most natural quantity to employ as the base of the solution, or, so to say, as the independent variable, and this admits of being determined by an equation of extraordinary simplicity.

For, let I be the discriminant of

$$\text{det.}\,(\lambda + \mu p + \nu q) = I = df + 2bce - c^2d - b^2f - e^2.$$

Then it will be seen by actual inspection that the equation found for λ takes the following form

$$e^{\lambda(2\delta_c - \delta_d)}\, I = 0,$$

that is

$$I + \left(2\frac{d}{dc} - \frac{d}{d.d}\right) I.\lambda + \frac{1}{2}\left(2\frac{d}{dc} - \frac{d}{d.d}\right)^2 I.\lambda^2 + \frac{1}{1.2.3}\left(2\frac{d}{dc} - \frac{d}{d.d}\right)^3 I.\lambda^3 = 0,$$

(the terms in the exponential function subsequent to the fourth term adding nothing to the value of the series).

If in the equation $x^2 - 2px + q = 0$, p and q be regarded as quaternions, then $\lambda = Sx^2 + Ip^2 - (Sp)^2$, $c = Sq$, $d = Ip^2$, and $I = \frac{1}{4}(pq - qp)^2$, which is a scalar quantity, and is to be regarded as an explicit function of Sp, Sq; Tp^2, $S(Vp\,Vq)$, Tq^2; it is in fact the discriminant of the form

$$X^2 + 2Sp\,XY + 2Sq\,XZ + Tp^2Y^2 + 2S(Vp\,Vq)\,YZ + Tq^2Z^2,$$

an identity unknown I believe to the geometrical quaternionists.

[As an example of it, let $p = i$, $q = j$, then

$$Sp = 0, \quad Sq = 0, \quad S(Vp\,Vq) = 0, \quad Tp^2 = -1, \quad Tq^2 = -1,$$

$\frac{1}{4}(pq - qp)^2 = 1 =$ the discriminant of $X^2 - Y^2 - Z^2$.]

With these definitions $e^{\lambda(2\delta_c - \delta_d)}\, I$ becomes identically zero.

The equation $x^2 - 2px + q = 0$ having six roots it is natural to inquire as to the value of their sum. This may be readily found as follows. We have found

$$x = -\frac{(p - b + u)(q - c - v)}{2\lambda}.$$

Also, if

$$x + x' = 2p,$$
$$x'^2 - 2x'p + q = 0,$$

and obviously

$$\Sigma x = \Sigma x'.$$

Hence

$$\Sigma x = -\Sigma\frac{(p - b + u)(q - c - v)}{2\lambda},$$

and

$$12p - \Sigma x = -\Sigma\frac{(q - c - v)(p - b + u)}{2\lambda}.$$

Therefore

$$\begin{aligned}\Sigma x &= 6p - \Sigma\frac{3}{2\lambda}(pq - qp)\\ &= 6p - 3I^{\frac{1}{2}}\Sigma\frac{1}{\lambda}\\ &= 6p + 3\frac{I^{\frac{1}{2}}(2\delta_c - \delta_d)I}{I}\\ &= 6\{p + (2\delta_c - \delta_d)I^{\frac{1}{2}}\},\end{aligned}$$

where the sign of $I^{\frac{1}{2}}$ must be so taken that it shall be equal to $\frac{1}{2}(pq - qp)$.

So again

$$\begin{aligned}\Sigma x^2 &= 2p\Sigma x - 6q\\ &= 12p^2 - 6q + 12(2\delta_c - \delta_d)I^{\frac{1}{2}}p.\end{aligned}$$

Thus the mean value of each root is ϵ in excess, and that of each square root ϵp in excess, of what these means would be if p and q were nominal quantities, ϵ denoting $(2\delta_c - \delta_d)I^{\frac{1}{2}}p$. Of course Σx^i may be found by the formula of derivation

$$\Sigma x^{i+1} = 2p\Sigma x^i - 9\Sigma x^{i-1}.$$

In conclusion it may be observed in regard to the equation $x^2 - 2px + q = 0$, (since in writing $x + x_1 = 2p$, we have $x_1^2 - 2x_1p + q = 0$) it follows that (whatever be the order of the quantities p and q) the roots of either equation must be associated in pairs; because, if the identical equation to p is $p^\omega - \omega bp^{\omega-1} + \dots$ and to x is $x^\omega - \omega Bx^{\omega-1} + \dots$, the equation for finding B must be of the form $T(B - b)^2 = 0$.

P.S.—Since the above was sent to press I have discovered the general solution of the unilateral equation of any degree in matrices of any order; see the *Comptes Rendus* of the Institute for Oct. 20, 1884 [pp. 197, 233 above], and *Nature* for Nov. 13, 1884*.

[* This paper contains the Theorem "Every latent root of every root of a given unilateral function in matrices of any order, is an algebraical root of the determinant of that function taken as if the unknown were an ordinary quantity, and conversely every algebraical root of the determinant so taken is a latent root of one of the roots of the given function."]

41.

INAUGURAL LECTURE AT OXFORD
12 December 1885.

ON THE METHOD OF RECIPROCANTS AS CONTAINING AN EXHAUSTIVE THEORY OF THE SINGULARITIES OF CURVES*.

[*Nature*, XXXIII. (1886), pp. 222—231.]

It is now two years and seven days since a message by the Atlantic cable containing the single word "Elected" reached me in Baltimore informing me that I had been appointed Savilian Professor of Geometry in Oxford, so that for three weeks I was in the unique position of filling the post and drawing the pay of Professor of Mathematics in each of two Universities: one, the oldest and most renowned, the other—an infant Hercules—the most active and prolific in the world, and which realises what only existed as a dream in the mind of Bacon—the House of Solomon in the New Atlantis.

To Johns Hopkins, who endowed the latter, and in conjunction with it a great Hospital and Medical School, between which he divided a vast fortune accumulated during a lifetime of integrity and public usefulness, I might address the words familiarly applied to one dear to all Wykehamists:—

"Qui condis lævâ, condis collegia dextrâ,
Nemo tuarum unam vicit utraque manû."

The chair which I have the honour to occupy in this University is made illustrious by the names and labours of its munificent and enlightened founder, Sir Henry Saville; of Thomas Briggs, the second inventor of logarithms; of Dr Wallis, who, like Leibnitz, drove three abreast to the temple of fame—being eminent as a theologian, and as a philologer, in addition to being illustrious as the discoverer of the theorem connected with the quadrature of the circle named after him, with which every schoolboy is supposed to be familiar, and as the author of the *Arithmetica Infinitorum*, the precursor of Newton's *Fluxions*; of Edmund Halley, the trusted friend and counsellor of Newton, whose work marks an epoch in the history of astronomy, the reviver of the study of Greek geometry and discoverer of the proper motions of the so-

[* The tables referred to in the text are given pp. 301, 302 below.]

called fixed stars; and by one in later times not unworthy to be mentioned in connection with these great names, my immediate predecessor, the mere allusion to whom will, I know, send a sympathetic thrill through the hearts of all here present, to whom he was no less endeared by his lovable nature than an object of admiration for his vast and varied intellectual acquirements, whose untimely removal, at the very moment when his fame was beginning to culminate, cannot but be regarded as a loss, not only to his friends and to the University for which he laboured so strenuously, but to science and the whole world of letters.

As I have mentioned, the first to occupy this chair was that remarkable man Thomas Briggs, concerning whose relation to the great Napier of Merchiston, the fertile nursery of heroes of the pen and the sword, an anecdote, taken from the *Life* of Lilly, the astrologer, has lately fallen under my eyes, which, with your permission, I will venture to repeat:—

"I will acquaint you (says Lilly) with one memorable story related unto me by John Marr, an excellent mathematician and geometrician, whom I conceive you remember. He was servant to King James and Charles the First. At first, when the lord Napier, or Marchiston, made public his logarithms, Mr Briggs, then reader of the astronomy lectures at Gresham College, in London, was so surprised with admiration of them, that he could have no quietness in himself until he had seen that noble person the lord Marchiston, whose only invention they were: he acquaints John Marr herewith, who went into Scotland before Mr Briggs, purposely to be there when those two so learned persons should meet. Mr Briggs appoints a certain day when to meet at Edinburgh; but failing thereof, the lord Napier was doubtful he would not come. It happened one day as John Marr and the lord Napier were speaking of Mr Briggs: 'Ah John (said Marchiston), Mr Briggs will not now come.' At the very moment one knocks at the gate; John Marr hastens down, and it proved Mr Briggs to his great contentment. He brings Mr Briggs up into my lord's chamber, where almost *one quarter of an hour was spent*, each beholding other almost with admiration *before one word was spoke*. At last Mr Briggs began: 'My lord, I have undertaken this long journey purposely to see your person, and to know by what engine of wit or ingenuity you came first to think of this most excellent help unto astronomy, namely, the logarithms; but, my lord, being by you found out, *I wonder nobody else found it out before*, when now known it is so easy.' He was nobly entertained by the lord Napier; and every summer after that, during the lord's being alive, this venerable man Mr Briggs went purposely into Scotland to visit him*."

* A very similar story is told of the meeting of Leopardi and Niebuhr in Rome. What Briggs said of logarithms may be said almost in the same words of the subject of this lecture:—"This most excellent help to geometry which, being found out, one wonders nobody else found it out

Some apology may be needed, and many valid reasons might be assigned, for the departure, in my case, from the usual course, which is that every professor on his appointment should deliver an inaugural lecture before commencing his regular work of teaching in the University. I hope that my remissness, in this respect, may be condoned if it shall eventually be recognised that I have waited, before addressing a public audience, until I felt prompted to do so by the spirit within me craving to find utterance, and by the consciousness of having something of real and more than ordinary weight to impart, so that those who are qualified by a moderate amount of mathematical culture to comprehend the drift of my discourse, may go away with the satisfactory feeling that their mental vision has been extended and their eyes opened, like my own, to the perception of a world of intellectual beauty, of whose existence they were previously unaware.

This is not the first occasion on which I have appeared before a general mathematical audience, as the messenger of good tidings, to announce some important discovery. In the year 1859 I gave a course of seven or eight lectures at King's College, London, at each of which I was honoured by the attendance of my lamented predecessor, on the subject of "The Partitions of Numbers and the Solution of Simultaneous Equations in Integers," in which it fell to my lot to show how the difficulties might be overcome which had previously baffled the efforts of mathematicians, and especially of one bearing no less venerable a name than that of Leonard Euler, and also laid the basis of a method which has since been carried out to a much greater extent in my "Constructive Theory of Partitions," published in the *American Journal of Mathematics*, in writing which I received much valuable co-operation and material contributions from many of my own pupils in the Johns Hopkins University*. Several years later, in the same place, I delivered a lecture on the well-known theorem of Newton, which fills a chapter in the *Arithmetica Universalis*, where it was stated without proof, and of which many celebrated mathematicians, including again the name of Euler, had sought for a proof in vain. In that lecture I supplied the missing demonstration, and owed my success, I believe, chiefly to merging the theorem to be proved, in one of

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before; when, now known, it is so easy." I quite entered into Briggs's feelings at his interview with Napier when I recently paid a visit to Poincaré in his airy perch in the Rue Gay-Lussac in Paris (will our grandchildren live to see an Alexander Williamson Street in the north-west quarter of London, or an Arthur Cayley Court in Lincoln's Inn, where he once abode?). In the presence of that mighty reservoir of pent-up intellectual force my tongue at first refused its office, my eyes wandered, and it was not until I had taken some time (it may be two or three minutes) to peruse and absorb as it were the idea of his external youthful lineaments that I found myself in a condition to speak.

* In one of those lectures, two hundred copies of the notes for which were printed off and distributed among my auditors, I founded and developed to a considerable extent the subject since rediscovered by M. Halphen under the name of the Theory of Aspects.

greater scope and generality. In mathematical research, reversing the axiom of Euclid, and converting the proposition of Hesiod, it is a continual matter of experience, as I have found myself over and over again, that the whole is less than its part. On a later occasion, taking my stand on the wonderful discovery of Peaucellier, in which he had realised that exact parallel motion which James Watt had believed to be impossible, and exhausted himself in contrivances to find an imperfect substitute for, in the steam-engine, I think I may venture to say that I brought into being a new branch of mechanico-geometrical science, which has been, since then, carried to a much higher point by the brilliant inventions of Messrs Kempe and Hart. I remember that my late lamented friend, the Lord Almoner's Reader of Arabic in this University, subsequently editor of the *Times*, Mr Chenery, who was present on that occasion in an unofficial capacity, remarked to me after the lecture, which was delivered before a crowded auditory at the Royal Institution, that when they saw two suspended opposite Peaucellier cells, coupled toe-and-toe together, swing into motion, which would have been impossible had not the two connected moving points each described an accurate straight line, "the house rose at you." (The lecture merely illustrated experimentally two or three simple propositions of Euclid, Book III.)

The matter that I have to bring before your notice this afternoon is one far bigger and greater, and of infinitely more importance to the progress of mathematical science, than any of those to which I have just referred. No subject during the last thirty years has more occupied the minds of mathematicians, or lent itself to a greater variety of applications, than the great theory of Invariants. The theory I am about to expound, or whose birth I am about to announce, stands to this in the relation not of a younger sister, but of a brother, who, though of later birth, on the principle that the masculine is more worthy than the feminine, or at all events, according to the regulations of the Salic law, is entitled to take precedence over his elder sister, and exercise supreme sway over their united realms. Metaphor apart, I do not hesitate to say that this theory, *minor natu potestate major*, infinitely transcends in the extent of its subject-matter, and in the range of its applications, the allied theory to which it stands in so close a relation. The very same letters of the alphabet which may be employed in the two theories, in the one may be compared to the dried seeds in a botanical cabinet, in the other to buds on the living branch ready to burst out into blossom, flower and fruit, and in their turn supply fresh seed for the maintenance of a continually self-perpetuating cycle of living forms. In order that I may not be considered to have lost myself in the clouds in making such a statement, let me so far anticipate what I shall have to say on the meaning of Reciprocants and their relation to the ordinary Invariantive or Covariantive forms by taking an instance which happens to be common (or at least, by a slight geometrical adjustment, may be made so) to the two theories. I ask you to compare the form

$$a^2d - 3abc + 2b^3$$

as it is read in the light of the one and in that of the other. In the one case the a, b, c, d stand for the coefficients of a so-called Binary Quantic, and its evanescence serves to express some particular relation between three points lying in a right line. In the other case the letters are interpreted* to mean the successive differential derivatives of the 2nd, 3rd, 4th, 5th orders of one Cartesian co-ordinate of a curve in respect to the other. The equation expressing this evanescence is capable of being integrated, and this integral will serve to denote a relation between the two co-ordinates which furnishes the necessary and sufficient condition in order that the point of the curve of any or no specified order (for it may be transcendental) to which the co-ordinates may refer, may admit of having, at the point where the condition is satisfied, a contact with a conic of a higher order than the common. In the one case the letters employed are dead and inert atoms; in the other they are germs instinct with motion, life, and energy.

A curious history is attached to the form which I have just cited, one of the simplest in the theory, of which the narrative may not be without interest to many of my hearers, even to those whose mathematical ambition is limited to taking a high place in the schools.

At pp. 19 and 20 of Boole's *Differential Equations* (edition of 1859) the author cites this form as the left-hand side of an equation which he calls the "Differential Equation of lines of the second order," and attributes it to Monge, adding the words, "But here our powers of geometrical interpretation fail, and results such as this can scarcely be otherwise useful than as a registry of integrable forms." In this vaticination, which was quite uncalled for, the eminent author, now unfortunately deceased, proved himself a false prophet, for the form referred to is among the first that attracts notice in crossing the threshold of the subject of Reciprocants, and is but one of a crowd of similar and much more complicated expressions, no less than it susceptible of geometrical interpretation and of taking their place on the register of integrable forms. A friend, with whom I was in communication on the subject, and whom I see by my side, remarked to me, in reference to this passage:—"I cannot help comparing a certain passage in Boole to Ezekiel's valley of the dry bones: 'The valley was full of bones, and lo, they were very dry.' The answer to the question, 'Can these bones live?' is supplied by the advent of the glorious idea of the Reciprocants; and the grand invocation, 'Come from the four winds, O breath, and breathe upon these slain, that they may live,' may well be used here. That they will

[* $a=\frac{1}{2!}\frac{d^2y}{dx^2}$, $b=\frac{1}{3!}\frac{d^3y}{dx^3}$, $c=\frac{1}{4!}\frac{d^4y}{dx^4}$, $d=\frac{1}{5!}\frac{d^5y}{dx^5}$.]

'live and stand up upon their feet an exceeding great army' is what we may expect to happen." This, as you will presently see, is just what actually has happened.

Not knowing where to look in Monge for the implied reference, I wrote to an eminent geometer in Paris to give me the desired information; he replied that the thing could not be in Monge, for that M. Halphen, who had written more than one memoir on the subject of the differential equation of a conic, had made nowhere any allusion to Monge in connection with the subject. Hereupon, as I felt sure that a reference contained in repeated editions of a book in such general use as Boole's *Differential Equations* was not likely to be erroneous, I addressed myself to M. Halphen himself, and received from him a reply, from which I will read an extract:—

"En premier lieu, c'est une chose nouvelle pour moi que l'équation différentielle des coniques se trouve dans Boole, dont je ne connais pas l'ouvrage. Je vais, bien entendu, le consulter avec curiosité. Ce fait a échappé à tout le monde ici, et l'on a cru généralement que j'avais le premier donné cette équation. *Nil sub sole novi!* Il m'est naturellement impossible de vous dire où la même équation est enfouie parmi les œuvres de Monge. Pour moi, c'est dans *Le Journal de Math.* (1876), p. 375, que j'ai eu, je crois, la première occasion de développer cette équation sous la forme même que vous citez; et c'est quand je l'ai employée, l'année suivante, pour le problème *sur les lois de Kepler* (*Comptes rendus*, 1877, t. LXXXIV. p. 939), que M. Bertrand l'a remarquée comme neuve. Ce qui vous intéresse plus, c'est de connaître la forme simplifiée sous laquelle j'ai donné plus tard cette équation dans le *Bulletin* de la Société Mathématique. C'est sous cette dernière forme que M. Jordan la donne dans son cours de l'École Polytechnique (t. I. p. 53)."

All my researches to obtain the passage in Monge referred to by Boole have been in vain*.

I will now proceed to endeavour to make clear to you what a Reciprocant means: the above form, which may be called the *Mongian*, would afford an example by which to illustrate the term; but I think it desirable to begin with a much easier one. Consider then the simple case of a single term, the second derivative of one variable, y, in respect to another, x. Every tyro in algebraical geometry knows that this, or rather the fact of its evanescence, serves to characterise one or more points in a curve which possess, so to say,

* Search has been made in the collected works of Monge and in manuscripts of his own or Prony in the library of the Institute, but without effect. I have also made application to the Universal Information Society, who undertake to answer "every conceivable question," but nothing has so far come of it. Perhaps until the citation from Monge is verified it will be safer in future to refer to the so-called Mongian as the Boole-Mongian. It may be regarded as the starting-point of the Differential Invariant Theory, as the Schwarzian is of the deeper-lying and more comprehensive Reciprocant Theory.

a certain indelible and intrinsic character, or what is technically called a singularity; in this case an inflexion such as exists in a capital letter *S*, or Hogarth's line of beauty.

If we invert the two variables, exchanging, that is to say, one with the other, the fact of this indelibility draws with it the consequence that in general these two reciprocal functions must vanish together, and as a fact each is the same as the other multiplied or divided by the third power of the first derivative of the one variable with respect to the other taken negatively. In this case we are dealing with a single derivative and its reciprocal. The question immediately presents itself whether there may not be a combination of derivatives possessing a similar property. We know that no *single* derivative except the second does.

Such a combination actually presents itself in a form which occurs in the solution of Differential Equations of the second order, the form

$$\frac{dy}{dx}\cdot\frac{d^3y}{dx^3}-\frac{3}{2}\left(\frac{d^2y}{dx^2}\right)^2,$$

which, after the name of its discoverer, Schwarz, we may agree to call a Schwarzian (Cayley's "Schwarzian Derivative*"). If in this expression the x and y be interchanged, its value, barring a factor consisting of a power of the first derivative, remains unaltered, or, to speak more strictly, merely undergoes a change of algebraical sign. We may now arrive at the generalised conception of an algebraical function of the derivatives of one variable in respect to another, which, if we agree to pay no regard to the algebraical sign, or to any power of the first derivative that may appear as a factor, will remain unaltered when the dependent and independent variables are interchanged one with another; and we may agree to call any such function a Reciprocant.

But here an important distinction arises—there are Reciprocants such as the one I first mentioned, $\frac{d^2y}{dx^2}$, or such as the Mongian to which allusion has

* More strictly speaking this is Cayley's Schwarzian derivative cleared of fractions—it may well be called the Schwarzian (see my note on it in the *Mathematical Messenger* for September or October past). Prof. Greenhill in regard to the Schwarzian derivative proper writes me as follows :—

"I found the reference in a footnote to p. 74 of Klein's *Vorlesungen über das Ikosaëder*, &c., in which Klein thanks Schwarz for sending him the reference to a paper by Lagrange, 'Sur la construction des cartes géographiques' in the *Nouveaux Mémoires de l'Académie de Berlin*, 1779. Compare also Schwarz's paper in Bd. 75 of *Borchardt's Journal*, where further literary notices are collected together. Klein says further that in the 'Sächsischen Gesellschaft von Januar 1883,' he has considered the inner meaning (*innere Bedeutung*) of the differential equation

$$\frac{\eta'''}{\eta'}-\frac{3}{2}\left(\frac{\eta''}{\eta'}\right)^2=f(\eta),\text{ where }\eta'=\frac{d\eta}{dz}\ldots."$$

There are two papers by Lagrange, one immediately following the other, "Sur la construction des cartes géographiques," but I have not been able to discover the Schwarzian derivative in either of them.

been made in the letter from M. Halphen, in which the second and higher differential derivatives alone appear, the first differential derivative not figuring in the expression. These may be termed Pure Reciprocants. Thus I repeat $\frac{d^2y}{dx^2}$, and

$$9\left(\frac{d^2y}{dx^2}\right)^2\cdot\frac{d^5y}{dx^5}-45\frac{d^2y}{dx^2}\cdot\frac{d^3y}{dx^3}\cdot\frac{d^4y}{dx^4}+40\left(\frac{d^3y}{dx^3}\right)^3$$

are pure reciprocants. Those from which the first derivative $\frac{dy}{dx}$ is not excluded may be called Mixed Reciprocants. An example of such kind of Reciprocants is afforded by the Schwarzian above referred to. This distinction is one of great moment, for a little attention will serve to make it clear that every pure reciprocant expressed in terms of x and y marks an intrinsic feature or singularity in the curve, whatever its nature may be, of which x and y are the co-ordinates; for if in place of the variables (x, y) any two linear functions of these variables be substituted, a pure reciprocant, by virtue of its reciprocantive character, must remain unaltered save as to the immaterial fact of its acquiring a factor containing merely the constants of substitution*.

The consequence is that every pure reciprocant corresponds to, and indicates, some singularity or characteristic feature of a curve, and *vice versâ* every such singularity of a general nature and of a descriptive (although not necessarily of a projective) kind, points to a pure reciprocant.

Such is not the case with mixed reciprocants. They will not in general remain unaltered when linear substitutions are impressed upon the variables. Is it then necessary, it may be asked, to pay any attention to mixed reciprocants; or may they not be formally excluded at the very threshold of the inquiry? Were I disposed to put the answer to this question on mere personal grounds, I feel that I should be guilty of the blackest ingratitude, that I should be kicking down the ladder by which I have risen to my present commanding point of view, if I were to turn my back on these humble mixed reciprocants, to which I have reason to feel so deeply indebted; for it was the putting together of the two facts of the substantial permanence under linear substitutions impressed upon the variables of the Schwarzian form and the simpler one which marks the inflexions of a curve—it was, if I may so say, the collision in my mind of these two facts—that kindled the spark and fired the train which set my imagination in a blaze by the light of which the whole horizon of Reciprocants is now illumined.

* The form as it stands shows that for y a linear function of x and y may be substituted; and the form *reciprocated* (by the interchange of x and y) shows that a similar substitution may be made for x. Hence arbitrary linear substitutions may be simultaneously impressed on x and y without inducing any change of form.

But it is not necessary for me to defend the retention of mixed reciprocants on any such narrow ground of personal predilection. The whole body of Reciprocants, pure and mixed, form one complete system, a single garment without rent or seam, a complex whole in which all the parts are inextricably interwoven with each other. It is a living organism, the action of no part of which can be thoroughly understood if dissevered from connection with the rest.

It was in fact by combining and interweaving mixed reciprocants that I was led to the discovery of the pure binomial reciprocant, which comes immediately after the trivial monomial one,—the earliest with which I became acquainted, and of the existence of compeers to which I was for some time in doubt, and only became convinced of the fact after the discovery of the Partial Differential Equation, the master-key to this portion of the subject, which gives the means of producing them *ad libitum* and ascertaining all that exist of any prescribed type. Of this partial differential equation I shall have occasion hereafter to speak; but this is not all, for, as we shall presently see, mixed reciprocants are well worthy of study on their own account, and lead to conclusions of the highest moment, whether as regards their applications to geometry or to the theory of transcendental functions and of ordinary differential equations.

The singularities of curves, taking the word in its widest acceptation, may be divided into three classes: those which are independent of homographic deformation and which remain unaltered in any perspective picture of the curve; those which, having an express or tacit reference to the line at infinity, are not indelible under perspective projection, but using the word descriptive with some little latitude may, in so far as they only involve a reference to the line at infinity as a line, be said to be of a purely descriptive character; and, lastly, those which are neither projective nor purely descriptive, having relation to the points termed, in ordinary parlance, "circular points at infinity"—for which the proper name is "centres of infinitely distant pencils of rays," that is, pencils, every ray of which is infinitely distant from every point external to it. Such, for instance, would be the character of points of maximum or minimum curvative, which, as we shall see, indicate, or are indicated by, that particular class of Mixed to which I give the name of "Orthogonal Reciprocants." All purely descriptive singularities alike, whether projective or non-projective, are indicated by pure reciprocants, and are subject to the same Partial Differential Equation; just as, in the Theory of Binary Quantics, Invariants, although under one aspect they may be regarded as a self-contained special class, admit of being and are most advantageously studied in connection with, and as forming a part of, the whole family of forms commonly known by the name of "semi-, or subinvariants," but which I find it conduces to much

greater clearness of expression and avoidance of ambiguity or periphrasis to designate as Binariants.

The question may here be asked, How, then, are projective and non-projective pure reciprocants to be discriminated by their external characters?

I believe that I know the answer to this question, which is, that the former are subject to satisfy a second partial differential equation of a certain simple and familiar type, but this is a matter upon which it is not necessary for me to enter on the present occasion*. It is enough for our present purpose to remark that every projective pure reciprocant must, so to say, be in essence a masked ternary covariant. For instance, if we take the simplest of all such, namely, a, that is $\frac{d^2y}{dx^2}$, we have, if $\phi(x, y) = 0$,

$$\frac{d^2y}{dx^2}\cdot\left(\frac{d\phi}{dy}\right)^3 = \begin{vmatrix} \frac{d^2\phi}{dx^2} & \frac{d^2\phi}{dxdy} & \frac{d\phi}{dx} \\ \frac{d^2\phi}{dxdy} & \frac{d^2\phi}{dy^2} & \frac{d\phi}{dy} \\ \frac{d\phi}{dx} & \frac{d\phi}{dy} & \cdot \end{vmatrix}$$

which, for facility of reference, let me call M. Obviously we might instead of $a = 0$ substitute $M = 0$ to mark an inflexion. And now if we write Φ as the completed form of ϕ, when made homogeneous by the substitution of z for unity; and if we suppose it to be of n dimensions in x, y, z, and call its Hessian H, we shall obtain the syzygy

$$(n-1)^2\left(\frac{d\phi}{dy}\right)^3 a + H + \left\{\frac{d^2\Phi}{dx^2}\cdot\frac{d^2\Phi}{dy^2} - \left(\frac{d^2\Phi}{dxdy}\right)^2\right\}\Phi = 0.$$

Hence the system $\Phi = 0$, $a = 0$, will be in effect the same as the system $\Phi = 0$, $H = 0$, and in this sense a may be said to carry H as it were in its bosom. And so in general every pure projective reciprocant may, in the language of insect transformation, be regarded as passing, so to say, first from the grub to the pupa or chrysalis, and from this again, divested of all superfluous integuments, to the butterfly or imago state.

Non-projective pure reciprocants undergo only one such change. There is no possibility of their ever emerging into the imago—their development being finally arrested at the chrysalis stage.

It would, I think, be an interesting and instructive task to obtain the imago or Hessianised transformation of the Mongian, but I am not aware

* In Paris, from which I correct the proofs, I have succeeded in reducing this conjecture to a certainty and in establishing the marvellous fact that every Projective Reciprocant, or, which is the same thing, every Differential Invariant, is, at the same time, an Ordinary Subinvariant. Thus a differential invariant (or projective reciprocant) may be regarded as *a single personality clothed with two distinct natures*—that of a reciprocant and that of a subinvariant.

that anyone has yet done, or thought of doing, this*. It seems to me that by substituting Reciprocants in lieu of Ternary Covariants we are as it were stealing a dimension from space, inasmuch as Reciprocants, that is, Ternary Covariants in their undeveloped state, are closely allied to, and march *pari passu* with, the familiar forms which appertain to merely binary quantics.

I will now proceed to bring before your notice the general partial differential equation which supplies the necessary and sufficient condition to which all pure reciprocants are subject.

It is highly convenient to denote the successive derivatives

$$\frac{d^2y}{dx^2},\ \frac{d^3y}{dx^3},\ \frac{d^4y}{dx^4},\ \dots$$

by the simple letters $a, b, c, \dots$.

The first derivative $\frac{dy}{dx}$ plays so peculiar a part in this theory that it is necessary to denote it by a letter standing aloof from the rest, and I call it t. This last letter, I need not say, does not make its appearance in any pure reciprocant. This being premised, I invite your attention to the equation in question, in which you will perceive the symbols of operation are separated from the object to be operated upon.

Writing $V = 3a^2\delta_b + 10ab\delta_c + (15ac + 10b^2)\,\delta_d + \dots$ and calling any pure reciprocant R,

$$VR = 0$$

is the equation referred to.

I cannot undertake, within the brief limits of time allotted to this lecture, to explain how this operation, or, as it may be termed, this annihilator V is arrived at. The table of binomial coefficients, or rather half series of binomial coefficients, shown† in Chart 4, will enable you to see what is the law of the numerical coefficients of its several terms. Let the words *weight, degree, extent* (extent, you will remember, means the number of places by which the most remote letter in the form is separated from the first letter in the alphabet) of a pure reciprocant signify the same things as they would do if the letters $a, b, c, \dots$ referred, according to the ordinary notation, to Binariants instead of to Reciprocants. The number of binariants linearly independent of each other whose weight, degree and extent or order are w, i, j is given by the partition formula $(w; i, j) - (w-1; i, j)$ where in general $(w; i, j)$ means the number of ways of partitioning w into i or fewer parts none greater than j.

* M. Halphen informs me that this has been done by Cayley in the *Phil. Trans.* for 1865, and subsequently in a somewhat simplified form by Painvin, *Comptes Rendus*, 1874. But neither of these authors seems to have had the Boole-Mongian objectively before him, so that a slight supplemental computation is wanting to establish the equation between it and the function which either of them finds to vanish at a *sextactic* point.

[† p. 302 below.]

It follows immediately from the mere form of V that the corresponding formula in the case of Reciprocants of a given type $w.i.j$ will be

$$(w; i, j) - (w-1; i+1, j)$$

the augmentation of i in the second term of the formula being due to the fact that, whereas in the partial differential equation for Binariants it is the letters themselves which appear as coefficients, it is quadratic functions of these in the case of Reciprocants. From the form of V we may also deduce a rigorous demonstration of the existence of Reciprocants strictly analogous to those with which you are familiar in the Binariant Theory, which are pictured in Chart 2, and are now usually designated as Protomorphs, as being the forms by the interweaving of which with one another (or rather by a sort of combined process of mixture and precipitation), all others, even the irreducible ones, are capable of being produced. The corresponding forms for Reciprocants you will see exhibited in the same table. Each series of Protomorphs may of course be indefinitely extended as more and more letters are introduced. In the table I have not thought it necessary to go beyond the letter g. You also know that besides Protomorphs there are other irreducible forms, the organic radicals, so to say, into which every compound form may be resolved, always limited in number, whatever the number of letters or primal elements we may be dealing with. The same thing happens to Reciprocants as you will notice in the comparative table in Chart 2. Without going into particulars, I will ask you to take from me upon faith the assurance that there is no single feature in the old familiar theory, whether it relates to Protomorphs, to Ground-forms, to Perpetuants, to Factorial constitution, to Generating Functions, or whatever else sets its stamp upon the one, which is not counterfeited by and reproduced in the parallel theory.

So much—for time will not admit of more—concerning pure reciprocants.

Let me now say a few words *en passant* on Mixed Reciprocants.

Pure Reciprocants, we have seen, are the analogues of Invariants, or else of the leading terms, for that is what are Semi- or Subinvariants, of Covariantive expansions; each is subject to its own proper linear partial differential equation. Mixed Reciprocants are the exact analogues of the coefficients in such expansions other than those of the leading terms. Starting from the leading terms as the unit point, the coefficients of rank ω are subject to a partial differential equation of order ω; and just so, mixed reciprocants, if involving t up to the power ω, are subject to a partial differential equation of that same order.

I have alluded to a peculiar class of mixed under the name of "Orthogonal Reciprocants." They are distinguished, as I have proved, by the beautiful property that, if differentiated with respect to t, the result must be itself a Reciprocant. In Chart 1 you will see this illustrated in the case of a mixed reciprocant $(1+t^2)b-3ta^2$, which serves to indicate the existence of points of maximum and minimum curvature. Its differential coefficient with respect to t is the oft-alluded-to Schwarzian, transliterated into the simpler notation. Proceeding in the inverse order—of Integration instead of Differentiation—I call your attention to a mixed reciprocant, of a very simple character, one which presents itself at the very outset of the theory, namely

$$tc-5ab,$$

which, integrated in respect to t between proper limits, yields the elegant orthogonal reciprocant

$$(t^2+1)c-10abt+15a^3.$$

Expressed in the ordinary notation, this, equated to zero, takes the form

$$\left\{\left(\frac{dy}{dx}\right)^2+1\right\}\frac{d^4y}{dx^4}-10\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}\cdot\frac{d^3y}{dx^3}+15\left(\frac{d^2y}{dx^2}\right)^3=0.$$

Mr Hammond has integrated this, treated as an ordinary differential equation, and has obtained the complete primitive expressed through the medium of two related Hyper-Elliptic Functions connecting the variables x and y (see* Chart 3). It may possibly turn out to be the case that every mixed reciprocant is either itself an Orthogonal Reciprocant, or by integration, in respect to t, leads to one.

It will of course be understood that, in interpreting equations obtained by equating to zero an Orthogonal Reciprocant, the variables must be regarded as representing not general but rectangular Cartesian co-ordinates.

Here seems to me to be the proper place for pointing out to what extent I have been anticipated by M. Halphen in the discovery of this new world of Algebraical Forms. When the subject first dawned upon my mind, about the end of October or the beginning of November last, I was not aware that it had been approached on any side by any one before me, and believed that I was digging into absolutely virgin soil. It was only when I received M. Halphen's letter, dated November 25, in relation to the Mongian business already referred to, accompanied by a presentation of his memoirs on Differential Invariants, that I became aware of there existing any link of connection between his work and my own. A Differential Invariant, in the sense in which the term is used by M. Halphen, is not what at first blush I supposed it to be, and as in my haste to repair what seemed to me an omission to be without loss of time supplied, I wrote to M. Hermite it was, in a letter which has been or is about to be inserted in the *Comptes Rendus* of the Institute of France; it is not, I say, identical with what I have termed a general pure reciprocant, but only with that peculiar species of Pure Reciprocants to which I have in a preceding part of this lecture referred as corresponding and pointing to Projective Singularities. In his

[* p. 302 below.]

splendid labours in this field Halphen has had no occasion to construct or concern himself with that new universe of forms viewed as a whole, whether of Pure or Mixed Reciprocants, which it has been the avowed and principal object of this lecture to bring under your notice.

I anticipate deriving much valuable assistance in the vast explorations remaining to be made in my own subject from the new and luminous views of M. Halphen, and possibly he may derive some advantage in his turn from the larger outlook brought within the field of vision by my allied investigations.

Let me return for a moment to that simplest class of pure reciprocants which I have called protomorphs. Each of these will be found (as may be shown either by a direct process of elimination, or by integrating the equations obtained by equating them severally to zero, regarded as ordinary differential equations between x and y) each of these, I say, will be found to represent some simple kind of singularity at the point (x, y) of the curve to which these co-ordinates are supposed to refer. Thus, for instance, No. 1 marks a single point of inflexion; No. 2, points of closest contact with a common parabola; No. 3, what our Cayley has called sextactic points, referring to a general conic; No. 4, points of closest contact with a common cubical parabola; and so on. The first and third, it will be noticed, represent projective singularities, and as such, in M. Halphen's language, would take the name of Differential Invariants. The second and fourth, having reference to the line at infinity in the plane of the curve, are of a non-projective character, and as such would not appear in M. Halphen's system of Differential Invariants. It is an interesting fact that every simple parabola, meaning one whose equation can be brought under the form $y=x^{\frac{m}{n}}$, corresponds to a linear function of a square of the third, and the cube of the second protomorph, and consequently will in general be of the sixth degree. In the particular case of the cubical parabola, the numerical parameter of this equation is such that the highest powers of b cancel each other so that the form sinks one degree, and becomes represented by the *Quasi-Discriminant*, No. 4.

This simple instance will serve to illustrate the intimate connection which exists between the projective and non-projective reciprocants, and the advantage, not to say necessity, of regarding them as parts of one organic whole.

It would take me too far to do more than make the most cursory allusion to an extension of this theory similar to that which happens when in the ordinary theory of invariants we pass from the consideration of a single Quantic to that of two or more. There is no difficulty in finding the partial differential equation to double reciprocants which, as far as I have as yet pursued the investigation, appear to be functions of $a, b, c, \ldots$; $a', b', c', \ldots$; and of $(t - t')$.

The theory of double reciprocants will then include as a particular case the question of determining the singularities of paired points of two curves at which their tangents are parallel, and consequently the theory of common tangents to two curves and of bi-tangents to a single one.

I think I may venture to say that a general pure multiple reciprocant which marks off relative singularities, whether projective or non-projective, of a group of curves, is a function of the second and higher differential derivatives appertaining to the several curves of the group, and of the differences of the first derivatives, whereas in a mixed multiple reciprocant these last-named differences are replaced by the first derivatives themselves. As a particular case, when the group dwindles to an individual and there is only one t, this letter disappears altogether from the form, for there are no differences of a single quantity.

In the chart (marked No. 2) you will see the table of Protomorphs carried on as far as the letter g inclusive, and will not fail to notice what may be termed the higher organisation of Reciprocantive as compared with ordinary Invariantive Protomorphs; the degrees of the latter oscillate or librate between the numbers 2 and 3, whereas in the former the degree is variable according to a certain transcendental law dependent on the solution of a problem in the Partition of Numbers. Another interesting difference between general Invariants and general Pure Reciprocants consists in the fact that, whilst the number of the former ultimately (that is, when the extent is indefinitely increased) becomes indefinitely great, that of the latter is determinate for any given degree even for an infinite number of letters.

In carrying on the table of protomorphs up to the letter h (see Chart 6) a new phenomenon presents itself, to which, however, there is a perfect parallel in the allied theory. An arbitrary constant enters into the form, its general value being a linear function of U and W (for which see Chart 6). But this is not all. If you examine the terms in both U and W (there are in all twelve such) you will find that these twelve do not comprise all of the same type to which they belong. There is a Thirteenth (a banished Judas), equally *à priori* entitled to admission to the group, but which does not make its appearance among them, namely, b^4d. I rather believe that a similar phenomenon of one or more terms, whose presence might be expected, but which do not appear, presents itself in the allied invariantive theory, but cannot speak with certainty as to this point, as the circumstance has not received, and possibly does not merit, any very particular attention.

Still, in the case before us, this unexpected absence of a member of the family, whose appearance might have been looked for, made an impression on my mind, and even went to the extent of acting on my emotions. I began to think of it as a sort of lost Pleiad in an Algebraical Constellation, and in the end, brooding over the subject, my feelings found vent, or sought relief, in a rhymed effusion, a *jeu de sottise*, which, not without some apprehension of appearing singular or extravagant, I will venture to rehearse. It will at least serve as an interlude, and give some relief to the strain upon your attention before I proceed to make my final remarks on the general theory.

TO A MISSING MEMBER

Of a Family Group of Terms in an Algebraical Formula.

Lone and discarded one! divorced by fate,
Far from thy wished-for fellows—whither art flown?
Where lingerest thou in thy bereaved estate,
Like some lost star, or buried meteor stone?
Thou mindst me much of that presumptuous one
Who loth, aught less than greatest, to be great,
From Heaven's immensity fell headlong down
To live forlorn, self-centred, desolate:
Or who, new Heraklid, hard exile bore,
Now buoyed by hope, now stretched on rack of fear,
Till throned Astræa, wafting to his ear
Words of dim portent through the Atlantic roar,
Bade him "the sanctuary of the Muse revere
And strew with flame the dust of Isis' shore."

Having now refreshed ourselves and bathed the tips of our fingers in the Pierian spring, let us turn back for a few brief moments to a light banquet of the reason, and entertain ourselves as a sort of after-course with some general reflections arising naturally out of the previous matter of my discourse. It seems to me that the discovery of Reciprocants must awaken a feeling of surprise akin to that which was felt when the galvanic current astonished the world previously accustomed only to the phenomena of machine or frictional electricity. The new theory is a ganglionic one: it stands in immediate and central relation to almost every branch of pure mathematics—to Invariants, to Differential Equations, ordinary and partial, to Elliptic and Transcendental Functions, to Partitions of Numbers, to the Calculus of Variations, and above all to Geometry (alike of figures and of complexes), upon whose inmost recesses it throws a new and wholly unexpected light. The geometrical singularities which the present portion of the theory professes to discuss are in fact the distinguishing *features* of curves; their *technical* name, if applied to the human countenance, would lead us to call a man's eyes, ears, nose, lips, and chin his singularities; but these singularities make up the character and expression, and serve to distinguish one individual from another. And so it is with the so-called singularities of curves.

Comparing the system of ground-forms which it supplies with those of the allied theory, it seems to me clear that some common method, some yet undiscovered, deep-lying, Algebraical principle remains to be discovered, which shall in each case alike serve to demonstrate the finite number of these forms (these organic radicals) for any specified number of letters. The road to it, I believe, lies in the Algebraical Deduction of ground-forms from the Protomorphs*. Gordan's method of demonstration, so difficult and so complicated, requiring the devotion of a whole University semester to master, is inapplicable to reciprocants, which, as far as we can at present see, do not lend themselves to symbolic treatment.

How greatly must we feel indebted to our Cayley, who while he was, to say the least, the joint founder of the symbolic method, set the first, and out of England little if at all followed, example of using as an engine that mightiest instrument of research ever yet invented by the mind of man—a Partial Differential Equation, to define and generate invariantive forms.

With the growth of our knowledge, and higher views now taken of invariantive forms, the old nomenclature has not altogether kept pace, and is in one or two points in need of a reform not difficult to indicate. I think that we ought to give a general name—I propose that of Binariants—to every rational integral form which is nullified by the general operator

$$\lambda a\delta_b + \mu b\delta_c + \nu c\delta_d + \ldots,$$

where $\lambda, \mu, \nu, \ldots$ are arbitrary numbers.

This operator, I think, having regard to the way in which its segments link on to one another, may be called the Vermicular.

Binariants corresponding to unit values of $\lambda, \mu, \nu, \ldots$ may be termed standard binariants. Those for which these numbers are the terms of the natural arithmetical series 1, 2, 3, ... Invariantive binariants, which may be either complete or incomplete invariants; these latter are what are usually termed semi- or sub-invariants. I may presently have to speak of a third class of binariants for which the arbitrary multipliers are the numbers 3, 8, 15, 24 ... (the squares of the natural numbers each diminished by unity) which, if the theorem I have in view is supported by the event, will have to be termed Reciprocantive Binariants. But first let me call attention to what seems a breach of the asserted parallelism between the Invariantive and the

* See the section on the Algebraical Deduction of the Ground-forms of the Quintic in my memoir on Subinvariants in the *American Journal of Mathematics*. [Vol. III. of this Reprint, p. 580.]

Reciprocantive theories. In the former we have complete and incomplete invariants, but we have drawn no such distinction between one set of pure reciprocants and another. A parallel distinction does however exist.

If we use w, i, j to signify the weight, degree, and extent of an invariantive form, w is never less than the half product of ij; when equal to it the form is complete. In the case of reciprocants certain observed facts seem to indicate that there exists an analogous but less simple inequality. If this conjecture is verified it is not merely $\frac{ij}{2} - w$, but $\frac{ij}{2} - (j-2) - w$, which is never negative: and when this is zero, the form may be said to be complete*. There would then be thus complete forms in each of the two theories; in the earlier one they take a special name: this is the only difference.

We have spoken of Pure Reciprocants as being either projective or non-projective, but so far have abstained from particularising the external characters by which the former may be distinguished from the latter. I have good reason to suspect that the former are distinguished from the latter by being Binariants; that, in addition to being subject to annihilation by the operator V, they are also subject to annihilation by the Vermicular operator when made special by the use of the numerical multipliers 3, 8, 15 ... above alluded to, or in other words (as previously mentioned incidentally) are subject to satisfy two simultaneous partial differential equations instead of only one†.

* If this should turn out to be true, the "crude generating function" for reciprocants would be almost identical with that of in- and co-variants of the same extent j. The denominators would be absolutely identical; as regards the numerators, while that for invariantive forms is $1 - a^{-1}x^{-2}$ the numerator for reciprocants would be $1 - a^{-2}x^{-2j}$. As I write abroad and from memory there is just a chance that the index of a here given may be erroneous.

† As already stated in a previous footnote this conjecture is fully confirmed, my own proof having been corroborated (if it needed corroboration) by another entirely different one invented by M. Halphen, who fully shares my own astonishment at the fact of there being forms (half-horse, half-alligator) at once reciprocants and sub-invariants, and as such satisfying two simultaneous partial differential equations.

If instead of denoting the successive differential derivatives (starting from the second) $a, b, c, \ldots$ we call them $1.2.a$, $1.2.3.b$, $1.2.3.4.c$, ... the two Annihilators will be

$$a\delta_b + 2b\delta_c + 3c\delta_d + 4d\delta_e + \ldots$$

and $$4\frac{a^2}{2}\delta_b + 5ab\delta_c + 6\left(ac + \tfrac{1}{2}b^2\right)\delta_d + 7\left(ad + bc\right)\delta_e + \ldots$$

the latter being my new operator, the Reciprocator V, accommodated to the above-stated change of notation for the successive differential derivatives.

Hardly necessary is it for me to point out in explanation of the semi-sums $\frac{1}{2}b^2$, ... that we may write the MacMahonised V under the form

$$4a^2\delta_b + 5(ab + ba)\delta_c + 6(ac + b^2 + ca)\delta_d + 7(ad + bc + cb + da)\delta_e + \ldots.$$

It is to be presumed that in addition to mixed reciprocants (the ocean into which flows the sea of pure reciprocants, as into that again empties itself the river of projective reciprocants) there may exist a theory of forms in which y as well as $\frac{dy}{dx}$ will appear, or, so to say, doubly mixed reciprocants, the most general of all, in which case we must speak of the content of these as the

Projective Reciprocants we have seen are disguised or masked Ternary Covariants—Covariants in the grub, the first undeveloped state. Now ternary covariants are capable, it may or may not be generally known, of satisfying 6 reducible to 2 simultaneous Partial Differential Equations, and at first sight it might be surmised that nothing would be gained by the substitution of the two new for the two old simultaneous partial differential equations. But the fact is not so, for the old partial differential equations are perfectly unmanageable, or at least have never, as far as I know, been handled by any one, for they have to do with a *triangular heap*, whereas the new ones are solely concerned with a *linear series* of coefficients.

I have alluded to there being a particular form common to the two theories. In the one theory it is the Mongian alluded to in the correspondence, which has been read, with M. Halphen. In the other it is the source of the skew covariant to the cubic. If the latter be subjected to a sort of MacMahonic numerical adjustment, it becomes absolutely identical with the former. Let us imagine that before the invention of Reciprocants an Algebraist happened to have had both forms present to his mind, and had thought of some contrivance for lowering the coefficients of the Mongian written out with the larger coefficients, and had thus stumbled upon this striking fact. It could not have failed to vehemently arouse his curiosity, and he would have set to work to discover, if possible, the cause of this coincidence. He would in all probability have addressed himself to the form which precedes the source alluded to in the natural order of genesis, and have applied a similar adjustment to the much simpler form, $ac-b^2$: having done so he would have tried to discover to what singularity it pointed—but his efforts to do so we know must have been fruitless, and he would have felt disposed to throw down his work in despair, for the intermediate ideas necessary to make out the parallelism would not have been present to his mind. So long as we confine ourselves to Differential Invariants, that is, to projective pure reciprocants, we are like men walking on those elevated ridges, those more than Alpine summits, such as I am told* exist in Thibet, where it may be the labour of days for two men who can see and speak to each other to come together. Reciprocants supply the bridge to span the yawning ravine and to bring allied forms into direct proximity.

ocean and of the others as sea, river, and brook. Curious is it to reflect that in the theory which as it exists comprises Invariantives, Reciprocants, and Invariantive Reciprocants or Reciprocant Invariantives, the order of discovery was (1) Invariantives (Eisenstein, Boole, &c.); (2) Invariantive Reciprocants (Monge and Halphen); (3) Reciprocants (Schwarz, the author of this lecture).

* I think my informant was my friend Dr Inglis, of the Athenæum Club, who some time ago undertook a journey in the Himalayas in the hopes of coming upon the traces of a lost religion which he thought he had reason to believe existed among mankind in the pre-Glacial period of the earth's history.

I have spoken of mixed reciprocants as being subject to satisfy not a linear partial differential equation, but one of a higher order dependent on the intensity, so to say, of its mixedness—the highest power, that is to say, of the first differential derivative which it contains, and it might therefore be supposed that these forms are much more difficult to be obtained than pure reciprocants. But the fact is just the reverse, for as I discovered in the very infancy of the inquiry, and have put on record in the September or October number* of the *Mathematical Messenger*, mixed reciprocants may be evolved in unlimited profusion by the application of simple and explicit processes of multiplication and differentiation. From any reciprocant whatever, be it mixed or pure, new mixed ones may be educed infinitely infinite in number, inasmuch as at each stage of the process, arbitrary functions of the first differential derivative may be introduced.

The wonderful fertility of this method of generation excited warm interest on the part of one of the greatest of living mathematicians, the expression of which acted as a powerful incentive to me to continue the inquiry. They may be compared with the shower of December meteors shooting out in all directions and covering the heavens with their brilliant trains, all diverging from one or more fixed radiant-points, the radiant-point in the theory before us being the particular form selected to be operated upon.

The new doctrine which I have endeavoured thus imperfectly to adumbrate has taken its local rise in this University, where it has already attracted some votaries to its side, and will, I hope, eventually obtain the cooperation of many more. I have ventured with this view to announce it as the subject of a course of lectures during the ensuing term.

When I lately had the pleasure of attending the new Slade Professor's inaugural discourse, I heard him promise to make his pupils participators in his work, by painting pictures in the presence of his class. I aspire to do more than this—not only to paint before the members of my class, but to induce them to take the palette and brush and contribute with their own hands to the work to be done upon the canvas. Such was the plan I followed at the Johns Hopkins University, during my connection with which I may have published scores of Mathematical articles and memoirs in the journals of America, England, France, and Germany, of which probably there was scarcely one which did not originate in the business of the class-room; in the composition of many or most of them I derived inestimable advantage from the suggestions or contributions of my auditors. It was frequently a chase, in which I started the fox, in which we all took a common interest, and in which it was a matter of eager emulation between my hearers and myself to try which could be first in at the death.

[* p. 255 above.]

During the past period of my professorship here, imperfectly acquainted with the usages and needs of the University, I do not think that my labours have been directed so profitably as they might have been either as regards the prosecution of my own work or the good of my hearers: my attention has been distracted between theories waiting to be ushered into existence and providing for the daily bread of class-teaching. I hope that in future I may be able to bring these two objects into closer harmony and correlation, and I think I shall best discharge my duty to the University by selecting for the material of my work in the class-room any subject on which my thoughts may, for the time being, happen to be concentrated, not too alien to, or remote from, that which I am appointed to teach; and thus, by example, give lessons in the difficult art of mathematical thinking and reasoning—how to follow out familiar suggestions of analogy till they broaden and deepen into a fertilising stream of thought—how to discover errors and to repair them, guided by faith in the existence and unity of that intellectual world which exists within us, and is at least as real as that with which we are environed.

The *American Mathematical Journal*, conducted under the auspices of the Johns Hopkins University, which has gained and retains a leading position among the most important of its class, whether measured by the value of its contents or the estimation in which it is held by the Mathematical world, bears as its motto—

πραγμάτων ἔλεγχος οὐ βλεπομένων.

I have the pleasure of seeing among my audience this day the most distinguished geometer of Holland, Prof. Schoute, who has done me the signal honour of coming over to England to be present at this lecture, who hospitably entertained me at Groningen (in a vacation visit which I recently paid to his country, the classic soil which has given birth to an Erasmus, a Grotius, a Boerhaave, a Spinoza, a Huyghens, and a Rembrandt), and who was kind enough, in proposing my health at a party where many of his colleagues were present, to say that he felt sure "that I should return to England cheered and invigorated, and would, ere long, light on some discovery which would excite the wonder of the Mathematical world."

I do not venture to affirm, nor to think, that this vaticination has been fulfilled in the terms in which it was uttered, but can most truly say that the discovery, which it has been my good fortune to be made the medium of revealing, has excited my own deepest feelings of ever-increasing wonder rising almost to awe, such as must have come over the revellers who saw the handwriting start out more and more plainly on the wall, or the *scienziati* crowding round the blurred palimpsest as they began to be able to decipher characters and piece together the sentences of the long lost and supposed irrecoverable *De Republicâ*.

When I was at Utrecht, on my way to Groningen, Mr Grinwis, the Professor of Mathematics at that University, showed me an English book on "Differential Equations," which had just appeared, of which he spoke in high terms of praise, and said it contained over 800 examples. I wrote at once for the book to England, and on seeing it on my arrival, forgetting that it had been ordered, mistook it for a present from the author or publisher, and, what is unusual with me, read regularly into it, until I came to the section on Hyper-geometrical series, where the Schwarzian Derivative (so named by Cayley after Prof. Schwarz) is spoken of.

Perhaps I ought to blush to own that it was new to me, and my attention was riveted by the property it possesses, in common with the more simple form which points to inflexions on curves, of remaining substantially unaltered, of persisting as a factor at least of its altered self, when the variables which enter it are interchanged. Following out this indication, I at once asked myself the question, "ought there not to exist combinations of derivatives of *all* orders possessing this property of reciprocation?" That question was soon answered, and the universe of mixed reciprocants stood revealed before me. These mixed reciprocants, by simple processes of combination, led me to the discovery of the first pure reciprocant, $3b^2 - 5ac$: whereupon I again put the question to myself, "are there, or are there not, others of this form, and if so, what are they?"

In an unexpected manner the question was answered, and my curiosity gratified to the utmost by the discovery of the partial differential equation which is the central point of the theory, and at once discloses the parallelism between it and the familiar doctrine of Invariants. Two principal exponents of that doctrine, who have infused new blood into it, and given it a fresh point of departure—Capt. MacMahon and Mr Hammond—I have the pleasure of seeing before me. Mr Kempe, who is also present, has lately entered into and signally distinguished himself in the same field, availing himself in so doing of his profound insight into the subject of linkages, his interest in which I believe I may say received its first impulse from the lecture which he heard me deliver upon it at the Royal Institution in January 1874, on the very night when the Prime Minister for the time being sent round letters to his supporters announcing his intention to dissolve Parliament. Of the two events I have ever regarded the lecture as by far the more important to the permanent interests of society. He has lately applied ideas founded upon linkages to produce a most original and remarkable scheme for explaining the nature of the whole pure body of Mathematical truth, under whatever different forms it may be clothed, in a memoir which has been recommended to be printed in the *Transactions* of the Royal Society, and which, I think, cannot fail when published to excite the deepest interest alike in the Mathematical and the Philosophical worlds*.

I also feel greatly honoured by the presence of Prof. Greenhill, who will be known to many in this room from his remarkable contributions to the theory of Hydrodynamics and Vortex Motion, and who has sufficient candour and largeness of mind to be able to appreciate researches of a different character from those in which he has himself gained distinction.

I should not do justice to my feelings if I did not acknowledge my deep obligations to Mr Hammond for the assistance which he has rendered me, not only in preparing this lecture which you have listened to with such exemplary patience, but in developing the theory; I am indebted to him for many valuable suggestions tending to enlarge its bounds, and believe have been saved, by my conversations with him, from falling into some serious errors of omission or oversight. Saving only our Cayley (who, though younger than myself, is my spiritual progenitor—who first opened my eyes and purged them of dross so that they could see and accept the higher mysteries of our common Mathematical faith), there is no one I can think of with whom I ever have conversed, from my intercourse with whom I have derived more benefit. It would be an immense gain to Science, and to the best interests of the University, if something could be done to bring such men as Mr Hammond (and, let me add, Mr Buchheim, who ought never to have been allowed to leave it) to come and live among us. I am sure that with their endeavours added to my own and those of that most able body of teachers and researchers with whom I have the good fortune to be associated—my brother Professors and the Tutorial Staff of the University—we could create such a School of Mathematics as might go some way at least to revive the old scientific renown of Oxford, and to light such a candle in England as, with God's grace, should never be put out†.

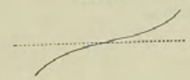
* In his memoir for the *Phil. Trans.* Mr Kempe contends that any whatever mathematical proposition or research is capable of being represented by some sort of simple or compound linkage. One would like to know by what sort of linkage he would represent the substance of the memoir itself.

† I have purposely confined myself in my lecture to reciprocants, indicative of properties of plane curves, but had in view to extend the theory to the case of higher dimensions in space leading to reciprocants involving the differential derivatives of any number of variables $y, z, \ldots$. M. Halphen, with whom I have had the great advantage of being in communication during my stay in Paris, has anticipated me in this part of my plan, and has found that the same method which I have used to obtain the Annihilator V applied to a system of variables leads to an Annihilator of a very similar form to V, and at my request will publish his results in a forthcoming number of the *Comptes Rendus*. Thus the dominion of reciprocants is already extended over the whole range of forms unlimited in their own number as well as in that of the variables which they contain.

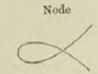
Tables of Singularities and Formulæ referred to in the Preceding Lecture.

Chart 1.

Inflexion

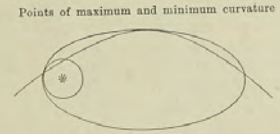


Node



Cusp

Points of maximum and minimum curvature



Bitangent

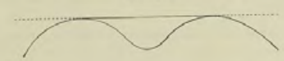


Chart 2.—Protomorphs.

Binariants.	*Reciprocants.*
a	a
$ac-b^2$	$3ac-5b^2$
$a^2d-3abc+2b^3$	$9a^2d-45abc+40b^3$
$ae-4bd+3c^2$	$5a^2e-35abd+7ac^2+35b^2c$
$a^2f+5abe+2acd+8b^2d-6bc^2$	$45a^3f-420a^2be-42a^2cd+1120ab^2d-315abc^2 - 1120b^3c$
$ag-6bf+15ce-10d^2$	$a^2g-12abf-450ace+792b^2e+588ad^2 - 2772bcd+1925c^3$

CHART 3.

No. 1. a

No. 2. $3ac-5b^2$

No. 3. $9a^2d-45abc+40b^3$

No. 4. $45a^3d^2-450a^2bc+192a^2c^3+400ab^3d+165ab^2c^2-400b^4c$

$$x=\int\frac{dt}{\sqrt{\{\kappa(1-15t^2+15t^4-t^6)+\lambda(3t-10t^3+3t^5)\}}}+\mu$$

$$y=\int\frac{t\,dt}{\sqrt{\{\kappa(1-15t^2+15t^4-t^6)+\lambda(3t-10t^3+3t^5)\}}}+\nu$$

$$V=3a^2\delta_b+10ab\delta_c+(15ac+10b^2)\delta_d+(21ad+35bc)\delta_e$$
$$+(28ae+56bd+35c^2)\delta_f+\dots$$

CHART 4.—COEFFICIENTS OF ANNIHILATOR V.

1	4	3			
1	5	10			
1	6	15	10		
1	7	21	35		
1	8	28	56	35	
1	9	36	84	126	
1	10	45	120	210	126

CHART 5.—RECIPROCANT TRANSFORMATIONS.

Grub $\qquad$ *Chrysalis* $\qquad$ *Imago*

$$\frac{d^2y}{dx^2}\qquad\begin{vmatrix}\frac{d^2\phi}{dx^2} & \frac{d^2\phi}{dxdy} & \frac{d\phi}{dx}\\ \frac{d^2\phi}{dxdy} & \frac{d^2\phi}{dy^2} & \frac{d\phi}{dy}\\ \frac{d\phi}{dx} & \frac{d\phi}{dy} & \cdot\end{vmatrix}\qquad\begin{vmatrix}\frac{d^2\Phi}{dx^2} & \frac{d^2\Phi}{dxdy} & \frac{d^2\Phi}{dxdz}\\ \frac{d^2\Phi}{dxdy} & \frac{d^2\Phi}{dy^2} & \frac{d^2\Phi}{dydz}\\ \frac{d^2\Phi}{dxdz} & \frac{d^2\Phi}{dydz} & \frac{d^2\Phi}{dz^2}\end{vmatrix}$$

(a) $\qquad$ (M) $\qquad$ (H)

$$(n-1)^2\left(\frac{d\phi}{dy}\right)^3a+H+\left\{\frac{d^2\Phi}{dx^2}\cdot\frac{d^2\Phi}{dy^2}-\left(\frac{d^2\Phi}{dxdy}\right)^2\right\}\Phi=0.$$

$\frac{dy}{dx}\frac{d^3y}{dx^3}-\frac{3}{2}\left(\frac{d^2y}{dx^2}\right)^2$ is the Schwarzian, otherwise written $tb-\frac{3a^2}{2}$.

CHART 6.—THE H RECIPROCANTIVE PROTOMORPH.

U	W
$65a^4h$	$120a^3cf$
$-975a^3bg$	$-200a^2b^2f$
$-990a^3cf$	$-195a^3de$
$+6200a^2b^2f$	$-145a^2bce$
$+4690a^2bce$	$+1000ab^3e$
$-1540ab^3e$	$+1365a^2bd^2$
$-2730a^2bd^2$	$-777a^2c^2d$
$+7161a^2c^2d$	$-22260ab^2cd$
$+3080ab^2cd$	$+2485abc^3$
$-24255abc^3$	$+105b^3c^2$
$+25410b^3c^2$	

$H+\Lambda U+MW$

The Vermicular Operator.

$\lambda a\delta_b+\mu b\delta_c+\nu c\delta_d+\pi d\delta_e+\dots$

Examples.

$a\delta_b+b\delta_c+c\delta_d+d\delta_e+\dots$

$a\delta_b+2b\delta_c+3c\delta_d+4d\delta_e+\dots$

$3a\delta_b+8b\delta_c+15c\delta_d+24d\delta_e+\dots$

☞ b^4d does not appear in either U or W.

Λ and M are arbitrary numbers.

42.

LECTURES ON THE THEORY OF RECIPROCANTS.

[*American Journal of Mathematics*, VIII. (1886), pp. 196—260; IX. pp. 1—37, 113—161, 297—352; X. pp. 1—16. Delivered in Oxford, 1886.]

THE lectures here reproduced were delivered, or are still in the course of delivery, before a class of graduates and scholars in the University of Oxford during the present year. They are to be regarded as easy lessons in the new Theory of Reciprocants of which an outline will be found in *Nature* for January 7, which contains a report of a Public Lecture on the subject delivered before the University of Oxford in December of the preceding year.

They are designed as a practical introduction to an enlarged theory of Algebraical Forms, and as such are not constructed with the rigorous adhesion to logical order which might be properly expected in a systematic treatise. The object of the lecturer was to rouse an interest in the subject, and in pursuit of this end he has not hesitated to state many results, by way of anticipation, which might, with stricter regard to method, have followed at a later period in the course.

There will be found also occasional repetitions and intercalations of allied topics which are to be justified by the same plea, and also by the fact that the lectures were not composed in their entirety previous to delivery, but gradually evolved and written between one lecture and another in the way that seemed most likely to the lecturer to secure the attention of his auditors.

Since the delivery of his public lecture in December last, papers have been contributed on the subject to the *Proceedings of the Mathematical Society of London* by Messrs Hammond, MacMahon, Elliott, Leudesdorf and Rogers, and one to the *Comptes Rendus de l'Institut* by M. George Perrin. It may therefore be inferred that the lectures have not altogether failed in attaining the desired end of drawing attention to a subject which, in the opinion of the lecturer, constitutes a very considerable extension of the previous limits of algebraical science.

LECTURE I.

A new world of Algebraical forms, susceptible of important geometrical applications, has recently come into existence, of which I gave a general account in a public lecture at the end of last term. I propose in the following brief course to go more fully into the subject and lay down the leading principles of the theory so far as they are at present known to me. The parallelism between the theory of what may be called pure reciprocants and that of invariants is so remarkable that it will be frequently expedient to pass from one theory to the other or to treat the two simultaneously. It may be as well therefore at once to give notice that the term invariant will hereafter be applied alike to invariants ordinarily so called and to those more general algebraical forms which have been termed sources of covariants, differentiants, seminvariants, or subinvariants. A form which is an invariant in the old sense will be termed, when necessary to specify it, a satisfied invariant, an expression which the chemico-graphical representation of invariants or covariants will serve to explain and justify.

In an elucidatory course of lectures such as the present, it will be advisable to follow a freer order of treatment than would be suitable to the presentation of it in a systematic memoir. My object is to make the theory known, to excite curiosity regarding it, and to invite co-operation in the task of its development.

By way of introduction to the subject, let us begin with an investigation of the properties of a differential expression involving only the first, second and third differential coefficients of either of two variables in respect to the other. For this purpose let us consider not what I have called the Schwarzian itself, which is an integral rational function of these three quantities, but the fractional expression

$$\frac{\dfrac{d^3y}{dx^3}}{\dfrac{dy}{dx}} - \frac{3}{2}\left(\frac{\dfrac{d^2y}{dx^2}}{\dfrac{dy}{dx}}\right)^2$$

which becomes the Schwarzian when cleared of fractions, and which after Cayley we may call the Schwarzian Derivative and denote by

$$(y, x);$$

(x, y) will then of course mean

$$\frac{\dfrac{d^3x}{dy^3}}{\dfrac{dx}{dy}} - \frac{3}{2}\left(\frac{\dfrac{d^2x}{dy^2}}{\dfrac{dx}{dy}}\right)^2.$$

It is easy to establish the identical equation

$$(y, x) = -\left(\frac{dy}{dx}\right)^2 (x, y). \qquad (1)$$

Using for brevity y', y'', y''' to denote, as usual,

$$\frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \frac{d^3y}{dx^3},$$

and $x_{,}, x_{,,}, x_{,,,}$ to denote

$$\frac{dx}{dy}, \quad \frac{d^2x}{dy^2}, \quad \frac{d^3x}{dy^3},$$

respectively, the relation to be verified is

$$\frac{2y'y''' - 3y''^2}{y'^2} = -y'^2 . \frac{2x_{,}x_{,,,} - 3x_{,,}^2}{x_{,}^2}.$$

Now,
$$x_{,} = \frac{1}{y'},$$

$$x_{,,} = \frac{d}{dy}(x_{,}) = \frac{1}{y'} \cdot \frac{d}{dx}\left(\frac{1}{y'}\right) = -\frac{y''}{y'^3},$$

and
$$x_{,,,} = \frac{d}{dy}(x_{,,}) = \frac{1}{y'} \cdot \frac{d}{dx}\left(-\frac{y''}{y'^3}\right) = -\frac{y'''}{y'^4} + \frac{3y''^2}{y'^5}.$$

Whence we obtain

$$2x_{,}x_{,,,} - 3x_{,,}^2 = \left(-\frac{2y'''}{y'^5} + \frac{6y''^2}{y'^6}\right) - \frac{3y''^2}{y'^6}$$

$$= -\frac{1}{y'^6}(2y'y''' - 3y''^2),$$

and the truth of (1) is manifest.

This may be put under the form

$$\frac{2y'y''' - 3y''^2}{y'^3} = -\frac{2x_{,}x_{,,,} - 3x_{,,}^2}{x_{,}^3},$$

showing that a certain function of the first, second and third derivatives of one variable in respect to another remains unaltered, save as to algebraical sign, when the variables are interchanged. An example of a similar kind with which we are all familiar is presented by the well-known function $\frac{d^2y}{dx^2} \div \left(\frac{dy}{dx}\right)^{\frac{3}{2}}$, which is equal to $-\frac{d^2x}{dy^2} \div \left(\frac{dx}{dy}\right)^{\frac{3}{2}}$.

We are thus led to inquire whether there may not be an infinite number of algebraical functions of differential derivatives which possess a similar property, and by prosecuting this inquiry to lay the foundations of the theory of Reciprocation or Reciprocants.

Having regard to the fact that the present theory originated in that of the Schwarzian Derivative, I shall proceed to demonstrate, although this is not strictly necessary for the theory of Reciprocants, the remarkable identity

$$(y, x) - (z, x) = \left(\frac{dz}{dx}\right)^2 . (y, z).$$

This identical relation is the fundamental property of Schwarzians, and from it every other proposition concerning their form is an immediate deduction.

In the following proof*, y and z are regarded as two given functions of any variable t, and x as a variable function of the same: so that y and z are functions of x for any given function that x is of t.

It will be seen that

$$\{(y, x) - (z, x)\} \left(\frac{dx}{dz}\right)^2$$

remains unaltered by any infinitesimal variation θ of x, that is, when x becomes $x + \epsilon\phi(x)$, ϵ being an infinitesimal constant and $\phi(x)$ an arbitrary finite function of x.

For brevity, let accents denote differential derivation in regard to x, and let any function of x enclosed in a square parenthesis signify the augmented value of that function when x becomes $x + \theta$. In calculating such augmented values, since we suppose that $\theta = \epsilon\phi(x)$, it is clear that $\theta, \theta', \theta'' \ldots$ are each of them infinitesimals of the first order, and consequently that all products, and all powers higher than the first of these quantities, may be neglected.

We have therefore

$$[y'] = \frac{dy}{dx + d\theta} = \frac{y'}{1 + \theta'} = y' - \theta' y'$$

$$[y''] = \frac{d[y']}{dx + d\theta} = \frac{\frac{d}{dx}(y' - \theta' y')}{1 + \theta'} = \frac{y''(1 - \theta') - \theta'' y'}{1 + \theta'}$$

$$= y'' - 2\theta' y'' - \theta'' y'$$

$$[y'''] = \frac{d[y'']}{dx + d\theta} = \frac{\frac{d}{dx}(y'' - 2\theta' y'') - \theta'' y'}{1 + \theta'} = \frac{y'''(1 - 2\theta') - 3\theta'' y'' - \theta''' y'}{1 + \theta'}$$

$$= y''' - 3\theta' y''' - 3\theta'' y'' - \theta''' y'.$$

Hence

$$[y'y'''] = y'y''' - 4\theta' y' y''' - 3\theta'' y' y'' - \theta''' y'^2$$

$$\tfrac{3}{2}[y''^2] = \tfrac{3}{2} y''^2 - 6\theta' y''^2 - 3\theta'' y' y''$$

$$[y'^2] = y'^2 - 2\theta' y'^2.$$

And since by definition

$$(y, x) = \frac{y'y''' - \frac{3}{2} y''^2}{y'^2},$$

* As originally given in the *Messenger of Mathematics*, Vol. xv., this was defaced by so many errata as to render expedient its reproduction in a corrected form.

we readily obtain

$$[(y, x)] = \frac{(y, x)}{1 - 2\theta'} - 4\theta'(y, x) - \theta''' = (y, x)(1 - 2\theta') - \theta'''.$$

So also

$$[(z, x)] = (z, x)(1 - 2\theta') - \theta'''.$$

Whence by subtraction

$$[(y, x) - (z, x)] = (1 - 2\theta')\{(y, x) - (z, x)\}.$$

Dividing the left-hand side of this by $[z'^2]$, and the right-hand side by $z'^2(1 - 2\theta')$ which is the equivalent of $[z'^2]$, our final result is

$$\left[\frac{(y, x) - (z, x)}{z'^2}\right] = \frac{(y, x) - (z, x)}{z'^2}.$$

Thus, then, we have seen that the expression

$$\frac{(y, x) - (z, x)}{\left(\frac{dz}{dx}\right)^2}$$

does not vary when x receives an infinitesimal variation $\epsilon\phi(x)$, from which it follows, by the general principle of successive continuous accumulation, that the same will be true when x undergoes any finite arbitrary variation, and consequently this expression has a value which is independent of the form of x regarded as a function of t; it will, of course, be remembered that y and z are supposed to be invariable functions of t. Let x become z, then (y, x) becomes (y, z), while at the same time (z, x) vanishes and $\frac{dz}{dx}$ becomes unity: so that we obtain

$$\frac{(y, x) - (z, x)}{\left(\frac{dz}{dx}\right)^2} = (y, z).$$

Hence, *whatever* function x may be of t,

$$(y, x) - (z, x) = \left(\frac{dz}{dx}\right)^2 . (y, z). \qquad (2)$$

To this fundamental proposition the equation marked (1), itself the important point in regard to the Theory of Reciprocants, is an immediate corollary. For if in (2) we interchange y and z, it becomes

$$(z, x) - (y, x) = \left(\frac{dy}{dx}\right)^2 . (z, y);$$

and now, making $x = z$, we have

$$-(y, z) = \left(\frac{dy}{dz}\right)^2 . (z, y).$$

which is the same as (1), except that z occupies the place of x.

But (1) may be obtained more immediately from (2) by substituting in it x for y and y for z, leaving x unaltered; when it becomes

$$-(y, x) = \left(\frac{dy}{dx}\right)^2 . (x, y).$$

This is equivalent to saying that

$$2y'y''' - 3y''^2 = -y'^6 (2x_{,}x_{,,,} - 3x_{,,}^2),$$

a verification of which has been given already.

Observe that $\dfrac{y'y''' - \frac{3}{2}y''^2}{y'^2}$ or (y, x) contains $\left(\dfrac{dy}{dx}\right)^2$ in its denominator and (x, y) contains $\left(\dfrac{dx}{dy}\right)^2$ in its denominator, which is the same as $\left(\dfrac{dy}{dx}\right)^2$ in the numerator. Thus it is that the *square* of $\dfrac{dy}{dx}$ enters three times.

Let me insist for a moment on the import of the fact brought to light in the course of this investigation, that $\dfrac{(y, x) - (z, x)}{\left(\dfrac{dz}{dx}\right)^2}$ is invariable when x, y and z being regarded as functions of t, x alters its form, but y and z retain theirs. Of course we might write $\left(\dfrac{dy}{dx}\right)^2$ in the denominator instead of $\left(\dfrac{dz}{dx}\right)^2$, and then make the same affirmation as before; as will be evident if we only remember that by hypothesis y and z are both of them constant functions of t, and that therefore $\left(\dfrac{dz}{dy}\right)^2$ must also be so. This is tantamount to saying that when the same conditions are fulfilled $\{(y, x) - (z, x)\} (dx)^2$ is invariable, that is, that when x becomes X in virtue of any substitution (including a homographic one) impressed upon it,

$$\{(y, x) - (z, x)\} (dx)^2 = \{(y, X) - (z, X)\} (dX)^2,$$

and thus we see that when x becomes X,

$$(y, x) - (z, x)$$

remains unaltered except that it takes to itself the factor $\left(\dfrac{dX}{dx}\right)^2$ which depends solely on the particular substitution impressed on x.

