MATHEMATICAL PAPERS

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Yours faithfully
J. J. Sylvester

THE COLLECTED MATHEMATICAL PAPERS

OF

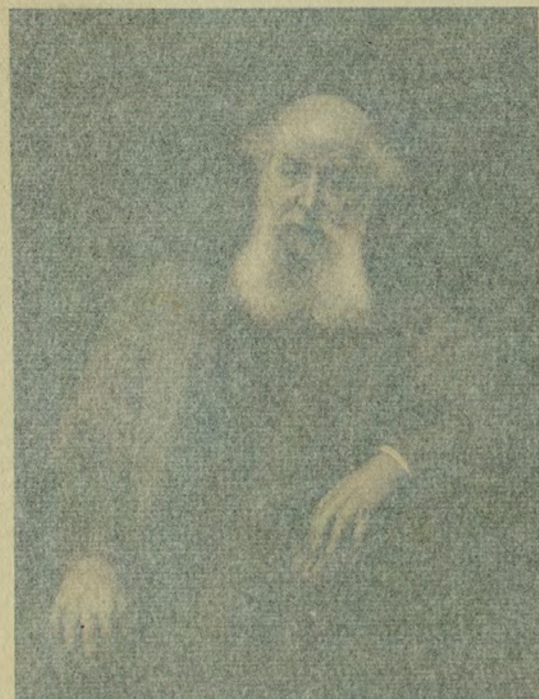
JAMES JOSEPH SYLVESTER

F.R.S., D.C.L., LL.D., Sc.D.,
Honorary Fellow of St John's College, Cambridge;
Sometime Professor at University College, London; at the University of Virginia;
at the Royal Military Academy, Woolwich; at the Johns Hopkins University, Baltimore
and Savilian Professor in the University of Oxford

VOLUME IV

(1882—1897)

Cambridge
At the University Press
1912



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PREFATORY NOTE

THE present volume contains Sylvester's Constructive Theory of Partitions, papers on Binary Matrices, and the Lectures on the Theory of Reciprocants. There is added an Index to the four volumes, and a Biographical Notice of Sylvester. The Mathematical Questions in the *Educational Times* are as yet unedited, but an Index to them is appended here. I have to acknowledge the kindness of Dr J. E. McTaggart, F.B.A., who secured for me the loan of the Essay on Canonical Forms, from the Library of Trinity College, Cambridge, for Vol. I, and that of Mr R. F. Scott, M.A., Master of St John's College, Cambridge, for the use of the volume called *The Laws of Verse*, from which the matter contained in the Appendix to Vol. II was reprinted, who supplied also the Autograph on the Frontispiece of this Volume. To the latter gentleman, as well as to Major P. A. MacMahon, Professor E. B. Elliott and Sir Joseph Larmor, I owe my best thanks for reading through the Biographical Notice. In carrying through the task of editing the Papers, I have, in general, thought it most fitting not to offer any remarks of my own in regard to Sylvester's text, though many times at a loss to know how best to act. In the Appendix to Vol. I I have departed from this rule, giving there an account of Sylvester's chief theorems in regard to determinants. For two other cases the reader may find notes, *Proceedings of the London Mathematical Society*, Vol. IV, Ser. II (1907), pp. 131—135, and Vol. VI (1908), pp. 122—140; these refer respectively to the paper No. 36, p. 229, and to the paper No. 74, p. 452, both in Vol. II of the Reprint. Many corrections of errors in the printing of algebraical formulae have been introduced, though many, it is to be feared, still remain; but no alterations of Sylvester's statements have been made without definite indication, by square brackets or otherwise. To the Readers and Staff of the University Press the very greatest credit and gratitude for their watchful carefulness are assuredly due, many of the corrections in the volumes being due to them.

H. F. BAKER.

June 1912.

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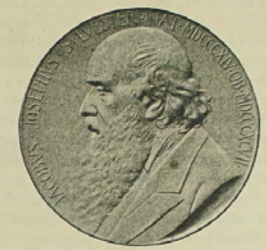


BIOGRAPHICAL NOTICE*.

Lord of himself and blest shall prove
 He who can boast "I've lived to-day,
To-morrow let dispensing Jove
 Cast o'er the skies what tint he may.

"Sunshine or cloud! the work begun
 And ended may his power defy,
He cannot change nor make undone
 What once swift Time has hurried by."

Laws of Verse, p. 73 (from Horace).

JAMES JOSEPH SYLVESTER was born in London on 3 September 1814, 1814
of a family said to have been originally resident in Liverpool. He was among the youngest of several brothers and sisters, and the last to survive. His father, whose name was Abraham Joseph, died while he was young. His eldest brother early in life established himself in America and assumed the name of Sylvester, an example followed by all the brothers.

If we attempt to realise the scientific circumstances of the time of Sylvester's birth by recalling the dates of some of those whose work might

* The chief authority for the outward facts of Sylvester's life used in this record is the Obituary Notice by Major P. A. MacMahon, R.A., F.R.S., *Royal Society Proceedings*, LXIII, 1898, p. ix. There is also an article in the *Dictionary of National Biography*, by Professor E. B. Elliott, F.R.S. and Mr P. E. Matheson, M.A., which gives a list of authorities, and an earlier article by Major MacMahon, *Nature*, 25 March 1897. Other sources of information are referred to in the course of the following.

naturally come before him, either in connexion with his subsequent career at Cambridge, or with his own later investigations, we find it difficult to make a choice. Of Englishmen Henry Cavendish (1731—1810) was dead, Thomas Young (1773—1829) was forty-one, Faraday (1791—1867) was twenty-three, and had just exchanged (in 1813) a bookbinder's workshop for the laboratory of the Royal Institution, Sir John Herschel (1792—1871) was twenty-two, and George Green (1793—1841), who was afterwards to be examined with Sylvester at Cambridge, was twenty-one. Cayley, with whom he was to be so much associated, was born in 1821, and was Senior Wrangler in 1842. The year 1814 was "the year of peace," and was the year in which Poncelet (1788—1867) returned to Paris from the Russian prison in which he had reconstructed the theory of conic sections; Lagrange (1736—1813) had just died, but there were living Laplace (1749—1827), Legendre (1752—1833), Fourier (1768—1830), Ampère (1775—1836), Poisson (1781—1840), Fresnel (1788—1827), Cauchy (1789—1857). J. C. F. Sturm (1803—1855), whose theorem was to have such an importance for Sylvester, was eleven years his senior; Hermite's life extended from 1822 to 1901. In Germany there were Gauss (1777—1855), whose *Disquisitiones Arithmeticae* is dated 1801, Steiner (1796—1863), von Staudt (1798—1867), Jacobi (1804—1851), W. Weber (1804—1891), Dirichlet (1805—1859), Kummer (1810—1893), while Weierstrass was born in 1815; and then there were Helmholtz (1821—1894), Kirchhoff (1824—1886), Riemann (1826—1866), and Clebsch (1833—1872). In Italy Brioschi, who took part in the development of the theory of invariants, was born in 1824 and died in 1897; and the name of Abel (1802—1829) cannot be omitted. All these, and many others, went to form the atmosphere in which Sylvester's life was spent.

Until Sylvester was fifteen years of age he was educated in London—from the age of six to the age of twelve with Mr Neumegen, at Highgate, subsequently, for a year and a half, with Mr Daniell at Islington, then, for five months, at the University of London (afterwards University College), where apparently he met Professor De Morgan, who (except from 1831 to 1835) taught at this institution from 1828 to 1867; for Sylvester speaks in 1840 (I 53) of having been a pupil of De Morgan's. His gift for Mathematics seems undoubtedly to have been apparent at this time; for Mr Neumegen sent him at the age of eleven to be examined in Algebra by Dr Olinthus Gregory, at the Royal Military Academy, Woolwich, and it is recorded that this gentleman was writing to Sylvester's father two years later to enquire for him, with a view to testing his progress in the interval.

1829 In 1829, at the age of fifteen, Sylvester went to Liverpool; here he attended the school of the Royal Institution, residing with aunts. The Institution, it appears, was founded in 1814, largely by the exertions of William Roscoe (1753—1831), and its school in 1819; it must not be confounded with the Liverpool Institute, which grew out of the Mechanics Institute, founded in

1825, by Mr Huskisson. The Head-master at this time was the Rev. T. W. Peile, afterwards Head-master of Repton, and the mathematical master was Mr Marratt. A contemporary at the school was Sir William Leece Drinkwater, afterwards First Deemster, Isle of Man. At this school Sylvester remained less than two years. In February 1830 he was awarded the first prize in the Mathematical School, and was so far beyond the other scholars that he could not be included in any class. While here, also, he was awarded a prize of 500 dollars for solving a question in arrangements, to the great satisfaction of the Contractors of Lotteries in the United States, the question being referred to him by the intervention of his elder brother in New York. At this early period of his life, too, he seems to have suffered for his Jewish faith at the hands of his young contemporaries; possibly this may account for the episode recorded, of his running away from school and sailing to Dublin. Here, with only a few shillings in his pocket, he was accidentally accosted by the Right Hon. R. Keatinge, Judge of the Prerogative Court of Ireland, who, having discovered him to be a first cousin of his wife, entertained him, and sent him back to Liverpool.

The indications were by now sufficient to encourage him to a mathe- 1831
matical career. After reading for a short time with the Rev. Dr Richard Wilson, sometime Fellow of St John's College, Cambridge, afterwards Head-master of St Peter's Collegiate School, Eaton Square, London, Sylvester was entered* at St John's College on 7 July, as a Sizar, commencing residence on 6 October 1831, when just over seventeen, his tutor being Mr Gwatkin. He resided continuously till the end of the Michaelmas Term, 1833, though he seems to have been seriously ill in June of this year. For two years from the beginning of 1834 his name does not appear as a member of the College, and apparently he was at home on account of illness. In January 1836 he was readmitted, this time as a Pensioner, and resided during the Lent and Michaelmas Terms, being also incapacitated in the intervening term. In January 1837 he underwent his final University examination, the Mathematical Tripos, and was placed second on the list. The first six names of that year were Griffin, St John's; Sylvester, St John's; Brumell, St John's; Green, Gonville and Caius; Gregory, Trinity, and Ellis, Trinity. Of these, George Green, born at Sneinton, near Nottingham, in 1793, was already the author of the famous paper, "An essay on the application of Mathematical Analysis to the theories of Electricity and Magnetism," which was published at Nottingham, by subscription, in 1828. He died in 1841, more than fifty years before Sylvester.

Of the general impression which Sylvester produced upon his contemporaries at Cambridge, it is difficult to judge. It is recorded that he attended the lectures of J. Cumming, Professor of Chemistry in the

* *The Eagle*, the College Magazine, XIX (1897), p. 603. A list of Sylvester's scientific distinctions is given in this place (p. 600).

University from 1815 to 1861, and, as required by College regulations, the Classical lectures of Bushby. We know how keen was his interest in Chemistry many years later in Baltimore (cf. his paper on *The New Atomic Theory*, III 148): and his writings furnish evidence of the pleasure he took in introducing a Classical allusion. When he became Editor of the *Quarterly Journal of Mathematics* in 1855 he secured the printing of a Greek motto on its title-page:

> ὅ τι οὐσία πρὸς γένεσιν, ἐπιστημὴ πρὸς πίστιν
> καὶ διάνοια πρὸς εἰκασίαν ἔστι;

later on, the *American Journal* under his care also had (IV 298) a Greek motto:

> πραγμάτων ἔλεγχος οὐ βλεπομένων;

in his older age the reading and translation of Classical authors was one of his resources.

He was, in later life at least, well acquainted with French, German and Italian, and rejoices (II 563) because these with Latin and English "may happily at the present day be regarded as the common property and inheritance of mathematical Europe." He was also much interested in Music. We are told that at one time he took lessons in singing from Gounod, and was known to sing at entertainments given to working men. "May not Music," he asks (II 419), "be described as the Mathematic of sense, Mathematic as Music of the reason?..." Or again (III 123), "It seems to me that the whole of aesthetic (...) may be regarded as a scheme having four centres, ..., namely Epic, Music, Plastic and Mathematic"; and he advocated "a new method of learning to read on the pianoforte" (III 8).

Of his interest in general literature, and his keen relish for a striking phrase, no reader of his papers needs to be reminded. To his first long paper on Syzygetic Relations, published in the *Philosophical Transactions of the Royal Society* (I 429), he prefixes the words

> How charming is divine philosophy!
> Not harsh and crabbed as dull fools suppose,
> But musical as is Apollo's lute
> And a perpetual feast of nectar'd sweets,
> Where no crude surfeit reigns!

In his paper on Newton's rule, also in the publications of the Royal Society (II 380), he quotes

> Turns them to shapes and gives to airy nothing
> A local habitation and a name.

In his *Constructive Theory of Partitions* (IV 1) he leads off with

> seeming parted,
> But yet a union in partition;

the Second Act, in which the Partitions are transformed by cunning operations performed on the diagrams which represent them, is introduced by

> Naturelly, by composiciouns
> Of anglis, and slie reflexiouns;

as the plot thickens he begins to feel more need of apology, and Act III begins with

> mazes intricate,
> Eccentric, intervolved, yet regular
> Then most, when most irregular they seem;

while, when he comes to the Exodion, and feels that, after fifty-eight pages, direct appeal may have lost its power, he takes refuge in Spenser's fairyland with the lines

> At which he wondred much and gan enquere
> What stately building durst so high extend
> Her lofty towres, unto the starry sphere.

Of his clever sayings we all remember many: "Symmetry, like the grace of an Eastern robe, has not unfrequently to be purchased at the expense of some sacrifice of freedom and rapidity of action" (I 309); or again, in support of the contention, that to say that a proposition is *little to the point* is not to be taken as *demurring to its truth* (II 725), "I should not hesitate to say, if some amiable youth wished to entertain his partner in a quadrille with agreeable conversation, that it would be *little to the point*, according to the German proverb, to regale her with such information as how

> Long are the days of summer-tide
> And tall the towers of Strasburg's fane,

but should be surprised to have it imputed to me on that account that I *demurred to the proposition* of the length of the days in summer, or the height of Strasburg's towers." More direct still (III 9), disclaiming the idea that the simplicity of Peaucellier's linkwork should discredit the difficulty of its discovery, "The idea of the facility of the result, by a natural mental illusion, gets transferred to the process of conception, as if a healthy babe were to be accepted as proof of an easy act of parturition." Some others will be found referred to in the index.

It is also recorded that among the friends of his earlier life was H. T. Buckle, author of the *History of Civilisation*, with whom, in addition to more serious reasons for sympathy, chess playing was a link of friendship.

Whether the many sides of Sylvester's character, indicated by these gleanings from his later life, were much in evidence at Cambridge, we do not know. The intellectual atmosphere of the place at the time was extremely vigorous in some ways. The Philosophical Society was founded in 1819, largely on the initiative of Adam Sedgwick and J. S. Henslow, and obtained a Charter in 1832; its early volumes are evidence of the great width and alertness of scientific interest in Cambridge at this time; papers of George Green were read at the Society in 1832, 1833, 1837 and 1839; James Cumming, whose chemical lectures Sylvester attended, Sir John Herschel, De Morgan, and Whewell are among the early contributors. Sir John Herschel's *Preliminary Discourse on the Study of Natural Philosophy* is dated 1831. The third meeting of the British Association was in Cambridge, on 24 June 1833. Whewell's *History of the Inductive Sciences* was published at Cambridge in 1837, the *Philosophy of the Inductive Sciences* in 1840. But we find* that in 1818 Sedgwick gave up his assistant tutorship, whose duties were mainly those of teaching the mathematical students of Trinity College, on the ground that "as far as the improvement of the mind is considered, I am at this moment doing nothing....I am...very sensibly approximating to that state of fatuity to which we must all come if we remain here long enough." This was before Sylvester's student time, and while mathematics at Cambridge was still suffering, partly from the long consequences of the controversy in regard to Leibniz and Newton, and more immediately from the loss of communication with the mathematicians of the Continent due to the war. Yet Sir John Herschel†, writing in 1833, feels compelled to speak very decidedly of the long-subsisting superiority of foreign mathematics to our own, as he phrases it, and there seems to be no doubt that mathematics, as distinct from physics, was then at a very low ebb in Cambridge, notwithstanding the success of the struggle, about a quarter of a century before, to introduce the analytical methods then in use on the Continent. C. Babbage, in his amusing *Passages from the Life of a Philosopher*, describes how he went (about 1812) to his public tutor to ask the solution of one of his mathematical difficulties and received the answer that it would not be asked in the Senate House, and was of no sort of consequence, with the advice to get up the earlier subjects of the university studies; and how, after two further attempts and similar replies from other teachers, he acquired a distaste for the routine of the place. His connexion with the translation of Lacroix's *Elementary Differential Calculus* (1816), and his association with George Peacock, Sir John Herschel and others in the Analytical Society, is well known; the title proposed by him for a volume of their *Transactions*, "The principles of pure D-ism in opposition to the Dot-age of the University," has often been quoted.

In addition to the better known accounts, there is an echo of what is usually said about Cambridge in this connexion in an Éloge on Sir John Herschel, read at the Royal Astronomical Society, 9 February 1872, by a writer who compares the work of Lagrange on the theory of equations with that of Waring, who was born in the same year, and was Senior Wrangler at Cambridge in 1757. We may add to this the bare titles of two continental

* *Life of Adam Sedgwick*, by J. W. Clark, I, p. 154.

† *Collected Essays*, Longmans, 1857, pp. 30—39.

publications of 1837, the year of Sylvester's Tripos Examination:—C. Lejeune Dirichlet, *Beweis des Satzes, dass jede unbegrenzte arithmetische Progression, deren erstes Glied und Differenz ganze Zahlen ohne gemeinschaftlichen Factor sind, unendlich viele Primzahlen enthält*; E. Kummer, *De aequatione* $x^{2\lambda} + y^{2\lambda} = z^{2\lambda}$ *per numeros integros resolvenda*. Augustus De Morgan, who was fourth Wrangler in 1827, speaking in 1865, at the inaugural meeting of the London Mathematical Society, pronounces that "The Cambridge Examination is nothing but a hard trial of what we must call problems—since they call them so—between the Senior Wrangler that is to be of this present January, and the Senior Wrangler of some three or four years ago. The whole object seems to be to produce problems—or, as I should prefer to call them, hard ten-minute conundrums....It is impossible in such an examination to propose a matter that would take a competent mathematician two or three hours to solve, and for the consideration of which it would be necessary for him to draw his materials from different sources, and see how he can put together his previous knowledge, so as to bring it to bear most effectually on this particular subject." This is the mathematician's criticism of the system then, and, to a large extent, still in vogue. A criticism from another point of view is found in a letter* of Sir Frederick Pollock, written in 1869, to De Morgan: "I believe the most valuable qualities for practical life cannot be got at by any examination—such as steadiness and perseverance....I think a Cambridge education has for its object to make good members of society—not to extend science and make profound mathematicians...." These criticisms appear to agree in one implication, the dominance of the examination in the training offered by the University; and they are necessary to a right appreciation of Sylvester's university life and subsequent work. Accordingly, we do not hear, as frequently we do in the case of young students at continental universities, of Sylvester being led to study for himself the great masters in Mathematics. We find him, in 1839 (I 39), disclaiming a first-hand knowledge of Gauss's works; there is no anecdote, known to me, to put with that he himself tells of Riemann. In a sheet of verses issued by himself, in February 1896—one of many such sheets, I believe—there is a footnote containing the following: "...the hotel on the river at Nuremberg, where I conversed outside with a Berlin bookseller, bound, like myself, for Prague....He told me he was formerly a fellow pupil of Riemann, at the University, and that, one day, after receipt of some numbers of the *Comptes rendus* from Paris, the latter shut himself up for some weeks, and when he returned to the society of his friends, said (referring to newly-published papers of Cauchy), 'This is a new mathematic.'" We find Sylvester, however, writing in 1839 of "the reflexions which Sturm's memorable theorem had originally excited" (I 44), and we know how much of his subsequent thought was given to this matter. Whether he read Sturm's paper of

* W. W. R. Ball, *History of Mathematics at Cambridge*, 1889, p. 113.

23 May 1829 (*Bulletin de Férussac*, XI, 1829, p. 419; *Mémoires par divers Savans*, VI, 1835, pp. 273—318), or in what way he learnt of the theorem, there seems to be no record. It is not referred to in the Report on Analysis by George Peacock, *Cambridge British Association Report*, 1833, pp. 185—352, which deals at length with Fourier's method. Sylvester records (II 655—6) that Sturm told him that the theorem originated in the theory of compound pendulums, but he makes no reference to Sturm's recognition of the application of his principles to certain differential equations of the second order.

Another aspect of Sylvester's time at Cambridge must be referred to. At this time, and indeed until 1871, it was necessary, in order to obtain the Cambridge degree, to subscribe to the Articles of the Church of England; one of the attempts, in 1834, to remove the restriction, is recorded in the *Life* of Adam Sedgwick, already referred to (I 418; Sedgwick writes a letter to the *Times*, 8 April 1834). Sylvester was, in his own subsequent bitter phrase (III 81), one of the first holding "the faith in which the Founder of Christianity was educated" to compete for high honours in the Mathematical Tripos; not only could he not obtain a degree, but he was excluded from the examination for Dr Smith's mathematical prizes, which, founded in 1769, was usually taken by those who had been most successful in the Mathematical Tripos. Most probably, too, had the facts been otherwise, he would have been shortly elected to a Fellowship at St John's College. To obtain a degree he removed to Trinity College, Dublin, from which, it appears, he received in turn the B.A. and the M.A. (1841). He finally received the B.A. degree at Cambridge, 29 February 1872, the M.A. (*honoris causa*) following 25 May of the same year.

1838 In the year succeeding his Tripos examination at Cambridge, he was elected to the Professorship of Natural Philosophy at (what is now) University College, London, and so became a colleague of Professor De Morgan. The list of the supporters of his candidature includes the names of Dr Olinthus Gregory, who had examined him in Algebra when a schoolboy of eleven, of Dr Richard Wilson, who had taught him before his entrance at St John's College, of the Senior Moderator and Senior Examiner in his Tripos examination, of Philip Kelland, of Queens' College, Senior Wrangler in 1834, afterwards Professor at Edinburgh, and of J. W. Colenso, afterwards Bishop of Natal; the two last had been private tutors of Sylvester at some portions of his career at Cambridge. He held the post of Professor of Natural Philosophy for a few years only; Professor G. B. Halsted (*Science*, 11 April 1897) makes a statement suggesting that the examination papers set by him during his tenure of the office are of a nature to indicate that he did not find his subject congenial. During these years he was elected a Fellow of the Royal Society (25 April 1839), at the early age of twenty-five. About this time also an oil-painting of him was made by Patten, of the Royal Scottish Academy, from the recorded description of which it appears that he had dark curly hair and

wore spectacles. It has been said that he took his Tripos examination in January 1837; he at once began to publish, in the *Philosophical Magazine* of 1837—38. The first four of his papers are on the analytical development of Fresnel's optical theory of crystals, and on the motion and rest of fluids and rigid bodies; but the papers immediately following contain the dialytic method of elimination, and the expression of Sturm's functions in terms of the roots of the equation, as well as many results afterwards included in the considerable memoir on the theory of the syzygetic relations of two polynomials, published in the *Philosophical Transactions* of 1853.