If $\qquad y = f(x),\ z = \phi(x),$ and $X = \omega(x),$

our formula becomes

$$\{(fx, x) - (\phi x, x)\} (dx)^2 = \{(f\omega^{-1}X, X) - (\phi\omega^{-1}X, X)\} (dX)^2,$$

so that, speaking of Quantics and Covariants with respect to a single variable x, $(fx, x) - (\phi x, x)$ is to all intents and purposes a Covariant to the simultaneous forms $f(x)$ and $\phi(x)$, in a sense comprehending but far transcending that in which the term is ordinarily employed; for it remains a persistent

factor of its altered self when for x any arbitrary function of x is substituted, the new factor taken on depending wholly and solely on the particular substitution impressed upon x. In the ordinary theory of invariants, the substitution impressed is limited to be homographic; in this case it is absolutely general. We might, moreover, add as a corollary that if (y, x), (z, x), (u, x) ... are regarded as roots of any Binary Quantic, every invariant of that Binary Quantic is a covariant in the extended sense in which the word has just been used, in respect to the system of simultaneous forms $f(x)$, $\phi(x)$, $\psi(x)$ .... For every such invariant will be a function of

$$(y, x) - (z, x),\quad (y, x) - (u, x),\quad (z, x) - (u, x),\ \ldots$$

and will therefore remain a persistent factor of its altered self, taking on a power of $\dfrac{dX}{dx}$ as its extraneous factor.

Calling (fx, x) the Schwarzian Derivative of $f(x)$, our theorem may be stated in general terms as follows:

All invariants of a Binary Quantic whose roots are the Schwarzian Derivatives of a given set of functions of the same variable are Covariants (in an extended sense) of that set of functions.

The theory of the Schwarzian derivative originates in that of the linear differential equation of the second order,

$$u'' + 2Pu' + Qu = 0,$$

which becomes, when we write $u = ve^{-\int P dx}$,

$$v'' + Iv = 0,$$

where $\qquad I = Q - P^2 - P'.$

Now, suppose that u_1 and u_2 are any two particular solutions of the first of these equations, and let z denote their mutual ratio; so that, when v_1 and v_2 are the corresponding particular solutions of the second equation, we readily obtain

$$z = \frac{u_2}{u_1} = \frac{v_2}{v_1},$$

and therefore, $\qquad z' = \dfrac{v_1 v_2' - v_2 v_1'}{v_1^2}.$

A second differentiation gives

$$z'' = \frac{v_1 v_2'' - v_2 v_1''}{v_1^2} - \frac{2v_1'(v_1 v_2' - v_2 v_1')}{v_1^3}.$$

But since $\qquad \dfrac{v_1''}{v_1} = \dfrac{v_2''}{v_2} = -I,$

the first term of the expression just found vanishes identically, and we have

$$z'' = -\frac{2v_1' z'}{v_1},$$

or,
$$v_1' = -\frac{z''v_1}{2z'}.$$

Differentiating this again, we find

$$-2v_1'' = \left(\frac{z'''}{z'} - \frac{z''^2}{z'^2}\right)v_1 + \frac{z''}{z'}v_1'$$
$$= \left(\frac{z'''}{z'} - \frac{3}{2}\frac{z''^2}{z'^2}\right)v_1.$$

Hence
$$\frac{z'''}{z'} - \frac{3}{2}\frac{z''^2}{z'^2} = 2I,$$

where the left-hand side of the equation is "the Schwarzian Derivative" with z written in the place of y.

LECTURE II.

The expression $2y'y''' - 3y''^2$, which we have called the Schwarzian, may be termed a reciprocant, meaning thereby that on interchanging y', y'', y''' with $x_,, x_{,,}, x_{,,,}$ its form remains unaltered, save as to the acquisition of what may be called an extraneous factor, which, in the case before us, is a power of y' (with a multiplier -1). Before we proceed to consider other examples of reciprocants it will be useful to give formulae by means of which the variables may be readily interchanged in any differential expression.

We shall write t for y' and τ for its reciprocal $x_,$, using the letters $a, b, c, \ldots$ to denote the second, third, fourth, etc., differential derivatives of y with respect to x, and $\alpha, \beta, \gamma, \ldots$ to denote those of x with respect to y. The advantage of this notation will be seen in the sequel.

The values of $\alpha, \beta, \gamma, \ldots$ in terms of $t, a, b, c, \ldots$ are given by the formulae

$$\alpha = -a \div t^3,$$
$$\beta = -bt + 3a^2 \div t^5,$$
$$\gamma = -ct^2 + 10abt - 15a^3 \div t^7,$$
$$\delta = -dt^3 + (15ac + 10b^2)t^2 - 105a^2bt + 105a^4 \div t^9,$$
$$\epsilon = -et^4 + (21ad + 35bc)t^3 - (210a^2c + 280ab^2)t^2 + 1260a^3bt - 945a^5 \div t^{11},$$
....

If, in these equations, we write

$$a = 1.2.a_0,\quad b = 1.2.3.a_1,\quad c = 1.2.3.4.a_2, \ldots$$

and
$$\alpha = 1.2.\alpha_0,\quad \beta = 1.2.3.\alpha_1,\quad \gamma = 1.2.3.4.\alpha_2, \ldots$$

they become

$$\alpha_0 = -a_0 \div t^3,$$
$$\alpha_1 = -a_1t + 2a_0^2 \div t^5,$$
$$\alpha_2 = -a_2t^2 + 5a_0a_1t - 5a_0^3 \div t^7,$$
$$\alpha_3 = -a_3t^3 + (6a_0a_2 + 3a_1^2)t^2 - 21a_0^2a_1t + 14a_0^4 \div t^9,$$
$$\alpha_4 = -a_4t^4 + (7a_0a_3 + 7a_1a_2)t^3 - (28a_0^2a_2 + 28a_1^2)t^2 + 84a_0^3a_1t - 42a_0^5 \div t^{11},$$
....

Any one of the formulae in either set may be deduced from the formula immediately preceding it by a simple process of differentiation.

Thus, since
$$\beta = \frac{-bt + 3a^2}{t^5} \text{ and } \frac{d}{dy} = \frac{1}{t}\cdot\frac{d}{dx},$$

we have
$$\frac{d\beta}{dy} = \frac{1}{t}\cdot\frac{d}{dx}\left(\frac{-bt + 3a^2}{t^5}\right).$$

But
$$\frac{d\beta}{dy} = \gamma \text{ and } \frac{d}{dx} = a\partial_t + b\partial_a + c\partial_b + \ldots,$$

so that
$$\gamma = \frac{1}{t}(a\partial_t + b\partial_a + c\partial_b + \ldots)\left(\frac{-bt + 3a^2}{t^5}\right)$$
$$= \frac{1}{t^7}(-ct^2 + 10abt - 15a^3).$$

By continually operating with $\frac{1}{t}(a\partial_t + b\partial_a + c\partial_b + \ldots)$ the table may be extended as far as we please, the expressions on the right-hand side being the successive values of

$$\left\{\frac{1}{t}(a\partial_t + b\partial_a + c\partial_b + \ldots)\right\}^n\left(-\frac{a}{t^3}\right)$$

found by giving to n the values $0, 1, 2, 3, \ldots$.

Precisely similar reasoning shows that, when the modified letters $a_0, a_1, a_2, \ldots$ are used,

$$(n+2)\alpha_n = \frac{1}{t}(2a_0\partial_t + 3a_1\partial_{a_0} + 4a_2\partial_{a_1} + \ldots)\alpha_{n-1},$$

and that
$$\alpha_n = \frac{\left\{\frac{1}{t}(2a_0\partial_t + 3a_1\partial_{a_0} + 4a_2\partial_{a_1} + \ldots)\right\}^n\left(-\frac{a_0}{t^3}\right)}{3.4.5\ldots(n+2)}.$$

A proof of the formula

$$\alpha_n = -t^{-n-3}(e^{-\frac{V}{t}})a_n,$$

obtained by Mr Hammond, in which

$$V = 4.\frac{a_0^2}{2}\partial_{a_1} + 5a_0a_1\partial_{a_2} + 6\left(a_0a_2 + \frac{a_1^2}{2}\right)\partial_{a_3} + 7(a_0a_3 + a_1a_2)\partial_{a_4} + \ldots,$$

will be given later on, when we treat of this operator, which, in the theory of Reciprocants, is the analogue of the operator $a\partial_b + 2b\partial_c + 3c\partial_d + \ldots$, with which we are familiarly acquainted in the theory of Invariants.

Consider the expression

$$ct - 5ab.$$

If, in $\gamma\tau - 5\alpha\beta$, which may be called its transform, we write

$$\tau = \frac{1}{t}, \quad \alpha = -\frac{a}{t^3}, \quad \beta = \frac{-bt + 3a^2}{t^5}, \quad \gamma = \frac{-ct^2 + 10abt - 15a^3}{t^7},$$

this becomes a fraction whose denominator is t^8, while its numerator is

$$-ct^2 + 10abt - 15a^3 + 5a(-bt + 3a^2) = -ct^2 + 5abt.$$

Removing the common factor t from the numerator and denominator of this fraction, we have

$$\gamma\tau - 5\alpha\beta = -\frac{ct - 5ab}{t^7}.$$

Here, then, as in the case of the well-known monomial for which

$$a = -t^3\alpha,$$

and the Schwarzian for which

$$2bt - 3a^2 = -t^6(2\beta\tau - 3\alpha^2),$$

the expression

$$ct - 5ab = -t^7(\gamma\tau - 5\alpha\beta)$$

changes its sign on reciprocation.

That reciprocation is not always accompanied with a change of sign will be clear if we consider the product of any pair of the three expressions given above. Or we may take, as an example of a reciprocant in which this change of sign does not occur, the form

$$3ac - 5b^2.$$

Here

$$3\alpha\gamma - 5\beta^2 = \frac{3a(ct^2 - 10abt + 15a^3) - 5(bt - 3a^2)^2}{t^{10}}.$$

In the fraction on the right-hand side the only surviving terms of the numerator are those containing the highest power of t, the rest destroying one another. Thus

$$3\alpha\gamma - 5\beta^2 = \frac{1}{t^8}(3ac - 5b^2).$$

Reciprocants which change their sign when the variables x and y are interchanged, will be said to be of odd character; those, on the contrary, which keep their sign unchanged will be said to be of even character. The distinction is an important one, and will be observed in what follows.

Forms such as the one just considered, where t does not appear in the form itself, but only in the extraneous factor, will be called Pure Reciprocants, in order to distinguish them from those forms (of which the Schwarzian $2tb - 3a^2$ is an example) into which t enters, which will be called Mixed Reciprocants. It will be seen hereafter that Pure Reciprocants are the analogues of the invariants of Binary Quantics.

With modified letters (that is, writing $a = 2a_0$, $b = 6a_1$, and $c = 24a_2$)

$$3ac - 5b^2 \text{ becomes } 144a_0a_2 - 180a_1^2 = 36(4a_0a_2 - 5a_1^2).$$

Operating on this with

$$V = 2a_0^2\partial_{a_1} + 5a_0a_1\partial_{a_2} + \ldots,$$

we have

$$V(4a_0a_2 - 5a_1^2) = 0.$$

We shall prove subsequently that all Pure Reciprocants are, in like manner, subject to annihilation by the operator V.

Hitherto we have only considered homogeneous forms; let us now take as an example of a non-homogeneous reciprocant the expression

$$(1 + t^2)b - 3a^2t.$$

Here

$$(1 + \tau^2)\beta - 3\alpha^2\tau = \left(1 + \frac{1}{t^2}\right)\left(\frac{-bt + 3a^2}{t^5}\right) - \frac{3a^2}{t^7}$$

$$= \frac{(1 + t^2)(-bt + 3a^2) - 3a^2}{t^7}.$$

In the numerator of this fraction the terms $+3a^2$ and $-3a^2$ cancel, a factor t divides out, and we have finally

$$(1 + \tau^2)\beta - 3\alpha^2\tau = -\frac{(1 + t^2)b - 3a^2t}{t^6}.$$

In general, a Reciprocant may be defined to be a function F of such a kind that $F(\tau, \alpha, \beta, \gamma, \ldots)$ contains $F(t, a, b, c, \ldots)$ as a factor. An important special case is that in which the other factor is merely numerical; the function F is then said to be an Absolute Reciprocant.

When we limit ourselves to the case where F is a rational integral function of the letters, it may be proved that

$$F(t, a, b, c, \ldots) = \pm t^{\mu} F(\tau, \alpha, \beta, \gamma, \ldots).$$

For, in the first place, since any one of the letters $\alpha, \beta, \gamma, \ldots$ is a rational function of $t, a, b, c, \ldots$ and integral with respect to all of them except t, containing only a power of this letter in the denominator, it is clear that any rational integral function of $\tau, \alpha, \beta, \gamma, \ldots$ such as $F(\tau, \alpha, \beta, \gamma, \ldots)$ is supposed to be, must be a rational integral function of $t, a, b, c, \ldots$ divided by some power of t. But since F is a reciprocant, $F(\tau, \alpha, \beta, \gamma, \ldots)$ must contain $F(t, a, b, c, \ldots)$ as a factor; and if we suppose the other factor to be

$$\frac{\phi(t, a, b, c, \ldots)}{t^{\lambda}},$$

we must have

$$F(\tau, \alpha, \beta, \gamma, \ldots) = \frac{\phi(t, a, b, c, \ldots)}{t^{\lambda}} F(t, a, b, c, \ldots),$$

where ϕ is rational and integral with respect to all the letters.

Moreover, $F(t, a, b, c, \ldots) = \dfrac{\phi(\tau, \alpha, \beta, \gamma, \ldots)}{\tau^\lambda} F(\tau, \alpha, \beta, \gamma, \ldots)$.

Hence we must have identically

$$\phi(t, a, b, c, \ldots)\, \phi(\tau, \alpha, \beta, \gamma, \ldots) = 1,$$

where, on the supposition that the functions ϕ contain other letters besides t and τ, $\phi(t, a, b, c, \ldots)$ is, and $\phi(\tau, \alpha, \beta, \gamma, \ldots)$ can be expressed as, a rational function integral as regards the letters $a, b, c, \ldots$. But this supposition is manifestly inadmissible, for the product of two integral rational functions of $a, b, c, \ldots$ cannot be identically equal to unity. Hence t is the only letter that can appear in the extraneous factor and we may write

$$F(\tau, \alpha, \beta, \gamma, \ldots) = \frac{\psi(t)}{t^\lambda} F(t, a, b, c, \ldots)$$

where $\psi(t)$ is a rational integral function.

The same reasoning as before shows that we must have identically

$$\psi(t)\, \psi(\tau) = 1.$$

But this cannot be true if $\psi(t)$ has any root different from zero; for if we give t such a value as will make $\psi(t)$ vanish, this value must also make $\psi(\tau)$ infinite; and since

$$\psi(\tau) = A + B\tau + C\tau^2 + \ldots + M\tau^m$$
$$= A + \frac{B}{t} + \frac{C}{t^2} + \ldots + \frac{M}{t^m},$$

the only value of t for which $\psi(\tau)$ becomes infinite is a zero value. Hence $\psi(t)$ is of the form Mt^m, and consequently $\psi(\tau) = M\tau^m$. Thus

$$\psi(t)\, \psi(\tau) = M^2 t^m \tau^m = 1,$$

and therefore $$M^2 = 1.$$

We have now proved that if F is a rational integral reciprocant,

$$F(t, a, b, c, \ldots) = \pm t^\mu F(\tau, \alpha, \beta, \gamma, \ldots),$$

or we may say, $$= (-)^\kappa t^\mu F(\tau, \alpha, \beta, \gamma, \ldots),$$

where $\kappa = 1$ or 0 according as the reciprocant is of odd or even character.

It obviously follows that the product or quotient of any two rational integral reciprocants is itself a reciprocant; but it must be carefully observed that this is not true of their sum or difference unless certain conditions are fulfilled. For if we write

$$F_1(t, a, \ldots) = (-)^{\kappa_1} t^{\mu_1} F_1(\tau, \alpha, \ldots)$$

and $$F_2(t, a, \ldots) = (-)^{\kappa_2} t^{\mu_2} F_2(\tau, \alpha, \ldots),$$

we see that

$$pF_1(t, a, \ldots) + qF_2(t, a, \ldots) = (-)^{\kappa_1} t^{\mu_1} pF_1(\tau, \alpha, \ldots) + (-)^{\kappa_2} t^{\mu_2} qF_2(\tau, \alpha, \ldots),$$

and consequently this expression will be a reciprocant if $\kappa_1 = \kappa_2$ and $\mu_1 = \mu_2$, but not otherwise. If we call the index of t in the extraneous factor the *characteristic*, what we have proved is that no linear function of two reciprocants can be a reciprocant, unless they have the same characteristic and are of the same character. In dealing with Absolute Reciprocants, since the characteristic of these is always zero, we need only attend to their character.

I propose for the present to confine myself to homogeneous and isobaric reciprocants*, that is, to such as are homogeneous and isobaric when the letters $t, a, b, c, \ldots$ are considered to be each of degree 1, their respective weights being $-1, 0, 1, 2, \ldots$. The letter w will be used to denote the weight of such a reciprocant, i its degree, and j its extent, that is, the weight of the most advanced letter which it contains.

Let any such reciprocant $F(t, a, b, c, \ldots)$ contain a term $At^v a^l b^m c^n \ldots$, then

$$v + l + m + n + \ldots = i,$$

and $$-v + m + 2n + \ldots = w.$$

The corresponding term in $F(\tau, \alpha, \beta, \gamma, \ldots)$ will be $A\tau^v \alpha^l \beta^m \gamma^n \ldots$ where

$$\tau = \frac{1}{t},\ \alpha = -\frac{a}{t^3},\ \beta = -\frac{b}{t^4} + \ldots,\ \gamma = -\frac{c}{t^5} + \ldots, \text{ etc.}$$

Now, if no term of F contains a smaller number of the letters $a, b, c, \ldots$ than are found in the term we are considering, the first terms of β, γ, etc., may be taken instead of these quantities themselves and $A\tau^v \alpha^l \beta^m \gamma^n \ldots$ may be replaced by

$$(-)^{l+m+n+\ldots} At^{-v-3l-4m-5n-\ldots} a^l b^m c^n \ldots = (-)^{i-v} At^{v-3i-w} a^l b^m c^n \ldots.$$

But since $$F(t, a, b, c, \ldots) = (-)^\kappa t^\mu F(\tau, \alpha, \beta, \gamma, \ldots)$$

we must have identically

$$At^v a^l b^m c^n \ldots = (-)^{i-v+\kappa} At^{\mu+v-3i-w} a^l b^m c^n \ldots.$$

Hence the character is even or odd according to the parity of $i - v$ (that is, of the smallest number of letters different from t in any term), and the characteristic $\mu = 3i + w$.

The type of a reciprocant depends on the *character*, weight, degree and extent. As the extraneous factor is always of the form $(-)^\kappa t^\mu$, where κ is 1 or 0, we may define the type of a reciprocant by

$$1 : w : i, j \quad \text{or} \quad 0 : w : i, j,$$

according as its character is odd or even.

For Pure Reciprocants the smallest number of letters different from t in any term is (since all the letters are different from t) the same as its degree.

* Here and elsewhere the word *reciprocant* is used in the sense of *rational integral reciprocant*: this will always be done when there is no danger of confusion arising from it.

Hence the character of a Pure Reciprocant is odd or even according to the parity of i, and for this reason the type of a Pure Reciprocant may be defined by $w:i, j$.

A linear combination of reciprocants of the same type will be a reciprocant, for when the type is known both the character and characteristic are given.

LECTURE III.

Let F be any function (not necessarily homogeneous or even algebraical) of the differential derivatives which acquires a numerical multiplier M, but is otherwise unchanged when the reciprocal substitution of x for y and y for x is effected. A second reciprocation multiplies the function again by M, and thus the total effect of both substitutions is to multiply F by M^2. But since the second reciprocation reproduces the original function, we must have $M^2 = 1$. Functions of this kind are therefore unaltered by reciprocation (except it may be in sign), and for this reason are called *Absolute Reciprocants.* These, as we shall presently see, play an important part in the general theory. Like all other reciprocants, they range naturally in two distinct classes, those of odd and those of even character.

It is perhaps worthy of notice that the extraneous factor of a general reciprocant is the exponential of an absolute reciprocant of odd character. For if

$$F(t, a, b, c, \ldots) = \phi(t, a, b, c, \ldots)\, F(\tau, \alpha, \beta, \gamma, \ldots),$$

we must still have, as before,

$$\phi(t, a, b, c, \ldots)\, \phi(\tau, \alpha, \beta, \gamma, \ldots) = 1;$$

that is

$$\log \phi(t, a, b, c, \ldots) = -\log \phi(\tau, \alpha, \beta, \gamma, \ldots);$$

or $\log \phi(t, a, b, c, \ldots)$ is an absolute reciprocant of odd character.

An absolute reciprocant may be obtained from any pair of rational integral reciprocants in the same way that an absolute invariant is found from two ordinary invariants. For let

$$F_1(t, a, b, c, \ldots) = (-)^{\kappa_1} t^{\mu_1} F_1(\tau, \alpha, \beta, \gamma, \ldots),$$

and

$$F_2(t, a, b, c, \ldots) = (-)^{\kappa_2} t^{\mu_2} F_2(\tau, \alpha, \beta, \gamma, \ldots),$$

then

$$\frac{\{F_1(t, a, b, c, \ldots)\}^{\mu_2}}{\{F_2(t, a, b, c, \ldots)\}^{\mu_1}} = (-)^{\kappa_1\mu_2 - \kappa_2\mu_1} \frac{\{F_1(\tau, \alpha, \beta, \gamma, \ldots)\}^{\mu_2}}{\{F_2(\tau, \alpha, \beta, \gamma, \ldots)\}^{\mu_1}};$$

or we may say that $F_1^{\mu_2} \div F_2^{\mu_1}$ is an absolute reciprocant of even or odd character according to the parity of $\kappa_1\mu_2 - \kappa_2\mu_1$.

Thus, for example, from

$$a = -t^3\alpha$$

and

$$3ac - 5b^2 = t^6(3\alpha\gamma - 5\beta^2)$$

we form $\dfrac{(3ac-5b^2)^3}{a^8}$, an absolute reciprocant of even character.

From a reciprocant F whose characteristic is μ we obtain an absolute reciprocant of the same character as F by dividing it by $t^{\frac{\mu}{2}}$.

For if we only remember that $\tau = \dfrac{1}{t}$, it obviously follows that

$$F(t, a, b, c, \ldots) = \pm t^{\mu} F(\tau, \alpha, \beta, \gamma, \ldots)$$

can be written in the form

$$\frac{F(t, a, b, c, \ldots)}{t^{\frac{\mu}{2}}} = \pm \frac{F(\tau, \alpha, \beta, \gamma, \ldots)}{\tau^{\frac{\mu}{2}}},$$

where the original character of the reciprocant F is preserved.

It may be noticed that a reciprocant of odd character cannot be divided by $\sqrt{(-1)}\, t^{\frac{\mu}{2}}$ so as to give an absolute reciprocant of even character; for, the reciprocal of F being $-t^{\mu}F'$, that of $F \div \sqrt{(-1)}\, t^{\frac{\mu}{2}}$ will still be $-F' \div \sqrt{(-1)}\,\tau^{\frac{\mu}{2}}$. The character of a reciprocant is thus seen to be one of its indelible attributes.

As simple examples of absolute reciprocants we may take $\dfrac{3ac-5b^2}{t^4}$, which becomes on reciprocation $\dfrac{3\alpha\gamma - 5\beta^2}{\tau^4}$, and $\dfrac{a}{t^{\frac{3}{2}}}$, which reciprocates into $-\dfrac{\alpha}{\tau^{\frac{3}{2}}}$. The character of the former is even, that of the latter odd.

Observing that

$$\log t = -\log \tau \text{ and } \frac{1}{\sqrt{t}} \cdot \frac{d}{dx} = \frac{1}{\sqrt{\tau}} \cdot \frac{d}{dy},$$

we have

$$\left(\frac{1}{\sqrt{t}} \cdot \frac{d}{dx}\right) \log t = -\left(\frac{1}{\sqrt{\tau}} \cdot \frac{d}{dy}\right) \log \tau.$$

From this, in like manner, we obtain

$$\left(\frac{1}{\sqrt{t}} \cdot \frac{d}{dx}\right)^2 \log t = -\left(\frac{1}{\sqrt{\tau}} \cdot \frac{d}{dy}\right)^2 \log \tau;$$

and so, in general,

$$\left(\frac{1}{\sqrt{t}} \cdot \frac{d}{dx}\right)^i \log t = -\left(\frac{1}{\sqrt{\tau}} \cdot \frac{d}{dy}\right)^i \log \tau.$$

Hence $\left(\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\right)^i \log t$ is an absolute reciprocant, and of an odd character, for all positive integral values of i. We thus obtain a series of fractions with rational integral homogeneous reciprocants in their numerators and powers of $t^{\frac{3}{2}}$ in their denominators. It will be sufficient, before proceeding to the more general theory of *Eduction*, as it may be called, to examine, by way of illustration, the cases in which $i=1$, 2 and 3.

Let $i=1$; then

$$\left(\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\right)\log t = \frac{a}{t^{\frac{3}{2}}}.$$

So that, in the case where $i=2$, we have

$$\left(\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\right)^2\log t = \left(\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\right)\frac{a}{t^{\frac{3}{2}}} = \frac{b}{t^2} - \frac{3}{2}\cdot\frac{a^2}{t^3}$$

$$= \frac{2bt-3a^2}{2t^3}.$$

The numerator of this fraction is the Schwarzian.

In like manner, when $i=3$,

$$\left(\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\right)^3\log t = \left(\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\right)\left(\frac{2bt-3a^2}{2t^3}\right) = \frac{2ct-4ab}{2t^{\frac{7}{2}}} - \frac{6abt-9a^3}{t^{\frac{9}{2}}} = \frac{2ct^2-10abt+9a^3}{2t^{\frac{9}{2}}}.$$

But here a reduction may be effected, for $\left(\frac{a}{t^{\frac{3}{2}}}\right)^3$, as well as $\frac{a}{t^{\frac{3}{2}}}$ itself, is an absolute reciprocant of the same character as the whole of the expression just found. Hence we may reject the term $\frac{9}{2}\cdot\frac{a^3}{t^{\frac{9}{2}}}$ without thereby affecting the reciprocantive property of the form, and thus obtain

$$\frac{ct-5ab}{t^{\frac{7}{2}}},$$

an absolute reciprocant of odd character. The corresponding rational integral reciprocant is

$$ct-5ab.$$

We have found that $\frac{a}{t^{\frac{3}{2}}}$ and $\frac{2bt-3a^2}{t^3}$ are each of them reciprocants. Why, then, by parity of reasoning, is not $\frac{2bt}{t^3}$, and therefore b, a reciprocant? It is because $\frac{a^2}{t^3}$, the square of $\frac{a}{t^{\frac{3}{2}}}$, is of even character, while $\frac{2bt-3a^2}{t^3}$ is of an odd character, so that no linear combination of the two would be *legitimate*.

If we differentiate any absolute reciprocant with respect to x, we shall obtain another reciprocant of the same character. For let R be any absolute reciprocant and R' its transform, then

$$R = \pm R';$$

and since $\frac{d}{dx} = t\frac{d}{dy}$ may be written in the equivalent but more symmetrical form

$$\frac{1}{\sqrt{t}}\cdot\frac{d}{dx} = \frac{1}{\sqrt{\tau}}\cdot\frac{d}{dy},$$

we have $$\left(\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\right)R = \pm\left(\frac{1}{\sqrt{\tau}}\cdot\frac{d}{dy}\right)R'.$$

On one side of this identical equation is a function of the differential derivatives of y with respect to x; on the other, a precisely similar function of those of x with respect to y. Hence $\frac{1}{\sqrt{t}}\cdot\frac{dR}{dx}$ is an absolute reciprocant, and therefore $\frac{dR}{dx}$ is a reciprocant, the character of each being the same as that of R.

I will avail myself of the conclusion just obtained, which is the cardinal property of absolute reciprocants, to give a general method of generating from any given Rational Integral Reciprocant an infinity of others—rational integral educts of it, we may say. Let F be such a reciprocant, and μ its characteristic; then $\frac{F}{t^{\frac{\mu}{2}}}$ is an absolute reciprocant, and consequently $\frac{d}{dx}\left(\frac{F}{t^{\frac{\mu}{2}}}\right)$ is a reciprocant, both of them of the same character as F; that is

$$\frac{t\frac{dF}{dx} - \frac{\mu}{2}\cdot aF}{t^{\frac{\mu}{2}+1}};$$

or we may say $$2t\frac{dF}{dx} - \mu aF$$

is a reciprocant of the same character as F.

This is even true for non-homogeneous reciprocants, for the only assumption made at present as to the nature of F is that it is a rational integral reciprocant. But if we further assume that it is homogeneous and isobaric*, we know that

$$\mu = 3i + w.$$

Now, Euler's equation gives

$$3i = 3\left(t\partial_t + a\partial_a + b\partial_b + c\partial_c + \ldots\right),$$

* It will subsequently be proved that every rational integral reciprocant which is homogeneous is also isobaric.

and from the similar equation for isobaric functions (remembering that the weights of $t, a, b, c, \ldots$ are $-1, 0, 1, 2, \ldots$) we obtain

$$w = -t\partial_t + b\partial_b + 2c\partial_c \ldots,$$

so that
$$\mu = 2t\partial_t + 3a\partial_a + 4b\partial_b + 5c\partial_c + \ldots .$$

And since
$$\frac{d}{dx} = a\partial_t + b\partial_a + c\partial_b + d\partial_c + \ldots,$$

we may in $\left(2t\dfrac{d}{dx} - \mu a\right) F$ replace $2t\dfrac{d}{dx} - \mu a$ by

$$2t\,(a\partial_t + b\partial_a + c\partial_b + d\partial_c + \ldots)$$
$$-\,a\,(2t\partial_t + 3a\partial_a + 4b\partial_b + 5c\partial_c + \ldots),$$

or by its equivalent

$$(2bt - 3a^2)\,\partial_a + (2ct - 4ab)\,\partial_b + (2dt - 5ac)\,\partial_c + \ldots .$$

The conclusion arrived at is that when F is a rational integral homogeneous reciprocant,

$$\{(2bt - 3a^2)\,\partial_a + (2ct - 4ab)\,\partial_b + (2dt - 5ac)\,\partial_c + \ldots\}\,F$$

is another, and that both are of the same character.

It will be convenient to use the letter G to denote the operator just found and to speak of it as the generator for mixed reciprocants. By the repeated operation of this generator on a we may obtain the series $Ga, G^2a, G^3a, \ldots$, whose terms will be mixed reciprocants, since each operation increases the highest power of t by unity. The forms thus obtained will, in general, not be irreducible. It is, in fact, easy to see that a reduction must always take place at every second step. Observing that GF only expresses the numerator of the absolute reciprocant $\dfrac{1}{\sqrt{t}}\cdot\dfrac{d}{dx}\left(\dfrac{F}{t^{\frac{\mu}{2}}}\right)$ in a convenient form, and that G^2F is equivalent to the numerator of $\left(\dfrac{1}{\sqrt{t}}\cdot\dfrac{d}{dx}\right)^2\left(\dfrac{F}{t^{\frac{\mu}{2}}}\right)$, we have

$$\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\left(\frac{F}{t^{\frac{\mu}{2}}}\right) = \frac{t\dfrac{dF}{dx} - \dfrac{\mu}{2}\cdot aF}{t^{\frac{\mu+3}{2}}};$$

so that
$$\left(\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\right)^2\left(\frac{F}{t^{\frac{\mu}{2}}}\right) = \frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\left(\frac{t\dfrac{dF}{dx} - \dfrac{\mu}{2}\cdot aF}{t^{\frac{\mu+3}{2}}}\right)$$
$$= \frac{t\dfrac{d}{dx}\left(t\dfrac{dF}{dx} - \dfrac{\mu}{2}\cdot aF\right) - \dfrac{\mu+3}{2}\cdot a\left(t\dfrac{dF}{dx} - \dfrac{\mu}{2}\cdot aF\right)}{t^{\frac{\mu}{2}+3}}.$$

The whole of this fraction is an absolute reciprocant of the same character as F; so also is $\dfrac{a^2F}{t^{\frac{\mu}{2}+3}}$ $\left(\text{the product of the } \textit{even} \text{ absolute reciprocant } \dfrac{a^2}{t^3} \text{ by } \dfrac{F}{t^{\frac{\mu}{2}}}\right)$. We may therefore reject the term $\dfrac{\mu}{2}\cdot\dfrac{\mu+3}{2}\cdot a^2F$ from the numerator, and the remaining fraction

$$\frac{\dfrac{d}{dx}\left(t\dfrac{dF}{dx} - \dfrac{\mu}{2}\cdot aF\right) - \dfrac{\mu+3}{2}\cdot a\dfrac{dF}{dx}}{t^{\frac{\mu}{2}+2}}$$

will still be an absolute reciprocant of the same character as F. Its numerator, which is one degree lower than G^2F, may be written in the form

$$t\frac{d^2F}{dx^2} - (\mu + \tfrac{1}{2})\,a\frac{dF}{dx} - \frac{\mu}{2}\,bF.$$

This, it may be noticed, is a reciprocant of the same character as F, even when F is non-homogeneous.

Starting with a, we have

$Ga = 2bt - 3a^2$ (the Schwarzian),

$G^2a = G\,(2bt - 3a^2) = -6a\,(2bt - 3a^2) + 2t\,(2ct - 4ab) = 4ct^2 - 20abt + 18a^3.$

But, for the reason previously given, $18a^3$ may be removed, so that rejecting this term and dividing out by $4t$ we obtain the form

$$ct - 5ab,$$

which may be called the Post-Schwarzian.

The next form is obtained by operating on the Post-Schwarzian with G; thus, we have to calculate the value of $G\,(ct - 5ab)$, where

$$G = (2bt - 3a^2)\,\partial_a + (2ct - 4ab)\,\partial_b + (2dt - 5ac)\,\partial_c.$$

The working may be arranged as follows:

	dt^2	act	b^2t	a^2b	
$t\,(2dt - 5ac) =$	2	-5	.	.	from $(2dt - 5ac)\,\partial_c$
$-5a\,(2ct - 4ab) =$	.	-10	.	20	„ $(2ct - 4ab)\,\partial_b$
$-5b\,(2bt - 3a^2) =$	.	.	-10	15	„ $(2bt - 3a^2)\,\partial_a$
	2	-15	-10	35	

The result should be read thus:

$$2dt^2 - 15act - 10b^2t + 35a^2b.$$

To obtain the next of this series of reciprocants, we have to operate on this with G and at the same time to take account of the reduction that has to be made at each alternate step. The arrangement of the work is similar to that of the former case.

	et^3	adt^2	bct^2	a^2ct	ab^2t	a^3b	
$2t^2(2et-6ad)=$	4	−12	.	.	.	.	from $(2et-6ad)\partial_d$
$-15at(2dt-5ac)=$	.	−30	.	75	.	.	„ $(2dt-5ac)\partial_c$
$(35a^2-20bt)(2ct-4ab)=$	.	.	−40	70	80	−140	„ $(2ct-4ab)\partial_b$
$(70ab-15ct)(2bt-3a^2)=$	.	.	−30	45	140	−210	„ $(2bt-3a^2)\partial_a$
	4	−42	−70	190	220	−350	
$-70a^2(ct-5ab)=$	.	.	.	−70	.	+350	
	4	−42	−70	120	220	.	

This divides by $2t$, giving the reduced value

$$2et^2-21adt-35bct+60a^2c+110ab^2.$$

The next obtained by this process will be seen by the following work to be

$$4ft^3-56aet^2-112bdt^2-70c^2t^2+309a^2dt+995abct+220b^3t-660a^3c-1210a^2b^2.$$

	ft^3	aet^2	bdt^2	c^2t^2	a^2dt	$abct$	b^3t	a^3c	a^2b^2	
$2t^2(2ft-7ae)=$	4	−14	.	.	.	.	.	.	.	from $(2ft-7ae)\partial_e$
$-21at(2et-6ad)=$	.	−42	.	.	126	.	.	.	.	„ $(2et-6ad)\partial_d$
$(-35bt+60a^2)(2dt-5ac)=$	.	.	−70	.	120	175	.	−300	.	„ $(2dt-5ac)\partial_c$
$(-35ct+220ab)(2ct-4ab)=$	.	.	.	−70	.	580	.	.	−880	„ $(2ct-4ab)\partial_b$
$(-21dt+120ac+110b^2)(2bt-3a^2)=$	.	.	−42	.	63	240	220	−360	−330	„ $(2bt-3a^2)\partial_a$
	4	−56	−112	−70	309	995	220	−660	−1210	

This cannot be reduced in the same manner as the preceding form, but it must not be supposed that the forms thus obtained are in general irreducible.

Having regard to the circumstance that the forms of the series

$$a,\ Ga,\ G^2a,\ \ldots$$

occur in the numerators of the successive values of $\left(\frac{1}{\sqrt{t}}\cdot\frac{d}{dx}\right)^n\log t$, they may be called the successive *educts*, and the reduced forms given above may be called the *reduced educts* and denoted by $E_1, E_2, E_3 \ldots$. Thus

$$E_1=a,$$
$$E_2=2bt-3a^2,$$
$$E_3=ct-5ab,$$
$$E_4=2dt^2-15act-10b^2t+35a^2b,$$
$$E_5=2et^2-21adt-35bct+60a^2c+110ab^2,$$
$$E_6=4ft^3-56aet^2-112bdt^2-70c^2t^2+309a^2dt+995abct$$
$$+220b^3t-660a^3c-1210a^2b^2.$$

LECTURE IV.

We have seen that when F is a rational integral homogeneous and isobaric reciprocant, GF is another of the same character. It will now appear that the condition of isobarism is implied in that of homogeneity; for let F be a rational integral homogeneous reciprocant, μ its characteristic and i its degree in the letters $t, a, b, c, \ldots$, then, in the identical equation

$$F(t, a, b, c, \ldots)=\pm t^{\mu}F(\tau, \alpha, \beta, \gamma, \ldots)$$

both members are homogeneous and of the same degree in the letters $t, a, b, c, \ldots$; that is, if $At^ka^lb^mc^n\ldots$ be any term of $F(t, a, b, c, \ldots)$, its degree must be the same as that of $t^{\mu}A\tau^k\alpha^l\beta^m\gamma^n\ldots$ when $\tau, \alpha, \beta, \gamma, \ldots$ are expressed in terms of $t, a, b, c, \ldots$. But

$$\tau=\frac{1}{t},\quad \alpha=-\frac{a}{t^3},\quad \beta=-\frac{b}{t^4}+\ldots,\quad \gamma=-\frac{c}{t^5}+\ldots,$$

and so on. The degrees of $\tau, \alpha, \beta, \gamma, \ldots$ are therefore $-1, -2, -3, -4, \ldots$ respectively. Hence

$$k+l+m+n+\ldots=\mu-k-2l-3m-4n-\ldots,$$

or $$\mu=2k+3l+4m+5n+\ldots.$$

And by hypothesis $$i=k+l+m+n+\ldots,$$

so that $$\mu-3i=-k+m+2n+\ldots.$$

Neither μ nor i is dependent for its value on the selection of a particular term of F, for all terms of $F(\tau, \alpha, \beta, \gamma, \ldots)$ are multiplied by the same extraneous factor $\pm t^{\mu}$, and all terms of $F(t, a, b, c, \ldots)$ are of the same degree i. Hence $-k+m+2n+\ldots$ must also be the same for each term of F; or, attributing the weights $-1, 0, 1, 2, \ldots$ to the letters $t, a, b, c, \ldots$, the function F is isobaric.

Next, suppose F to be fractional, and let it be the ratio of the two rational integral homogeneous reciprocants F_1 and F_2. The operation of G on F will, in this case also, generate another reciprocant of the same character as F. For, since G is linear in the differential operative symbols $\partial_a, \partial_b, \partial_c, \ldots$, its operation will be precisely analogous to that of differentiation, so that, operating with G on

$$F=\frac{F_1}{F_2},$$

we have $$GF=\frac{F_2GF_1-F_1GF_2}{F_2^2}.$$

In order to prove that this is a reciprocant, we have to show that the character and characteristic are the same for both terms of the numerator. But GF_1 is a reciprocant of the same character as F_1, and GF_2 is one of the same character as F_2; thus the two terms of the numerator are of the same character as F_1F_2. As regards the characteristic, it should be noticed that G, that is, the operator $(2bt-3a^2)\partial_a+(2ct-4ab)\partial_b+\ldots$, increases the degree by unity, but does not alter the weight, so that it increases the characteristic of any rational integral homogeneous reciprocant by 3. Thus the characteristic of each term in the numerator exceeds by 3 that of F_1F_2. Hence GF is a reciprocant, and, taking account of its denominator as well as its numerator, we see that the operation of G on a rational homogeneous reciprocant, whether fractional or integral, produces another in which the original character is preserved while the characteristic is increased by three units.

More generally, let F_1, F_2, F_3, ... be any rational homogeneous reciprocants whose extraneous factors are $(-)^{\kappa_1}t^{\mu_1}$, $(-)^{\kappa_2}t^{\mu_2}$, $(-)^{\kappa_3}t^{\mu_3}$, ... respectively; and suppose Φ to consist of a series of terms of the form $AF_1^{\lambda_1}F_2^{\lambda_2}F_3^{\lambda_3}\ldots$, such that the extraneous factor for each term is $(-)^{\kappa}t^{\mu}$. Then Φ is a reciprocant, but not necessarily a rational one; for the indices λ_1, λ_2, λ_3, ... may be supposed fractional, provided only that they satisfy the conditions

$$\kappa_1\lambda_1+\kappa_2\lambda_2+\kappa_3\lambda_3+\ldots-\kappa = \text{a positive or negative } \textit{even} \text{ integer},$$

and

$$\mu_1\lambda_1+\mu_2\lambda_2+\mu_3\lambda_3+\ldots-\mu = 0.$$

We proceed to show that $G\Phi$ is also a reciprocant, and that its extraneous factor is $(-)^{\kappa}t^{\mu+3}$. Since

$$G\Phi=\frac{d\Phi}{dF_1}\cdot GF_1+\frac{d\Phi}{dF_2}\cdot GF_2+\frac{d\Phi}{dF_3}\cdot GF_3+\ldots,$$

we have to prove not only that each term of this expression is a reciprocant, but also that all of them have the same extraneous factor; otherwise their sum would not be a reciprocant.

Now, in

$$\Phi=\Sigma AF_1^{\lambda_1}F_2^{\lambda_2}F_3^{\lambda_3}\ldots,$$

the extraneous factor for each term is by hypothesis $(-)^{\kappa}t^{\mu}$, so that the extraneous factor for each term of

$$\frac{d\Phi}{dF_1}=\Sigma A\lambda_1F_1^{\lambda_1-1}F_2^{\lambda_2}F_3^{\lambda_3}\ldots,$$

is $(-)^{\kappa-\kappa_1}t^{\mu-\mu_1}$, and therefore $\dfrac{d\Phi}{dF_1}$ is a reciprocant. Also, GF_1 is a reciprocant whose extraneous factor is $(-)^{\kappa_1}t^{\mu_1+3}$. Hence $\dfrac{d\Phi}{dF_1}.GF_1$ is a reciprocant having $(-)^{\kappa}t^{\mu+3}$ for extraneous factor, and in exactly the same way we see that every other term of $G\Phi$ is also a reciprocant with the same extraneous factor.

Thus G, operating on *any* homogeneous reciprocant whose extraneous factor is $(-)^{\kappa}t^{\mu}$, generates another whose extraneous factor is $(-)^{\kappa}t^{\mu+3}$.

If, in the generator for mixed reciprocants,

$$G=(2bt-3a^2)\partial_a+(2ct-4ab)\partial_b+(2dt-5ac)\partial_c+\ldots,$$

we write $\quad a=1\,.\,2\,.\,a_0,\quad b=1\,.\,2\,.\,3\,.\,a_1,\quad c=1\,.\,2\,.\,3\,.\,4\,.\,a_2\,\ldots,$

(that is, if we use the system of modified letters previously mentioned), its expression assumes a more elegant form. Substituting for a, b, c, ... their values in terms of the modified letters, we have

$$2bt-3a^2=2\,.\,1\,.\,2\,.\,3a_1t-3\,(1\,.\,2)^2a_0^2=1\,.\,2^2\,.\,3\,(a_1t-a_0^2),$$

and

$$\partial_a=\frac{1}{1\,.\,2}\cdot\partial_{a_0};$$

so that

$$(2bt-3a^2)\,\partial_a=1\,.\,2\,.\,3\,(a_1t-a_0^2)\,\partial_{a_0}.$$

Again,

$$(2ct-4ab)=1\,.\,2^2\,.\,3\,.\,4\,(a_2t-a_0a_1)$$

and

$$\partial_b=\frac{1}{1\,.\,2\,.\,3}\,\partial_{a_1};$$

so that

$$(2ct-4ab)\,\partial_b=1\,.\,2\,.\,4\,(a_2t-a_0a_1)\,\partial_{a_1}.$$

Similarly,

$$(2dt-5ac)\,\partial_c=1\,.\,2\,.\,5\,(a_3t-a_0a_2)\,\partial_{a_2}.$$

Thus the modified generator for mixed reciprocants is

$$1\,.\,2\,.\,3\,(a_1t-a_0^2)\,\partial_{a_0}+1\,.\,2\,.\,4\,(a_2t-a_0a_1)\,\partial_{a_1}+1\,.\,2\,.\,5\,(a_3t-a_0a_2)\,\partial_{a_2}+\ldots,$$

in which the general term is

$$1\,.\,2\,(n+3)\,(a_{n+1}t-a_0a_n)\,\partial_{a_n}.$$

The factor $1\,.\,2$ may, of course, be rejected, and our modified generator may be written in the simple form

$$3\,(a_1t-a_0^2)\,\partial_{a_0}+4\,(a_2t-a_0a_1)\,\partial_{a_1}+5\,(a_3t-a_0a_2)\,\partial_{a_2}+\ldots.$$

Operating with this on the homogeneous reciprocant $F(t, a_0, a_1, a_2, \ldots)$, the result will be another homogeneous reciprocant of the same character as F. When we start with a_0 and make the reductions which, as we have seen, occur at every second step, we find a system of reduced educts corresponding in every particular with those formerly given, but expressed in terms of the modified letters a_0, a_1, a_2, ... instead of a, b, c, ... These are as follows:

a_0,

*$a_1t-a_0^2$,

$2a_2t-5a_0a_1$,

*$2a_3t^2-6a_0a_2t-3a_1^2t+7a_0^2a_1$,

$2a_4t^2-7a_0a_3t-7a_1a_2t+8a_0^2a_2+11a_0a_1^2$,

*$14a_5t^3-56a_0a_4t^2-56a_1a_3t^2-28a_2^2t^2+103a_0^2a_3t+199a_0a_1a_2t$
$\qquad+33a_1^3t-88a_0^3a_2-121a_0^2a_1^2$.

* It will be observed that in the unreduced forms, marked with an asterisk, the sum of the numerical coefficients is zero. This is a direct consequence, as may be easily seen, of the form of the modified generator, in which the sum of the numerical coefficients in each term is also zero.

It will be found on trial that these modified educts are obtained with greater ease and with less liability to error by a direct application of the generator

$$3(a_1t-a_0^2)\partial_{a_0}+4(a_2t-a_0a_1)\partial_{a_1}+5(a_3t-a_0a_2)\partial_{a_2}+\ldots,$$

than by making the substitution of $1.2.a_0$, $1.2.3.a_1$, $1.2.3.4.a_2$, ... for a, b, c, ... in the system of educts already given. For this reason the working by the former method is here performed, instead of being merely indicated.

From a_0 we obtain immediately

$$a_1t-a_0^2.$$

Operating on this with the generator, there results

$$4t(a_2t-a_0a_1)-6a_0(a_1t-a_0^2)=4a_2t^2-10a_0a_1t+6a_0^3.$$

This, when reduced by removing its last term and dividing the others by $2t$, gives

$$2a_2t-5a_0a_1.$$

The next form is found from this by a simple operation, without subsequent reduction, and is therefore

$$10t(a_3t-a_0a_2)-20a_0(a_2t-a_0a_1)-15a_1(a_1t-a_0^2).$$

Or, collecting the terms and rejecting the numerical factor 5,

$$2a_3t^2-6a_0a_2t-3a_1^2t+7a_0^2a_1.$$

The operation of the generator on this gives

$$12t^2(a_4t-a_0a_3)-30a_0t(a_3t-a_0a_2)+4(7a_0^2-6a_1t)(a_2t-a_0a_1)$$
$$+3(14a_0a_1-6a_2t)(a_1t-a_0^2).$$

The collection of terms and subsequent reduction is shown below:

	a_4t^3	$a_0a_3t^2$	$a_1a_2t^2$	$a_0^2a_2t$	$a_0a_1^2t$	$a_0^3a_1$
	12	−12	.	.	.	.
	.	−30	.	30	.	.
	.	.	−24	28	24	−28
	.	.	−18	18	42	−42
	12	−42	−42	76	66	−70
$-14a_0^2(2a_2t-5a_0a_1)=$	.	.	.	−28	.	+70
	12	−42	−42	48	66	.

Removing the factor $6t$, the reduced form is

$$2a_4t^2-7a_0a_3t-7a_1a_2t+8a_0^2a_2+11a_0a_1^2.$$

Operating on this with the generator, we have

$$14t^2(a_5t-a_0a_4)-42a_0t(a_4t-a_0a_3)+5(8a_0^2-7a_1t)(a_3t-a_0a_2)$$
$$+4(22a_0a_1-7a_2t)(a_2t-a_0a_1)+3(11a_1^2+16a_0a_2-7a_3t)(a_1t-a_0^2)$$
$$=14a_5t^3-56a_0a_4t^2-56a_1a_3t^2-28a_2^2t^2+103a_0^2a_3t+199a_0a_1a_2t$$
$$+33a_1^3t-88a_0^3a_2-121a_0^2a_1^2,$$

which cannot be reduced in the same manner as the preceding form.

To obtain a generator for passing from pure to pure reciprocants a process is employed similar to that which gave the generator for mixed reciprocants which we have just been using. I state the results before giving the proof, and then proceed to speak of generators in the theory of Invariants. The generator for pure reciprocants is

$$(3ac-4b^2)\partial_b+(3ad-5bc)\partial_c+(3ae-6bd)\partial_d+\ldots;$$

or, expressed in terms of the modified letters,

$$4(a_0a_2-a_1^2)\partial_{a_1}+5(a_0a_3-a_1a_2)\partial_{a_2}+6(a_0a_4-a_1a_3)\partial_{a_3}+\ldots.$$

By operating with this on any pure reciprocant R, we generate another pure reciprocant of opposite character to that of R.

The connection between the two theories of Reciprocants and Invariants is so close, and these brother-and-sister theories throw so much light upon each other, that I began to inquire whether, in the latter, there did not exist a theory of Generators parallel to that of the former.

Fortunately, Mr Hammond was able to recall a correspondence in which Prof. Cayley had given such a theory, which he regarded, and justly, as an important invention. Its substance has been subsequently incorporated in the *Quarterly Journal* (Vol. XX. p. 212). It offers itself spontaneously in the Reciprocantive Theory; in the Invariantive one it calls for a distinct act of invention. Prof. Cayley has discovered two generators similar in form with those for reciprocants, and one of them strikingly so; in a letter to me he calls these P and Q. As given by him,

$$P=ab\partial_a+ac\partial_b+ad\partial_c+\ldots-ib,$$
$$Q=\qquad ac\partial_b+2ad\partial_c+\ldots-2wb,$$

where i is the degree and w the weight, the weights of a, b, c, d, ... being taken to be 0, 1, 2, 3, ... (I supply the a which Cayley turns into unity.) As an example he takes the "Invariant" $a^2d-3abc+2b^3=I$, suppose. We have then

$$PI=(ab\partial_a+ac\partial_b+ad\partial_c+ae\partial_d-3b)I$$
$$=ab(2ad-3bc)+ac(-3ac+6b^2)-3a^2bd+a^3e-3b(a^2d-3abc+2b^3)$$
$$=a^3e-4a^2bd-3a^2c^2+12ab^2c-6b^4$$
$$=a^2(ae-4bd+3c^2)-6(ac-b^2)^2,$$

and $QI=(ac\partial_b+2ad\partial_c+3ae\partial_d-6b)I$

$$=ac(-3ac+6b^2)-6a^2bd+3a^3e-6b(a^2d-3abc+2b^3)$$
$$=3a^3e-12a^2bd-3a^2c^2+24ab^2c-12b^4$$
$$=3a^2(ae-4bd+3c^2)-12(ac-b^2)^2.$$

P and Q may be transformed by means of Euler's equation and the similar one for isobaric functions, which enable us to write

$$i=a\partial_a+b\partial_b+c\partial_c+d\partial_d+\ldots,$$

and

$$w=\qquad b\partial_b+2c\partial_c+3d\partial_d+\ldots;$$

P thus becomes

$$\begin{aligned} & ab\partial_a + ac\partial_b + ad\partial_c + ae\partial_d + \ldots \\ & - ab\partial_a - b^2\partial_b - bc\partial_c - bd\partial_d - \ldots \\ = & (ac - b^2)\,\partial_b + (ad - bc)\,\partial_c + (ae - bd)\,\partial_d + \ldots, \end{aligned}$$

the same in *form* as either of our generators, except that the arithmetical coefficients are all made units; $a, b, c, \ldots$ taking the place of the $t, a, b, \ldots$ of the generator for mixed reciprocants.

In like manner, Q becomes

$$(ac - 2b^2)\,\partial_b + 2\,(ad - 2bc)\,\partial_c + 3\,(ae - 2bd)\,\partial_d + \ldots,$$

where the arithmetical series 1, 2, 3, ... takes the place of 3, 4, 5, ... or of 4, 5, 6, ... in the two Reciprocant Generators.

The effect of P and of Q is obviously to raise the degree and the weight of the operand I each by one unit. But if we take $R = \frac{1}{a}(2wP - iQ)$, the terms in Cayley's original formulae containing b cancel, so that $2wP - iQ$ divides out by a and the weight is raised one unit without the degree being affected. This is mentioned in the *Quarterly Journal* (*loc. cit.*); but it may also be remarked that when I is a *satisfied invariant*, it is annihilated by the operation of R; when the *invariant* is *unsatisfied*, each of the three operators P, Q and R increases its extent by an unit, that is, introduces an additional letter. For let j denote the extent, then, writing $a_0, a_1, a_2, \ldots a_j$ for $a, b, c, \ldots$, we have

$$\begin{aligned} P &= a_0 a_1 \partial_{a_0} + a_0 a_2 \partial_{a_1} + \ldots + a_0 a_{j+1} \partial_{a_j} - i a_1, \\ Q &= \qquad\qquad a_0 a_2 \partial_{a_1} + 2a_0 a_3 \partial_{a_2} + \ldots + j a_0 a_{j+1} \partial_{a_j} - 2w a_1; \end{aligned}$$

whence we find

$$\begin{aligned} R &= \frac{1}{a_0}(2wP - iQ) \\ &= 2wa_1\partial_{a_0} + (2w - i)\,a_2\partial_{a_1} + \ldots + (2w - ij + i)\,a_j\partial_{a_{j-1}} + (2w - ij)\,a_{j+1}\partial_{a_j}. \end{aligned}$$

But for a *satisfied invariant*

$$2w = ij;$$

and substituting this value for $2w$ in the above expression for R, it becomes

$$i\,\{ja_1\partial_{a_0} + (j - 1)\,a_2\partial_{a_1} + \ldots + a_j\partial_{a_{j-1}}\},$$

which, as is well known, annihilates any satisfied invariant.

LECTURE V.

It will be desirable to fill up some of the previous investigations by discussing some points in them that have not yet received our consideration.

There may be some to whom it may appear tedious to watch the complete exposition of the algebraical part of the Theory, who are impatient to rush on to its applications. But it is my duty to consider what may be expected to be most useful to the great majority of the class, and for that purpose to make the ground sure under our feet as I proceed. To the greater number it will, I think, be of advantage to have their memories refreshed on the kindred subject of invariants, and probably made acquainted with some important points of that theory which are new to them.

I confess that, to myself, the contemplation of this relationship—the spectacle of a new continent rising from the waters, resembling yet different from the old, familiar one—is a principal source of interest arising out of the new theory. I do not regard Mathematics as a science purely of calculation, but one of ideas, and as the embodiment of a Philosophy. An eminent colleague of mine, in a public lecture in this University, magnifying the importance of classical over mathematical studies, referred to a great mathematician as one who might possibly know every foot of distance between the earth and the moon; and when I was a member, at Woolwich, of the Government Committee of Inventions, one of my colleagues, appealing to me to answer some question as to the number of cubic inches in a pipe, expressed his surprise that I was not prepared with an immediate answer, and said he had supposed that I had all the tables of weights and measures at my fingers' ends.

I hope that in any class which I may have the pleasure of conducting in this University, other ideas will prevail as to the true scope of mathematical science as a branch of liberal learning; and it will be my endeavour to regulate the pace in a manner which seems to me most conducive to real progress in the order of ideas and philosophical contemplation, thus bringing our noble science into harmony and in a line with the prevailing tone and studies of this University. Faraday, at the end of his experimental lectures, was accustomed to say—I have myself heard him do so—"We will now leave that to the calculators." So long as we are content to be regarded as mere calculators we shall be the Pariahs of the University, living here on sufferance, instead of being regarded, as is our right and privilege, as the real leaders and pioneers of thought in it.

That Cayley's two operators, which have been called P and Q, are in fact generators, may be proved as follows †:

Let $$\Omega = a\partial_b + 2b\partial_c + 3c\partial_d + 4d\partial_e + \ldots,$$
and $$\Theta = a(\lambda b\partial_a + \mu c\partial_b + \nu d\partial_c + \ldots) - \kappa b,$$
where $\kappa, \lambda, \mu, \nu, \ldots$ are numbers.

When κ is the degree of the operand, and $\lambda = \mu = \nu = \ldots = 1$, the operator Θ is identical with P; but Θ is identical with Q when κ is twice the weight of the operand and $\lambda = 0, \mu = 1, \nu = 2, \ldots$.

If now we use $*$ to signify the act of pure differential operation, it is obvious that

$$\Omega\Theta = (\Omega \times \Theta) + (\Omega * \Theta),$$
$$\Theta\Omega = (\Omega \times \Theta) + (\Theta * \Omega),$$
so that $$\Omega\Theta - \Theta\Omega = (\Omega * \Theta) - (\Theta * \Omega).$$
But since $$\Omega a = 0, \quad \Omega b = a, \quad \Omega c = 2b, \ldots$$
we have $$\Omega * \Theta = a(\lambda a\partial_a + 2\mu b\partial_b + 3\nu c\partial_c + \ldots - \kappa)$$
and $$\Theta * \Omega = a(\lambda b\partial_b + 2\mu c\partial_c + 3\nu d\partial_d + \ldots).$$
Hence $$\Omega\Theta - \Theta\Omega = a\{\lambda a\partial_a + (2\mu - \lambda) b\partial_b + (3\nu - 2\mu) c\partial_c + \ldots - \kappa\};$$
now if the operand I be any *invariant* (satisfied or unsatisfied), we have $\Omega I = 0$, and therefore $\Theta\Omega I = 0$; so that we find

$$\Omega\Theta I = a\{\lambda a\partial_a + (2\mu - \lambda) b\partial_b + (3\nu - 2\mu) c\partial_c + \ldots - \kappa\} I.$$

If in this we write $\lambda = \mu = \nu = \ldots = 1$, and $\kappa = i$, where i is the degree of the operand, Θ becomes P and we have

$$\Omega PI = a(a\partial_a + b\partial_b + c\partial_c + \ldots - i) I.$$

But, by Euler's theorem, the right-hand side of this vanishes, and therefore

$$\Omega PI = 0.$$

Similarly, by means of the corresponding theorem for isobaric functions, we may prove that

$$\Omega QI = 0.$$

For if, in the general formula, we write $\lambda = 0, \mu = 1, \nu = 2, \ldots$ and $\kappa = 2w$, where w is the weight of the operand, we find

$$\Omega QI = a(2b\partial_b + 4c\partial_c + 6d\partial_d + \ldots - 2w) I = 0.$$

Thus, when Θ stands either for P or for Q, it is either an annihilator or a generator (that is, ΘI is either identically zero or else an invariant). But if l be the most advanced, or say the *radical letter* of I, no term of $m\partial_l I$ can cancel with any other term of ΘI; and since, for this reason, ΘI cannot vanish identically, it must be an invariant, and the operators P and Q must be generators.

† In the *Quarterly Journal* (Vol. xx. p. 212) Prof. Cayley only considers a special example, and has not given the proof of the general theorem.

The generators previously given for reciprocants also possess this property of introducing a fresh radical letter at each step. The radical letter, on its first introduction, enters in the first degree only, and in the case of the educts of $\log t$, whose values have been calculated, its multiplier is seen to be a power of t. The form of the generator for mixed reciprocants

$$3(a_1 t - a_0^2)\partial_{a_0} + 4(a_2 t - a_0 a_1)\partial_{a_1} + \ldots + (n+3)(a_{n+1} t - a_0 a_n)\partial_{a_n}$$

shows this, or it may be seen by considering the successive values of

$$\left(\frac{1}{\sqrt{t}} \cdot \frac{d}{dx}\right)^i \log t.$$

For let $\dfrac{F(t, a_0, a_1, a_2, \ldots)}{t^\lambda}$ denote this expression, and let its radical letter be a_n; then, on differentiating again with respect to x, the new letter introduced arises solely from a term in the numerator

$$\frac{d}{da_n} F(t, a_0, a_1, a_2, \ldots a_n) \cdot \frac{da_n}{dx}.$$

But $$a_n = \frac{d^n y}{dx^n} \div 2 \cdot 3 \ldots n+2; \text{ so that } \frac{da_n}{dx} = (n+3) a_{n+1}.$$

Hence, if when a_n is the radical letter, it occurs in the first degree only and multiplied by a power of t, it follows that, since $\dfrac{dF}{da_n}$ will be a power of t, the derived expression which contains the radical letter a_{n+1} will contain it in the first degree only and multiplied by a power of t. And since this is true for the case $i = 1$, when $\dfrac{1}{\sqrt{t}} \cdot \dfrac{d}{dx} \log t = \dfrac{a_0}{t^{\frac{3}{2}}}$, it is true universally.

Observe that for $i = 1, 2, 3, \ldots$ the radical letter is $a_0, a_1, a_2, \ldots$ respectively.

It will be remembered that $\left(\dfrac{1}{\sqrt{t}} \cdot \dfrac{d}{dx}\right)^i \log t$ is an absolute reciprocant. It may be called the ith absolute educt, to distinguish it from the rational integral educts $E_1, E_2, E_3, \ldots$ whose values have already been calculated.

Let $R(t, a_0, a_1, a_2, \ldots a_n)$ be any homogeneous rational integral reciprocant, and let the educts be $A_0, A_1, A_2, \ldots A_n$; then obviously

a_n may be expressed rationally in terms of A_n and $a_{n-1}, a_{n-2}, \ldots a_0, t$,
a_{n-1} „ „ „ „ A_{n-1} and $a_{n-2}, \ldots a_0, t$,
....
a_1 „ „ „ „ A_1, a_0 and t,
a_0 „ „ „ „ A_0 and t,

where observe that the denominators in these expressions are all powers of t. Hence, by successive substitutions, $R(t, a_0, a_1, \ldots a_n)$ may be expressed rationally in terms of $A_n, \ldots A_1, A_0$, and t. Thus any rational integral homogeneous reciprocant is a rational function of educts, and is of the form $\frac{E}{t^{\theta}}$, where E is a rational *integral* function of the educts.

Does not this prove too much, it may be asked, namely, that any function F of the letters is a rational function of the educts, which are themselves reciprocants, and will therefore be a reciprocant? But this is not so; for observe that although F will be expressed as a sum of products of educts, such products will not in general be all of the same character, and their linear combination will be an illicit one, such as is seen in the illicit combination of a_0^2 with the Schwarzian $(a_1 t - a_0^2)$.

We have seen that by differentiating an absolute reciprocant, or by the use of a generator, we obtain a fresh reciprocant. But there are other methods of finding reciprocants; as, for example, if the transform of

$$\phi(t, a, b, c, \ldots)$$

is

$$\psi(\tau, \alpha, \beta, \gamma, \ldots),$$

that is, if

$$\phi(t, a, b, c, \ldots) = \psi(\tau, \alpha, \beta, \gamma, \ldots),$$

then

$$\psi(t, a, b, c, \ldots) = \phi(\tau, \alpha, \beta, \gamma, \ldots).$$

Whence, by multiplication,

$$\phi(t, a, b, c, \ldots)\,\psi(t, a, b, c, \ldots) = \phi(\tau, \alpha, \beta, \gamma, \ldots)\,\psi(\tau, \alpha, \beta, \gamma, \ldots).$$

Thus $\phi \cdot \psi$ is a reciprocant, and, moreover, an absolute one of even character, although neither ϕ, which is a perfectly arbitrary function, nor ψ, its transform, is a reciprocant.

Herein a mixed reciprocant differs from an invariant, which cannot be resolved into non-invariantive factors. It is worth while to give a proof of this proposition; but first I prove its converse, that if $p, q, r, \ldots$ are all invariants, their product must be so too. This is an immediate consequence of the well-known theorem that

$$\Omega I = 0$$

is the necessary and sufficient condition that I may be an invariant where, as usual, Ω is the operator

$$a\partial_b + 2b\partial_c + 3c\partial_d + \ldots,$$

and the word invariant has been used in the same extended sense as formerly.

For
$$\Omega(pqrs\ldots) = \left(\frac{\Omega p}{p} + \frac{\Omega q}{q} + \frac{\Omega r}{r} + \ldots\right) pqrs \ldots.$$

But since $p, q, r, \ldots$ are all invariants, we have

$$\Omega p = 0, \quad \Omega q = 0, \quad \Omega r = 0, \ldots,$$

and therefore
$$\Omega(pqrs\ldots) = 0.$$

Next, suppose that
$$I = P_1 Q_1,$$

where I is but Q_1 is not an invariant.

To meet the case in which P_1 and Q_1 are not prime to one another, Q_1, if resolved into its factors, must contain one Q^i where Q is not an invariant.

Suppose that P_1 contains Q^j, and let $i + j = k$; then we may write

$$I = PQ^k,$$

where P is prime to Q. But since I is an invariant by hypothesis,

$$\Omega I = 0,$$

and therefore,
$$Q^k \Omega P + kPQ^{k-1}\Omega Q = 0;$$

or,
$$\frac{Q}{P} = -k\frac{\Omega Q}{\Omega P}.$$

Now P is prime to Q, so that the fraction $\frac{Q}{P}$ is in its lowest terms; therefore ΩQ contains Q; but this is impossible, for the weight of ΩQ is less than that of Q. Hence I cannot contain any non-invariantive factor Q_1.

All this will be equally true for a general function J annihilated by any operator Ω which is *linear* in the differential operators $\partial_a, \partial_b, \partial_c, \ldots$ no matter what its degree in the letters $a, b, c, \ldots$ themselves; that is, we shall still have

$$J = PQ^k$$

and
$$\frac{Q}{P} = -k\frac{\Omega Q}{\Omega P},$$

where P and Q are prime to each other, and, as before, ΩQ will contain Q as a factor. But if Ω is an operator which diminishes either the degree or the weight, ΩQ is either of lower degree or of lower weight than Q, and so cannot contain it as a factor. Hence J cannot contain a factor Q not subject to annihilation by Ω.

If, however, Ω does not diminish either the degree or the weight, it may be objected that ΩQ might conceivably contain the factor Q; and were it so, there would be nothing to show the impossibility, in this case, of a function J subject to annihilation by Ω containing a factor Q, which is not so. But *quaere*: Is it possible, when J is a general homogeneous and isobaric function of $a, b, c, \ldots$, for ΩJ to contain J and at the same time the quotient to be other than a number*? *Valde dubitor.* But I reserve the point. Setting aside this doubtful case, and considering only such *linear* partial differential operators as *diminish* either the degree or the weight of the operand, we see that there cannot exist any universal operator of this kind whose effect in annihilating a form is the necessary and sufficient condition of that form being a reciprocant. But this does not preclude the possibility of the existence of such annihilators for special classes of reciprocants, and in fact

* If $\Omega = pa\partial_a + qb\partial_b + rc\partial_c + \ldots$, where $p, q, r, \ldots$ are in Arithmetical Progression, $\frac{\Omega J}{J}$ is a number; but then Ω could not be an annihilator.

(as we have already stated and shall hereafter prove) Pure Reciprocants are definable by means of the Partial Differential Annihilator

$$V = 4 \cdot \frac{a_0^2}{2} \partial_{a_1} + 5a_0 a_1 \partial_{a_2} + 6\left(a_0 a_2 + \frac{a_1^2}{2}\right) \partial_{a_3} + \dots,$$

which is *linear* in the differential operators, and *diminishes* the weight.

The generator for mixed reciprocants, which we have called G, will not assist us in obtaining pure reciprocants, but generates a mixed reciprocant in every case, even when the one we start with is pure. Thus, starting with the pure reciprocant R, our formula

$$GR = \{3(a_1 t - a_0^2)\partial_{a_0} + 4(a_2 t - a_0 a_1)\partial_{a_1} + 5(a_3 t - a_0 a_2)\partial_{a_2} + \dots\} R$$

may be written thus

$$\begin{aligned} GR &= t(3a_1\partial_{a_0} + 4a_2\partial_{a_1} + 5a_3\partial_{a_2} + \dots) R \\ &\quad - a_0(3a_0\partial_{a_0} + 4a_1\partial_{a_1} + 5a_2\partial_{a_2} + \dots) R. \end{aligned}$$

Here R being *pure*, that is, a function of $a_0, a_1, a_2, \dots$ (without t), we see that

$$\begin{aligned} &(3a_0\partial_{a_0} + 4a_1\partial_{a_1} + 5a_2\partial_{a_2} + 6a_3\partial_{a_3} + \dots) R \\ &\quad = 3(a_0\partial_{a_0} + a_1\partial_{a_1} + a_2\partial_{a_2} + \dots) R \\ &\quad + (a_1\partial_{a_1} + 2a_2\partial_{a_2} + 3a_3\partial_{a_3} + \dots) R \\ &\quad = (3i + w) R, \end{aligned}$$

where i is the degree and w the weight of R. Hence

$$GR = t(3a_1\partial_{a_0} + 4a_2\partial_{a_1} + 5a_3\partial_{a_2} + \dots) R - (3i + w) a_0 R,$$

where it should be noticed that $a_0 R$ is of opposite character to R (for a_0 is of odd character), while GR has been proved to be of the same character as R. Thus we cannot infer that $t(3a_1\partial_{a_0} + 4a_2\partial_{a_1} + 5a_3\partial_{a_2} + \dots) R$ is a reciprocant. The mixed reciprocant GR cannot therefore be resolved into the sum of two terms, one of which is a pure reciprocant and the other a pure reciprocant multiplied by t.

LECTURE VI.

Before proceeding to prove that, as was stated in anticipation in Lecture IV, the operator

$$(3ac - 4b^2)\partial_b + (3ad - 5bc)\partial_c + (3ae - 6bd)\partial_d + \dots,$$

or, when the modified letters are used,

$$4(a_0 a_2 - a_1^2)\partial_{a_1} + 5(a_0 a_3 - a_1 a_2)\partial_{a_2} + 6(a_0 a_4 - a_1 a_3)\partial_{a_3} + \dots,$$

will serve to generate a pure reciprocant from a pure one, it may be useful to briefly recapitulate what has been said concerning the character and

characteristic of reciprocants. It will be remembered that the extraneous factor of any rational integral reciprocant is of the form $(-)^\kappa t^\mu$, that the character is determined by the parity (oddness or evenness) of κ, and that μ is what has been called the characteristic.

For homogeneous reciprocants it has been proved that $\mu = 3i + w$, where i is the degree of the reciprocant and w its weight, the weights of the letters $t, a, b, c, \dots$ being taken to be $-1, 0, 1, 2, \dots$ respectively. The character is odd or even according as the number of letters other than t in the principal term or terms is odd or even. By a principal term is to be understood one in which t is contained the greatest number of times. So that, in other words, the character is governed by the parity of the smallest number of non-t letters that can be found in any term. For pure reciprocants, there being no t in any term, the character is determined by the parity of the number of letters in any one term.

Let R be any pure reciprocant, and suppose its characteristic to be μ; then $\frac{R}{t^{\frac{\mu}{2}}}$ is an absolute reciprocant. If, however, we differentiate this with respect to x, and thus obtain another reciprocant, the resulting form will not be pure, for its numerator will be identical with the form obtained by the direct operation on R of the generator for mixed reciprocants, and its denominator will be a power of t. But, remembering that $\frac{a}{t^{\frac{3}{2}}}$, and therefore $\frac{a^{\frac{\mu}{3}}}{t^{\frac{\mu}{2}}}$, is an absolute reciprocant, we see that $\frac{R}{a^{\frac{\mu}{3}}}$, which is the quotient of the two absolute reciprocants $\frac{R}{t^{\frac{\mu}{2}}}$ and $\frac{a^{\frac{\mu}{3}}}{t^{\frac{\mu}{2}}}$, is so also. Hence $\frac{d}{dx}\left(\frac{R}{a^{\frac{\mu}{3}}}\right)$ is a reciprocant, and, since it no longer contains t, a pure one. Now,

$$\frac{d}{dx}\left(\frac{R}{a^{\frac{\mu}{3}}}\right) = \frac{a\frac{dR}{dx} - \frac{\mu}{3} \cdot bR}{a^{\frac{\mu}{3}+1}}$$

remains a reciprocant when multiplied by any power of the reciprocant a. Hence the numerator of this expression, or

$$\left(3a\frac{d}{dx} - \mu b\right) R,$$

is a reciprocant. The general value of $\frac{d}{dx}$ has been seen to be

$$a\partial_t + b\partial_a + c\partial_b + d\partial_c + \dots,$$

but, since R is supposed to be *pure*, $\partial_t R = 0$.

We may therefore, in $3a\dfrac{d}{dx}-\mu b$, replace $\dfrac{d}{dx}$ by

$$b\partial_a+c\partial_b+d\partial_c+e\partial_d+\ldots.$$

Now, remembering that $\mu=3i+w$, and that by Euler's theorem and the similar one for isobaric functions

$$i=a\partial_a+b\partial_b+c\partial_c+d\partial_d+\ldots$$

and

$$w=\quad b\partial_b+2c\partial_c+3d\partial_d+\ldots,$$

we see that μ is equivalent to

$$3a\partial_a+4b\partial_b+5c\partial_c+6d\partial_d+\ldots.$$

Hence,

$$\begin{aligned}3a\frac{d}{dx}-\mu b&=3a\,(b\partial_a+c\partial_b+d\partial_c+e\partial_d+\ldots)\\&\quad-b\,(3a\partial_a+4b\partial_b+5c\partial_c+6d\partial_d+\ldots)\\&=(3ac-4b^2)\,\partial_b+(3ad-5bc)\,\partial_c+(3ae-6bd)\,\partial_d+\ldots.\end{aligned}$$

Thus, if R be any *pure* reciprocant,

$$\{(3ac-4b^2)\,\partial_b+(3ad-5bc)\,\partial_c+(3ae-6bd)\,\partial_d+\ldots\}\,R$$

is also a pure reciprocant. If the type of R be w; i, j, that of the form derived from it will clearly be $w+1$; $i+1$, $j+1$. Its character (which, for pure reciprocants, depends solely on the degree) will therefore be opposite to that of R, and its characteristic will be $\mu+4$, that of R being μ.