Leaving University College in the session of 1840—41, he proceeded 1841
as Professor of Mathematics across the Atlantic, to the University of Virginia, founded in 1824 at Charlottesville, Albemarle Co., where* his colleague, Key, of University College, had previously occupied the chair of Mathematics. Such a considerable change deserved a better fate than befell; in Virginia at this time the question of slavery was a subject of bitter contention, and Sylvester had a horror of slavery. The outcome was his almost immediate return; apparently he had intervened vigorously in a quarrel between two of his students.

On his return from America Sylvester seems to have abandoned mathe- 1844
matics for a time. In 1844 he accepted the post of Actuary to the Legal and Equitable Life Assurance Company, and threw himself into the work with great energy. He did not accept another teaching post for ten years, until 1854, but seems to have given some private instruction, as it is related† that he had, what was unusual at that time, a lady among his pupils—whose name was afterwards famous—Miss Florence Nightingale. He entered at the Inner Temple 29 July 1846, and was called to the Bar 22 November 1850.
He also founded the Law Reversionary Interest Society. It was in 1846 1846
that Cayley, who had been Senior Wrangler in 1842, left Cambridge and became a pupil of the famous conveyancer, Mr Christie, entering at Lincoln's Inn. He was already an author, and had in fact entered upon one of the main activities of his life; for in 1845 he had published his fundamental paper "On the Theory of Linear Transformations," in which he discusses Boole's discovery of the invariance of a discriminant. To us, knowing how pregnant with consequences the meeting was, it would be interesting to have some details of the introduction of Cayley and Sylvester; the latter lived, then or soon after, in Lincoln's Inn Fields, and we are told‡ that during the following years they might often be found walking together round the Courts of Lincoln's Inn, discussing no doubt many things but among them assuredly the Theory of Invariants. Perhaps it was particularly of this time that Sylvester was thinking when he described Cayley (I 376) as "habitually

* J. J. Walker, *Proc. Lond. Math. Soc.* XXVIII (1896—97), p. 582.

† *The Eagle*, XIX (1897), p. 597.

‡ Biographical notice of Arthur Cayley, Cayley's *Collected Papers*, Volume VIII.

1846 discoursing pearls and rubies," or when, much later (IV 300), he spoke of "Cayley, who, though younger than myself, is my spiritual progenitor—who first opened my eyes and purged them of dross so that they could see and accept the higher mysteries of our common mathematical faith." It is in a paper published in 1851 (I 246) that we find him saying, "The theorem above enunciated was in part suggested in the course of a conversation with Mr Cayley (to whom I am indebted for my restoration to the enjoyment of mathematical life)"; and Sylvester's productiveness during the latter part of this period is remarkable. In particular there are seven papers whose date of publication is 1850, including the paper on the intersections, contacts and other correlations of two conics, wherein he was on the way to establish the properties of the invariant factors of a determinant, afterwards recognised by Weierstrass; and there are thirteen papers whose date is 1851, including the sketch of a memoir on elimination, transformation and canonical forms, in which the remarkable expression of a cubic surface by five cubes is given, the essay on Canonical Forms, and the paper on the relation between the minor determinants of linearly equivalent quadratic functions, in which the notion of invariant factors is implicit; while in 1852 is dated the first of the papers "On the principles of the Calculus of Forms." Dr Noether remarks* how important for the history of mathematics these years were in other respects; Kummer's memoir, "Ueber die Zerlegung der aus Wurzeln der Einheit gebildeten complexen Zahlen in ihre Primfactoren," appeared in 1847 (*Crelle*, XXXV); Weierstrass's "Beitrag zur Theorie der Abel'schen Integrale" (*Beilage zum Jahresbericht über das Gymnasium zu Braunsberg*) is dated 1849; Riemann's Inaugural-dissertation, "Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse," is dated 1851. Referring to the discovery of the Canonical Forms in order to enforce the statement that observation, induction, invention and experimental verification all play a part in mathematical discovery (II 714), Sylvester tells an anecdote which has a personal interest: "I discovered and developed the whole theory of canonical binary forms for odd degrees, and, as far as yet made out, for even degrees too, at one evening sitting, with a decanter of port wine to sustain nature's flagging energies, in a back office in Lincoln's Inn Fields. The work was done, and well done, but at the usual cost of racking thought—a brain on fire, and feet feeling, or feelingless, as if plunged in an ice-pail. *That night we slept no more.*"

To Englishmen, in whose minds the modern developments of physical mathematics are associated with many familiar names, who recall Thomas Young, Faraday, Herschel, George Green, Stokes, Adams, Kelvin, Maxwell, the activity of Cayley and Sylvester may at first sight seem very natural. But in fact the aim of such men as those first named was primarily the coordination of the phenomena of Nature, not the development of any

* Charles Hermite, *Math. Annalen*, LV, p. 343.

mathematical theory. And if we think of such names as those of De Morgan, 1846
Warren, Peacock, their interest perhaps was either systematic or didactic; their endeavours were necessarily largely directed to criticising, and expounding to their countrymen, the proposals of continental mathematicians. But Cayley and Sylvester were in a different position at the time of which we are speaking; neither of them had any official duties as teacher of mathematics; to Cayley, as he afterwards said (in 1883) to the British Association, mathematics was "a tract of beautiful country seen at first in the distance, but which will bear to be rambled through and studied in every detail of hillside and valley, stream, rock, wood, and flower." To him and to Sylvester, Pure Mathematics was an opportunity for unceasing exploration; or, in another figure, a challenge to carve from the rough block a face whose beauty should for all time tell of the joy there was in the making of it; or again, it was the discernment and identification of high peaks of which the climbing might be in the years to come the task of those to whom strenuous labour is a delight and fine air an intoxication. And this spirit was a new one in England at this time, of which we may easily miss the significance. It may therefore help if we quote, without expressing any opinion as to its proportionate justice, the impression of an American observer, Dr Fabian Franklin, who succeeded Sylvester as Professor at Baltimore. Speaking* at the memorial meeting held immediately after Sylvester's death, 2 May 1897, he says of Sylvester, "His influence upon the development of mathematical science rests chiefly, of course, upon his work in the Theory of Invariants. Apart from Sir William Rowan Hamilton's invention and development of Quaternions, this theory is the one great contribution made by British thought to the progress of Pure Mathematics in the present century, or indeed since the days of the contemporaries of Newton. From about the middle of the eighteenth century, until near the middle of the nineteenth, English mathematics was in a condition of something like torpor....And, accordingly, it proved to be the case that in the magnificent extension of the bounds of mathematics which was effected by the continental mathematicians during the first four decades of the present century, England had no share. It is almost literally correct to say that the history of mathematics for about a hundred years might be written without serious defect with English mathematics left entirely out of account.

"That a like statement cannot be made in regard to the past fifty years is due pre-eminently to the genius and labours of three men: Hamilton, Cayley and Sylvester....Not only did other English mathematicians join in the work, but Hermite in France, Aronhold and Clebsch in Germany, Brioschi in Italy, and other continental mathematicians, seized upon the new ideas, and the theory of invariants was for three decades one of the leading objects of mathematical research throughout Europe. It is impossible to apportion

* *Johns Hopkins University Circulars*, June 1897.

between Cayley and Sylvester the honour of the series of brilliant discoveries which marked the early years of the theory of invariants...."

It would not be right to omit reference to another factor in the mathematical life of the time we are dealing with—the influence of George Salmon. At what time Sylvester first became acquainted with him, I have not ascertained; but we know that the theory of the straight lines lying upon a cubic surface was worked out in a correspondence between Cayley and Salmon in 1849. Readers of Salmon are aware of the intimate way in which he followed Sylvester's work, while Sylvester, in his papers, makes frequent reference to Salmon's books. There is a personal letter* from Salmon to Sylvester, of date 1 May 1861, which exhibits the relations of the two men in an interesting light, "...I should be very glad if there was any chance of your preparing an edition of your opuscula. There have been, of course, occasional little statements in your papers requiring verification. Written, as they were, in the very heat of discovery, they are rather to be compared to the hurried bulletins written by a general on the field of battle than to the cool details of the historian. Honestly, however, I don't think there is the least chance of your going back to these former studies. I shall be content to let you off some of these if you will do justice to what you have done on the subject of partitions. I wish you would seriously consider whether it is not a duty everyone owes to Society, when one brings a child into the world, to look to the decent rearing of it. I must say that you have to a reprehensible degree, a cuckoo-like fashion of dropping eggs and not seeming to care what becomes of them. Your procreative instincts ought to be more evenly balanced by such instincts as would inspire greater care of your offspring and more attention to providing for them in life, and producing them to the world in a presentable form.

"Hoping you will meditate on this homily and be the better for it, I remain, yours sincerely, GEO. SALMON."

Salmon himself did a great deal for the rearing of many of Sylvester's offspring, and I suppose it would be hard to estimate how much of Sylvester's and Cayley's reputation in their lifetime was due to his large-minded and genial exposition.

Sylvester himself, in a paper of 1863 (II 337), supplies some answer to such criticisms as this of Salmon's: "in consequence of the large arrears of algebraical and arithmetical speculations waiting in his mind their turn to be called into outward existence, he [the author] is driven to the alternative of leaving the fruits of his meditations to perish...or venturing to produce from time to time such imperfect sketches as the present, calculated to evoke the mental cooperation of his readers...."

1854 It was not until 10 June 1863 that Cayley returned to Cambridge, as Sadlerian Professor of Pure Mathematics. In 1854, Sylvester was a

* Printed in the *Eagle*, the Magazine of St John's College, XXIX (1908), p. 380.

candidate for the Professorship of Mathematics at the Royal Military Academy, Woolwich. At this time he had published the papers now reprinted in Volume I, the Theory of Invariants had an existence firmly established, and Sylvester had an European reputation. But his candidature was unsuccessful. This was in August of 1854. In December of the same year he gave his Probationary lecture on Geometry before the Electors to the Professorship of Geometry in Gresham College, London (II 2). In this he was also unsuccessful. Professor G. B. Halsted has recorded that Sylvester often deplored the time he had lost "fighting the world," and he would feel these disappointments keenly. However, the successful candidate at Woolwich died a few months after being appointed, and Sylvester was again a candidate. A letter on his behalf by Lord Brougham, of date 28 August 1855, speaks of him as my "learned and excellent friend and brother mathematician Mr Sylvester." This time he was elected. He took up the appointment on 15 September 1855, being, for a year, lecturer in Natural Philosophy as well as Professor of Mathematics. There is record of the exact emoluments of the post, a salary of £550, a Government Residence (K Quarters, Woolwich Common), medical attendance and right of pasturage on the Common. The house was a pleasant one, with a good garden, in which he could enjoy the shade of his own walnut tree, we are told, and he was able to entertain his scientific friends. The conversations with Cayley still went on; we hear of them walking to meet one another, Cayley from 2 Stone Buildings and he from his home, their meeting point falling near Lewisham. Sylvester retained this post until July 1870, sometimes justifying, we are led to believe, the original hesitation of the electors in regard to his efficiency as an elementary teacher; there are stories such as that of his housekeeper pursuing him from home carrying his collar and necktie. His publications during this time are, approximately, those reprinted in Volume II.

Sylvester gave seven lectures on the Theory of Partitions at King's College, London, in 1859 (II 119), not published until 1897, and then only from outlines privately circulated at the time of delivery; Capt. (now Sir Andrew) Noble collaborated with him in an important degree in his work on the Theory of Partitions. He wrote the paper on the involution of lines in space considered as axes of rotation (II 236). The long paper on Newton's rule and the invariantive discrimination of the roots of a quintic was published in the *Philosophical Transactions*, 1864 (II 376). His work on the proof of Newton's rule made its appeal in various directions—Todhunter remarks in his *Theory of Equations*, "If we consider the intrinsic beauty of the theorem, the interest which belongs to the rule associated with the great name of Newton, and the long lapse of years during which the reason and extent of that rule remained undiscovered by mathematicians—among whom Maclaurin, Waring and Euler are explicitly included—we must regard

Professor Sylvester's investigations as among the most important contributions made to the Theory of Equations in modern times, justly to be ranked with those of Fourier, Sturm and Cauchy."

1855 Sylvester's outward life also contained points to be remarked. In April 1855 appeared the first number of the *Quarterly Journal of Pure and Applied Mathematics*, edited by J. J. Sylvester, M.A., F.R.S. and N. M. Ferrers, M.A.; this replaced the *Cambridge and Dublin Mathematical Journal* which had first been edited by W. Thomson, M.A. (the late Lord Kelvin) and then by W. Thomson, M.A. and N. M. Ferrers, M.A. In the Preface, the plea is put forward that a more ambitious journal was necessary in view of the growing state of the subject, and might render British mathematicians less dependent on the courtesy of the editors of Foreign journals. Assisted by Stokes, Cayley and Hermite, this joint editorship continued unchanged until June 1877.

1856 In 1856 Sylvester was elected* to the Athenaeum Club, under the special Rule II. The fact is worth recording. Sylvester was never married, and in subsequent years this was the address he frequently appended to his writings.

1859 In 1859 he delivered seven lectures on the Partition of Numbers, at King's College, London, as noted above.

1861 In 1861 he was awarded a Royal Medal by the Royal Society, Cayley having received that honour in 1859.

1863 On 7 Dec.† 1863 he was chosen correspondent in mathematics by the French Academy of Sciences, in place of the great geometer Steiner, who had died in the preceding April. We notice that he had just commenced (in 1861) what was to be a long series of communications to the Academy, and his paper on Involutions of lines in space had been presented to the Academy by M. Chasles (II 236). His closely following paper on the Double Sixes of lines on a Cubic surface (II 242) he himself afterwards (II 451) notes as being an unconscious plagiarism from a paper of Schläfli, which he had read as editor before its publication in the *Quarterly Journal* (Vol. II (1858), p. 116).

1864 His memoir in the *Phil. Trans.* on Newton's rule is of date 1864 (II 376). In 1865 he delivered a lecture on the subject at King's College, London (II 498). A syllabus of this lecture forms the first mathematical paper published by the London Mathematical Society. This Society was inaugurated by a speech of Professor De Morgan 16 Jan. 1865, with "the great aim of the cultivation of pure Mathematics and their most immediate applications." The Society consisted at its formation of twenty-seven members, nearly all of whom were members of University College. Sylvester was elected the second President at the Annual General Meeting held at Burlington

* As I have been able to verify by the courtesy of the Secretary.

† J. J. Walker, *Proc. Lond. Math. Soc.* XXVIII (1896—97), p. 585.

House on 8 November 1866 (in the rooms of the Chemical Society), and held office until November 1868. He served on the Council for many years.

In 1869 Sylvester was President of the Mathematical and Physical 1869
Section of the British Association at Exeter. He took as the subject of his Presidential address the charge that Huxley had brought against Mathematics, of being the study that knew nothing of observation or induction (II 650), nothing of experiment or causation. An incidental reference in this address to Kant's doctrine of space and time led to a lively controversy in the pages of *Nature*, in which Sylvester's trenchant style and wide range of intellectual alertness may be well seen (II Appendix). Characteristically enough Sylvester reprinted the address, with annotations, and the correspondence in regard to Kant, as an Appendix to his volume on the *Laws of Verse* (Longmans, 1870)—a volume which should be consulted for an appreciation of a side of Sylvester's activity which occupied him to the end of his life.

In 1870 Sylvester retired from his post at Woolwich, in consequence 1870
of what he regarded as an unfair change in the regulations. As may be seen in the article of G. B. Halsted, above quoted, *Science*, 11 April 1897, and in the Leading Article which appeared in the *Times*, 17 August 1871 (see also Sylvester's own letter to the *Times*, 24 August 1871, and *Nature*, Vol. IV (1871), pp. 324, 326), there was much bitterness as to the question of pension, which was however finally secured to him, if not on the scale desired. For the next few years Sylvester resided near the Athenaeum Club, apparently somewhat undecided as to his course in life. We hear of him as reciting and singing at Penny Readings (cf. his remarks on the utility of these in the *Laws of Verse*, p. 70), and as being a candidate for the London School Board*, and, in *The Gentleman's Magazine* for February 1871, there appears "The Ballad of Sir John de Courcy," translated from the German by "Syzygeticus."

In 1874 Sylvester gave a Friday evening discourse at the Royal Insti- 1874
tution on Peaucellier's link bar motion. He was then sixty years old, yet, even in the abstract of the lecture which remains (III 7), the vivacity with which he dealt with the matter is very striking. His enthusiasm evoked a wide interest in the subject.

In 1875 the Johns Hopkins University was founded at Baltimore. A 1875
letter to Sylvester from the celebrated Joseph Henry, of date 25 August 1875, seems to indicate that Sylvester had expressed at least a willingness to share in forming the tone of the young university; the authorities seem to have felt that a Professor of Mathematics and a Professor of Classics could inaugurate the work of an University without expensive buildings or

* Sylvester's election address as candidate for the London School Board for Marylebone in the place of Professor Huxley, with a list of his scientific supporters, is found in *Nature*, 21 March 1872, p. 410.

elaborate apparatus. It was finally agreed that Sylvester should go, securing, besides his travelling expenses, an annual stipend of 5000 dollars "paid in gold." And so, at the age of sixty-one, still full of fire and enthusiasm, as appears abundantly from the work he devoted to the papers here reprinted in Volume III, he again crossed the Atlantic, and did not relinquish the post for eight years, until 1883. It was an experiment in educational method; Sylvester was free to teach whatever he wished in the way he thought best; so far as one can judge from the records, if the object of an University be to light a fire of intellectual interests, it was a triumphant success. His foibles no doubt caused amusement, his faults as a systematic lecturer must have been a sore grief to the students who hoped to carry away note-books of balanced records for future use; but the moral effect of such earnestness as we see him shewing for instance in the papers 19—21 of Volume III (on the true number of irreducible concomitants for the cubic and biquadratic), and in paper 34 (on the system for two cubics), must have been enormous. "His first pupil, his first class," was Professor George Bruce Halsted; he it was who, as recorded in the Commemoration-day Address (III 76) "would have the New Algebra." How the consequence was that Sylvester's brain "took fire," is recorded in the pages of the *American Journal of Mathematics*. Others have left records of his influence and methods. Major MacMahon quotes the impressions of Dr E. W. Davis, Mr A. S. Hathaway and Dr W. P. Durfee. Professor Halsted's Article in *Science* has already been quoted. From Dr Fabian Franklin's long commemorative address*, already referred to, another paragraph may be given: "One of the most striking of Sylvester's achievements was his demonstration and extension of Newton's improved rule concerning the number of the imaginary roots of an algebraic equation....We who knew him well in later years can find no difficulty in understanding the hold this problem had upon him. It was the good fortune of his early hearers in this University to be present when he came into the lecture-room, flushed with the achievement of a somewhat similar task. A certain fundamental theorem in the Theory of Invariants (III 117, 232), which had formed the basis of an important section of Cayley's work, had never been completely demonstrated. The lack of this demonstration had always been, to Sylvester's mind, a most serious blemish in the structure. He had, however, he told us, years ago given up the attempt to find the proof, as hopeless. But, upon coming fresh to the subject in connection with his Baltimore Lectures, he again grappled with the problem, and by a fortunate inspiration, succeeded in solving it. It was with a thrill of sympathetic pleasure that his young hearers thus found themselves in some measure associated with an intellectual feat, by which had been overcome a difficulty that had successfully resisted assault for a quarter of a century."

* *Johns Hopkins University Circulars*, June 1897.

The same writer gives an anecdote illustrating another side of the picture, which may be repeated here. "The reading of the Rosalind poem at the Peabody Institute was the occasion of an amusing exhibition of absence of mind. The poem consisted of no less than 400 lines, all rhyming with the name Rosalind (the long and short sound of i both being allowed). The audience quite filled the hall, and expected to find much interest or amusement in listening to this unique experiment in verse. But Professor Sylvester had found it necessary to write a large number of explanatory footnotes, and he announced that in order not to interrupt the poem he would read the footnotes in a body, first. Nearly every footnote suggested some additional extempore remark, and the reader was so interested in each one that he was not in the least aware of the flight of time, or of the amusement of the audience. When he had dispatched the last of the notes, he looked up at the clock, and was horrified to find that he had kept the audience an hour and a half before beginning to read the poem they had come to hear. The astonishment on his face was answered by a burst of good-humoured laughter from the audience; and then, after begging all his hearers to feel at perfect liberty to leave if they had engagements, he read the Rosalind poem." It may be noted here that it was at Baltimore he wrote "Spring's Début, a Town Idyll," two centuries of lines all rhyming with "Winn." (January 1880.)

Sylvester's own account of his life at Baltimore, and many other matters, are sufficiently given in the Commemoration-day Address, 22 February 1877 (III 72); it is not necessary to dwell on this further here.

In 1878 appeared the first volume of the *American Journal of Mathe-* 1878
matics established by the University under Sylvester's care. His first paper is a long account of the application of the new atomic theory to the graphical representation of the concomitants of binary quantics (III 148).

In 1880 he was awarded by the Royal Society the highest honour 1880
possible, the Copley Medal; on 11 June 1880, he was elected Honorary Fellow of his old College of St John at Cambridge, Benjamin Hall Kennedy, the famous schoolmaster and Greek scholar, being elected on the same day. Their portraits are now both preserved in the College.

It is to this period of his life we must refer also the beginning of his investigations in regard to matrices, especially binary matrices. He says (IV 209) "my memoir on Tchebycheff's method concerning the totality of prime numbers within certain limits, was the indirect cause of turning my attention to the subject, as (through the systems of difference equations therein employed to contract Tchebycheff's limits) I was led to the discovery of the properties of the latent roots of matrices, and had made considerable progress in developing the theory of matrices considered as quantities, when on writing to Professor Cayley upon the subject he referred me to [his own] memoir." Here also, in the interesting communications to the Mathematical

Club reprinted in the *Johns Hopkins University Circulars*, arose a new interest in developing the Theory of Partitions, which issued in the *Constructive Theory of Partitions* (IV 1—83) printed in the *American Journal* (1883). In the course of the year 1883 the University of Oxford conferred upon Sylvester the honorary degree of D.C.L.; and in December of that year, soon after his sixty-ninth birthday, his great distinction was recognised further in the same University by his election to succeed the illustrious H. J. S. Smith as occupant of the chair of Savilian Professor of Geometry. The Professorship had been founded in 1619 by Sir Henry Savile, Warden of Merton College, the first professor being obtained by promoting Henry Briggs from the post which Sylvester had vainly sought in 1854, that of Gresham Professor of Geometry in London, so that, as Mr Rouse Ball remarks, Briggs held in succession the two earliest chairs of mathematics that were founded in England—the college founded by Sir Thomas Gresham having been opened in 1596. Other holders of the Savilian chair were John Wallis, 1649, and Edmund Halley, 1704. The companion chair at Oxford, of Savilian Professorship of Astronomy, was held from 1870 to 1893 by the Rev. Charles Pritchard, who was also an alumnus at St John's College, Cambridge. These two were now to be again members of the same house, as Fellows of New College.

The election of Sylvester to Oxford was a matter of importance at Baltimore. On 20 December 1883, a goodbye meeting was held in Hopkins' Hall, Baltimore, by invitation of the President, the guests including Mr Matthew Arnold, Professor Newcomb and others. The following address was agreed to, in Professor Sylvester's presence*.

"The teachers of the Johns Hopkins University, in bidding farewell to their illustrious colleague, Professor Sylvester, desire to give united expression to their appreciation of the eminent services he has rendered the University from the beginning of its actual work. To the new foundation he brought the assured renown of one of the great mathematical names of our day, and by his presence alone made Baltimore a great center of mathematical research.