Beginning with the form $3ac-5b^2$, which was given as an example in Lecture II, a series of pure "educts" may be obtained by the repeated use of the above generator; and it will be noticed that the successive educts thus formed are alternately of even and odd character, whereas those previously given, namely, a, $2bt-3a^2\ldots$, were all negative. A reduction similar to that which formerly took place when the generator for mixed reciprocants was used, may be effected at each second step in the present case. For, since the characteristic of $\left(3a\dfrac{d}{dx}-\mu b\right)R$ is $\mu+4$, the next operation will give

$$\left(3a\frac{d}{dx}-(\mu+4)\,b\right)\left(3a\frac{d}{dx}-\mu b\right)R.$$

Performing the indicated differentiations, this becomes

$$\begin{aligned}&3a\frac{d}{dx}\left(3a\frac{dR}{dx}-\mu bR\right)-3\,(\mu+4)\,ab\frac{dR}{dx}+\mu\,(\mu+4)\,b^2R\\&=9a^2\frac{d^2R}{dx^2}+9ab\frac{dR}{dx}-3\mu ab\frac{dR}{dx}-3\mu acR-3\,(\mu+4)\,ab\frac{dR}{dx}+\mu\,(\mu+4)\,b^2R\\&=9a^2\frac{d^2R}{dx^2}-3\,(2\mu+1)\,ab\frac{dR}{dx}-3\mu acR+\mu\,(\mu+4)\,b^2R.\end{aligned}$$

Adding $\mu\,(\mu+4)\,(3ac-5b^2)\,R$ to 5 times the above expression, we obtain

$$45a^2\frac{d^2R}{dx^2}-15\,(2\mu+1)\,ab\frac{dR}{dx}+3\mu\,(\mu-1)\,acR,$$

which, when divided by $3a$, gives the pure reciprocant

$$15a\frac{d^2R}{dx^2}-5\,(2\mu+1)\,b\frac{dR}{dx}+\mu\,(\mu-1)\,cR.$$

This form is one degree lower than the second educt from R, the depression of degree being due to the removal of a factor a by division.

When the modified letters a_0, a_1, a_2, a_3, ... are used, the generator

$$(3ac-4b^2)\,\partial_b+(3ad-5bc)\,\partial_c+(3ae-6bd)\,\partial_d+\ldots \qquad (1)$$

is easily transformed by writing in it

$$a=2a_0,\quad b=2.3.a_1,\quad c=2.3.4.a_2,\quad d=2.3.4.5\;a_3\ldots,$$

and consequently

$$\partial_b=\frac{\partial_{a_1}}{2.3},\quad \partial_c=\frac{\partial_{a_2}}{2.3.4},\quad \partial_d=\frac{\partial_{a_3}}{2.3.4.5}\ldots,$$

when it becomes

$$\begin{aligned}&\frac{2^2.3^2.4}{2.3}(a_0a_2-a_1^2)\,\partial_{a_1}+\frac{2^2.3^2.4.5}{2.3.4}(a_0a_3-a_1a_2)\,\partial_{a_2}\\&\qquad\qquad+\frac{2^2.3^2.4.5.6}{2.3.4.5}(a_0a_4-a_1a_3)\,\partial_{a_3}+\ldots.\end{aligned}$$

Dividing each term of this by 2.3, and writing the numerical coefficients in their simplest form, we have

$$4\,(a_0a_2-a_1^2)\,\partial_{a_1}+5\,(a_0a_3-a_1a_2)\,\partial_{a_2}+6\,(a_0a_4-a_1a_3)\,\partial_{a_3}+\ldots, \qquad (2)$$

which is the modified generator previously mentioned.

The generators formerly used in the theory of mixed reciprocants were

$$(2tb-3a^2)\,\partial_a+(2tc-4ab)\,\partial_b+(2td-5ac)\,\partial_c+\ldots \qquad (3)$$

and

$$3\,(ta_1-a_0^2)\,\partial_{a_0}+4\,(ta_2-a_0a_1)\,\partial_{a_1}+5\,(ta_3-a_0a_2)\,\partial_{a_2}+\ldots. \qquad (4)$$

The memory will be assisted in retaining these formulae if we observe that (1) is obtainable from (3), or (2) from (4), by increasing at the same time each numerical coefficient and the weight of each letter by unity.

It will, I think, be instructive to see how the form $3ac-5b^2$ was found originally by combining mixed reciprocants. The degree alone of a pure reciprocant suffices, as we have seen, to determine its character; but when we are dealing with mixed reciprocants their character does not depend either on the degree or the weight, so that we require a notation to discriminate between forms of the same degree-weight, but of opposite character. In what follows, (+) placed before any form signifies that it is a reciprocant of *even* character, while (−) signifies that its character is *odd*.

I have previously given the three *odd* reciprocants

$$(-)\quad a, \qquad \text{(A)}$$
$$(-)\quad 2bt-3a^2, \qquad \text{(B)}$$
$$(-)\quad ct-5ab. \qquad \text{(C)}$$

From these we obtain *even* reciprocants; thus the product of (A) and (C) is

$$(+)\quad act-5a^2b, \qquad \text{(D)}$$

and the square of (B) is

$$(+)\quad 4b^2t^2-12a^2bt+9a^4.$$

After subtracting the *even* reciprocant $9a^4$ from this, we may remove the factor $4t$ from the remainder without thereby affecting its character. These reductions give

$$(+)\quad b^2t-3a^2b,$$

which may be combined with the *even* reciprocant (D) in such a manner that the combination contains a factor t. In fact,

$$3\,(act-5a^2b)-5\,(b^2t-3a^2b)=(3ac-5b^2)\,t,$$

so that a *legitimate* combination of mixed reciprocants can be made to give the pure one

$$3ac-5b^2.$$

Similarly we might find the known form

$$9a^2d-45abc+40b^3,$$

which equated to zero expresses Sextactic Contact at a point x, y. But it is more readily obtained by operating with the generator on $3ac-5b^2$; thus,

$$\{(3ac-4b^2)\,\partial_b+(3ad-5bc)\,\partial_c\}\,(3ac-5b^2)=-10b\,(3ac-4b^2)+3a\,(3ad-5bc)$$
$$=9a^2d-45abc+40b^3.$$

An *orthogonal reciprocant* may be defined as a mixed reciprocant whose form remains invariable (save as to the acquisition of an extraneous factor when the reciprocant is not absolute) when any orthogonal substitution is impressed on the variables x and y. Concerning such reciprocants, we have the very beautiful theorem: *If R and $\frac{dR}{dt}$ are both of them reciprocants, then R is an orthogonal reciprocant.*

For suppose R to be an absolute reciprocant; that is, let

$$R=qR'\ (q=\pm 1),$$

where R is a function of $t, a, b, c, \ldots$ and R' the same function of $\tau, \alpha, \beta, \gamma, \ldots$; then, denoting by ΔR the variation of R due to the variation of y by ϵx, and by DR the variation of R due to the variation of x by $-\epsilon y$, we have

$$\Delta R=\epsilon\frac{dR}{dt}.$$

For the variation of t is ϵ and the variations of $a, b, c, \ldots$ vanish. Similarly

$$DR'=-\epsilon\frac{dR'}{d\tau}.$$

Now, since $$R=qR',$$

$$DR=qDR'=-\epsilon q\frac{dR'}{d\tau},$$

therefore $$DR+\Delta R=\epsilon\left(\frac{dR}{dt}-q\frac{dR'}{d\tau}\right);$$

that is, the total variation of R (due to the change of x into $x-\epsilon y$ and of y into $y+\epsilon x$) vanishes if

$$\frac{dR}{dt}=q\frac{dR'}{d\tau}.$$

Hence, if R be an absolute orthogonal reciprocant, $\frac{dR}{dt}$ is also an absolute reciprocant (though it is not orthogonal) of the same character as R.

If R be not absolute, suppose its characteristic to be μ; then it can be made absolute by dividing it by $a^{\frac{\mu}{3}}$. The application of the foregoing method of variations will now prove that $\frac{d}{dt}\left(\frac{R}{a^{\frac{\mu}{3}}}\right)$ is an absolute reciprocant of the same character as $\frac{R}{a^{\frac{\mu}{3}}}$. But $\frac{d}{dt}\left(\frac{R}{a^{\frac{\mu}{3}}}\right)=\frac{1}{a^{\frac{\mu}{3}}}\frac{dR}{dt}$. Hence $\frac{dR}{dt}$ is a reciprocant whose characteristic is μ, and character the same as that of R.

The simplest Orthogonal Reciprocant is the form

$$(1+t^2)\,b-3a^2t,$$

which occurs on p. 19 of Boole's *Differential Equations*. When equated to zero it is the general differential equation of a circle. It is noticeable that although Boole obtains this form by equating to zero the differential of the radius of curvature

$$\frac{(1+t^2)^{\frac{3}{2}}}{a},$$

he does not recognise the fact that it vanishes at points of maximum or minimum curvature of any plane curve, but says that the "geometrical property which this equation expresses is the invariability of the radius of curvature."

Taking this form as an example of our general theorem, let

$$R=(1+t^2)\,b-3a^2t;$$

then $$\frac{dR}{dt}=2bt-3a^2,$$

which is the familiar Schwarzian. Observe that

$$(1+t^2)\,b - 3a^2t = -t^6\,\{(1+\tau^2)\,\beta - 3\alpha^2\tau\}$$

and

$$2bt - 3a^2 = -t^6\,(2\beta\tau - 3\alpha^2),$$

so that the characteristic and character are the same for both these forms.

The form $ct-5ab$, which we have called the Post-Schwarzian, when multiplied by 2 and integrated with respect to t, gives

$$ct^2 - 10abt + \phi\,(a, b, \ldots).$$

In order that this may be a reciprocant, we must have

$$\phi\,(a, b, \ldots) = c + 15a^3.$$

In this way the Orthogonal Reciprocant

$$(1+t^2)\,c - 10abt + 15a^3$$

was obtained originally.

It will be easy to verify that this is a reciprocant by means of the identical relations

$$t = \frac{1}{\tau},$$

$$a = -\frac{\alpha}{\tau^3},$$

$$b = -\frac{\beta\tau - 3\alpha^2}{\tau^5},$$

$$c = -\frac{\gamma\tau^2 - 10\alpha\beta\tau + 15\alpha^3}{\tau^7}.$$

We shall find that

$$(1+t^2)\,c - 10abt + 15a^3 = -t^7\,\{(1+\tau^2)\,\gamma - 10\alpha\beta\tau + 15\alpha^3\},$$

and comparing this with

$$ct - 5ab = -t^7\,(\gamma\tau - 5\alpha\beta),$$

it will be noticed that both forms have the same character and the same characteristic.

The complete primitive of the differential equation

$$c\,(1+t^2) - 10abt + 15a^3 = 0$$

has been found by Mr Hammond and Prof. Greenhill. The solution may be written in the following forms:

$$\left.\begin{aligned} x &= \int \frac{dt}{\sqrt{\{\kappa\,(1-15t^2+15t^4-t^6)+\lambda\,(6t-20t^3+6t^5)\}}} + \mu \\ y &= \int \frac{t\,dt}{\sqrt{\{\kappa\,(1-15t^2+15t^4-t^6)+\lambda\,(6t-20t^3+6t^5)\}}} + \nu \end{aligned}\right\},$$

$$\left.\begin{aligned} x &= \int \frac{\cos(\theta - A)\,d\theta}{\sqrt{\{B\cos 6\,(\theta - A)\}}} + \text{const.} \\ y &= \int \frac{\sin(\theta - A)\,d\theta}{\sqrt{\{B\cos 6\,(\theta - A)\}}} + \text{const.} \end{aligned}\right\}.$$

$$k'^2 tn^2\,(X, k) = k^2 tn^2\,(Y, k'),$$

where

$$k = \sin 15^\circ, \quad k' = \sin 75^\circ,$$

and

$$X = lx + my + n_1,$$

$$Y = mx - ly + n_2,$$

l, m, n_1, n_2 being arbitrary constants.

The last two forms of solution are due to Prof. Greenhill.

LECTURE VII.

I have frequently referred to, and occasionally dilated on, the analogy between pure reciprocants and invariants. A new bond of connection between the two theories has been established by Capt. MacMahon, which I will now explain. Let me, by way of preface, so far anticipate what I shall have to say on the Theorem of Aggregation in Invariants (that is, the theorem concerning the number of linearly independent invariants of a given type) as to remark that the proof of this theorem, first given by me in *Crelle's Journal* and subsequently in the *Phil. Mag.* for March, 1878, depends on the fact that if we take two operators, namely, the Annihilator, say

$$\Omega = a_0\partial_{a_1} + 2a_1\partial_{a_2} + 3a_2\partial_{a_3} + \ldots + ja_{j-1}\partial_{a_j}$$

and its opposite, say

$$O = a_j\partial_{a_{j-1}} + 2a_{j-1}\partial_{a_{j-2}} + 3a_{j-2}\partial_{a_{j-3}} + \ldots + ja_1\partial_{a_0},$$

then $(\Omega O - O\Omega)\,I$ is a multiple of I.

Thus, if I stands for any invariant (that is, if $\Omega I = 0$), it follows immediately that ΩOI is a multiple of I, and consequently $\Omega^m O^m I$ is also a multiple of I. We may call Ω and O, which are exact opposites to each other, reversing operators.

Now, MacMahon has found out the reversor to V, the Annihilator of pure reciprocants. His reversing operator is no longer of a similar, though opposite, form to V, as O is to Ω, but is simply $\frac{d}{dx}$; nor is the effect of operating with $V\frac{d}{dx}$ on any pure reciprocant R equivalent to multiplication by a merely numerical factor, as was the case with ΩOI, but $\left(V\frac{d}{dx}\right)R$ is a numerical multiple of aR, and as a consequence of this $\left(V^m\frac{d^m}{dx^m}\right)R$ is a numerical multiple of a^mR. Thus the parallelism is like that between the two sexes, the same with a difference, as is usually the case in comparing the two theories.

This remarkable relation between the operators V and $\frac{d}{dx}$ may be seen *a priori* if we assume that (as we shall hereafter prove) to each pure reciprocant R there is an annihilator V of the form

$$3a^2\partial_b + (\ldots)\partial_c + (\ldots)\partial_d + (\ldots)\partial_e + \ldots,$$

not containing ∂_a and linear in the remaining differential operators $\partial_b, \partial_c, \partial_d, \ldots$. For if we call the characteristic μ, by differentiating the absolute pure reciprocant $\frac{R}{a^{\frac{\mu}{3}}}$ with respect to x we obtain, as was shown in the last lecture, the pure reciprocant

$$3a\frac{dR}{dx} - \mu bR.$$

Since this is annihilated by V, we have

$$3a\left(V\frac{d}{dx}\right)R - \mu RVb - \mu bVR = 0.$$

But, since R is a pure reciprocant, $VR = 0$; and from the assumed form of V it follows that

$$Vb = 3a^2.$$

Hence
$$3a\left(V\frac{d}{dx}\right)R - 3\mu a^2R = 0,$$

or
$$\left(V\frac{d}{dx}\right)R = \mu aR.$$

Thus the operation of $V\frac{d}{dx}$ is equivalent to multiplication by μa, so that (barring the introduction of a) V restores to $\frac{dR}{dx}$ the form it had antecedent to the operation of $\frac{d}{dx}$, and may be called a qualified reversor to $\frac{d}{dx}$.

For example, suppose that

$$R = 3ac - 5b^2.$$

Since we are using *natural* letters for the derivatives of y with respect to x, we have

$$\frac{d}{dx} = b\partial_a + c\partial_b + d\partial_c + \ldots,$$

and, as we shall presently see,

$$V = 3a^2\partial_b + 10ab\partial_c + (15ac + 10b^2)\partial_d + \ldots.$$

Now, $\frac{dR}{dx} = (b\partial_a + c\partial_b + d\partial_c)(3ac - 5b^2) = 3bc - 10bc + 3ad = 3ad - 7bc.$

Operating on this with V, we find

$$V\frac{dR}{dx} = V(3ad - 7bc) = -21a^2c - 70ab^2 + 3a(15ac + 10b^2) = 24a^2c - 40ab^2;$$

that is
$$V\frac{d}{dx}(3ac - 5b^2) = 8a(3ac - 5b^2).$$

Let us now inquire whether it is possible so to determine an operator V that the relation

$$\left(V\frac{d}{dx} - \frac{d}{dx}V\right)F = (3i + w)aF$$

may be satisfied identically when F is any homogeneous isobaric function of the letters $a, b, c, \ldots$ of degree i and weight w. If so, we must be able to satisfy each of the equations

$$\left(V\frac{d}{dx} - \frac{d}{dx}V\right)a = 3a^2,$$
$$\left(V\frac{d}{dx} - \frac{d}{dx}V\right)b = 4ab,$$
$$\left(V\frac{d}{dx} - \frac{d}{dx}V\right)c = 5ac,$$
$$\left(V\frac{d}{dx} - \frac{d}{dx}V\right)d = 6ad,$$

....

which are found by writing $a, b, c, d, \ldots$ successively in the place of F.

Now $\frac{da}{dx} = b$, $\frac{db}{dx} = c$, $\frac{dc}{dx} = d$, ... so that the above equations may be written

$$Vb = 3a^2 + \frac{d}{dx}(Va),$$
$$Vc = 4ab + \frac{d}{dx}(Vb),$$
$$Vd = 5ac + \frac{d}{dx}(Vc),$$
$$Ve = 6ad + \frac{d}{dx}(Vd),$$

....

These equations are sufficient to completely determine V on the supposition previously made that it is linear in the differential operators and does not contain ∂_a; for, since V is linear, it must be of the form

$$(Va)\partial_a + (Vb)\partial_b + (Vc)\partial_c + \ldots,$$

and, since it does not contain ∂_a, we must have $Va = 0$, and therefore

$$Vb = 3a^2,$$
$$Vc = 4ab + \frac{d}{dx}(3a^2) = 4ab + 6ab = 10ab,$$
$$Vd = 5ac + \frac{d}{dx}(10ab) = 5ac + 10b^2 + 10ac = 15ac + 10b^2,$$
$$Ve = 6ad + \frac{d}{dx}(15ac + 10b^2) = 6ad + 15bc + 20bc + 15ad = 21ad + 35bc,$$

....

Hence $V = 3a^2\partial_b + 10ab\partial_c + (15ac + 10b^2)\partial_d + (21ad + 35bc)\partial_e + \ldots.$

When the modified letters $a_0, a_1, a_2, \ldots$ are used, we shall have, in consequence of the change of notation, $\left(V\frac{d}{dx}\right)R = 2\mu a_0 R$ (instead of $\mu a R$). If, as before, we seek to satisfy the equation

$$\left(V\frac{d}{dx} - \frac{d}{dx}V\right)F = 2(3i + w)a_0 F, \qquad (1)$$

we shall find, on writing a_n in the place of F,

$$\left(V\frac{d}{dx} - \frac{d}{dx}V\right)a_n = 2(3 + n)a_0 a_n. \qquad (2)$$

This condition will be sufficient, as well as necessary, for the satisfaction of (1) when V is linear; for then

$$V\frac{d}{dx} - \frac{d}{dx}V$$

will also be linear, its general term being

$$\left(V\frac{da_n}{dx} - \frac{d}{dx}Va_n\right)\partial_{a_n},$$

which is equal to $2(3+n)a_0 a_n \partial_{a_n}$ by equation (2). Hence

$$\begin{aligned}\left(V\frac{d}{dx} - \frac{d}{dx}V\right)F &= \text{a sum of terms of the form } 2(3+n)a_0 a_n \partial_{a_n} F \\ &= 2a_0(3a_0\partial_{a_0} + 3a_1\partial_{a_1} + 3a_2\partial_{a_2} + \ldots)F \\ &\quad + 2a_0(a_1\partial_{a_1} + 2a_2\partial_{a_2} + \ldots)F;\end{aligned}$$

that is, equation (1) is satisfied whenever (2) is. Writing in (2)

$$\frac{da_n}{dx} = (n+3)a_{n+1},$$

we obtain

$$(n+3)Va_{n+1} = 2(n+3)a_0 a_n + \frac{d}{dx}(Va_n), \qquad (3)$$

from which the values of Va_n may be successively determined.

When $Va_0 = 0$, the value of Va_n, which satisfies (3), is

$$Va_n = \frac{n+3}{2}(a_0 a_{n-1} + a_1 a_{n-2} + \ldots + a_{n-2}a_1 + a_{n-1}a_0);$$

thus

$$Va_1 = \frac{4}{2}\cdot a_0^2,\quad Va_2 = 5a_0 a_1,\quad Va_3 = 6a_0 a_2 + 3a_1^2, \ldots$$

and the value of V is therefore

$$\frac{4}{2}\cdot a_0^2\partial_{a_1} + 5a_0 a_1\partial_{a_2} + 6\left(a_0 a_2 + \frac{1}{2}a_1^2\right)\partial_{a_3} + 7(a_0 a_3 + a_1 a_2)\partial_{a_4} + \ldots.$$

Now that we are on the subject of parallelism between the old and new worlds of Algebraical Form, I feel tempted to point out yet another very interesting bond of connection between them. There is a theorem concerning Invariants which I am not aware that any one but myself has noticed, or at

all events I do not remember ever seeing it in print*, which is this: If we take any "invariant" and regard its most advanced letter as a variable, or say rather as the ratio of two variables $u:v$, by multiplying by a proper power of v we obtain a new Quantic in u, v; or, if we take any number of such invariants with the same most advanced letter (or, as we may call it in a double sense, the same radical letter) in common, we shall have a system of binary Quantics in u, v. My theorem is, or was, that an Invariant of any one or more of such Quantics is an Invariant of the original Quantic. I recently found a similar proposition to be true for Reciprocants, namely, forming as before a system of *pure* Reciprocants into Quantics in u, v, any "Invariant" of such system is itself a Reciprocant.

The two theorems may be stated symbolically thus:

$$\left.\begin{aligned}II' &= I''\\ IR &= R'\end{aligned}\right\}.$$

On mentioning this to Mr L. J. Rogers, he sent me next day a proof which, although only stated as applicable to Reciprocants, is equally so, *mutatis mutandis*, to Invariants. Although given for a single invariant, it applies equally to a system.

I give Mr Rogers' proof that any invariant of a *pure* reciprocant (the proof will not hold for impure ones) is a pure reciprocant; or rather I use his method to prove the analogous theorem that any invariant of an invariant is itself an invariant. It will be seen hereafter that this same proof applies to *pure* reciprocants with only trifling changes; but the proof as given by Mr Rogers requires some further considerations to be gone into for which we are not yet ripe.

Consider, for the sake of simplicity, the binary Quintic

$$(a, b, c, d, e, f\between x, y)^5,$$

and let I be any invariant of it (satisfied or unsatisfied); then

$$I = a_0 f^n + a_1 f^{n-1} + a_2 f^{n-2} + \ldots + a_n,$$

where $a_0, a_1, a_2, \ldots a_n$ do not contain f, but are functions of a, b, c, d, e alone.

Let the Protomorphs for our Quintic be denoted by A, B, C, D, E, F; then

$$F = a^2 f - 5abe + 2acd + 8b^2 d + 6bc^2.$$

Eliminating f from I by means of this equation, we have

$$Ia^{2n} = A_0 F^n + A_1 F^{n-1} + A_2 F^{n-2} + \ldots + A_n,$$

where $A_0, A_1, A_2, \ldots A_n$ are all of them invariants (not necessarily integral

* The theorem is, however, given in Vol. XI. p. 98 of the *Bulletin de la Société Mathématique de France*, in a paper by M. Perrin, which has only recently come under the lecturer's notice.

forms, but this is immaterial to the proof, for Ω annihilates fractional and integral invariants alike). For

$$\Omega (Ia^{2n}) = \Omega (A_0F^n + A_1F^{n-1} + \ldots + A_n),$$

and, in consequence of Ia^{2n} and F being invariants, so that, as regards Ω, F may be treated as if it were a constant, this becomes

$$0 = F^n\Omega A_0 + F^{n-1}\Omega A_1 + F^{n-2}\Omega A_2 + \ldots + \Omega A_n,$$

in which the coefficients of the several powers of F must be separately equated to zero. In other words, $A_0, A_1, A_2, \ldots A_n$ are all of them invariants. Now, any invariant of

$$A_0F^n + A_1F^{n-1} + A_2F^{n-2} + \ldots + A_n$$

is a function of $A_0, A_1, A_2, \ldots A_n$, and therefore an invariant.

(N.B.—We cannot assume that any function of general reciprocants is itself a reciprocant.)

Again, since $A_0F^n + \ldots + A_n$, and $a_0f^n + \ldots + a_n$

are connected by the substitution

$$F = a^2f - 5abe + \ldots,$$

which is *linear* in respect to the letters F and f, any invariant of

$$A_0F^n + \ldots + A_n$$

is (to a factor *près*, that factor being a power of a which is itself an invariant) equal to the corresponding invariant of

$$a_0f^n + \ldots + a_n.$$

But every invariant of the former has been shown to be an invariant of the original quantic, and therefore every invariant of the latter is so also.

I add some examples in illustration of this theorem:

Ex. 1. Take the invariant of the Quintic

$$a^2f^2 - 10abef + 4acdf + 16b^2df - 12bc^2f + 16ace^2 + 9b^2e^2 - 12ad^2e - 76bcde + 48c^3e + 48bd^3 - 32c^2d^2.$$

The discriminant of this, considered as a quadratic in f, is

$$a^2(16ace^2 + 9b^2e^2 - 12ad^2e - 76bcde + 48c^3e + 48bd^3 - 32c^2d^2) - (5abe - 2acd - 8b^2d + 6bc^2)^2$$
$$= 16a^3ce^2 - 16a^2b^2e^2 - 12a^3d^2e - 56a^2bcde + 48a^2c^3e + 80ab^3de - 60ab^2c^2e + 48a^2bd^3 - 36a^2c^2d^2 - 32ab^2cd^2 - 64b^4d^2 + 24abc^3d + 96b^3c^2d - 36b^2c^4.$$

It will be found on trial that this is divisible by the invariant

$$4(ae - 4bd + 3c^2),$$

the quotient being

$$4a^2ce - 4ab^2e - 3a^2d^2 + 2abcd + 4b^3d - 3b^2c^2$$
$$= 3a(ace - b^2e - ad^2 + 2bcd - c^3) + (ac - b^2)(ae - 4bd + 3c^2).$$

Thus the discriminant of the quadratic in f, that is, of the invariant

$$a^2f^2 - 2f(5abe - 2acd + 8b^2d - 6bc^2) + \ldots,$$

is shown to be an invariant. It will further illustrate the proof of the theorem if we remark that precisely the same invariant is obtained by eliminating f between the above form and the protomorph

$$a^2f - 5abe + 2acd + 8b^2d - 6bc^2.$$

Ex. 2. If we take the pure reciprocant

$$45a^2d^2 - 450a^2bcd + 400ab^3d + 192a^2c^3 + 165ab^2c^2 - 400b^4c,$$

which, from its similarity to the Discriminant of the Cubic, I have called the Quasi-Discriminant, and form *its* discriminant, when regarded as a quadratic in d, we find

$$45a^2(192a^2c^3 + 165ab^2c^2 - 400b^4c) - (225a^2bc - 200ab^3)^2.$$

If, in this expression, we write $P = 3ac - 5b^2$, so that $3ac = P + 5b^2$, it becomes

$$5 \cdot 64a^2(P + 5b^2)^3 + 5 \cdot 165a^2b^2(P + 5b^2)^2 - 15 \cdot 400a^2b^4(P + 5b^2) - 625a^2b^2(3P + 7b^2)^2.$$

On performing the calculation it will be found that all the terms involving b will disappear from this result, and there will remain the single term $320a^2P^3$, that is, $320a^2(3ac - 5b^2)^3$, which is a reciprocant.

LECTURE VIII.

In my last lecture the complete expression, both in terms of the modified and unmodified letters, was obtained for V, the annihilator for pure reciprocants assuming its existence and its form. These assumptions I shall now make good by proving, from first principles, the fundamental theorem that the satisfaction of the equation

$$VR = 0$$

is a necessary and sufficient condition in order that R may be a pure reciprocant.

It will be advantageous to use the modified system of letters, in which

$$t, a_0, a_1, a_2, \ldots \text{ stand for } \frac{dy}{dx}, \frac{1}{1 \cdot 2} \cdot \frac{d^2y}{dx^2}, \frac{1}{1 \cdot 2 \cdot 3} \cdot \frac{d^3y}{dx^3}, \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{d^4y}{dx^4}, \ldots$$

and $\alpha_0, \alpha_1, \alpha_2, \ldots$ for $\frac{1}{1 \cdot 2} \cdot \frac{d^2x}{dy^2}, \frac{1}{1 \cdot 2 \cdot 3} \cdot \frac{d^3x}{dy^3}, \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{d^4x}{dy^4}, \ldots$

respectively. Let the variation due to the change of x into $x + \epsilon y$, where ϵ

is an infinitesimal number, be denoted by Δ. Obviously this change leaves the value of each of the quantities $\alpha_0, \alpha_1, \alpha_2, \ldots$ unaltered, and therefore

$$\Delta R(\alpha_0, \alpha_1, \alpha_2, \ldots) = 0,$$

whatever the nature of R may be. But when R is a pure reciprocant,

$$R(a_0, a_1, a_2, \ldots) = \pm t^\mu R(\alpha_0, \alpha_1, \alpha_2, \ldots),$$

whence it immediately follows that

$$\Delta t^{-\mu} R(a_0, a_1, a_2, \ldots) = 0^*.$$

Before proceeding to determine the values of

$$\Delta t, \ \Delta a_0, \ \Delta a_1, \ \Delta a_2, \ \ldots$$

it will be useful to remark that since

$$\frac{dy}{dx} = t, \ \frac{d^2y}{dx^2} = 1 \cdot 2 \cdot a_0, \ \frac{d^3y}{dx^3} = 1 \cdot 2 \cdot 3 \cdot a_1, \ldots,$$

we have $$\frac{dt}{dx} = 2a_0, \ \frac{da_0}{dx} = 3a_1, \ldots,$$

and generally $$\frac{da_n}{dx} = (n+3)\, a_{n+1}.$$

Now let $[t]$ denote the augmented value of t, and in general let [ ] be used to signify that the augmented value of the quantity enclosed in it is to be taken. Then

$$[t] = \frac{dy}{d[x]} = \frac{dy}{d(x+\epsilon y)} = \frac{dy}{dx\left(1 + \epsilon \dfrac{dy}{dx}\right)} = \frac{t}{1+\epsilon t} = t - \epsilon t^2;$$

so also $$2[a_0] = [2a_0] = \frac{d[t]}{d[x]} = \frac{d[t]}{d(x+\epsilon y)} = \frac{d[t]}{dx(1+\epsilon t)} = (1-\epsilon t)\frac{d[t]}{dx}$$

$$= (1-\epsilon t)\frac{d}{dx}(t - \epsilon t^2) = (1-\epsilon t)(2a_0 - 4\epsilon t a_0)$$

$$= 2a_0 - 6\epsilon t a_0;$$

that is $$[a_0] = a_0 - 3\epsilon t a_0.$$

Reasoning precisely similar to that which gave

$$2[a_0] = (1-\epsilon t)\frac{d}{dx}[t],$$

leads to the formula

$$(n+3)[a_{n+1}] = (1-\epsilon t)\frac{d}{dx}[a_n],$$

* It has been suggested by Mr J. Chevallier that the proof might be simplified by considering the variation $\Delta a_0^{-\frac{\mu}{3}} R(a_0, a_1, a_2, \ldots)$ instead of $\Delta t^{-\mu} R(a_0, a_1, a_2, \ldots)$.

from which the augmented values of $a_1, a_2, a_3, \ldots$ may be found by giving to n the values 0, 1, 2, ... in succession. Thus, writing $n = 0$, we have

$$3[a_1] = (1-\epsilon t)\frac{d}{dx}[a_0] = (1-\epsilon t)\frac{d}{dx}(a_0 - 3\epsilon t a_0)$$

$$= (1-\epsilon t)(3a_1 - 9\epsilon t a_1 - 6\epsilon a_0^2) = 3a_1 - \epsilon(12 t a_1 + 6a_0^2),$$

or $$[a_1] = a_1 - \epsilon(4ta_1 + 2a_0^2).$$

Similarly, when $n = 1$,

$$4[a_2] = (1-\epsilon t)\frac{d}{dx}[a_1] = (1-\epsilon t)\frac{d}{dx}(a_1 - 4\epsilon t a_1 - 2\epsilon a_0^2)$$

$$= (1-\epsilon t)(4a_2 - 16\epsilon t a_2 - 20\epsilon a_0 a_1)$$

$$= 4a_2 - 20\epsilon t a_2 - 20\epsilon a_0 a_1,$$

and $$[a_2] = a_2 - 5\epsilon(ta_2 + a_0 a_1).$$

Again, $$5[a_3] = (1-\epsilon t)\frac{d}{dx}[a_2] = (1-\epsilon t)\frac{d}{dx}(a_2 - 5\epsilon t a_2 - 5\epsilon a_0 a_1)$$

$$= (1-\epsilon t)(5a_3 - 25\epsilon t a_3 - 30\epsilon a_0 a_2 - 15\epsilon a_1^2)$$

$$= 5a_3 - 30\epsilon t a_3 - 30\epsilon a_0 a_2 - 15\epsilon a_1^2,$$

so that $$[a_3] = a_3 - \epsilon(6ta_3 + 6a_0 a_2 + 3a_1^2).$$

In like manner we shall find

$$[a_4] = a_4 - 7\epsilon(ta_4 + a_0 a_3 + a_1 a_2).$$

These results may be written in a more symmetrical form; thus:

$$2[t] = 2t - 2\epsilon t^2,$$
$$2[a_0] = 2a_0 - 3\epsilon(ta_0 + a_0 t),$$
$$2[a_1] = 2a_1 - 4\epsilon(ta_1 + a_0^2 + a_1 t),$$
$$2[a_2] = 2a_2 - 5\epsilon(ta_2 + a_0 a_1 + a_1 a_0 + a_2 t),$$
$$2[a_3] = 2a_3 - 6\epsilon(ta_3 + a_0 a_2 + a_1^2 + a_2 a_0 + a_3 t),$$
$$2[a_4] = 2a_4 - 7\epsilon(ta_4 + a_0 a_3 + a_1 a_2 + a_2 a_1 + a_3 a_0 + a_4 t).$$

The general law

$$2[a_n] = 2a_n - (n+3)\,\epsilon(ta_n + a_0 a_{n-1} + \ldots + a_{n-1}a_0 + a_n t),$$

or, as it may also be written,

$$\Delta a_n = -\frac{n+3}{2}\,\epsilon(ta_n + a_0 a_{n-1} + \ldots + a_{n-1}a_0 + a_n t),$$

admits of an easy inductive proof.

Assuming the truth of the theorem for $[a_n]$, and writing for brevity in what follows,

$$S_n = ta_n + a_0 a_{n-1} + a_1 a_{n-2} + \ldots + a_{n-2}a_1 + a_{n-1}a_0 + a_n t,$$

we have $$[a_n] = a_n - \frac{n+3}{2}\epsilon S_n.$$

Now,

$$\begin{aligned}\frac{dS_n}{dx} &= (n+3)\,ta_{n+1} + 2a_0a_n \\ &\quad + (n+2)\,a_0a_n + 3a_1a_{n-1} \\ &\quad + (n+1)\,a_1a_{n-1} + 4a_2a_{n-2} \\ &\quad + \ldots\ldots\ldots\ldots + \ldots\ldots \\ &\quad + 4a_{n-2}a_2 + (n+1)\,a_{n-1}a_1 \\ &\quad + 3a_{n-1}a_1 + (n+2)\,a_na_0 \\ &\quad + 2a_na_0 + (n+3)\,a_{n+1}t \\ &= (n+4)\,(ta_{n+1} + a_0a_n + a_1a_{n-1} + \ldots \\ &\qquad + a_{n-1}a_1 + a_na_0 + a_{n+1}t) - 2ta_{n+1} \\ &= (n+4)\,S_{n+1} - 2ta_{n+1}.\end{aligned}$$

Hence $$\frac{d}{dx}[a_n] = (n+3)\,a_{n+1} - \frac{n+3}{2}\,\epsilon\,\{(n+4)\,S_{n+1} - 2ta_{n+1}\}.$$

But, as we have already seen,

$$(n+3)\,[a_{n+1}] = (1-\epsilon t)\,\frac{d}{dx}[a_n];$$

consequently,

$$[a_{n+1}] = (1-\epsilon t)\,a_{n+1} - \frac{n+4}{2}\,\epsilon S_{n+1} + \epsilon t a_{n+1} = a_{n+1} - \frac{n+4}{2}\,\epsilon S_{n+1};$$

that is, the theorem holds for $[a_{n+1}]$ when it holds for $[a_n]$. But we know that it is true for the cases $n=0, 1, 2, 3, 4$, and therefore it is true universally.

Resuming the proof of the main theorem, it has been shown that

$$\Delta t^{-\mu} R\,(a_0, a_1, a_2, \ldots) = 0;$$

that is $$-\mu t^{-1}\Delta t + R^{-1}\Delta R = 0,$$

or $$-\mu R t^{-1}\Delta t + \frac{dR}{da_0}\,\Delta a_0 + \frac{dR}{da_1}\,\Delta a_1 + \frac{dR}{da_2}\,\Delta a_2 + \ldots = 0.$$

But

$$\begin{aligned}\Delta t &= -\epsilon t^2, \\ \Delta a_0 &= -3\epsilon t a_0, \\ \Delta a_1 &= -\epsilon\,(4ta_1 + 2a_0^2), \\ \Delta a_2 &= -\epsilon\,(5ta_2 + 5a_0a_1), \\ \Delta a_3 &= -\epsilon\,(6ta_3 + 6a_0a_2 + 3a_1^2), \\ \Delta a_4 &= -\epsilon\,(7ta_4 + 7a_0a_3 + 7a_1a_2), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

and consequently

$$\begin{aligned}t\,(\mu - 3a_0\partial_{a_0} - 4a_1\partial_{a_1} - 5a_2\partial_{a_2} - 6a_3\partial_{a_3} - 7a_4\partial_{a_4} - \ldots)\,R \\ - \left\{4\left(\frac{a_0^2}{2}\right)\partial_{a_1} + 5\,(a_0a_1)\,\partial_{a_2} + 6\left(a_0a_2 + \frac{a_1^2}{2}\right)\partial_{a_3} \right. \\ \left. + 7\,(a_0a_3 + a_1a_2)\,\partial_{a_4} + \ldots\right\} R = 0.\end{aligned}$$

This is equivalent to the two conditions

$$(3a_0\partial_{a_0} + 4a_1\partial_{a_1} + 5a_2\partial_{a_2} + \ldots)\,R = \mu R$$

and $$VR = 0,$$

where

$$V = 4\left(\frac{a_0^2}{2}\right)\partial_{a_1} + 5\,(a_0a_1)\,\partial_{a_2} + 6\left(a_0a_2 + \frac{a_1^2}{2}\right)\partial_{a_3} + 7\,(a_0a_3 + a_1a_2)\,\partial_{a_4} + \ldots.$$

For greater simplicity I confine what I have to say to the only essential case, to which every other may be reduced, of a *homogeneous* pure reciprocant. The equation

$$(3a_0\partial_{a_0} + 4a_1\partial_{a_1} + 5a_2\partial_{a_2} + \ldots)\,R = \mu R$$

shows that for every term $w + 3i$ is constant; that is, w is constant and therefore the function R is isobaric. This is also immediately deducible from the form of the relations between $a_0, a_1, a_2, \ldots$; $\alpha_0, \alpha_1, \alpha_2, \ldots$, and, what is important to notice, for future purposes,

$$F(a_0, a_1, a_2, \ldots) - t^{\mu} F(\alpha_0, \alpha_1, \alpha_2, \ldots),$$

when F is a homogeneous isobaric function, and $\mu = w + 3i$ is itself a homogeneous function of $(a_0, a_1, a_2, \ldots)$, whose degree is the same as that of F.

The only condition affecting R, a function of $a_0, a_1, a_2, \ldots$, supposed homogeneous and isobaric, is

$$VR = 0.$$

I shall now prove the converse, that if $R = F(a_0, a_1, a_2, \ldots)$ (being homogeneous and isobaric) has V for its annihilator, then R is a pure reciprocant. Let D be the value of $F(a_0, a_1, a_2, \ldots) - t^{\mu} F(\alpha_0, \alpha_1, \alpha_2, \ldots)$ expressed as a function of $a_0, a_1, a_2, \ldots$ alone. Then D will be a function of the same type as $F(a_0, a_1, a_2, \ldots)$.

Suppose that $$\Delta D = 0;$$

that is, that the variation of D due to the change of x into $x + \epsilon y$ vanishes in virtue of the equation $VR = 0$.

Let D become D' when y receives an arbitrary variation $y + \eta u$, where η is an infinitesimal constant and u an arbitrary function of x; then the variation of D' will vanish when x is changed into $x + \epsilon y + \epsilon\eta u$, and consequently when x is changed into $x + \epsilon y$ the variation of D' will also vanish. Hence

$$\Delta D' = 0,$$

and if we take the difference of the variations of D and D', we shall find

$$\Delta\left(u''\frac{d}{da_0}D + u'''\frac{d}{da_1}D + u^{\text{IV}}\frac{d}{da_2}D + \ldots\right) = 0.$$

Now, the arbitrary nature of the function u shows that we must have

$$\Delta\frac{d}{da_0}D = 0, \quad \Delta\frac{d}{da_1}D = 0, \quad \Delta\frac{d}{da_2}D = 0, \quad \ldots$$

and if we reason on $\frac{d}{da_0}D$, $\frac{d}{da_1}D$, ... in the same way as we have on D, we see that the variation Δ of each of the second differential derivatives of D will also vanish; and, pursuing the same argument further, it will be evident that the Δ of any derivative of D, of any order whatever, with respect to $a_0, a_1, a_2, \ldots$ will vanish. Hence

$$D = 0;$$

for if this is not so we may, supposing D to be a function of degree i in the letters $a_0, a_1, a_2, \ldots$, take the Δ of each of the differential derivatives of D of the order $i-1$; each of these variations would vanish by what precedes; that is, the variation due to the change of x into $x + \epsilon y$ of each of the letters $a_0, a_1, a_2, \ldots$ contained in D would be identically zero, which is absurd. We see, therefore, that when $\Delta D = 0$ (that is, when R is annihilated by V), $D = 0$, or

$$F(a_0, a_1, a_2, \ldots) = t^{\mu} F(\alpha_0, \alpha_1, \alpha_2, \ldots),$$

which proves the converse proposition.

It will not fail to be noticed how much language, and as a consequence algebraical thought (for words are the tools of thought), is facilitated by the use of the concept of annihilation in lieu of that of equality as expressed by a partial differential equation.

It is somewhat to the point that in the recent two grand determinations of the order of precedence among the so-called fixed stars relative to our planet, as approximately represented by the intensities of the light from them which reaches the eye, the one is directed by the principle of annihilation, the other by that of equality. Prof. Pritchard's method essentially consists in determining what relative thicknesses of an interposed glass screen, effected by means of a sliding wedge of glass, will serve to extinguish the light of a star; that employed by Prof. Pickering depends on finding what degree of rotation of an interposed prism of Iceland spar (a Nicol Prism) will serve to bring to an equality the ordinary image of one star with the extraordinary one of another. As these intensities depend on the squared sines and cosines of this angle of rotation measured from the position of non-visibility of one of them, it follows that the tangent squared of the twist measures the relative intensities by this method.

Hereafter it will be shown that if F is a homogeneous isobaric function of

$$y, y', y'', y''', \ldots,$$

whose weights are reckoned as

$$-2, -1, 0, 1, \ldots$$

then, when x becomes $x + hy$, where h is any constant quantity, F becomes

$$(1 + ht)^{-\mu} e^{-\frac{hV_1}{1+ht}} F,$$

where $\quad t = y', \; V_1 = -t^2\partial_t + V,$ and $\mu = 3i + w,$

i being the degree and w the weight of F.

From this, by an obvious course of reasoning, could be deduced as a particular case the condition of $F(a_0, a_1, a_2, \ldots)$ remaining a factor of its altered self when *any* linear substitutions are impressed on x and y; namely, the necessary and sufficient condition is that F has V for its annihilator.

LECTURE IX.

The prerogative of a Pure Reciprocant is that it continues a factor of its altered self when the variables x and y are subjected to any linear substitution. Its form, like that of any other reciprocant, is of course persistent when the variables are interchanged; that is, when in the general substitution, in which y is changed into

$$fy + gx + h$$

and x into

$$f'y + g'x + h',$$

we give the particular values $h = 0$, $h' = 0$, $f = 0$, $g' = 0$, $f' = 1$, $g = 1$, to the constants. Stated geometrically, the theorem is that the evanescence of any pure reciprocant R indicates a property independent of transformation of axes in a plane. We suppose R to be homogeneous and isobaric in $a, b, c, \ldots$. (If it were not, the theorem could not hold, for either the change of y into κy or that of x into λx would destroy the form.)

The persistence, under any linear substitution, of the form of pure reciprocants may be easily established as follows:

By a *semi-substitution* understand one where one of the variables remains unaltered. There are two such semi-substitutions, namely, where x remains unaltered, and where y does.

(1) Let x remain unaltered and y become $fy + gx + h$; then $a, b, c, \ldots$ become $fa, fb, fc, \ldots$ respectively; and therefore

$$R(a, b, c, \ldots) \text{ becomes } f^i R(a, b, c, \ldots),$$

where i is the degree of R.

(2) Let y remain unchanged and x become $f'y + g'x + h'$. Then, instead of R, I look to its equal

$$qt^{\mu} R(\alpha, \beta, \gamma, \ldots) \; (q = \pm 1);$$

that is, to

$$q\tau^{-\mu} R(\alpha, \beta, \gamma, \ldots),$$

which becomes

$$q(f' + g'\tau)^{-\mu} g'^i R(\alpha, \beta, \gamma, \ldots).$$

Since R is a reciprocant, this is equal to

$$\frac{\tau^{\mu}}{(f' + g'\tau)^{\mu}} g'^i R(a, b, c, \ldots),$$

or, replacing τ by its equivalent $\frac{1}{t}$,

$$(f't + g')^{-\mu} g'^i R(a, b, c, \ldots).$$

Thus we see that the proposition is true for a semi-substitution of either kind. Consider now the complete substitution made by changing y into

$$fy + gx + h$$

and x into

$$Fy + Gx + H.$$

If $f = 0$ and $G = 0$, then $\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots$ become $\frac{g}{F^2} \cdot \frac{d^2x}{dy^2}, \frac{g}{F^3} \cdot \frac{d^3x}{dy^3}, \ldots$; so that $R(a, b, c, \ldots)$ becomes $\frac{g^i}{F^{2i+w}} \cdot R(\alpha, \beta, \gamma, \ldots)$; and since this is equal to

$$\frac{g^i}{F^{2i+w}} \cdot qt^{-\mu} R(a, b, c, \ldots),$$

the proposition is true.

But if either of the two letters f, G (say f) is not zero, we may combine two semi-substitutions so as to obtain the complete substitution, in which y changes into

$$fy + gx + h,$$

and x changes into

$$Fy + Gx + H.$$

(1) Substitute $y_1 (= fy + gx + h)$ for y, and $x_1 (= x)$ for x.

(2) Then substitute $y_2 (= y_1)$ for y_1, and $x_2 (= f'y_1 + g'x_1 + h')$ for x_1.

By the first of these semi-substitutions

$$R(a, b, c, \ldots)$$

takes up an extraneous factor f^i. By the second it acquires the factor

$$\left(f' \frac{dy_1}{dx_1} + g'\right)^{-\mu} g'^i, \text{ where } \frac{dy_1}{dx_1} = f\frac{dy}{dx} + g = ft + g.$$

Hence we see that the extraneous factor is a negative power of a linear function of t, which we shall presently particularize, though it is not essential to the present demonstration to do so.

It only remains to show how the combination of these two semi-substitutions can be made to give the complete one in question. We have

$$y_2 = y_1 = fy + gx + h$$

and

$$x_2 = f'y_1 + g'x_1 + h' = f'(fy + gx + h) + g'x + h'$$
$$= ff'y + (f'g + g')x + (f'h + h').$$

In order that this may be equal to $Fy + Gx + H$, we must be able to satisfy the equations

$$f' = \frac{F}{f}, \quad g' = G - \frac{gF}{f}, \quad h' = H - \frac{hF}{f},$$

which is always possible, since by hypothesis f is not zero. Similarly it may be shown that when f vanishes, but G does not, by substituting

(1) $x_1 (= Fy + Gx + H)$ for x, and $y_1 (= y)$ for y,

(2) $x_2 (= x_1)$ for x_1, and $y_2 (= f''y_1 + g''x_1 + h'')$ for y_1,

we may so determine f'', g'', h'' as to get the complete substitution as before.

In every case, therefore, any linear substitution impressed upon the variables x and y will leave $R(a, b, c, \ldots)$ unaltered, barring the acquisition of an extraneous factor which is a negative power of a linear function of t.

Now, the first semi-substitution introduces, as we have seen, the constant factor

$$f^i;$$

the second introduces the factor

$$\left(f' \frac{dy_1}{dx_1} + g'\right)^{-\mu} g'^i,$$

where

$$\frac{dy_1}{dx_1} = ft + g.$$

The complete extraneous factor is the product of these two, and is therefore

$$f^i g'^i (ff't + f'g + g')^{-\mu}.$$

To express f' and g' in terms of the constants of the complete substitution we have

$$f' = \frac{F}{f}, \quad g' = G - \frac{gF}{f}.$$

Writing these values for f' and g' in the expression just found, we obtain

$$(fG - gF)^i (Ft + G)^{-\mu},$$

which is the extraneous factor acquired by R when the complete substitution is made. For example, if x becomes

$$Fy + Gx + H,$$

and y becomes

$$fy + gx + h,$$

the altered value of a $\left(\text{that is, of } \frac{d^2y}{dx^2}\right)$ is

$$(fG - gF)(Ft + G)^{-3} a.$$

Corresponding to the simple interchange of the variables, we have

$$F = 1, \; G = 0, \; H = 0; \; f = 0, \; g = 1, \; h = 0,$$

so that

$$fG - gF = -1,$$

and the altered value of a is $\frac{d^2x}{dy^2}$, or

$$\alpha = -\frac{a}{t^3},$$

which is right. In this case the general value of the acquired extraneous factor

$$(fG - gF)^i (Ft + G)^{-\mu} \text{ becomes } (-)^i t^{-\mu},$$

thus showing, what we have already proved from other considerations, that the character of a pure reciprocant is odd or even according as its degree is odd or even.

We saw in the last lecture that *every* pure reciprocant necessarily satisfied the two conditions

$$(3a_0\partial_{a_0} + 4a_1\partial_{a_1} + 5a_2\partial_{a_2} + \dots)\, R = \mu R$$

(where μ is the characteristic), and

$$VR = 0.$$

We also saw that $VR = 0$ was a sufficient as well as necessary condition that *any homogeneous function* R of $a_0, a_1, a_2, \dots$ should be a pure reciprocant. It will now be shown that every pure reciprocant is either homogeneous and isobaric, or else resoluble into a sum of homogeneous and isobaric reciprocants. Non-homogeneous mixed ones, it may be observed, are not so resoluble, so that the theorem only holds for pure reciprocants.

(1) Let us suppose that R (a pure reciprocant) is homogeneous in $a_0, a_1, a_2 \dots$; then it must be isobaric also. For, if i is the degree of R, Euler's theorem shows that

$$(3a_0\partial_{a_0} + 3a_1\partial_{a_1} + 3a_2\partial_{a_2} + 3a_3\partial_{a_3} + \dots)\, R = 3iR\,;$$

and since R is a pure reciprocant, the condition

$$(3a_0\partial_{a_0} + 4a_1\partial_{a_1} + 5a_2\partial_{a_2} + 6a_3\partial_{a_3} + \dots)\, R = \mu R$$

is necessarily satisfied. Hence

$$(a_1\partial_{a_1} + 2a_2\partial_{a_2} + 3a_3\partial_{a_3} + \dots)\, R = (\mu - 3i)\, R = \text{a constant multiple of } R,$$

which is the distinctive property of isobaric functions.

And, *vice versâ*, if R is homogeneous and isobaric of weight w and degree i, then

$$(3a_0\partial_{a_0} + 4a_1\partial_{a_1} + 5a_2\partial_{a_2} + \dots)\, R = (w + 3i)\, R = \mu R.$$

Thus homogeneous pure reciprocants are also isobaric and their characteristic is $3i + w$. (This property is also true for mixed reciprocants, as we have previously shown.)

(2) Suppose that R is not homogeneous, but made up of the homogeneous parts

$$R_{,},\ R_{,,},\ R_{,,,},\ \dots.$$

Then, since
$$V(R_{,} + R_{,,} + R_{,,,} + \dots) = 0$$

is satisfied identically, it is obvious that

$$VR_{,} + VR_{,,} + VR_{,,,} + \dots = 0$$

must also be satisfied identically.

But since all the terms are of different degrees, the only way in which this can happen is by making $VR_{,}, VR_{,,}, VR_{,,,}, \dots$ separately vanish. Now, $R_{,}, R_{,,}, R_{,,,}, \dots$ are by hypothesis *homogeneous* functions of $a_0, a_1, a_2, \dots$, and it has just been shown that each of them is annihilated by V, which has been shown to be a sufficient condition that any homogeneous function of $a_0, a_1, a_2, \dots$ may be a pure reciprocant. Thus each part $R_{,}, R_{,,}, R_{,,,}, \dots$ of R is a pure reciprocant.

Also, the condition

$$(3a_0\partial_{a_0} + 4a_1\partial_{a_1} + 5a_2\partial_{a_2} + \dots)\, R = \mu R$$

shows that if $i_1, w_1;\ i_2, w_2;\ i_3, w_3; \dots$ are the deg. weights of $R_{,}, R_{,,}, R_{,,,}, \dots$, we must have

$$3i_1 + w_1 = \mu,\ 3i_2 + w_2 = \mu,\ 3i_3 + w_3 = \mu,\ \dots.$$

Thus non-homogeneous pure reciprocants are severable into parts each of which is a homogeneous and isobaric pure reciprocant, the characteristic of each part being equal to the same quantity μ, which is the characteristic of the whole.

I will now explain what information concerning the number of pure reciprocants of a given type is afforded by the equation $VR = 0$. Let

$$Aa_0^{\lambda_0}a_1^{\lambda_1}a_2^{\lambda_2} \dots a_j^{\lambda_j}$$

be a term of a homogeneous isobaric function (with its full number of terms) of $a_0, a_1, a_2, \dots a_j$, whose degree is i, extent j, and weight w, and which we will call R.

Then in the entire function there are as many terms as there are solutions in integers of the equations

$$\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \dots + \lambda_j = i,$$
$$\lambda_1 + 2\lambda_2 + 3\lambda_3 + \dots + j\lambda_j = w.$$

In other words, the number of terms in R is equal to the number of ways in which w can be made up of i or fewer parts, none greater than j. This number will be denoted by

$$(w;\ i, j).$$

Since the function R is the sum of every possible term of the form

$$Aa_0^{\lambda_0}a_1^{\lambda_1} \dots a_j^{\lambda_j},$$

each multiplied by an arbitrary constant, the number of these arbitrary constants is also

$$(w;\ i, j).$$

Now, suppose R to be a reciprocant; this imposes the condition

$$VR = 0.$$

Consider the effect produced by the operation of any term of

$$V = 4\left(\frac{a_0^2}{2}\right)\partial_{a_1} + 5a_0a_1\partial_{a_2} + 6\left(a_0a_2 + \frac{a_1^2}{2}\right)\partial_{a_3} + \dots,$$

say $\left(a_0a_2 + \frac{a_1^2}{2}\right)\partial_{a_3}$ (rejecting the numerical coefficient 6).

Operating on R with ∂_{a_3} decreases its weight by 3 and its degree by 1 unit. The subsequent multiplication by $a_0a_2 + \frac{a_1^2}{2}$, on the other hand, increases the weight by 2 and the degree by 2 units. Hence the total effect of $\left(a_0 a_2 + \frac{a_1^2}{2}\right)\partial_{a_3}$ is to increase the degree by 1 and to diminish the weight by 1 unit. The same is evidently true for any other term of V. Thus the total effect of V operating on the general homogeneous isobaric function R of weight w, degree i, extent j, is to change it into another homogeneous isobaric function whose weight, degree and extent are respectively $w-1$, $i+1, j$. Observe that the extent is not altered by the operation of V.

It is easily seen that the coefficients of VR are linear functions of the coefficients of R; for example, if

$$R = Aa_0^2 a_3 + Ba_0 a_1 a_2 + Ca_1^3,$$

$$VR = a_0^3 a_2 (6A + 2B) + a_0^2 a_1^2 (3A + 5B + 6C).$$

Hence the condition $VR = 0$ gives us $(w-1;\ i+1, j)$ *linear* equations between the $(w;\ i, j)$ coefficients of R; so that, *assuming that these equations of condition are all independent*, after they have been satisfied the number of arbitrary constants remaining in R (that is, the number of linearly independent reciprocants of the type $w;\ i, j$) is equal to

$$(w;\ i, j) - (w-1;\ i+1, j),$$

when this difference is positive; but when it is zero or negative there are no reciprocants of the given type.

If, however, any r of the $(w-1;\ i+1, j)$ equations of condition should not be independent of the rest, these equations would be equivalent to $(w-1;\ i+1, j) - r$ *independent* conditions, and therefore the number of linearly independent reciprocants of the type $w;\ i, j$ would be

$$(w;\ i, j) - (w-1;\ i+1, j) + r.$$

It is therefore certain that this number *cannot be less* than

$$(w;\ i, j) - (w-1;\ i+1, j).$$

We shall assume provisionally that $r = 0$, or in other words that the above partition formula is exact, instead of merely giving an inferior limit. Though it would be unsafe to rely on its accuracy, no positive grounds for doubting its exactitude have been revealed by calculation.

Such attempts as I have hitherto made to demonstrate the theorem have proved infructuous, but it must be remembered that more than a quarter of a century elapsed between the promulgation of Cayley's analogous theorem and its final establishment by myself on a secure basis of demonstration.

LECTURE X.

I will commence this lecture with a proof of Capt. MacMahon's theorem that if R is any pure reciprocant and μ its characteristic (that is, its weight added to three times its degree),

$$\left(V^m \frac{d^m}{dx^m}\right) R = 1 \,.\, 2 \,.\, 3 \ldots m \{\mu(\mu+2)(\mu+4)\ldots(\mu+2m-2)\}(y'')^m R,$$

where y'' may be replaced by either $2a_0$ or a, according as the modified or unmodified system of letters is employed.

Instead of a pure reciprocant, let us consider any homogeneous isobaric function F of degree i and weight w; and $\left(\text{for the sake of simplicity writing } \partial_x \text{ for } \frac{d}{dx}\right)$ instead of the operator $V^m \partial_x^m$ let us consider $V^m \partial_x^n - \partial_x^n V^m$. We have identically

$$\begin{aligned}(V^m \partial_x^n - \partial_x^n V^m) F = {} & V^{m-1}(V\partial_x - \partial_x V)\partial_x^{n-1} F \\ & + V^{m-2}(V\partial_x - \partial_x V)\, V\partial_x^{n-1} F \\ & + V^{m-3}(V\partial_x - \partial_x V)\, V^2\partial_x^{n-1} F \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + V(V\partial_x - \partial_x V)\, V^{m-2}\partial_x^{n-1} F \\ & + \quad (V\partial_x - \partial_x V)\, V^{m-1}\partial_x^{n-1} F \\ & + \partial_x (V^m \partial_x^{n-1} - \partial_x^{n-1} V^m) F.\end{aligned}$$

Now, the operation of $(V\partial_x - \partial_x V)$ on any homogeneous isobaric function whose characteristic is μ_1 is equivalent, as we have seen in Lecture VII, to multiplication by $\mu_1 y''$; so that if the characteristics of

$$\partial_x^{n-1}F,\quad V\partial_x^{n-1}F,\quad V^2\partial_x^{n-1}F,\ \ldots\ V^{m-1}\partial_x^{n-1}F$$

are

$$\mu_1 \quad,\quad \mu_2 \quad,\quad \mu_3 \quad,\ \ldots \quad \mu_m,$$

it follows that

$$(V^m\partial_x^n - \partial_x^n V^m) = (\mu_1 + \mu_2 + \mu_3 + \ldots + \mu_m)\, y'' V^{m-1}\partial_x^{n-1} F + \partial_x (V^m \partial_x^{n-1} - \partial_x^{n-1} V^m) F.$$

Observe that

$$V^{m-1}(V\partial_x - \partial_x V)\partial_x^{n-1}F = V^{m-1}\mu_1 y'' \partial_x^{n-1} F = \mu_1 y'' V^{m-1} \partial_x^{n-1} F,$$

where the transposition of the y'' is permissible because V does not act on it; but if y'' were preceded by ∂_x it could not be similarly transposed.

The numbers $\mu_1, \mu_2, \mu_3, \ldots$ form an arithmetical progression, for each operation of V increases the degree by unity and diminishes the weight by unity, so that

$$\mu_1 = 3i_1 + w_1 \text{ becomes } \mu_2 = 3(i_1 + 1) + (w_1 - 1) = \mu_1 + 2.$$

Similarly $\mu_3 = \mu_1 + 4,\ \ \mu_4 = \mu_1 + 6, \ldots \mu_m = \mu_1 + 2m - 2.$

The characteristic of F being

$$\mu = 3i + w, \text{ that of } \partial_x^{n-1}F \text{ is } \mu_1 = \mu + n - 1;$$

for each operation of ∂_x leaves the degree unaltered, but adds an unit to the weight; hence

$$\mu_1 + \mu_2 + \mu_3 + \ldots + \mu_m = m(\mu + m + n - 2);$$

so that

$$(V^m\partial_x^n - \partial_x^n V^m)F = m(\mu+m+n-2)y'' V^{m-1}\partial_x^{n-1}F + \partial_x(V^m\partial_x^{n-1} - \partial_x^{n-1}V^m)F. \quad (1)$$

When $F = R$, a pure reciprocant, so that $VR = 0$, our formula becomes

$$V^m\partial_x^n R = m(\mu + m + n - 2)\, y'' V^{m-1}\partial_x^{n-1}R + \partial_x V^m \partial_x^{n-1}R. \quad (2)$$

Suppose that in (2) $m > n$, then $V^m\partial_x^n R = 0$. This is obviously true when $n = 0$, and when $n = 1$. When $n = 2$ we find

$$V^m\partial_x^2 R = m(\mu + m)\, y'' V^{m-1}\partial_x R + \partial_x V^m \partial_x R$$
$$= 0 \text{ if } m > 2.$$

Similarly the case $n = 3$, $m > 3$ can be made to depend on $n = 2$, $m > 2$, and in general each case depends on the one immediately preceding it. Next let $n = m$ in (2); then, remembering that $V^m\partial_x^{m-1}R = 0$, we have

$$V^m\partial_x^m R = m(\mu + 2m - 2)\, y'' V^{m-1}\partial_x^{m-1}R,$$

from which MacMahon's theorem that

$$V^m\partial_x^m R = 1 \,.\, 2 \,.\, 3 \ldots m\, \{\mu(\mu+2)(\mu+4)\ldots(\mu+2m-2)\}\,(y'')^m R$$

is an immediate consequence.

Another special case of Formula (1) is worthy of notice, namely, that in which we take $n = 1$, when we obtain the simple formula

$$(V^m\partial_x - \partial_x V^m)F = m(\mu + m - 1)\, y'' V^{m-1}F. \quad (3)$$

If in this we write a_n in the place of F, and (the modified system of letters being used) $2a_0$ for y'', μ becomes $3 + n$, and we have

$$(V^m\partial_x - \partial_x V^m)\, a_n = 2m(m + n + 2)\, a_0 V^{m-1}a_n,$$

or, as it may also be written,

$$\frac{V^m\partial_x a_n}{1 \,.\, 2 \,.\, 3 \ldots m} = \frac{\partial_x V^m a_n}{1 \,.\, 2 \,.\, 3 \ldots m} + \frac{2(m+n+2)\, a_0 V^{m-1}a_n}{1 \,.\, 2 \,.\, 3 \ldots (m-1)}. \quad (4)$$

Mr Hammond remarks that this last formula may be used to prove the theorem

$$\alpha_n = -t^{-n-3}(e^{-\frac{V}{t}})\, a_n,$$

which was given without proof in Lecture II. Assuming that

$$\alpha_n = -t^{-n-3}a_n + t^{-n-4}Va_n - t^{-n-5}\frac{V^2 a_n}{1 \,.\, 2} + \ldots,$$

we have to prove that the theorem is also true when n is increased by unity. Differentiating both sides of the assumed identity with respect to x, we find

$$\partial_x \alpha_n = \partial_x\left(-t^{-n-3}a_n + t^{-n-4}Va_n - t^{-n-5}\frac{V^2 a_n}{1 \,.\, 2} + \ldots\right)$$
$$= -t^{-n-3}\partial_x a_n + t^{-n-4}\{\partial_x V a_n + 2(n+3)\, a_0 a_n\}$$
$$- t^{-n-5}\left\{\frac{\partial_x V^2 a_n}{1 \,.\, 2} + 2(n+4)\, a_0 V a_n\right\}$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots;$$

the general term being

$$(-)^{m+1}t^{-n-m-3}\left\{\frac{\partial_x V^m a_n}{1 \,.\, 2 \,.\, 3 \ldots m} + \frac{2(m+n+2)\, a_0 V^{m-1}a_n}{1 \,.\, 2 \,.\, 3 \ldots (m-1)}\right\}$$

which, by means of (4), reduces to

$$(-)^{m+1}t^{-n-m-3}\frac{V^m\partial_x a_n}{1 \,.\, 2 \,.\, 3 \ldots m}.$$

Hence $$\partial_x \alpha_n = -t^{-n-3}\partial_x a_n + t^{-n-4}V\partial_x a_n - t^{-n-5}\frac{V^2\partial_x a_n}{1 \,.\, 2} + \ldots,$$

or, more concisely,

$$\partial_x \alpha_n = -t^{-n-3}(e^{-\frac{V}{t}})\, \partial_x a_n.$$

But $$\partial_x a_n = (n+3)\, a_{n+1}, \text{ and } \partial_x \alpha_n = t\partial_y \alpha_n = (n+3)\, t\alpha_{n+1},$$

and therefore

$$(n+3)\, t\alpha_{n+1} = -(n+3)\, t^{-n-3}(e^{-\frac{V}{t}})\, a_{n+1},$$

or $$\alpha_{n+1} = -t^{-n-4}(e^{-\frac{V}{t}})\, a_{n+1}.$$

The theorem is easily seen to be true, for $n = 0, 1, 2$, and is thus proved to be true universally.

I will now return to the point at which I left off in my previous lecture. We saw that the exactitude of the formula

$$(w; i, j) - (w - 1; i + 1, j)$$

for the number of pure reciprocants of the type $w; i, j$ could not be inferred with certainty unless we were able to prove that the $(w - 1; i + 1, j)$ linear equations between the coefficients of R, found by equating VR to zero, were all of them independent. A similar difficulty presents itself in the proof of the corresponding formula $(w; i, j) - (w - 1; i, j)$ in the invariantive theory; but in that case I succeeded in making out a proof of the independence of the equations of condition founded on the fact that $\Omega^m O^m I$ is a numerical multiple of I, where I is any invariant, and Ω, O are the well-known operators

$$a_0\partial_{a_1} + 2a_1\partial_{a_2} + 3a_2\partial_{a_3} + \ldots + ja_{j-1}\partial_{a_j}$$
$$a_j\partial_{a_{j-1}} + 2a_{j-1}\partial_{a_{j-2}} + 3a_{j-2}\partial_{a_{j-3}} + \ldots + ja_1\partial_{a_0}.$$

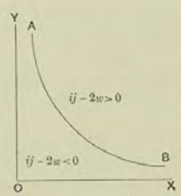
I have since discovered a second proof of the theorem for invariants which, though very interesting, is less simple than my first; but neither of these methods can be extended to the case of reciprocants.

It was suggested by Capt. MacMahon that the fact that $V^m\partial_x{}^m R$ is a numerical multiple of $a^m R$ ought to lead to a proof of the theorem for reciprocants similar to that obtained for invariants by my first method, alluded to above, but this I find is not the case; and indeed it is capable of being shown *a priori* that it cannot lead to a proof. One great distinction between the two theories, which is fatal to the success of the proposed method, is well worthy of notice.

If $$(w;i,j)-(w-1;i,j)=>0$$
(I shall sometimes call this positive), then
$$(w';i,j)-(w'-1;i,j)=>0$$
for all values of w' less than w; the condition that this difference, say $\Delta(w;i,j)$ shall be positive being simply that $ij-2w$ is positive (that is, $ij-2w=>0$). This is not the case with the difference
$$(w;i,j)-(w-1;i+1,j),$$
say $E(w;i,j)$; it by no means follows that if this is positive for a given value of w (i,j being kept constant), it will be so for any inferior value of w.

We may illustrate geometrically the condition $ij-2w=>0$, which holds when $\Delta(w;i,j)$ is non-negative.

Let (i,j) be co-ordinates of a point in a plane and draw the positive branch of the rectangular hyperbola
$$ij-2w=0.$$



Then, $ij-2w<0$ for all points in the area $YOXBA$ between the curve and its asymptotes; but for points on the curve AB,
$$ij-2w=0,$$
and for all points of the infinite area on the side of AB remote from the origin,
$$ij-2w>0.$$
Thus, for all points which lie either on or beyond the curve AB, $\Delta(w;i,j)$ is non-negative, and for all points between the curve and the asymptotes $\Delta(w;i,j)$ is non-positive.

We have here considered w as constant and i,j as variable, but in the case where all three are variable we should have to consider the hyperbolic paraboloid
$$ij-2w=0,$$
of which the curve AB is a section, by the plane $w=\text{const.}$; and the condition

of $\Delta(w;i,j)$ being non-negative or non-positive depends on the variable point (i,j,w) lying in the one case on or beyond the surface, and in the other between the surface and the planes of reference.

The function of i,j,w, whose positive or negative sign determines in like manner that of $E(w;i,j)$, cannot be linear in w. What its form is, or whether it is an Algebraical or Transcendental function, no one at present can say. Indeed, except for the light shed on the subject by the Algebraical Theory of Invariants, it would have been exceedingly difficult (as I know from vain efforts made by myself and others in Baltimore) to prove the much simpler theorem that $\Delta(w;i,j)$ is positive (that is, non-negative) when $ij-2w$ is so. It amounts to the assertion that the coefficient of a^ix^w in the expansion of
$$\frac{1-x}{(1-a)(1-ax)(1-ax^2)\dots(1-ax^j)}$$
is always non-negative, provided that $ij-2w$ is non-negative.

This is a theorem of great importance in the ordinary Theory of Invariants, and may be seen to be a consequence of the fact, which I have proved, that (using $[w;i,j]$ to denote a function of the type $w;i,j$ having its full number of arbitrary coefficients) there are no linear connections between the coefficients of $\Omega\,[w;i,j]$ when $ij-2w=>0$; but no one, as far as I know, has ever found a *direct* proof of it.

Viewing the connection between the two theories of Invariants and Reciprocants, I think it desirable to recapitulate with some improvements the proof, given in the *Phil. Mag.* for March, 1878, of the theorem that the number of linearly independent invariants of the type $w;i,j$ is exactly $\Delta(w;i,j)$ when this quantity is positive, and exactly zero when it is 0 or negative.

As regards reciprocants, at present we can only say that the number of linearly independent ones of the type $w;i,j$ is never less than $E(w;i,j)$, leaving to some gifted member of the class to prove or disprove that the first is always exactly equal to the second. The *exact* theorem to be proved in the theory of invariants is as follows:

If $ij-2w=>0$, the number of linearly independent invariants of the type $w;i,j$ is $\Delta(w;i,j)$.

If $ij-2w<0$, the number of such invariants is zero; that is, there are none. The proof is made to depend on the properties of
$$\Omega=a_0\partial_{a_1}+2a_1\partial_{a_2}+3a_2\partial_{a_3}+\dots+ja_{j-1}\partial_{a_j}$$
and of $$O=a_j\partial_{a_{j-1}}+2a_{j-1}\partial_{a_{j-2}}+3a_{j-2}\partial_{a_{j-3}}+\dots+ja_1\partial_{a_0}.$$

If U be any homogeneous isobaric function of degree i and weight w in the letters $a_0, a_1, a_2, \dots a_j$, it is easy to prove that
$$(\Omega O-O\Omega)\,U=(ij-2w)\,U,$$

and consequently, if U is an invariant I, so that $\Omega I = 0$,

$$\Omega O I = (ij - 2w)\, I.$$

I call $ij - 2w$ the *excess* and denote it by η, and shall first show that if η is negative $I = 0$; that is, there exists no invariant with a negative excess. This will prove that when $\Delta(w\,; i, j)$ is negative, that is, when

$$(w - 1\,; i, j) > (w\,; i, j),$$

the number of independent functions of the coefficients of $[w\,; i, j]$ which appear in $\Omega\,[w\,; i, j]$ is exactly equal to $(w\,; i, j)$, which is the number of the coefficients themselves. Clearly it cannot be greater; for, no matter what the number of linear functions of n quantities may be, only n at the utmost can be independent; there might be fewer, there cannot possibly be more. The complete theorem is that the number of independent coefficients in $\Omega\,[w\,; i, j]$ is the *subdominant* of two numbers: one the number of terms of the type $w\,; i, j$, the other the number of terms of the type $w - 1\,; i, j$.

N.B. That one of two numbers which is not greater than the other is called the subdominant.

LECTURE XI.

We may write for the Annihilator of an Invariant

$$\Omega = a_0 \dot{a}_1 + 2a_1 \dot{a}_2 + 3a_2 \dot{a}_3 + \ldots + j a_{j-1} \dot{a}_j$$

and for its opposite

$$O = j a_1 \dot{a}_0 + (j-1)\, a_2 \dot{a}_1 + (j-2)\, a_3 \dot{a}_2 + \ldots + a_j \dot{a}_{j-1},$$

where the pointed letters $\dot{a}_0, \dot{a}_1, \dot{a}_2, \ldots \dot{a}_j$ stand for the partial differential operators

$$\partial_{a_0}, \partial_{a_1}, \partial_{a_2}, \ldots \partial_{a_j}.$$

Suppose Ω and O to operate on any function $U(a_0, a_1, a_2, \ldots a_j)$; then

$$\Omega O U = (\Omega\,.\,O + \Omega * O)\, U$$

and

$$O \Omega U = (O\,.\,\Omega + O * \Omega)\, U,$$

where the full stop between O and Ω signifies multiplication, and the asterisk operation on the unpointed letters only. Thus,

$$\Omega\,.\,O = O\,.\,\Omega,$$

and, consequently, $\quad (\Omega O - O\Omega)\, U = (\Omega * O - O * \Omega)\, U.$

Now,

$$\Omega * O U = \{1\,.\,j a_0 \dot{a}_0 + 2(j-1)\, a_1 \dot{a}_1 + 3(j-2)\, a_2 \dot{a}_2 + \ldots + j\,.\,1 a_{j-1} \dot{a}_{j-1}\}\, U,$$

and

$$O * \Omega U = \{1\,.\,j a_1 \dot{a}_1 + 2(j-1)\, a_2 \dot{a}_2 + \ldots + (j-1)\, 2 a_{j-1} \dot{a}_{j-1} + j\,.\,1 a_j \dot{a}_j\}\, U,$$

whence we readily obtain

$$(\Omega O - O\Omega)\, U = j\,(a_0 \dot{a}_0 + a_1 \dot{a}_1 + a_2 \dot{a}_2 + \ldots + a_j \dot{a}_j)\, U$$
$$- 2\,(a_1 \dot{a}_1 + 2a_2 \dot{a}_2 + 3a_3 \dot{a}_3 + \ldots + j a_j \dot{a}_j)\, U.$$

Introducing the conditions of homogeneity and isobarism, namely,

$$(a_0 \dot{a}_0 + a_1 \dot{a}_1 + a_2 \dot{a}_2 + \ldots + a_j \dot{a}_j)\, U = iU$$

and

$$(a_1 \dot{a}_1 + 2a_2 \dot{a}_2 + 3a_3 \dot{a}_3 + \ldots + j a_j \dot{a}_j)\, U = wU,$$

where i and w denote the degree and weight of U, supposed now to be a rational integral homogeneous and isobaric function (or, to avoid a tedious periphrasis, say a *gradient*), we see that if the complete type of the gradient U is $w\,; i, j$,

$$(\Omega O - O\Omega)\, U = (ij - 2w)\, U = \eta U,$$

where η is the excess.