"To the work of his own department he brought an energy and a devotion that have quickened and informed mathematical study not only in America, but all over the world; to the workers of the University, whether within his own field or without, the example of reverent love of truth and of knowledge for its own sake, the example of a life consecrated to the highest intellectual aims. To the presence, the work, the example of such a master as Professor Sylvester, the teachers of the Johns Hopkins University all owe, each in his own measure, guidance, help, inspiration; and in grateful recognition of all that he has done for them and through them for the University, they wish for him a long and happy continuance of his work in his native land, for

* *Johns Hopkins University Circulars*, January 1884, p. 31.

themselves the power of transmitting to others that reverence for the ideal which he has done so much to make the dominant characteristic of this University."

And thus at length, crowned with the gratitude of his American colleagues, 1884
Sylvester was acknowledged in one of the two ancient English Universities, though not his own. And to this, at the age of seventy years, he did not come without something new to say! On 12 December 1885, he delivered an Inaugural lecture, On the Method of Reciprocants (IV 278), that is of functions of differential coefficients whose form is unaltered by certain linear transformations of the variables. This he followed up by a course of lectures which, as finally edited, extend over more than two hundred pages of the present Reprint. The matter evidently appealed to him as a generalisation of the theory of concomitants, and he worked hard and enthusiastically at the relations of the two theories, gathering round him a school of advanced students. This was the last great continent of thought to be won by him, though he wrote, in 1886, for the centenary volume of "the leading Mathematical Journal in the world," *Crelle's Journal*, a paper on the so-called Tschirnhausen Transformation, which he ascribed to the inspiration of Bring (1786) (IV 531), and a paper on a funicular solution of Buffon's "problem of the needle" in 1890 (IV 663), besides other papers. In the Theory of Reciprocants he had been anticipated in detail by Halphen (*Thèse*, 1878), as he acknowledges. The general idea of differential invariants had been already formulated by Sophus Lie (see his paper on Differential Invariants, *Math. Ann.* XXIV (1884) in which he states that his investigations go back to 1869—72), as an application of his theory of Continuous Groups; to this Sylvester paid but scant attention. On the other hand it may be recalled that Sylvester had himself in cooperation with Cayley long before stated and frequently employed the principle of infinitesimal transformations, and in his first paper on Schwarzian Derivatives (IV 252) he employs the idea of "extended" infinitesimal variations without remark.

One striking note in his Inaugural address at Oxford is the fulness of his references to his colleagues in mathematical work—and of these, what he said about Hammond, fully borne out as it was by the help he gave in the Theory of Reciprocants, seems worthy of being recalled: "I should not do justice to my feelings if I did not acknowledge my deep obligations to Mr Hammond for the assistance which he has rendered me, not only in preparing this lecture which you have listened to with such exemplary patience, but in developing the theory;...saving only our Cayley (...) there is no one I can think of with whom I ever have conversed, from my intercourse with whom I have derived more benefit..." (IV 300)*.

* Another worker to whom he referred in warm terms was Arthur Buchheim. It was his melancholy duty a few years later to write an Obituary Notice of this distinguished young mathematician, who died at the age of twenty-nine. *Nature*, 27 September 1888, p. 515.

1887 In 1887 the Council of the London Mathematical Society made the second award of the De Morgan medal to Sylvester, the first award (in 1884) having been made to Cayley.

1889 In 1889, at the request of a few College friends at Cambridge and elsewhere, he sat to A. E. Emslie for an oil-painting, now hanging in the Hall of St John's College, which was exhibited in the Academy of that year*. It is stated to be a good portrait, though, as he himself writes (*Eagle*, Vol. XIX, 1897, p. 604), "I was in much trouble at that time...and could scarcely keep awake from the effect of the light on my wearied eyes." This portrait is reproduced at the commencement of the present volume. A copy of it is at New College, Oxford. An oil-painting by Patten, made when he was twenty-six, has already been referred to. An engraving by G. J. Stodart, from a photograph by Messrs I. Stilliard & Co., Oxford, appeared in *Nature*, accompanying an appreciation by Cayley (*Nature*, Vol. XXXIX, 1889; Cayley's *Collected Papers*, XIII, p. 43 gives the appreciation); he himself is said to have much prized a particular photograph taken at Venice. On the occasion of his leaving Baltimore a medal was struck in his honour, of which an exemplar is in the library of St John's College, Cambridge, giving in profile an idea of powerful features. Another medal, struck shortly after his death, is now awarded triennially by the Royal Society of London, for the encouragement of Mathematical Research. This also is a profile with the same impression of strength. It is one side of this medal which is reproduced at the beginning of this Notice (p. xv).

1890 On 10 June 1890 he was awarded the Honorary Degree of Sc.D. by the University of Cambridge. Honorary degrees were conferred on this occasion upon Benjamin Jowett, Henry Parry Liddon, Andrew Clark, Jonathan Hutchinson, George Richmond, John Evans, James Joseph Sylvester and Alexander John Ellis. The speech delivered upon Sylvester by the Public Orator, with his own footnotes, is as follows (*Orationes et Epistolae Cantabrigienses* (1876—1909), Macmillan, 1910, p. 83):

"Plus quam tres et quinquaginta anni interfuerunt, ex quo Academiae nostrae inter silvas adulescens quidem errabat, populi sacri antiquissima stirpe oriundus, cuius maiores ultimi, primum Chaldaeorum in campis, deinde Palestinae in collibus, caeli nocturni stellas innumerabiles, prolis futurae velut imaginem referentes†, non sine reverentia quadam suspiciebant. Ipse numerorum peritia praeclarus, primum inter Londinienses Academiae nostrae studia praecipua ingenii sui lumine illustrabat. Postea trans aequor Atlanticum plus quam semel honorifice vocatus, fratribus nostris transmarinis doctrinae mathematicae facem praeferebat‡. Nuper professoris insignis in locum electus, et Britanniae non sine laude redditus, in Academia Oxoniensi

* Graves' *Catalogue of the Royal Academy*, 1769—1904.
† *Genesis*, xv. 5.
‡ University of Virginia, 1841—45; Johns Hopkins University, 1877—83.

scientiae flammam indies clariorem excitat*. Ubicumque incedit, exemplo suo nova studia semper accendit. Sive numerorum θεωρίαν explicat, sive Geometriae recentioris terminos extendit, sive regni sui velut in puro caelo regiones prius inexploratas pererrat, scientiae suae inter principes ubique conspicitur. Nonnulla quae Newtonus noster, quae Fresnelius, Iacobius, Sturmius, alii, imperfecta reliquerunt, Sylvester noster aut elegantius explicavit, aut argumentis veris comprobavit. Quam parvis ab initiis argumenta quam magna evolvit; quotiens res prius abditas exprimere conatus, sermonem nostrum ditavit, et nova rerum nomina audacter protulit†! Arte quali numerorum leges non modo poëtis antiquis interpretandis sed etiam carminibus novis pangendis accommodat‡! Neque surdis canit, sed 'respondent omnia silvae§,' si quando, inter rerum graviorum curas, aevi prioris pastores aemulatus,

'Silvestrem tenui musam meditatur avena‖.'

Duco ad vos Collegii Divi Ioannis Socium, trium simul Academiarum Senatorem, quattuor deinceps Academiarum Professorem, *Iacobum Iosephum Sylvester*."

During his residence at Oxford he founded the Oxford Mathematical Society. "Members of that Society, even more perhaps than the attendants at his formal lectures, have been impressed and excited to emulation as they have seen his always commanding face grow handsome with enthusiasm, and his eyes flash out irresistible fascination, while he expounded his latest discovery or brilliant anticipation," writes the *Oxford Magazine* (5 May 1897). From the same source we gather that, "despondent over his lecturing work he undoubtedly was, and the feeling of discouragement grew upon him." In
1893 his eyesight began to be a serious trouble to him, and in 1894 he applied 1893
for leave to resign the active duties of his chair. After that he lived mainly in London or at Tunbridge Wells, sad and dejected because his mathematical
power was failing. About August 1896 a revival of energy took place and 1896
he worked at the theory of Compound Partitions, and the Goldbach-Euler conjecture of the expression of every even number as a sum of two primes. He was present at a meeting of the London Mathematical Society on 14 January 1897, and spoke at some length of his work, answering questions put to him in regard to it. On 12 February he sent a paper, on the number of fractions in their lowest terms that can be formed with limited integers, to the editor of the *Messenger of Mathematics*, and corrected the proofs about the end of
the month (IV 742). At the beginning of March, he had a paralytic seizure 1897
while working in his rooms at Hertford Street, Mayfair. He never spoke again, and died 15 March 1897. He was buried with simple ceremonial at

* Succeeded H. J. S. Smith as Savilian Professor, 1883—97.
† Prof. Cayley in *Nature*, 3 Jan. 1889.
‡ *The Laws of Verse*, 1870; *Eagle*, XIV 251, XV 251, XIX 601 f., 604.
§ Virgil, *Ecl.* X 8. ‖ *ib. Ecl.* I 2.

the Jewish Cemetery at Dalston on March 19, the Royal Society, the London Mathematical Society, and New College, Oxford, being represented (*Nature*, 25 March 1897).

One rises from the task of editing Sylvester's mathematical writings for the Press, with a feeling that here was a great personality as well as a remarkable mathematician, wide and accurate in thought, deep and sensitive in feeling, and inspired with a great faith in things spiritual. "...is a very great genius," he is reported to have said when pressed on one occasion, "I only wish he would stick to mathematics, instead of talking atheism."

Of the detailed relations of his work with that of contemporary writers, especially for the Theory of Equations, Dr M. Noether has written a masterly and easily accessible account (*Math. Annalen*, Bd L, 1898). In his Presidential address to the London Mathematical Society (*Proceedings*, XXVIII, 1896—97) Major MacMahon has given an appreciation of his work on the Theory of Partitions, which should be consulted. Sylvester's long devotion to the Theory of Invariants, in conjunction with Cayley, transforming the whole analysis of Projective Geometry, has left an ineffaceable mark on Mathematics; but in all questions of algebraical form, working more often by divination than by computation, he is wonderful—his theorems in regard to Sturm's Functions, Canonical Forms, and Determinants suggest themselves at once. So general are some of his results that even the recognition of other theorems as particular cases of them may sometimes be difficult, as very distinguished writers have found.

But another aspect of his mathematical work must, I think, be referred to, if only to place in due proportion what has been said already. It would seem that the multiplicity of the ideas which pressed upon Sylvester's mind left him little leisure to read, more than cursorily, the writings of other mathematicians. He gives a proof of the theorem for six points lying upon a conic section, known as Pascal's theorem, by the method of indeterminate coordinates, and no theorem of analytical geometry seems strange to him, but he makes no reference to the philosophical interest of Poncelet's imaginary elements at infinity. He deals with von Staudt's formulae for the mensuration of pyramids, but von Staudt's scheme for dispensing with the notion of length in geometrical theory does not attract him. The ferment and broad conclusions as to the foundations of geometry, surely one of the most important of nineteenth century speculations, stir no echo in his pages. Again, he gives remarkable formulae in the Theory of Numbers, but Kummer's investigations in regard to ideal numbers, and the vast new regions opened by them, even Gauss's consideration of complex integers, he does not speak of. His silence as to Lie's theory of continuous groups has already been remarked; he is also silent as to the theory of systems of linear partial differential equations; and though he gives important results as to the permutations of

an assigned number of elements, he does not refer to the question of the algebraic solution of the quintic equation, and writes nothing as to the abstract theory of groups. Most remarkable of all, though he gives, and evidently values, an evaluation of an elliptic integral, and proves, in a wonderful way, by partitions, formulae of theta-functions, the majesty of the new world which we associate with such names as those of Cauchy, Abel and Jacobi, of Riemann and Weierstrass and others, does not greatly stir his longing, so far as his writings declare. Indeed the abstract notion of a function whether for a real, or a complex variable, never occurs in his papers; such a definite instance as Fourier's use of trigonometric series in the Theory of Heat, of 1822, fails to draw him from his combinatorial standpoint; to him the solution of a differential equation is its solution in explicit form; and his formula for the quotity of a partition is an isolated result. For an ordinary man, trained in a country where the importance attached to time examinations tends to discourage the study of all mathematical doctrine, this might be easy to understand; but in Sylvester's case it is very noticeable, and should not be passed over without mention.

Sylvester's position however is secure. As the physicist glories in the interest of his contact with concrete things, so Sylvester loved to mark his progress with definite formulae. He was however before all an abstract thinker, his admiration was ever for intellectual triumphs, his constant worship was of the things of the mind. This it was which seems to have most impressed those who knew him personally. And because of this, his work will endure, according to its value,—mingling with the stream fed by the toil of innumerable men,—of which the issue is as the source. He is of those to whom it is given to renew in us the sanity which is called faith.

H. F. BAKER.

1.

A CONSTRUCTIVE THEORY OF PARTITIONS, ARRANGED IN THREE ACTS, AN INTERACT AND AN EXODION.

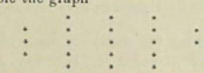
[*American Journal of Mathematics*, v. (1882), pp. 251—330; vi. (1884), pp. 334—336.]

ACT I. ON PARTITIONS REGARDED AS ENTITIES.

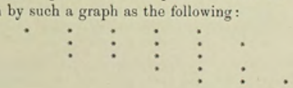
. . . seeming parted,
But yet a union in partition.
Twelfth-night.

(1) IN the new method of partitions it is essential to consider a partition as a *definite thing*, which end is attained by regularization of the succession of its parts according to some prescribed law. The simplest law for the purpose is that the arrangement of the parts shall be according to their order of magnitude. A leading idea of the method is that of correspondence between different complete systems of partitions regularized in the manner aforesaid. The perception of the correspondence is in many cases greatly facilitated by means of a graphical method of representation, which also serves *per se* as an instrument of transformation.

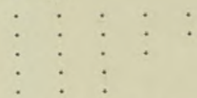
(2) The most obvious mode of graphically representing a partition is by means of a network or web formed by two systems of parallel lines or filaments. Any continuous portion of such web will serve to represent a partition, as for example the graph



will represent the partition 3 5 5 4 3 of 20 by reading off the successive numbers of nodes parallel to the horizontal lines of the web. This, however, is not a regularized partition; the partition will be represented in its regularized form by such a graph as the following:



which corresponds to the order 5 5 4 3 3, but it may be represented much more advantageously by the figure



which is a portion of the web bounded by lines of nodes belonging to the two systems of parallel filaments. Any such portion becomes then subject to the important condition that the two transverse parallel readings will each give a regularized partition, one being in the present example 5 5 4 3 3, and the other 5 5 5 3 2. Any such graph as this will be termed a *regular* partition-graph, and the two partitions which it represents will be said to be conjugate to one another. The mere conception of a regular graph serves at once by effecting an interchange (so to say) between the warp and the woof, through the principle of correspondence, to establish a well-known fundamental theorem of reciprocity. In the last figure, the extent* of (meaning the number of nodes contained by) the uppermost horizontal line or filament is the maximum magnitude of any element (or part) of the partition, and the extent of the first vertical line is the number of the parts. Hence, every regularized partition beginning with i and containing j parts is conjugate to another beginning with j and containing i parts. The content of the graph (that is, the sum of the parts) of the partition is the same in both cases (it will sometimes be convenient to speak of the *partible number* as the content of the elements of the partition). From the above correspondence it is clear that if two complete partition-systems be formed with the same content in one of which the largest part is i and the number of parts j, and in the other the largest part is j and the number of the parts i, the order (that is, the number of partitions) of the first system will be identical with the order of the second: so that calling the content n, it follows that $n-i$ may be decomposed in as many ways into $j-1$ parts as $n-j$ into $i-1$ parts.

(3) This, however, is not the usual nor the more convenient mode of expressing the reciprocity in question. We may, for the two partition systems spoken of, substitute two others of larger inclusion, taking for the first, all partitions of n in which no one part is greater than i, and the number of parts is not greater than j (that is, is j or fewer), and for the second system, one subject to the same conditions as just stated, but with i and j (as before) interchanged: it is obvious that each regularized partition

* *Extent* may be used to denote the number of nodes on a line or column or angle of a graph; *content* the number of nodes in the graph itself; but I have by inadvertence in what follows frequently applied *content* alike to designate areal and linear numerosity.

of one system will be conjugate to one regularized partition of the other system, and accordingly the order of the two systems will be the same*.

(4) When $i=\infty$ it follows from the general theorem of reciprocity last established, that the number of partitions of n into j parts or fewer will be the same as the number of ways of composing n with the integers $1, 2, \ldots j$, and is therefore the coefficient of x^n in the expansion of

$$\frac{1}{1-x\,.\,1-x^2\ldots 1-x^j}.$$

Thus, then, we can at once find the general term in

$$\frac{1}{(1-a)(1-ax)(1-ax^2)\ldots},$$

expanded according to ascending powers of a; for, if the above fraction be regarded as the product of an infinite number of infinite series arising from the expansion of the several factors

$$\frac{1}{1-a},\ \frac{1}{1-ax},\ \frac{1}{1-ax^2},\ \ldots$$

it will readily be seen that the coefficient of $x^n a^j$ will be the number of ways in which n can be resolved into j parts or fewer, that is, by what has been just shown is the coefficient of x^n in

$$\frac{1}{1-x\,.\,1-x^2\ldots 1-x^j};$$

and this being true for all values of n, it follows that the entire coefficient of a^j is the fraction last written developed in ascending powers of x; so that

$$\frac{1}{(1-a)(1-ax)(1-ax^2)\ldots}$$
$$=1+\frac{1}{1-x}a+\frac{1}{1-x\,.\,1-x^2}a^2+\frac{1}{1-x\,.\,1-x^2\,.\,1-x^3}a^3\ldots$$

as is well known.

The general term in

$$\frac{1}{(1-a)(1-ax)\ldots(1-ax^i)}$$

is also well known to be

$$\frac{1-x^{i+1}\,.\,1-x^{i+2}\ldots 1-x^{i+j}}{1-x\,.\,1-x^2\ldots 1-x^j}a^j,$$

* The above proof of the theorem of reciprocity is due to Dr Ferrers, the present head of Gonville and Caius College, Cambridge. It possesses the double merit of having set the first example of graphical construction and of putting into salient relief the principle of correspondence, applied to the theory of partitions. It was never made public by its author, but first promulgated by myself in the *Lond. and Edin. Phil. Mag.* for 1853. [Vol. I. of this Reprint, p. 597.]

or in other words, the number of ways of resolving n into j parts none greater than i is the coefficient of x^n in the fraction

$$\frac{1-x^{i+1}\,.\,1-x^{i+2}\,\ldots\,1-x^{i+j}}{1-x\,.\,1-x^2\,\ldots\,1-x^j},$$

which [denoting $1-x^q$ by (q)] is the same as

$$\frac{(1)\,(2)\ldots(i+j)}{(1)\,(2)\ldots(i)\,.\,(1)\,(2)\ldots(j)},$$

and furnishes, if I am not mistaken, Euler's proof of the theorem of reciprocity already established by means of the correspondence of conjugate partitions.

(5) [It may be as well to advert here to the practical method of obtaining the conjugate to a given partition. For this purpose it is only necessary to call a_i the number of parts in the given partition not less than i; $a_1, a_2, a_3, \ldots a_i \ldots$ continued to infinity (or which comes to the same thing until i is equal to the maximum part), will be the required conjugate.]

(6) The following very beautiful method of obtaining the general term in question by the constructive method is due to Mr F. Franklin of the Johns Hopkins University* :

He, as it were, interpolates between the theorem to be established in general and the theorem for $i=\infty$, and attaches a definite meaning to the above fraction regarded as a generating function when the factors in the numerator are limited to the first q of them, q being any number not exceeding i, so that in fact the theorem to be proved, according to this view, is only the extreme case of (the last link in the chain to) a new and more general one with which he has enriched the theory of partitions. The method will be most easily understood by means of an example or two: the proof and use to be made of the construction will be given towards the end of the Act.

Let $n=10$, $i=5$, $j=4$.

Write down the indefinite partitions of 10 into 4 or fewer parts, or say rather into 4 parts, among which zeros are admissible: they will be

(1)	10.0.0.0	5.5.0.0
(1)	9.1.0.0	5.4.1.0
(1)	8.2.0.0	5.3.2.0
(1)	8.1.1.0	5.3.1.1
(2)	7.3.0.0	5.2.2.1
(2)	7.2.1.0	4.4.2.0
(1)	7.1.1.1	4.4.1.1
(2)	6.4.0.0	4.3.3.0
(3)	6.3.1.0	4.3.2.1
(3)	6.2.2.0	4.2.2.2
(4)	6.2.1.1	3.3.3.1
		3.3.2.2

* For a vindication of the constructive method applied to this and an allied theorem, see p. [18] *et seq.*

The partitions to which (1) is prefixed are those in which the *first excess*, that is, the excess of the first (the dominant) part over the next is *too great* (meaning greater than i, here 5); those to which (2) is prefixed are those in which after the batch marked with (1) are removed, the second excess, that is, the excess of the first over the third element is "too great"; those to which (3) is prefixed are those in which after the batches marked (1) and (2) are removed, the third excess is "too great," and lastly those (only one as it happens) marked with j (here 4) are those in which, so to say, the *absolute excess* of the dominant, that is its actual value, is "too great," that is, exceeding i (here 5); the partitions that are left over will be the partitions of n (here 10) into 4 parts, none exceeding i (here 5) in magnitude.

It is easy to see from this how to *construct* the partitions which are to be *eliminated* from the indefinite partitions of the n (10) into 4 (j) parts so as to obtain the quaternary partitions in which no part superior to 5 (i) appears. To obtain the first batch we must subtract $i+1$ (6) from n (10) and form the system of indefinite partitions of 4 into four parts, namely:

4.0.0.0
3.1.0.0
2.2.0.0
2.1.1.0
1.1.1.1

and adding to each of these 6.0.0.0 (term-to-term addition) batch (1) will be obtained.

To obtain the second batch, form the quaternary partitions of $n-(i+2)$, that is, 3, namely:

3.0.0.0
2.1.0.0
1.1.1.0

[but omit those in which the first excess is "too great" (greater than i); here there are none such to be omitted] and bring the second element into the first place; thus we shall obtain the system

0 3 0 0
1 2 0 0
1 1 1 0

The *augments* of those obtained by adding 6.1.0.0 to each of them will reproduce batch (2).

Again, form the quaternary partition-system of $n-(i+3)$, rejecting all those (here there are none such) in which the second excess is "too great." We thus obtain

2 0 0 0
1 1 0 0

and now bringing the third element in each of these into the first place so as to obtain

0 2 0 0
0 1 1 0

The augments of these last partitions obtained by adding 6 . 1 . 1 . 0 to each of them will give the third batch, and finally taking the quaternary partition-system to $n-(i+j)$, that is, 1, rejecting (if there should be any such) those in which the third excess is "too great," we obtain 1 . 0 . 0 . 0, and bringing the fourth element to the first place so as to get 0 . 1 . 0 . 0, and adding 6 . 1 . 1 . 1, the fourth batch 6 . 2 . 1 . 1 is reconstructed.

As another example take $n=15$, $i=3$, $j=3$.