Since the operation of O increases the weight of the operand by unity, but does not alter either its degree or its extent, it is clear that the type of $O^\theta U$ is $w + \theta\,; i, j$. The excess of $O^\theta U$ is therefore

$$ij - 2(w + \theta) = \eta - 2\theta,$$

and the theorem just proved shows that

$$(\Omega O - O\Omega)\, O^\theta U = (\eta - 2\theta)\, O^\theta U.$$

From this we pass on to prove that $\Omega O^q - O^q \Omega$, acting on any gradient as its objective, is equivalent to $q(\eta - q + 1)\, O^{q-1}$; that is, when q is any positive integer, we shall show that

$$(\Omega O^q - O^q \Omega)\, U = q(\eta - q + 1)\, O^{q-1} U.$$

The subsequent consideration of a special case of this formula, in which U is replaced by any invariant I, will enable us to prove that there can be no invariants for which the excess $ij - 2w$ is negative. Let

$$O^{q-\theta} \Omega O^\theta U = P_\theta U;$$

then

$$O^{q-\theta-1} \Omega O^{\theta+1} U = P_{\theta+1} U,$$

and therefore

$$(P_{\theta+1} - P_\theta)\, U = O^{q-\theta-1} (\Omega O - O\Omega)\, O^\theta U.$$

Substituting in this for

$$(\Omega O - O\Omega)\, O^\theta U \text{ its value } (\eta - 2\theta)\, O^\theta U,$$

we have

$$(P_{\theta+1} - P_\theta)\, U = (\eta - 2\theta)\, O^{q-1} U.$$

Hence

$$(P_q - P_0)\, U = \{(P_1 - P_0) + (P_2 - P_1) + (P_3 - P_2) + \ldots + (P_q - P_{q-1})\}\, U$$
$$= \{\eta + (\eta - 2) + (\eta - 4) + \ldots + (\eta - 2q + 2)\}\, O^{q-1} U$$
$$= q(\eta - q + 1)\, O^{q-1} U.$$

But since $P_q = \Omega O^q$ and $P_0 = O^q \Omega$, this result may be written

$$(\Omega O^q - O^q \Omega)\, U = q(\eta - q + 1)\, O^{q-1} U.$$

If now $U = I$, an invariant, we have $\Omega U = 0$, and our formula becomes

$$\Omega O^q I = q(\eta - q + 1) O^{q-1} I.$$

Writing in succession $q = m, m-1, \dots 1$, we obtain

$$m(\eta - m + 1) O^{m-1} I = \Omega O^m I$$
$$(m-1)(\eta - m + 2) O^{m-2} = \Omega O^{m-1} I$$
$$(m-2)(\eta - m + 3) O^{m-3} I = \Omega O^{m-2} I$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$1 . \eta I = \Omega O I.$$

By assigning to m a sufficiently large value we are able to make $O^m I$ vanish as well as ΩI; for, the type of I being $w; i, j$, that of $O^m I$ is $w + m; i, j$. But it is evident that no *gradient* can have a greater weight than ij, the product of its degree and extent, for each term is a product of i letters none of them having a weight greater than j. If, then, we suppose that $m = ij - w + 1$, the weight of $O^m I$ is

$$w + m = ij + 1.$$

Therefore $$O^m I = 0.$$

Again, $$\eta - m + 1 = ij - 2w - (ij - w + 1) + 1 = -w.$$

If, then, η is negative, every term in the series

$$m(\eta - m + 1),\ (m-1)(\eta - m + 2), \dots 2(\eta - 1),\ 1 . \eta$$

is negative and can never vanish. Hence we have successively

$$O^{m-1} I = 0,\ O^{m-2} I = 0, \dots I = 0;$$

that is, when $ij - 2w < 0$ no invariant of the type $w; i, j$ exists.

Observe that the elenchus of the demonstration consists in the fact that the successive numerical factors $\eta - m + 1, \eta - m + 2, \eta - m + 3, \dots \eta$ are all non-zero on account of η being negative; but if η were positive we should eventually come to a factor $\eta - \mu$ which would be zero, and we could not conclude from $(\mu + 1)(\eta - \mu) O^\mu I$ being zero that $O^\mu I = 0$. Since $\eta - (m - 1)$ passes from $\eta - (ij - w)$ to η, that is, from $-w$ to η, it passes through zero when η is positive.

The second part of Cayley's completed theorem remains to be proved, namely, that when $ij - 2w => 0$, the number of linearly independent invariants of the type $w; i, j$ is precisely equal to $\Delta(w; i, j)$; that is, to

$$(w; i, j) - (w - 1; i, j).$$

I show this by proving that if $D(w; i, j)$ is the number in question, keeping i and j constant and taking $w <= \frac{ij}{2}$,

$$D(w; i, j) + D(w-1; i, j) + D(w-2; i, j) + \dots + D(0; i, j)$$

cannot be greater than

$$\Delta(w; i, j) + \Delta(w-1; i, j) + \Delta(w-2; i, j) + \dots + \Delta(0; i, j),$$

and consequently, since we know that no single $D(w; i, j)$ can possibly be less than the corresponding $\Delta(w; i, j)$, it follows that

$$D(w; i, j) + D(w-1; i, j) + D(w-2; i, j) + \dots + D(0; i, j)$$
$$= \Delta(w; i, j) + \Delta(w-1; i, j) + \Delta(w-2; i, j) + \dots + \Delta(0; i, j);$$

and, furthermore, that each

$$D(w; i, j) = \Delta(w; i, j).$$

For if any D were greater than its corresponding Δ, some other D would have to be less, which is impossible.

This principle of reasoning may be illustrated by imagining a row of ballot-boxes and supposing it to be ascertained that no single box contains fewer white balls than black ones. If, then, there are not more white than black balls altogether, the total number of whites must be the same as that of the blacks. And since there are just as many whites as blacks distributed among the ballot-boxes, the number of white and black balls must be the same in each box; for otherwise some box must contain fewer whites than blacks, which is contrary to the hypothesis.

Observe that the sum of these Δ's is $(w; i, j)$; for

$$(w; i, j) - (w-1; i, j) + (w-1; i, j) - (w-2; i, j) + \dots + (0; i, j) - (-1; i, j)$$
$$= (w; i, j) - (-1; i, j)$$

and $$(-1; i, j) = 0,$$

since there is no way of composing -1 with parts $0, 1, 2, \dots j$. Hence what I have to show is that

$$D(w; i, j) + D(w-1; i, j) + \dots + D(1; i, j) + D(0; i, j) = (w; i, j).$$

I want preliminarily to express $\Omega^q O^q I$ as a multiple of I*.

This can be done by a formula previously demonstrated, namely,

$$\Omega O^q I = q(\eta - q + 1) O^{q-1} I,$$

which gives

$$\Omega^2 O^q I = q(\eta - q + 1) \Omega O^{q-1} I = q(\eta - q + 1)(q-1)(\eta - q + 2) O^{q-2} I;$$

similarly

$$\Omega^3 O^q I = q(\eta - q + 1)(q-1)(\eta - q + 2)(q-2)(\eta - q + 3) O^{q-3} I;$$

and finally, changing the order of the numerical factors,

$$\Omega^q O^q I = 1 . 2 . 3 \dots q \{\eta(\eta - 1)(\eta - 2) \dots (\eta - q + 1)\} I.$$

This shows that $\Omega^q O^q I$ and *à fortiori* $O^q I$ can never vanish unless $\eta - q + 1$ becomes negative.

* The result of operating on I with O and Ω each q times, the two operations following each other according to any law of distribution whatever, will always be a numerical multiple of I; but the value of this multiple will differ for different laws of distribution.

Suppose now that I^q means an invariant of the type $w-q; i, j$; its excess is $ij-2(w-q)$, and consequently O^qI_q cannot vanish unless

$$ij-2(w-q)-q+1$$

becomes negative, which is impossible. For

$$ij-2(w-q)-q+1=ij-2w+q+1,$$

and $ij-2w=>0$ by hypothesis.

By taking O^qI_q as an *image*, so to say, of I_q we shall be able to obtain a limit to the number of I_q's by obtaining a limit to the number of their images. In fact, taking the *image* O^qI_q of each of the $D(w-q; i, j)$ linearly independent invariants of the type $w-q; i, j$ (this is what is meant by the I_q's) and giving q all possible values from 0 to w inclusive, the total number of these images is obviously

$$D(w; i, j)+D(w-1; i, j)+\dots+D(0; i, j).$$

Each of them will be a gradient of the weight $w-q+q$ (that is, of weight w), and will consist of terms of weight w, degree i, and extent j. The total number of such terms will be the number of ways of making up w with i of the numbers 0, 1, 2, 3, ... j, or with the usual notation $(w; i, j)$. If, then, it can be shown that none of these forms are linearly connected, then, inasmuch as they are all functions of the same $(w; i, j)$ arguments, it will follow that their total number cannot exceed $(w; i, j)$. That is, we shall have shown that

$$D(w; i, j)+D(w-1; i, j)+D(w-2; i, j)+\dots+D(0; i, j)$$

cannot exceed

$$\Delta(w; i, j)+\Delta(w-1; i, j)+\Delta(w-2; i, j)+\dots+\Delta(0; i, j),$$

and by the ballot-box principle, as already stated (inasmuch as no D is less than its corresponding Δ), it will follow that each D is the same as the corresponding Δ, and the theorem to be proved is established.

The proof of this independence is easy. For (1) suppose that there is any linear relation between the forms

$$O^qI_q,\ O^qI_q',\ O^qI_q'',\ \dots,$$

for each of which the value of q is the same. Denoting these forms by

$$P_q,\ P_q',\ P_q'',$$

let the relation in question be

$$\lambda P_q+\lambda' P_q'+\lambda'' P_q''+\dots=0.$$

Then $$\lambda\Omega^qP_q+\lambda'\Omega^qP_q'+\lambda''\Omega^qP_q''+\dots=0.$$

But each argument Ω^qP_q is of the form $\Omega^qO^qI_q$, and since this is equal

to I_q multiplied by a number which does not vanish*, we have a linear relation between I_q, I_q', I_q'', ..., namely

$$\lambda I_q+\lambda' I_q'+\lambda'' I_q''+\dots=0;$$

that is, the I_q's would not be linearly independent, contrary to hypothesis. Thus the *images* (O^qI_q, O^qI_q', O^qI_q'' ...) belonging to invariants of the same type $w-q; i, j$ cannot be linearly connected.

(2) I say that the images of invariants of different types cannot be linearly connected. For let q, q', q'', ... arranged in descending order of magnitude, be the different values of q in the images supposed to be linearly related. The result of operating with Ω^q on any image of the form $O^{q'}I_{q'}$ is to bring it to the form $\Omega^{q-q'}\Omega^{q'}O^{q'}I_{q'}$, which is a multiple of $\Omega^{q-q'}I_{q'}$, and therefore vanishes. But Ω^q, acting on any of the images O^qI_q, O^qI_q', ..., will, as we have seen, bring back the multiple of I_q; thus the operation of Ω^q on the supposed relation will give a linear equation connecting $I_q, I_q', I_q'', \dots$, and for the same reason as before this is impossible. Hence there can be no linear relation whatever between the images of the invariants whose types extend from $w; i, j$ to $0; i, j$, and the number of these images will accordingly be not greater than $(w; i, j)$, as was to be proved.

It is well worthy of notice that $D(w; i, j)$ may be zero, but obviously cannot be negative, as it denotes a number of things which may have any value from zero upwards. Hence follows a remarkable theorem in the pure theory of partitions which it would be extremely difficult to prove from first principles, namely, that the difference between the two partition numbers

$$(w; i, j)-(w-1; i, j)$$

can never be negative when $ij-2w=>0$. It may be zero, but cannot be less than zero. This explains what I said about the hyperbolic paraboloid $ij-2w=0$, where i, j, w are treated as co-ordinates of a point in space. We might call the value of $(w; i, j)-(w-1; i, j)$ the density of any point i, j, w, and the theorem may then be expressed by saying that at points within or upon the hyperbolic paraboloid the density can never be negative; for points outside this surface it can never be positive.

As regards the analogous formula in the Theory of Reciprocants

$$(w; i, j)-(w-1; i+1, j),$$

we do not know that any algebraical surface can be constructed which will enable us to discriminate between the cases in which this difference, say $E(w; i, j)$, is positive or negative. Should such a surface exist, its equation must contain w in a higher degree than the first. Supposing that the above

* In fact, remembering that the excess of the type $w-q$; i, j is $ij-2(w-q)=\eta+2q$, we find

$$\Omega^qO^qI_q=1.2.3\dots q\,\{(\eta+2q)(\eta+2q-1)\dots(\eta+q+1)\}\,I_q,$$

in which both η and q are *positive* integers.

formula represents the actual number of reciprocants, it will follow (and this is confirmed by experience) that there can be no reciprocants to a type of negative excess. For

$$\begin{aligned} &(w;i,j)-(w-1;i+1,j) \\ &=(w;i,j)-(w-1;i,j)-[(w-1;i+1,j)-(w-1;i,j)] \\ &=(w;i,j)-(w-1;i,j)-(w-i-2;i+1,j-1). \end{aligned}$$

But if $ij-2w$ is negative, $(w;i,j)-(w-1;i,j)$ is zero or negative. Hence $(w;i,j)-(w-1;i+1,j)$ is non-positive.

For *satisfied* invariants (those ordinarily so called) $w=\frac{ij}{2}$, and the formula for their number becomes $\left(\frac{ij}{2};i,j\right)-\left(\frac{ij}{2}-1;i,j\right)$.

As these form a well-defined class apart, it would have seemed very natural to begin with them in endeavouring to establish the theorem, reserving the theory of unsatisfied invariants (sources of covariants) for future consideration. But to all appearance it would have been very difficult, if not impossible, to have succeeded in dealing with them alone.

This is another example of the law in Heuristic that the whole is easier of deglutition than its part.

LECTURE XII.

Before proceeding further with the development of the pure analytical theory of reciprocants, it may be useful to point out some instances of its relations and applications to geometrical questions.

Using $y_1, y_2, y_3, \ldots y_n$ to denote the successive derivatives of y with respect to x*, let the complete primitive of the differential equation

$$F(x, y, y_1, y_2, \ldots y_n)=0$$

be

$$\phi(x, y, \lambda, \mu, \nu, \ldots)=0.$$

We can in general so determine the n constants $\lambda, \mu, \nu, \ldots$ that the curve ϕ may pass through n given points, and if we take these to be consecutive points on the curve

$$\Phi(x, y)=0,$$

ϕ and Φ will have a contact of the $(n-1)$th order at a given point of Φ. In order that the curves may have a contact of the nth order at a point

* In future $y_1, y_2, y_3, \ldots y_n$ will always have this meaning, the derivatives of x with respect to y will be denoted by $x_1, x_2, x_3, \ldots$, and whenever the letters $t, a, b, c, \ldots$ are used they will stand for $y_1, \frac{y_2}{1.2}, \frac{y_3}{1.2.3}, \frac{y_4}{1.2.3.4}, \ldots$ respectively.

whose abscissa is x, the ordinates of Φ and ϕ at that point and their 1st, 2nd, ... nth derivatives with respect to x must be the same for both curves. But at every point of ϕ its differential equation

$$F(x, y, y_1, y_2, \ldots y_n)=0$$

has to be satisfied, and therefore the $x, y, y_1, y_2, \ldots y_n$ of any point on Φ, at which contact of the nth order with ϕ is possible, must also satisfy the same equation.

Now, suppose that for x and y we substitute given functions of them, X and Y; the curves ϕ and Φ become

$$\phi(X, Y, \lambda, \mu, \nu, \ldots)=0 \text{ and } \Phi(X, Y)=0.$$

Contact of the nth order with the transformed ϕ will therefore be possible at any point of the transformed Φ for which

$$F(X, Y, Y_1, Y_2, \ldots Y_n)=0,$$

where $Y_1, Y_2, Y_3, \ldots Y_n$ are the derivatives of Y with respect to X.

But, unless the function F and the substitutions $X=f_1(x, y)$, $Y=f_2(x, y)$ are so related that the transformed differential equation

$$F(X, Y, Y_1, Y_2, \ldots Y_n)=0$$

is identical with the untransformed one, the property marked by the contact of the transformed curves will not be identical with that marked by the contact of the untransformed ones.

For example, let $F=y_2$; then the relation between $\phi\equiv y+\lambda x+\mu=0$ (the complete primitive of $y_2=0$) and an arbitrary curve Φ is that the constants λ and μ may be so chosen that the line $y+\lambda x+\mu=0$ may have a contact of the second order at any point of Φ for which $y_2=0$; and the property marked is an inflexion on Φ. But if we make the substitution $X=x^2$, $Y=y^2$, so that the differential equation $y_2=0$ is transformed into $\left(\frac{d}{dx^2}\right)^2 y^2=0$ and its complete primitive into $y^2+\lambda x^2+\mu=0$, it will still be possible so to choose λ and μ that $y^2+\lambda x^2+\mu=0$ may have a contact of the second order at any point of an arbitrary curve for which $\left(\frac{d}{dx^2}\right)^2 y^2=0$, but the property marked, instead of being an inflexion, will be a *contact of the second order with a conic having a pair of conjugate diameters coincident with the co-ordinate axes.*

The property remains unaltered when the co-ordinate axes are interchanged, and therefore the differential equation $\left(\frac{d}{dx^2}\right)^2 y^2=0$ will be identical with $\left(\frac{d}{dy^2}\right)^2 x^2=0$, in which the variables x and y have changed places. The identity of the two differential equations is easily verified, for

$$\left(\frac{d}{dx^2}\right)^2 y^2 = \frac{1}{2x}\cdot\frac{d}{dx}\left(\frac{y}{x}\cdot\frac{dy}{dx}\right) = \frac{1}{2x}\cdot\left\{\frac{y}{x}\cdot\frac{d^2y}{dx^2} + \frac{1}{x}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x^2}\cdot\frac{dy}{dx}\right\}$$
$$= \frac{1}{2x^3}(xyy_2 + xy_1^2 - yy_1);$$

so that the differential equation may be written

$$xyy_2 + xy_1^2 - yy_1 = 0.$$

Interchanging x and y in this, we have

$$yxx_2 + yx_1^2 - xx_1 = 0,$$

in which, if we write $x_1 = \frac{dx}{dy} = \frac{1}{y_1}$, and $x_2 = \frac{d^2x}{dy^2} = -\frac{y_2}{y_1^3}$, it follows immediately that

$$yxx_2 + yx_1^2 - xx_1 = -\frac{1}{y_1^3}(xyy_2 + xy_1^2 - yy_1),$$

and the identity in question is established.

Such a form as the above, which merely acquires an extraneous factor when the variables are interchanged, might be called a reciprocant, if it were not convenient to restrict the use of the word to forms in which the variables x and y do not appear explicitly. With this limitation, the geometrical property indicated by the evanescence of a reciprocant will be independent of the position of the origin, but not in general independent of the directions of the co-ordinate axes. Thus, we may prove that the equation

$$2y_1y_3 - 3y_2^2 = 0$$

indicates the possibility of 4-point contact with a hyperbola whose asymptotes are *parallel to the co-ordinate axes*. To do this it is sufficient to show that its complete primitive is the equation to such a hyperbola.

Writing the equation in the form

$$\frac{y_3}{y_2} = \frac{3}{2}\cdot\frac{y_2}{y_1},$$

we see that its first integral is

$$\log y_2 = \frac{3}{2}\log y_1 + \text{const.};$$

or, when prepared for a second integration,

$$-\frac{1}{2}\cdot y_1^{-\frac{3}{2}} y_2 = \lambda.$$

Hence $$y_1^{-\frac{1}{2}} = \lambda x + \mu,$$
$$y_1 = (\lambda x + \mu)^{-2},$$

and finally we obtain the complete primitive

$$\lambda(\nu - y) = (\lambda x + \mu)^{-1},$$

which proves the proposition.

With the notation previously explained, in which $y_1 = t$, $y_2 = 2a$, $y_3 = 6b$, the differential equation is $bt - a^2 = 0$. We have therefore proved that at all points of a general curve for which the Schwarzian $(bt - a^2)$ vanishes, 4-point contact with a hyperbola whose asymptotes are parallel to the co-ordinate axes is possible.

We now consider the important case in which the conditioning differential equation remains unchanged when the axes are orthogonally transformed, and is therefore found by equating to zero an orthogonal reciprocant. The simplest example of this class of equations is that which marks the points of maximum or minimum curvature on a curve. Since these points are points of 4-point contact with a circle, the conditioning differential equation will be that of the circle

$$(x + \lambda)^2 + (y + \mu)^2 + \nu = 0.$$

Differentiating this three times in succession, we have

$$x + \lambda + (y + \mu)t = 0,$$
$$1 + t^2 + 2a(y + \mu) = 0,$$
$$at + b(y + \mu) = 0.$$

Eliminating μ from the last two of these equations, y will disappear at the same time, and the condition for points of maximum or minimum curvature is found to be

$$2a^2t - b(1 + t^2) = 0.$$

In Salmon's *Higher Plane Curves* (2nd edition, p. 357) the "aberrancy of curvature" is given by the formula

$$\tan\delta = y_1 - \frac{(1 + y_1^2)y_3}{3y_2^2} = t - \frac{(1 + t^2)b}{2a^2}.$$

The above differential equation is therefore equivalent to $\delta = 0$.

If we differentiate the radius of curvature $\rho = \frac{(1 + y_1^2)^{\frac{3}{2}}}{y_2} = \frac{(1 + t^2)^{\frac{3}{2}}}{2a}$, we find

$$\frac{d\rho}{dx} = \frac{6a^2t(1 + t^2)^{\frac{1}{2}} - 3b(1 + t^2)^{\frac{3}{2}}}{2a^2} = 3(1 + t^2)^{\frac{1}{2}}\tan\delta = 3\tan\delta\,.\,\frac{ds}{dx}.$$

Hence it follows that

$$\tan\delta = \frac{1}{3}\cdot\frac{d\rho}{ds}.$$

The conditioning equation for points at which $\frac{d\rho}{ds}$ or $\tan\delta$ is a maximum or minimum is $\frac{d^2\rho}{ds^2} = 0$; or the same condition may be expressed by

$$\frac{d\tan\delta}{dx} = 0.$$

Now

$$\frac{d\tan\delta}{dx} = \frac{d}{dx}\left\{t - \frac{b(1 + t^2)}{2a^2}\right\} = 2a - \frac{2c(1 + t^2)}{a^2} - \frac{2abt}{a^2} + \frac{3b^2(1 + t^2)}{a^3}$$

is an orthogonal reciprocant, for it can be expressed in terms of legitimate combinations of $1+t^2$, which is an orthogonal reciprocant of even character, with the three orthogonal reciprocants of odd character,

$$a,\ b(1+t^2)-2a^2t,\ c(1+t^2)-5abt+5a^3.$$

In fact, the above expression for $\dfrac{d\tan\delta}{dx}$, when multiplied by a^3 to clear of fractions, becomes

$$2a^4-2a^2bt+3b^2(1+t^2)-2ac(1+t^2)$$
$$=\frac{3}{1+t^2}\{b(1+t^2)-2a^2t\}^2+\frac{12a^4}{1+t^2}-2a\{c(1+t^2)-5abt+5a^3\},$$

where the right-hand side is a linear function of orthogonal reciprocants of the same (even) character, so that the combination is legitimate.

Quantities such as $\rho,\ \dfrac{d\rho}{ds},\ \dfrac{d^2\rho}{ds^2},\ \ldots$, or $\rho,\ \dfrac{d\rho}{d\phi},\ \dfrac{d^2\rho}{d\phi^2},\ \ldots$, where $d\phi$ is the angle subtended by the arc ds at the centre of curvature, have values independent of the particular position of the co-ordinate axes (supposed rectangular), and consequently these values, expressed in terms of $t, a, b, c, \ldots$ will be absolute orthogonal reciprocants. A differential equation expressing the condition that any one of these quantities vanishes, or that any one of them has a maximum or minimum value, will also be independent of the position of the rectangular axes, and must therefore be expressible in the form of an orthogonal reciprocant equated to zero.

Mr Hammond remarks that, since the radii of curvature at corresponding points of a curve and its evolute are ρ and $\dfrac{d\rho}{d\phi}$, the radius of curvature of its nth evolute is $\dfrac{d^n\rho}{d\phi^n}$. The radius of curvature of the nth evolute of any nth involute of a circle is constant, and, consequently, the differential equation of an nth involute to a circle is

$$\frac{d^{n+1}\rho}{d\phi^{n+1}}=0.$$

Writing this in the form

$$\left(\frac{1+t^2}{a}\cdot\frac{d}{dx}\right)^{n+1}\cdot\frac{(1+t^2)^{\frac{3}{2}}}{a}=0,$$

to which it is easily reduced, since

$$\frac{d}{d\phi}=\rho\cdot\frac{d}{ds}=\frac{\rho}{(1+t^2)^{\frac{1}{2}}}\cdot\frac{d}{dx}=\frac{(1+t^2)}{2a}\cdot\frac{d}{dx},$$

we see by what precedes that the left-hand member of the differential equation is an orthogonal reciprocant.

As an example of the class of singularities which next presents itself for consideration, let us find the differential condition which holds at points of

contact of the fourth order with a common parabola. This condition is expressible by the differential equation whose complete primitive is

$$(y+\kappa x)^2+2\lambda x+2\mu y+\nu=0.$$

Differentiating three times in succession, we obtain

$$(y+\kappa x)(t+\kappa)+\lambda+\mu t=0,$$
$$2a(y+\kappa x+\mu)+(t+\kappa)^2=0,$$
$$b(y+\kappa x+\mu)+a(t+\kappa)=0.$$

The arbitrary constants ν and λ do not appear in the last two of these equations, from which, if we eliminate μ, the variables x and y disappear at the same time, and we find

$$2a^2-b(t+\kappa)=0.$$

A final differentiation and elimination give

$$10ab-4c(t+\kappa)=0,$$
$$4ac-5b^2=0.$$

Points of 5-point contact with a parabola are therefore indicated by the evanescence of the pure reciprocant $4ac-5b^2$. And in general the differential equation $R=0$, where R is any pure reciprocant, indicates a property of a curve which may be called a descriptive singularity, since it is totally unaffected by the arbitrary choice of any two lines on the plane for the axes of co-ordinates. For it was proved in Lecture IX of the present course that if i be the degree and μ the characteristic of R, the substitution of $ly+mx+n$ for x and $l'y+m'x+n'$ for y changes R into $(l'm-lm')^i(lt+m)^{-\mu}R$, so that the differential equation $R=0$ and the geometrical property corresponding to it are left unchanged by the substitution.

Six-point contact with a cubical parabola is another example of a descriptive singularity. Its defining differential equation may be written in any of the following forms:

$$45y_2^3y_5^2-450y_2^2y_3y_4y_5+192y_2^2y_4^3+400y_2y_3^3y_5+165y_2y_3^2y_4^2-400y_3^4y_4=0,$$
$$125a^3d^2-750a^2bcd+256a^2c^3+500ab^3d+165ab^2c^2-300b^4c=0,$$
$$5(9y_2^2y_5-45y_2y_3y_4+40y_3^3)^2+64(3y_2y_4-5y_3^2)^3=0,$$
$$125(a^2d-3abc+2b^3)^2+4(4ac-5b^2)^3=0;$$

or, if we make $a^2d-3abc+2b^3=A$ and $ac-\frac{5}{4}b^2=M$, the equation may be put in the form

$$\left(\frac{A}{16}\right)^2+\left(\frac{M}{5}\right)^3=0.$$

In the theory of Binary Forms, when the numerical parameter κ in

$$(a^2d-3abc+2b^3)^2+\kappa(ac-b^2)^3$$

is so chosen that the highest powers of b cancel each other, the form divides by a^2 and gives the Discriminant of the Cubic

$$a^2d^2 - 6abcd + 4b^3d + 4ac^3 - 3b^2c^2.$$

In the parallel theory of Reciprocants the form

$$125A^2 + 256M^3$$

is divisible by a (instead of by a^2), giving

$$125a^3d^2 - 750a^2bcd + 500ab^3d + 256a^2c^3 + 165ab^2c^2 - 300b^4c,$$

which may be called the Quasi-Discriminant.

A complete discussion of the differential equation

$$A^2 + \kappa M^3 = 0$$

is reserved for the next ensuing lecture, in the course of which it will appear that the Quasi-Discriminant equated to zero is the differential equation of the cubical parabola.

LECTURE XIII.

We may integrate the general homogeneous equation in reciprocants extending to d, inclusive, as follows:

Calling $\quad ac - \frac{5}{4}b^2 = M$ and $a^2d - 3abc + 2b^3 = A$,

the equation in question will be of the form

$$A^2 + \kappa M^3 = 0.$$

But if we write $\quad \beta = \Lambda\alpha^\lambda$,

where β, α are general linear functions of the co-ordinates, say

$$y + mx + n,\ y + m'x + n',$$

we may eliminate the five constants m, n, m', n', Λ, and the result will evidently be a pure reciprocant extending to d, inclusive, and, being homogeneous and isobaric, can only be of the form

$$A^2 + \kappa M^3 = 0,$$

so that it remains only to determine κ in terms of λ, or, which is the same thing, λ in terms of κ.

The solution $\beta = \Lambda\alpha^\lambda$ implies $\alpha = \Lambda^{-\frac{1}{\lambda}}\beta^{\frac{1}{\lambda}}$. Hence the equation between M and A must be of the form

$$\theta\,\{(\lambda + p)(p\lambda + 1)\}^i M^3 + \{(\lambda + q)(q\lambda + 1)\}^j A^2 = 0,$$

where θ is a constant, for otherwise there would be more than one general solution to it. It only remains then to determine the values of p, q, θ, i, j, which may be affected by considering the particular solution $y = x^\lambda$.

When $\lambda = 2$, M and A both vanish, and if $\lambda = 2 + \epsilon$, where ϵ is an infinitesimal, M and A will each be of the same order as ϵ (that the first power of ϵ does not vanish in M or A may be easily verified). Hence $2 + q + \epsilon$ is of the order ϵ, and therefore $q = -2$ and $j = 1$.

When $\lambda = -1 + \epsilon$, M remains finite and A is of the order ϵ. Hence $p = 1$ and $i = 1$. Thus, the equation is

$$\theta\,(\lambda + 1)^2 M^3 + (\lambda - 2)(2\lambda - 1) A^2 = 0.$$

To find θ, let $\lambda = 3$ and $y = x^3$; then

$$a = 3x,\ b = 1,\ c = 0,\ d = 0,\ M = -\frac{5}{4},\ A = 2,$$

so that $$-\theta \,.\, \frac{5^3}{4} + 5\,.\,4 = 0,\quad \theta = \frac{16}{25},$$

and finally $$16\,(\lambda + 1)^2 M^3 + 25\,(2\lambda^2 - 5\lambda + 2)\,A^2 = 0$$

has for its integral $$\beta = \Lambda\alpha^\lambda.$$

If $\lambda = \infty$, we may make

$$y = \left(1 + \frac{x}{\lambda}\right)^{\lambda l} = e^{lx},$$

and, consequently, $\beta = e^{l\alpha}$, which contains five independent arbitrary constants, will be the general integral.

For a parallel method of deducing the Integral of $A^8 + \kappa\Delta^3 = 0$, where Δ (our future $AC - B^2$) is the projective reciprocant whose letters go up to f, see Halphen's *Thèse sur les Invariants Différentiels*, Paris, 1878.

Mr Hammond has succeeded in deducing the equation between A and M from the primitive $\beta = \Lambda\alpha^\lambda$ by direct elimination, as shown in what follows. Possibly he, or some other algebraist, may eventually succeed in the more difficult task of obtaining the Differential Equation to $\gamma = \beta^\lambda\alpha^{1-\lambda}$ (that is, the linear relation between A^8 and Δ^3) by some similar direct process.

Differentiating the equation $\beta\alpha^{-\lambda} = \Lambda$ three times in succession, and observing that, since $\alpha = y + mx + n$ and $\beta = y + m'x + n'$,

$$\alpha'' = \beta'' = \frac{d^2y}{dx^2} = y_2,$$

we have $$\alpha\beta' - \lambda\alpha'\beta = 0,$$

$$y_2(\alpha - \lambda\beta) + (1 - \lambda)\,\alpha'\beta' = 0,$$

$$y_3(\alpha - \lambda\beta) + y_2\,\{(2 - \lambda)\,\alpha' + (1 - 2\lambda)\,\beta'\} = 0.$$

From the last two of these three equations we obtain, by eliminating $(\alpha - \lambda\beta)$,

$$y_3\,(1 - \lambda)\,\alpha'\beta' - y_2^2\,\{(2 - \lambda)\,\alpha' + (1 - 2\lambda)\,\beta'\} = 0;$$

or, writing

$$y_2 = 2a,\ y_3 = 6b,\ 2 - \lambda = 3q^2,\ 1 - 2\lambda = -3r^2,\ 1 - \lambda = q^2 - r^2,$$

and dividing by $\alpha'\beta'$, the equation assumes the form

$$\frac{b}{2a^2}(q^2-r^2)=\frac{q^2}{\beta'}-\frac{r^2}{\alpha'}.$$

Differentiating again, remembering that

$$\alpha''=\beta''=2a, \text{ and } \frac{da}{dx}=3b,\ \frac{db}{dx}=4c,$$

we find

$$\frac{4ac-6b^2}{4a^4}(q^2-r^2)=-\frac{q^2}{\beta'^2}+\frac{r^2}{\alpha'^2}.$$

The elimination of β' between this and the equation immediately preceding it gives

$$\frac{4ac-6b^2}{4a^4}(q^2-r^2)\,q^2+\left\{\frac{b}{2a^2}(q^2-r^2)+\frac{r^2}{\alpha'}\right\}^2-\frac{q^2r^2}{\alpha'^2}=0.$$

Writing in this $4ac-5b^2=4M$, we obtain by an easy reduction

$$4q^2M\alpha'^2=r^2\{2a^2-b\alpha'\}^2,$$

and, taking the square root of each side,

$$\alpha'(2q\sqrt{M}+rb)-2a^2r=0.$$

A final differentiation gives

$$\alpha'\left(\frac{qM'}{\sqrt{M}}+4cr\right)+2a(2q\sqrt{M}-5br)=0.$$

Finally, eliminating α', we obtain

$$(2q\sqrt{M}+rb)(2q\sqrt{M}-5rb)+ar\left(4cr+\frac{qM'}{\sqrt{M}}\right)=0.$$

Hence $$4Mq^2+qr\left(\frac{aM'}{\sqrt{M}}-8b\sqrt{M}\right)+r^2(4ac-5b^2)=0;$$

or, $$4(q^2+r^2)M^{\frac{3}{2}}+qr(aM'-8bM)=0.$$

Now $$M'=\frac{dM}{dx}=\frac{d}{dx}\left(ac-\frac{5b^2}{4}\right)=5ad-7bc,$$

and, consequently,

$$aM'-8bM=a(5ad-7bc)-b(8ac-10b^2)=5(a^2d-3abc+2b^3)=5A;$$

so that we may write

$$4(q^2+r^2)M^{\frac{3}{2}}=-qr(aM'-8bM)=-5qrA;$$

or, $$16(q^2+r^2)^2M^3-25q^2r^2A^2=0,$$

where $$3q^2=2-\lambda \text{ and } -3r^2=1-2\lambda.$$

Replacing q^2 and r^2 by their expressions in terms of λ, the differential equation becomes

$$16(\lambda+1)^2M^3+25(2\lambda^2-5\lambda+2)A^2=0.$$

Some special cases may be noticed.

When $\lambda=2$ or $\frac{1}{2}$, the equation reduces to $M=0$, which is the differential equation of the common parabola previously obtained.

When $\lambda=3$ or $\frac{1}{3}$, we obtain $256M^3+125A^2=0$ for the equation of the cubical parabola, where the expression on the left-hand side is the Quasi-Discriminant.

When $\lambda=-1$, we find $A=0$ for the differential equation of the general conic.

When λ is an imaginary cube root of negative unity, so that $\lambda^2-\lambda+1=0$, we have

$$(\lambda+1)^2+(2\lambda^2-5\lambda+2)=0,$$

and the differential equation becomes

$$16M^3-25A^2=0.$$

We shall subsequently avail ourselves of this result in finding the complete primitive of the Halphenian Δ.

In the case where λ is infinite, from the complete primitive $\beta=e^{l\alpha}$ we first eliminate the exponential function and afterwards the arbitrary constant l.

Thus we find $$\beta'=l\alpha'\beta \text{ and } \frac{y_2}{\beta'}=\frac{y_2}{\alpha'}+\frac{\beta'}{\beta};$$

or, $$y_2\beta(\alpha'-\beta')-\alpha'\beta'^2=0.$$

Hence $$y_3\beta(\alpha'-\beta')-y_2\beta'(\alpha'+2\beta')=0.$$

The elimination of β gives

$$y_3\alpha'\beta'-y_2^2(\alpha'+2\beta')=0;$$

or, $$\frac{3b}{2a^2}=\frac{1}{\beta'}+\frac{2}{\alpha'}.$$

Comparing this with the equation previously obtained,

$$\frac{b}{2a^2}(q^2-r^2)=\frac{q^2}{\beta'}-\frac{r^2}{\alpha'},$$

we see that $q^2=1$ and $r^2=-2$. Substituting these values in the differential equation

$$16(q^2+r^2)^2M^3-25q^2r^2A^2=0,$$

it becomes $$8M^3+25A^2=0,$$

which is the differential equation corresponding to the complete primitive $\beta=e^{l\alpha}$.

We shall hereafter consider in detail the theory of that special class of pure reciprocants (M. Halphen's Differential Invariants) which retain their form when any homographic substitution is impressed on the variables; that is, when, instead of x and y, we write

$$\frac{lx+my+n}{l''x+m''y+n''} \text{ and } \frac{l'x+m'y+n'}{l''x+m''y+n''}.$$

Since perspective projection is the geometrical equivalent of homographic substitution, it follows from the definition of Differential Invariants that they are connected with the properties and relations of curves which remain unaffected by perspective projection. For this reason Differential Invariants are sometimes called Projective Reciprocants. Two reciprocants with which we are familiar belong to this important class. One of them, y_2 or a, vanishes at points of inflexion on the curve $y=f(x)$; the other,

$$9y_2^2y_5-45y_2y_3y_4+40y_3^3, \text{ or } a^2d-3abc+2b^3,$$

which, for reasons given below, we shall call the Mongian, vanishes at sextactic points; that is, at points where a conic can be drawn having 6-point contact with the given curve.

To illustrate the distinction between a projective and a merely descriptive singularity, consider for an instant the pure reciprocant $4ac-5b^2$, which, as we have seen, vanishes at all points of a general curve where 5-point contact with a parabola is possible. Now, 5-point contact with a parabola is a descriptive but not a projective singularity; after projection the parabola becomes a general conic, and 5-point contact with it becomes 5-point contact with a general conic, which is not a singularity at all. But inflexions and sextactic points are indelible by projection, and thus belong to the class of projective singularities.

The differential equation to a conic was originally obtained by Monge in the form

$$9y_2^2y_5-45y_2y_3y_4+40y_3^3=0$$

(see Monge, "Sur les Équations différentielles des Courbes du Second Degré," *Corresp. sur l'École Polytech.*, Paris, II. 1809–13, pp. 51–54, and *Bulletin de la Soc. Philom.*, Paris, 1810, pp. 87, 88). At the end of the first chapter of his *Differential Equations*, Boole mentions this form of equation as due to Monge, but without any reference, and adds the remark: "But here our powers of geometrical interpretation fail, and results such as this can scarcely be otherwise useful than as a registry of integrable forms." The theory of Reciprocants, however, furnishes both a simple interpretation of the Mongian equation and an obvious method of integrating it.

To see that the differential equation of a conic is satisfied at the sextactic points of a given curve, we have only to remember that at such points the derivatives of y with respect to x, up to the fifth order, inclusive, are the same for the given curve as for a conic.

We proceed to show how the Mongian may be integrated. Writing in the above equation

$$y_2=2a,\ y_3=2.3b,\ y_4=2.3.4c,\ y_5=2.3.4.5d,$$

it becomes

$$a^2d-3abc+2b^3=0,$$

where it can hardly fail to be noticed that the left-hand member of the equation is an ordinary Invariant as well as a Reciprocant. It will be proved hereafter that all Differential Invariants possess this double nature.

Now, if $\mu=3i+w$, where i is the degree and w the weight of any pure reciprocant R, the ordinary theory of eduction shows that

$$\frac{d}{dx}\left(\frac{R}{a^{\frac{\mu}{3}}}\right)=\frac{a\dfrac{dR}{dx}-\mu bR}{a^{\frac{\mu}{3}+1}}$$

is another pure reciprocant.

When we consider the letters $a, b, c, \ldots$ in any invariant I to mean $\frac{y_2}{2}, \frac{y_3}{2.3}, \frac{y_4}{2.3.4}, \ldots$ the parallel theory of generation for Invariants gives the corresponding theorem that if $\nu=3i+2w$, where i is the degree and w the weight of I,

$$\frac{d}{dx}\left(\frac{I}{a^{\frac{\nu}{3}}}\right)=\frac{a\dfrac{dI}{dx}-\nu bI}{a^{\frac{\nu}{3}+1}}$$

is also an invariant.

A strict proof of this theorem will subsequently be given. For present purposes it is sufficient to notice the easily verified special cases of the two theorems

$$\frac{d}{dx}\left(\frac{4ac-5b^2}{a^{\frac{8}{3}}}\right)=\frac{20\,(a^2d-3abc+2b^3)}{a^{\frac{11}{3}}},$$

and

$$\frac{d}{dx}\left(\frac{ac-b^2}{a^{\frac{10}{3}}}\right)=\frac{5\,(a^2d-3abc+2b^3)}{a^{\frac{13}{3}}}.$$

It follows as an immediate consequence that the equation

$$a^2d-3abc+2b^3=0$$

admits of the two first integrals

$$a^{-\frac{8}{3}}(4ac-5b^2)=\text{const.}$$

and

$$a^{-\frac{10}{3}}(ac-b^2)=\text{const.}$$

Now,

$$a^{-\frac{8}{3}}(4ac-5b^2)=\frac{d}{dx}(a^{-\frac{5}{3}}b)=-\frac{1}{2}\frac{d^2}{dx^2}(a^{-\frac{2}{3}});$$

so that the Mongian equation is equivalent to

$$\frac{d^3}{dx^3}(a^{-\frac{2}{3}})=0, \text{ or to } \frac{d^3}{dx^3}(y_2^{-\frac{2}{3}})=0.$$

We thus obtain an integral of the form

$$y_2^{-\frac{2}{3}}=l+2mx+nx^2,$$

from which the complete primitive may be found by two easy integrations. Thus,

$$y_1 + p = \int \frac{dx}{(l + 2mx + nx^2)^{\frac{3}{2}}} = \frac{m + nx}{(ln - m^2)(l + 2mx + nx^2)^{\frac{1}{2}}}$$

gives

$$y + px + q = \frac{1}{ln - m^2}(l + 2mx + nx^2)^{\frac{1}{2}},$$

which is the equation of a general conic.

By first interchanging the variables x, y in the Mongian equation (whose form remains unaltered by this interchange, since $a^2d - 3abc + 2b^3$ is a reciprocant) and then integrating three times with respect to x, we should find another integral of the form

$$x_2^{-\frac{2}{3}} = l' + 2m'y + n'y^2.$$

The solution may be completed by two integrations, as in the former method.

Mr Hammond remarks that $\dfrac{2(ac - b^2)}{a^{\frac{10}{3}}} = \dfrac{d^2}{dt^2}(a^{\frac{2}{3}})$, where $t = y_1$. For, since

$$\frac{d}{dt} = \frac{dx}{dt} \cdot \frac{d}{dx} = \frac{1}{2a} \cdot \frac{d}{dx},$$

we have

$$\frac{d}{dt}(a^{\frac{2}{3}}) = \frac{1}{2a} \cdot \frac{2}{3} \cdot a^{-\frac{1}{3}} . 3b = \frac{b}{a^{\frac{4}{3}}},$$

and, consequently,

$$\frac{d^2}{dt^2}(a^{\frac{2}{3}}) = \frac{1}{2a} \cdot \frac{d}{dx}(a^{-\frac{4}{3}}b) = 2a^{-\frac{10}{3}}(ac - b^2).$$

Hence the integral $a^{-\frac{10}{3}}(ac - b^2) = \text{const.}$ previously obtained for the Mongian is equivalent to $\dfrac{d^2}{dt^2}(a^{\frac{2}{3}}) = \text{constant}$; that is, to $\dfrac{d^2}{dy_1^2}(y_2^{\frac{2}{3}}) = \text{const.}$ Thus we have another integral of the form

$$y_2^{\frac{2}{3}} = \lambda + 2\mu y_1 + \nu y_1^2,$$

from which it is also easy to pass to the complete primitive.

I add a few general remarks relating to the subject-matter of this and the preceding lecture. Instead of the cumbrous terms Projective Reciprocants or Differential Invariants, it may be better to use the single word Principiants to denominate that crowning class or order of Reciprocants which remain, to a factor *près*, unaltered for any homographic substitutions impressed on the variables. This is the *species princeps*. If we go back to the *species infima*, we see the beginning of life in the subject. In general Reciprocants, all that is affirmed is that there exist forms-functions of the derivatives of y in regard to x which (to a factor *près*) remain unaltered when the variables x and y are interchanged, so that $f(y_1, y_2, y_3, \ldots)$ becomes

$\phi(x_1, x_2, x_3, \ldots)$. The function ϕ only differs from f by the acquisition of an extraneous factor $(-)^\kappa y_1^\mu$; that is,

$$f(y_1, y_2, y_3, \ldots) = (-)^\kappa y_1^\mu f(x_1, x_2, x_3, \ldots).$$

A particular species of these general (mixed) reciprocants arises when $f(y_1, y_2, y_3, \ldots)$, differentiated in regard to y_1, gives a reciprocant. These are Orthogonal Reciprocants, and in them we see the first dawn of free continuous motion as distinguished from mere displacement (or mere interchange of axes). Orthogonal Reciprocants, when x, y are rectangular co-ordinates, remain unaltered (save as to a factor) when the orthogonal axes are moved continuously. A quarter of a revolution of course will reverse their original positions, so that we see the condition of mutual displacement is fulfilled. Thirdly, Reciprocants into whose form the first derivative y_1 does not enter are called Pure. Their form is invariable when the axes (now taken generally) undergo separate displacement (instead of turning round together) in a plane. Here there is a further development, so to say, of life in the subject.

Finally, in Principiants, a particular species of Pure Reciprocants, the invariance remains good, not merely for any position of the axes of reference, but for any homographic deformation of the plane in which they lie, so that the evanescence of a Principiant corresponds to some property of a curve not only intrinsic but indelible by projection, as, for example, an inflexion, or a double point, or a sextactic point, and so on.

It is clear from this review that the Theory as we have given it goes to the root of the subject, and that the word Reciprocant is rightly chosen as conveying the notion of a property which is common to the entire continuous series of forms bearing that name. All the links of this connected chain are thus comprehended under the general name of Reciprocants.

LECTURE XIV.

The remaining lectures of the course will be devoted to the theory of Pure and Projective Reciprocants. I shall first treat of the existence and properties of the Protomorphs of Invariants and Reciprocants, using the latter system of protomorphs to obtain all the fundamental forms of Reciprocants in the letters a, b, c, d, e. I shall then pass on to the theory of Projective Reciprocants, or Principiants, with its applications contained in M. Halphen's *Thèse pour obtenir le grade de docteur ès sciences* (Paris, Gauthier-Villars, 1878). It will be seen that M. Halphen's very ingenious methods become greatly simplified when his results are read by the light of an important discovery in the theory of Principiants recently made by myself and Mr Hammond working conjointly, arising out of a theorem put forward by one of my hearers. This theorem, on examination, we found was necessarily erroneous and would fail at the very first step of its application. But although the proposition stated was wrong, it contained an Idea which survives and may be incorporated in a valid and extremely important theorem, which I will endeavour to explain.

A Principiant, besides being an Invariant in the original letters $a, b, c, d, \ldots$ is also an Invariant in the letters $a, A, B, C, D, \ldots$ where each capital letter is itself a Reciprocant; and, conversely, every invariant in the capital letters $A, B, C, D, \ldots$ is a Principiant. The invariants in the capital letters form a system of protomorphs for Principiants, so that every Principiant is either some such invariant simply, or a rational integral function of such invariants provided by some power of a. Thus, for example, it will be proved that the Cubic Criterium (that is, the Principiant which gives, when equated to zero, the differential equation of a cubic curve) may be expressed as the quotient of

$$\frac{9}{64}A^3+\frac{5}{4}A(A^2D-3ABC+2B^3)-(ACE-AD^2-B^2E+2BCD-C^3)$$

by the fifth power of a.

The proof of this theorem is based upon the fact that we can form a series of terms beginning with the Mongian (namely, $a^2d-3abc+2b^3$), say $A, B, C, D, \ldots$ such that

$$\Omega A=0,$$
$$\Omega B=A\times\frac{a}{2},$$
$$\Omega C=2B\times\frac{a}{2},$$
$$\Omega D=3C\times\frac{a}{2},$$
$$\cdots\cdots\cdots\cdots$$

where $$\Omega=a\partial_b+2b\partial_c+3c\partial_d+\ldots,$$

coupled with the fact that every Principiant must be a function of the letters in such series and the small a.

Each consequent of the series $A, B, C, D, \ldots$ is, so to say, an Invariant relative to its antecedent; it becomes an actual Invariant when its antecedent vanishes.

In the theorem as originally proposed, each letter of the series was derived by the operation of an eductive generator upon the one which precedes. In the true theorem the scale of relation is between three and not two consecutive terms. Calling the letters $u_0, u_1, u_2, \ldots u_i$, we have

$$(i+7)\,u_{i+2}-Gu_{i+1}+(i+1)\,Mu_i=0,$$

where G is the ordinary eductive generator,

$$4(ac-b^2)\partial_b+5(ad-bc)\partial_c+6(ae-bd)\partial_d+\ldots,$$

M is the first pure reciprocant after the monomial a, namely, $M=ac-\frac{5}{4}b^2$, $u_0=A=a^2d-3abc+2b^3$, and $6u_1=GA$.

But although, as I have said, the theorem in the form proposed was absolutely erroneous, its proposer has rendered an invaluable service to the theory by the mere suggestion of what turns out to be true, namely, that every Principiant is an Invariant in regard to a known series of Reciprocants considered as simple elements.

To this theorem there is a correlative one, for it will be shown that there exists a series of invariants $A_0, A_1, A_2, \ldots$, the first term of which, A_0, is the same as the Mongian A, each of the other terms of the series being a Reciprocant relative to the one that precedes it. In fact, we have

$$VA_0=0,$$
$$VA_1=-a^2A_0,$$
$$VA_2=-2a^2A_1,$$
$$\cdots\cdots\cdots\cdots$$
$$VA_n=-na^2A_{n-1},$$

where $$V=4\left(\frac{a^2}{2}\right)\partial_b+5ab\partial_c+6\left(ac+\frac{b^2}{2}\right)\partial_d+\ldots,$$

and, as a consequence, every Principiant will be an Invariant in respect to these Invariants and the first small letter a.

Thus, speaking symbolically, we have not only

$$P=R+I$$

(a logical equation meaning that P has the same qualities as both R and I, or that a Principiant is both a Reciprocant and an Invariant), but also

$$P=IR \text{ and } P=II,$$

meaning that a Principiant is an Invariant of Reciprocantive elements, and an Invariant whose elements are themselves Invariants.

I may add that the invariantive elements $A_0, A_1, A_2, A_3, \ldots$ are defined by the equations

$$A_0=A,$$
$$A_1=B-\frac{b}{2}A,$$
$$A_2=C-2\left(\frac{b}{2}\right)B+\left(\frac{b}{2}\right)^2A,$$
$$A_3=D-3\left(\frac{b}{2}\right)C+3\left(\frac{b}{2}\right)^2B-\left(\frac{b}{2}\right)^3A,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

so that any invariant in the reciprocantive elements $A, B, C, D, \ldots$ is equal to the corresponding invariant in $A_0, A_1, A_2, A_3, \ldots$. Thus,

$$A = A_0,$$
$$AC - B^2 = A_0A_2 - A_1^2,$$
$$A^2D - 3ABC + 2B^3 = A_0^2A_3 - 3A_0A_1A_2 + 2A_1^3,$$
$$AE - 4BD + 3C^2 = A_0A_4 - 4A_1A_3 + 3A_2^2,$$

....

M. Halphen appears not to have noticed the Principiant $AE - 4BD + 3C^2$, which presents itself naturally when the theory is viewed from our present ground of vantage, but A, $AC - B^2$ and $A^2D - 3ABC + 2B^3$ occur in his *Thèse* in connection with the curve

$$\alpha = \beta^\lambda \gamma^{1-\lambda},$$

in which α, β, γ are any linear functions of $x, y, 1$.

When $\lambda = -1$ the differential equation of this curve (the conic $\alpha\beta = \gamma^2$) is $A = 0$, but it is

$$AC - B^2 = 0$$

when λ is a cube root of negative unity, and

$$A^2D - 3ABC + 2B^3 = 0$$

when λ has an arbitrary value.

Before making out an exhaustive table of all the irreducible forms of pure reciprocants in the letters a, b, c, d, e similar to, but not identical with, the corresponding table for invariants, it seems to me desirable to say something of Protomorphs in general; and this will be better understood if we devote a short space to the protomorphs of Invariants. The simplest forms of these are the following well-known ones of alternately the second and third degrees:

$$P_2 = ac - b^2,$$
$$P_3 = a^2d - 3abc + 2b^3,$$
$$P_4 = ae - 4bd + 3c^2,$$
$$P_5 = a^2f - 5abe + 2acd + 8b^2d - 6bc^2,$$
$$P_6 = ag - 6bf + 15ce - 10d^2,$$
$$P_7 = a^2h - 7abg + 9acf - 5ade + 12b^2f - 30bce + 20bd^2,$$

....

The quadratic Protomorphs $P_2, P_4, P_6, \ldots$, are absolutely unique, for the number of invariants of the type $j; 2, j$ is $(j; 2, j) - (j-1; 2, j) = 1$ if j is even, and $= 0$ if j is odd. Their form is so well known that there is no need to dilate upon it here.

The cubic ones $P_3, P_5, P_7, \ldots$, may be derived from the quadratic ones by means of Cayley's generators, given early in the course, namely,

$$P = (ac - b^2)\partial_b + (ad - bc)\partial_c + (ae - bd)\partial_d + \ldots,$$
$$Q = (ac - 2b^2)\partial_b + 2(ad - 2bc)\partial_c + 3(ae - 2bd)\partial_d + \ldots.$$

Let us first use the P generator

$$P(ac - b^2) = a(ad - bc) - 2b(ac - b^2) = a^2d - 3abc + 2b^3,$$
$$P(ae - 4bd + 3c^2) = a(af - be) - 4b(ae - bd) + 6c(ad - bc) - 4d(ac - b^2)$$
$$= a^2f - 5abe + 2acd + 8b^2d - 6bc^2.$$

Similarly, we find

$$P(ag - 6bf + 15ce - 10d^2) = a^2h - 7abg + 9acf - 5ade + 12b^2f - 30bce + 20bd^2,$$

and so on.

Let I be any invariant whatever of the type $w; i, j$ (satisfied or unsatisfied); then using the original forms of the generators P and Q as given by Cayley (see Lecture IV), we have

$$PI = a(b\partial_a + c\partial_b + d\partial_c + \ldots)I - ibI,$$
$$QI = a(c\partial_b + 2d\partial_c + 3e\partial_d + \ldots)I - 2wbI,$$

and, consequently,

$$(jP - Q)I = a\{jb\partial_a + (j-1)c\partial_b + (j-2)d\partial_c + \ldots\}I - (ij - 2w)bI.$$

If in this formula we write

$$O = jb\partial_a + (j-1)c\partial_b + (j-2)d\partial_c + \ldots,$$

it becomes

$$(jP - Q)I = aOI - (ij - 2w)bI,$$

which, when I is a satisfied invariant, so that $ij - 2w = 0$ and $OI = 0$, reduces to

$$(jP - Q)I = 0,$$

showing that the forms obtained by operating with either P or Q on any satisfied invariant are the same to a numerical factor *près*.

Now, each quadratic protomorph is a satisfied invariant (for when $w = j$ and $i = 2$, $ij - 2w = 0$), and therefore the cubic protomorphs found by operating on the quadratic ones with Q will only differ by a numerical factor from those already obtained by the operation of P. But we must not conclude from this that the cubic protomorphs are unique. Their number is in fact given by the formula

$$(j; 3, j) - (j-1; 3, j),$$

where it is obvious that

$$(j-1; 3, j) = (j-1; 3, j-1);$$

so that the above formula may be written

$$(j; 3, j) - (j-1; 3, j-1), \text{ or say } \Delta(j; 3, j).$$

Now, there is a simple rule for finding $(j; 3, j)$; it is the nearest integer to $\frac{(j+3)^2}{12}$. From the following table, obtained by the use of this rule,

$j =$	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$(j; 3, j) =$	2	3	4	5	7	8	10	12	14	16	19	21	24	27
$\Delta (j; 3, j) =$		1		1		1		2		2		2		3

it may be seen that for any odd number $j = > 9$ there are two or more forms of extent j equally entitled to rank as protomorphs. If l be the last letter which occurs in one of these forms, its first term will of course be $a^2 l$; the difference between any two such forms will not involve the letter l, and will only extend to k, but will still be of the same (potential) extent as l.

The property of the protomorphs $a, P_2, P_3, P_4, \ldots$ is that every invariant is a rational integral function of them divided by some power of a, as appears from the fact that Q, any given rational integral function whatever of the letters $a, b, c, d, e, \ldots$, may obviously be expressed as a rational integral function of $a, b, P_2, P_3, P_4, \ldots$ divided by some power of a. Thus,

$$Q = a^{-m} \phi (a, b, P_2, P_3, P_4, \ldots).$$

Suppose Q to be an invariant I; then

$$Ia^m = \phi (a, b, P_2, P_3, P_4, \ldots),$$

and, consequently,

$$\Omega (Ia^m) = \frac{d\phi}{da} \Omega a + \frac{d\phi}{db} \Omega b + \frac{d\phi}{dP_2} \Omega P_2 + \frac{d\phi}{dP_3} \Omega P_3 + \ldots,$$

where Ω is the annihilator for invariants; so that

$$\Omega (Ia^m) = 0,\ \Omega a = 0,\ \Omega P_2 = 0,\ \Omega P_3 = 0, \ldots.$$

We have therefore

$$\frac{d\phi}{db} \Omega b = a \frac{d\phi}{db} = 0.$$

Hence ϕ does not contain b, but is a rational integral function of the protomorphs alone, and

$$I = a^{-m} \phi (a, P_2, P_3, P_4, \ldots).$$

I shall show how to obtain a similar scale of forms possessing like properties for pure reciprocants.

LECTURE XV.

A Protomorph may be defined as a form whose weight is equal to its actual extent, so that its type is $j; i, j$. The first protomorph is a, which corresponds to $j = 0$. For higher values of j it follows immediately from the definition that every protomorph will contain a term $a^{j-1} l$, in which the letter of highest extent appears only in the first degree multiplied by a

power of the first letter. The existence of this term enables us to instantly recognize a protomorph. As in the case of invariants, it will be shown that every pure reciprocant is either a rational integral function of protomorphs or else such a function divided by some power of a. But first it will be better to prove *à priori* their existence and exhibit examples of them for the earlier values of j.

It was proved, in Lecture IX, that the number of pure reciprocants of the type $w; i, j$ is at least equal to

$$(w; i, j) - (w-1; i+1, j).$$

Now, obviously, the number of partitions of w into i parts not exceeding $w + \epsilon$ is the same as the number of partitions of w into i parts not exceeding w, so that

$$(w; i, w+\epsilon) = (w; i, w);$$

and since, by a well-known theorem, $(w; i, j) = (w; j, i)$, we see that

$$(w; w+\epsilon, j) = (w; j, w+\epsilon) = (w; j, w) = (w; w, j),$$

a result which follows more immediately from the consideration that the partitions of $w; w+\epsilon, j$ differ only from those of $w; w, j$ by ϵ columns of zeros, as we see in the annexed example:

3; 5, 3	3; 3, 3
30000	300
21000	210
11100	111

Hence, if $w = j$, and $i = > j$, we have

$$(w; i, j) = (j; j, j)$$

and

$$(w-1; i+1, j) = (j-1; j-1, j-1).$$

Thus, the number of pure reciprocants of the type $j; j, j$ is

$$(j; j, j) - (j-1; j-1, j-1),$$

in other words, the difference between the indefinite partitions of j and those of $j-1$. Expressed by means of generating functions, this difference is the coefficient of x^j in

$$\frac{1-x}{(1-x)(1-x^2)(1-x^3)\ldots(1-x^j)}$$

= coefficient of x^j in the expansion of

$$\frac{1}{(1-x^2)(1-x^3)\ldots(1-x^j)}.$$

This coefficient is a positive integer for all values of j (except $j = 1$, when it is zero), which proves the existence of reciprocants of the type $j; j, j$ when j has any value except unity.

But we wish to prove the existence of one or more reciprocants of the type $j; j, j$ which actually contain a term of the form $a^{j-1} l$, where the letter l

is of extent j. The number of such forms is the difference between the number of pure reciprocants of the types $j; j, j$ and $j; j, j-1$.

Now, the number of linearly independent pure reciprocants of the type $j; j, j$ has just been shown to be

$$(j;\, j,\, j) - (j-1;\, j-1,\, j-1).$$

And, in like manner, that of the linearly independent reciprocants of the type $j; j, j-1$ is

$$(j;\, j,\, j-1) - (j-1;\, j+1,\, j-1)$$
$$= (j;\, j,\, j-1) - (j-1;\, j-1,\, j-1).$$

The difference between these two numbers is therefore

$$(j;\, j,\, j) - (j;\, j,\, j-1) = 1.$$

For the only partition not common to the two types is $j \,.\, 0^{j-1}$, made up of one j and $j-1$ zeros, which belongs to the first type, but not to the second. Hence reciprocants of the type $j; j, j$ contain one term which those of the type $j; j, j-1$ do not, and which can only be $a^{j-1}l$. This proves the existence of protomorphs.

In the latter part of the above proof we have assumed the truth of the theorem, which, however probable, is not demonstrated, that the number of reciprocants of the type $w; i, j$ is $(w; i, j) - (w-1; i+1, j)$ and *no more* [that concerns the subtrahend, namely, $(j; j, j-1) - (j-1; j-1, j-1)$].

We shall, however, have an independent method of arriving at Protomorphs by direct generation, just as we saw that all the cubic protomorphs to invariants were derivable by direct operation of generators from the quadratic ones.

The difference between the two cases is that the lowest degree of Invariantive Protomorphs fluctuates alternately between 2 and 3. For Reciprocantive Protomorphs the lowest degree corresponding to a given extent fluctuates, but has a tendency to rise, and goes on progressing until it exceeds any assignable number.

It is interesting to find what the degrees are for successive values of j. The calculations required are greatly facilitated by an extensive table of partitions given by Euler in 1750, and partly reproduced by Cayley in the *American Journal of Mathematics*, Vol. IV., Part III. In the table as presented by Cayley, the number in column j and line i means the number of ways of partitioning j into exactly i parts (zeros excluded). Hence, to find the number of ways of partitioning j into i parts or fewer, that is, to find $(j;\, i,\, \infty)$ or its equivalent $(j;\, i,\, j)$, we must add up the numbers in the 1st, 2nd, 3rd, ... ith lines of column j.

When these summations are made we obtain the subjoined table:

Degree $i=$ \ Extent $j=$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10
3	1	1	2	3	4	5	7	8	10	12	14	16	19	21	24	27	30	33	37
4	1	1	2	3	5	6	9	11	15	18	23	27	34	39	47	54	64	72	84
5	1	1	2	3	5	7	10	13	18	23	30	37	47	57	70	84	101	119	141
6	1	1	2	3	5	7	11	14	20	26	35	44	58	71	90	110	136	163	199
7	1	1	2	3	5	7	11	15	21	28	38	49	65	82	105	131	164	201	248
8	1	1	2	3	5	7	11	15	22	29	40	52	70	89	116	146	186	230	288

The number of pure reciprocants of the type $j; i, j$ is

$$(j;\, i,\, j) - (j-1;\, i+1,\, j) = (j;\, i,\, j) - (j-1;\, i+1,\, j-1).$$

To find the minimum degree for protomorphs of extent j we have therefore only to see for what value of i any figure in the j column first becomes greater than the figure in the column to the left one place lower down. The fluctuations of the minimum degree are indicated by the dark irregularly waving line which runs through the table.

Accordingly, we find that the types of the protomorphs, omitting w, which is always equal to j, are as follows:

(2, 2), (3, 3), (3, 4), (4, 5), (3, 6), (4, 7), (4, 8), (5, 9), (5, 10), (5, 11), (5, 12), ...,

whereas for invariants they are

(2, 2), (3, 3), (2, 4), (3, 5), (2, 6), (3, 7), (2, 8), (3, 9), (2, 10), (3, 11), (2, 12), ...

Corresponding to the extents

2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, ...,

the lowest degrees of the Reciprocantive Protomorphs are

2, 3, 3, 4, 3, 4, 4, 5, 5, 5, 5, ...

Contrast this with the regularly fluctuating series

2, 3, 2, 3, 2, 3, 2, 3, 2, 3, 2, 3, ...,

which shows the minimum degrees of invariantive protomorphs for successive extents.

It may be proved, from known formulae in the theory of partitions, that as the extent increases the minimum degree of reciprocantive protomorphs increases (on the whole) and ultimately becomes infinite when the extent is so.

The apparent number of protomorphs to the several types is

(2, 2),	(3, 3),	(3, 4),	(4, 5),	(3, 6),	(4, 7),	(4, 8),	(5, 9),	(5, 10),	(5, 11),	(5, 12),	....
1	1	1	1	1	1	2	3	4	2	3	

The explanation of this multiplicity is the same as that previously given for the case of invariants: the difference between any two protomorphs of a given type j; i, j will be a reciprocant (no longer a protomorph) of the type j; i, $j-1$.

For the only term containing the letter l (of extent j) will disappear from the result of subtraction; and, accordingly, the above numbers, each diminished by unity, will give the numbers of a set of reciprocants of the same degree-weight as the protomorphs, but of a smaller (actual) extent.

Assuming that the number of pure reciprocants of the type w; i, j is correctly given by the formula

$$(w;\ i,\ j) - (w-1;\ i+1,\ j),$$

Euler's great table of partitions, already referred to, enables us to carry on the determination of the minimum degree and multiplicity of protomorphs for all extents as far as 59.

If m is the multiplicity corresponding to the minimum degree i of a reciprocantive protomorph whose extent is j, we form without difficulty, using only the principles explained above, the following table:

$j=$	0	1	2	3	4	5	6	7	8	9	10	11
$i=$	1	–	2	3	3	4	3	4	4	5	5	5,
$m=$	1	0	1	1	1	1	1	1	2	3	4	2
$j=$	12	13	14	15	16	17	18	19	20	21	22	23
$i=$	5	6	6	6	6	7	7	7	7	7	8	8,
$m=$	3	6	8	5	5	15	18	12	12	2	40	32
$j=$	24	25	26	27	28	29	30	31	32	33	34	35
$i=$	8	8	8	9	9	9	9	10	10	10	10	10,
$m=$	32	14	6	84	82	58	45	207	211	180	161	102
$j=$	36	37	38	39	40	41	42	43	44	45	46	47
$i=$	10	11	11	11	11	11	11	12	12	12	12	12,
$m=$	45	482	469	391	320	167	13	1126	1064	881	687	337
$j=$	48	49	50	51	52	53	54	55	56	57	58	59
$i=$	13	13	13	13	13	13	13	14	14	14	14	14.
$m=$	2829	2666	2492	2097	1643	892	26	6394	6017	5227	4266	2755

Notice the repetitions of i indicated by the series

$$1^1,\ 0^1,\ 2^1,\ 3^2,\ 4^1,\ 3^1,\ 4^2,\ 5^4,\ 6^4,\ 7^5,\ 8^5,\ 9^4,\ 10^6,\ 11^6,\ 12^5,\ 13^7,\ 14^{5+?}.$$

It will be observed that there is a general tendency of the number of equal values of i to increase, but that this is subject to occasional fluctuations. When $j=5$, $i=4$; but when $j=6$, $i=3$, so that the minimum value of i recedes. After this point is reached, i either advances or remains stationary, but never recedes.

In order actually to find the protomorphs, we may use the annihilator V. This was my original method of obtaining them; a shorter way, analogous to that used by Halphen for differential invariants (principiants), has been previously mentioned, but it will be instructive to begin with the method of indeterminate coefficients. In the first place we have the form a of weight 0, which is annihilated by

$$V = 2a^2\partial_b + 5ab\partial_c + (6ac+3b^2)\,\partial_d + (7ad+7bc)\,\partial_e + \dots.$$

For weight 1 there is no pure reciprocant. We could not make $R=\lambda a^{i-1}b$, for then $VR = 2\lambda a^{i+1}$, which cannot vanish unless $\lambda = 0$ and consequently $R=0$.

To find the Protomorph of extent 2, assume $R = \lambda ac + \mu b^2$; then

$$VR = 4\mu a^2 b + 5\lambda a^2 b = (4\mu + 5\lambda)\, a^2 b.$$

Hence λ and μ are proportional to 4 and -5, and we may write

$$R = 4ac - 5b^2.$$

For extent 3, assuming $R = \lambda a^2 d + \mu abc + \nu b^3$, we have

$$VR = 2\mu a^3 c + 6\nu a^2 b^2 + 5\mu a^2 b^2 + 6\lambda a^3 c + 3\lambda a^2 b^2,$$

which vanishes when

$$2\mu + 6\lambda = 0,\ 6\nu + 5\mu + 3\lambda = 0.$$

We may therefore write $\lambda = 1$, $\mu = -3$, $\nu = 2$, and thus obtain

$$R = a^2 d - 3abc + 2b^3.$$

For extent 4 the table of minimum degrees indicates the existence of a protomorph of degree 3. To find its value we assume

$$R = \kappa a^2 e + \lambda abd + \mu ac^2 + \nu b^2 c.$$

Operating with V, we find

$$\begin{array}{rccc} & a^3d & a^2bc & ab^3 \\ VR = & 2\lambda & 4\nu & . \\ & . & 10\mu & 5\nu \\ & . & 6\lambda & 3\lambda \\ & 7\kappa & 7\kappa & . \end{array}$$

In order that VR may vanish, we must have

$$2\lambda + 7\kappa = 0,\ 4\nu + 10\mu + 6\lambda + 7\kappa = 0,\ \text{and}\ 5\nu + 3\lambda = 0.$$

To avoid fractions, let $\kappa = 50$; then $\lambda = -175$, $\nu = 105$, and $\mu = 28$; thus,

$$R = 50a^2e - 175abd + 28ac^2 + 105b^2c;$$

whereas, the protomorph of extent 4 for Invariants is $ae - 4bd + 3c^2$. There is no reciprocant of degree 2 weight 4 to correspond to this.

LECTURE XVI.

By using the generator for pure reciprocants instead of the annihilator V, we readily obtain the protomorph of extent 5 and of the fourth degree whose existence is indicated in the previously given table of minimum degrees. We have only to operate on the protomorph of degree 3 and extent 4 with

$$G = 4(ac-b^2)\partial_b + 5(ad-bc)\partial_c + 6(ae-bd)\partial_d + 7(af-be)\partial_e + \ldots$$

Thus,

$$\begin{aligned} G(50a^2e - 175abd + 28ac^2 + 105b^2c) &= 4(ac-b^2)(-175ad+210bc) \\ &+ 5(ad-bc)(56ac+105b^2) \\ &+ 6(ae-bd)(-175ab) \\ &+ 7(af-be)(50a^2). \end{aligned}$$

Rejecting the numerical factor 35, which is common to all the terms in the result, and at the same time writing the terms themselves in reverse order, we have

$$\begin{aligned} &10a^2(af-be) - 30ab(ae-bd) + (ad-bc)(8ac+15b^2) + 4(ac-b^2)(-5ad+6bc) \\ &= 10a^3f - 40a^2be - 12a^2cd + 65ab^2d + 16abc^2 - 39b^3c, \end{aligned}$$

which is the protomorph in question.

The form just found is irreducible, as indeed it ought to be, since the minimum degree for extent 5 is greater than that for extent 4 by unity, which exactly corresponds with the unit increase of degree due to the operation of G. But if we use G to generate a protomorph of extent 4 from that of extent 3, the resulting form will be reducible. In fact

$$\begin{aligned} &G(a^2d - 3abc + 2b^3) \\ &= 4(ac-b^2)(-3ac+6b^2) + 5(ad-bc)(-3ab) + 6(ae-bd)a^2 \\ &= 3(2a^3e - 7a^2bd - 4a^2c^2 + 17ab^2c - 8b^4). \end{aligned}$$

If now we write

$$ac - \frac{5}{4}b^2 = M,$$

$$a^2d - 3abc + 2b^3 = A,$$

$$a^3e - \frac{7}{2}a^2bd - 2a^2c^2 + \frac{17}{2}ab^2c - 4b^4 = B,$$

we have shown that

$$GA = 6B.$$

But

$$\begin{aligned} 50B + 128M^2 &= 25(2a^3e - 7a^2bd - 4a^2c^2 + 17ab^2c - 8b^4) + 8(4ac-5b^2)^2 \\ &= a(50a^2e - 175abd + 28ac^2 + 105b^2c); \end{aligned}$$

so that B is reducible, being expressible as a rational integral function of a, M, and the previously obtained protomorph of degree 3 and extent 4.

The general theory of the generator G is contained in that of the differentiation of absolute reciprocants, in which, if $\mu = 3i + w$, where w is the weight and i the degree of any pure reciprocant R, we have

$$\frac{R}{a^{\frac{\mu}{3}}} = \pm \frac{R_1}{a_1^{\frac{\mu}{3}}},$$

and, consequently,

$$\frac{d}{dx}\left(\frac{R}{a^{\frac{\mu}{3}}}\right) = \pm \frac{dy}{dx}\cdot\frac{d}{dy}\left(\frac{R_1}{a_1^{\frac{\mu}{3}}}\right),$$

where R_1 and a_1 are what R and a become when x and y are interchanged. Hence

$$\frac{a\dfrac{dR}{dx} - \dfrac{\mu}{3}R\dfrac{da}{dx}}{a^{\frac{\mu}{3}+1}},$$

and therefore also the numerator of this fraction is a reciprocant.

Remembering that

$$\frac{da}{dx} = 3b,\ \frac{db}{dx} = 4c,\ \frac{dc}{dx} = 5d,\ \ldots,$$

the numerator may be written

$$a\frac{dR}{dx} - \mu bR = GR.$$

The ordinary expression for G is found by writing

$$\begin{aligned} a\frac{d}{dx} - \mu b &= a(3b\partial_a + 4c\partial_b + 5d\partial_c + \ldots) \\ &- b(3a\partial_a + 4b\partial_b + 5c\partial_c + \ldots). \end{aligned}$$

If the actual extent of R is j, that of GR is $j+1$; for the operation of G introduces an additional letter. Both the weight and degree are also increased by unity. Thus, the type of R being $w;\ i, j$, that of GR is

$$w+1;\ i+1,\ j+1.$$

Suppose the weight of R to be equal to its actual extent; then R is a protomorph of the type $j;\ i, j$, and GR, whose type is $j+1;\ i+1, j+1$, is also a protomorph. This proves the existence of protomorphs for every possible extent. Starting with the form $4ac - 5b^2$ we obtain, by successive eduction, a series of protomorphs of the type $j;\ j, j$ for which the general expression is

$$G^{j-2}(4ac - 5b^2),$$

where j has any of the values 2, 3, 4, ...

If R is a protomorph of minimum degree, GR (if irreducible) will also be a protomorph of minimum degree. Hence the minimum degree can never increase by more than one unit when the extent is increased by unity.

The second educt G^2R is always reducible; for

$$G^2R = \left\{a\frac{d}{dx} - (\mu+4)\,b\right\}\left(a\frac{d}{dx} - \mu b\right)R$$
$$= \left\{a^2\frac{d^2}{dx^2} - (2\mu+1)\,ab\frac{d}{dx} - 4\mu ac + \mu(\mu+4)\,b^2\right\}R.$$

Combining this with $M = ac - \frac{5}{4}b^2$, we have

$$5G^2R + 4\mu(\mu+4)\,MR = a\left\{5a\frac{d^2}{dx^2} - 5(2\mu+1)\,b\frac{d}{dx} + 4\mu(\mu-1)\,c\right\}R,$$

where the right-hand side is divisible by a, showing that the degree of G^2R is always depressible by unity. R being a protomorph of degree i and extent j,

$$\left\{5a\frac{d^2}{dx^2} - 5(2\mu+1)\,b\frac{d}{dx} + 4\mu(\mu-1)\,c\right\}R$$

is one of degree $i+1$ and extent $j+2$. Hence we may conclude that an increase in the minimum degree for protomorphs cannot be immediately followed by another increase; for, if this were possible, the minimum degree for extent $j+2$ would be $i+2$, instead of being $i+1$ at most.

This conclusion is in accordance with the sequence of the values of i in the table of minimum degrees, and as far as it goes confirms the exactitude of the formula $(w; i, j) - (w-1; i+1, j)$ for the number of pure reciprocants which was assumed in calculating the table.

The method previously employed to prove that every invariant is a rational integral function of protomorphs, or such function divided by a power of a, may be very easily extended to the case of reciprocants.

In the first place, it is obvious that every rational integral function of the letters $a, b, c, d, \ldots$ is by successive substitutions reducible to the form

$$a^{-\theta}\Phi(a, b, P_2, P_3, P_4, \ldots P_j),$$

where P_j means the protomorph of extent j.

Let any reciprocant R be put under this form; then

$$a^{\theta}R = \Phi(a, b, P_2, P_3, P_4, \ldots P_j),$$

and, consequently,

$$V(a^{\theta}R) = \frac{d\Phi}{da}Va + \frac{d\Phi}{db}Vb + \frac{d\Phi}{dP_2}VP_2 + \ldots + \frac{d\Phi}{dP_j}VP_j.$$

Now, V annihilates $R, a, P_2, P_3, \ldots P_j$, since these are all pure reciprocants. Hence the above identity reduces to $\frac{d\Phi}{db}Vb = 0$, from which (since Vb does not vanish) we conclude that Φ does not contain b explicitly. Thus,

$$a^{\theta}R = \Phi(a, P_2, P_3, P_4, \ldots P_j),$$

and the theorem is established for reciprocants.

The Protomorphs for Reciprocants as far as extent 8 are as follows:

$P_2 = 4ac - 5b^2,$

$P_3 = a^2d - 3abc + 2b^3,$

$P_4 = 50a^2e - 175abd + 28ac^2 + 105b^2c,$

$P_5 = 10a^3f - 40a^2be - 12a^2cd + 65ab^2d + 16abc^2 - 39b^3c,$

$P_6 = 14a^3g - 63abf - 1350ace + 1782b^2e + 1470ad^2 - 4158bcd + 2310c^3,$

$$P_7 = 7a^3h - 35a^2bg - 539a^2cf + 735ab^2f + 605a^2de + 306abce - 1485b^3e$$
$$- 2135abd^2 + 1001ac^2d + 3465b^2cd - 1925bc^3,$$

$$P_8 = 420a^3i - 2310a^2bh - 25648a^2cg + 9240a^2df + 21780a^2e^2 + 36680ab^2g$$
$$+ 85386abcf - 191730abde - 59220ac^2e + 120540acd^2$$
$$- 126945b^3f + 252126b^2ce + 169260b^2d^2 - 419034bc^2d$$
$$+ 129360c^4.$$

The work necessary for obtaining the first four of these, P_2, P_3, P_4, P_5, has been fully set out. Since P_4 is of degree 3, its second educt, G^2P_4, is of degree 5 and its reduced second educt of degree 4. A linear combination of this with a form whose leading term is a^3ce becomes divisible by a and gives P_6; but as this requires the preliminary calculation of the form (a^3ce), it is simpler to find P_6 directly by the method of indeterminate coefficients, and thence by eduction to get P_7 and P_8. Thus (to a numerical factor *près*) P_7 is the educt and P_8 the reduced second educt of P_6. Beyond this point the calculation of protomorphs has not at present been carried.

Referring to the table which gives the minimum degree and multiplicity for a Protomorph of any extent, we see that the multiplicity exceeds unity when the extent $j = > 8$, and is exactly equal to 2 when $j = 8$, 11, or 21.

Hence the protomorphs as far as P_7 inclusive are unique; but there are two forms of extent 8 and degree 4, any linear combination of which (provided it contains the term a^3i) may be regarded as a protomorph. One of these forms is P_8, whose value is given above; the other is a linear combination of P_8 with a form, whose leading term is a^3cg, hereafter to be set forth.

The irreducible forms for extent 2 are a and P_2; every other form must be simply a power of P_2 multiplied by a power of a. We proceed to the calculation of all the Irreducible Forms for the extents 3 and 4 respectively. When $j = 3$, we may combine the protomorphs

$$4ac - 5b^2$$

and

$$a^2d - 3abc + 2b^3$$

with one another.

Adding 125 times the square of the latter to 4 times the cube of the former and dividing by a, there results the form

$$125a^3d^2 - 750a^2bcd + 500b^3d + 256a^2c^3 + 165ab^2c^2 - 300b^4c.$$

This form is analogous to the discriminant of the cubic, but is of a higher degree by one unit. Its type is 6; 5, 3, whereas that of the discriminant is 6; 4, 3.

In the case of invariants, we have to combine $ac-b^2$ with $a^2d-3abc+2b^3$. The square of the second, added to 4 times the cube of the first, gives

$$a^4d^2-6a^3bcd+4a^2b^3d+9a^2b^2c^2-12ab^4c+4b^6+4a^3c^3-12a^2b^2c^2+12ab^4c-4b^6.$$

Here the term $12ab^4c$ is nullified by $-12ab^4c$, so that the result contains a^2, the other factor being the discriminant

$$a^2d^2-6abcd+4b^3d+4ac^3-3b^2c^2,$$

which is of the type 6; 4, 3.

We may show *à priori*, assuming the problematical but highly probable formula $(w;\ i,\ j)-(w-1;\ i+1,\ j)$, that the type 6; 4, 3 does not belong to any reciprocant.

For, as seen in the partitionments set out below,

$$(6;\ 4,\ 3)-(5;\ 5,\ 3)=5-5=0$$

3.3	3.2
3.2.1	3.1.1
3.1.1.1	2.2.1
2.2.2	2.1.1.1
2.2.1.1	1.1.1.1.1

We can by no other means combine the protomorphs with one another or with the Quasi-Discriminant $(125a^3d^2\ldots)$ so as to obtain additional fundamental forms. Every Rational Integral Pure Reciprocant of extent 3 is therefore necessarily a rational integral function of the four forms

deg. wt.		
1 . 0	a,	
2 . 2	$4M$	$=4ac-5b^2$,
3 . 3	A	$=a^2d-3abc+2b^3$,
5 . 6	(a^3d^2)	$=125a^3d^2-750a^2bcd+500b^3d+256a^2c^3+165ab^2c^2-300b^4c$.

These are connected by a syzygy of degree-weight 6 . 6, namely

$$125A^2+256M^3=a\,(a^3d^2),$$

analogous to the syzygy of the same degree-weight, in the Theory of the Binary Cubic, which connects the Discriminant with a and the Protomorphs of extent 2 and 3.

It will be clearly seen from an inspection of the fundamental forms that there is no law for the coefficients of Reciprocants akin to that of their algebraical sum being zero in Invariants.

LECTURE XVII.

The fundamental reciprocants for extent 3, given in the last lecture, agree with the irreducible invariants of a binary cubic both in number and type, with the single exception that the degree of the cubic discriminant is lower by unity than that of the reciprocant corresponding to it. When the extent is raised to 4, both the discriminant and its analogue cease to rank among the irreducible forms, the former being expressible as a rational integral function of invariants of lower degree, and the latter as a similar function of reciprocants. But the increase of extent introduces three additional reciprocants whose leading terms are a^2e, a^2ce and a^3e^2, whereas the additional invariants are only two in number and begin with ae and ace respectively.

The irreducible reciprocants of extent 4 are as follows:

deg. wt.		
1 . 0	a,	
2 . 2	$4M$	$=4ac-5b^2$,
3 . 3	A	$=a^2d-3abc+2b^3$,
3 . 4	P_4	$=50a^2e-175abd+28ac^2+105b^2c$*,
4 . 6	(a^2ce)	$=800a^2ce-1000ab^2e-875a^2d^2+2450abcd-1344ac^3-35b^2c^2$,
5 . 8	(a^3e^2)	$=625a^3e^2-4375a^2bde-49700a^2c^2e+128625ab^2ce-78750b^4e$ $+55125a^2cd^2-61250ab^2d^2-156800abc^2d+183750b^3cd$ $+84868ac^4-102165b^2c^3$.

The similar list of invariants for the quartic is

deg. wt.	
1 . 0	a,
2 . 2	$ac-b^2$,
3 . 3	$a^2d-3abc+2b^3$,
2 . 4	$ae-4bd+3c^2$,
3 . 6	$ace-b^2e-ad^2+2bcd-c^3$.

To obtain the fundamental forms of extent 4 we have to combine M, A and the Quasi-Discriminant

$$(a^3d^2)=125a^3d^2-750a^2bcd+500ab^3d+256a^2c^3+165ab^2c^2-300b^4c$$

with the additional Protomorph

$$P_4=50a^2e-175abd+28ac^2+105b^2c$$

* P_4 is the protomorph of minimum degree; the other protomorph, B, which will be used when we treat of Principiants, is, when expressed in terms of the irreducible forms,

$$B=\frac{1}{50}(aP_4-128M^2).$$

in such a manner that the combination contains a factor a. The removal of this factor gives rise to a form of lower degree, and the process is repeated as often as possible.

Calling that portion of any form which does not contain a its residue, the residue of $4M$ is $-5b^2$, that of (a^3d^2) being $-300b^4c$, and that of P_4 being $105b^2c$. Thus

$$16MP_4 - 7(a^3d^2)$$

contains the factor a, and leads to (a^2ce) of the type 6; 4, 4, which is the analogue to the Catalecticant

$$\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}.$$

The form (a^3d^2) now ceases to be a groundform (= irreducible form) and is replaced by the Quasi-Catalecticant (a^2ce), for

$$(a^3d^2) = \frac{16}{7} MP_4 - \frac{1}{7} a (a^2ce).$$

Similarly, the Cubic Discriminant, a groundform *quâ* the letters a, b, c, d, becomes reducible when a new letter, e, is introduced, and is then replaced by the Catalecticant.

We now come to an extra form which has no analogue in invariants. The residue of the Quasi-Catalecticant (a^2ce) is $-35b^2c^2$, and consequently

$$P_4^2 - 252M(a^2ce)$$

divides by a numerical multiple of a (as it happens by $4a$) and yields the form (a^3e^2), whose type is 8; 5, 4.

Here the deduction of new fundamental forms comes to an end on account of the appearance of e in the residue of (a^3e^2). It would have ended sooner but for the apparently accidental non-appearance of the term b^3d (of the same type 6; 4, 4 as b^2c^2) in the residue of (a^2ce). Had this term appeared, no combination could have been made leading to a new groundform after (a^2ce). We are able to show from *à priori* considerations that it cannot exist.

For the arguments in the annihilator V, up to ∂_e inclusive, are

$$a^2\partial_b,\ ab\partial_c,\ ac\partial_d,\ b^2\partial_d,\ ad\partial_e, \text{ and } bc\partial_e.$$

If, now, the term μb^3d were to form part of a Pure Reciprocant, $b^2\partial_d$ operating upon it would give μb^5; but every other portion of the operator would necessarily give terms containing one or other of the letters a, c. Since such terms cannot destroy μb^5, we must have $\mu b^5 = 0$. Hence the term in question is necessarily non-existent.

The method of combining the protomorphs which we have followed shows that the fundamental reciprocants of extent 4 are connected *inter se* by the two relations or syzygies

$$7(256M^3 + 125A^2) - 16aMP_4 + a^2(a^2ce) = 0,$$
$$P_4^2 - 252M(a^2ce) - 4a(a^3e^2) = 0.$$

The invariants of the binary quartic are connected by only one syzygy, similar to the first of these; the second has no analogue in the theory of Invariants. It has been shown that the irreducible reciprocants of extent 3 are connected by the syzygy

$$256M^3 + 125A^2 - a(a^3d^2) = 0.$$

Substituting in this for the Quasi-Discriminant (a^3d^2) its value expressed in terms of the fundamental forms of extent 4, by means of the equation

$$16MP_4 - 7(a^3d^2) = a(a^2ce),$$

we obtain the first of the above syzygies. By a precisely similar substitution, the syzygy connecting the invariants of the quartic is derived from the one which connects the invariants of the cubic.

Every reciprocant of extent 4 is a rational integral function of the six fundamental forms given in the table; and, by means of the syzygies, powers, but not products, of A and P_4 can be removed from this function. For the first syzygy gives A^2 and the second gives P_4^2 as a rational integral function of the four remaining forms a, M, (a^2ce), and (a^3e^2). Hence every reciprocant of extent 4 is of one or other of the forms

$$\Phi,\ A\Phi,\ P_4\Phi,\ AP_4\Phi,$$

where Φ does not contain either A or P_4, but is a rational integral function of the other four fundamental forms.

Let the four forms which appear in Φ occur raised to the powers $\kappa, \lambda, \mu, \nu$, respectively, in one of its terms. Since the degree-weights of these four forms are

$$1.0,\quad 2.2,\quad 4.6 \text{ and } 5.8,$$

any such term may be represented by

$$a^\kappa (a^2x^2)^\lambda (a^4x^6)^\mu (a^5x^8)^\nu.$$

Thus the totality of the terms in Φ will be represented by

$$\Sigma\, a^\kappa (a^2x^2)^\lambda (a^4x^6)^\mu (a^5x^8)^\nu = \frac{1}{(1-a)(1-a^2x^2)(1-a^4x^6)(1-a^5x^8)}.$$

Now, A, P_4 and AP_4 have the degree-weights

$$3.3,\quad 3.4 \text{ and } 6.7,$$

and consequently the totality of terms in

$$\Phi,\ A\Phi,\ P_4\Phi \text{ and } AP_4\Phi$$

(that is, the totality of the pure reciprocants of extent 4) will be represented by

$$(1 + a^3x^3 + a^3x^4 + a^6x^7)\,\Sigma\, a^\kappa(a^2x^2)^\lambda(a^4x^6)^\mu(a^5x^8)^\nu$$
$$= \frac{1 + a^3x^3 + a^3x^4 + a^6x^7}{(1-a)(1-a^2x^2)(1-a^4x^6)(1-a^5x^8)}.$$

Hence the number of Pure Reciprocants of the type $w;\ i,\ 4$ is the coefficient of a^ix^w in the expansion of a fraction whose numerator is

$$1 + a^3x^3 + a^3x^4 + a^6x^7,$$

with the denominator

$$(1-a)(1-a^2x^2)(1-a^4x^6)(1-a^5x^8).$$

This fraction is called the Representative Form of the Generating Function, in contradistinction to the Crude Form, which is a fraction with the numerator

$$1 - a^{-1}x,$$

having for its denominator

$$(1-a)(1-ax)(1-ax^2)(1-ax^3)(1-ax^4).$$

The crude form expresses the fact that the number of pure reciprocants of the type

$$w;\, i, j$$

is $$(w;\, i, j) - (w-1;\, i+1, j).$$

Its numerator is $1 - a^{-1}x$ for all extents; for the general case in which the extent is j, its denominator consists of the $j+1$ factors

$$(1-a)(1-ax)(1-ax^2)\ldots(1-ax^j).$$

The removal of the negative terms [corresponding to cases in which $(w;\, i, j) < (w-1;\, i+1, j)$] from the crude form would give either the representative form or one equivalent to it, according as the representative form is or is not in its lowest terms. In the parallel theory of Invariants the terms to be rejected are those for which $ij - 2w < 0$; but we do not at present know of any similar criterion for reciprocants, and are thus unable to pass directly from the crude to the representative form of their generating function.

Knowing both the crude and the representative form for reciprocants of extent 4, we may verify that the difference between these two forms of the generating function is omninegative. It will be found that

$$\frac{1 - a^{-1}x}{(1-a)(1-ax)(1-ax^2)(1-ax^3)(1-ax^4)}$$
$$= \frac{1 + a^3x^3 + a^3x^4 + a^6x^7}{(1-a)(1-a^2x^2)(1-a^4x^6)(1-a^5x^8)}$$
$$- \frac{1}{(1-ax^2)(1-ax^3)(1-ax^4)}\left(\frac{a^{-1}x + a^2x^5}{1-a^4x^6} + \frac{x^3 + a^2x^6}{1-a^5x^8}\right)$$
$$- \frac{1}{(1-ax^4)(1-a^4x^6)(1-a^5x^8)}\left(\frac{x + a^5x^{10}}{1-ax^2} + \frac{a^3x^5 + a^2x^7}{1-ax^3}\right).$$

Thus the crude form is seen to consist of an omnipositive part, equal to the representative form, and an omninegative part.

There is no difficulty in obtaining the representative form of the generating function for pure reciprocants of extents 2 and 3. In the one case every reciprocant is a rational integral function of two forms of degree-weight, 1 . 0 and 2 . 2 respectively. The generating function is therefore

$$\frac{1}{(1-a)(1-a^2x^2)}.$$

In the other case (that is, for extent 3) every pure reciprocant can be expressed as a rational integral function of four forms, of which the degree-weights are 1 . 0, 2 . 2, 3 . 3 and 5 . 6, no higher power than the first of the form 3 . 3 occurring in the function. Thus the representative form is

$$\frac{1 + a^3x^3}{(1-a)(1-a^2x^2)(1-a^5x^6)}.$$

LECTURE XVIII.

The number of Pure Reciprocants of a given degree is finite; the number of Invariants of the same degree is infinite. Thus, for example, we have the well-known series of invariants

$$ac - b^2,\ \ ae - 4bd + 3c^2,\ \ldots,$$

all of degree 2, but of weights and extents proceeding to infinity. This may be proved from the theory of partitions (see *American Journal of Mathematics*, Vol. v., No. 1, "On Subinvariants," Excursus on Rational Fractions and Partitions). It will be seen in that article that if $N(w:i)$ is the number of ways in which w can be divided into i parts, and if P is the least common multiple of 2, 3, 4, ..., i, then $N(w:i)$ can be expressed under the form

$$F(w, i) + F'(w, i, p),$$

where p is the residue of w in respect of P.

Writing $$w + \frac{i(i+1)}{4} = \nu,$$

$F(w, i)$ is of the form

$$\frac{\nu^{i-1}}{2^2 . 3^2 \ldots (i-1)^2 . i} + \ldots,$$

all the succeeding indices of the powers of ν in $F(w, i)$ decreasing by 2, and their coefficients being transcendental functions of i which involve Bernoulli's Numbers.

In $F'(w, i, p)$ the highest index of ν is one unit less than the number of times that i is divisible by 2, that is, is $\frac{i-2}{2}$ or $\frac{i-3}{2}$, according as i is even or odd.

Thus, for the partitions of w into 3 parts, we have the formula

$$N(w:3)=\left\{\frac{\nu^2}{12}-\frac{7}{72}\right\}+\left\{\frac{1}{8}(-1)^{\nu+1}+\frac{1}{9}(\rho_1^{\nu}+\rho_2^{\nu})\right\},$$

where $$\nu=w+\frac{1+2+3}{2}=w+3.$$

And, for the partitions of w into 4 parts,

$$N(w:4)=\left\{\frac{\nu^3}{144}-\frac{5\nu}{96}\right\}+\left\{\frac{1}{32}(-)^{\nu+1}\nu+\frac{1}{27}(\rho_1^{\nu+1}+\rho_2^{\nu+1}-\rho_1^{\nu-1}-\rho_2^{\nu-1})\right.$$
$$\left.-\frac{1}{32}(i_1^{\nu+1}+i_2^{\nu+1}-i_1^{\nu-1}-i_2^{\nu-1})\right\},$$

where $$\nu=w+\frac{1+2+3+4}{2}=w+5,$$

and ρ_1, ρ_2 are the roots of $\rho^2+\rho+1=0$,

i_1, i_2 " " $i^2+1 \quad =0$;

in other words, ρ_1 and ρ_2 are primitive cube roots, and i_1, i_2 primitive fourth roots of unity.

The principal term of $N(w:3)$, regarded as a function of w, is

$$\frac{w^2}{12}=\frac{w^2}{2^2 . 3}, \text{ that of } N(w:4) \text{ being } \frac{w^3}{144}=\frac{w^3}{2^2 . 3^2 . 4}.$$

And in general the principal term of $N(w:i)$ is

$$\frac{w^{i-1}}{2^2 . 3^2 . 4^2 \ldots (i-1)^2 . i}.$$

Hence it follows, from a general algebraical principle, that for all values of w above a certain limit, which depends on the value of i and may be determined by the aid of partition tables, $(w;\, i, \infty)-(w-1;\, i+1, \infty)$ must become negative.

Ultimately, $\dfrac{(w-1;\; i+1, \infty)}{(w;\; i, \infty)}=\dfrac{w}{i(i+1)}$, which must eventually be greater than unity. This shows that beyond a certain value of w there can be no pure reciprocant, and consequently that the number of pure reciprocants of a given degree i is finite.

Mr Hammond remarks that the formulae for $N(w:3)$ and $N(w:4)$ may, by the substitution of trigonometrical expressions for the roots of unity, accompanied by some easy reductions, be transformed into

$$N(w:3)=\frac{\nu^2}{12}+\frac{1}{4}\sin^2\frac{\nu\pi}{2}-\frac{4}{9}\sin^2\frac{\nu\pi}{3},$$

and $$N(w:4)=\frac{\nu^3}{144}-\frac{\nu}{12}+\frac{\nu}{16}\sin^2\frac{\nu\pi}{2}+\frac{1}{8}\sin\frac{\nu\pi}{2}-\frac{2}{9\sqrt{3}}\sin\frac{\nu\pi}{3},$$

where, in the first formula, $\nu=w+3$, and in the second $\nu=w+5$. He also obtains the principal term of $N(w:i)$ from first principles as follows:

The partitions of w into i parts may be separated into two sets, the first containing at least one zero part in each of its partitions, the second consisting of partitions in which no zero part occurs.

Suppressing one zero part in each partition of the first set, we see that the number of partitions in which 0 occurs is $N(w:i-1)$. Diminishing each part by unity in those partitions which contain no zeros, their number is seen to be $N(w-i:i)$. The sum of these two numbers is $N(w:i)$, which is the total number of partitions, and consequently

$$N(w:i)=N(w:i-1)+N(w-i:i).$$

Let the principal term of $N(w:i-1)$ be αw^{i-2}, where α is independent of w, and write

$$w=ix,\; N(w:i)=u_x,\; N(w-i:i)=u_{x-1}.$$

Then $$u_x-u_{x-1}=\alpha w^{i-2}+\ldots=\alpha i^{i-2}x^{i-2}+\ldots.$$

Hence, by a simple summation, we find

$$u_x=\alpha i^{i-2}\{x^{i-2}+(x-1)^{i-2}+(x-2)^{i-2}+\ldots\}+\ldots.$$

But, since only the principal term of u_x is required, this summation may be replaced by an integration. Thus the principal term of u_x is

$$\alpha i^{i-2}\int x^{i-2}dx=\frac{\alpha i^{i-2}x^{i-1}}{i-1}.$$

Restoring $w=ix$ and $N(w:i)=u_x$,

we see that the principal term of $N(w:i)$ is $\dfrac{\alpha w^{i-1}}{(i-1)i}$. Thus the principal term of $N(w:i)$ is found from that of $N(w:i-1)$ by multiplying it by

$$\frac{w}{(i-1)i}.$$

When $i=3$, the principal term is $\dfrac{w^2}{2^2 . 3}$; it is therefore $\dfrac{w^3}{2^2 . 3^2 . 4}$ when $i=4$; and for the general case it is $\dfrac{w^{i-1}}{2^2 . 3^2 . 4^2 \ldots (i-1)^2 . i}$.

The value of $N(w:i)$ is given in line i and column w of the following table:

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
2	1	2	2	3	3	4	4	5	5	6	6	7	7	8
3	1	2	3	4	5	7	8	10	12	14	16	19	21	24
4	1	2	3	5	6	9	11	15	18	23	27	34	39	47
5	1	2	3	5	7	10	13	18	23	30	37	47	57	70
6	1	2	3	5	7	11	14	20	26	35	44	58	71	90

From an inspection of the tabulated values of $N(w:i)$ we see that

$$\begin{array}{llll} N(w:2)-N(w-1:3) & \text{is negative or zero when} & & w>2, \\ N(w:3)-N(w-1:4) & \quad ,, & ,, & w>6, \\ N(w:4)-N(w-1:5) & \quad ,, & ,, & w>8, \\ N(w:5)-N(w-1:6) & \quad ,, & ,, & w>12. \end{array}$$

Hence for pure reciprocants of indefinite extent, whose degrees are

2, 3, 4, 5,

the highest possible weights are 2, 6, 8 and 12, respectively.

In like manner, from Euler's table, in his memoir "De Partitione Numerorum" (published in 1750), it will be found that

for degrees	2	3	4	5	6	7	8	9	10	11	12	13
the highest weights are	2	6	8	12	16	21	26	30	36	42	49	55

Further than this the table, which goes up to $w=59$, will not enable us to proceed.

The actual number of pure reciprocants of degree i, weight w, and of indefinite extent, is seen in the following table, which gives the value of $N(w:i)-N(w-1:i+1)$ when positive, blank spaces being left in the table when this difference is zero or negative.

Degree $i=$ \ Weight $w=$	2	3	4	5	6	7	8	9	10	11	12	13	14
2	1												
3	1	1	1		1								
4	1	1	2	1	2	1	2						
5	1	1	2	2	3	2	4	3	4	2	3		

Thus, for degree 2, there is only one pure reciprocant, namely

$$(ac)=4ac-5b^2.$$

For degree 3 the table shows that, in addition to the compound form

$$a(ac)=a(4ac-5b^2),$$

there are three others whose weights are 3, 4 and 6 respectively.

These are the three protomorphs,

$$(a^2d)=a^2d-3abc+2b^3,$$

$$(a^2e)=50a^2e-175abd+28ac^2+105b^2c,$$

$$(a^2g)=14a^2g-63abf-1350ace+1782b^2e+1470ad^2-4158bcd+2310c^3.$$

With the above forms and a we are able to form the following compounds of degree 4:

$$a^2(ac),\quad a(a^2d),\quad (ac)^2,\quad a(a^2e),\quad a(a^2g),$$

whose weights are 2, 3, 4, 4, 6.

The forms of degree 4 and weights 5, 7, 8, and one of the forms of weight 6, cannot be similarly made up of forms of inferior degree, and are therefore groundforms. Three of them are the protomorphs (a^3f), (a^3h) and (a^3i) of weights 5, 7 and 8, whose values were given in Lecture XVI. The groundform of weight 6 is the Quasi-Catalecticant given in the last lecture. All the forms of degree 4 have thus been accounted for except one of the two forms of weight 8, which will be seen to be of extent 6, and to have a^2cg for its leading term.

We know from Euler's table that $N(8:4)-N(7:5)=2$; that is,

$$(8;4,8)-(7;5,8)=2.$$

Now, $(8;4,7)=N(8:4)-1$, the omitted partition being 8.0.0.0,

$(8;4,6)=N(8:4)-2$, the partition 7.1.0.0 being also left out,

$(8;4,5)=N(8:4)-4$, {for 6.2.0.0 and 6.1.1.0 are excluded from (8; 4, 5), but make their appearance in (8; 4, 6).

Similarly,

$$(7;5,7)=N(7:5),$$

$$(7;5,6)=N(7:5)-1,$$

$$(7;5,5)=N(7:5)-2.$$

We have, therefore,

$$(8;4,8)-(7;5,8)=2,$$

$$(8;4,7)-(7;5,7)=1,$$

$$(8;4,6)-(7;5,6)=1,$$

$$(8;4,5)-(7;5,5)=0.$$

Hence we may draw the following inferences:

(1) No pure reciprocant exists whose type is 8; 4, 5.

(2) The one whose type is 8; 4, 6 must contain the letter g.

(3) No fresh form is found by making the extent 7 instead of 6, so that there is no pure reciprocant of weight 8 and degree 4 whose *actual extent* is 7.

(4) There is a pure reciprocant (the Protomorph whose leading term is a^3i) whose actual extent is 8.

(5) This, with the one whose actual extent is 6, makes up the two given by $(8;4,8)-(7;5,8)=2$.

LECTURE XIX.

The following is a complete list of the irreducible reciprocants of indefinite extent for the degrees 2, 3 and 4:

Deg. wt.	
2 . 2	(ac),
3 . 3	(a^2d),
3 . 4	(a^2e),
3 . 6	(a^2g),
4 . 5	(a^3f),
4 . 6	(a^2ce),
4 . 7	(a^3h),
4 . 8	(a^3i), (a^2cg).

The values of all of them except (a^2cg) have been given in previous lectures, and the method of obtaining them sufficiently indicated. Thus (ac), (a^2d), (a^2e), (a^3f), (a^2g), (a^3h) and (a^3i) are the Protomorphs of minimum degree P_2, P_3, P_4, P_5, P_6, P_7 and P_8, respectively; and (a^2ce) is the Quasi-Catalecticant whose value has been set forth in the table of irreducible forms of extent 4. It will be remembered that (a^2ce) was found by combining the Quasi-Discriminant (a^3d^2) with P_2P_4 linearly in such a manner that the combination, which is of the 5th degree, divides by a and gives (a^2ce) of the 4th degree. If we try to find (a^2cg) by a similar process, it will be necessary to rise as high as the 7th degree, and then to drop down by successive divisions by a to the fourth.

In fact, since to a numerical factor *près* the residues of

	$P_2,\ P_3,\ P_4,\ P_5$
are	$b^2,\ b^3,\ b^2c,\ b^3c$,
that of	P_3P_5 will be b^6c,
and that of	$P_2^2P_4$ will be b^6c.

Thus a linear combination of P_3P_5 and $P_2^2P_4$ will be divisible by a, and, taking account of the numerical coefficients, we shall find

$$26P_2^2P_4 + 875P_3P_5 \equiv 0 \text{ (mod. } a).$$

As a result of calculation, it will be seen that the above combination of the protomorphs divided by a,

$$\frac{1}{a}(26P_2^2P_4 + 875P_3P_5),$$

has (to a numerical factor *près*) the same residue as P_4^2.

Making a second combination and division by a, we find

$$7\left(\frac{26P_2^2P_4 + 875P_3P_5}{a}\right) - 25P_4^2 \equiv 0 \text{ (mod. } a) = aS, \text{ suppose.}$$

Then, by actual calculation, the residue of S is found to be

$$-262500b^4e + 612500b^3cd - 339080b^2c^3.$$

Two reductions have already been made in obtaining this form S of the 5th degree. A final combination of S with P_2P_6 and the form (a^3e^2), whose value was given in a former lecture, enables us to divide out once more by a and thus get the form (a^2cg) of the 4th degree.

It is the fact that P_2P_6 and (a^3e^2) have residues which are not the same to a numerical factor *près* which necessitates the long calculation above described. No linear combination of P_2P_6 and (a^3e^2) with one another is divisible by a, and it is necessary to find a third form S a linear combination of which with both P_2P_6 and (a^3e^2) will divide by a.

There is, however, another way of arriving at the form (a^2cg) by using the eductive generator

$$G = 4(ac - b^2)\partial_b + 5(ad - bc)\partial_c + 6(ae - bd)\partial_d + \ldots.$$

Starting with the Quasi-Catalecticant

$$(a^2ce) = 800a^2ce - 1000ab^2e - 875a^2d^2 + 2450abcd - 1344ac^3 - 35b^2c^2,$$

and operating on it with G, we have

$$\begin{aligned} G(a^2ce) = {} & 4(ac - b^2)(-2000abe + 2450acd - 70bc^2) \\ & + 5(ad - bc)(800a^2e + 2450abd - 4032ac^2 - 70b^2c) \\ & + 6(ae - bd)(-1750a^2d + 2450abc) \\ & + 7(af - be)(800a^2c - 1000ab^2). \end{aligned}$$

The terms of this expression contain the common numerical factor 10, which may be rejected; thus we have

$$G(a^2ce) = 10(a^3cf),$$

where

$$\begin{aligned} (a^3cf) = {} & 560a^3cf - 700a^2b^2f - 650a^3de - 290a^2bce + 1500ab^3e \\ & + 2275a^2bd^2 - 1036a^2c^2d - 3710ab^2cd + 1988abc^3 + 63b^3c^2. \end{aligned}$$

This form (a^3cf) is the first educt of (a^2ce), and is irreducible (but, being of the fifth degree, does not appear in our list, which contains no forms of higher degree than the fourth). Operating on it with G, we obtain the educt of (a^3cf), which is the second educt of (a^2ce). This second educt will be of the 6th degree (its leading term will be a^4cg), but is reducible to the 5th when combined with

$$(4ac - 5b^2)(a^2ce),$$

as we know from the general theorem concerning the reduction of second educts. We shall thus obtain a form (a^3cg), the reduced second educt of (a^2ce), of the 5th degree, and a final combination of (a^3cg) with one or both of the forms $P_2 P_6$ and (a^3e^2) will enable us to divide once more by a and thus arrive at (a^2cg) of the 4th degree.

By either of these methods we obtain

$$\begin{aligned}(a^2cg) = {} & 1176a^2cg - 8085a^2df + 7040a^2e^2 - 1470ab^2g + 18963abcf \\ & - 16940abde - 27160ac^2e + 26460acd^2 - 9555b^3f \\ & + 28098b^2ce + 12740b^2d^2 - 52822bc^2d + 21560c^4;\end{aligned}$$

but the second way, besides being more direct, gives us at the same time the value of the irreducible form (a^2cf).

Every Pure Reciprocant is an Invariant of a Binary Quantic whose coefficients $A, B, C, D, \ldots$ are functions of the original elements $a, b, c, d, \ldots$ such that

$$\begin{aligned} VA &= 0, \\ VB &= A, \\ VC &= 2B, \\ VD &= 3C, \\ &\cdots\cdots \end{aligned}$$

and conversely, every Invariant of this Binary Quantic, or of a system of such Binary Quantics, is a Pure Reciprocant.

This is a particular case of the more general theorem, due to Mr Hammond, that if Θ is the operator,

$$\phi_1(a)\partial_b + \phi_2(a, b)\partial_c + \phi_3(a, b, c)\partial_d + \ldots,$$

where $\phi_1, \phi_2, \phi_3, \ldots$ are arbitrary rational integral functions, and if

$$A, B, C, D, \ldots, A', B', C', D', \ldots, A'', B'', C'', \ldots$$

be any rational integral functions of the original letters $a, b, c, \ldots$ which satisfy the conditions

$$\begin{array}{lll} \Theta A = 0, & \Theta A' = 0, & \Theta A'' = 0, \\ \Theta B = A, & \Theta B' = A', & \Theta B'' = A'', \\ \Theta C = 2B, & \Theta C' = 2B', & \Theta C'' = 2B'', \\ \Theta D = 3C, & \Theta D' = 3C', & \Theta D'' = 3C'', \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \end{array}$$

then every invariant in respect to the elements

$$A, B, C, D, \ldots, A', B', C', D', \ldots, A'', B'', C'', D'', \ldots$$

is a rational integral solution of the equation

$$\Theta = 0.$$

Obviously, every rational integral solution of $\Theta = 0$ is an invariant in the above elements, so that the converse of the proposition is true. For the only

conditions imposed upon $A, A', A'', \ldots$ are that they shall be rational integral functions of $a, b, c, d, \ldots$ annihilated by Θ. Let

$$\Phi(A, B, C, D, \ldots, A', B', C', D', \ldots, A'', B'', C'', D'', \ldots)$$

be any invariant in the large letters. We have to show that

$$\Theta\Phi = 0.$$

Now,

$$\begin{aligned}\Theta\Phi = {} & \frac{d\Phi}{dA}\Theta A + \frac{d\Phi}{dB}\Theta B + \frac{d\Phi}{dC}\Theta C + \ldots \\ & + \frac{d\Phi}{dA'}\Theta A' + \frac{d\Phi}{dB'}\Theta B' + \frac{d\Phi}{dC'}\Theta C' + \ldots \\ & + \cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

Hence, writing for $\Theta A, \Theta B, \Theta C, \ldots$, their values given above, we have

$$\begin{aligned}\Theta\Phi = {} & (A\partial_B + 2B\partial_C + 3C\partial_D + \ldots)\Phi \\ & + (A'\partial_{B'} + 2B'\partial_{C'} + 3C'\partial_{D'} + \ldots)\Phi \\ & + \cdots\cdots\cdots\cdots\cdots\cdots \\ = {} & 0 \text{ (since } \Phi \text{ is an invariant)};\end{aligned}$$

which proves the proposition.

Confining our attention to a single set of letters, the Binary Quantic

$$(A, B, C, \ldots J, K, L \between x, y)^n,$$

whose coefficients are formed from one another by the successive operation of Θ as above, may be called a Quasi-Covariant; and it will follow immediately from the Theory of Binary Forms that every Covariant of a Quasi-Covariant is itself a Quasi-Covariant, and that every Invariant of any Quasi-Covariant (or system of Quasi-Covariants) is an Invariant in respect to the letters $A, B, C, \ldots$, and therefore, by what precedes, a rational integral solution of $\Theta = 0$.

Writing the terms of

$$(A, B, C, \ldots J, K, L \between x, y)^n$$

in reverse order, we have

$$Ly^n + nKxy^{n-1} + \frac{n(n-1)}{1 \cdot 2} Jxy^{n-2} + \ldots + Ax^n,$$

where

$$\Theta L = nK, \ \Theta K = (n-1)J, \ \ldots \Theta A = 0.$$

Thus the Quasi-Covariant may be written

$$Ly^n + \Theta Lxy^{n-1} + \frac{\Theta^2 L}{1 \cdot 2} x^2 y^{n-2} + \ldots + \frac{\Theta^n L}{1 \cdot 2 \cdot 3 \ldots n} x^n = y^n \left(e^{\frac{x\Theta}{y}}\right) L,$$

where $\Theta^{n+1} L = 0$.

This is the general symbolic expression for a Quasi-Covariant. An example of a Quasi-Covariant has already been given in Lecture II. [p. 310, above], where it was stated, and afterwards proved [p. 360], that the reciprocal of the nth modified derivative could be put under the form

$$-t^{-n-3}\left(e^{-\frac{V}{t}}\right)a_n.$$

The numerator of this reciprocal expression, which may be called the reciprocal function, is

$$t^n\left(e^{-\frac{V}{t}}\right)a_n,$$

which is identical with the general expression

$$y^n\left(e^{\frac{x\Theta}{y}}\right)L,$$

if $\qquad x=-1,\ y=t,\ L=a_n$ and $\Theta=V$.

Hence every Invariant of the reciprocal function is a Pure Reciprocant.

This property of the reciprocal function was discovered independently by Mr C. Leudesdorf, who published his results in the *Proceedings of the London Mathematical Society* (Vol. XVII. p. 208). Mr Hammond's results were given in two letters to me dated January 15th and January 20th, 1886, and were briefly alluded to by him at a meeting of the London Mathematical Society. They are here published for the first time.

Recalling the form of the operator

$$\Theta=\phi_1(a)\partial_b+\phi_2(a,b)\partial_c+\phi_3(a,b,c)\partial_d+\ldots,$$

where $\phi_1, \phi_2, \phi_3, \ldots$ are rational integral functions, we can form a Quasi-Covariant of extent j by a finite number of successive operations on a single letter of that extent.

To fix the ideas, take the letter d of extent 3, and operate on it with Θ; then

$$\Theta d=\phi_3(a, b, c).$$

Since $\phi_1, \phi_2, \phi_3, \ldots$ are by definition rational integral functions, we can, by operating a finite number of times with Θ, remove first c and then b from $\phi_3(a, b, c)$, and thus obtain

$$\Theta^n d=\text{funct. } a,$$

where n denotes a finite number of operations. Since $\Theta a=0$, we have

$$\Theta^{n+1}d=0.$$

In this manner we form the Quasi-Covariant of the nth order

$$y^n\left(e^{\frac{x\Theta}{y}}\right)d.$$

If $\phi_2, \phi_3, \phi_4, \ldots$ do not contain higher powers than the first of the last letter in each, the order of the above Quasi-Covariant will be the same as its extent. This is the case with the reciprocal function, which is a co-reciprocant (that is, a Quasi-Covariant relative to V).

Ex. $\qquad y^2\left(e^{\frac{xV}{y}}\right)c=cy^2+Vcxy+\dfrac{V^2c}{1.2}x^2=cy^2+5abxy+5a^3x^2.$

The discriminant of this is the pure reciprocant

$$5a^3\left(ac-\frac{5b^2}{4}\right).$$

As an additional example, consider the pair of linear co-reciprocants

$$4a(4ac-5b^2)x+(5ad-7bc)y,$$
$$50a(a^2d-3abc+2b^3)x+(25abd-32ac^2+5b^2c)y.$$

The resultant of this pair is

$$2a(125a^3d^2-750a^2bcd+500ab^3d+256a^2c^3+165ab^2c^2-300b^4c),$$

that is, is the Quasi-Discriminant multiplied by $2a$.

LECTURE XX

"Quintessenced into a finer substance."—*Drummond of Hawthornden.*

Before proceeding with the proper subject of this day's lecture, I should like to mention a geometrical theorem which has fallen in my way, and which, *inter alia*, gives an immediate proof of the existence of 27 straight lines on a general cubic surface. It is proved by means of a Lemma (itself of quasi-geometrical origin) which finds its principal application in an extension of Bring's or Tschirnhausen's method, and shows how any number of specified terms, reckoning from either end, can be taken away from any equation of a sufficiently high degree*.

Subjectively speaking, I was led to the Lemma by considering the question, closely connected with Differential Invariants, of the method of depriving a linear differential equation of several terms.

Let ϕ be a cubic and u a linear function in x, y, z, t, say

$$\phi=ax^3+\ldots+fx^2y+\ldots,$$
$$u=lx+my+nz+pt.$$

Then, if ψ is a scroll which contains all the straight lines on $\phi+\lambda u^3$, when the parameter λ has any arbitrary numerical value from $+\infty$ to $-\infty$, I prove that

$$\psi=\phi^2A+\phi u^3B+u^6C,$$

* I recover all Hamilton's results contained in his Report to the British Association, 1836, "On Jerrard's Method," in a much more clear and concise manner, and make important additions to his theory.

where ψ is of the degree 15 in the variables x, y, z, t,
.... 6 in the coefficients (l, m, n, p) of u,
.... 11 .... $(a, \ldots\ldots)$ of ϕ.

Or, more briefly, in	x	l	a
ψ is of degree	15	6	11, and consequently
C...............	9	0	11.

The intersections of ϕ with ψ are its intersections with u^6 and with C, of which the intersections with the arbitrary plane u^6 are clearly foreign to the question, but the cubic ϕ and the $9^{ic}C$ intersect in 27 straight lines, which are the 27 ridges on ϕ.

C is identical with the covariant found by Clebsch and given in Salmon's *Geometry of Three Dimensions* at the end of the chapter on Cubic Surfaces. It may with propriety be called the Clebschian.

By giving the parameter λ (which occurs in $\phi + \lambda u^3$) an infinitesimal variation, it is easily proved that

$$B = -2EC, \quad A = E^2C, \quad E^3C = 0,$$

where E is the operator $l^3\partial_a + \ldots + 3l^2m\partial_f + \ldots$, which may be simply and completely defined by its property of changing the general cubic ϕ into $(lx + my + nz + pt)^3$.

The equation $E^3C = 0$ expresses a new property of the Clebschian: it shows that if a, f are the coefficients of x^3 and any other term in ϕ containing x^2, neither a^3 nor a^2f can occur in any one of the terms of C. Defining a principal term in ϕ as one which contains the cube of one of the variables, and a term adjacent to it as one which contains the square of the same variable, this is equivalent to saying that neither the cube of the coefficient of a principal term nor its square multiplied by the coefficient of any adjacent term can appear in any of the terms of C.

An interesting special case of the general theorem is when the arbitrary plane u is taken to be one of the planes of reference, say $u = x$. Then

$$l = 1, \quad m = 0, \quad n = 0, \quad p = 0,$$

and the operator E becomes simply $\frac{d}{da}$. Thus we learn that

$$\phi^2 \frac{d^2C}{da^2} - 2x^3\phi \frac{dC}{da} + x^6C$$

is a Scroll of the fifteenth order which contains all the Ridges on

$$\phi + \lambda x^3$$

for any arbitrary value of the parameter λ.

It also contains 6 times over the curve of intersection of $\phi = 0$ with $x = 0$.

I now propose to give the substance, with a brief commentary, of some very interesting letters I have recently received from Capt. MacMahon. I abstain from giving a proof of his results, as I am informed that he intends to do this himself at an early meeting of the London Mathematical Society.

Using V to signify the Reciprocant Annihilator and Ω the Annihilator of Invariants, we have studied the properties of

$$V\frac{d}{dx} - \frac{d}{dx}V$$

and those of

$$\Omega\frac{d}{dx} - \frac{d}{dx}\Omega.$$

These may be written in the form

$$\begin{vmatrix} V & \frac{d}{dx} \\ V & \frac{d}{dx} \end{vmatrix} \quad \begin{vmatrix} \Omega & \frac{d}{dx} \\ \Omega & \frac{d}{dx} \end{vmatrix},$$

and may be called alternants to $V, \frac{d}{dx}$ and to $\Omega, \frac{d}{dx}$ respectively.

It has been shown in Lecture VII. [p. 341, above] that

$$V\frac{d}{dx} - \frac{d}{dx}V = 2(3i + w)a.$$

The corresponding formula is

$$\Omega\frac{d}{dx} - \frac{d}{dx}\Omega = 3i + 2w,$$

as may be seen by writing $\kappa = 0, \lambda = 3, \mu = 4, \nu = 5, \ldots$ in a more general formula given in Lecture V. [p. 329, above].

Observe that operating with the alternant to $\Omega, \frac{d}{dx}$ is equivalent to multiplication by a number, and that operating with the alternant to $V, \frac{d}{dx}$ merely introduces a numerical multiple of a as a factor. No such property exists for the Alternant

$$V\Omega - \Omega V,$$

but one much more extraordinary.

MacMahon has found that this alternant, which he calls J, is a generator to a Reciprocant and a generator to an Invariant; that is, it converts a Reciprocant into another Reciprocant, and an Invariant into another Invariant. As regards a Differential Invariant, which is at once an Invariant and a Reciprocant, it is an Annihilator. He shows, in fact, that

$$\Omega J - J\Omega = 0$$

and

$$VJ - JV = 0.$$

If, then, $\Omega R = 0$, it follows immediately that $\Omega(JR) = 0$; that is, if R is an invariant, JR is so too. And in like manner, if

$$VR = 0, \quad V(JR) = 0,$$

that is, if R is a reciprocant, so is JR.

Of course, if M is a Differential Invariant,

$$JM = V(\Omega M) - \Omega(VM) = 0.$$

Let me here give a caution which may be necessary: The fact that a form is annihilated by J is not sufficient to show that it is a Differential Invariant, though all Differential Invariants are necessarily annihilated by J. Forms exist which are subject to annihilation by

$$J = a^2\partial_c + 3ab\partial_d + \ldots,$$

but are, notwithstanding, *neither* invariants nor reciprocants.

Such a form is the monomial b, which is obviously annihilated by J. Another is $ad - 3bc$. For, since

$$a^2d - 3abc + 2b^3$$

is a Differential Invariant, we have

$$J(a^2d - 3abc + 2b^3) = 0.$$

But $\qquad Jb^3 = 0$ and $Ja = 0$;

therefore, also, $\qquad aJ(ad - 3bc) = 0.$

The general theorem is as follows, and is a most remarkable one: If we write

$$\begin{aligned} mP(m, \mu, v, n) = \mu a^m\partial_{a_n} &+ (\mu + v)\, m a^{m-1} b\partial_{a_{n+1}} \\ &+ (\mu + 2v)\left(m a^{m-1}c + \frac{m(m-1)}{2} a^{m-2}b^2\right)\partial_{a_{n+2}} \\ &+ (\mu + 3v)\left\{m a^{m-1}d + m(m-1)a^{m-2}bc \right. \\ &\left. + \frac{m(m-1)(m-2)}{6} a^{m-3}b^3\right\}\partial_{a_{n+3}} + \ldots, \end{aligned}$$

where the coefficients of the terms inside the brackets are the same as those of the corresponding terms in the expansion of $(a + b + c + \ldots)^m$, and where a_n stands for the nth letter of the series $a, b, c, d, \ldots$, then Capt. MacMahon establishes that *the alternant of any two P's is another P.*

A question here suggests itself naturally: What would be the alternant of three or more P's? For instance, would the alternant

$$\begin{vmatrix} P_1 & P_2 & P_3 \\ P_1 & P_2 & P_3 \\ P_1 & P_2 & P_3 \end{vmatrix} = P_1P_2P_3 - P_1P_3P_2 + P_2P_3P_1 - P_2P_1P_3 + P_3P_1P_2 - P_3P_2P_1$$

be another P?*

* In my Multiple Algebra investigations, which I hope some day to resume, I have made important use of similar Alternants, which, it may be noticed, do not vanish when their elements are non-commutative. In this connection it is well worthy of observation that the P's (as indeed would be true of any operators linear in the differential inverses) obey the *associative* law.

It would be interesting to ascertain under what arithmetical conditions, if any, other than MacMahon's, *any* two linear operators of the same general form as his P's become commutative.

Perhaps it would also be worthy of inquiry whether the P theory might not admit of extension in some form to operators non-linear in the differential inverses, and whether to every such operator of degrees i and j in the letters and their differential inverses there is not correlated another in which i and j are interchanged.

Moreover, he obtains expressions for the parameters m, μ, v, n of the resulting P in terms of the parameters of its two components. He proves that if P_1, P_2 are the two components whose alternant is P, supposing

m_1, μ_1, v_1, n_1 to be the parameters of P_1,

m_2, μ_2, v_2, n_2 .... P_2,

then the parameters m, μ, v, n of their resultant P are given by the equations

$$\begin{aligned} m &= m_1 + m_2 - 1, \\ \mu &= (m_1 + m_2 - 1)\left\{\frac{\mu_2}{m_2}(\mu_1 + n_2v_1) - \frac{\mu_1}{m_1}(\mu_2 + n_1v_2)\right\}, \\ v &= (n_2 - n_1)v_1v_2 - \frac{m_2 - 1}{m_1}\mu_1v_2 + \frac{m_1 - 1}{m_2}\mu_2v_1, \\ n &= n_1 + n_2. \end{aligned}$$

It will be seen that Ω and V are special forms of P. Thus,

$$\Omega = P(1, 1, 1, 1),$$
$$V = P(2, 4, 1, 1).$$

Now, if the second and third parameters are zero, every term of P vanishes, and MacMahon finds that in the following two cases the second and third parameters of the resultant above given vanish.

(1) Supposing $\frac{\mu_1}{m_1v_1}$ to be an integer, this takes place when the two component systems of parameters are

$$m_1, \mu_1, \quad v_1, \quad n_1,$$
$$m_2, \mu_1m_2, m_1v_1, n_1 + \frac{\mu_1}{m_1v_1}(m_2 - m_1).$$

(2) When they are

$$m_1, \mu_1, \quad v_1, \quad n_1,$$
$$m_2, n_1m_2, m_1 - 1, \frac{\mu_1}{m_1v_1}(m_2 - 1).$$

Now, $\qquad P(1, 1, 1, 1) = \Omega,$

$\qquad P(2, 4, 1, 1) = V,$

and by the law of composition

$$J = \Omega V - V\Omega = P(2, 2, 1, 2).$$

Also, $\left.\begin{matrix} 2, 2, 1, 2 \\ 1, 1, 1, 1 \end{matrix}\right\}$ will be found to come under the first case;

and $\left.\begin{matrix} 2, 2, 1, 2 \\ 2, 4, 1, 1 \end{matrix}\right\}$ .... the second.

Hence, $\Omega J - J\Omega = 0$ and $VJ - JV = 0$.

The above theorem is one of extraordinary beauty, and must play an important part in the future of Algebra.

In another letter Capt. MacMahon calls my attention to the fact that the operator called by me Cayley's generator P, in Lecture IV. of this course [p. 323, above], is a particular case of one of a much more general character given by him in the *Quarterly Mathematical Journal* (Vol. XX., p. 362).

He also states that every pure reciprocant, when multiplied by the needful power of a, is an invariant of the binary quantic

$$\{2.(2n+1)!\}\,a^{n+1} - n\,\{1!\,(2n+1)!\}\,a^{n-1}bt$$
$$+\frac{n(n-1)}{1.2}\{2!\,(2n)!\}\left\{a^{n-2}c + \frac{n-2}{2}a^{n-3}b^2\right\}t^2$$
$$-\frac{n(n-1)(n-2)}{1.2.3}\{3!\,(2n-1)!\}\left\{a^{n-3}d + (n-3)a^{n-4}bc + \frac{(n-3)(n-4)}{1.2.3}a^{n-5}b^3\right\}t^3$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

which I have written in the non-homogeneous form.

But this expression is (to a numerical factor *près*) identical with the numerator of $\frac{d^{n+2}x}{dy^{n+2}}$ when $t, a, b, \ldots$ are taken to be the modified differential derivatives $\frac{dy}{dx}, \frac{1}{2}\frac{d^2y}{dx^2}, \frac{1}{2.3}\frac{d^3y}{dx^3}, \ldots$. See my note on Burman's law for the Inversion of the Independent Variable [Vol. II. of this Reprint, p. 44].

The property that its invariants are pure reciprocants has already been proved in the lectures [above, p. 412].

LECTURE XXI.

I take blame to myself for not earlier communicating to the class the substance of a note of Mr Hammond's under date of January 20th, 1886, in which he makes an interesting application of the theorem that any invariant of the form

$$y^n\left(e^{\frac{x}{y}V}\right)F(a, b, c, \ldots),$$

in which the function F is subject to the condition

$$V^{n+1}F = 0,$$

or of any combination of such forms, is a pure reciprocant.

Forms such as the above, whose invariants are pure reciprocants, he calls *co-reciprocants*. It follows that any covariant of one or more co-reciprocants is itself a co-reciprocant, for any invariant of a covariant is an invariant.

Taking F to be a single letter b, c, d, he forms the functions

$$by + 2a^2x, \qquad (1)$$
$$cy^2 + 5abxy + 5a^3x^2, \qquad (2)$$
$$dy^3 + 3(2ac+b^2)xy^2 + 21a^2bx^2y + 14a^4x^3, \qquad (3)$$

in which
$$2a^2 = Vb,$$
$$5ab = Vc,\quad 5a^3 = \frac{V^2c}{1.2},$$
$$3(2ac+b^2) = Vd,\quad 21a^2b = \frac{V^2d}{1.2},\quad 14a^4 = \frac{V^3d}{1.2.3}.$$

On writing $y = t$, $x = -1$, it will be observed that these three forms are the numerators of

$$\frac{1}{3!}\frac{d^3x}{dy^3},\quad \frac{1}{4!}\frac{d^4x}{dy^4},\quad \frac{1}{5!}\frac{d^5x}{dy^5}.$$

The Jacobian of (1) and (2) is

$$(4ac - 5b^2)\,ay;$$

the coefficient of ay is the familiar pure reciprocant $4ac - 5b^2$.

The Jacobian of (1) and (3) is the determinant

$$\begin{vmatrix} b & 2a^2 \\ dy^2 + (4ac-5b^2)xy & (2ac+b^2)y^2 \end{vmatrix},$$

which is divisible by y, giving the quotient

$$(2a^2d - 2abc - b^3)\,y + 2a^2(4ac - 5b^2)\,x. \qquad (4)$$

This is
$$y\left(e^{\frac{x}{y}V}\right)(2a^2d - 2abc - b^3),$$

the terms involving $\frac{x^2}{y}, \frac{x^3}{y^2}, \ldots$ vanishing identically.

Looking at $2a^2d - 2abc - b^3$ as the anti-source to a Co-reciprocant*, we might at first sight expect that it would give rise to a co-reciprocant of the third order in x, y, whereas we see it is the anti-source of a linear co-reciprocant.

* What differentiates Reciprocants from Invariants is that we have no reverser to V as O is to Ω in the theory of Invariants, that is, no reverser which does not introduce an additional letter.

The coefficients of a covariant are obtained either from the source by continually operating with O, or from the anti-source by continually operating with Ω. But in the case of a co-reciprocant, we are only able to proceed in one direction (namely from the anti-source, or coefficient of the highest power of y, to the source), as we have only one operator, V, at our disposal.

We have $$V(2a^2d - 2abc - b^3) = 2a^2(4ac - 5b^2).$$

Combining this with
$$V(a^2d - 3abc + 2b^3) = 0 \quad \text{(the well-known Mongian),}$$
and dividing by a, he obtains
$$V(5ad - 7bc) = 4a(4ac - 5b^2).$$

Hence $$(5ad - 7bc)\,y + 4a(4ac - 5b^2)\,x \qquad (5)$$
is a co-reciprocant. It is in fact (4) reduced in degree.

The Jacobian of (5) and of $cy^2 + 5abxy + 5a^3x^2$, that is,
$$\begin{vmatrix} 5ad - 7bc & 4a(4ac - 5b^2) \\ 2cy + 5abx & 5aby + 10a^3x \end{vmatrix},$$
will divide by a, and gives the new linear co-reciprocant
$$(25abd - 32ac^2 + 5b^2c)\,y + 50a(a^2d - 3abc + 2b^3)\,x. \qquad (6)$$

The coefficient of y is of weight 4, but instead of giving rise to a co-reciprocant of the 4th order, we see that this again is the anti-source of a linear co-reciprocant.

The resultant of the two linear co-reciprocants (4) and (6) divided by a numerical multiple of a gives the well-known Quasi-Discriminant $125a^2d^2 + \ldots$, as was stated at the end of Lecture XIX [above, p. 413].

The noticeable fact is that (including $by + 2a^2x$) there exist 3 linear independent co-reciprocants of extent 3. Probably there are no more, but this requires proof.

The promised land of Differential Invariants or Projective Reciprocants is now in sight, and the remainder of the course will be devoted to its elucidation. Twenty lectures have been given on the underlying matter, and probably ten more, at least, will have to be expended on this higher portion of the theory.

One is surprised to reflect on the change which has come over the face of Algebra in the last quarter of a century. It is now possible to enlarge to an almost unlimited extent on any branch of it. These thirty lectures, embracing only a fragment of the theory of reciprocants, might be compared to an unfinished epic in thirty cantos. Does it not seem as if Algebra had attained to the character of a fine art, in which the workman has a free hand to develop his conceptions as in a musical theme or a subject for painting? Formerly it consisted almost exclusively of detached theorems, but now-a-days it has reached a point in which every properly developed algebraical composition, like a skilful landscape, is expected to suggest the notion of an infinite distance lying beyond the limits of the canvas.

It is quite conceivable that the results we have been investigating may be descended upon from a higher and more general point of view. Many circumstances point to such a consummation being probable. But man must creep before he can walk or run, and a house cannot be built downwards from the roof. I think the mere fact that our work enables us to simplify and extend the results obtained by so splendid a genius as M. Halphen, is sufficient to convey to us the assurance that we have not been beating the wind or chasing a phantom, but doing solid work. Let me instance one single point: M. Halphen has succeeded, by a prodigious effort of ingenuity, in obtaining the differential equation to a cubic curve with a given absolute invariant. His method involves the integration of a complicated differential equation. In the method which I employ the same result is obtained by a simple act of substitution in an exceedingly simple special form of Aronhold's S and T, capable of being executed in the course of a few minutes on half a sheet of paper, without performing any integration whatever. This will be seen to be a simple inference from the theorem invoked under three names, to which allusion has been made in a preceding lecture and the demonstration of which will shortly occupy our attention.

Before entering upon the theory of Differential Invariants, I think it desirable to bring forward the exceedingly valuable and interesting communication with which I have been favoured by M. Halphen establishing *à priori* the existence of *invariants* in general.

Sur l'Existence des Invariants.

(Extracted from a Letter of M. Halphen to Professor Sylvester.)

Dans des théories diverses on a rencontré des Invariants sans qu'on ait pénétré la cause générale de leur existence. C'est cette lacune qu'il s'agit ici de faire disparaître.

1. Soient $A, B, \ldots, L$ des quantités auxquelles on puisse attribuer des valeurs *ad libitum*.

Une *substitution* consiste à remplacer ces quantités $(A, B, \ldots, L)$ par d'autres $(a, b, \ldots, l)$.

Les substitutions, que l'on doit considérer ici, sont définies par des relations algébriques, de forme supposée donnée, mais contenant des *paramètres* arbitraires $p, q, \ldots$.
$$\left.\begin{array}{l} a = f(A, B, \ldots, L;\ p, q, \ldots) \\ b = f_1(A, B, \ldots, L;\ p, q, \ldots) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\}. \qquad (1)$$

Soit maintenant une seconde substitution, de même espèce, mais avec d'autres paramètres $\pi, \chi, \ldots$, et donnant lieu à $(\alpha, \beta, \ldots, \lambda)$, en sorte qu'on ait
$$\left.\begin{array}{l} \alpha = f(A, B, \ldots, L;\ \pi, \chi, \ldots) \\ \beta = f_1(A, B, \ldots, L;\ \pi, \chi, \ldots) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\}. \qquad (1\ \text{bis})$$

2. Définition. Les substitutions dont il s'agit forment un groupe, si, quels que soient les paramètres $p, q, \ldots, \pi, \chi, \ldots$, ainsi que $A, B, \ldots, L$, il existe des quantités $P, Q, \ldots$ vérifiant les égalités semblables

$$\left.\begin{array}{l} \alpha = f\,(a, b, \ldots, l;\ P, Q, \ldots) \\ \beta = f_1(a, b, \ldots, l;\ P, Q, \ldots) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\}. \qquad (1\ \text{ter})$$

Les invariants sont l'apanage exclusif des substitutions formant groupe. On va le montrer. Mais auparavant, pour éviter toute confusion, on doit faire une remarque sur la définition.

3. Dans les diverses théories où l'on a rencontré des Invariants, les substitutions forment groupe, en effet, suivant cette définition; mais il s'y rencontre encore une circonstance particulière de plus, c'est que les paramètres $P, Q, \ldots$ de la substitution composée (1 ter) dépendent uniquement des paramètres $p, q, \ldots, \pi, \chi, \ldots$ des substitutions composantes (1) et (1 bis). Cette propriété *n'est pas nécessaire* à l'existence des Invariants, et nous ne la supposerons pas ici. Il sera donc entendu que $P, Q, \ldots$ peuvent dépendre, non seulement de $p, q, \ldots, \pi, \chi, \ldots$, mais aussi de $A, B, \ldots, L$.

Exemples:

I. $a = Ap^2,\quad b = Apq + Bp,\quad c = Aq^2 + 2Bq + C;$

$\alpha = A\pi^2,\quad \beta = A\pi\chi + B\pi,\quad \gamma = A\chi^2 + 2B\chi + C;$

$\alpha = aP^2,\quad \beta = aPQ + bP,\quad \gamma = aQ^2 + 2bQ + c;$

$$P = \frac{\pi}{p},\quad Q = \frac{\chi - q}{p}.$$

P et Q ne dépendent pas de A, B, C.

II. $a = A^3p^2,\quad b = A^2pq + ABp,\quad c = Aq^2 + 2Bq + C;$

$\alpha = A^3\pi^2,\quad \beta = A^2\pi\chi + AB\pi,\quad \gamma = A\chi^2 + 2B\chi + C;$

$\alpha = a^3P^2,\quad \beta = a^2PQ + abP,\quad \gamma = aQ^2 + 2bQ + c;$

$$P = \frac{\pi}{A^2p^2},\quad Q = \frac{\chi - q}{Ap}.$$

P et Q dépendent de A.

Dans ces deux exemples, il y a un invariant absolu, $\dfrac{B^2 - AC}{A}$.

4. Dans la substitution (1) nous supposerons que le nombre des paramètres soit inférieur au nombre des quantités $A, B, \ldots, L$.

Soient ainsi m le nombre des paramètres $p, q, \ldots$,

n le nombre des quantités $A, B, \ldots, L$,

on suppose $$m < n.$$

Cela étant, on peut éliminer les paramètres entre les équations (1), et il reste $(n - m)$ équations

$$\left.\begin{array}{l} F\,(a, b, \ldots, l;\ A, B, \ldots, L) = 0 \\ F_1(a, b, \ldots, l;\ A, B, \ldots, L) = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\}. \qquad (2)$$

Théorème: *Si les substitutions considérées forment* groupe, *les $(n - m)$ équations* (2) *peuvent être mises sous la forme*

$$\left.\begin{array}{l} \Phi\,(a, b, \ldots, l) = \Phi\,(A, B, \ldots, L) \\ \Phi_1(a, b, \ldots, l) = \Phi_1(A, B, \ldots, L) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\}, \qquad (3)$$

en d'autres termes, *il y a $(n - m)$ invariants absolus.*

Réciproquement, s'il y a $(n - m)$ invariants absolus (distincts), *les substitutions forment groupe.*

5. Démonstration. Prouvons d'abord la seconde partie, ou réciproque. Voici l'hypothèse: des équations (1), par élimination de $p, q, \ldots$ resultent les équations (3).

Par conséquent, $A, B, \ldots, L$ et $a, b, \ldots, l$ étant quelconques, mais satisfaisant aux équations (3), on peut déterminer p, q, au moyen des équations (1).

Soient $A, B, \ldots, L, p, q, \ldots, \pi, \chi, \ldots$ pris arbitrairement, et $a, b, \ldots, l$, $\alpha, \beta, \ldots, \lambda$ déterminés par (1) et (1 bis). Suivant l'hypothèse, on a

$$\Phi(a, b, \ldots, l) = \Phi(A, B, \ldots, L) \quad \text{et} \quad \Phi(\alpha, \beta, \ldots, \lambda) = \Phi(A, B, \ldots, L);$$

donc $$\Phi(a, b, \ldots, l) = \Phi(\alpha, \beta, \ldots, \lambda), \text{ etc.}$$

Donc on peut déterminer $P, Q, \ldots$ par les équations (1 ter), ce qu'il fallait démontrer.

Démontrons maintenant la première partie, ou théorème direct. Par hypothèse, $A, B, \ldots, L, p, q, \ldots, \pi, \chi, \ldots$ étant pris à volonté et $a, b, \ldots, l$, $\alpha, \beta, \ldots, \lambda$ déterminés au moyen de (1) et (1 bis), il en résulte les relations (1 ter).

Des équations (1) résulte le système (2); de même, de (1 bis) et de (1 ter) resultent

$$\left.\begin{array}{l} F\,(\alpha, \beta, \ldots, \lambda;\ A, B, \ldots, L) = 0 \\ F_1(\alpha, \beta, \ldots, \lambda;\ A, B, \ldots, L) = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\}, \qquad (2\ \text{bis})$$

$$\left.\begin{array}{l} F\,(\alpha, \beta, \ldots, \lambda;\ a, b, \ldots, l)\ \ = 0 \\ F_1(\alpha, \beta, \ldots, \lambda;\ a, b, \ldots, l)\ \ = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\}. \qquad (2\ \text{ter})$$

Je dis que le système (2 ter) résulte de (2) et de (2 bis).

En effet, $a, b, \ldots, l$ et $\alpha, \beta, \ldots, \lambda$ n'étant definis que par (1) et (1 bis), le système (2 ter) résulte de (1) et de (1 bis) par l'élimination de $p, q, \ldots, \pi, \chi, \ldots$ et $A, B, \ldots, L$. Mais l'élimination de $p, q, \ldots$ remplace le système (1) par le système (2), celle de $\pi, \chi, \ldots$ remplace le système (1 bis) par (2 bis); donc (2 ter) résulte de l'élimination de $A, B, \ldots, L$ entre (2) et (2 bis).

Le système (2), (2 bis) est formé par $2(n-m)$ équations, et cependant l'élimination de n lettres $A, B, \ldots, L$, au lieu de donner $(n-2m)$ équations, en donne $(n-m)$, les équations (2 ter). Si donc on élimine seulement $(n-m)$ lettres $A, B, \ldots, G$, les m autres $H, \ldots, L$ disparaîtront d'elles-mêmes. Tirons $A, B, \ldots, G$ des équations (2), et nous aurons

$$A = \Psi\,(a, b, \ldots, l;\ H, \ldots, L),$$
$$B = \Psi_1(a, b, \ldots, l;\ H, \ldots, L),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Tirons de même $A, B, \ldots, G$ des équations (2 bis), et nous aurons

$$A = \Psi\,(\alpha, \beta, \ldots, \lambda;\ H, \ldots, L),$$
$$B = \Psi_1(\alpha, \beta, \ldots, \lambda;\ H, \ldots, L),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Le résultat de l'élimination est donc représenté par $(n-m)$ équations telles que

$$\left.\begin{array}{l}\Psi\,(a, b, \ldots, l;\ H, \ldots, L) = \Psi\,(\alpha, \beta, \ldots, \lambda;\ H, \ldots, L)\\ \Psi_1(a, b, \ldots, l;\ H, \ldots, L) = \Psi_1(\alpha, \beta, \ldots, \lambda;\ H, \ldots, L)\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\end{array}\right\}, \qquad (4)$$

et l'on sait que $H, \ldots, L$ disparaissent, d'eux-mêmes, de ces équations.

En assignant donc à $H, \ldots, L$ des valeurs numériques à volonté, on voit donc bien que les équations résultantes, équivalentes à (2 ter), ont la forme

$$\Phi\,(a, b, \ldots, l) = \Phi\,(\alpha, \beta, \ldots, \lambda),$$
$$\Phi_1(a, b, \ldots, l) = \Phi_1(\alpha, \beta, \ldots, \lambda),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

C'est ce qu'il fallait démontrer.

6. Remarques. Si les équations (4) sont rationnelles, la disparition de $H, \ldots, L$ exige que Ψ ait la forme suivante

$$\Psi = \Phi(a, b, \ldots, l)\,\Theta(H, \ldots, L) + \theta(H, \ldots, L),$$

et de même pour Ψ_1, etc. Sous cette forme, on voit que Θ et θ disparaissent dans les équations (4), et l'invariant résultant est Φ.

Mais, si les équations (4) sont irrationnelles, la disparition de $H, \ldots, L$ peut n'être pas immédiate. En assignant à $H, \ldots, L$ des valeurs numériques à volonté, comme on l'a dit dans la démonstration, c'est-à-dire en considérant $H, \ldots, L$ comme des *constantes arbitraires*, on voit les invariants se présenter

avec des constantes arbitraires. Ceci ne doit pas étonner, puisqu'il s'agit ici d'invariants *absolus*, que l'on peut effectivement modifier en leur ajoutant des constantes arbitraires ou en les multipliant par des constantes arbitraires, sans troubler la propriété d'invariance.

L'analyse employée dans la démonstration fournit un moyen régulier de former les invariants; ce moyen consiste à éliminer les paramètres dans les équations (1), puis à résoudre par rapport à $(n-m)$ quantités $A, B, \ldots, G$. Mais, les substitutions formant groupe, on peut aussi résoudre par rapport à $a, b, \ldots, g$, en éliminant les paramètres.

Exemple: $a = Ap^2,\ b = Apq + Bp,\ c = Aq^2 + 2Bq + C.$

En résolvant par rapport à c, c'est-à-dire en tirant p, q des deux premières, on obtient

$$c = A\left(\frac{b - Bp}{Ap}\right)^2 + 2B\frac{b - Bp}{Ap} + C = \frac{b^2}{Ap^2} + C - \frac{B^2}{A} = \frac{b^2}{a} + C - \frac{B^2}{A}.$$

Voici l'invariant $C - \dfrac{B^2}{A}$.

En résolvant par rapport à b, on trouve $b = \sqrt{a}\sqrt{\left(\dfrac{B^2 - AC}{A}\right)} + c$, ce qui donne l'invariant $\dfrac{B^2 - AC}{A} + c$, où c est une constante arbitraire.

LECTURE XXII.

E pur si muove.

The theory still moves on. We have now emerged from the narrows and are entering on the mid-ocean of Differential Invariants, or of Principiants, as I have called them. These, it will now be seen, are perfectly defined by their property of being at one and the same time invariants and pure reciprocants. In other words, if P be a Principiant, it has both Ω and V for its annihilators. Thus, for example, the Mongian

$$A = a^2d - 3abc + 2b^3$$

is necessarily a Principiant. For

$$\Omega A = (a\partial_b + 2b\partial_c + 3c\partial_d)(a^2d - 3abc + 2b^3) = 0,$$

and at the same time

$$VA = \{2a^2\partial_b + 5ab\partial_c + (6ac + 3b^2)\,\partial_d\}\,(a^2d - 3abc + 2b^3) = 0.$$

Among Pure Reciprocants, those only are entitled to rank as Principiants whose form is persistent (merely taking up an extraneous factor, but otherwise unchanged) under the most general homographic substitution (see

Lecture XIII. [pp. 379, 382 above]. We have therefore to show that such reciprocants and no others are subject to annihilation by Ω.

With this end in view, let us consider the effect of substituting $\frac{x}{1+hx}$ for x and $\frac{y}{1+hx}$ for y in any rational integral function of y and its derivatives with respect to x. Suppose that, in consequence of this substitution, the function

$$F(y, y_1, y_2, y_3, \dots y_n)$$

becomes changed into

$$F_1(x, y, y_1, y_2, y_3, \dots y_n);$$

then the transformed function will be

$$F(Y, Y_1, Y_2, Y_3, \dots Y_n),$$

where $X = \frac{x}{1+hx}$, $Y = \frac{y}{1+hx}$, and $Y_1, Y_2, Y_3, \dots Y_n$ are the successive derivatives of Y with respect to X.