The indefinite ternary partitions of 15 are

15 . 0 . 0	(1)	9 . 4 . 2	(1)
14 . 1 . 0	(1)	9 . 3 . 3	(1)
13 . 2 . 0	(1)	8 . 7 . 0	(2)
13 . 1 . 1	(1)	8 . 6 . 1	(2)
12 . 3 . 0	(1)	8 . 5 . 2	(2)
12 . 2 . 1	(1)	8 . 4 . 3	(1)
11 . 4 . 0	(1)	7 . 7 . 1	(2)
11 . 3 . 1	(1)	7 . 6 . 2	(2)
11 . 2 . 2	(1)	7 . 5 . 3	(2)
10 . 5 . 0	(1)	7 . 4 . 4	(3)
10 . 4 . 1	(1)	6 . 6 . 3	(3)
10 . 3 . 2	(1)	6 . 5 . 4	(3)
9 . 6 . 0	(2)	5 . 5 . 5	(3)
9 . 5 . 1	(1)		

There are, of course, no partitions left in which no part exceeds 3, as the maximum content subject to that condition would be only 9.

The partitions marked (1) (2) (3) are those in which the first, second and absolute excess respectively exceed 3.

Firstly, the indefinite ternary partitions of $15-4$ or 11 augmented by 4 . 0 . 0 will obviously reproduce the system of partitions marked (1).

Secondly, taking the indefinite ternary partitions of 10 in which the *first* excess, and those of 9 in which the *second* excess, does not exceed 3, we shall obtain

6 . 4 . 0	and 5 . 2 . 2
6 . 3 . 1	4 . 4 . 1
5 . 5 . 0	4 . 3 . 2
5 . 4 . 1	3 . 3 . 3
5 . 3 . 2	
4 . 4 . 2	
4 . 3 . 3	

which by *metastasis* become

4 . 6 . 0	2 . 5 . 2
3 . 6 . 1	1 . 4 . 4
5 . 5 . 0	2 . 4 . 3
4 . 5 . 1	3 . 3 . 3
3 . 5 . 2	
4 . 4 . 2	
3 . 4 . 3	

and adding to each term of these two groups 4 . 1 . 0 and 4 . 1 . 1 respectively, the systems of partitions marked (2) and (3) respectively result.

(7) It may, I think, be desirable to give here my own construction for the case of repeated partitions, which, having regard to its features of resemblance to the one preceding, it is proper to state preceded it in the date of its discovery and promulgation. The problem which I propose to myself is to construct a system of partitions of a given number into parts limited in number and magnitude, by means of partitions of itself and other numbers into parts limited in number but not in magnitude.

As before, let i be the limit of magnitude, j the number of parts (zeros admissible), and n the partible number; form a square matrix of the jth order in which the diagonal elements are all $i+1$, the elements below the diagonal all of them unity, and those above the diagonal all of them zero, say M_1.

From this matrix construct $M_1, M_2, M_3, \ldots M_j$, such that the lines in M_q (q being any integer from 1 to j inclusive) are the sums of those in M_1, added (term-to-term) q and q together.

Let (r, q) be the rth line in M_q and $[r, q]$ the sum of the numbers which it contains.

Form the complete system of the partitions of $n-[r, q]$ into j parts, and to each such add (term-to-term) (r, q).

In this way, by giving r all possible values we shall obtain a system of partitions of n into j parts corresponding to M_q, which may be called P_q. I say that $P_1 - P_2 + P_3 \ldots + (-)^{j-1} P_j$ will be the complete system of partitions of n into j parts in which one element at least exceeds i; where it is to be observed that having regard to the effect of the − and + signs (which are used here to indicate the addition and subtraction, or say rather the adduction and sub-duction not of numbers but of things), each such partition will occur once and once only; so that calling P the complete system of indefinite partitions of n into j parts, the complete system of partitions of n into j parts in which no part exceeds i in magnitude will be

$$P - P_1 + P_2 \ldots + (-)^j P_j\,*.$$

* It must, however, be understood that the same partition is liable to appear in more than one, and not exclusively in its regularized phase, or as it may be expressed, the regularized partition undergoes *metastasis*.

(8) This construction, which I will illustrate by two examples, proceeds upon the fact which, although confirmed by a multitude of instances, *remains to be proved*, that if $k_1, k_2, \ldots k_j$ be any partition of n into j parts and the number of *unequal* parts greater than i be μ, then the number of times in which this partition, in its regular or any other phase, appears in P_q is $\frac{\mu(\mu-1)\ldots(\mu-q+1)}{1.2\ldots q}$ (interpreted to mean 1 when $q=0$), and consequently its total number of appearances in $P-P_1+P_2\ldots$ is $(1-1)^\mu$, that is, is 0.

From this it follows that the total number of partitions of n into j parts none exceeding i in magnitude will be $C-C_1+C_2-\ldots$, where C_q is the sum of the number of ways in which the various numbers $n_1, n_2, n_3\ldots$ can be decomposed into j parts, the numbers $n_1, n_2, n_3, \ldots$ being n diminished by the sums of the quantities $i+1, i+2, \ldots, i+j$ added q and q together; C_q is therefore the coefficient of x^n in $\frac{x^{n-n_1}+x^{n-n_2}+x^{n-n_3}+\ldots}{(1-x)(1-x^2)\ldots(1-x^j)}$; and consequently the number of partitions of n into j parts none exceeding i in magnitude will be the coefficient of x^n in $\frac{1-x^{i+1}.1-x^{i+2}\ldots1-x^{i+j}}{1-x.1-x^2\ldots1-x^j}$ as was to be shown.

(9) As a first example let $i=2$, $j=3$, $n=12$, the matrices and the partitions corresponding to their several lines will be as underwritten; the indefinite partitions of the reduced contents, $n-[r, q]$, are written opposite to the respective matrix lines to which they correspond, and their augments, found by adding the line to this partition system, are written immediately under them. The zeros are omitted for the sake of brevity.

3.0.0	9	8.1	7.2	7.1.1	6.3	6.2.1	5.4	5.3.1	5.2.2	4.4.1	4.3.2	3.3.3
	12	11.1	10.2	10.1.1	9.3	9.2.1	8.4	8.3.1	8.2.2	7.4.1	7.3.2	6.3.3
1.3.0	8	7.1	6.2	6.1.1	5.3	5.2.1	4.4	4.3.1	4.2.2	3.3.2		
	9.3	8.4	7.5	7.4.1	6.6	6.5.1	5.7	5.6.1	5.5.2	4.6.2		
1.1.3	7	6.1	5.2	5.1.1	4.3	4.2.1	3.3.1	3.2.2				
	8.1.3	7.2.3	6.3.3	6.2.4	5.4.3	5.3.4	4.4.4	4.3.5				
—												
4.3.0	5	4.1	3.2	3.1.1	2.2.1							
	9.3	8.4	7.5	7.4.1	6.5.1							
4.1.3	4	3.1	2.2	2.1.1								
	8.1.3	7.2.3	6.3.3	6.2.4								
2.4.3	3	2.1	1.1.1									
	5.4.3	4.5.3	3.5.4									
—												
5.4.3	0											
	5.4.3											

In 6.3.3 there are two unlike elements greater than 2; accordingly 6.3.3 occurs 2 times in P_1 and 1 time in P_2.

In 7.3.2 there are again two unlike elements greater than 2, and 7.3.2, 7.2.3 (the metastatic equivalent to the former) are found in P_1 and 7.2.3 in P_2.

Again, in 5.4.3 there are 3 unlike elements greater than 2, and we find

5.4.3	5.3.4	4.3.5	in P_1
5.4.3	4.5.3	3.5.4	„ P_2
5.4.3			„ P_3.

But such terms as 11.1 10.1.1 9.2.1 8.2.2 in which there is only one distinct element greater than 2 are found 1 time only in P_1 and not at all in P_2 or P_3.

As another example let $n=12$, $i=4$, $j=3$, then a similarly constructed table to the foregoing will be as follows, in which, however, all matrices or lines of matrices which have a sum too large to give rise to partition systems are omitted.

5.0.0	7	6.1	5.2	5.1.1	4.3	4.2.1	3.3.1	3.2.2
	12	11.1	10.2	10.1.1	9.3	9.2.1	8.3.1	8.2.2
1.5.0	6	5.1	4.2	4.1.1	3.3	3.2.1	2.2.2	
	7.5	6.6	5.7	5.6.1	4.8	4.7.1	3.7.2	
1.1.5	5	4.1	3.2	3.1.1	2.2.1			
	6.1.5	5.2.5	4.3.5	4.2.6	3.3.6			
—								
6.5.0	1							
	7.5							
6.1.5	0							
	6.1.5							

7.5 and 6.5.1 are the only two partitions of 12 into 3 parts in which there are two unlike parts greater than 4; each of these accordingly is found twice (in one or another phase) in P_1 and once in P_2. Every other partition of 12 into 3 parts in which one of them at least is greater than 4 will be found exclusively and only once in P_1.

(10) The two expansions for $(1-ax)(1-ax^2)\ldots(1-ax^i)$ and its reciprocal may readily be obtained from one another by the method of correspondence.

The coefficient of $x^n a^j$ in the former is the number of partitions of n into j *unequal*, and in the latter into j *equal or unequal* parts none greater than i or less than unity. The correspondence to be established has been given by Euler for the case of $i=\infty$ (*Comm. Arith.*, 1849, Tom. I. p. 88), and is probably known for the general case, but as coming strictly within the purview of the essay, seems to deserve mention here.

If $k_1, k_2, k_3, \ldots, k_j$ be a partition of n into j equal or unequal parts written in ascending order, none exceeding i, on adding to it $0, 1, 2 \ldots (j-1)$, it becomes a partition of $n+\frac{j^2-j}{2}$ into j parts none exceeding $i+j-1$, and conversely, if $\lambda_1, \lambda_2, \ldots, \lambda_j$ be a partition of $n+\frac{j^2-j}{2}$ into j unequal parts none exceeding $i+j-1$, written in ascending order, on subtracting from it $0, 1, 2 \ldots (j-1)$, it becomes a partition of n into equal or unequal (say relatively independent) parts none exceeding i.

Hence the complete system of partitions of n into j unlike parts none exceeding i has a one-to-one correspondence with the complete system of the partitions of $n-\frac{j^2-j}{2}$ into j parts none exceeding $i-j+1$. Consequently the coefficient of a^j in the expansion of $(1-ax) \ldots (1-ax^i)$ may be found from that of a^j in the expansion of its reciprocal by changing i into $i-j+1$ and introducing the factor $x^{\frac{j^2-j}{2}}$.

(11) The expansion of the reciprocal may of course be found algebraically from the multiplication of the expansion which has been given of $\frac{1}{(1-a)(1-ax)\ldots(1-ax^i)}$ by $(1-a)$, or immediately by the correspondence between partitions into an exact number j of parts limited not to exceed i, and partitions into j or fewer parts so limited.

By subtracting a unit from each term of $k_1, k_2, \ldots, k_j$, a partition of n where no k exceeds i, results a partition $q_1, q_2, \ldots q_j$, a partition of $n-j$ where no q exceeds $i-1$. Hence the coefficient of a^j in

$$\frac{1}{1-ax \,.\, 1-ax^2 \ldots 1-ax^i}$$

may be found from that in

$$\frac{1}{1-a \,.\, 1-ax \ldots 1-ax^i}$$

by introducing the factor x^j and changing i into $i-1$, so that choosing for the latter the alternative form

$$\frac{1-x^{j+1} \,.\, 1-x^{j+2} \ldots 1-x^{j+i}}{1-x \,.\, 1-x^2 \ldots 1-x^i},$$

the former becomes

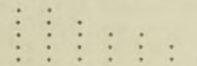
$$\frac{1-x^{j+1} \,.\, 1-x^{j+2} \ldots 1-x^{j+i-1}}{1-x \,.\, 1-x^2 \ldots 1-x^{i-1}} x^j,$$

and consequently the coefficient of a^j in $1-ax \,.\, 1-ax^2 \ldots 1-ax^i$ will be

$$\frac{1-x^{j+1} \,.\, 1-x^{j+2} \ldots 1-x^i}{1-x \,.\, 1-x^2 \ldots 1-x^{i-j}} x^{\frac{j^2+j}{2}}.$$

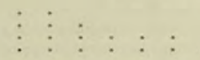
(12) Before quitting this part of the subject it is desirable to make mention of Dr F. Franklin's remarkable method of proving Euler's celebrated expansion of $(1-x)(1-x^2)(1-x^3)\ldots$ *ad inf.* by the method of correspondence. This has been given by Dr Franklin himself in the *Comptes Rendus* of the Institut (1880), and by myself in some detail in the last February Number of the *J. H. U. Circular**. The method is in its essence absolutely independent of graphical considerations, but as it becomes somewhat easier to apprehend by means of graphical description and nomenclature, I shall avail myself here of graphical terminology to express it.