If, for the moment, we agree to consider h as an infinitesimal (we shall afterwards give it a finite value), neglecting squares and higher powers of h, we may write

$$X = x - hx^2,$$
$$Y = y - hxy.$$

Hence, by n successive differentiations of Y with respect to X, neglecting squares of h whenever they occur, we deduce

$$Y_1 = y_1 + hxy_1 - hy,$$
$$Y_2 = y_2 + 3hxy_2,$$
$$Y_3 = y_3 + 5hxy_3 + 3hy_2,$$
$$Y_4 = y_4 + 7hxy_4 + 8hy_3,$$
$$Y_5 = y_5 + 9hxy_5 + 15hy_4,$$
$$\dots\dots\dots\dots\dots\dots$$
$$Y_{n-1} = y_{n-1} + (2n-3)\,hxy_{n-1} + (n-1)(n-3)\,hy_{n-2},$$
$$Y_n = y_n + (2n-1)\,hxy_n + n(n-2)\,hy_{n-1}.$$

The last of these, for instance, is obtained as follows:

We have $$Y_n = \frac{dY_{n-1}}{dX}.$$

But $$\frac{d}{dX} = \frac{1}{1-2hx}\cdot\frac{d}{dx} = (1+2hx)\frac{d}{dx},$$

and $$\frac{dY_{n-1}}{dx} = \frac{d}{dx}\{y_{n-1} + (2n-3)\,hxy_{n-1} + (n-1)(n-3)\,hy_{n-2}\}$$
$$= y_n + (2n-3)\,hxy_n + n(n-2)\,hy_{n-1}.$$

Consequently, $$Y_n = (1+2hx)\frac{dY_{n-1}}{dx}$$
$$= (1+2hx)\{y_n + (2n-3)\,hxy_n + n(n-2)\,hy_{n-1}\}$$
$$= y_n + (2n-1)\,hxy_n + n(n-2)\,hy_{n-1}.$$

On substituting the above values of $Y, Y_1, Y_2, \dots Y_n$ in the transformed function, we find immediately

$$F(Y, Y_1, Y_2, \dots Y_n) = (1 + hx\nu + h\Theta)\,F(y, y_1, y_2, \dots y_n),$$

where ν and Θ are the partial differential operators

$$\nu = -y\partial_y + y_1\partial_{y_1} + 3y_2\partial_{y_2} + 5y_3\partial_{y_3} + 7y_4\partial_{y_4} + \dots,$$
$$\Theta = -y\partial_{y_1} + 3y_2\partial_{y_3} + 8y_3\partial_{y_4} + 15y_4\partial_{y_5} + \dots + n(n-2)\,y_{n-1}\partial_{y_n}.$$

Changing to our usual notation, we write

$$y_1 = t,\ y_2 = 2a,\ y_3 = 2.3b,\ y_4 = 2.3.4c, \dots,$$

and then if F_1 is what F (a rational integral function of $a, b, c, \dots$) becomes when we substitute $\frac{x}{1+hx}$, $\frac{y}{1+hx}$ for x, y (regarding h as *infinitesimal*), we have

$$F_1 = (1 + hx\nu + h\Theta)\,F,$$

where $$\nu = -y\partial_y + t\partial_t + 3a\partial_a + 5b\partial_b + 7c\partial_c + 9d\partial_d + \dots,$$

and $$\Theta = -y\partial_t + a\partial_b + 2b\partial_c + 3c\partial_d + 4d\partial_e + \dots.$$

In general ν is merely the partial differential operator written above; but when its subject, F, is homogeneous, of degree i, and isobaric, of weight w, in

the letters	y,	t,	a,	b,	c,	d,	… supposed to be
of degrees	1,	1,	1,	1,	1,	1,	…
and of weights	-2,	-1,	0,	1,	2,	3,	…,

its operation is equivalent to multiplication by the number $3i + 2w$. For in this case we have

$$y\partial_y + t\partial_t + a\partial_a + b\partial_b + c\partial_c + d\partial_d + \dots = i,$$

and $$-2y\partial_y - t\partial_t + b\partial_b + 2c\partial_c + 3d\partial_d + \dots = w;$$

so that we may regard ν as a number, simply writing

$$\nu = 3i + 2w$$

when we have occasion to do so.

We are now able to show that if F is a persistent form, we must necessarily have

$$\Theta F = 0.$$

For $$\frac{F_1}{F} = 1 + \nu hx + \frac{h\Theta F}{F};$$

and consequently, if F_1 is divisible by F (this is what is meant by saying that F is a persistent form), unless ΘF vanishes, $\frac{\Theta F}{F}$ must be a rational integral function of $y, t, a, b, c, \ldots$. But since the operation of Θ diminishes the weight by unity without altering the degree, $\frac{\Theta F}{F}$ must be of degree 0 and weight -1. The impossibility of the existence of such a function leads to the necessary conclusion that

$$\Theta F = 0.$$

Let us apply this result to the case of a pure reciprocant. We have

$$\Theta = -y\partial_t + a\partial_b + 2b\partial_c + 3c\partial_d + \ldots = -y\partial_t + \Omega.$$

Thus when F is a pure reciprocant, or indeed any function in which t does not appear, $y\partial_t F = 0$ and Θ reduces to Ω. We have therefore shown, in what precedes, that the condition

$$\Omega F = 0$$

is necessary to ensure the persistence of the form of F under a particular homographic substitution; *à fortiori*, this condition is also necessarily satisfied when the form of F is persistent under the most general homographic substitution $\left(\text{in which } x, y \text{ are changed into } \frac{lx + my + n}{l''x + m''y + n''}, \frac{l'x + m'y + n'}{l''x + m''y + n''}\right)$.

The satisfaction of $\Omega F = 0$ is of itself inadequate to ensure persistence under the general homographic substitution; the necessary and sufficient condition of pure reciprocants

$$VF = 0$$

must also be satisfied. This follows from the fact that the general linear substitution, for which all pure reciprocants are persistent, is merely a particular case of the most general homographic substitution.

It only remains to be proved that the two conditions $VF = 0$, $\Omega F = 0$, taken conjointly, are sufficient as well as necessary.

In what follows I use a method which may be termed that of composition of variations. Its nature and value will be better understood if I first apply it to the rigorous demonstration of the theorem that the substitution of $x + hy$ for x in the Quantic

$$(a, b, c, \ldots \between x, y)^n$$

changes any function whatever of its coefficients, say

$$F(a, b, c, \ldots), \text{ into } e^{h\Omega} F(a, b, c, \ldots).$$

This is not proved, but only verified up to terms of the second order of differentiation, in Salmon's *Modern Higher Algebra* (3rd ed. 1876, p. 59). Remembering that, whatever the order n of the Quantic may be, the changed values of the coefficients $a, b, c, d, \ldots$ are

$$\begin{aligned} a' &= a, \\ b' &= b + ah, \\ c' &= c + 2bh + ah^2, \\ d' &= d + 3ch + 3bh^2 + ah^3, \\ &\ldots\ldots\ldots\ldots\ldots\ldots, \end{aligned}$$

what we have to prove is that, for all values of h,

$$F(a', b', c', d', \ldots) = e^{h\Omega} F(a, b, c, d, \ldots).$$

In other words, if for brevity we write

$$F(a, b, c, \ldots) = F,$$

and

$$F(a', b', c', \ldots) = F_1,$$

it is required to show that

$$F_1 = F + h\Omega F + \frac{h^2}{1.2}\Omega^2 F + \frac{h^3}{1.2.3}\Omega^3 F + \ldots,$$

where

$$\Omega = a\partial_b + 2b\partial_c + 3c\partial_d + \ldots.$$

When h is infinitesimal, it is obvious that

$$F_1 = F + h\Omega F.$$

Hence, when h has a general value, we may assume

$$F_1 = F + h\Omega F + \frac{h^2}{1.2}P + \frac{h^3}{1.2.3}Q + \frac{h^4}{1.2.3.4}R + \ldots.$$

Let h be increased by the infinitesimal quantity ϵ; then, considering this increase as resulting from a second substitution similar to the first, we see that F_1 becomes

$$F_1 + \epsilon\Omega F_1.$$

But it also becomes

$$\begin{aligned} F + (h+\epsilon)\Omega F + \frac{(h+\epsilon)^2}{1.2}P + \frac{(h+\epsilon)^3}{1.2.3}Q + \ldots &= F_1 + \epsilon\frac{dF_1}{dh} \\ &= F_1 + \epsilon\left(\Omega F + hP + \frac{h^2}{1.2}Q + \frac{h^3}{1.2.3}R + \ldots\right). \end{aligned}$$

Equating this to $F_1 + \epsilon\Omega F_1$, we obtain

$$\Omega F_1 = \Omega F + hP + \frac{h^2}{1.2}Q + \frac{h^3}{1.2.3}R + \ldots.$$

But

$$\Omega F_1 = \Omega\left(F + h\Omega F + \frac{h^2}{1.2}P + \frac{h^3}{1.2.3}Q + \ldots\right).$$

The comparison of these two expressions gives

$$P = \Omega^2 F,$$
$$Q = \Omega P = \Omega^3 F,$$
$$R = \Omega Q = \Omega^4 F,$$
$$\cdots\cdots\cdots\cdots\cdots$$

Substituting these values in the assumed expansion for F_1, there results

$$F_1 = F + h\Omega F + \frac{h^2}{1\,.\,2}\Omega^2 F + \frac{h^3}{1\,.\,2\,.\,3}\Omega^3 F + \ldots,$$

which is the expanded form of

$$F_1 = e^{h\Omega} F.$$

A similar method of procedure will enable us to establish the corresponding but more elaborate formula

$$F_1 = (1 + hx)^\nu e^{\frac{h\Theta}{1+hx}} F,$$

in which F is any *homogeneous* and *isobaric* function* of degree i and weight w in y and its modified derivatives $(t, a, b, c, \ldots)$ with respect to x; the operator $\Theta = -y\partial_t + a\partial_b + 2b\partial_c + 3c\partial_d + \ldots$; the function F_1 is what F becomes in consequence of the substitution of $\frac{x}{1+hx}$, $\frac{y}{1+hx}$ for x, y; h is any finite quantity, and $\nu = 3i + 2w$.

Before giving the proof of this theorem, I will show that, upon the assumption of its truth, two inverse finite substitutions will, as they ought, nullify each other, leaving the function operated upon unaltered in form.

To avoid needless periphrasis, we call the substitution of $\frac{x}{1+hx}$, $\frac{y}{1+hx}$ for x, y the substitution h.

Either of the two substitutions, $h, -h$, reverses the effect of the other; for the substitution $-h$ turns

$$\frac{x}{1+hx} \text{ into } \frac{x}{1-hx} \div 1 + \frac{hx}{1-hx} = x,$$

and

$$\frac{y}{1+hx} \text{ into } \frac{y}{1-hx} \div 1 + \frac{hx}{1-hx} = y.$$

The two substitutions $h, -h$, performed successively on F, ought therefore to leave its value unaltered. But by hypothesis the substitution h converts F into F_1; consequently the substitution $-h$ performed on F_1 ought to change it back again into F.

* F need not be integral or even rational; whenever it is homogeneous or isobaric, ν will be a number.

It must be carefully observed that (since the operation of Θ decreases the weight by unity, leaving the degree unchanged) the weight of $\Theta^\kappa F$ is κ units lower than that of F, whilst the degree is the same for both.

Thus for F we have $3i + 2w = \nu$,

and for $\Theta^\kappa F$ $\quad 3i + 2(w - \kappa) = \nu - 2\kappa$.

Hence the substitution $-h$, which changes

$$F \text{ into } (1 - hx)^\nu e^{-\frac{h\Theta}{1-hx}} F,$$

also changes

$$\Theta F \;\;\text{,,}\;\; (1 - hx)^{\nu-2} e^{-\frac{h\Theta}{1-hx}} \Theta F,$$
$$\Theta^2 F \;\;\text{,,}\;\; (1 - hx)^{\nu-4} e^{-\frac{h\Theta}{1-hx}} \Theta^2 F,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

and in general

$$\Theta^\kappa F \text{ into } (1 - hx)^{\nu-2\kappa} e^{-\frac{h\Theta}{1-hx}} \Theta^\kappa F.$$

Moreover, $1 + hx$ becomes $1 + \frac{hx}{1-hx} = (1 - hx)^{-1}$, so that

$$(1 + hx)^{\nu-\kappa}\Theta^\kappa F \text{ becomes } (1 - hx)^{-(\nu-\kappa)}(1 - hx)^{\nu-2\kappa} e^{-\frac{h\Theta}{1-hx}} \Theta^\kappa F$$
$$= (1 - hx)^{-\kappa} e^{-\frac{h\Theta}{1-hx}} \Theta^\kappa F$$
$$= e^{-\frac{h\Theta}{1-hx}} (1 - hx)^{-\kappa} \Theta^\kappa F \text{ (since } \Theta \text{ does not act on } x).$$

Consequently,

$$(1 + hx)^\nu F \quad \text{becomes} \quad e^{-\frac{h\Theta}{1-hx}} F,$$
$$(1 + hx)^{\nu-1}\Theta F \quad \text{,,} \quad e^{-\frac{h\Theta}{1-hx}} (1 - hx)^{-1}\Theta F,$$
$$(1 + hx)^{\nu-2}\Theta^2 F \quad \text{,,} \quad e^{-\frac{h\Theta}{1-hx}} (1 - hx)^{-2}\Theta^2 F,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

And since, by the formula to be verified,

$$F_1 = (1 + hx)^\nu F + h(1 + hx)^{\nu-1}\Theta F + \frac{h^2}{1\,.\,2}(1 + hx)^{\nu-2}\Theta^2 F + \ldots,$$

$$F_1 \text{ becomes } e^{-\frac{h\Theta}{1-hx}} \left\{1 + h(1 - hx)^{-1}\Theta + \frac{h^2}{1\,.\,2}(1 - hx)^{-2}\Theta^2 + \ldots\right\} F$$
$$= e^{-\frac{h\Theta}{1-hx}} e^{\frac{h\Theta}{1-hx}} F = F.$$

LECTURE XXIII.

We now proceed to show how the composition of variations can be made to furnish a strict proof of the formula

$$F_1 = (1 + hx)^\nu e^{\frac{h\Theta}{1+hx}} F,$$

which was set forth in the preceding lecture.

As before, calling the change of x, y into $\frac{x}{1+hx}, \frac{y}{1+hx}$, the substitution h, it is easy to see that the *product* of two substitutions, h, ϵ, is the substitution $h + \epsilon$. For

$$\frac{x}{1+hx} \div 1 + \epsilon \frac{x}{1+hx} = \frac{x}{1+(h+\epsilon)x},$$

$$\frac{y}{1+hx} \div 1 + \epsilon \frac{x}{1+hx} = \frac{y}{1+(h+\epsilon)x}.$$

This shows that if

F_1 is what F becomes on making the substitution h,

and F_2 ,, F_1 ,, ,, ,, ,, ϵ,

then F_2 ,, F ,, ,, ,, ,, $h + \epsilon$.

Thus we can find two expressions for F_2, the comparison of which will enable us to assign the coefficients of all the powers of h in the expanded values of F_1.

The first two terms of this expansion were obtained, in the preceding lecture, by treating h as an infinitesimal. We may therefore write

$$F_1 = F + h(\nu x + \Theta) F + \frac{h^2}{1.2} N_2 + \frac{h^3}{1.2.3} N_3 + \ldots.$$

Changing h into $h + \epsilon$, we deduce

$$F_2 = F + (h+\epsilon)(\nu x + \Theta) F + \frac{(h+\epsilon)^2}{1.2} N_2 + \frac{(h+\epsilon)^3}{1.2.3} N_3 + \ldots.$$

For greater simplicity, let ϵ be an infinitesimal, and write

$$\frac{F_2 - F_1}{\epsilon} = \Delta F_1.$$

Then $$\Delta F_1 = (\nu x + \Theta) F + hN_2 + \frac{h^2}{1.2} N_3 + \ldots.$$

Now look at each term in the expansion of F_1 and find its increment (that is, its Δ) when x, y undergo the substitution ϵ. We thus obtain

$$\Delta F_1 = \Delta F + h\Delta(\nu x + \Theta) F + \frac{h^2}{1.2} \Delta N_2 + \frac{h^3}{1.2.3} \Delta N_3 + \ldots.$$

Comparing these two values of ΔF_1, we find

$$N_2 = \Delta(\nu x + \Theta) F,$$
$$N_3 = \Delta N_2,$$
$$N_4 = \Delta N_3,$$
$$\ldots\ldots\ldots\ldots$$

and generally $$N_r = \Delta N_{r-1}.$$

These equations are sufficient to determine all the coefficients of F_1; it only remains to show how the operations Δ may be performed.

We have in fact

$$F_1 = F + h\Delta F + \frac{h^2}{1.2}\Delta^2 F + \frac{h^3}{1.2.3}\Delta^3 F + \ldots,$$

where $$\Delta F = (\nu x + \Theta) F.$$

But we must not from this rashly infer that

$$\Delta^n F = (\nu x + \Theta)^n F.$$

To do so would be tantamount to regarding ν as a constant number, whereas its value depends on the degree and weight of the subject of operation.

This will be clearly seen in the calculation which follows*. We first generalize the formula

$$\Delta F = (\nu x + \Theta) F$$

by making $\Theta^\kappa F$ the operand instead of F.

Then, since i is the degree and $w - \kappa$ the weight of $\Theta^\kappa F$, instead of

$$3i + 2w = \nu,$$

we have $$3i + 2(w - \kappa) = \nu - 2\kappa.$$

Thus, $$\Delta\Theta^\kappa F = \{(\nu - 2\kappa)x + \Theta\}\Theta^\kappa F.$$

Again, since $$\Delta x = \left(\frac{x}{1+\epsilon x} - x\right) \div \epsilon = -x^2,$$

we find

$$\Delta x^\lambda \Theta^\kappa F = \lambda x^{\lambda-1}\Theta^\kappa F . \Delta x + x^\lambda \Delta\Theta^\kappa F = -\lambda x^{\lambda+1}\Theta^\kappa F + x^\lambda\{(\nu - 2\kappa)x + \Theta\}\Theta^\kappa F.$$

Hence we obtain the general formula

$$\Delta x^\lambda \Theta^\kappa F = x^\lambda\{(\nu - 2\kappa - \lambda)x + \Theta\}\Theta^\kappa F,$$

* If our sole object were to show that $\Theta F = 0$ is a sufficient as well as necessary condition of the persistence of F, we might dispense with all further calculation. Thus it is obvious that, since $\Delta F = (\nu x + \Theta) F$, $\Delta^n F$ must be of the form $(x, \Theta)^n F$; for the dependence of ν on the degree-weight of the operand will not affect the *form* of Δ^n, but only its numerical coefficients. Hence we conclude that F_1 is of the form $\phi(x, \Theta) F$; and remembering that $\Theta^2 F = 0$, $\Theta^3 F = 0$, ... whenever $\Theta F = 0$, it is at once seen that not only (as was shown in the last lecture) must ΘF vanish when F is persistent under the substitution h, but, conversely, that when $\Theta F = 0$, the altered value of F contains the original value as a factor (the other factor being in this case a function of x only); that is, F is persistent.

by means of which we calculate in succession the values of $\Delta^2 F$, $\Delta^3 F$, ...
Thus,

$$\begin{aligned}\Delta^2 F &= \Delta(\nu x + \Theta) F \\ &= \nu \Delta x F + \Delta \Theta F \\ &= \nu x \{(\nu - 1) x + \Theta\} F + \{(\nu - 2) x + \Theta\} \Theta F \\ &= \{\nu(\nu-1) x^2 + 2(\nu - 1) x\Theta + \Theta^2\} F.\end{aligned}$$

Hence

$$\begin{aligned}\Delta^3 F &= \nu(\nu-1)\Delta x^2 F + 2(\nu-1)\Delta x \Theta F + \Delta\Theta^2 F \\ &= \nu(\nu-1) x^2 \{(\nu-2)x + \Theta\} F + 2(\nu-1) x \{(\nu-3) x + \Theta\} \Theta F \\ &\qquad + \{(\nu-4)x + \Theta\} \Theta^2 F \\ &= \{\nu(\nu-1)(\nu-2) x^3 + 3(\nu-1)(\nu-2) x^2\Theta + 3(\nu-2) x\Theta^2 + \Theta^3\} F.\end{aligned}$$

If $[\nu]^n$ is used to denote $\nu(\nu-1)(\nu-2)\ldots$ to n factors ($[\nu]^1$ will of course mean ν), we have shown that

$$\begin{aligned}\Delta F &= ([\nu]^1 x + \Theta) F, \\ \Delta^2 F &= ([\nu]^2 x^2 + 2[\nu-1]^1 x\Theta + \Theta^2) F, \\ \Delta^3 F &= ([\nu]^3 x^3 + 3[\nu-1]^2 x^2 \Theta + 3[\nu-1]^1 x\Theta^2 + \Theta^3) F,\end{aligned}$$

and by induction it may be proved that in general

$$\Delta^n F = \left\{[\nu]^n x^n + n[\nu-1]^{n-1} x^{n-1}\Theta + \frac{n(n-1)}{1.2}[\nu-2]^{n-2} x^{n-2}\Theta^2 + \ldots + \Theta^n\right\} F.$$

That the last term of this expression is $\Theta^n F$ is sufficiently obvious; what we wish to prove is that, when m is any positive integer less than n, the term in $\Delta^n F$ which involves Θ^m will be

$$\frac{n(n-1)\ldots(n-m+1)}{1.2.3\ldots m}[\nu-m]^{n-m} x^{n-m} \Theta^m F.$$

To find the term involving Θ^m in $\Delta^{n+1}F$, we need only consider the operation of Δ on two consecutive terms of $\Delta^n F$; none of the remaining terms will affect the result. Suppose, then, that

$$\Delta^n F = \ldots + p x^{n-m} \Theta^m F + q x^{n-m+1} \Theta^{m-1} F + \ldots.$$

Operating with Δ, we find

$$\begin{aligned}\Delta^{n+1} F &= \ldots + p\Delta x^{n-m}\Theta^m F + q\Delta x^{n-m+1}\Theta^{m-1} F + \ldots \\ &= \ldots + p x^{n-m}\{(\nu-n-m)x + \Theta\}\Theta^m F \\ &\qquad + q x^{n-m+1}\{(\nu-n-m+1)x + \Theta\}\Theta^{m-1}F + \ldots \\ &= \ldots + \{p(\nu-n-m) + q\} x^{n+1-m}\Theta^m F + \ldots.\end{aligned}$$

Now, assuming the general term of $\Delta^n F$ to be as written above, we have

$$p = \frac{n(n-1)\ldots(n-m+1)}{1.2.3\ldots m}[\nu-m]^{n-m},$$

$$q = \frac{n(n-1)\ldots(n-m+2)}{1.2.3\ldots(m-1)}[\nu-m+1]^{n-m+1};$$

so that

$$q = p\left\{\frac{m(\nu-m+1)}{n-m+1}\right\}.$$

Thus the general term of $\Delta^{n+1}F$ has for its numerical coefficient

$$\begin{aligned}p(\nu-n-m) + q &= p\left\{\frac{m(\nu-m+1) + (\nu-n-m)(n-m+1)}{n-m+1}\right\} \\ &= p\left\{\frac{(n+1)(\nu-n)}{n-m+1}\right\} = \frac{(n+1)n\ldots(n-m+2)}{1.2.3\ldots m}[\nu-m]^{n+1-m},\end{aligned}$$

which shows that the numerical coefficients in $\Delta^{n+1}F$ obey the same law as those in $\Delta^n F$; and as this law is true for $n = 1, 2, 3$, it is also true universally.

We have thus shown that the general term in $\Delta^n F$ is

$$\frac{n(n-1)\ldots(n-m+1)}{1.2.3\ldots m}[\nu-m]^{n-m} x^{n-m}\Theta^m F,$$

and, consequently, the corresponding general term in

$$\frac{h^n \Delta^n F}{1.2.3\ldots n} \text{ is } \frac{[\nu-m]^{n-m}}{1.2.3\ldots(n-m)} h^{n-m} x^{n-m} \cdot \frac{h^m \Theta^m F}{1.2.3\ldots m}.$$

Now, as we have already seen,

$$F_1 = \left(1 + h\Delta + \frac{h^2}{1.2}\Delta^2 + \frac{h^3}{1.2.3}\Delta^3 + \ldots\right) F,$$

which, by merely expressing the symbolic factor as a series of powers of Θ, may be transformed into

$$\begin{aligned}F_1 &= \left(1 + [\nu]^1 hx + \frac{[\nu]^2}{1.2} h^2 x^2 + \frac{[\nu]^3}{1.2.3} h^3 x^3 + \ldots\right) F \\ &\quad + \left(1 + [\nu-1]^1 hx + \frac{[\nu-1]^2}{1.2} h^2 x^2 + \frac{[\nu-1]^3}{1.2.3} h^3 x^3 + \ldots\right) h\Theta F \\ &\quad + \left(1 + [\nu-2]^1 hx + \frac{[\nu-2]^2}{1.2} h^2 x^2 + \frac{[\nu-2]^3}{1.2.3} h^3 x^3 + \ldots\right) \frac{h^2\Theta^2 F}{1.2} \\ &\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

where, remembering that $[\nu]^n$ stands for $\nu(\nu-1)(\nu-2)\ldots$ to n factors, it is evident that the functions of x which multiply F, $h\Theta F$, $\frac{h^2}{1.2}\Theta^2 F$, ... are all of them binomial expansions. Hence we immediately obtain

$$\begin{aligned}F_1 &= (1+hx)^\nu F + (1+hx)^{\nu-1} h\Theta F + (1+hx)^{\nu-2}\frac{h^2}{1.2}\Theta^2 F + \ldots \\ &= (1+hx)^\nu \left\{1 + (1+hx)^{-1} h\Theta + (1+hx)^{-2}\frac{h^2\Theta^2 F}{1.2} + \ldots\right\} F,\end{aligned}$$

and finally,

$$F_1 = (1+hx)^\nu e^{\frac{h\Theta}{1+hx}} F.$$

Mr Hammond has remarked that, with a slight modification, the foregoing demonstration will serve to establish the analogous theorem, that

$$F_1 = (1+ht)^{-\mu} e^{-\frac{hV_1}{1+ht}} F,$$

where, as before, F means any homogeneous and isobaric function of degree i and weight w in the letters $y, t, a, b, c, \ldots$; and F_1 is what F becomes when, leaving y unaltered, we change x into $x + hy$, where h is any finite quantity. Instead of the operator

$$\Theta = -y\partial_t + a\partial_b + 2b\partial_c + 3c\partial_d + \ldots = -y\partial_t + \Omega$$

we have $$-V_1 = yt\partial_y + t^2\partial_t - 2a^2\partial_b - 5ab\partial_c - \ldots = yt\partial_y + t^2\partial_t - V\text{ [*]};$$

and instead of $\nu = 3i + 2w$, a different number, $\mu = 3i + w$ (which I have called the characteristic), taken negatively.

If we suppose that

F_1 is what F becomes on changing x into $x + hy$,

and F_2 ,, F ,, ,, ,, x ,, $x + \epsilon y$,

then F_2 ,, F ,, ,, ,, x ,, $x + (h + \epsilon) y$.

Hence, if $$F_1 = F + hP + \frac{h^2}{1.2}Q + \frac{h^3}{1.2.3}R + \ldots,$$

we must have $$F_2 = F + (h + \epsilon) P + \frac{(h + \epsilon)^2}{1.2}Q + \frac{(h + \epsilon)^3}{1.2.3}R + \ldots$$

$$= F_1 + \epsilon\frac{dF_1}{dh} + \ldots.$$

Thus, if ϵ be regarded as infinitesimal, and we write

$$\frac{F_2 - F_1}{\epsilon} = \Delta F_1,$$

it follows that $$\Delta F_1 = P + hQ + \frac{h^2}{1.2}R + \ldots.$$

But, by the direct operation of Δ, we find

$$\Delta F_1 = \Delta F + h\Delta P + \frac{h^2}{1.2}\Delta Q + \ldots,$$

and, comparing these two values of ΔF_1,

$$P = \Delta F,$$
$$Q = \Delta P = \Delta^2 F,$$
$$R = \Delta Q = \Delta^3 F,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

Hence it follows that

$$F_1 = F + h\Delta F + \frac{h^2}{1.2}\Delta^2 F + \frac{h^3}{1.2.3}\Delta^3 F + \ldots.$$

[*] This theorem was stated without proof in Lecture VIII, where, through inadvertence, the term $yt\partial_y$ in the expression for V_1 was omitted [p. 352, above].

It remains to find the value of $\Delta^n F$. This can be effected by means of formulae given in Lecture VIII. [p. 350, above], where it is shown that

$$\Delta x = y,$$
$$\Delta y = 0,$$
$$\Delta t = -t^2,$$
$$\Delta a = -3at,$$
$$\Delta b = -4bt - 2a^2,$$
$$\Delta c = -5ct - 5ab,$$
$$\Delta d = -6dt - 6ac - 3b^2,$$
$$\Delta e = -7et - 7ad - 7bc,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

We now show that $$\Delta F = -(\mu t + V_1) F,$$

where $$V_1 = V - t^2\partial_t - yt\partial_y,$$

just as in the cognate theorem we had

$$\Delta F = (\nu x + \Theta) F.$$

Since F is a function of $y, t, a, b, c, \ldots$ without x, it is evident that

$$\Delta F = \frac{dF}{dy}\Delta y + \frac{dF}{dt}\Delta t + \ldots$$
$$= -t(t\partial_t + 3a\partial_a + 4b\partial_b + 5c\partial_c + \ldots) F$$
$$- \{2a^2\partial_b + 5ab\partial_c + (6ac + 3b^2)\partial_d + \ldots\} F,$$

where the part of ΔF which is independent of t is $-VF$.

Now, $$y\partial_y + t\partial_t + a\partial_a + b\partial_b + c\partial_c + \ldots = i$$

and $$-2y\partial_y - t\partial_t + b\partial_b + 2c\partial_c + \ldots = w;$$

so that $$t\partial_t + 3a\partial_a + 4b\partial_b + 5c\partial_c + \ldots = 3i + w - y\partial_y - t\partial_t.$$

Hence, writing $3i + w = \mu$,

$$\Delta F = -t(\mu - y\partial_y - t\partial_t) F - VF$$
$$= -(\mu t + V_1) F,$$

where $$V_1 = V - t^2\partial_t - yt\partial_y.$$

Observing that $V_1^\kappa F$ is of degree $i + \kappa$ and weight $w - \kappa$; since

$$3(i + \kappa) + (w - \kappa) = \mu + 2\kappa,$$

we see that $$\Delta V_1^\kappa F = -\{(\mu + 2\kappa) t + V_1\} V_1^\kappa F.$$

Again, $$\Delta t^\lambda V_1^\kappa F = \lambda t^{\lambda - 1} V_1^\kappa F . \Delta t + t^\lambda \Delta V_1^\kappa F$$
$$= -\lambda t^{\lambda + 1} V_1^\kappa F - t^\lambda \{(\mu + 2\kappa) t + V_1\} V_1^\kappa F.$$

We thus obtain the formula

$$\Delta t^\lambda V_1^\kappa F = -t^\lambda \{(\mu + \lambda + 2\kappa) t + V_1\} V_1^\kappa F, \qquad (1)$$

analogous to the one previously employed,

$$\Delta x^\lambda \Theta^\kappa F = x^\lambda \{(\nu - 2\kappa - \lambda) x + \Theta\} \Theta^\kappa F, \qquad (2)$$

The remainder of the work will be step for step the same for this as for the previous theorem. In fact, by using (1) just as we used (2), we shall deduce

$$F_1 = (1+ht)^{-\mu} e^{-\frac{hV_1}{1+ht}} F, \qquad (3)$$

just as we deduced the analogous formula

$$F_1 = (1+hx)^{\nu} e^{\frac{h\Theta}{1+hx}} F. \qquad (4)$$

The reason of this is obvious: by interchanging x and t, μ and $-\nu$, Θ and $-V_1$, we interchange the formulae (1) and (2), (3) and (4).

It may be well to observe that if we use S_h to denote a substitution of such a nature that

$$S_\epsilon S_h = S_{h+\epsilon},$$

and if (regarding ϵ as an infinitesimal) we write

$$\frac{S_\epsilon - 1}{\epsilon} = \Delta,$$

then in general $$S_h F = e^{h\Delta} F.$$

The proof of this proposition is virtually contained in what precedes.

LECTURE XXIV.

Whenever a rational integral function of $x, y, t, a, b, c, \ldots$ is persistent in form under the general linear substitution, it cannot contain explicitly either x, y or t, but must be a function of the remaining letters $a, b, c, \ldots$ (the successive modified derivatives, beginning with the second, of y with respect to x) alone.

For if, keeping y unaltered, we change x into $x+\alpha$, where α is any arbitrary constant which may be regarded as an infinitesimal, the derivatives $t, a, b, c, \ldots$ are not affected by this change, and consequently the function

$$F = F(x, y, t, a, b, c, \ldots) \text{ becomes } F + \alpha\frac{dF}{dx},$$

which cannot be divisible by F unless $\frac{dF}{dx} = 0$.

(The alternative hypothesis of $\frac{dF}{dx}$ being divisible by F is inadmissible, because F is a rational integral function.)

Hence F cannot contain x explicitly; and if we write $y+\beta$ for y, keeping x unchanged, we see, in like manner, that F cannot contain y explicitly.

Again, if in the function

$$F = F(t, a, b, c, \ldots)$$

we change x, y into $x+\alpha$, $y+\beta_{,}x+\beta$, the effect of this substitution will be to increase t by the arbitrary constant $\beta_{,}$, without altering any of the remaining derivatives $a, b, c, \ldots$.

Hence, in order that the form of F may still be persistent, we must have $\frac{dF}{dt} = 0$; the reasoning being just the same as that by which $\frac{dF}{dx}$ was seen to vanish. Thus, F does not contain t explicitly. Moreover, the function

$$F = F(a, b, c, \ldots)$$

must be both homogeneous and isobaric.

For the substitution of $\alpha_{,}x+\alpha$, $\beta_{,,}y+\beta_{,}x+\beta$ for x, y, respectively, will multiply the letters

$$a\ ,\quad b\ ,\quad c\ ,\quad d\ ,\ \ldots$$

by $$\beta_{,,}\alpha_{,}^{-2},\quad \beta_{,,}\alpha_{,}^{-3},\quad \beta_{,,}\alpha_{,}^{-4},\quad \beta_{,,}\alpha_{,}^{-5},\ \ldots.$$

Each term of F will therefore be multiplied by a positive power of $\beta_{,,}$ and a negative power of $\alpha_{,}$.

Let one of the terms of F be $a^{\lambda_0}b^{\lambda_1}c^{\lambda_2}d^{\lambda_3}\ldots$. It will be multiplied by

$$\beta_{,,}^{\lambda_0+\lambda_1+\lambda_2+\lambda_3+\ldots}\,\alpha_{,}^{-(2\lambda_0+3\lambda_1+4\lambda_2+5\lambda_3+\ldots)}.$$

In order that F may retain its form, this multiplier must be the same for every term of F, no matter what arbitrary values are assigned to $\alpha_{,}$ and $\beta_{,,}$. This can only happen when, for all terms of the function F, we have

$$\lambda_0+\lambda_1+\lambda_2+\lambda_3+\ldots = \text{const.}$$

and $$\lambda_1+2\lambda_2+3\lambda_3+\ldots = \text{const.},$$

that is, when F is homogeneous and isobaric.

We have thus proved that among all the rational integral functions of $x, y, t, a, b, c, \ldots$ the only ones persistent under the substitution of $\alpha+\alpha_{,}x$, $\beta+\beta_{,}x+\beta_{,,}y$ for x, y, respectively, are such as simultaneously satisfy the conditions of not explicitly containing x, y or t, and of being homogeneous and isobaric in the remaining letters $a, b, c, \ldots$.

If F, any function satisfying these conditions, merely acquires an extraneous factor when, leaving y unaltered, we change x into $x+hy$, the form of F will be persistent under the general linear substitution. For both $\alpha+\alpha_{,}(x+hy)$ and $\beta+\beta_{,}(x+hy)+\beta_{,,}y$ are general linear functions of $x, y, 1$.

Now, the change of x into $x+hy$ converts (as was shown in the preceding lecture) F into

$$F_1 = (1+ht)^{-\mu} e^{-\frac{hV_1}{1+ht}} F,$$

where $$V_1 = V - t^2\partial_t - yt\partial_y.$$

But, since neither y nor t occurs in F, we must have

$$\partial_y F = 0 \text{ and } \partial_t F = 0.$$

Consequently, $$V_1 F = VF, \quad V_1^2 F = V^2 F,$$

and so on. Hence

$$F_1 = (1+ht)^{-\mu} e^{-\frac{hV}{1+ht}} F$$
$$= (1+ht)^{-\mu} F - (1+ht)^{-\mu-1} hVF + (1+ht)^{-\mu-2} \frac{h^2 V^2}{1 \cdot 2} F - \dots .$$

Unless VF, V^2F, V^3F, ... all of them vanish, F_1 cannot contain F as a factor. If it could, VF, V^2F, ... would all have to be divisible by F. But this is impossible; for VF, a rational integral function of $a, b, c, \dots$ whose weight is $w-1$, cannot be divisible by F, a rational integral function of weight w.

We must therefore have

$$VF = 0$$

(which implies $V^2F = 0$, etc.) as the necessary and sufficient condition of the persistence of the form of F under the general linear substitution. In other words, F must be a pure reciprocant.

In order that F may also be persistent in form under the general homographic substitution, it must (besides being a pure reciprocant) be subject to annihilation by the operator

$$\Omega = a\partial_b + 2b\partial_c + 3c\partial_d + \dots .$$

For it was seen, in the preceding lecture, that the special homographic substitution in which $\frac{x}{1+hx}$, $\frac{y}{1+hx}$ are written instead of x, y, respectively, has the effect of changing any homogeneous and isobaric function F into F_1, where

$$F_1 = (1+hx)^{\nu} e^{\frac{h\Theta}{1+hx}} F,$$
$$\Theta = \Omega - y\partial_t.$$

When the letter t does not occur in F, we may write $\partial_t F = 0$, so that Θ becomes simply Ω, and the above formula becomes

$$F_1 = (1+hx)^{\nu} e^{\frac{h\Omega}{1+hx}} F.$$

Hence it follows immediately that, when F is a rational integral function of the letters $a, b, c, \dots$, the condition $\Omega F = 0$ is sufficient as well as necessary to ensure the persistence of the form of F under the special homographic substitution we have employed.

But when F is a pure reciprocant it also satisfies the condition $VF = 0$, and it is the simultaneous satisfaction of $\Omega F = 0$ and $VF = 0$ that ensures

the persistence of the form of F under the most general homographic substitution. This may be shown by combining the substitution $\frac{x}{1+hx}$, $\frac{y}{1+hx}$ (for which F is persistent when, and only when, $\Omega F = 0$) with the general linear substitution (for which $VF = 0$ is the necessary and sufficient condition of the persistence of the form of F), so as to obtain the most general homographic substitution. Thus the linear substitution

$$\left.\begin{aligned} x &= lx_, + my_, + n \\ y &= l'x_, + m'y_, + n' \end{aligned}\right\},$$

when combined with

$$x_, = \frac{x_{,,}}{1+hx_{,,}}, \quad y_, = \frac{y_{,,}}{1+hx_{,,}},$$

gives the substitution

$$\left.\begin{aligned} x &= \frac{lx_{,,} + my_{,,} + n(1+hx_{,,})}{1+hx_{,,}} \\ y &= \frac{l'x_{,,} + m'y_{,,} + n'(1+hx_{,,})}{1+hx_{,,}} \end{aligned}\right\},$$

in which both the numerators are general linear functions.

By combining the substitution just obtained with the linear substitution

$$x_{,,} = \lambda x_{,,,} + \mu y_{,,,} + \nu, \quad y_{,,} = y_{,,,},$$

the denominator of each fraction is changed into a general linear function, and thus, by combining the special homographic substitution $\frac{x}{1+hx}$, $\frac{y}{1+hx}$ with two linear substitutions, we arrive at the most general homographic substitution.

This proves that the necessary and sufficient condition of F being a *homographically persistent* form is the coexistence of the two conditions

$$VF = 0, \quad \Omega F = 0.$$

Thus a Projective Reciprocant, or Principiant, or Differential Invariant, combines the natures of a Pure Reciprocant and Invariant *in respect of the elements*.

Notice that every Pure Reciprocant is an Invariant of the Reciprocal Function (that is, the numerator of the expression for $\frac{d^n x}{dy^n}$ in terms of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, ..., or what is the same in terms of the modified derivatives $t, a, b, \dots$), but the elements of such invariants are *not* the original simple elements, but more or less complicated functions of them.

What has just been stated is obvious from the fact that all invariants of the "reciprocal function" have been shown to be pure reciprocants (*vide** Lecture XIX.). The ordinary protomorph invariants of this function will

[* above, p. 412.]

have for their leading term a power of a multiplied by a single letter. Consequently, by reasoning previously employed in these lectures, every pure reciprocant will be a rational function of invariants of the Reciprocal Function divided by some power of a. Thus, for example, the Reciprocal Function

$$14a^4 - 21a^2bt + 3(2ac + b^2)t^2 - dt^3 = (\alpha, \beta, \gamma, \delta \between 1, -t)^3$$

if $$\alpha = 14a^4,\ \beta = 7a^2b,\ \gamma = 2ac + b^2,\ \delta = d.$$

The two protomorph invariants of this reciprocal function are

$$\alpha\gamma - \beta^2 = 7a^4(4ac - 5b^2)$$

and $$\alpha^2\delta - 3\alpha\beta\gamma + 2\beta^3 = 196a^6(a^2d - 3abc + 2b^3).$$

All other pure reciprocants of extent 3 may be rationally expressed in terms of a and the two protomorphs $4ac - 5b^2$, $a^2d - 3abc + 2b^3$; that is, all pure reciprocants of extent 3 are invariants of the reciprocal function of extent 3.

The reasoning employed can be applied with equal facility to the general case of extent n.

Instead of $\frac{x}{1+hx}$, $\frac{y}{1+hx}$, let us consider the special homographic substitution $\frac{1}{x}$, $\frac{y}{x}$ employed by M. Halphen.

Writing $$X = \frac{1}{x} \text{ and } Y = \frac{y}{x},$$

let $Y_1, Y_2, Y_3, \ldots$ denote the successive derivatives of Y with respect to X, and $y_1, y_2, y_3, \ldots$ those of y with respect to x. Then

$$Y = x^{-1}y,$$

$$Y_1 = -x\left(y_1 - \frac{1}{x}y\right),$$

$$Y_2 = x^3y_2,$$

$$Y_3 = -x^5\left(y_3 + \frac{3}{x}y_2\right),$$

$$Y_4 = x^7\left(y_4 + \frac{8}{x}y_3 + \frac{12}{x^2}y_2\right),$$

$$Y_5 = -x^9\left(y_5 + \frac{15}{x}y_4 + \frac{60}{x^2}y_3 + \frac{60}{x^3}y_2\right),$$

....

Hence, if $a, b, c, d, \ldots$ are the successive modified derivatives (beginning with the second) of y with respect to x, and $a', b', c', d', \ldots$ the corresponding

modified derivatives of Y with respect to X, it follows immediately that

$$a' = x^3a,$$

$$b' = -x^5\left(b + \frac{1}{x}a\right),$$

$$c' = x^7\left(c + \frac{2}{x}b + \frac{1}{x^2}a\right),$$

$$d' = -x^9\left(d + \frac{3}{x}c + \frac{3}{x^2}b + \frac{1}{x^3}a\right),$$

....

Attributing the weights 0, 1, 2, 3, ... to the letters $a, b, c, d, \ldots$, it is very easily seen that if F is any homogeneous and isobaric function of degree i and weight w,

$$F(a', b', c', \ldots) = (-)^w x^{3i+2w} F\left(a,\ b + \frac{1}{x}a,\ c + \frac{2}{x}b + \frac{1}{x^2}a,\ \ldots\right).$$

But we proved (in Lecture XXII.) [above, p. 429] that for all values of h

$$F(a, b + ah, c + 2bh + ah^2, \ldots) = e^{h\Omega}F(a, b, c, \ldots).$$

Hence, making $h = \frac{1}{x}$, we obtain

$$F(a', b', c', d', \ldots) = (-)^w x^{3i+2w} e^{\frac{\Omega}{x}} F(a, b, c, \ldots),$$

which proves that the satisfaction of

$$\Omega F(a, b, c, \ldots) = 0$$

is the necessary and sufficient condition for the persistence of the form of F under the Halphenian substitution $\frac{1}{x}$, $\frac{y}{x}$.

Similarly we might prove that $F(y, t, a, b, c, \ldots)$, which contains y and t, but not x, is changed by the substitution $\frac{1}{x}$, $\frac{y}{x}$ into

$$(-)^w x^\nu e^{\frac{\Theta}{x}} F(y, t, a, b, c, \ldots),$$

where $$\Theta = -y\partial_t + a\partial_b + 2b\partial_c + \ldots = \Omega - y\partial_t;$$

or we may deduce this result from the formula, demonstrated in the preceding lecture of this course,

$$F_1 = (1 + hx)^\nu e^{\frac{h\Theta}{1+hx}} F,$$

in which F_1 is what F becomes in consequence of the substitution $\frac{x}{1+hx}$, $\frac{y}{1+hx}$ impressed on the variables.

Let i be the degree and ω the weight measured by the sum of the orders of differentiation in each term of

$$F(y, t, a, b, c, \ldots).$$

If we measure the weight by the sum of the orders of differentiation of every term of F diminished by 2 units for each letter in the term, then

$$w = \omega - 2i \text{ and } 2\omega - i = 3i + 2w = \nu.$$

Let $\quad F(y, t, a, b, c, \ldots)$ become $F'(y, t, a, b, c, \ldots)$,

when we change

$$x \text{ into } qx + p \text{ and } y \text{ into } ry;$$

then $\quad F'(y, t, a, b, c, \ldots) = r^i q^{-\omega} F(y, t, a, b, c, \ldots).$

A further substitution $\dfrac{x}{1+hx}, \dfrac{y}{1+hx}$, impressed on the variables in F', will convert the original variables into

$$\frac{qx}{1+hx} + p \text{ and } \frac{ry}{1+hx},$$

that is, into $\quad \dfrac{p(1+hx)+qx}{1+hx}$ and $\dfrac{ry}{1+hx}$.

The function F' is at the same time changed into

$$r^i q^{-\omega} (1+hx)^\nu e^{\frac{h\Theta}{1+hx}} F(y, t, a, b, c, \ldots).$$

If now, in the above, we write $p = h$, $q = -h^2$, $r = h$, we shall have changed the original variables x, y into $\dfrac{h}{1+hx}, \dfrac{hy}{1+hx}$, and the original function F into

$$h^i(-h^2)^{-\omega}(1+hx)^\nu e^{\frac{h\Theta}{1+hx}} F = (-)^\omega h^{i-2\omega}(1+hx)^\nu e^{\frac{h\Theta}{1+hx}} F = (-)^w \left(\frac{1+hx}{h}\right)^\nu e^{\frac{h\Theta}{1+hx}} F.$$

Let h become infinite; then $\dfrac{h}{1+hx}, \dfrac{hy}{1+hx}$ and $(-)^w\left(\dfrac{1+hx}{h}\right)^\nu e^{\frac{h\Theta}{1+hx}} F$ become $\dfrac{1}{x}, \dfrac{y}{x}$ and $(-)^w x^\nu e^{\frac{\Theta}{x}} F$, showing that the substitution $\dfrac{1}{x}, \dfrac{y}{x}$ changes F into $(-)^w x^\nu e^{\frac{\Theta}{x}} F$.

LECTURE XXV.

In a letter to me dated June 14th, 1886, M. Halphen calls forms which are persistent under the substitution $\dfrac{1}{x}, \dfrac{y}{x}$, *Invariants d'homologie.* He uses the letters

$$a_0, a_1, a_2, a_3, \ldots a_n,$$

to denote y and its successive modified derivatives with respect to x; and, supposing them to become

$$A_0, A_1, A_2, A_3, \ldots A_n,$$

in consequence of the substitution $\dfrac{1}{x}, \dfrac{y}{x}$, gives, in the briefest possible manner, two very ingenious proofs of the formula

$$A_n = (-)^n x^{2n-1} \left\{ a_n + \frac{n-2}{1 \,.\, x} a_{n-1} + \frac{(n-2)(n-3)}{1 \,.\, 2 \,.\, x^2} a_{n-2} + \ldots \right\},$$

from which he deduces the theorem that the substitution in question changes any homogeneous and isobaric function f, of degree i and weight ω in

$$a_0, a_1, a_2, a_3, \ldots a_n,$$

into $$F = (-)^\omega x^{2\omega - i} e^{\frac{\Theta}{x}} f,$$

where Θ is the partial differential operator

$$-a_0\partial_{a_1} + a_2\partial_{a_3} + 2a_3\partial_{a_4} + \ldots + (n-2)a_{n-1}\partial_{a_n}.$$

I give the two proofs mentioned above in M. Halphen's own words, adding occasional footnotes, and making slight changes in the literation of his formulae when it seems desirable to do so.

Soient $\quad X = \dfrac{1}{x}, \; Y = \dfrac{y}{x}.$

Par une formule connue (Schlömilch, Compendium II.)

$$\frac{d^n y}{dX^n} = (-1)^n x^{n+1} \frac{d^n}{dx^n}(x^{n-1}y)^*$$

* An easy inductive proof of this may be obtained as follows:

Since $\quad \dfrac{d}{dX} = -x^2 \dfrac{d}{dx}$ we have $\dfrac{d^{\kappa+1}y}{dX^{\kappa+1}} = -x^2 \dfrac{d}{dx}\left(\dfrac{d^\kappa y}{dX^\kappa}\right)$.

Hence, assuming the truth of the formula when $n = \kappa$, we find

$$\begin{aligned}\frac{d^{\kappa+1}y}{dX^{\kappa+1}} &= (-)^{\kappa+1} x^2 \frac{d}{dx}\left\{x^{\kappa+1}\frac{d^\kappa}{dx^\kappa}(x^{\kappa-1}y)\right\}\\ &= (-)^{\kappa+1} x^2 \left\{x^{\kappa+1}\frac{d^{\kappa+1}}{dx^{\kappa+1}}(x^{\kappa-1}y) + (\kappa+1)x^\kappa \frac{d^\kappa}{dx^\kappa}(x^{\kappa-1}y)\right\}\\ &= (-)^{\kappa+1} x^{\kappa+2} \left\{x\frac{d^{\kappa+1}}{dx^{\kappa+1}}(x^{\kappa-1}y) + (\kappa+1)\frac{d^\kappa}{dx^\kappa}(x^{\kappa-1}y)\right\}\\ &= (-)^{\kappa+1} x^{\kappa+2} \frac{d^{\kappa+1}}{dx^{\kappa+1}}(x^\kappa y).\end{aligned}$$

Thus, if the formula is true for $n = \kappa$, it will be equally so when $n = \kappa + 1$. But it is obviously true when $n = 1$ $\left(\text{when it becomes } \dfrac{dy}{dX} = -x^2 \dfrac{dy}{dx}\right)$, and therefore holds universally.

et puisque $$Y = Xy,$$
il en resulte
$$\frac{d^nY}{dX^n} = X\frac{d^ny}{dX^n} + n\frac{d^{n-1}y}{dX^{n-1}} = (-1)^n x^n\left\{\frac{d^n}{dx^n}(x^{n-1}y) - n\frac{d^{n-1}}{dx^{n-1}}(x^{n-2}y)\right\}$$
$$= (-1)^n x^{2n-1}\left\{y_n + \frac{n(n-2)}{1\,.\,x}y_{n-1} + \frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,x^2}y_{n-2} + \dots\right\}^*.$$

Si l'on pose $$\frac{d^nY}{dX^n} = n!\,A_n,\quad y_n = n!\,a_n,$$

il vient
$$A_n = (-1)^n x^{2n-1}\left\{a_n + \frac{n-2}{1\,.\,x}a_{n-1} + \frac{(n-2)(n-3)}{1\,.\,2\,.\,x^2}a_{n-2} + \dots\right\}. \qquad \text{(I)}$$

Soit $$\Theta f = \Sigma\,(n-2)\,a_{n-1}\frac{\partial f}{\partial a_n}\,\dagger$$

on aura
$$\Theta a_n = (n-2)\,a_{n-1},$$
$$\Theta^2 a_n = (n-2)(n-3)\,a_{n-2},$$
$$\dots\dots\dots\dots\dots\dots\dots$$
$$A_n = (-1)^n x^{2n-1}\left\{a_n + \frac{1}{1\,.\,x}\Theta a_n + \frac{1}{1\,.\,2\,.\,x^2}\Theta^2 a_n + \dots\right\}.$$

Par conséquent, pour une fonction contenant $a_0, a_1, a_2, \dots$, de degré i et de poids ω, à chaque terme, on aura
$$F = (-1)^\omega x^{2\omega-i}\left\{f + \frac{1}{1\,.\,x}\Theta f + \frac{1}{1\,.\,2\,.\,x^2}\Theta^2 f + \dots\right\}\ddagger. \qquad \text{C.Q.F.D.}$$

* For, expanding by Leibnitz's Theorem,
$$\frac{d^n}{dx^n}(x^{n-1}y) - n\frac{d^{n-1}}{dx^{n-1}}(x^{n-2}y) = x^{n-1}y_n + n(n-1)x^{n-2}y_{n-1} + \frac{n(n-1)}{1\,.\,2}(n-1)(n-2)x^{n-3}y_{n-2} + \dots$$
$$- n\{x^{n-2}y_{n-1} + (n-1)(n-2)x^{n-3}y_{n-2} + \dots\}$$
$$= x^{n-1}y_n + n(n-2)x^{n-2}y_{n-1} + \frac{n(n-1)(n-2)(n-3)}{1\,.\,2}x^{n-3}y_{n-2} + \dots.$$

† The summation extending to all positive integral values of n, from 1 to ∞, so that
$$\Theta = -a_0\partial_{a_1} + a_2\partial_{a_3} + 2a_3\partial_{a_4} + 3a_4\partial_{a_5} + \dots.$$
Remembering that Halphen's $a_0, a_1, a_2, a_3, \dots$ have the same meaning as our $y, t, a, b, \dots$, this operator is $-y\partial_t + a\partial_b + 2b\partial_c + 3c\partial_d + \dots$ identical with the Θ used in previous lectures.

‡ We may show without much difficulty that, when $\Theta_1, \Theta_2, \Theta_3, \dots$ are each of them equivalent to Θ, but Θ_1 acts on u only, Θ_2 on v, Θ_3 on w, and so on, $\Theta uvw \dots = (\Theta_1 + \Theta_2 + \Theta_3 + \dots)\,uvw \dots$. From this it can be deduced that $\Theta^\kappa uvw \dots = (\Theta_1 + \Theta_2 + \Theta_3 + \dots)^\kappa\,uvw \dots$, when κ is any positive integer. Now let the number of the functions $u, v, w, \dots$ be i, and suppose that
$$u = a_n,\quad v = a_p,\quad w = a_q,\ \dots;$$
suppose, also, that the weight $n + p + q + \dots = \omega$. Then
$$A_nA_pA_q \dots = (-)^\omega x^{2\omega-i}\left(e^{\frac{\Theta}{x}}a_n\right)\left(e^{\frac{\Theta}{x}}a_p\right)\left(e^{\frac{\Theta}{x}}a_q\right)\dots = (-)^\omega x^{2\omega-i}e^{\frac{1}{x}(\Theta_1+\Theta_2+\Theta_3+\dots)}a_na_pa_q\dots$$
$$= (-)^\omega x^{2\omega-i}e^{\frac{\Theta}{x}}a_na_pa_q\dots$$
(for by what precedes $\Theta_1 + \Theta_2 + \Theta_3 + \dots$ may be replaced by Θ). Taking $a_na_pa_q\dots$ and $A_nA_pA_q\dots$ to be corresponding terms of f and F, we see at once that
$$F = (-)^\omega x^{2\omega-i}e^{\frac{\Theta}{x}}f.$$

Autre Demonstration de la Formule (I)*.

Si l'on change X et x en $X + H$ et $x + h$, on a
$$h = -\frac{H}{X(X+H)}.$$

Maintenant la formule
$$y = a_0 + ha_1 + h^2a_2 + \dots + h^na_n + \dots$$
écrite *symboliquement*†
$$y = \frac{1}{1 - ah}$$
devient $$y = \frac{X(X+H)}{X^2 + H(X+a)}.$$

D'ailleurs $$Y = (X + H)\,y;$$

donc *symboliquement*
$$Y = \frac{X(X+H)^2}{X^2 + H(X+a)}. \qquad \text{(II)}$$

Si l'on développe le second membre (II) suivant les puissances ascendants de H, le coefficient de H^n est A_n. Or ce développement est
$$Y = X\left\{1 + \left(1 - \frac{a}{X}\right)\frac{H}{X} + \left(\frac{a}{X}\right)^2\left(\frac{H}{X}\right)^2\right.$$
$$\left. + \dots + (-1)^n\left(\frac{H}{X}\right)^n\left(1 + \frac{a}{X}\right)^{n-2}\left(\frac{a}{X}\right)^2 + \dots\right\}$$

* If x becomes $x + h$ in consequence of the augmentation of X by an arbitrary quantity H, the increment of x will not be a constant, but will depend on X as well as on H. The value of h may be found at once by eliminating x between $X = \frac{1}{x}$ and $X + H = \frac{1}{x+h}$, when we obtain $X + H = \frac{X}{1 + hX}$, and consequently $h = -\frac{H}{X(X+H)}$.

This increase of X also changes y and Y (functions of x and X, whose original values were a_0 and A_0 before the augmentation of X took place) into
$$y = a_0 + ha_1 + h^2a_2 + \dots + h^na_n + \dots$$
and into $$Y = A_0 + HA_1 + H^2A_2 + \dots + H^nA_n + \dots.$$
These altered values of y and Y are the ones used in this second proof; the other letters retain their original signification.

† The word *symboliquement* indicates, whenever it is used, that powers of a are to be replaced by suffixes of corresponding value. For example, in the final result
$$A_n = (-)^n x^{2n-1}\left(a^n + \frac{n-2}{x}a^{n-1} + \dots\right)$$
is to be replaced by $$A_n = (-)^n x^{2n-1}\left(a_n + \frac{n-2}{x}a_{n-1} + \dots\right).$$

In our notation the final result is $A_{n+2} = (-)^n x^{2n+3}\left(a, b, c, d, \dots \between \frac{1}{x}, 1\right)^n$.

donc *symboliquement*

$$A_n = (-1)^n \frac{1}{X^{n+1}} \left(1 + \frac{a}{X}\right)^{n-2} a^2 = (-1)^n x^{2n-1} \left(a + \frac{1}{x}\right)^{n-2} a^2$$

ce qui est justement la formule (1).

We may regard the coefficients $a, b, c, \ldots$ of the ordinary binary Quantic in u, v,

$$(a, b, c, \ldots \between u, v)^n,$$

as the successive modified derivatives, beginning with the second, of a new variable y with respect to another new variable x.

Any invariant I of this Quantic will then retain its form unaltered, or at most merely acquire an extraneous factor, if

(1) leaving x, y, v unaltered we change u into $u + \lambda v$,

(2) „ u, v „ „ „ x, y „ $\dfrac{x}{1+hx}, \dfrac{y}{1+hx}$,

(3) „ u, v „ „ „ x, y „ $\dfrac{1}{x}, \dfrac{y}{x}$,

where λ and h are arbitrary constants.

For we have seen that these three substitutions will severally convert any homogeneous and isobaric function F, of degree i and weight w in the letters $a, b, c, \ldots$, into

$$e^{\lambda\Omega} F, \quad (1+hx)^{\nu} e^{\frac{h\Omega}{1+hx}} F, \text{ and } (-)^w x^{\nu} e^{\frac{\Omega}{x}} F,$$

where, in each case, $\Omega = a\partial_b + 2b\partial_c + 3c\partial_d + \ldots$, and $\nu = 3i + 2w$. From our point of view an invariant is defined as a homogeneous and isobaric solution of the equation

$$\Omega I = 0.$$

Hence the above substitutions convert the invariant I into

$$I, \quad (1+hx)^{\nu} I, \text{ and } (-)^w x^{\nu} I, \text{ respectively.}$$

An *absolute invariant* with respect to any substitution is one which, disregarding its sign, remains unchanged in absolute value by that substitution. Thus, any invariant for which

$$\nu = 3i + 2w = 0$$

is an absolute invariant with respect to each of the three substitutions here considered.

An invariant is of odd or even character with respect to any substitution according as its sign is or is not changed by that substitution. Thus, invariants are of odd or even character with respect to the substitution $\dfrac{1}{x}, \dfrac{y}{x}$ according as their *weights* are odd or even.

This corresponds to the theorem that the character (with respect to the interchange of x and y) of a pure reciprocant is odd or even according as its degree is odd or even [p. 316, above].

From any two invariants for which ν has the same value we can form an absolute invariant (that is, one for which $\nu = 0$) by taking their ratio, and then by differentiating the absolute invariant thus formed obtain another invariant.

Suppose I_1 to be an invariant of degree i_1 and weight w_1,

I_2 „ „ „ i_2 „ w_2,

and let $3i_1 + 2w_1 = \nu_1, \quad 3i_2 + 2w_2 = \nu_2;$

then the ν for $I_1^{\nu_2}$ is the same as that for $I_2^{\nu_1}$, and consequently $I_1^{\nu_2} I_2^{-\nu_1}$ is an absolute invariant.

We proceed to show that $\dfrac{d}{dx}(I_1^{\nu_2} I_2^{-\nu_1})$ is an invariant, though not an absolute one.

Using accents to denote differential derivation with respect to x, we have

$$\frac{d}{dx}(I_1^{\nu_2} I_2^{-\nu_1}) = I_1^{\nu_2 - 1} I_2^{-\nu_1 - 1} (\nu_2 I_1' I_2 - \nu_1 I_1 I_2').$$

If, then, we can prove that $\nu_2 I_1' I_2 - \nu_1 I_1 I_2'$ is an invariant, it will follow that $\dfrac{d}{dx}(I_1^{\nu_2} I_2^{-\nu_1})$ will be one also, and the proposition will be established. It may be very easily shown that this is the case by using Cayley's generators P and Q. For [p. 327, above], I being any invariant of degree i and weight w, PI and QI are also invariants where

$$P = a(b\partial_a + c\partial_b + d\partial_c + e\partial_d + \ldots) - ib,$$

and

$$Q = a(c\partial_b + 2d\partial_c + 3e\partial_d + \ldots) - 2wb.$$

Hence $(3P + Q)\, I$ is an invariant.

Now, since $3b\partial_a + 4c\partial_b + 5d\partial_c + \ldots = \dfrac{d}{dx}$,

and $3i + 2w = \nu$,

$$(3P + Q)\, I = a(3b\partial_a + 4c\partial_b + 5d\partial_c + \ldots)\, I - (3i + 2w)\, bI = aI' - \nu bI.$$

Consequently $aI_1' - \nu_1 bI_1$ and $aI_2' - \nu_2 bI_2$

are both of them invariants. Hence the combination

$$\nu_2 I_2 (aI_1' - \nu_1 bI_1) - \nu_1 I_1 (aI_2' - \nu_2 bI_2) = a(\nu_2 I_1' I_2 - \nu_1 I_1 I_2')$$

is also an invariant; that is

$$\nu_2 I_1' I_2 - \nu_1 I_1 I_2'$$

is one; which is the theorem to be demonstrated.

The invariant $aI' - \nu bI$, which we generated from I, is of degree $i+1$ and weight $w+1$; its ν is therefore the original ν increased by 5 units, three for the unit increase in the degree and two for the unit increase in the weight. Hence, on repeating the process of generation, we obtain the invariant

$$\left\{a\frac{d}{dx} - (\nu+5)\,b\right\}(aI' - \nu bI) = a^2I'' - 2\,(\nu+1)\,abI' - 4\nu acI + \nu\,(\nu+5)\,b^2I.$$

By adding on the invariant $\nu\,(\nu+5)\,(ac-b^2)\,I$ and dividing the sum by a, the above invariant is reduced to

$$aI'' - 2\,(\nu+1)\,bI' + \nu\,(\nu+1)\,cI,$$

which is an invariant of lower degree by unity than the unreduced form.

The results obtained above may be compared with the corresponding ones in the theory of reciprocants.

Thus to the invariants	correspond the reciprocants
I (deg. i, wt. w),	R (deg. i, wt. w),
$aI' - \nu bI$,	$aR' - \mu bR$,
$\nu_2 I_1' I_2 - \nu_1 I_1 I_2'$,	$\mu_2 R_1' R_2 - \mu_1 R_1 R_2'$,
$aI'' - 2\,(\nu+1)\,bI' + \nu\,(\nu+1)\,cI$,	$5aR'' - 5\,(2\mu+1)\,bR' + 4\mu\,(\mu-1)\,cR$,
where $\nu = 3i + 2w$,	where $\mu = 3i + w$.

Defining a *plenarily absolute* form to be one whose degree and weight are both zero ($i=0$, $w=0$), the theorem I shall now prove may be stated as follows:

By differentiating a plenarily absolute principiant we obtain another principiant.

Let P be any principiant of degree i and weight w. Then, by what precedes, since P is both an invariant and a reciprocant,

$$a\frac{dP}{dx} - \nu bP \text{ is an invariant,}$$

and

$$a\frac{dP}{dx} - \mu bP \text{ is a reciprocant.}$$

Hence, when $\nu = 0$ (that is, when $3i + 2w = 0$),

$$\frac{dP}{dx} \text{ is an invariant,}$$

and when $\mu = 0$ (that is, when $3i + w = 0$),

$$\frac{dP}{dx} \text{ is a reciprocant.}$$

When both $\mu = 0$ and $\nu = 0$ (which happens when $i = 0$, $w = 0$),

$$\frac{dP}{dx} \text{ is both a reciprocant and an invariant;}$$

that is,

$$\frac{dP}{dx} \text{ is a principiant.}$$

LECTURE XXVI.

In the theory of Invariants the annihilator Ω has two independent reversors any linear combination of which will also be a reversor. To each of these reversors there corresponds a generator for invariants. Thus Cayley's two generators

$$a\,(b\partial_a + c\partial_b + d\partial_c + e\partial_d + \ldots) - ib,$$
$$a\,(c\partial_b + 2d\partial_c + 3e\partial_d + \ldots) - 2wb,$$

correspond to the two reversors

$$b\partial_a + c\partial_b + d\partial_c + e\partial_d + \ldots,$$
$$c\partial_b + 2d\partial_c + 3e\partial_d + \ldots.$$

The only linear combination of these which does not increase the extent j as well as the weight of the operand is

$$O = jb\partial_a + (j-1)\,c\partial_b + (j-2)\,d\partial_c + \ldots.$$

It is convenient to take this for one of our reversors, and for the other

$$\frac{d}{dx} = 3b\partial_a + 4c\partial_b + 5d\partial_c + \ldots,$$

which is a reversor to V, the annihilator for reciprocants, as well as to Ω, the annihilator for invariants.

We saw in Lecture XI. [p. 364, above] that when F is any homogeneous and isobaric function of degree i and weight w in the $j+1$ letters $a, b, c, \ldots$

$$(\Omega O - O\Omega)\,F = (ij - 2w)\,F.$$

The method employed in proving this can also be applied to show that

$$\left(\Omega\frac{d}{dx} - \frac{d}{dx}\Omega\right)F = \nu F,$$

where $\nu = 3i + 2w$.

Corresponding to the reversors O and $\frac{d}{dx}$ we have the two generators for invariants

$$a\frac{d}{dx} - \nu b \quad\text{and}\quad aO - (ij - 2w)\,b,$$

which are linear combinations of Cayley's generators.

Thus, if I be any invariant,

$$\left(a\frac{d}{dx} - \nu b\right)I \quad\text{and}\quad \{aO - (ij - 2w)\,b\}\,I$$

are also invariants.

29—2

The operator $\frac{d}{dx}$ has, but O has not, analogous properties in the theory of Reciprocants; namely, $\frac{d}{dx}$ is a reversor to V and $a\frac{d}{dx} - \mu b$ is a generator for reciprocants. Thus, we have shown in previous lectures that

$$\left(V\frac{d}{dx} - \frac{d}{dx}V\right)F = 2\mu a F,$$

where F is any homogeneous and isobaric function, and $\mu = 3i + w$, and that if R is any pure reciprocant $\left(a\frac{d}{dx} - \mu b\right)R$ is one also.

Now, Mr Hammond has found that if

$$W = \frac{b}{a}\partial_a + \frac{2ac - b^2}{a^2}\partial_b + \frac{3a^2d - 3abc + b^3}{a^3}\partial_c + \ldots,$$

W is a reversor to V, and $a^2W - ib$ is a generator for pure reciprocants. In fact we have

$$\begin{aligned} VW - WV = {} & V\left(\frac{b}{a}\right)\partial_a \\ & + \left\{V\left(\frac{2ac - b^2}{a^2}\right) - W(2a^2)\right\}\partial_b \\ & + \left\{V\left(\frac{3a^2d - 3abc + b^3}{a^3}\right) - W(5ab)\right\}\partial_c \\ & + \ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

But, since

$$\begin{aligned} V\left(\frac{b}{a}\right) &= 2a, \\ V\left(\frac{2ac - b^2}{a^2}\right) &= 10b - 4b = 6b, \\ V\left(\frac{3a^2d - 3abc + b^3}{a^3}\right) &= \left(18c + 9\frac{b^2}{a}\right) - \left(15\frac{b^2}{a} + 6c\right) + 6\frac{b^2}{a} = 12c, \\ & \ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

and

$$\begin{aligned} W(2a^2) &= 4b, \\ W(5ab) &= 5\frac{b^2}{a} + 5\left(\frac{2ac - b^2}{a}\right) = 10c, \\ & \ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

it follows that

$$VW - WV = 2a\partial_a + 2b\partial_b + 2c\partial_c + \ldots = 2i.$$

Thus W is a reversor to V. Moreover, $a^2W - ib$ acting on any pure reciprocant generates another.

Let R be a pure reciprocant of degree i; then, by what precedes,

$$(VW - WV)R = 2iR.$$

But, since R is a pure reciprocant, $VR = 0$, and consequently $VWR = 2iR$.

Now, $\quad V(a^2W - ib)R = a^2VWR - iRVb = a^2 . 2iR - iR . 2a^2 = 0.$

Hence

$$(a^2W - ib)R$$

is a pure reciprocant; that is $\quad a^2W - ib$

is a generator for pure reciprocants.

Mr Hammond shows that W is a reversor to V in the following manner:

Let

$$\begin{aligned} u &= a_0 + a_1e^{\theta} + a_2e^{2\theta} + a_3e^{3\theta} + \ldots, \\ \phi(u) &= A_0 + A_1e^{\theta} + A_2e^{2\theta} + A_3e^{3\theta} + \ldots, \\ \psi(u) &= A_0' + A_1'e^{\theta} + A_2'e^{2\theta} + A_3'e^{3\theta} + \ldots, \end{aligned}$$

and consider the operators

$$\begin{aligned} P &= \lambda A_0\partial_{a_n} + (\lambda + \mu)A_1\partial_{a_{n+1}} + (\lambda + 2\mu)A_2\partial_{a_{n+2}} + \ldots, \\ Q &= \lambda' A_0'\partial_{a_{n'}} + (\lambda' + \mu')A_1'\partial_{a_{n'+1}} + (\lambda' + 2\mu')A_2'\partial_{a_{n'+2}} + \ldots. \end{aligned}$$

Regarding e^{θ} as an operative symbol defined by the equation

$$e^{\kappa\theta}[\partial_{a_0}] = \partial_{a_\kappa},$$

we may write

$$\begin{aligned} P &= \{\lambda A_0e^{n\theta} + (\lambda + \mu)A_1e^{(n+1)\theta} + (\lambda + 2\mu)A_2e^{(n+2)\theta} + \ldots\}[\partial_{a_0}] \\ &= e^{n\theta}\lambda(A_0 + A_1e^{\theta} + A_2e^{2\theta} + \ldots)[\partial_{a_0}] \\ &\quad + e^{n\theta}\mu(A_1e^{\theta} + 2A_2e^{2\theta} + \ldots)[\partial_{a_0}] \\ &= e^{n\theta}\left(\lambda + \mu\frac{d}{d\theta}\right)\phi(u)[\partial_{a_0}]. \end{aligned}$$

Similarly,

$$Q = e^{n'\theta}\left(\lambda' + \mu'\frac{d}{d\theta}\right)\psi(u)[\partial_{a_0}].$$

Now,

$$\begin{aligned} PQ - QP &= \left\{Pe^{n'\theta}\left(\lambda' + \mu'\frac{d}{d\theta}\right)\psi(u) - Qe^{n\theta}\left(\lambda + \mu\frac{d}{d\theta}\right)\phi(u)\right\}[\partial_{a_0}] \\ &= \left\{e^{n'\theta}\left(\lambda' + \mu'\frac{d}{d\theta}\right)P\psi(u) - e^{n\theta}\left(\lambda + \mu\frac{d}{d\theta}\right)Q\phi(u)\right\}[\partial_{a_0}]. \end{aligned}$$

For

$$Q\phi(u) = QA_0 + e^{\theta}QA_1 + e^{2\theta}QA_2 + \ldots;$$

so that

$$e^{n\theta}\frac{d}{d\theta}Q\phi(u) = e^{n\theta}(e^{\theta}QA_1 + 2e^{2\theta}QA_2 + \ldots)$$

and

$$e^{n\theta}\frac{d}{d\theta}\phi(u) = e^{n\theta}(e^{\theta}A_1 + 2e^{2\theta}A_2 + \ldots);$$

so that

$$\begin{aligned} Qe^{n\theta}\frac{d}{d\theta}\phi(u) &= e^{n\theta}(e^{\theta}QA_1 + 2e^{2\theta}QA_2 + \ldots) \\ &= e^{n\theta}\frac{d}{d\theta}Q\phi(u). \end{aligned}$$

Similarly,

$$Pe^{n'\theta}\frac{d}{d\theta}\psi(u) = e^{n'\theta}\frac{d}{d\theta}P\psi(u).$$

Moreover,

$$P\psi(u) = \psi'(u)\,Pu = \psi'(u)\,P(a_0 + a_1e^{\theta} + a_2e^{2\theta} + \dots)$$
$$= \psi'(u)\,\{e^{n\theta}\lambda A_0 + e^{(n+1)\theta}(\lambda+\mu)\,A_1 + e^{(n+2)\theta}(\lambda+2\mu)\,A_2 + \dots\}$$
$$= e^{n\theta}\psi'(u)\left(\lambda + \mu\frac{d}{d\theta}\right)\phi(u).$$

Similarly, $$Q\phi(u) = e^{n'\theta}\phi'(u)\left(\lambda' + \mu'\frac{d}{d\theta}\right)\psi(u).$$

Hence

$$PQ - QP = \left\{e^{n'\theta}\left(\lambda' + \mu'\frac{d}{d\theta}\right)e^{n\theta}\psi'(u)\left(\lambda + \mu\frac{d}{d\theta}\right)\phi(u)\right.$$
$$\left. - e^{n\theta}\left(\lambda + \mu\frac{d}{d\theta}\right)e^{n'\theta}\phi'(u)\left(\lambda' + \mu'\frac{d}{d\theta}\right)\psi(u)\right\}[\partial_{a_0}]$$
$$= e^{(n+n')\theta}\left\{\left(\lambda' + \mu'n + \mu'\frac{d}{d\theta}\right)\psi'(u)\left(\lambda + \mu\frac{d}{d\theta}\right)\phi(u)\right.$$
$$\left. - \left(\lambda + \mu n' + \mu\frac{d}{d\theta}\right)\phi'(u)\left(\lambda' + \mu'\frac{d}{d\theta}\right)\psi(u)\right\}[\partial_{a_0}].$$

If in this we write

$$\phi = \frac{u^2}{2},\qquad \lambda = 4,\ \mu = 1,\ n = 1,$$
$$\psi = \log u,\ \lambda' = 0,\ \mu' = 1,\ n' = -1,$$

we have

$$PQ - QP = \left\{\left(1 + \frac{d}{d\theta}\right)u^{-1}\left(4 + \frac{d}{d\theta}\right)\frac{u^2}{2} - \left(3 + \frac{d}{d\theta}\right)u\frac{d}{d\theta}\log u\right\}[\partial_{a_0}]$$
$$= \left\{\left(1 + \frac{d}{d\theta}\right)\left(2u + \frac{du}{d\theta}\right) - \left(3 + \frac{d}{d\theta}\right)\frac{du}{d\theta}\right\}[\partial_{a_0}]$$
$$= \left\{\left(1 + \frac{d}{d\theta}\right)\left(2 + \frac{d}{d\theta}\right) - \left(3 + \frac{d}{d\theta}\right)\frac{d}{d\theta}\right\}u\,[\partial_{a_0}]$$
$$= 2u\,[\partial_{a_0}].$$

Now, $$2u\,[\partial_{a_0}] = 2\,(a_0 + a_1e^{\theta} + a_2e^{2\theta} + \dots)\,[\partial_{a_0}]$$
$$= 2\,(a_0\partial_{a_0} + a_1\partial_{a_1} + a_2\partial_{a_2} + \dots).$$

Also $$P = 4A_0\partial_{a_1} + 5A_1\partial_{a_2} + 6A_2\partial_{a_3} + \dots,$$
$$Q = A_1'\partial_{a_0} + 2A_2'\partial_{a_1} + 3A_3'\partial_{a_2} + \dots,$$

where $$\frac{1}{2}(a_0 + a_1e^{\theta} + a_2e^{2\theta} + \dots)^2 = A_0 + A_1e^{\theta} + A_2e^{2\theta} + \dots$$

and $$\log(a_0 + a_1e^{\theta} + a_2e^{2\theta} + \dots) = \log a_0 + A_1'e^{\theta} + A_2'e^{2\theta} + \dots.$$

Equating coefficients, we have

$$A_0 = \frac{1}{2}a_0^2,\ A_1 = a_0a_1,\ A_2 = a_0a_2 + \frac{a_1^2}{2},\ \dots$$
$$A_1' = \frac{a_1}{a_0},\ A_2' = \frac{2a_0a_2 - a_1^2}{2a_0^2},\ \dots.$$

It is easily seen by expanding the logarithm that the general value of A_n' is $(-)^{n+1}\frac{S_n}{n}$ where S_n denotes the sum of the nth powers of the roots of

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n.$$

Thus we have shown that if

$$P = 2a_0^2\partial_{a_1} + 5a_0a_1\partial_{a_2} + (6a_0a_2 + 3a_1^2)\,\partial_{a_3}$$

and $$Q = \frac{a_1}{a_0}\partial_{a_0} + \frac{2a_0a_2 - a_1^2}{a_0^2}\partial_{a_1} + \frac{3a_0^2a_3 - 3a_0a_1a_2 + a_1^3}{a_0^3}\partial_{a_2} + \dots,$$

then $$PQ - QP = 2\,(a_0\partial_{a_0} + a_1\partial_{a_1} + a_2\partial_{a_2} + \dots) = 2i.$$

The general formula obtained for $PQ - QP$ is an extension of a result of Capt. MacMahon's, who considers the case in which

$$\phi(u) = \frac{u^m}{m},\ \psi(u) = \frac{u^{m'}}{m'}.$$

When $\phi(u)$ and $\psi(u)$ have these values, the general formula becomes

$$PQ - QP = e^{(n+n')\theta}\left\{\left(\lambda' + \mu'n + \mu'\frac{d}{d\theta}\right)\left(\frac{\lambda u^{m+m'-1}}{m} + \mu u^{m+m'-2}\frac{du}{d\theta}\right)\right.$$
$$\left. - \dots\dots\dots\dots\right\}[\partial_{a_0}].$$

But $$\left(\lambda' + \mu'n + \mu'\frac{d}{d\theta}\right)\left(\frac{\lambda}{m}u^{m+m'-1} + \mu u^{m+m'-2}\frac{du}{d\theta}\right)$$
$$= \left(\lambda' + \mu'n + \mu'\frac{d}{d\theta}\right)\left(\frac{\lambda}{m} + \frac{\mu}{m+m'-1}\frac{d}{d\theta}\right)u^{m+m'-1}.$$

Consequently

$$PQ - QP = e^{(n+n')\theta}\left\{\left(\lambda' + \mu'n + \mu'\frac{d}{d\theta}\right)\left(\frac{\lambda}{m} + \frac{\mu}{m+m'-1}\frac{d}{d\theta}\right)\right.$$
$$\left. - \dots\dots\dots\dots\right\}u^{m+m'-1}[\partial_{a_0}].$$

In Capt. MacMahon's notation

$$P = (m, \lambda, \mu, n),\ Q = (m', \lambda', \mu', n');$$

in our notation

$$P = e^{n\theta}\left(\lambda + \mu\frac{d}{d\theta}\right)\frac{u^m}{m}[\partial_{a_0}],$$
$$Q = e^{n'\theta}\left(\lambda' + \mu'\frac{d}{d\theta}\right)\frac{u^{m'}}{m'}[\partial_{a_0}].$$

If now we write

$$PQ - QP = e^{(n+n')\theta}\left(\lambda_1 + \mu_1\frac{d}{d\theta}\right)\frac{u^{m+m'-1}}{m+m'-1}[\partial_{a_0}],$$

which is equivalent to

$$PQ - QP = (m + m' - 1,\ \lambda_1,\ \mu_1,\ n + n'),$$

we have
$$\left(\lambda' + \mu' n + \mu' \frac{d}{d\theta}\right)\left\{\frac{\lambda}{m}(m+m'-1) + \mu\frac{d}{d\theta}\right\}$$
$$-\left(\lambda + \mu n' + \mu\frac{d}{d\theta}\right)\left\{\frac{\lambda'}{m'}(m+m'-1) + \mu'\frac{d}{d\theta}\right\} = \lambda_1 + \mu_1\frac{d}{d\theta}.$$

Hence we obtain
$$\lambda_1 = (m+m'-1)\left\{\frac{\lambda}{m}(\lambda' + \mu' n) - \frac{\lambda'}{m'}(\lambda + \mu n')\right\},$$
$$\mu_1 = \mu\mu'(n-n') + \frac{\lambda\mu'}{m}(m'-1) - \frac{\lambda'\mu}{m'}(m-1).$$

This agrees with Capt. MacMahon's result, a statement of which was given in Lecture XX. [above, p. 417].

Let Q be a reversor to the operator $P = \lambda a^m \partial_b + (\ldots)\partial_c + (\ldots)\partial_d + \ldots$, and suppose that
$$(PQ - QP)F = \kappa a^{m-1}F,$$
where F is any homogeneous and isobaric function and κ some number depending on its degree and weight. Then $\lambda aQ - \kappa b$ will be the generator corresponding to Q. In other words, we have to prove that
$$P(\lambda aQ - \kappa b)F = 0 \text{ whenever } PF = 0.$$

Now, by hypothesis, $Pa = 0$, $Pb = \lambda a^m$, and when $PF = 0$,
$$PQF = \kappa a^{m-1}F.$$

Thus,
$$P(\lambda aQ - \kappa b)F = \lambda a PQF - \kappa F . Pb$$
$$= \lambda\kappa a^m F - \lambda\kappa a^m F = 0.$$

As an example, consider the case of the reversor $\frac{d}{dx}$ in the theory of reciprocants. Here
$$P = V,\ \lambda = 2,\ m = 2;$$
and since
$$\left(V\frac{d}{dx} - \frac{d}{dx}V\right)F = 2\mu a F,$$
we have $\kappa = 2\mu$. Hence the corresponding generator is $2\left(a\frac{d}{dx} - \mu b\right)$; or, disregarding the numerical factor 2, we may take $a\frac{d}{dx} - \mu b$ for the generator in question, which is usually denoted by the letter G.

We may also write G in the equivalent form
$$G = 4(ac-b^2)\partial_b + 5(ad-bc)\partial_c + 6(ae-bd)\partial_d + \ldots,$$
which it is sometimes more convenient to use.

I shall now show that
$$\Omega G - G\Omega = aw - b\Omega,$$
where w is the weight of the operand.

It is very easily seen that
$$\Omega(ac - b^2) = 0,$$
$$\Omega(ad - bc) = 2(ac - b^2),$$
$$\Omega(ae - bd) = 3(ad - bc),$$
$$\Omega(af - be) = 4(ae - bd),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

Hence it follows, by a direct and very simple calculation, that
$$\Omega G - G\Omega = 2(ac-b^2)\partial_c + 3(ad-bc)\partial_d + 4(ae-bd)\partial_e + \ldots.$$

But, since
$$b\partial_b + 2c\partial_c + 3d\partial_d + 4e\partial_e + \ldots = w,$$
and
$$a\partial_b + 2b\partial_c + 3c\partial_d + 4d\partial_e + \ldots = \Omega,$$
$$aw - b\Omega = 2(ac-b^2)\partial_c + 3(ad-bc)\partial_d + 4(ae-bd)\partial_e + \ldots.$$

Consequently
$$\Omega G - G\Omega = aw - b\Omega.$$

The use of this formula will be seen in a subsequent lecture.

We may also prove an analogous theorem relating to the invariant generator $a\frac{d}{dx} - \nu b$, which we shall call G'.

Let the operand be F, a homogeneous and isobaric function of degree i and weight w. Then VF is of degree $i+1$ and weight $w-1$; its ν is therefore
$$3(i+1) + 2(w-1) = \nu + 1.$$

Thus,
$$(VG' - G'V)F = \left\{V\left(a\frac{d}{dx} - \nu b\right) - \left(a\frac{d}{dx} - \nu b - b\right)V\right\}F$$
$$= a\left(V\frac{d}{dx} - \frac{d}{dx}V\right)F - \nu(Vb - bV)F + bVF.$$

But
$$\left(V\frac{d}{dx} - \frac{d}{dx}V\right)F = 2\mu a F = 2(3i + w)aF,$$
and
$$VbF = bVF + 2a^2F.$$

Consequently
$$VG' - G'V = 2(3i+w)a^2F - 2\nu a^2F + bVF$$
$$= 2(3i + w - \nu)a^2F + bVF$$
$$= -2wa^2F + bVF.$$

It is perhaps worthy of notice that if I is an invariant of weight w and R a pure reciprocant, also of weight w, then
$$\Omega GI = awI \text{ and } VG'R = -2a^2wR;$$
whereas
$$\Omega G'I = 0 \text{ and } VGR = 0.$$

LECTURE XXVII.

I should like to make a momentary pause in the development of the theory which now engages our attention and to revert to the proof of Cayley's theorem for the enumeration of linearly independent invariants contained in Lecture XI. and expressed by the formula $(w; i, j) - (w - 1; i, j)$.

Since that proof was written out I have endeavoured to obtain one that might be capable of being extended to the supposed analogous theorem, regarding pure reciprocants, expressed by the formula $(w; i, j) - (w-1; i+1, j)$, but all my efforts and those of another and most skilful algebraist in this direction have hitherto proved ineffectual.

In aiming at this object, however, I obtained a second proof of Cayley's theorem, less compendious than the previous one, and subject to the drawback that it assumes the law of Reciprocity, but which possesses the advantage over it of being more direct and of looking the question, so to say, more squarely in the face. The forms of thought employed in it seem to me too peculiar and precious to be consigned to oblivion. I am not one of those who look upon Analysis as only valuable for the positive results to which it leads, and who regard proofs as almost a superfluity, thinking it sufficient that mathematical formulae should be obtained, no matter how, and duly entered on a register.

I look upon Mathematics not merely as a language, an art, and a science, but also as a branch of Philosophy, and regard the forms of reasoning which it embodies and enshrines as among the most valuable possessions of the human mind. Add to this that it is scarcely possible that a well-reasoned mathematical proof shall not contain within itself subordinate theorems—germs of thought of intrinsic value and capable of extended application.

That such was the opinion of our High Pontiff is shown by the publication of his seven proofs of the Theorem of Reciprocity, a number to which subsequent researches have made almost annual additions (like so many continually augmenting asteroids in the Arithmetical Firmament) to such an extent that it would seem to be an interesting task for some one to undertake to form a corolla of these various proofs and to construct a reasoned bibliography, a *catalogue raisonnée*, of this one single theorem. For these reasons, I shall venture to put on record (*valeat quantum*) the following Second Proof of Cayley's Theorem.

The notation which I proceed to explain will be found very convenient. A rational integral homogeneous isobaric function will be called a *gradient;* its weight, degree, extent (extent meaning the number of letters after the first) will be denoted by $w; i, j$ and spoken of as the *type* of the gradient. Either a single letter, such as ϕ, will be employed to denote a gradient, or

else its type enclosed in a parenthesis thus $[w; i, j]$. The abbreviation $T\phi$ signifies the type of ϕ; thus, $T\phi = w; i, j$.

The number of terms in the most general gradient whose type is the same as that of ϕ will be spoken of as the *denumerant* of ϕ. The letter N will be used to denote such a denumerant; thus, $N\phi$ signifies the denumerant of ϕ.

In like manner, the letter Δ will be used to denote the number of linear relations between the coefficients of any gradient, whenever such relations exist. Hence $N\phi - \Delta\phi$ expresses the number of terms in ϕ whose coefficients are left arbitrary. Obviously, when ϕ is the most general gradient of its type, we have

$$\Delta\phi = 0.$$

We also use E to denote the $ij - 2w$, which may be called the *excess*, of the gradient of type $w; i, j$. Thus, if $T\phi = w; i, j$, we write $E\phi = ij - 2w$.

The operators which we shall employ, namely, Ω and Ω', are defined by the equations

$$\Omega = a_0\partial_{a_1} + a_1\partial_{a_2} + a_2\partial_{a_3} + \dots,$$
$$\Omega' = \qquad a_1\partial_{a_2} + a_2\partial_{a_3} + \dots.$$

The first of these is of course an equivalent, but for present purposes more convenient, form of $a\partial_b + 2b\partial_c + 3c\partial_d + \dots$, the ordinary invariant annihilator Ω $\left(\text{as will be evident on writing } a_0 = a,\ a_1 = \frac{b}{1},\ a_2 = \frac{c}{1\,.\,2},\ \dots\right)$; the second of them, Ω', is merely Ω deprived of its first term.

We may now give the following enunciation of the theorem to be proved:

If ϕ is the most general gradient of its type, $\Omega\phi$ is also the most general gradient of its type whenever $E\phi$ is not negative. In other words, we shall prove that, subject to the condition stated above, $\Delta\Omega\phi = 0$ whenever $\Delta\phi = 0$. This is equivalent to Cayley's Theorem on the number of linearly independent invariants. For the number of forms of the same type as ϕ, and subject to annihilation by Ω, is

$$N\phi - N\Omega\phi + \Delta\Omega\phi;$$

and Cayley's Theorem states that the number of such forms is $N\phi - N\Omega\phi$, which will be the case when

$$\Delta\Omega\phi = 0.$$

The theorem of Reciprocity enables us to dispense with the discussion of those cases in which the extent j is greater than the degree i. For since [Vol. III. of this Reprint, p. 151] the number of linearly independent invariants for the type $w; j, i$ is the same as for the type $w; i, j$, we can substitute the first of these types for the second, using ψ, whose type is $w; j, i$, instead of ϕ, whose type is $w; i, j$. Thus we have

$$N\psi - N\Omega\psi + \Delta\Omega\psi = N\phi - N\Omega\phi + \Delta\Omega\phi.$$

But by Ferrers' proof of Euler's Theorem (*vide* "A Constructive Theory of Partitions" [p. 1, above]),

$$N\psi = N\phi \text{ and } N\Omega\psi = N\Omega\phi.$$

It obviously follows that

$$\Delta\Omega\psi = \Delta\Omega\phi.$$

Cases for which the extent is greater than the degree may therefore be made to depend on those for which the degree is greater than the extent. Hence Cayley's Theorem depends on the proof that $\Delta\Omega\phi = 0$ when $i = > j$ and $ij = > 2w$.

In the course of the demonstration, the following Lemma will be used:

If $\quad T\phi = w; i, j$ and $T\psi = ij - w; i, j$, then $N\phi = N\psi$.

The types of the two gradients we are now considering may be said to be *complementary*, and then the Lemma may be enunciated in words as follows:

The denumerants of two gradients are equal when the types of the gradients are complementary.

The proof consists in showing that to each term of the type $w; i, j$ there corresponds a term of the type $ij - w; i, j$. Let $a_0^{\lambda_0} a_1^{\lambda_1} a_2^{\lambda_2} \ldots a_j^{\lambda_j}$ be any term of the type $w; i, j$; then

$$w = \lambda_1 + 2\lambda_2 + 3\lambda_3 + \ldots + j\lambda_j$$

and

$$i = \lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_j.$$

Writing the suffixes of the letters $a_0, a_1, a_2, \ldots a_j$ in reverse order, everything else being kept unchanged, we obtain the term $a_j^{\lambda_0} a_{j-1}^{\lambda_1} a_{j-2}^{\lambda_2} \ldots a_0^{\lambda_j}$, whose weight we will call w'. Then

$$\begin{aligned} w' &= j\lambda_0 + (j-1)\lambda_1 + (j-2)\lambda_2 + \ldots + \lambda_{j-1} \\ &= j(\lambda_0 + \lambda_1 + \lambda_2 + \ldots + \lambda_j) - (\lambda_1 + 2\lambda_2 + 3\lambda_3 + \ldots + j\lambda_j) \\ &= ij - w. \end{aligned}$$

The degree of the transformed term is still i, and its extent is still j, while its weight has become $ij - w$; its type is therefore complementary to that of the original term. Hence to each term of any given type there corresponds a term of the complementary type, and consequently the total number of possible terms (that is, the Denumerant) for each type is the same.

By means of this Lemma it can be shown that $\Delta\Omega\phi = 0$ when $E\phi = -1$. Let

$$T\phi = w; i, j \text{ where } ij - 2w = -1;$$

then, since $T\Omega\phi = w - 1; i, j$, the types $T\phi$ and $T\Omega\phi$ are complementary (the sum of the weights being $w + w - 1 = ij$).

It follows from the Lemma that the Denumerants of ϕ and $\Omega\phi$ are equal. Hence

$$\Delta\Omega\phi = 0.$$

For if not, the number of independent terms in $\Omega\phi$ being less than the denumerant of $\Omega\phi$, will also be less than its equal, the denumerant of ϕ, and therefore there will be one or more invariants of the type $w; i, j$ for which the excess is negative. Since this is known to be impossible, we must have

$$\Delta\Omega\phi = 0.$$

We next prove that, in all cases for which $i = > w$, the number of linearly independent invariants of the type $w; i, j$ is correctly given by the formula

$$(w; i, j) - (w-1; i, j),$$

which is equivalent (as we showed at the beginning of Lecture XV.) to

$$(w; w, j) - (w-1; w, j),$$

or, what is the same thing, to the coefficient of $a^w x^w$ in the expansion of

$$F = \frac{1-x}{(1-a)(1-ax)(1-ax^2)(1-ax^3)\ldots(1-ax^j)}.$$

Let the expansion of

$$G = \frac{1-x}{(1-ax)(1-ax^2)(1-ax^3)\ldots(1-ax^j)}$$

be

$$1 + (a-1)x + A_2 x^2 + \ldots + A_w x^w + \ldots.$$

The expansion of F is obtained by multiplying that of G by the infinite geometrical series

$$1 + a + a^2 + a^3 + \ldots.$$

But we only require the coefficient of $a^w x^w$ in the expansion of F, so that we need only retain the portion

$$A_w x^w (1 + a + a^2 + \ldots + a^w)$$

of the above product instead of its complete expression.

It is of importance to notice here that A_w, which is independent of x, cannot contain any higher power of a than a^w. (That this is so will be evident from the constitution of the fraction G, for clearly no power of a in the expansion of G can be associated with a lower power of x.) Thus we see that

$$A_w = \alpha a^w + \beta a^{w-1} + \gamma a^{w-2} + \ldots + \kappa a + \lambda,$$

and consequently

$$A_w x^w (1 + a + a^2 + \ldots + a^w) = \ldots + a^w x^w (\alpha + \beta + \gamma + \ldots + \kappa + \lambda) + \ldots.$$

Hence the coefficient of $a^w x^w$ in the expansion of F is

$$\alpha + \beta + \gamma + \ldots + \kappa + \lambda,$$

which is the value assumed by A_w when in it we write $a = 1$. Call this value A_w', and let the value of G when $a = 1$ be denoted by G'. Then A_w' is the coefficient of x^w in

$$G' = \frac{1}{(1-x^2)(1-x^3)\ldots(1-x^j)}.$$

Hence we see that, when $i = > w$, the value of $(w; i, j) - (w-1; i, j)$ is the total number of ways in which w can be made up of the parts $2, 3, \ldots j$.

We have yet to show that this number is the same as that of the linearly independent invariants of the type $w; i, j$ when $i = > w$.

This follows from the known theorem that every invariant is either a rational integral function of the Protomorphs $a, P_2, P_3, \ldots P_j$ (meaning the invariant a and those of the second and third degrees alternately whose first terms are $ac, a^2d, ae, a^2f, \ldots$), or can be made so by multiplying it by a suitable power of a. Thus, if I be any invariant of degree i and weight w,

$$Ia^{w-i} = \Phi(a, P_2, P_3, \ldots P_j),$$

where Φ, which is of degree-weight $w.w$ when expressed in terms of $a, b, c, \ldots$, is rational and integral as regards the protomorphs.

When $i = > w$, writing

$$I = a^{i-w}\Phi(a, P_2, P_3, \ldots P_j),$$

Φ consists of a series of terms of the form $Aa^{\theta}P_2^{\lambda}P_3^{\mu}\ldots P_j^{\rho}$, *each with an arbitrary coefficient*, where, since

$$2\lambda + 3\mu + 4\nu + \ldots + j\rho = w,$$

the number of arbitrary constants in Φ is the total number of partitions of w into parts $2, 3, \ldots j$. Hence the number of linearly independent invariants of the type $w; i, j$ is also this number of partitions, that is, by what precedes is $(w; i, j) - (w-1; i, j)$. This proves Cayley's theorem for cases in which $i = > w$.

But when $i < w$, the equation

$$Ia^{w-i} = \Phi(a, P_2, P_3, \ldots P_j)$$

shows that the coefficients of Φ are not all arbitrary, but must be so chosen that Φ may be divisible by a^{w-i}, and the reasoning employed in the case of $i = > w$ no longer holds.

It will be convenient at this point of the investigation to review the results we have hitherto obtained and to see what remains to be proved.

Cayley's Theorem has been demonstrated for cases in which the degree is not less than the weight. This will be expressed by saying that

$$\Delta\Omega\,[w; i, j] = 0 \text{ when } i = > w.$$

We have also proved that

$$\Delta\Omega\,[w; i, j] = 0 \text{ when } ij - 2w = -1.$$

The law of reciprocity has been expressed in the form

$$\Delta\Omega\,[w; i, j] = \Delta\Omega\,[w; j, i],$$

where $[w; i, j]$ denotes the most general gradient of the type $w; i, j$.

The theorem to be proved is that

$$\Delta\Omega\,[w; i, j] = 0 \text{ when } ij - 2w = > 0;$$

but we may at once dismiss those cases in which $i = > w$, and (assuming the theorem to have been proved for Quantics of order inferior to j) those in which $i < j$, for these depend on the truth of the theorem for a Quantic of order i.

It remains, then, to prove that, when $ij - 2w = > 0$, $\Delta\Omega\,[w; i, j] = 0$ for values of i inferior to w, but not inferior to j. This may be effected as follows:

Let ϕ be the most general gradient of the type $w; i+1, j$, and suppose

$$\phi = P + Qa + Ra^2 + Sa^3,$$

where P, Q and R do not contain the letter a, though S may do so. Then, writing

$$\phi_1 = Q + Ra + Sa^2,$$

ϕ_1 is the most general gradient of the type $w; i, j$.

Now, if $\Omega = a\partial_b + b\partial_c + c\partial_d + \ldots$, and $\Omega' = b\partial_c + c\partial_d + \ldots$, we have

$$\Omega\phi = \Omega'P + \left(\Omega'Q + \frac{dP}{db}\right)a + \left(\Omega'R + \frac{dQ}{db}\right)a^2 + \left(\Omega S + \frac{dR}{db}\right)a^3, \quad (1)$$

and $$\Omega\phi_1 = \Omega'Q + \left(\Omega'R + \frac{dQ}{db}\right)a + \left(\Omega S + \frac{dR}{db}\right)a^2.$$

Confining our attention for the present to $\Omega\phi_1$, it is clear that if no linear relations exist among the coefficients of $\Omega'R$ (that is, if $\Delta\Omega'R = 0$) the coefficients of $\Omega'Q$ are not connected with those of $\Omega'R + \frac{dQ}{db}$ by any linear relation. For the coefficient of each term of $\Omega'R + \frac{dQ}{db}$ is the sum of a single coefficient of Q and an independent linear function of the coefficients of R. Moreover, obviously the coefficients of $\Omega'Q$ are unconnected with those of $\Omega S + \frac{dR}{db}$.

If, then, the coefficients of $\Omega'Q$ are not related *inter se* (that is, if $\Delta\Omega'Q = 0$), we have

$$\Delta\Omega\phi_1 = \Delta\left\{\left(\Omega'R + \frac{dQ}{db}\right)a + \left(\Omega S + \frac{dR}{db}\right)a^2\right\}. \quad (2)$$

Looking now to the expression (1) for $\Omega\phi$, we see immediately from (2) that any linear relation subsisting between the coefficients of $\Omega\phi_1$ will also subsist between those of $\Omega\phi$, and therefore that $\Delta\Omega\phi_1$ is not greater than $\Delta\Omega\phi$.

If, then, $\Delta\Omega\phi = 0$, it follows that $\Delta\Omega\phi_1 = 0$, provided that both the supplementary conditions $\Delta\Omega'Q = 0$ and $\Delta\Omega'R = 0$ are also satisfied.

Now, since $\phi_1 = Q + Ra + Sa^2$ is the most general gradient of the type $w; i, j$,

Q will be the most general gradient of the type $w - i; i, j-1$,
and R „ „ „ „ „ $w - i + 1; i - 1, j - 1$,
when in Q and R we change $b, c, d, \ldots$ into $a, b, c, \ldots$. This change converts $\Omega' = b\partial_c + c\partial_d + \ldots$ into $\Omega = a\partial_b + b\partial_c + \ldots$. Hence the conditions $\Delta\Omega' Q = 0$ and $\Delta\Omega' R = 0$ are respectively equivalent to

$$\Delta\Omega\,[w - i; i, j-1] = 0 \text{ and } \Delta\Omega\,[w - i + 1; i - 1, j - 1] = 0.$$

Supposing these supplementary conditions to be satisfied, what we have proved is that when

$$\Delta\Omega\,[w; i+1, j] = 0 \text{ (that is, } \Delta\Omega\phi = 0),$$

then also

$$\Delta\Omega\,[w; i, j] \quad = 0 \text{ (that is, } \Delta\Omega\phi_1 = 0).$$

Now,

$T\phi = w; i+1, j$, so that $E\phi = (i+1)j - 2w = (ij - 2w) + j$,
$TQ = w - i; i, j - 1$, so that $EQ = i(j-1) - 2(w - i) = (ij - 2w) + i$,
$TR = w - i + 1; i - 1, j - 1$, so that $ER = (i-1)(j-1) - 2(w - i + 1)$
$= (ij - 2w) + i - j - 1$.

Thus, when $ij - 2w = > 0$ and $i = > j$,

$E\phi$ and EQ are both positive.

ER is in general $= > 0$, but in the special case where $ij - 2w = 0$ and $i = j$, we have $ER = -1$. Except in this case (which gives us no trouble, since we have seen that $\Delta\Omega R = 0$ in consequence of $ER = -1$), we have never to deal with a type of which the excess is negative.

Hence, if we assume Cayley's Theorem to have been proved for all extents up to $j - 1$ inclusive, we have

$$\Delta\Omega\,[w - i; i, j - 1] = 0,$$

and

$$\Delta\Omega\,[w - i + 1; i - 1, j - 1] = 0,$$

(that is, the two supplementary conditions are satisfied).

We wish to extend the theorem to the extent j.

Subject to the conditions $i = > j$ and $ij - 2w = > 0$, we have

$$\Delta\Omega\,[w; i, j] = 0 \text{ if } \Delta\Omega\,[w; i + 1, j] = 0.$$

But we need consider no value of i greater than w, as we have proved that

$$\Delta\Omega\,[w; w, j] = 0 = \Delta\Omega\,[w; w + \kappa, j];$$

therefore

$$\Delta\Omega\,[w; w - 1, j] = 0,$$
$$\Delta\Omega\,[w; w - 2, j] = 0,$$
$$\cdots\cdots\cdots\cdots$$
$$\Delta\Omega\,[w; j, j] \quad = 0.$$

As previously shown, the theorem is true for all values of i inferior to j if it is true for all Quantics of inferior order. Thus the theorem is true for a Quantic of order j and for every value of i if it is true for all Quantics of order inferior to j. But it is true for the Quadric (where $j = 2$)*; therefore also for the Cubic ($j = 3$); therefore also for the Quartic ($j = 4$), and so universally. Hence the theorem to be proved is demonstrated.

LECTURE XXVIII.

We now resume the theory of Principiants and proceed to prove the important theorem that every Principiant is either simply an invariant in respect to a known series of pure reciprocants, which we call $A, B, C, D, \ldots$, or else becomes such an invariant when multiplied by a^{w-i}, where w is the weight and i the degree of the Principiant in question. Using the letter M to denote the pure reciprocant $ac - \frac{5}{4}b^2$, and G the ordinary eductive generator,

$$4(ac - b^2)\partial_b + 5(ad - bc)\partial_c + 6(ae - bd)\partial_d + 7(af - be)\partial_e + \ldots$$

(which, it will be remembered, is only another form of $a\frac{d}{dx} - \mu b$, with the advantage of the μ being suppressed, that is, only implicitly contained), we obtain in succession the values of $A, B, C, D, \ldots$ from the following equations:

$$5A = GM,$$
$$6B = GA,$$
$$7C = GB - MA,$$
$$8D = GC - 2MB,$$
$$9E = GD - 3MC,$$
$$\cdots\cdots\cdots\cdots$$

On performing the calculations indicated by these equations we shall find

$$A = a^2d - 3abc + 2b^3,$$
$$B = a^3e - 2a^2c^2 - \frac{7}{2}a^2bd + \frac{17}{2}ab^2c - 4b^4,$$
$$C = a^4f - 5a^3cd - 4a^3be + 13a^2bc^2 + \frac{45}{4}a^2b^2d - \frac{103}{4}ab^3c + \frac{19}{2}b^5,$$
$$D = a^5g - \frac{25}{8}a^4d^2 - 6a^4ce + 7a^3c^3 + \text{terms involving } b,$$
$$E = a^6h - \frac{15}{2}a^5de - 7a^5cf + 29a^4c^2d + \text{terms involving } b.$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

* When $j = 2$ the condition $ij = > 2w$ becomes identical with $i = > w$; but we have already seen that the theorem is true whenever $i = > w$.

The fact that D is a pure reciprocant enables us to calculate the terms in E which are independent of b without a previous knowledge of the values of those terms in D which involve b. For, since

$$G = 4(ac - b^2)\partial_b + \dots \text{ and } V = 2a^2\partial_b + \dots,$$

$$a^2G - 2(ac - b^2)V \text{ does not contain } \partial_b.$$

Hence the operation of $a^2G - 2(ac - b^2)V$ on terms involving b cannot give rise to terms independent of b. But,

$$D \text{ being a pure reciprocant, } VD = 0;$$

so that $$\{a^2G - 2(ac - b^2)V\}D = a^2GD,$$

and the terms of a^2GD which do not involve b are found by operating with

$$[a^2G - 2(ac - b^2)V]_{b=0}$$

on the terms of D which do not involve b.

If, now, we use $M_0, A_0, B_0, C_0, \dots$ to denote those portions of $M, A, B, C, \dots$ which are independent of b, and write

$$[a^2G - 2(ac - b^2)V]_{b=0} = a^2G_0,$$

we shall still have $$9E_0 = G_0D_0 - 3M_0C_0;$$

and in general the law of successive derivation for $A_0, B_0, C_0, D_0, \dots$ is the same as that for $A, B, C, D, \dots$ except that G_0 takes the place of G.

We have

$$\begin{aligned} a^2G_0 &= [a^2G - 2(ac - b^2)V]_{b=0} \\ &= a^2(5ad\partial_c + 6ae\partial_d + 7af\partial_e + 8ag\partial_f + 9ah\partial_g + \dots) \\ &\quad - 2ac\{6ac\partial_d + 7ad\partial_e + (8ae + 4c^2)\partial_f + (9af + 9cd)\partial_g + \dots\}; \end{aligned}$$

so that

$$G_0 = 5ad\partial_c + 6(ae - 2c^2)\partial_d + 7(af - 2cd)\partial_e + \frac{8}{a}(a^2g - 2ace - c^3)\partial_f + \frac{9}{a}(a^2h - 2acf - 2c^2d)\partial_g + \dots;$$

and consequently (since $M_0 = ac$),

$$\begin{aligned} &5A_0 = G_0M_0 && \text{gives } A_0 = a^2d, \\ &6B_0 = G_0A_0 && \quad ,, \quad B_0 = a^3e - 2a^2c^2, \\ &7C_0 = G_0B_0 - M_0A_0 && \quad ,, \quad C_0 = a^4f - 5a^3cd, \\ &8D_0 = G_0C_0 - 2M_0B_0 && \quad ,, \quad D_0 = a^5g - \frac{25}{8}a^4d^2 - 6a^4ce + 7a^3c^3, \\ &9E_0 = G_0D_0 - 3M_0C_0 && \quad ,, \quad E_0 = a^6h - \frac{15}{2}a^5de - 7a^5cf + 29a^4c^2d, \\ &\dots\dots\dots && \quad ,, \quad \dots\dots\dots \end{aligned}$$

Thus, for example,

$$\begin{aligned} 8D_0 &= G_0(a^4f - 5a^3cd) - 2ac(a^3e - 2a^2c^2) \\ &= -25a^4d^2 - 30a^3c(ae - 2c^2) + 8a^3(a^2g - 2ace - c^3) - 2ac(a^3e - 2a^2c^2); \end{aligned}$$

whence $$D_0 = a^5g - \frac{25}{8}a^4d^2 - 6a^4ce + 7a^3c^3.$$

Again, $$\begin{aligned} 9E_0 &= G_0\left(a^5g - \frac{25}{8}a^4d^2 - 6a^4ce + 7a^3c^3\right) - 3ac(a^4f - 5a^3cd) \\ &= 5ad(-6a^4e + 21a^3c^2) - \frac{75}{2}(ae - 2c^2)a^4d - 42(af - 2cd)a^4c \\ &\quad + 9(a^2h - 2acf - 2c^2d)a^4 - 3ac(a^4f - 5a^3cd), \end{aligned}$$

gives $$E_0 = a^6h - \frac{15}{2}a^5de - 7a^5cf + 29a^4c^2d.$$

Similarly, from the known values of D_0 and E_0 we may deduce that of the next letter, F_0, and so on to any extent.

It may be noticed that each of the pure reciprocants $A, B, C, D, \dots$ can be determined without ambiguity, by means of the annihilator V, when the portions of them, $A_0, B_0, C_0, D_0, \dots$ independent of b are known.

For suppose R and R' to be two reciprocants, of weight w, for each of which the terms independent of b are the same. Then their difference is divisible by b. Let

$$R - R' = b\phi; \text{ then } V(b\phi) = 0; \text{ that is, } 2a^2\phi + bV\phi = 0.$$

Hence ϕ is divisible by b, and $R - R'$ is divisible by b^2; say $R - R' = b^2\psi$. Then

$$V(b^2\psi) = 4a^2b\psi + b^2V\psi = 0,$$

showing that ψ is divisible by b, and $R - R'$ by b^3.

By continually reasoning in this manner, we prove that $R - R'$ must be divisible by b^w; and then the remaining factor (being of weight 0) is necessarily of the form λa^θ, where λ and θ are numerical constants. Thus

$$R - R' = \lambda a^\theta b^w, \text{ and consequently } V(\lambda a^\theta b^w) = 0.$$

This is impossible unless $\lambda = 0$, when the two reciprocants R, R' become equal, showing that there cannot be two different reciprocants for which the terms independent of b are the same. When, therefore, the terms which do not involve b of any pure reciprocant are known, the complete expression of that reciprocant can be determined without ambiguity.

Each reciprocant of the series $A, B, C, D, \dots$ possesses the property of being, so to say, an Invariant relative to the one which precedes it, meaning that the operation of $\Omega = a\partial_b + 2b\partial_c + 3c\partial_d + \dots$ on any letter gives (to a factor *près*) the one immediately preceding it. The first letter, A, is an Invariant in the ordinary sense. We can in fact show that

$$\Omega A = 0,$$

$$\Omega B = A \times \frac{a}{2},$$

$$\Omega C = 2B \times \frac{a}{2},$$

$$\Omega D = 3C \times \frac{a}{2},$$

$$\Omega E = 4D \times \frac{a}{2},$$

....

The proof depends on a formula established in Lecture XXVI. of this course [p. 457, above], namely

$$\Omega G - G\Omega = wa - b\Omega,$$

where G is the generator $4(ac-b^2)\partial_b + 5(ad-bc)\partial_c + \ldots$, and w is the weight of the operand.

Thus, observing that the weights of $A, B, C, D, \ldots$ are 3, 4, 5, 6, ... respectively, we have

$$(\Omega G - G\Omega) A = (3a - b\Omega) A,$$

$$(\Omega G - G\Omega) B = (4a - b\Omega) B,$$

$$(\Omega G - G\Omega) C = (5a - b\Omega) C,$$

....

Now, since A is the well-known invariant $a^2d - 3abc + 2b^3$, we may write $\Omega A = 0$ in the first of these equations, which then reduces to

$$\Omega G A = 3aA.$$

But, since $6B = GA,$

we have $6\Omega B = \Omega G A = 3aA.$

Thus $\Omega B = A \times \frac{a}{2}.$

Again, substituting for ΩB in the formula

$$(\Omega G - G\Omega) B = (4a - b\Omega) B,$$

we find $$\Omega G B - G\left(\frac{aA}{2}\right) = 4aB - \frac{ab}{2} A,$$

where, since G (which is linear in $\partial_b, \partial_c, \ldots$ and does not contain ∂_a) does not operate on a,

$$G\left(\frac{aA}{2}\right) = \frac{a}{2} GA = 3aB,$$

and consequently $$\Omega G B + \frac{ab}{2} A = 7aB.$$

Now, $7C = GB - MA;$

so that $7\Omega C = \Omega G B - A\Omega M - M\Omega A.$

But, since $\Omega M = \Omega\left(ac - \frac{5b^2}{4}\right) = -\frac{ab}{2}$ and $\Omega A = 0,$

$$7\Omega C = \Omega G B + \frac{ab}{2} A = 7aB.$$

Thus $\Omega C = 2B \times \frac{a}{2}.$

We may, in exactly the same way, prove that

$$\Omega D = 3C \times \frac{a}{2},$$

$$\Omega E = 4D \times \frac{a}{2},$$

and so on to any extent.

In the following inductive proof it will be convenient to denote the letters

$$A,\ B,\ C,\ D,\ E,\ \ldots$$

by $$u_0,\ u_1,\ u_2,\ u_3,\ u_4,\ \ldots,$$

and then the theorem to be proved is that

$$\Omega u_n = n u_{n-1} \times \frac{a}{2}.$$

When this notation is used, the law of successive derivation which defines the capital letters is expressed by the equation

$$(n+7)\, u_{n+2} - G u_{n+1} + (n+1)\, M u_n = 0, \qquad (1)$$

where G is the generator

$$4(ac-b^2)\partial_b + 5(ad-bc)\partial_c + \ldots, \text{ and } M = ac - \frac{5b^2}{4}.$$

Operating with Ω on the above equation, we obtain

$$(n+7)\, \Omega u_{n+2} - \Omega G u_{n+1} + (n+1)(M\Omega u_n + u_n \Omega M) = 0. \qquad (2)$$

Now, the weights of $u_0, u_1, u_2, \ldots$ are 3, 4, 5, ... respectively, and consequently the operation of

$$\Omega G - G\Omega = wa - b\Omega$$

on u_{n+1} (whose weight is $n+4$) gives

$$(\Omega G - G\Omega)\, u_{n+1} = (n+4)\, a u_{n+1} - b\Omega u_{n+1}.$$

Or, assuming that $\Omega u_\kappa = \kappa u_{\kappa-1} \times \frac{a}{2}$ for all values of κ as far as $n+1$ inclusive (it has previously been shown that $\Omega B = A \times \frac{a}{2}$ and $\Omega C = 2B \times \frac{a}{2}$, so that the theorem is true for $\kappa = 1$ and $\kappa = 2$),

$$\Omega G u_{n+1} = G\Omega u_{n+1} + (n+4)\, a u_{n+1} - b\Omega u_{n+1}$$

$$= (n+1)\, G\left(\frac{a}{2} u_n\right) + (n+4)\, a u_{n+1} - (n+1)\frac{ab}{2} u_n.$$

But (remembering that G does not operate on a, so that $G \cdot \frac{a}{2} u_n = \frac{a}{2} G u_n$) we have, in virtue of equation (1),

$$G\left(\frac{a}{2} u_n\right) = \frac{a}{2}\{(n+6)\, u_{n+1} + nMu_{n-1}\}.$$

Hence it follows that

$$\Omega G u_{n+1} = \frac{n+1}{2} a \{(n+6)\, u_{n+1} + nMu_{n-1}\} + (n+4)\, a u_{n+1} - (n+1)\frac{ab}{2} u_n$$

$$= \frac{(n+2)(n+7)}{2} a u_{n+1} + \frac{n(n+1)}{2} aMu_{n-1} - (n+1)\frac{ab}{2} u_n.$$

On substituting this in (2) we obtain

$$(n+7)\left\{\Omega u_{n+2} - (n+2)\frac{a}{2} u_{n+1}\right\}$$

$$+ (n+1)\, M \left\{\Omega u_n - n\frac{a}{2} u_{n-1}\right\}$$

$$+ (n+1)\, u_n \left\{\Omega M + \frac{ab}{2}\right\} = 0.$$

This reduces to $$\Omega u_{n+2} = (n+2)\frac{a}{2} u_{n+1}.$$

For, according to the assumption previously made in the course of the demonstration,

$$\Omega u_n = n\frac{a}{2} u_{n-1};$$

so that the second term vanishes; and the third term vanishes because

$$\Omega M = \Omega\left(ac - \frac{5b^2}{4}\right) = -\frac{ab}{2}.$$

We have therefore proved that if the theorem is true for Ωu_κ, when κ has any value up to $n+1$ inclusive, it is also true for Ωu_{n+2}. But the theorem holds for $\kappa = 1$, and for $\kappa = 2$. It therefore holds universally for any positive integer value of κ.

Recalling the known values of the reciprocants $M, A, B, C, D, \ldots$ we observe that their principal terms are $ac, a^2d, a^3e, a^4f, a^5g, \ldots$, where it is to be noticed that the most advanced of the small letters in the expression for any capital letter occurs only in the first degree multiplied by a power of a. In other words, $M, A, B, C, D, \ldots$ form a series of Protomorphs, and consequently every Pure Reciprocant can, as we have already seen (vide [p. 384, above]), be expressed as a function of $a, M, A, B, C, D, \ldots$ rational in all of them and integral in all except a.

But it is further to be noticed that whereas

a	is of degree	1	and weight	0,
M	,,	2	,,	2,
A	,,	3	,,	3,
B	,,	4	,,	4,

and in fact that every capital letter is of equal weight and degree.

From this it will follow that every Pure Reciprocant will be the product of a power of a into a function of the capital letters alone.

For let i be the degree and w the weight of any pure reciprocant expressed in terms of $a, M, A, B, C, \ldots$, and suppose one of its terms to be

$$a^\eta M^\theta A^\kappa B^\lambda C^\mu \ldots;$$

then $$\eta + 2\theta + 3\kappa + 4\lambda + 5\mu + \ldots = i,$$

and $$2\theta + 3\kappa + 4\lambda + 5\mu + \ldots = w.$$

Hence $$\eta = i - w,$$

which is the same for every term of the pure reciprocant in question. Thus each term contains a^{i-w} as a factor, and the reciprocant is of the form

$$a^{i-w}\Phi(M, A, B, C, D, \ldots).$$

Let us now consider any Principiant P; since P is a pure reciprocant, we must have

$$P = a^{i-w}\Phi(M, A, B, C, D, \ldots).$$

But Principiants are subject to annihilation by Ω, and consequently $\Omega P = 0$, which gives

$$\frac{d\Phi}{dM}\Omega M + \frac{d\Phi}{dA}\Omega A + \frac{d\Phi}{dB}\Omega B + \frac{d\Phi}{dC}\Omega C + \ldots = 0.$$

On writing for $$\Omega M, \quad \Omega A, \quad \Omega B, \quad \Omega C, \ldots$$

their values $$-b \times \frac{a}{2}, \quad 0, \quad A \times \frac{a}{2}, \quad 2B \times \frac{a}{2}, \ldots$$

we obtain $$\frac{a}{2}(-b\partial_M + A\partial_B + 2B\partial_C + 3C\partial_D + \ldots)\,\Phi = 0.$$

From this it would follow that Φ is an invariant in the two sets of letters

$$-b, M \text{ and } A, B, C, D, \ldots;$$

but it is easy to see that it is an invariant in the latter set exclusively. For M and $A, B, C, D, \ldots$ being all of them pure reciprocants,

$$\Phi \text{ and } \partial_M\Phi, \partial_B\Phi, \partial_C\Phi, \partial_D\Phi, \ldots,$$

which are functions of $M, A, B, C, \ldots$ exclusively, must also be pure reciprocants.

If, then, we operate with V on

$$(-b\partial_M + A\partial_B + 2B\partial_C + 3C\partial_D, \ldots)\,\Phi = 0,$$

we shall find $V(-b\partial_M)\Phi = 0$ (every other term being annihilated by V). Thus

$$V(b\partial_M)\,\Phi = (\partial_M\Phi)\,Vb = 2a^2\partial_M\Phi = 0,$$

and consequently $\partial_M\Phi = 0$. Hence

$$(A\partial_B + 2B\partial_C + 3C\partial_D + \ldots)\,\Phi = 0.$$

The equation $\partial_M\Phi = 0$ shows that M does not appear in the expression for any principiant in terms of the capital letters, while

$$(A\partial_B + 2B\partial_C + 3C\partial_D + \ldots)\,\Phi = 0$$

shows that Φ is an invariant in $A, B, C, D, \ldots$.

We have thus shown that every invariant of

$$(A, B, C, \ldots)(x, y)^j$$

is a principiant, and conversely that every principiant is an invariant of

$$(A, B, C, \ldots)(x, y)^j,$$

or such an invariant multiplied by a power of a.

LECTURE XXIX.

From the theorem that every Principiant is (to a power of *a près*) an Invariant in the reciprocantive elements $A, B, C, \ldots$ we readily deduce its correlative in which, everything else remaining unchanged, the *reciprocantive* elements $A, B, C, \ldots$ are replaced by a set of *invariantive* elements which we call $A_0, A_1, A_2, \ldots$. The equations connecting the new elements with the old ones are as follows:

$$A_0 = A,$$

$$A_1 = B - \left(\frac{b}{2}\right)A,$$

$$A_2 = C - 2\left(\frac{b}{2}\right)B + \left(\frac{b}{2}\right)^2 A,$$

$$A_3 = D - 3\left(\frac{b}{2}\right)C + 3\left(\frac{b}{2}\right)^2 B - \left(\frac{b}{2}\right)^3 A,$$

$$A_4 = E - 4\left(\frac{b}{2}\right)D + 6\left(\frac{b}{2}\right)^2 C - 4\left(\frac{b}{2}\right)^3 B + \left(\frac{b}{2}\right)^4 A,$$

....

We have, in the first place, to prove that $A_0, A_1, A_2, \ldots$ are all of them invariants in the small letters $a, b, c, \ldots$. This is an immediate consequence of the identities

$$\Omega A = 0,$$

$$\Omega B = A \times \frac{a}{2},$$

$$\Omega C = 2B \times \frac{a}{2},$$

....

established in the preceding Lecture, coupled with the fact that $\Omega b = a$. Thus

$$\Omega A_0 = \Omega A = 0,$$

$$\Omega A_1 = -\frac{b}{2}\,\Omega A + \left(\Omega B - A \times \frac{a}{2}\right) = 0,$$

$$\Omega A_2 = \left(\frac{b}{2}\right)^2 \Omega A - 2\left(\frac{b}{2}\right)\left(\Omega B - A \times \frac{a}{2}\right) + \left(\Omega C - 2B \times \frac{a}{2}\right) = 0;$$

and in general, writing the equation which gives A_n in the form

$$A_n = \left(-\frac{b}{2}\right)^n A + n\left(-\frac{b}{2}\right)^{n-1} B + \frac{n(n-1)}{1.2}\left(-\frac{b}{2}\right)^{n-2} C + \frac{n(n-1)(n-2)}{1.2.3}\left(-\frac{b}{2}\right)^{n-3} D + \ldots,$$

and operating on it with Ω, we find

$$\Omega A_n = \left(-\frac{b}{2}\right)^n \Omega A + n\left(-\frac{b}{2}\right)^{n-1}\left(\Omega B - A \times \frac{a}{2}\right) + \frac{n(n-1)}{1.2}\left(-\frac{b}{2}\right)^{n-2}\left(\Omega C - 2B \times \frac{a}{2}\right) + \frac{n(n-1)(n-2)}{1.2.3}\left(-\frac{b}{2}\right)^{n-3}\left(\Omega D - 3C \times \frac{a}{2}\right) + \ldots$$

$= 0$ (each term vanishing separately).

We next observe that

$(A_0, A_1, A_2, \ldots)(x, y)^j$, being equal to $(A, B, C, \ldots)\left(x - \frac{b}{2}y, y\right)^j$,

is a linear transformation of $(A, B, C, \ldots)(x, y)^j$,

and that the determinant of the transformation $\begin{vmatrix} 1 & -\frac{b}{2} \\ 0 & 1 \end{vmatrix}$ is equal to unity.

Hence every invariant in $A_0, A_1, A_2, \ldots$ is equal to the corresponding invariant in $A, B, C, \ldots$, which proves the theorem in question.

Each of the invariantive elements $A_0, A_1, A_2, \ldots$ is, so to say, a *reciprocant* relative to the one which immediately precedes it, just as in the cognate theorem each of the capital letters $A, B, C, \ldots$ was an *invariant* relative to its antecedent. It is in fact easily seen that

$$VA_0 = 0,$$
$$VA_1 = -A_0a^2,$$
$$VA_2 = -2A_1a^2,$$
$$VA_3 = -3A_2a^2,$$
$$\ldots\ldots\ldots\ldots$$

and in general $$VA_n = -nA_{n-1}a^2.$$

Thus, for example, if we operate with V on

$$A_3 = D - 3\left(\frac{b}{2}\right)C + 3\left(\frac{b}{2}\right)^2 B - \left(\frac{b}{2}\right)^3 A,$$

remembering that A, B, C, D are pure reciprocants, we shall find

$$VA_3 = -\frac{3}{2}\left\{C - 2\left(\frac{b}{2}\right)B + \left(\frac{b}{2}\right)^2 A\right\} VB.$$

But $$C - 2\left(\frac{b}{2}\right)B + \left(\frac{b}{2}\right)^2 A = A_2 \text{ and } Vb = 2a^2;$$

so that $$VA_3 = -3A_2a^2.$$

In like manner, operating with V on

$$A_n = (A, B, C, \ldots)\left(-\frac{b}{2}, 1\right)^n,$$

we obtain $$VA_n = -\frac{n}{2}(A, B, C, \ldots)\left(-\frac{b}{2}, 1\right)^{n-1} Vb$$
$$= -nA_{n-1}a^2.$$

This property enables us to give a proof (exactly similar to the proof of the cognate theorem in the preceding Lecture) of the theorem that every principiant is expressible as the product of an invariant in $A_0, A_1, A_2, \ldots$ by a suitable power of a. We first observe that, using N to denote $ac - b^2$,

$$N, A_0, A_1, A_2, \ldots$$

form a series of invariantive protomorphs of equal degree and weight.

Hence it follows that any invariant of degree i and weight w can be expressed in the form

$$a^{i-w}\Phi(N, A_0, A_1, A_2, \ldots),$$

and consequently that every Principiant can be expressed in this form, provided only that

$$V\Phi = 0.$$

Substituting for $VA_0, VA_1, VA_2, \ldots$ their values given above, and at the same time observing that

$$VN = V(ac - b^2) = 5a^2b - 4a^2b = a^2b,$$

we find $$V\Phi = a^2(b\partial_N - A_0\partial_{A_1} - 2A_1\partial_{A_2} - 3A_2\partial_{A_3} - \ldots)\Phi = 0.$$

Finally, we prove that Φ does not contain N, but is an invariant in $A_0, A_1, A_2, \ldots$ alone, by operating with Ω on

$$(b\partial_N - A_0\partial_{A_1} - 2A_1\partial_{A_2} - 3A_2\partial_{A_3} - \ldots)\Phi = 0,$$

when it is easily seen that every term vanishes except the first, which gives

$$\Omega(b\partial_N\Phi) = \Omega b \times \partial_N\Phi = 0,$$

where, $\Omega b = a$ being different from zero, we must have $\partial_N\Phi = 0$.

The invariants $N, A_0, A_1, A_2, \ldots$ obey a law of successive derivation similar to that which holds for the reciprocants $M, A, B, C, \ldots$.

Starting with $N = ac - b^2$ and operating continually with

$$G' = a\frac{d}{dx} - (3i + 2w)b = (4ac - 5b^2)\partial_b + (5ad - 7bc)\partial_c + \ldots,$$

we shall find $$G'N = 5A_0,$$
$$G'A_0 = 6A_1,$$
$$G'A_1 = 7A_2 - NA_0,$$
$$G'A_2 = 8A_3 - 2NA_1,$$
$$G'A_3 = 9A_4 - 3NA_2,$$
$$\ldots\ldots\ldots\ldots$$

and generally $$G'A_n = (n + 6)A_{n+1} - nNA_{n-1}.$$

These equations are exactly analogous to

$$GM = 5A,$$
$$GA = 6B,$$
$$GB = 7C + MA,$$
$$GC = 8D + 2MB,$$
$$GD = 9E + 3MC,$$
$$\ldots\ldots\ldots\ldots$$

in which $M = ac - \frac{5}{4}b^2$, and $GM, GA, GB, \ldots$ are the educts of $M, A, B, \ldots$ obtained by operating with

$$G = a\frac{d}{dx} - (3i + w)b = 4(ac - b^2)\partial_b + 5(ad - bc)\partial_c + \ldots.$$

It should be noticed that the two generators G and G' are connected by the relation

$$G' = G - wb,$$

where w is the weight of the operand.

Also, that

$$Gb = 4(ac - b^2) = 4N, \text{ and } G'b = 4ac - 5b^2 = 4M.$$

We may easily verify that

$$G'N = 5A_0 = 5(a^2d - 3abc + 2b^3)$$

by operating with $G' = (4ac - 5b^2)\partial_b + (5ad - 7bc)\partial_c$ on $N = ac - b^2$.

To prove that $$G'A_0 = 6A_1,$$
we operate on $$A_0 = A,$$
for which the weight is 3, with

$$G' = G - 3b.$$

Thus $$G'A_0 = (G - 3b)A = 6B - 3bA = 6A_1.$$

For by definition $$A_1 = B - \left(\frac{b}{2}\right)A.$$

In general, to find $G'A_n$, we have by definition

$$A_n = (A, B, C, \ldots)\left(-\frac{b}{2}, 1\right)^n,$$

and, since the weight of A_n is $n + 3$,

$$G'A_n = GA_n - (n+3)bA_n.$$

Now

$$GA_n = G(A, B, C, \ldots)\left(-\frac{b}{2}, 1\right)^n$$
$$= (GA, GB, GC, \ldots)\left(-\frac{b}{2}, 1\right)^n - \frac{n}{2}(A, B, C, \ldots)\left(-\frac{b}{2}, 1\right)^{n-1} Gb.$$

Substituting for GA, GB, GC, ... their known values, and remembering that $Gb = 4N$ and that $(A, B, C, \ldots)\left(-\frac{b}{2}, 1\right)^{n-1} = A_{n-1}$, we have

$$GA_n = (6B, 7C, 8D, \ldots)\left(-\frac{b}{2}, 1\right)^n$$
$$+ M(0, A, 2B, 3C, \ldots)\left(-\frac{b}{2}, 1\right)^n - 2nNA_{n-1}$$
$$= 6(B, C, D, \ldots)\left(-\frac{b}{2}, 1\right)^n + (0, C, 2D, 3E, \ldots)\left(-\frac{b}{2}, 1\right)^n$$
$$+ M(0, A, 2B, 3C, \ldots)\left(-\frac{b}{2}, 1\right)^n - 2nNA_{n-1}.$$

But $$(0, C, 2D, 3E, \ldots)\left(-\frac{b}{2}, 1\right)^n$$
$$= nC\left(-\frac{b}{2}\right)^{n-1} + n(n-1)D\left(-\frac{b}{2}\right)^{n-2} + \frac{n(n-1)(n-2)}{1.2}E\left(-\frac{b}{2}\right)^{n-3} + \ldots$$
$$= n(C, D, E, \ldots)\left(-\frac{b}{2}, 1\right)^{n-1};$$

and similarly

$$(0, A, 2B, 3C, \ldots)\left(-\frac{b}{2}, 1\right)^n = n(A, B, C, \ldots)\left(-\frac{b}{2}, 1\right)^{n-1} = nA_{n-1}.$$

Hence

$$GA_n = 6(B, C, D, \ldots)\left(-\frac{b}{2}, 1\right)^n + n(C, D, E, \ldots)\left(-\frac{b}{2}, 1\right)^{n-1}$$
$$+ n(M - 2N)A_{n-1}.$$

Now let $$U = (A, B, C, \ldots)(u, v)^n;$$

then $$\frac{dU}{du} = n(A, B, C, \ldots)(u, v)^{n-1},$$

and $$\frac{dU}{dv} = n(B, C, D, \ldots)(u, v)^{n-1};$$

whence it follows that

$$U = (A, B, C, \ldots)(u, v)^n = u(A, B, C, \ldots)(u, v)^{n-1}$$
$$+ v(B, C, D, \ldots)(u, v)^{n-1}. \qquad (1)$$

Similarly, we see that

$$(B, C, D, \ldots)(u, v)^n = u(B, C, D, \ldots)(u, v)^{n-1}$$
$$+ v(C, D, E, \ldots)(u, v)^{n-1}. \qquad (2)$$

Writing $u = -\frac{b}{2}$ and $v = 1$ in the above equations, and remembering that

$$(A, B, C, \ldots)\left(-\frac{b}{2}, 1\right)^n = A_n,$$

we obtain immediately from (1)

$$(B, C, D, \ldots)\left(-\frac{b}{2}, 1\right)^{n-1} = A_n + \frac{b}{2}A_{n-1},$$

and then (2) gives

$$(C, D, E, \ldots)\left(-\frac{b}{2}, 1\right)^{n-1} = \left(A_{n+1} + \frac{b}{2}A_n\right) + \frac{b}{2}\left(A_n + \frac{b}{2}A_{n-1}\right)$$
$$= A_{n+1} + bA_n + \frac{b^2}{4}A_{n-1}.$$

But it has been shown that

$$GA_n = 6(B, C, D, \ldots)\left(-\frac{b}{2}, 1\right)^n + n(C, D, E, \ldots)\left(-\frac{b}{2}, 1\right)^{n-1}$$
$$+ n(M - 2N)A_{n-1}.$$

Hence, by substitution,

$$GA_n = 6\left(A_{n+1} + \frac{b}{2}A_n\right) + n\left(A_{n+1} + bA_n + \frac{b^2}{4}A_{n-1}\right) + n(M - 2N)A_{n-1}$$
$$= (n+6)A_{n+1} + (n+3)bA_n + n\left(M + \frac{b^2}{4} - 2N\right)A_{n-1}.$$

Now, $G'A_n = GA_n - (n+3)\,bA_n = (n+6)\,A_{n+1} + n\left(M + \frac{b^2}{4} - 2N\right)A_{n-1}$,

where $$M + \frac{b^2}{4} = ac - \frac{5}{4}b^2 + \frac{b^2}{4} = ac - b^2 = N.$$

Thus $$G'A_n = (n+6)\,A_{n+1} - nNA_{n-1},$$

which proves the law of successive derivation for the invariantive elements $A_0, A_1, A_2, \ldots$*.

We now proceed to explain the method of transforming a Principiant, given in terms of the small letters $a, b, c, \ldots$, into one expressed in terms of $a, A, B, C, \ldots$.

Remembering that the expressions for

$$A, B, C, D, E, \ldots$$

have for their most advanced small letters

$$d, e, f, g, h, \ldots,$$

and that, in each capital letter, the most advanced letter occurs only in the first degree, multiplied by a power of a, it follows, as an immediate consequence, that we may, by continually substituting for the most advanced letter, eliminate $d, e, f, g, h, \ldots$ from any rational integral function

$$\phi\,(a, b, c, d, e, f, g, h, \ldots)$$

and thus transform it into another function whose arguments are

$$a, b, c, A, B, C, D, E, \ldots$$

and which is rational in all its arguments, and integral in all of them, with the possible exception of the first argument, a.

But (see Lecture XXVIII.) [above, p. 471] the result of this elimination is known to be

$$a^{i-w}\,\Phi\,(A, B, C, D, E, \ldots)$$

in the case where ϕ is a Principiant of known degree i and weight w. Hence b and c must disappear spontaneously during the process of elimination.

This being so, we can give b and c any arbitrary values, without thereby affecting the result, and it will greatly simplify the work to take $b=0$ and $c=0$.

It is also permissible to take $a=1$; for, although the factor a^{i-w} is thereby lost, it can always be restored in the final result because both i and

* The establishment of the scale of relation between the terms of the $A_0, A_1, A_2, \ldots$ series, and the above proof of it, is due exclusively to Mr Hammond.

w are known numbers. Now, if we write $a=1, b=0, c=0$ in the known expressions for $A, B, C, D, \ldots$, we shall find

$$A = d,$$
$$B = e,$$
$$C = f,$$
$$D = g - \frac{25}{8}d^2,$$
$$E = h - \frac{15}{2}de,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

Hence we have to eliminate $d, e, f, g, h, \ldots$ between the above equations and

$$P = \phi\,(1, 0, 0, d, e, f, g, h, \ldots),$$

where P stands for the given Principiant. In other words, we have to substitute for

$$\begin{array}{cccccccccc} a, & b, & c, & d, & e, & f, & g, & h, & \ldots \\ 1, & 0, & 0, & A, & B, & C, & D + \frac{25}{8}A^2, & E + \frac{15}{2}AB, & \ldots \end{array}$$

in $$P = \phi\,(a, b, c, d, e, f, g, h, \ldots).$$

The result of this substitution will be

$$P = \Phi\,(A, B, C, D, E, \ldots),$$

where, to compensate for the factor lost by taking $a=1$, we must multiply Φ by a^{i-w}. As an easy example, consider the Principiant which Halphen calls Δ, and for which he obtains the expression

$$\begin{vmatrix} b & c & d & e & f \\ a & b & c & d & e \\ -a^2 & 0 & b^2 & 2bc & 2bd + c^2 \\ 0 & a^2 & 2ab & 2ac + b^2 & 2ad + 2bc \\ 0 & 0 & a^2 & 3ab & 3b^2 + 3ac \end{vmatrix}.$$

Here the degree $i=8$ and the weight $w=8$; so that $i-w=0$, and no factor has to be restored. On making the substitutions spoken of, the determinant becomes

$$\begin{vmatrix} 0 & 0 & A & B & C \\ 1 & 0 & 0 & A & B \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2A \\ 0 & 0 & 1 & 0 & 0 \end{vmatrix},$$

which immediately reduces to $AC - B^2$ by striking out the first three columns and the last three rows.

Of this Principiant we shall have more to say hereafter.

LECTURE XXX.

The method of substituting large letters for small ones will be better understood if we employ it to obtain an expression of the form

$$a^{i-w}\Phi(M, A, B, C, D, E, \ldots)$$

for any pure reciprocant

$$\phi(a, b, c, d, e, f, g, h, \ldots)$$

of known degree i and weight w in the small letters.

The transformation is effected by substituting in ϕ for $c, d, e, f, g, h, \ldots$ their values (which are perfectly definite) in terms of $a, b, M, A, B, C, D, E, \ldots$ But since b does not appear in the final result, we are at liberty to give it any arbitrary value, and it will be convenient to take $b=0$, for then (see Lecture XXVIII.) [above, p. 465] we have

$$\begin{aligned}
M &= ac,\\
A &= a^2d,\\
B &= a^3e - 2a^2c^3,\\
C &= a^4f - 5a^3cd,\\
D &= a^5g - \frac{25}{8}a^4d^2 - 6a^4ce + 7a^3c^3,\\
E &= a^6h - \frac{15}{2}a^5de - 7a^5cf + 29a^4c^2d,
\end{aligned}$$

....

There is an additional advantage in taking $b=0$, namely, that then the values of the *invariants* $N, A_0, A_1, A_2, \ldots$ (see their definition at the beginning of* Lecture XXIX.) exactly coincide with those of the *reciprocants* $M, A, B, C, \ldots$ set forth above. Hence, merely interchanging the capital letters, the same substitutions enable us to express any invariant in terms of $a, N, A_0, A_1, \ldots$, as well as any reciprocant in terms of $a, M, A, B, \ldots$.

The solution of the above equations will give $\frac{c}{a}, \frac{d}{a}, \frac{e}{a}, \ldots$ in terms of $\frac{M}{a^2}, \frac{A}{a^3}, \frac{B}{a^4}, \ldots$; but we can, without loss of generality, put $a=1$, when we shall find

$$\begin{aligned}
a &= 1,\\
b &= 0,\\
c &= M,\\
d &= A,\\
e &= B + 2M^2,\\
f &= C + 5MA,\\
g &= D + \frac{25}{8}A^2 + 6MB + 5M^3,\\
h &= E + \frac{15}{2}AB + 7MC + 6MA^2,
\end{aligned}$$

....

[* p. 472, above.]

The substitution of these values in the pure reciprocant

$$\phi(a, b, c, d, e, f, g, h, \ldots)$$

will convert it into

$$\Phi(M, A, B, C, D, E, \ldots).$$

We have written $a=1$ for the sake of simplicity; but without doing this we have, since ϕ is homogeneous of degree i,

$$\phi(a, 0, c, d, e, \ldots) = a^i\phi\left(1, 0, \frac{c}{a}, \frac{d}{a}, \frac{e}{a}, \ldots\right).$$

Hence, substituting for $\frac{c}{a}, \frac{d}{a}, \frac{e}{a}, \ldots$ in terms of $\frac{M}{a^2}, \frac{A}{a^3}, \frac{B}{a^4}, \ldots$,

$$\phi(a, 0, c, d, e, \ldots) = a^i\Phi\left(\frac{M}{a^2}, \frac{A}{a^3}, \frac{B}{a^4}, \ldots\right);$$

or, since $M, A, B, \ldots$ are of weights $2, 3, 4, \ldots$ and Φ is of weight w,

$$\phi(a, 0, c, d, e, \ldots) = a^{i-w}\Phi(M, A, B, \ldots).$$

Thus, in consequence of writing $a=1$, the factor a^{i-w} has been lost; but this factor can always be restored, both i and w being known numbers.

When ϕ is a Principiant, M will not appear in the final result, which will be identical with that obtained by the simpler substitutions of the preceding Lecture. If, for example, we substitute for

$$\begin{array}{llllll}
a, & b, & c, & d, & e, & f,\\
1, & 0, & M, & A, & B+2M^2, & C+5MA,
\end{array}$$

instead of

$$\begin{array}{llllll}
1, & 0, & 0, & A, & B, & C,
\end{array}$$

in the determinant expression for Halphen's Δ, previously given, it becomes

$$\begin{vmatrix}
0 & M & A & B+2M^2 & C+5MA\\
1 & 0 & M & A & B+2M^2\\
-1 & 0 & 0 & 0 & M^2\\
0 & 1 & 0 & 2M & 2A\\
0 & 0 & 1 & 0 & 3M
\end{vmatrix}.$$

Subtracting the 4th row multiplied by M from the first, the determinant reduces to

$$\begin{vmatrix}
0 & A & B & C+3MA\\
1 & M & A & B+2M^2\\
-1 & 0 & 0 & M^2\\
0 & 1 & 0 & 3M
\end{vmatrix}.$$

Again, subtracting the 2nd column multiplied by $3M$ from the last, and reducing, the determinant becomes

$$\begin{vmatrix}
0, & B, & C\\
1, & A, & B-M^2\\
-1, & 0, & M^2
\end{vmatrix} = AC - B^2,$$

where M disappears, as it ought to do, because Δ is a Principiant.

In what follows we shall have frequent occasion to make use of the fact that if R_a is an absolute pure reciprocant, $\dfrac{dR_a}{a^{\frac{1}{3}}dx}$, which we know is a pure reciprocant, is also an absolute one.

This is very easily proved. For let R be any pure reciprocant, of degree i and weight w, which becomes R_a when made absolute by division by a power of a, then

$$R_a = \frac{R}{a^{\frac{\mu}{3}}}, \text{ where } \mu = 3i + w,$$

and, using G as usual to denote the generator for pure reciprocants,

$$\frac{dR_a}{dx} = \frac{GR}{a^{\frac{\mu}{3}+1}}.$$

Hence $$\frac{dR_a}{a^{\frac{1}{3}}dx} = \frac{GR}{a^{\frac{\mu+4}{3}}},$$

which is an absolute pure reciprocant because GR, which is of degree $i+1$ and weight $w+1$, must be divided by $a^{\frac{\mu+4}{3}}$ in order to make it absolute. Thus, if $M_a, A_a, B_a, C_a, \ldots$ are what $M, A, B, C, \ldots$ become when each of them is made absolute by division by a power of a, we have

$$a^{-\frac{1}{3}}\frac{d}{dx}M_a = 5A_a,$$

$$a^{-\frac{1}{3}}\frac{d}{dx}A_a = 6B_a,$$

$$a^{-\frac{1}{3}}\frac{d}{dx}B_a = 7C_a + M_aA_a,$$

....

We shall use these results in deducing the complete primitive of the differential equation

$$AC - B^2 = 0$$

from that of the equation in pure reciprocants,

$$25A^2 - 16M^3 = 0.$$

This equation may be written in the form

$$25A_a^2 = 16M_a^3;$$

whence, by differentiation, we obtain

$$50A_a\left(a^{-\frac{1}{3}}\frac{d}{dx}A_a\right) = 48M_a^2\left(a^{-\frac{1}{3}}\frac{d}{dx}M_a\right),$$

which gives $$50A_a \,.\, 6B_a = 48M_a^2 \,.\, 5A_a;$$

that is, $$5B_a = 4M_a^2.$$

Differentiating this result, we find

$$5(7C_a + M_aA_a) = 40M_aA_a;$$

which gives $$C_a = M_aA_a.$$

We now restore the non-absolute reciprocants M, A, B, C; that is, we write

$$5B = 4M^2 \text{ and } C = MA.$$

Hence $25(AC - B^2) = M(25A^2 - 16M^3) = 0$ (because $25A^2 = 16M^3$).

Now, the equation $AC - B^2 = 0$ remains unaltered by any homographic substitution, so that it will be satisfied not only by any solution of the equation in pure reciprocants $25A^2 - 16M^3 = 0$, but also by any homographic transformation of such solution. But it has been shown (in Lecture XIII., [p. 379, above]) that the complete primitive of $25A^2 - 16M^3 = 0$ is a linear transformation of $y = x^\lambda$, where $\lambda^2 - \lambda + 1 = 0$ (that is, where λ is a cube root of negative unity).

Consequently any homographic transformation of $y = x^\lambda$ is a solution of

$$AC - B^2 = 0.$$

Moreover, this is its complete primitive; for the highest letter, f, which occurs in $AC - B^2$, corresponds to the seventh order of differentiation, and if we write

$$y = \frac{Y}{Z}, \quad x = \frac{X}{Z},$$

where X, Y, Z are general linear functions of $x, y, 1$ (that is, if we make the most general homographic substitution), $y = x^\lambda$ becomes $Y = X^\lambda Z^{1-\lambda}$, which will be found to contain exactly 7 independent arbitrary constants. Thus the complete primitive of $AC - B^2 = 0$ is $Y = X^\lambda Z^{1-\lambda}$, where X, Y, Z are general linear functions of $x, y, 1$, and λ is a cube root of negative unity.

Observe that although any solution of $M = 0$ also makes $A, B, C, \ldots$ all vanish, and so satisfies $AC - B^2 = 0$, we cannot from this infer that a homographic transformation of the parabola $y = x^2$ will be the complete primitive of $AC - B^2 = 0$. For, though $YZ = X^2$ is a solution of $AC - B^2 = 0$, it only contains 5 independent arbitrary constants, and therefore cannot be its complete primitive. Neither can $YZ = X^2$ be obtained from the complete primitive by giving special values to the arbitrary constants. Hence $YZ = X^2$ is a singular solution of $AC - B^2 = 0$.

We may also deduce the differential equation of the curve $Y = X^\lambda Z^{1-\lambda}$, where λ has a general value, from the corresponding equation in pure reciprocants,

$$25(2\lambda^2 - 5\lambda + 2)A^2 + 16(\lambda + 1)^2 M^3 = 0,$$

which has (see [p. 377, above]) for its complete primitive any linear transformation of the general parabola $y = x^\lambda$.

Writing for shortness

$$2\lambda^2-5\lambda+2=p \text{ and } (\lambda+1)^2=q,$$

and at the same time making both A and M absolute, the above equation becomes

$$25pA_a^2+16qM_a^3=0.$$

Hence, by differentiation, we obtain

$$50pA_a \, . \, 6B_a+48qM_a^2 \, . \, 5A_a=0,$$

which gives $$5pB_a+4qM_a^2=0.$$

After a second differentiation we find

$$5p\,(7C_a+M_aA_a)+40qM_aA_a=0\,;$$

that is, $$7pC_a+(p+8q)\,M_aA_a=0.$$

We now replace the absolute reciprocants M_a, A_a, B_a, C_a by M, A, B, C, and thus write the original equation and its two differentials in the form

$$25pA^2=-16qM^3,$$
$$5pB=-4qM^2,$$
$$7pC=-(p+8q)\,MA.$$

Hence we find

$$5^2 \, . \, 7 \, . \, p^2\,(AC-B^2)=-25p\,(p+8q)\,MA^2-16 \, . \, 7q^2M^4$$
$$=16q\,(p+q)\,M^4,$$
$$5^6 \, . \, 7^3 \, . \, p^6\,(AC-B^2)^3=16^3q^3\,(p+q)^3\,M^{12},$$
$$5^8p^4A^8=16^4q^4M^{12},$$

and, eliminating M from the two last equations,

$$2^4 \, . \, 7^3 \, . \, p^2q\,(AC-B^2)^3=5^2\,(p+q)^3\,A^8.$$

Now restoring $$p=2\lambda^2-5\lambda+2=(\lambda-2)\,(2\lambda-1)$$

and $$q=(\lambda+1)^2,$$

we have $$p+q=3\,(\lambda^2-\lambda+1)\,;$$

so that the final equation becomes

$$2^4 \, . \, 7^3\,(\lambda+1)^2\,(\lambda-2)^2\,(2\lambda-1)^2\,(AC-B^2)^3=3^3 \, . \, 5^2\,(\lambda^2-\lambda+1)^3\,A^8.$$

The same reasoning as before will show that, for a general value of λ, the complete primitive of this equation is the general homographic transformation $Y=X^\lambda Z^{1-\lambda}$ of the curve $y=x^\lambda$.

There is, however, a special exceptional case in which the differential equation becomes

$$2^6 \, . \, 7^3\,(AC-B^2)^3=3^3 \, . \, 5^2A^8,$$

the corresponding value of the parameter λ being either 0, 1 or ∞, as may be seen by solving the equation

$$(\lambda+1)^2\,(\lambda-2)^2\,(2\lambda-1)^2=4\,(\lambda^2-\lambda+1)^3.$$

In the case where $\lambda=0$ or ∞ we can, in the same manner as before, show that the complete primitive is a homographic transformation of the curve $y=e^x$ by deducing the differential equation from the corresponding equation in pure reciprocants,

$$25A^2+8M^3=0,$$

whose complete primitive is (see Lecture XIII.) [p. 379 above] a linear transformation of $y=e^x$.

When $\lambda=1$ the corresponding equation in pure reciprocants is

$$25A^2-64M^3=0,$$

whose complete primitive may be shown to be a linear transformation of $y=x\log x$. The reason why these two distinct equations in pure reciprocants lead to the same equation in principiants is that the two curves $y=e^x$ and $y=x\log x$ are *homographically* equivalent but not *linearly* transformable into one another. For we may write the equation $y=x\log x$ in the form $x=e^{\frac{y}{x}}$, which is a homographic transformation of $y=e^x$.

Besides the special case just considered, in which the complete primitive of the equation in Principiants is $\frac{Y}{Z}=e^{\frac{X}{Z}}$, we may notice that in which the parameter λ is either -1, 2, or $\frac{1}{2}$, the differential equation reducing to $A=0$ simply, and its complete primitive $Y=X^\lambda Z^{1-\lambda}$ being the equation to a conic, as it should be. The case where $\lambda^2-\lambda+1=0$ and the differential equation reduces to $AC-B^2=0$ has been considered already. There remains the case in which $\lambda=3$, when the complete primitive becomes $YZ^2=X^3$ (the equation of the general cuspidal cubic) and the differential equation assumes the simple form

$$\left(\frac{AC-B^2}{3}\right)^3=\left(\frac{A}{2}\right)^8,$$

which is therefore the differential equation of cuspidal cubics.

We shall hereafter show that in this case the Principiant

$$2^8\,(AC-B^2)^3-3^3A^8,$$

which is apparently of the 24th degree, loses a factor a^4 and so sinks to the 20th degree. It is, however, generally difficult to determine the power of a contained as a factor in a Principiant given in terms of the large letters.

The results obtained in the present Lecture agree with those of M. Halphen contained in his *Thèse sur les Invariants différentiels* (Paris, Gauthier-Villars, 1878), which contains a complete investigation of the properties of the Principiant $AC-B^2$, which he calls Δ. But our point of view is different from his. He obtains Δ in the form of a determinant from geometrical considerations. With him $\Delta = 0$ is the differential equation which expresses the condition that, at a point x, y on any curve, a nodal cubic shall exist, having its node at x, y, and such that *one of its branches* shall have 8-point contact with the curve at that point. With us $AC - B^2$ is the simplest example, after the Mongian A, of an invariant in the capital letters $A, B, C, \ldots$.

LECTURE XXXI.

We may include λ among the arbitrary constants in the primitive equation $Y = X^\lambda Z^{1-\lambda}$, which can also be written in the form

$$\lambda \log X - \log Y + (1 - \lambda) \log Z = 0,$$

or (X, Y, Z being general linear functions of x, y, 1) in the equivalent form

$$\lambda \log (y + \alpha x + \beta) - \log (y + \alpha' x + \beta') + (1 - \lambda) \log (y + \alpha'' x + \beta'') = \text{const.},$$

which evidently contains 8 independent arbitrary constants.

One of these will be made to disappear by differentiation, and thus we shall obtain a differential equation of the first order, containing 7 arbitrary constants, identical (when the constants are rearranged) with

$$(y - xt)(lx + my) + t(l'x + m'y + n') + l''x + m''y + n'' = 0,$$

which is known as Jacobi's Equation.

For, by differentiating the primitive equation, we obtain

$$\lambda (t + \alpha)(y + \alpha x + \beta)^{-1} - (t + \alpha')(y + \alpha' x + \beta')^{-1} + (1 - \lambda)(t + \alpha'')(y + \alpha'' x + \beta'')^{-1} = 0,$$

which, when cleared of negative indices by multiplication, becomes

$$\lambda (y + \alpha' x + \beta') \{(y + \alpha'' x + \beta'')(t + \alpha) - (y + \alpha x + \beta)(t + \alpha'')\} + (y + \alpha x + \beta)\{(y + \alpha' x + \beta')(t + \alpha'') - (y + \alpha'' x + \beta'')(t + \alpha')\} = 0.$$

Writing this equation in the equivalent form

$$\lambda (y + \alpha' x + \beta') \{(\alpha - \alpha'')(y - xt) + (\beta'' - \beta) t + (\alpha\beta'' - \alpha''\beta)\} + (y + \alpha x + \beta)\{(\alpha'' - \alpha')(y - xt) + (\beta' - \beta'') t + (\alpha''\beta' - \alpha'\beta'')\} = 0,$$

it is easily seen to be identical with Jacobi's equation given above.

The seven arbitrary constants which occur in Jacobi's equation are the mutual ratios of the eight coefficients l, m, l', m', n', l'', m'', n'', any one of which may have an arbitrarily chosen value assigned to it.

Taking $m = -1$, the equation may be written in the form

$$Pt + lxy - y^2 + l''x + m''y + n'' = 0,$$

where $$P = l'x + m'y + n' - lx^2 + xy.$$

In order to eliminate n'' and l'', we differentiate the above equation twice. The first differentiation gives

$$2aP + t(P' + lx - 2y + m'') + ly + l'' = 0,$$

where $P' = \dfrac{dP}{dx} = l' + m't - 2lx + y + xt$, and the second differentiation gives

$$6bP + 2a(2P' + lx - 2y + m'') + t(P'' + 2l - 2t) = 0.$$

Now, $P'' = \dfrac{dP'}{dx} = 2a(m' + x) + 2(t - l)$; so that, on substituting this value, the above equation becomes

$$3bP + aQ = 0, \qquad (1)$$

where $$Q = 2P' + lx - 2y + m'' + m't + xt = 2l' + 3m't - 3lx + 3xt + m''.$$

Differentiating (1) we have

$$12cP + 3bP' + 3bQ + aQ' = 0,$$

where $$Q' = 3(t - l) + 6a(x + m') = 3R + 6aS, \text{ suppose.}$$

Thus we have $$4cP + bP' + bQ + aR + 2a^2S = 0. \qquad (2)$$

Differentiating this 4 times in succession, and at each step substituting for

$$P'', \quad Q', \quad R', \quad S',$$

their values $$2R + 2aS, \quad 3R + 6aS, \quad 2a, \quad 1,$$

we obtain 4 more equations, from which, combined with the 2 previously obtained, we can eliminate

$$P, P', Q, R, S.$$

Thus, differentiating (2), we find

$$20dP + 8cP' + b(2R + 2aS) + 4cQ + b(3R + 6aS) + 3bR + 2a^2 + 12abS + 2a^2 = 0;$$

that is, $$5dP + 2cP' + cQ + 2bR + 5abS + a^2 = 0, \qquad (3)$$

and continuing the same process,

$$6eP + 3dP' + dQ + 3cR + (6ac + 3b^2) S + 3ab = 0, \qquad (4)$$

$$7fP + 4eP' + eQ + 4dR + (7ad + 7bc) S + (4ac + 2b^2) = 0, \qquad (5)$$

$$8gP + 5fP' + fQ + 5eR + (8ae + 8bd + 4c^2) S + (5ad + 5bc) = 0. \qquad (6)$$

The result of elimination is

$$\begin{vmatrix} 3b & 0 & a & 0 & 0 & 0 \\ 4c & b & b & a & 2a^2 & 0 \\ 5d & 2c & c & 2b & 5ab & a^2 \\ 6e & 3d & d & 3c & 6ac + 3b^2 & 3ab \\ 7f & 4e & e & 4d & 7ad + 7bc & 4ac + 2b^2 \\ 8g & 5f & f & 5e & 8ae + 8bd + 4c^2 & 5ad + 5bc \end{vmatrix} = 0,$$

where the determinant equated to zero is a Principiant.

In his *Thèse sur les Invariants différentiels*, p. 42, M. Halphen states that this equation can be found by eliminating the constants from Jacobi's equation, but he does not set out the work. When in the above determinant twice the 3rd column is added to the second, it becomes exactly identical with the one given by Halphen, which he calls T.

We proceed to express the above result in terms of the capital letters, using the method explained in Lecture XXIX., and observing that the determinant is of degree 8 and of weight 12; so that in this case $i-w=8-12=-4$, showing that the final result has to be multiplied by a^{-4}.

Substituting in the determinant for

$$\begin{matrix} a & b & c & d & e & f & g \\ 1 & 0 & 0 & A & B & C & D+\frac{25}{8}A^2, \end{matrix}$$

it becomes

$$\begin{vmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 5A & 0 & 0 & 0 & 0 & 1 \\ 6B & 3A & A & 0 & 0 & 0 \\ 7C & 4B & B & 4A & 7A & 0 \\ 8D+25A^2 & 5C & C & 5B & 8B & 5A \end{vmatrix}.$$

Subtracting the last column multiplied by $5A$ from the first, and the 4th column multiplied by 2 from the 5th, and then striking out rows and columns, we obtain

$$\begin{vmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 6B & 3A & A & 0 & 0 & 0 \\ 7C & 4B & B & 4A & -A & 0 \\ 8D & 5C & C & 5B & -2B & 5A \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 6B & 3A & 0 & 0 & 0 \\ 7C & 4B & 4A & -A & 0 \\ 8D & 5C & 5B & -2B & 5A \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 & 1 \\ 6B & 3A & 0 & 0 \\ 7C & 4B & -A & 0 \\ 8D & 5C & -2B & 5A \end{vmatrix} = \begin{vmatrix} 6B & 3A & 0 \\ 7C & 4B & A \\ 8D & 5C & 2B \end{vmatrix}$$

$$= 24\,(A^2D-3ABC+2B^3).$$

If, using Halphen's notation, we call the principiant now under consideration T, what we have proved is that

$$T = 24a^{-4}\,(A^2D-3ABC+2B^3),$$

and consequently that $A^2D-3ABC+2B^3$ is divisible by a^4.

The differential equation $T=0$ corresponds, as we have seen, to the complete primitive $Y=X^{\lambda}Z^{1-\lambda}$, in which λ is counted as one of the arbitrary constants.

This result may be otherwise obtained. For we have shown in the preceding Lecture that the differential equation of the seventh order, from which all the arbitrary constants except λ have disappeared, has the form

$$(AC-B^2)^3 = \kappa A^8,$$

where κ depends solely on λ.

Writing this equation in the form

$$(AC-B^2)\,A^{-\frac{8}{3}} = \text{const.},$$

and differentiating with respect to x, we remove the remaining arbitrary constant, and thus obtain the differential equation of the 8th order free from all arbitrary constants, a result which, to a factor *près*, must coincide with

$$T=0.$$

We proceed to show how this differentiation may be performed without introducing any of the small letters. In the first place, it is clear that since

$$G = 4\,(ac-b^2)\,\partial_b + 5\,(ad-bc)\,\partial_c + 6\,(ae-bd)\,\partial_d + \dots$$

does not contain ∂_a and is linear in the other differential reciprocals $\partial_b, \partial_c, \dots,$

$$Ga^\theta\Phi\,(A, B, C, \dots) = a^\theta G\Phi\,(A, B, C, \dots)$$

$$= a^\theta\left(\frac{d\Phi}{dA}\,GA + \frac{d\Phi}{dB}\,GB + \frac{d\Phi}{dC}\,GC + \dots\right).$$

And since we have

$$GA = 6B,$$
$$GB = 7C + MA,$$
$$GC = 8D + 2MB,$$
$$\dots\dots\dots\dots\dots\dots$$

it follows immediately that

$$Ga^\theta\Phi\,(A, B, C, \dots) = a^\theta\,(6B\partial_A + 7C\partial_B + 8D\partial_C + \dots)\,\Phi$$
$$+ a^\theta M\,(A\partial_B + 2B\partial_C + 3C\partial_D + \dots)\,\Phi.$$

This is true for any function of the capital letters, whatever its nature may be; but when Φ is a principiant, it is also an invariant in the large letters; so that in this case we have

$$(A\partial_B + 2B\partial_C + 3C\partial_D + \dots)\,\Phi = 0$$

and

$$Ga^\theta\Phi = a^\theta\,(6B\partial_A + 7C\partial_B + 8D\partial_C + \dots)\,\Phi.$$

Now, the operation of G on a function of degree i and weight w is equivalent to that of $a\frac{d}{dx}-(3i+w)\,b$, or to that of $a\frac{d}{dx}$, when both $i=0$ and $w=0$ (which happens in the case of a plenarily absolute form). Hence, if we suppose Φ to be a plenarily absolute principiant, $G\Phi$ is also a principiant, though not a plenarily absolute one.

For a is a principiant, and $\frac{d\Phi}{dx}$ is a principiant; therefore $a\frac{d\Phi}{dx}$ or $G\Phi$ is one also*. Thus

$$6B\partial_A+7C\partial_B+8D\partial_C+\ldots,$$

acting on any plenarily absolute principiant, generates another principiant, but not a plenarily absolute one.

We now resume the consideration of the equation

$$(AC-B^2)\,A^{-\frac{8}{3}}=\text{const.}$$

Differentiating and multiplying by a, we have

$$a\frac{d}{dx}\,\{(AC-B^2)\,A^{-\frac{8}{3}}\}=0.$$

Hence, by what precedes,

$$(6B\partial_A+7C\partial_B+8D\partial_C)\,\{(AC-B^2)\,A^{-\frac{8}{3}}\}=0\,;$$

or, using Θ to denote the operator,

$$6B\partial_A+7C\partial_B+8D\partial_C+\ldots,$$

$$A^{-\frac{8}{3}}\,\Theta\,(AC-B^2)-\frac{8}{3}\,A^{-\frac{11}{3}}\,(AC-B^2)\,\Theta A=0\,;$$

or, observing that $\Theta A=6B$,

$$A\Theta\,(AC-B^2)-16B\,(AC-B^2)=0.$$

This gives $\quad A\,(6BC-14BC+8AD)-16B\,(AC-B^2)=0\,;$

or finally $\quad A^2D-3ABC+2B^3=0.$

We may find a generator for principiants expressed in terms of the large letters similar to the expression for the reciprocant generator G in terms of

* See the concluding paragraph of Lecture XXV. [p. 450 above], where it was shown that P, being a principiant (of degree i and weight w), $a\frac{dP}{dx}-(3i+w)\,bP$ is a reciprocant, and $a\frac{dP}{dx}-(3i+2w)\,bP$ an invariant. This proves, what we omitted to mention there, that P being a *zero-weight* principiant,

$$GP=\left(a\frac{d}{dx}-3ib\right)P \text{ is a principiant.}$$

It may here be remarked that a principiant of degree i and of *zero weight* is equal to the corresponding plenarily absolute principiant (which is a function of the large letters only) multiplied by the factor a^i, on which the operator G does not act.

the small letters. For let P be any principiant, of weight w, which, when reduced to zero weight by division by $A^{\frac{w}{3}}$, becomes $PA^{-\frac{w}{3}}$; then

$$\Theta\,(PA^{-\frac{w}{3}})$$

is a principiant. But

$$\Theta\,(PA^{-\frac{w}{3}})=A^{-\frac{w}{3}-1}\,(A\Theta-2wB)\,P,$$

where, remembering that $A^{-\frac{w}{3}-1}$ is a principiant, $(A\Theta-2wB)\,P$ is one also.

Now, the weights of $\quad A, B, C, D, \ldots$

being $\quad 3, 4, 5, 6, \ldots,$

we may write $\quad w=3A\partial_A+4B\partial_B+5C\partial_C+6D\partial_D+\ldots,$

and consequently

$$\begin{aligned}A\Theta-2wB&=A\,(6B\partial_A+7C\partial_B+8D\partial_C+9E\partial_D+\ldots)\\&\quad-2B\,(3A\partial_A+4B\partial_B+5C\partial_C+6D\partial_D+\ldots)\\&=(7AC-8B^2)\,\partial_B+(8AD-10BC)\,\partial_C+(9AE-12BD)\,\partial_D+\ldots,\end{aligned}$$

which is the generator in question.

As an easy example of its use, suppose it to operate on $AC-B^2$; then

$$\begin{aligned}&\{(7AC-8B^2)\,\partial_B+(8AD-10BC)\,\partial_C\}\,(AC-B^2)\\&=-2B\,(7AC-8B^2)+A\,(8AD-10BC)\\&=8\,(A^2D-3ABC+2B^3).\end{aligned}$$

The generator just obtained,

$$(7AC-8B^2)\,\partial_B+(8AD-10BC)\,\partial_C+(9AE-12BD)\,\partial_D+\ldots,$$

is a linear combination of Cayley's two generators (given in Lecture IV., [p. 327, above]), which, when we write $A, B, C, \ldots$ instead of the corresponding small letters, become

$$(AC-B^2)\,\partial_B+(AD-BC)\,\partial_C+(AE-BD)\,\partial_D+\ldots$$

and $\quad (AC-2B^2)\,\partial_B+(2AD-4BC)\,\partial_C+(3AE-6BD)\,\partial_D+\ldots.$

Thus we shall obtain the principiant generator by adding the second of Cayley's generators to six times the first. Either of Cayley's generators acting on a principiant would of course give an invariant in the large letters (that is, a principiant), but the combination we have used has special relation to the theory of the generation of principiants by differentiation.

LECTURE XXXII.

I will now pass on to the consideration of the Principiant which, when equated to zero, gives the Differential Equation to the most general Algebraic Curve of any order.

The Differential Equation to a Conic (see the reference given [p. 380, above]) was obtained by Monge in the first decade of this century. This was followed by the determination, in 1868, by Mr Samuel Roberts, of the Differential Equation to the general Cubic (see Vol. X. p. 47 of Mathematical Questions and Solutions from the *Educational Times*). I do not consider that any substantial advance was made upon this by Mr Muir, in the *Philosophical Magazine* for February, 1886, except that he sets out explicitly the quantities to be eliminated in obtaining the final result. These may, of course, be collected from the processes indicated by Mr Roberts, but are not set forth by him. In speaking of the history of this part of the subject, I pass over M. Halphen's process for obtaining the Differential Equation to a Conic. It is very ingenious, like everything that proceeds from his pen, but, being founded on the solution of a quadratic equation, does not admit of being extended to forms of a higher degree, and consequently, viewed in the light of subsequent experience, must be regarded as faulty in point of method.

Let the Differential Equation to a curve of any order, when written in its simplest form, containing no extraneous factor, be $\chi = 0$. It is convenient to give χ a single name; I call it the Criterion. The integral of the Criterion to a curve of order n must contain as many arbitrary constants as there are ratios between the coefficients of a curve of the nth order. The number of these ratios being $\dfrac{n^2+3n+2}{2} - 1$, the order of the Criterion ought to be $\dfrac{n^2+3n}{2}$.

It must be independent of Perspective Projection, because projection does not affect the order of a curve. Hence it is a Principiant, and as such ought not (when y is regarded as the dependent and x as the independent variable) to contain either x, y or $\dfrac{dy}{dx}$ (see Lecture XXIV. [p. 438, above]).

Let $U = 0$ be an algebraical equation of the nth order between x, y. I write symbolically

$$U = (p + qx + y)^n = u^n,$$

where the different powers and products of $p, q, 1$ which occur in the expan-

sion of u^n are considered as representing the different coefficients in U; so that, for example, if $n = 3$ the coefficients of

$$y^3,\ 3y^2x,\ 3y^2,\ 3yx^2,\ 6yx,\ 3y,\ x^3,\ 3x^2,\ 3x,\ 1$$

are represented by

$$1,\ q,\ p,\ q^2,\ pq,\ p^2,\ q^3,\ pq^2,\ p^2q,\ p^3.$$

The number of terms in U is

$$1 + 2 + 3 + \dots + (n+1) = \frac{(n+1)(n+2)}{2}.$$

The number of these containing y is

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}.$$

To obtain the Differential Equation we equate to zero the Differential Derivatives of U of all orders from $n+1$ to $\frac{1}{2}(n^2+3n)$ inclusive, and from the $\frac{1}{2}(n^2+n)$ equations thus formed eliminate the $\frac{1}{2}(n^2+n)$ coefficients of the terms in U containing y.

All the coefficients of pure powers of x will obviously disappear under differentiation; for no power of x higher than x^n occurs in U, and no differential derivative of U of lower order than $n+1$ is taken.

We thus find a differential equation of the order $\frac{1}{2}(n^2+3n)$, free from all the $\frac{1}{2}(n^2+3n+2)$ coefficients of U. This equation might conceivably contain x, y and *all* the successive differential derivatives of y with respect to x. But we know *à priori* that it ought not to contain either x, y or $\dfrac{dy}{dx}$; and in fact we shall be able so to conduct the elimination that x, y and $\dfrac{dy}{dx}$ appear only in the quantities to be eliminated and not in the final result.

Treating $u = p + qx + y$ as an ordinary algebraical quantity, we have, by Taylor's theorem,

$$\frac{1}{1.2.3\dots r}\cdot\frac{d^r u^n}{dx^r} = \text{co. } h^r \text{ in } \left(u + u_1 h + u_2\frac{h^2}{1.2} + u_3\frac{h^3}{1.2.3} + \dots\right)^n,$$

where $u_1, u_2, u_3, \dots$ are the successive differential derivatives of u with respect to x. And this result will remain true when for u^n we write U, meaning thereby that $\dfrac{1}{1.2.3\dots r}\cdot\dfrac{d^r U}{dx^r}$ will be the quantitative interpretation of the function of $u, u_1, u_2, \dots$ which multiplies h^r in the expansion of

$$\left(u + u_1 h + u_2\frac{h^2}{1.2} + \dots\right)^n,$$

subject to the condition that this function shall be *linear* in the coefficients of U. This condition can be fulfilled in only one way, so that there is no ambiguity in such interpretation. Hence the equations obtained by equating to zero the successive differential derivatives of U of all orders from $n+1$ to $\frac{1}{2}(n^2+3n)$ inclusive may be written under the form

$$\text{co. } h^r \text{ in } \left(u + u_1 h + u_2 \frac{h^2}{1.2} + u_3 \frac{h^3}{1.2.3} + \dots\right)^n = 0,$$

where $$r = n+1,\ n+2,\ n+3,\ \dots\ \tfrac{1}{2}(n^2+3n).$$

Now, using $y_1, y_2, y_3, \dots$ to denote the successive differential derivatives of y with respect to x, we have

$$u_1 = q + y_1, \quad u_2 = y_2, \quad u_3 = y_3, \dots,$$

and, in general, $u_i = y_i$ when i is any positive integer greater than 1. Thus

$$\text{co. } h^r \text{ in } \left(u + u_1 h + y_2 \frac{h^2}{1.2} + y_3 \frac{h^3}{1.2.3} + \dots\right)^n = 0;$$

or, employing the usual modified derivatives $a, b, c, \dots$,

$$\text{co. } h^r \text{ in } (u + u_1 h + ah^2 + bh^3 + ch^4 + \dots)^n = 0.$$

Writing now $$Q = ah^2 + bh^3 + ch^4 + \dots,$$

and expanding $(u + u_1 h + Q)^n$ in ascending powers of Q, we have

$$\text{co. } h^r \text{ in } \left\{(u+u_1h)^n + n(u+u_1h)^{n-1}Q + \frac{n(n-1)}{1.2}(u+u_1h)^{n-2}Q^2 + \dots\right\} = 0,$$

where, remembering that $r > n$, the value of co. h^r in $(u+u_1h)^n$ is zero; so that, omitting this term, we may write

$$\text{co. } h^r \text{ in } \left\{n(u+u_1h)^{n-1}Q + \frac{n(n-1)}{1.2}(u+u_1h)^{n-2}Q^2 + \dots + Q^n\right\} = 0.$$

The quantities to be eliminated will now be combinations of the various powers of u, u_1 and 1. Their number will be the same as that of the terms in $(u, u_1, 1)^{n-1}$, which is $\frac{1}{2}(n^2+n)$, the same number as that of the equations between which the elimination is to be performed.

We now use $(m.\mu)$ to denote the coefficient of h^m in Q^μ (which, since

$$Q = ah^2 + bh^3 + ch^4 + \dots,$$

will be independent of the combinations of u and u_1 to be eliminated), and in writing out the $\frac{1}{2}(n^2+n)$ equations which result from making the coefficients of $h^{n+1}, h^{n+2}, \dots h^{\frac{n^2+3n}{2}}$ in

$$n(u+u_1h)^{n-1}Q + \frac{n(n-1)}{1.2}(u+u_1h)^{n-2}Q^2 + \dots + Q^n$$

vanish, we arrange their terms according to ascending values of m and μ. Thus, making the coefficient of h^{n+1} vanish, we find

$$nu_1^{n-1}(2.1) + n(n-1)u_1^{n-2}u(3.1) + \frac{n(n-1)}{1.2}u_1^{n-2}(3.2) + \dots + (n+1.n) = 0,$$

and similarly, making the coefficient of h^{n+2} vanish,

$$nu_1^{n-1}(3.1) + n(n-1)u_1^{n-2}u(4.1) + \frac{n(n-1)}{1.2}u_1^{n-2}(4.2) + \dots + (n+2.n) = 0.$$

So in general the equation obtained by making the coefficient of $h^{n+\kappa}$ vanish consists of a series of numerical multiples (which are independent of the value of κ) of $u_1^{n-\theta}u^{\theta-\eta}(\theta+\kappa, \eta)$ where η has all values from 1 to θ inclusive, and θ all values from 1 to n inclusive. Hence, by elimination, we find

$$\begin{vmatrix}
(2.1) & (3.1) & (3.2) & (4.1) & (4.2) & (4.3) & (5.1) & (5.2) & (5.3) & (5.4)\dots \\
(3.1) & (4.1) & (4.2) & (5.1) & (5.2) & (5.3) & (6.1) & (6.2) & (6.3) & (6.4)\dots \\
(4.1) & (5.1) & (5.2) & (6.1) & (6.2) & (6.3) & (7.1) & (7.2) & (7.3) & (7.4)\dots \\
(5.1) & (6.1) & (6.2) & (7.1) & (7.2) & (7.3) & (8.1) & (8.2) & (8.3) & (8.4)\dots \\
(6.1) & (7.1) & (7.2) & (8.1) & (8.2) & (8.3) & (9.1) & (9.2) & (9.3) & (9.4)\dots \\
(7.1) & (8.1) & (8.2) & (9.1) & (9.2) & (9.3) & (10.1) & (10.2) & (10.3) & (10.4)\dots \\
(8.1) & (9.1) & (9.2) & (10.1) & (10.2) & (10.3) & (11.1) & (11.2) & (11.3) & (11.4)\dots \\
(9.1) & (10.1) & (10.2) & (11.1) & (11.2) & (11.3) & (12.1) & (12.2) & (12.3) & (12.4)\dots \\
(10.1) & (11.1) & (11.2) & (12.1) & (12.2) & (12.3) & (13.1) & (13.2) & (13.3) & (13.4)\dots \\
(11.1) & (12.1) & (12.2) & (13.1) & (13.2) & (13.3) & (14.1) & (14.2) & (14.3) & (14.4)\dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots
\end{vmatrix} = 0,$$

where the determinant on the left-hand side, consisting of $\frac{1}{2}(n^2+n)$ rows and columns, is the Criterion of the curve of the nth order.

Thus in the case of the Cubic Criterion, which we shall specially consider, we have $n = 3$, and the elimination of $3u_1^2$, $6u_1u$, $3u_1$, $3u^2$, $3u$ and 1 between the six equations

$$\begin{aligned}
3u_1^2(2.1) + 6u_1u(3.1) + 3u_1(3.2) + 3u^2(4.1) + 3u(4.2) + (4.3) &= 0, \\
3u_1^2(3.1) + 6u_1u(4.1) + 3u_1(4.2) + 3u^2(5.1) + 3u(5.2) + (5.3) &= 0, \\
3u_1^2(4.1) + 6u_1u(5.1) + 3u_1(5.2) + 3u^2(6.1) + 3u(6.2) + (6.3) &= 0, \\
3u_1^2(5.1) + 6u_1u(6.1) + 3u_1(6.2) + 3u^2(7.1) + 3u(7.2) + (7.3) &= 0, \\
3u_1^2(6.1) + 6u_1u(7.1) + 3u_1(7.2) + 3u^2(8.1) + 3u(8.2) + (8.3) &= 0, \\
3u_1^2(7.1) + 6u_1u(8.1) + 3u_1(8.2) + 3u^2(9.1) + 3u(9.2) + (9.3) &= 0,
\end{aligned}$$

gives the Cubic Criterion in the form of the determinant

$$\begin{vmatrix}
(2.1) & (3.1) & (3.2) & (4.1) & (4.2) & (4.3) \\
(3.1) & (4.1) & (4.2) & (5.1) & (5.2) & (5.3) \\
(4.1) & (5.1) & (5.2) & (6.1) & (6.2) & (6.3) \\
(5.1) & (6.1) & (6.2) & (7.1) & (7.2) & (7.3) \\
(6.1) & (7.1) & (7.2) & (8.1) & (8.2) & (8.3) \\
(7.1) & (8.1) & (8.2) & (9.1) & (9.2) & (9.3)
\end{vmatrix}.$$

Remembering that

$$(m\,.\,\mu) = \text{co. } h^m \text{ in } (ah^2 + bh^3 + ch^4 + \ldots)^\mu,$$

it is easy to express the Criterion explicitly in terms of $a, b, c, \ldots$.

Thus, since

$$(ah^2 + bh^3 + ch^4 + \ldots)^2 = a^2h^4 + 2abh^5 + (2ac + b^2)h^6 + (2ad + 2bc)h^7 + (2ae + 2bd + c^2)h^8 + (2af + 2be + 2cd)h^9 + \ldots$$

and

$$(ah^2 + bh^3 + ch^4 + \ldots)^3 = a^3h^6 + 3a^2bh^7 + (3a^2c + 3ab^2)h^8 + (3a^2d + 6abc + b^3)h^9 + \ldots,$$

the Cubic Criterion may be written in the form

$$\begin{vmatrix} a & b & 0 & c & a^2 & 0 \\ b & c & a^2 & d & 2ab & 0 \\ c & d & 2ab & e & 2ac + b^2 & a^3 \\ d & e & 2ac + b^2 & f & 2ad + 2bc & 3a^2b \\ e & f & 2ad + 2bc & g & 2ae + 2bd + c^2 & 3a^2c + 3ab^2 \\ f & g & 2ae + 2bd + c^2 & h & 2af + 2be + 2cd & 3a^2d + 6abc + b^3 \end{vmatrix}$$

in which it was originally obtained by Mr Roberts.

M. Halphen has remarked that the minor of h in the Cubic Criterion is the Principiant which he calls Δ (our $AC - B^2$) multiplied by a (see p. 50 of his *Thèse*).

We proceed to determine the degree and weight of the Criterion of the curve of the nth order. These are the same as the degree and weight of its diagonal

$$(2\,.\,1)(4\,.\,1)(5\,.\,2)(7\,.\,1)(8\,.\,2)(9\,.\,3)(11\,.\,1)(12\,.\,2)(13\,.\,3)(14\,.\,4)\ldots,$$

which consists of $\frac{1}{2}(n^2 + n)$ factors, separable into n groups,

$$(2\,.\,1),\ (4\,.\,1)(5\,.\,2),\ (7\,.\,1)(8\,.\,2)(9\,.\,3),\ (11\,.\,1)(12\,.\,2)(13\,.\,3)(14\,.\,4), \ldots$$

containing $1, 2, 3, 4, \ldots n$ factors respectively. Now,

$$(m\,.\,\mu) = \text{co. } h^m \text{ in } (ah^2 + bh^3 + ch^4 + \ldots)^\mu$$
$$= \text{co. } h^{m-2\mu} \text{ in } (a + bh + ch^2 + \ldots)^\mu,$$

and consequently $(m\,.\,\mu)$ is of degree μ and weight $m - 2\mu$. Hence the degree of the Criterion (found by adding together the second numbers of the duads which occur in the diagonal) is

$$1 + (1 + 2) + (1 + 2 + 3) + (1 + 2 + 3 + 4) + \ldots + (1 + 2 + 3 + \ldots + n)$$
$$= 1 + 3 + 6 + 10 + \ldots + \frac{n^2 + n}{2}$$
$$= \frac{n(n+1)(n+2)}{6}.$$

To find the weight of the Criterion, we begin by arranging the factors of its diagonal according to their weight. This is done by writing each group of factors in reverse order, so that the diagonal is written thus:

$$(2\,.\,1)(5\,.\,2)(4\,.\,1)(9\,.\,3)(8\,.\,2)(7\,.\,1)(14\,.\,4)(13\,.\,3)(12\,.\,2)(11\,.\,1)\ldots.$$

The weights of the factors are now seen to be $0, 1, 2, 3, \ldots \frac{n^2+n}{2} - 1$; there being $\frac{1}{2}(n^2+n)$ factors in the diagonal, one of them of zero weight. Hence the weight of the Criterion is

$$1 + 2 + 3 + \ldots + \left(\frac{n^2+n}{2} - 1\right)$$
$$= \frac{\left(\frac{n^2+n}{2} - 1\right)\frac{n^2+n}{2}}{2} = \frac{(n-1)n(n+1)(n+2)}{8}.$$

If, in the above formulae, we make $n = 2$, we shall find that the degree is 4 and the weight 3, whereas the Mongian $a^2d - 3abc + 2b^3$ (which is the Criterion of the second order) is of degree 3 and weight 3.

To account for this discrepancy, observe that in this case

$$\begin{vmatrix} (2\,.\,1) & (3\,.\,1) & (3\,.\,2) \\ (3\,.\,1) & (4\,.\,1) & (4\,.\,2) \\ (4\,.\,1) & (5\,.\,1) & (5\,.\,2) \end{vmatrix} = \begin{vmatrix} a & b & 0 \\ b & c & a^2 \\ c & d & 2ab \end{vmatrix},$$

which is divisible by a, the other factor being the Mongian, as may easily be verified. This is the only case in which the determinant expression for the Criterion contains an irrelevant factor.

To express the Cubic Criterion in terms of a, A, B, C, D, E, we first remark that its degree is $\frac{3\,.\,4\,.\,5}{6} = 10$, and its weight $\frac{2\,.\,3\,.\,4\,.\,5}{8} = 15$. Thus the Cubic Criterion is expressible as the product of $a^{-5}(\overset{\text{deg.}}{10} - \overset{\text{wt.}}{15} = -5)$ into a function of the capital letters, which we determine by the usual method of substituting for

$$\begin{matrix} a, & b, & c, & d, & e, & f, & g & , & h & , \\ 1, & 0, & 0, & A, & B, & C, & D + \frac{25}{8}A^2, & & E + \frac{15}{2}AB. & \end{matrix}$$

When these substitutions are made, the Cubic Criterion becomes

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & A & 0 & 0 \\ 0 & A & 0 & B & 0 & 1 \\ A & B & 0 & C & 2A & 0 \\ B & C & 2A & D + \frac{25}{8}A^2 & 2B & 0 \\ C & D + \frac{25}{8}A^2 & 2B & E + \frac{15}{2}AB & 2C & 3A \end{vmatrix}.$$

Subtracting the first column of this determinant from the fifth and reducing, we obtain

$$\begin{vmatrix} 0 & 1 & A & 0 & 0 \\ A & 0 & B & 0 & 1 \\ B & 0 & C & A & 0 \\ C & 2A & D+\frac{25}{8}A^2 & B & 0 \\ D+\frac{25}{8}A^2 & 2B & E+\frac{15}{2}AB & C & 3A \end{vmatrix}.$$

Again, subtracting the second column multiplied by A from the third and reducing, there results

$$-\begin{vmatrix} A & B & 0 & 1 \\ B & C & A & 0 \\ C & D+\frac{9}{8}A^2 & B & 0 \\ D+\frac{25}{8}A^2 & E+\frac{11}{2}AB & C & 3A \end{vmatrix},$$

which, after subtracting the first row multiplied by $3A$ from the last and reducing, becomes

$$\begin{vmatrix} B & C & A \\ C & D+\frac{9}{8}A^2 & B \\ D+\frac{1}{8}A^2 & E+\frac{5}{2}AB & C \end{vmatrix}$$

$$= B\left(CD+\frac{9}{8}A^2C-BE-\frac{5}{2}AB^2\right)+C\left(BD+\frac{1}{8}A^2B-C^2\right)$$
$$+A\left(CE+\frac{5}{2}ABC-D^2-\frac{5}{4}A^2D-\frac{9}{64}A^4\right)$$

$$=(ACE-B^2E-AD^2+2BCD-C^3)-\frac{5}{4}A\,(A^2D-3ABC+2B^3)-\frac{9}{64}A^5.$$

This expression, which is of degree-weight 15 . 15, instead of 10 . 15, must be divided by a^5 to give the correct value of the Cubic Criterion.

LECTURE XXXIII.

In this Lecture it is proposed to investigate the differential equation of a cubic curve having a given absolute invariant $\frac{S^3}{T^2}$.

Since the value of $\frac{S^3}{T^2}$ is the same for any homographic transformation of the cubic as for the original curve, the differential equation in question must be of the form

$$\text{Plenarily absolute principiant} = \frac{S^3}{T^2}.$$

This equation is (as we see at once by differentiating it) the integral of another of the form

$$\text{Principiant} = 0,$$

which is satisfied, independently of the value of the absolute invariant, at all points on a perfectly general cubic.

Now, the differential equation of the general cubic is of the 9th order, and when expressed in terms of $A, B, C, \ldots$ contains no letter beyond E. Hence the integral of this equation, which we are in search of, will be of the 8th order and will contain no capital letter beyond D.

When no letters beyond D are involved, all plenarily absolute principiants are functions of the two fundamental, or protomorphic, ones,

$$\frac{AC-B^2}{A^{\frac{8}{3}}},\quad \frac{A^2D-3ABC+2B^3}{A^4}.$$

Thus the differential equation of a cubic with a given absolute invariant is of the form

$$F\left(\frac{AC-B^2}{A^{\frac{8}{3}}},\quad \frac{A^2D-3ABC+2B^3}{A^4}\right)=\frac{S^3}{T^2}.$$

M. Halphen actually integrates the differential equation of the general cubic, which he shows (on p. 52 of his *Thèse sur les Invariants Différentiels*) may be put under the form

$$\xi\zeta d\xi+\left\{\zeta-\frac{3}{8}(\xi+3)(\xi+27)\right\}d\zeta=0,$$

where, in our notation,

$$\xi=\frac{24(A^2D-3ABC+2B^3)}{A^4},\quad \zeta=\frac{288\,(AC-B^2)^3}{A^8}.$$

The integral of this equation, which M. Halphen obtains partly from geometrical considerations, involves an arbitrary parameter depending on $\frac{S^3}{T^2}$. His result is as follows:

$$R^2 = hQ^3,$$

where

$$2^9 R = 2^9 \zeta^3 + 2^6 . 3 [(\xi - 3^3)^2 + 2^4 . 3^4] \zeta^2 + 2^3 . 3 (\xi + 3^3)^3 (\xi - 3^2 . 5) \zeta + (\xi + 3^3)^6,$$

$$2^6 Q = 2^6 \zeta^2 + 2^4 (\xi + 3^3)(\xi - 3^2 . 5) \zeta + (\xi + 3^3)^4,$$

and

$$T^2 - 64hS^3 = 0.$$

(Two misprints, which are here corrected, occur in the expression for R as given on p. 54 of the *Thèse*.)

In this result the invariant S differs in sign from the invariant usually denoted by that letter. Thus the discriminant is $T^2 - 64S^3$ instead of $T^2 + 64S^3$.

When $h = 1$ the discriminant vanishes and the differential equation becomes

$$R^2 - Q^3 = 0.$$

This is divisible by a numerical multiple of ζ^3; in fact,

$$R^2 = Q^3 + 2^3 . 3^5 \zeta^3 P,$$

where

$$2^6 P \equiv (2^3 \zeta + \xi^2 - 2 . 3^3 \xi - 3^5)^2 + 2^6 . 3 \xi^3 = 0$$

is the differential equation of a nodal cubic, previously obtained by Halphen.

It is from a knowledge of the fact that $P = 0$ and another algebraic relation between ξ and ζ, which he finds by trial to be $Q = 0$, constitute two particular integrals of the differential equation to the general cubic, that he arrives, not by any regular method but by repeated strokes of penetrative genius, at the general integral

$$R^2 = hQ^3.$$

In establishing the relation $T^2 - 64hS^3 = 0$ he supposes that, by means of the equation to the cubic and its differentials as far as the 8th order inclusive, the coefficients of the cubic have been expressed in terms of the variables x, y and the derivatives of y with respect to x up to the 8th order, and that the values thus obtained for the coefficients have been substituted in Aronhold's S and T.

The abbreviations introduced by the use of our notation enable us to actually perform this calculation, which would otherwise be impracticable in consequence of the enormous amount of labour required; and we shall use this method to obtain the plenarily absolute principiant which, equated to $\frac{S^3}{T^2}$, gives the differential equation to a cubic with a known absolute invariant.

Using the symbolic notation explained in Lecture XXXII. [above, p. 492], the equation of the cubic and its first eight differentials are

$$u^3 = 0,$$
$$u^2 u_1 = 0,$$
$$2uu_1^2 + u^2 u_2 = 0,$$
$$2u_1^3 + 6uu_1 u_2 + u^2 u_3 = 0,$$
$$3u_1^2 (2 . 1) + 6u_1 u (3 . 1) + 3u_1 (3 . 2) + 3u^2 (4 . 1) + 3u (4 . 2) + (4 . 3) = 0,$$
$$3u_1^2 (3 . 1) + 6u_1 u (4 . 1) + 3u_1 (4 . 2) + 3u^2 (5 . 1) + 3u (5 . 2) + (5 . 3) = 0,$$
$$3u_1^2 (4 . 1) + 6u_1 u (5 . 1) + 3u_1 (5 . 2) + 3u^2 (6 . 1) + 3u (6 . 2) + (6 . 3) = 0,$$
$$3u_1^2 (5 . 1) + 6u_1 u (6 . 1) + 3u_1 (6 . 2) + 3u^2 (7 . 1) + 3u (7 . 2) + (7 . 3) = 0,$$
$$3u_1^2 (6 . 1) + 6u_1 u (7 . 1) + 3u_1 (7 . 2) + 3u^2 (8 . 1) + 3u (8 . 2) + (8 . 3) = 0,$$

where $u = p + qx + y, \quad u_1 = q + t, \quad u_2 = 2a, \quad u_3 = 6b$;

as usual, $t = \frac{dy}{dx}, \quad a = \frac{1}{2} \cdot \frac{d^2y}{dx^2}, \quad b = \frac{1}{6} \cdot \frac{d^3y}{dx^3}, \ldots$;

$(m . \mu)$ denotes the coefficient of h^m in $(ah^2 + bh^3 + ch^4 + \ldots)^\mu$; and if, as in Salmon's *Higher Plane Curves* (2nd edit., p. 187), the equation of the cubic is taken to be

$$r + 3a_0 x + 3a_1 y + 3b_0 x^2 + 6b_1 xy + 3b_2 y^2 + c_0 x^3 + 3c_1 x^2 y + 3c_2 xy^2 + c_3 y^3 = 0,$$

then, in the above equations, the symbols

$$p^3, \; p^2q, \; p^2, \; pq^2, \; pq, \; p, \; q^3, \; q^2, \; q, \; 1$$

stand for

$$r, \; a_0, \; a_1, \; b_0, \; b_1, \; b_2, \; c_0, \; c_1, \; c_2, \; c_3.$$

These nine equations are sufficient to determine the values of the coefficients of the cubic which have to be substituted in $\frac{S^3}{T^2}$ in order to obtain our differential equation, which will be, as we have seen, of the form

$$F\left(\frac{AC - B^2}{A^{\frac{8}{3}}}, \; \frac{A^2D - 3ABC + 2B^3}{A^4}\right) = \frac{S^3}{T^2}.$$

Since this equation *contains nothing which involves x, y, or t*, these letters must have disappeared spontaneously in the process of forming it, and consequently we may, at any stage of the work, *give x, y, and t any arbitrary values without thereby affecting the result.* Let, then,

$$x = 0, \; y = 0, \; t = 0, \; \text{so that} \; u = p, \; u_1 = q, \; u_2 = 2a, \; u_3 = 6b,$$

and the first four equations become

$$u^3 = p^3 = r = 0,$$
$$u^2 u_1 = p^2 q = a_0 = 0,$$
$$\frac{1}{2}(2uu_1^2 + u^2 u_2) = pq^2 + p^2 a = b_0 + a_1 a = 0,$$
$$\frac{1}{2}(2u_1^3 + 6uu_1 u_2 + u^2 u_3) = q^3 + 6pqa + 3p^2 b = c_0 + 6b_1 a + 3a_1 b = 0.$$

Writing in the last five equations

$$u_1^2 = q^2 = c_1,$$
$$u_1 u = pq = b_1,$$
$$u_1 = q = c_2,$$
$$u^2 = p^2 = a_1,$$
$$u = p = b_2,$$
$$1 = c_3,$$

we have

$$3c_1(2.1) + 6b_1(3.1) + 3c_2(3.2) + 3a_1(4.1) + 3b_2(4.2) + c_3(4.3) = 0,$$
$$3c_1(3.1) + 6b_1(4.1) + 3c_2(4.2) + 3a_1(5.1) + 3b_2(5.2) + c_3(5.3) = 0,$$
$$3c_1(4.1) + 6b_1(5.1) + 3c_2(5.2) + 3a_1(6.1) + 3b_2(6.2) + c_3(6.3) = 0,$$
$$3c_1(5.1) + 6b_1(6.1) + 3c_2(6.2) + 3a_1(7.1) + 3b_2(7.2) + c_3(7.3) = 0,$$
$$3c_1(6.1) + 6b_1(7.1) + 3c_2(7.2) + 3a_1(8.1) + 3b_2(8.2) + c_3(8.3) = 0^*.$$

Substituting in $\frac{S^3}{T^2}$ for r, a_0, b_0, c_0 their values given by the equations

$$r = 0, \quad a_0 = 0, \quad b_0 + a_1 a = 0, \quad c_0 + 6b_1 a + 3a_1 b = 0,$$

and for the mutual ratios of $a_1, b_1, b_2, c_1, c_2, c_3$ their values found by solving the last five equations, we obtain the differential equation required.

Referring to Salmon's *Higher Plane Curves*, p. 188, we see that, when $r = 0$,

$$S = (c^2a^2) + (cb^2a) - (b^2)^2,$$
$$T = 4(c^3a^3) - 3(c^2b^2a^2) - 12(b^2)(cb^2a) + 8(b^2)^3,$$

where (c^2a^2), (cb^2a), ... are functions of $a_0, a_1, b_0, b_1, b_2, c_0, c_1, c_2, c_3$, which, when $a_0 = 0$, become

$$(c^2a^2) = (c_0c_2 - c_1^2)\,a_1^2,$$
$$(cb^2a) = (b_0^2c_3 - 3b_0b_1c_2 + b_0b_2c_1 + 2b_1^2c_1 - b_1b_2c_0)\,a_1,$$
$$(b^2) = b_0b_2 - b_1^2,$$
$$(c^3a^3) = (c_0^2c_3 - 3c_0c_1c_2 + 2c_1^3)\,a_1^3,$$
$$(c^2b^2a^2) = (c_0^2b_2^2 - 4c_0c_1b_1b_2 - 2c_0c_2b_0b_2 - 4c_0c_2b_1^2 + 8c_0c_3b_0b_1$$
$$+ 8c_1^2b_1^2 + 4c_1^2b_0b_2 - 12c_1c_2b_0b_1 - 8c_1c_3b_0^2 + 9c_2^2b_0^2)\,a_1^2.$$

We have now reached a point at which the work will be greatly facilitated by the introduction of the capital letters A, B, C, D. This is usually done by writing for

$$a,\ b,\ c,\ d,\ e,\ f,\ g,$$
$$1,\ 0,\ 0,\ A,\ B,\ C,\ D + \frac{25}{8}A^2.$$

* These equations are only set out for the sake of distinctness; when our abbreviations are introduced, only two terms survive in the first three, and only three terms in the last two of these five equations.

But in the present instance we may make a further simplification by writing

$$A = 1, \quad B = 0, \quad C = C_1, \quad D = D_1,$$

for the only effect of this will be to make the final result take the form

$$F(C_1, D_1) = \frac{S^3}{T^2}$$

instead of
$$F\left(\frac{AC - B^2}{A^{\frac{8}{3}}},\ \frac{A^2D - 3ABC + 2B^3}{A^4}\right) = \frac{S^3}{T^2}.$$

The form of the function will not be affected by writing in it $A = 1$, $B = 0$, and the letters A, B can be restored at pleasure by making

$$C_1 = \frac{AC - B^2}{A^{\frac{8}{3}}}, \quad D_1 = \frac{A^2D - 3ABC + 2B^3}{A^4}.$$

Hence we may write for

$$a,\ b,\ c,\ d,\ e,\ f,\ g,$$
$$1,\ 0,\ 0,\ 1,\ 0,\ C_1,\ D_1 + \frac{25}{8}.$$

Instead of the coefficient of

$$h^m \text{ in } (ah^2 + bh^3 + ch^4 + \dots)^\mu,$$

$(m.\mu)$ will now signify

$$\text{co. } h^m \text{ in } \left\{h^2 + h^5 + C_1h^7 + \left(D_1 + \frac{25}{8}\right)h^8\right\}^\mu.$$

Thus we have

$$(2.1) = 1$$
$$(3.1) = 0 \qquad (3.2) = 0$$
$$(4.1) = 0 \qquad (4.2) = 1 \qquad (4.3) = 0,$$
$$(5.1) = 1 \qquad (5.2) = 0 \qquad (5.3) = 0,$$
$$(6.1) = 0 \qquad (6.2) = 0 \qquad (6.3) = 1,$$
$$(7.1) = C_1 \qquad (7.2) = 2 \qquad (7.3) = 0,$$
$$(8.1) = D_1 + \frac{25}{8} \qquad (8.2) = 0 \qquad (8.3) = 0.$$

Hence the equations which give $a_1, b_1, b_2, c_1, c_2, c_3$ become

$$c_1 + b_2 = 0,$$
$$c_2 + a_1 = 0,$$
$$6b_1 + c_3 = 0,$$
$$c_1 + a_1C_1 + 2b_2 = 0,$$
$$2b_1C_1 + 2c_2 + a_1\left(D_1 + \frac{25}{8}\right) = 0.$$

From the first four of these, coupled with the equations

$$b_0 + a_1 = 0, \quad c_0 + 6b_1 = 0,$$

obtained by making $a=1$ and $b=0$ in the original equations which give b_0, c_0, we find

$$c_0 = \quad c_3 = -6b_1,$$
$$c_1 = -b_2 = -C_1^2,$$
$$c_2 = \quad b_0 = -a_1 = C_1,$$

by assuming $a_1 = -C_1$ (which we are at liberty to do since any one of the coefficients may be chosen arbitrarily).

The last equation then gives

$$b_1 = \frac{D_1}{2} + \frac{9}{16}.$$

Substituting these values in the previously given expressions for (c^2a^2), $(cb^2a), \ldots$ we have

$$(c^2a^2) = -(6b_1 + C_1^3)\,C_1^3,$$
$$(cb^2a) = -(4b_1^2 - 9b_1 - C_1^3)\,C_1^3,$$
$$(b^2) = C_1^3 - b_1^2,$$
$$(c^3a^3) = (216b_1^3 + 18b_1C_1^3 + 2C_1^6)\,C_1^3,$$
$$(c^2b^2a^2) = (312b_1^3 + 20b_1^2C_1^3 - 24b_1C_1^3 + 9C_1^3 + 4C_1^6)\,C_1^3.$$

Hence $$S = (c^2a^2) + (cb^2a) - (b^2)^2$$
$$= -C_1^6 + 3b_1C_1^3 - 2b_1^2C_1^3 - b_1^4,$$
and $$T = 4(c^3a^3) - 3(c^2b^2a^2) - 12(b^2)(cb^2a) + 8(b^2)^3$$
$$= -8C_1^9 - 3(8b_1^2 - 12b_1 + 9)\,C_1^6 - 12b_1^3(2b_1 - 3)\,C_1^3 - 8b_1^6.$$

To express S and T in terms of A, B, C, D, we write

$$C_1 = \frac{AC - B^2}{A^{\frac{8}{3}}}, \quad b_1 = \frac{D_1}{2} + \frac{9}{16} = \frac{A^2D - 3ABC + 2B^3}{2A^4} + \frac{9}{16},$$

or, if we use Halphen's notation in which

$$\zeta = \frac{288\,(AC - B^2)^3}{A^8}, \quad \xi = \frac{24\,(A^2D - 3ABC + 2B^3)}{A^4},$$

we have $$2^5 . 3^2 C_1^3 = \zeta, \quad 2^4 . 3b_1 = \xi + 3^3,$$
and consequently,

$$2^3 . 3\,(2b_1 - 3) = \xi - 3^2 . 5,$$
$$2^5 . 3^2 (8b_1^2 - 12b_1 + 9) = (2^4 . 3b_1 - 2^2 . 3^2)^2 + 2^4 . 3^4 = (\xi - 3^2)^2 + 2^4 . 3^4.$$

Hence

$$-2^{16} . 3^4 S = 2^{16} . 3^4 C_1^6 + 2^{16} . 3^4 b_1 (2b_1 - 3)\,C_1^3 + 2^{16} . 3^4 b_1^4$$
$$= 2^6\zeta^2 + 2^4(\xi + 3^3)(\xi - 3^2 . 5)\,\zeta + (\xi + 3^3)^4,$$
$$-2^{21} . 3^6 T = 2^{24} . 3^6 C_1^9 + 2^{23} . 3^7 (8b_1^2 - 12b_1 + 9)\,C_1^6$$
$$+ 2^{23} . 3^7 b_1^3 (2b_1 - 3)\,C_1^3 + 2^{24} . 3^6 b_1^6$$
$$= 2^9\zeta^3 + 2^6 . 3\,[(\xi - 3^2)^2 + 2^4 . 3^4]\,\zeta^2$$
$$+ 2^3 . 3\,(\xi + 3^3)^3 (\xi - 3^2 . 5)\,\zeta + (\xi + 3^3)^6,$$

where the expressions on the right-hand side are 2^6Q and 2^9R in Halphen's notation. Thus

$$-2^{10} . 3^4 S = Q, \quad -2^{12} . 3^6 T = R;$$

so that $$\frac{Q^3}{R^2} = -\frac{2^{30} . 3^{12} S^3}{2^{24} . 3^{12} T^2} = -\frac{64S^3}{T^2}.$$

This result agrees exactly with Halphen's, if we remember that his S is taken with a different sign from ours.

Since $$b_1 = \frac{D_1}{2} + \frac{9}{16} = \frac{A^2D - 3ABC + 2B^3}{2A^4} + \frac{3^2}{2^4},$$
we may write

$$\Phi = 2^4A^4b_1 = 2^3(A^2D - 3ABC + 2B^3) + 3^2A^4,$$

and in like manner

$$\Psi = A^6C_1^3 = (AC - B^2)^3.$$

Now $$2^8A^8(b_1^2 + C_1^3) = \Phi^2 + 2^8\Psi,$$
which is divisible by A^2. Hence if

$$\Phi^2 + 2^8\Psi = A^2\Theta,$$

we have $$\Theta = 2^8A^6(b_1^2 + C_1^3)$$
$$= 2^6(A^2D^2 - 6ABCD + 4AC^3 + 4B^3D - 3B^2C^2)$$
$$+ 2^4 . 3^2A^2(A^2D - 3ABC + 2B^3) + 3^4A^8.$$

The equations which give S and T in terms of b_1 and C_1 may be written

$$-S = (b_1^2 + C_1^3)^2 - 3b_1C_1^3,$$
$$-T = 2^3(b_1^2 + C_1^3)^3 - 2^2 . 3^2(b_1^2 + C_1^3)\,b_1C_1^3 + 3^3C_1^6,$$

and consequently,

$$-2^{16}A^{12}S = \Theta^2 - 2^{12} . 3\Phi\Psi,$$
$$-2^{23}A^{18}T = \Theta^3 - 2^{11} . 3^2\Theta\Phi\Psi + 2^{21} . 3^3A^2\Psi^2,$$

where Θ, Φ, Ψ are the rational integral principiants

$$\Theta = 2^6(A^2D^2 - 6ABCD + 4AC^3 + 4B^3D - 3B^2C^2)$$
$$+ 2^4 . 3^2A^2(A^2D - 3ABC + 2B^3) + 3^4A^8,$$
$$\Phi = 2^3(A^2D - 3ABC + 2B^3) + 3^2A^4,$$
$$\Psi = (AC - B^2)^3,$$

which, as we have seen, are connected by the relation

$$\Phi^2 + 2^8\Psi = A^2\Theta.$$

The differential equation of cubics with a given absolute invariant is

$$\frac{(\Theta^2 - 2^{12} . 3\Phi\Psi)^3}{(\Theta^3 - 2^{11} . 3^2\Theta\Phi\Psi + 2^{21} . 3^3A^2\Psi^2)^2} = -\frac{2^6S^3}{T^2},$$

or, as it may also be written,

$$(\Theta^2 - 2^{12} . 3\Phi\Psi)^3\,T^2 + 2^6S^3(\Theta^3 - 2^{11} . 3^2\Theta\Phi\Psi + 2^{21} . 3^3A^2\Psi^2)^2 = 0.$$

For a nodal cubic, the discriminant $T^2 + 2^6S^3$ vanishes. Hence the differential equation of a nodal cubic is

$$(\Theta^3 - 2^{11}.3^2\Theta\Phi\Psi + 2^{21}.3^3A^2\Psi^2)^2 - (\Theta^2 - 2^{12}.3\Phi\Psi)^3 = 0.$$

When expanded, and divided by $2^{22}.3^3\Psi^2$, this reduces to

$$A^2\Theta^3 - \Theta^2\Phi^2 - 2^{11}.3^2A^2\Theta\Phi\Psi + 2^{14}\Phi^3\Psi + 2^{20}.3^3A^4\Psi^2 = 0,$$

which (since $A^2\Theta - \Phi^2 = 2^8\Psi$) divides out by $2^8\Psi$, giving

$$\Theta^2 - 2^3.3^2A^2\Theta\Phi + 2^6\Phi^3 + 2^{12}.3^3A^4\Psi = 0,$$

or, what is the same thing,

$$\Theta^2 - 2^3.3^2A^2\Theta\Phi + 2^6\Phi^3 + 2^4.3^3A^4(A^2\Theta - \Phi^2) = 0.$$

This may also be written in the form

$$(\Theta - 2^2.3^2A^2\Phi + 2^3.3^3A^6)^2 + 2^6(\Phi - 3^2A^4)^3 = 0,$$

or, replacing Θ and Φ by their values in terms of A, B, C, D,

$$\{2^6(A^2D^2 - 6ABCD + 4AC^3 + 4B^3D + 3B^2C^2)$$
$$- 2^4.3^2A^2(A^2D - 3ABC + 2B^3) - 3^3A^6]^2 + 2^{15}(A^2D - 3ABC + 2B^3)^3 = 0.$$

For a cubic whose invariant S vanishes, the differential equation is

$$\Theta^2 - 2^{12}.3\Phi\Psi = 0,$$

and for a cubic whose invariant T vanishes,

$$\Theta^3 - 2^{11}.3^2\Theta\Phi\Psi + 2^{21}.3^3A^2\Psi^2 = 0.$$

For the cuspidal cubic, both S and T vanish, so that the algebraic equation of the cuspidal cubic is a particular solution of each of these equations. We can, however, replace the system

$$\Theta^2 - 2^{12}.3\Phi\Psi = 0, \qquad (1)$$

$$\Theta^3 - 2^{11}.3^2\Theta\Phi\Psi + 2^{21}.3^3A^2\Psi^2 = 0, \qquad (2)$$

by another pair of equations, for one of which the cuspidal cubic is a particular solution, and for the other the complete primitive.

Multiplying the first equation by Θ and subtracting the second from it, we have, after dividing by $2^{11}.3\Psi$,

$$\Theta\Phi - 2^{10}.3^2A^2\Psi = 0. \qquad (3)$$

From (1) and (3) we obtain

$$\Theta^2\Phi^2 = 2^{12}.3\Phi^3\Psi = 2^{20}.3^4A^4\Psi^2.$$

Hence $\qquad \Phi^3 = 2^8.3^3A^4\Psi. \qquad (4)$

But $\qquad A^2\Theta = \Phi^2 + 2^8\Psi,$

so that $\qquad A^2\Theta\Phi = \Phi^3 + 2^8\Phi\Psi.$

Substituting in this the values of $\Theta\Phi$ and Φ^3 found from (3) and (4) and dividing by Ψ, we have

$$2^{10}.3^2A^4 = 2^8.3^3A^4 + 2^8\Phi,$$

which gives $\qquad \Phi = 3^2A^4. \qquad (5)$

Substituting this value of Φ in (4) and rejecting the factor 3^2A^4, we obtain

$$3^3A^8 = 2^8\Psi;$$

that is $\qquad \left(\frac{A}{2}\right)^8 = \left(\frac{AC - B^2}{3}\right)^3.$

In the course of the work we have only rejected powers of Ψ (that is of $AC - B^2$) and of A, of which neither corresponds to the cuspidal cubic.

Since $\Phi = 3^2A^4$, it follows that $A^2D - 3ABC + 2B^3 = 0$. The equation to the cuspidal cubic above obtained is a particular solution of this, its complete primitive being (see Lecture XXXI. [above, p. 486]), $Y = X^\lambda Z^{1-\lambda}$, where λ is an arbitrary constant.

LECTURE XXXIV.

The preceding 33 lectures contain the substance of the lectures on Reciprocants actually delivered, entire or in abstract, in the course of three terms, to a class at the University of Oxford.

A good deal of material remains over which the lecturer has lacked leisure or energy to throw into form, which he hopes to be able to recover and annex to what has gone before as supplemental matter in the convenient form of lectures numbered on from those which have already appeared.

The one that follows is entirely due to Mr Hammond, who has rendered invaluable aid in compiling, and in many cases bettering, the lectures previously published.

It constitutes probably the most difficult problem in elimination which has been effected up to the present time. J. J. S.

The problem in question is to obtain the differential equation corresponding to the complete primitive

$$(l'x + m'y + n') = (lx + my + n)^\lambda (l''x + m''y + n'')^{1-\lambda}$$

(say $Y = X^\lambda Z^{1-\lambda}$) by the process of eliminating all the arbitrary constants except λ.

The eliminations to be performed become greatly simplified by aid of the following Lemma. If X be any linear function of x and y, and M_a the absolute pure reciprocant corresponding to M; then

$$X_3 - 4M_aX_1 = 0,$$

where $\qquad \frac{dX}{dx} = a^{\frac{1}{3}}X_1, \quad \frac{dX_1}{dx} = a^{\frac{1}{3}}X_2, \quad \frac{dX_2}{dx} = a^{\frac{1}{3}}X_3.$

For if we suppose $\qquad X = lx + my + n,$

two successive differentiations give

$$a^{\frac{1}{3}}X_1 = l + mt$$

and $\qquad a^{\frac{2}{3}}X_2 + a^{-\frac{2}{3}}bX_1 = 2ma.$

Writing the second of these equations in the form

$$a^{-\frac{1}{3}}X_2 + a^{-\frac{2}{3}}bX_1 = 2m,$$

and differentiating again, we find

$$X_3 - a^{-\frac{4}{3}}bX_2 + a^{-\frac{4}{3}}bX_2 + (4ac - 5b^2)\,a^{-\frac{5}{3}}X_1 = 0,$$

or, since $4M_a = (4ac - 5b^2)\,a^{-\frac{5}{3}}$,

$$X_3 + 4M_aX_1 = 0.$$

N.B.—Throughout the following work all letters with numerical suffixes are to be considered as derived from the corresponding unsuffixed letters in the same way as, in what precedes, X_1, X_2, and X_3 are derived from X; namely by successive differentiations, each of which is accompanied by a division by $a^{\frac{1}{3}}$.

Writing the equation

$$Y = X^{\lambda}Z^{1-\lambda}$$

(in which X, Y, Z denote any three linear functions of x, y) in the form

$$\log Y = \lambda \log X + (1-\lambda)\log Z,$$

we obtain by differentiation and division by $a^{\frac{1}{3}}$,

$$\frac{Y_1}{Y} = \lambda\frac{X_1}{X} + (1-\lambda)\frac{Z_1}{Z}. \qquad (1)$$

Let now

$$X_1 = uX,$$
$$Y_1 = vY,$$
$$Z_1 = wZ,$$

so that (1) takes the form

$$v = \lambda u + (1-\lambda)\,w,$$

and consequently

$$v_1 = \lambda u_1 + (1-\lambda)\,w_1,$$
$$v_2 = \lambda u_2 + (1-\lambda)\,w_2.$$

By means of the Lemma it can be shown that

$$u^3 + 3uu_1 + u_2 + 4M_au = 0, \qquad (2)$$
$$v^3 + 3vv_1 + v_2 + 4M_av = 0, \qquad (3)$$
$$w^3 + 3ww_1 + w_2 + 4M_aw = 0. \qquad (4)$$

For, since $\quad X_1 = Xu,$

we have $\quad X_2 = X_1u + Xu_1 = X(u^2 + u_1)$

and $\quad X_3 = X_2u + 2X_1u_1 + Xu_2 = X(u^3 + 3uu_1 + u_2).$

Substituting these values for X_1 and X_3 in

$$X_3 + 4M_aX_1 = 0,$$

we obtain

$$u^3 + 3uu_1 + u_2 + 4M_au = 0,$$

which proves equation (2). The equations (3) and (4) connecting v, v_1, v_2 and w, w_1, w_2 are similarly established. We now write

$$\left.\begin{aligned} u + v + w &= 3\omega \\ u - w &= 3z \end{aligned}\right\}.$$

These, combined with

$$v = \lambda u + (1-\lambda)\,w,$$

give

$$\left.\begin{aligned} u &= \omega - (\lambda - 2)\,z \\ v &= \omega - (1 - 2\lambda)\,z \\ w &= \omega - (\lambda + 1)\,z \end{aligned}\right\},$$

which, when operated on by $a^{-\frac{1}{3}}\dfrac{d}{dx}$ twice in succession, yield

$$\left.\begin{aligned} u_1 &= \omega_1 - (\lambda - 2)\,z_1 \\ v_1 &= \omega_1 - (1 - 2\lambda)\,z_1 \\ w_1 &= \omega_1 - (\lambda + 1)\,z_1 \end{aligned}\right\}, \qquad \left.\begin{aligned} u_2 &= \omega_2 - (\lambda - 2)\,z_2 \\ v_2 &= \omega_2 - (1 - 2\lambda)\,z_2 \\ w_2 &= \omega_2 - (\lambda + 1)\,z_2 \end{aligned}\right\}.$$

When expressed in terms of $\omega, \omega_1, \omega_2$ and z, z_1, z_2, equations (2), (3), and (4) become transformed into

$$P - (\lambda - 2)\,Q + (\lambda - 2)^2R - (\lambda - 2)^3z^3 = 0, \qquad (5)$$
$$P - (1 - 2\lambda)\,Q + (1 - 2\lambda)^2R - (1 - 2\lambda)^3z^3 = 0, \qquad (6)$$
$$P - (\lambda + 1)\,Q + (\lambda + 1)^2R - (\lambda + 1)^3z^3 = 0, \qquad (7)$$

where, for the sake of brevity, we write

$$\omega^3 + 3\omega\omega_1 + \omega_2 + 4M_a\omega = P,$$
$$3\omega^2z + 3\omega z_1 + 3\omega_1z + z_2 + 4M_az = Q,$$
$$3\omega z^2 + 3zz_1 = R.$$

In order to simplify (5), (6), and (7), we multiply the first of them by λ, the second by -1, and the third by $1-\lambda$, and take their sum, which is obviously independent of P, and from which it is easily seen that the terms containing Q and z^3 will also disappear. For

$$\lambda(\lambda - 2) - (1 - 2\lambda) + (1 - \lambda)(\lambda + 1) = 0,$$

and

$$\lambda(\lambda - 2)^3 - (1 - 2\lambda)^3 + (1 - \lambda)(\lambda + 1)^3 = 0.$$

We are thus left with

$$\{\lambda(\lambda - 2)^2 - (1 - 2\lambda)^2 + (1 - \lambda)(\lambda + 1)^2\}\,R = 0,$$

which, on restoring the value of R and reducing, becomes

$$\lambda(\lambda - 1)\,z\,(\omega z + z_1) = 0.$$

Now the values of u, v, w, which are equal to $\dfrac{X_1}{X}, \dfrac{Y_1}{Y}, \dfrac{Z_1}{Z}$ respectively, being distinct from each other, z cannot vanish; for $z = 0$ would imply $u = v = w$. Hence, considering λ to have any finite numerical value except 1 or 0, we may write

$$\omega z + z_1 = 0$$

in equations (5), (6), (7), which will then become

$$P-(\lambda-2)\ (3\omega_1 z+z_2+4M_a z)-(\lambda-2)^3 z^3 = 0, \tag{8}$$

$$P-(1-2\lambda)(3\omega_1 z+z_2+4M_a z)-(1-2\lambda)^3 z^3 = 0, \tag{9}$$

$$P-(\lambda+1)\ (3\omega_1 z+z_2+4M_a z)-(\lambda+1)^3 z^3 = 0. \tag{10}$$

Adding these together, we find

$$3P=\{(\lambda-2)^3+(1-2\lambda)^3+(\lambda+1)^3\} z^3$$
$$=3(\lambda-2)(1-2\lambda)(\lambda+1) z^3.$$

Restoring the value of P, and writing for shortness

$$(\lambda-2)(\lambda+1)(2\lambda-1)=p,$$

there results $\omega^3+3\omega\omega_1+\omega_2+4M_a\omega+pz^3=0.$

From any pair of the equations (8), (9), (10) we obtain by subtraction

$$3\omega_1 z+z_2+4M_a z+3(\lambda^2-\lambda+1) z^3=0.$$

Thus, for example, subtracting (10) from (8), we have

$$3(3\omega_1 z+z_2+4M_a z)=\{(\lambda-2)^3-(\lambda+1)^3\} z^3=-9(\lambda^2-\lambda+1) z^3.$$

Collecting our results, we see that equations (5), (6), (7) may be replaced by

$$\omega^3+3\omega\omega_1+\omega_2+4M_a\omega+pz^3=0, \tag{11}$$

$$3\omega_1 z+z_2+4M_a z+3qz^3=0, \tag{12}$$

$$\omega z+z_1=0, \tag{13}$$

where $p=(\lambda-2)(\lambda+1)(2\lambda-1),$

and $q=\lambda^2-\lambda+1.$

Differentiating (13), we obtain

$$\omega_1 z+\omega z_1+z_2=0.$$

Subtracting this from (12) and adding (13) multiplied by ω, the result divides by z, and we find

$$\omega^2+2\omega_1+4M_a+3qz^2=0, \tag{14}$$

which, when multiplied by ω and subtracted from (11), reduces it to

$$\omega\omega_1+\omega_2+pz^3-3qz^2\omega=0. \tag{15}$$

Now it has been shown in Lecture XXX. [above, p. 482] that

$$a^{-\frac{1}{3}}\frac{d}{dx}M_a=5A_a,$$

$$a^{-\frac{1}{3}}\frac{d}{dx}A_a=6B_a,$$

$$a^{-\frac{1}{3}}\frac{d}{dx}B_a=7C_a+M_aA_a,$$

whence it follows that (14) gives on differentiation

$$\omega\omega_1+\omega_2+10A_a+3qzz_1=0.$$

Combining this with (15) we have

$$10A_a=pz^3-3qz(\omega z+z_1),$$

or, finally, since $\omega z+z_1=0$,

$$10A_a=pz^3.$$

Differentiating this, we have

$$20B_a=pz^2z_1=-pz^3\omega;$$

that is $2B_a+A_a\omega=0, \quad (16)$

whence, by differentiation,

$$14C_a+2M_aA_a+6B_a\omega+A_a\omega_1=0.$$

Subtracting (14) multiplied by A_a from the double of this, we have

$$28C_a-A_a\omega^2+12B_a\omega-3qz^2A_a=0.$$

Substituting in this for ω its value $-\frac{2B_a}{A_a}$, found from (16), there results

$$28(A_aC_a-B_a^2)=3qz^2A_a^2.$$

But it has been shown that

$$10A_a=pz^3.$$

Hence the elimination of z gives

$$28^3p^2(A_aC_a-B_a^2)^3=3^3q^3p^2z^6A_a^6=10^2 3^3q^3A_a^8.$$

Or restoring for p and q their values in terms of λ, and replacing the absolute reciprocants A_a, B_a, C_a by the non-absolute ones A, B, C (which is effected by merely multiplying throughout by a power of a), we have

$$2^4.7^3(\lambda-2)^2(\lambda+1)^2(2\lambda-1)^2(AC-B^2)^3=3^3.5^2(\lambda^2-\lambda+1)^3A^8. \tag{17}$$

For other methods of obtaining this differential equation see Halphen's *Thèse sur les Invariants Différentiels*, p. 30, and Lecture XXX. of the present course. It corresponds *in general* (that is unless $\lambda=0, 1, \infty$) to the complete primitive

$$Y=X^\lambda Z^{1-\lambda}.$$

When $\lambda=0, 1, \infty$, the differential equation (17) becomes

$$28^3(AC-B^2)^3=3^3.5^2A^8, \tag{18}$$

which corresponds to the complete primitive

$$Y=Xe^{\frac{Z}{X}}. \tag{19}$$

This case has been discussed in the *Thèse* and in Lecture XXX. [above, p. 480].

We may obtain (18) from (19) by a method of elimination similar to that employed in deducing (17) from its complete primitive. Thus the first differential of (19) may be written

$$\frac{Y_1}{Y}=\frac{X_1}{X}+\frac{Z_1X-ZX_1}{X^2},$$

which becomes $v=u+3z$

when we assume $X_1=Xu,\quad Y_1=Yv,\quad Z_1=Zu+3Xz.$

By means of the Lemma we obtain

$$u^3 + 3uu_1 + u_2 + 4M_a u = 0, \qquad (20)$$

$$v^3 + 3vv_1 + v_2 + 4M_a v = 0, \qquad (21)$$

$$3u^2z + 3u_1z + 3uz_1 + z_2 + 4M_a z = 0. \qquad (22)$$

The first two of these are identical with (2) and (3) previously given; the third is found as follows. Since

$$Z_1 = Zu + 3Xz,$$

$$Z_2 = Z_1u + Zu_1 + 3X_1z + 3Xz_1$$

$$= Z(u^2 + u_1) + 3X(2uz + z_1).$$

Hence

$$Z_3 = Z_1(u^2 + u_1) + Z(2uu_1 + u_2) + 3X_1(2uz + z_1) + 3X(2u_1z + 2uz_1 + z_2)$$

$$= Z(u^3 + 3uu_1 + u_2) + 3X(3u^2z + 3u_1z + 3uz_1 + z_2).$$

Thus we have

$$Z_3 + 4M_aZ_1 = Z(u^3 + 3uu_1 + u_2 + 4M_a u) + 3X(3u^2z + 3u_1z + 3uz_1 + z_2 + 4M_a z).$$

But $Z_3 + 4M_aZ_1 = 0$, and $u^3 + 3uu_1 + u_2 + 4M_a u = 0$, which shows that

$$3u^2z + 3u_1z + 3uz_1 + z_2 + 4M_a z = 0.$$

Equations (20), (21), and (22), of which we have just proved the last, are merely convenient expressions of the fact that X, Y, Z are linear functions of x, y. We combine them with the first, second, and third differentials of the primitive equation (19) by writing

$$\left.\begin{aligned} v &= u + 3z \\ v_1 &= u_1 + 3z_1 \\ v_2 &= u_2 + 3z_2 \end{aligned}\right\}.$$

When this is done (21) becomes

$$(u^3 + 3uu_1 + u_2 + 4M_a u) + 3(3u^2z + 3uz_1 + 3u_1z + z_2 + 4M_a z)$$
$$+ 27z(uz + z^2 + z_1) = 0,$$

which, in consequence of the identities (20) and (22), reduces to

$$(u + z)z + z_1 = 0.$$

Let now $u = \omega - z$ (so that $\omega z + z_1 = 0$). Substituting in (20) and (22) we find

$$\omega^3 + 3\omega\omega_1 + \omega_2 + 4M_a\omega - 3(\omega - z)(\omega z + z_1) - z^3 - 3\omega_1 z - z_2 - 4M_a z = 0,$$

and $$(3\omega - 6z)(\omega z + z_1) + 3z^3 + 3\omega_1 z + z_2 + 4M_a z = 0$$

respectively. Adding both equations together, and remembering that

$$\omega z + z_1 = 0,$$

we obtain $$\omega^3 + 3\omega\omega_1 + \omega_2 + 4M_a\omega + 2z^3 = 0, \qquad (23)$$

$$3\omega_1 z + z_2 + 4M_a z + 3z^3 = 0, \qquad (24)$$

which, combined with $$\omega z + z_1 = 0, \qquad (25)$$

replace the system (20), (21), (22).

Comparing these equations with (11), (12), (13), we see that the two sets are identical if we make $\lambda = 0$, when p becomes 2 and $q = 1$. Hence, by performing exactly the same work as in the previous case, we shall find

$$5A_a = z^3 \qquad \text{(instead of } 10A_a = pz^3\text{)}$$

and $$28(A_aC_a - B_a^2) = 3z^2A_a^2 \text{ (instead of } 3qz^2A_a^2\text{)}.$$

And, finally, eliminating z between this pair of equations, at the same time replacing the absolute reciprocants A_a, B_a, C_a by the corresponding non-absolute ones A, B, C, we have

$$28^3(AC - B^2)^3 = 3^3 . 5^2 A^8,$$

which is what (17) becomes when λ has any of the values 0, 1, or ∞.