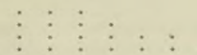
If a regular graph represent a partition with unequal elements, the lines of magnitude must continually increase or decrease. Let the annexed figures be such graphs written in ascending order from above downwards.



(*A*)

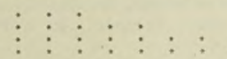
(*B*)





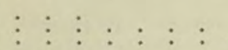
(*C*)

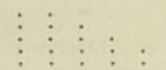
In A and B the graphs may be transformed without altering their content or regularity by removing the nodes at the summit and substituting for them a new slope line at the base. In C the slope line at the base may be removed and made to form a new summit; the graphs so transformed will be as follows:



(*A′*)

(*B′*)





(*C′*)

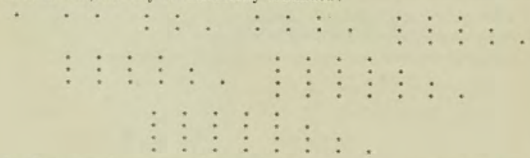
A' and B' may be said to be derived from A, B by a process of contraction, and C' from C by one of protraction.

Contraction could not now be applied to A' and B', nor protraction to C' without destroying the regularity of the graph; but the inverse processes may of course be applied, namely, of protraction to A' and B' and contraction to C', so as to bring back the original graph A, B, C.

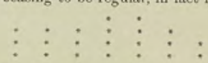
In general (but as will be seen not universally), it is obvious that when the number of nodes in the summit is inferior or equal to the number in the base-slope, contraction may be applied, and when superior to that number, protraction: each process alike will alter the number of parts from even to

[* Vol. III. of this Reprint, p. 664.]

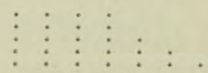
odd or from odd to even, so that barring the exceptional cases which remain to be considered where neither protraction nor contraction is feasible, there will be a one-to-one correspondence between the partitions of n into an odd number and the partitions of n into an even number of unrepeated parts; the exceptional cases are those shown below where the summit meets the base-slope line, and contains either the same number or one more than the number of nodes in that line; in which case neither protraction nor contraction will be possible, as seen in the annexed figures which are written in regular order of succession, but may be indefinitely continued:



for the protraction process which *ought*, for example, according to the general rule, to be applicable to the last of the above graphs, cannot be applied to it, because on removing the nodes in the slope line and laying them on the summit, in the very act of so doing the summit undergoes the loss of a node and is thereby incapacitated to be surmounted by the nodes in the slope, which will have not now a less, but the same number of nodes as itself; and in like manner, in the last graph but one, the nodes in the summit cannot be removed and a slope line be added on containing the same number of nodes without the transformed graph ceasing to be regular, in fact it would take the form



and so the last graph transformed according to rule [by protraction] would become:



which, although regular, would cease to represent a partition into unlike numbers.

The excepted cases then or unconjugate partitions are those where the number of parts being j, the successive parts form one or the other of the two arithmetical series

$$j, j+1, j+2, \ldots 2j-1 \text{ or } j+1, j+2, \ldots 2j,$$

in which cases the contents are $\frac{3j^2-j}{2}$ and $\frac{3j^2+j}{2}$ respectively, and consequently

since in the product of $1-x \,.\, 1-x^2 \,.\, 1-x^3 \ldots$ the coefficient of x^n is the number of ways of composing n with an even less the number of ways of composing it with an odd number of parts, the product will be completely represented by $\sum\limits_{j=+\infty}^{j=-\infty} (-)^j x^{\frac{3j^2+j}{2}}$ *.

(13) It has been well remarked by Prof. Cayley that barring the unconjugate partitions, the rest really constitute 4 classes, which using c and x to signify contractile and extensile and e and o to signify of-an-even or of-an-odd order, may be denoted by

$$\begin{matrix} c\,.\,e & c\,.\,o \\ x\,.\,e & x\,.\,o. \end{matrix}$$

Hence as each $c\,.\,e$ is conjugate to an $x\ o$ and *vice versâ*, and each $c\,.\,o$ to an $x\,.\,e$ and *vice versâ*, the theorem established really splits up into two, one affirming that the number of contractile partitions of an odd order is the same as the number of extensile ones of an even order, the other that the number of contractiles of an even is equal to the number of extensiles of an odd order. It might possibly be worth while to investigate the difference between the number of partitions which each set of one couple and the number of partitions which each set of the sub-contrary couple contain: the sets which belong to the same couple and contain the same number of partitions being those *both* of whose characters are dissimilar.

(14) There are one or two other simple cases of correspondence which are interesting, inasmuch as the construction employed to effect the correspondence involves the operations of division and multiplication, which have not occurred previously.

If $$fx = (1-x)(1-x^3)(1-x^5)(1-x^7)(1-x^9)\ldots$$
and $$\phi x = (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)\ldots$$
$$fx \,.\, \phi x = 1,$$
from which we obtain $\phi x = 1/fx$ and $1/\phi x = fx$.

The first of these equations has been noticed by Euler as involving the elegant theorem that a number may be partitioned in as many ways into only-once-occurring odd-or-even integers as into any-number-of-times-occurring only-odd integers.

* Another proof of this theorem, deduced as an immediate algebraical consequence of a more general one, obtained by graphical dissection, will be given in Act 2; and in the Exodion I furnish a purely arithmetical proof by the method of correspondence of Jacobi's series for

$$(1 \pm x^{n-m})(1 \pm x^{n+m})(1-x^{2n})(1 \pm x^{3n-m})(1 \pm x^{3n+m})(1-x^{4n})\ldots$$

(which includes Euler's theorem as a particular case). I prove this theorem in a more extended sense than was probably intended by its immortal author, inasmuch as I regard m and n as absolutely general symbols.

The second, which I think he does not dwell upon, expresses that the difference between the number of partitions with an even number of parts and that of partitions with an odd number of parts of the same number n is the same as the number of partitions of n into exclusively odd [unrepeated] numbers (such difference being in favour of the partitions of even or of odd order, according as the partible number is even or odd).

This latter theorem brings out a point of analogy between repetitional and non-repetitional partition systems which appears to me worthy of notice.

Any one of the former contains a class of what may be termed singular partitions, in the sense that they are their own associates, or more briefly, *self-conjugate* in respect to the Ferrers transformation. Any one system of the latter may also be *said* to contain a set of singular partitions (0 or 1 in number) in the sense of being *unconjugate* in respect to the Franklin process of transformation. Since then in this case the difference between the number of partitions of an odd and those of an even order of the same number is equal to the number (1 or 0) of singular partitions of that number, so we might anticipate as not improbable that the like difference for the repetitional partitions of a number should be equal to the number of singular partitions of that number—and such is actually the case; for it will be shown in a future section that the number of self-conjugate partitions of a number is the same as the number of ways in which it can be composed with odd integers.

(15) The correspondence indicated by the equation $\phi x = 1/fx$ can be established as follows:

Let $2^\lambda . l$, $2^\mu . m$, $2^\nu . n$, ... be any partition of unrepeated general numbers, where l, m, n ... are any odd integers not exceeding unity; and let $k^{[q]}$ in general denote q parts k, then without changing its content the above partition can be converted into $l^{[2^\lambda]}$, $m^{[2^\mu]}$, $n^{[2^\nu]}$, ... which consists exclusively of odd numbers.

It will of course be understood that the original partition may contain any the same odd number as l multiplied by different powers 2^λ, $2^{\lambda'}$, $2^{\lambda''}$ ... of 2, with the sole restriction that the λ, λ', λ'', ... must be all unequal.

Conversely, any such partitions as $l^{[\sigma]}$, $m^{[\tau]}$, $n^{[\upsilon]}$ may be converted back into one and only one partition of the former kind. For there will be one and but one way of resolving σ into the sum of powers of 2 (the zero power not excluded), and supposing σ to be equal to $2^\lambda + 2^{\lambda'} + 2^{\lambda''} + \ldots$, $l^{[\sigma]}$ may be replaced by $2^\lambda l$, $2^{\lambda'} l$, $2^{\lambda''} l$, and the same process of conversion may be simultaneously applied to each of the other products $m^{[\tau]}$, $n^{[\upsilon]}$, ...

Hence each partition of either one kind is conjugate to one of the other, and the number of partitions in the two systems will be the same, as was to be shown.

(16) But we have here another example of the fact that the theory of correspondence reaches far deeper than that of mere numerical congruity with which it is associated as the substance with the shadow. For a correspondence exists of a much more refined nature than that above demonstrated between the two systems, and which, moreover (it is important to notice) does not bring the same individuals into correlation as does the former method.

The partition system made up of unrepeated general numbers may be divided into groups of the first, second, ... ith ... class respectively, those of the ith class containing i distinct sequences of consecutive numbers having no term in common, with the understanding that no two sequences must form part of a single sequence (so that the largest term of one sequence and the smallest one of the next sequence must differ by more than a single unit), and that a single number unpreceded and unfollowed by a consecutive number is to count as a sequence.

The partition system, made up of repeatable odd numbers may, in like manner, be resolved into groups of the 1st, 2nd, ... ith, ... class respectively, those of the ith class containing i distinct numbers; and the new theorem of correspondence is that there is a correlation between the numbers of the ith class of one system and the ith class of the other; so that the number of partitions in a class of the same name must be the same to whichever system it belongs; and thus Euler's theorem becomes a corollary to this deeper-reaching one, obtained from it by *adding together* the number of partitions in all the several classes in the one system and in the other.

(17) As regards the first class, the theorem amounts to the statement that the number of single sequences of consecutive numbers into which n may be resolved is equal to the number of odd factors which n contains; so that if $N = 2^c . l^\lambda . m^\mu . n^\nu \ldots$ where l, m, n, ... are odd numbers, N can be represented by $(\lambda + 1)(\mu + 1)(\nu + 1) \ldots$ such sequences; thus, for example, if $N = 15 = 3 . 5$ we have

$$1 + 2 + 3 + 4 + 5 = 4 + 5 + 6 = 7 + 8 = 15.$$

So

$$30 = 4 + 5 + 6 + 7 + 8 = 6 + 7 + 8 + 9 = 9 + 10 + 11,$$

$$27 = 2 + 3 + 4 + 5 + 6 + 7 = 8 + 9 + 10 = 13 + 14,$$

$$45 = 1 + 2 + 3 + \ldots + 9 = 5 + 6 + 7 + 8 + 9 + 10$$

$$= 7 + 8 + 9 + 10 + 11 = 14 + 15 + 16 = 22 + 23.$$

So too if N is a prime number it can only be resolved into the two sequences $\frac{N-1}{2} + \frac{N+1}{2}$ and N. More generally N can be resolved into as many different sets of i distinct sequences as there are solutions in positive integers of the equation $2(x_1y_1 + x_2y_2 + \dots + x_iy_i) + x_1 + x_2 + \dots + x_i = N$, of the truth of which remarkable theorem, in its general form, I have for the present only obtained empirical evidence, but may possibly be able to discover the proof in time to annex it in the form of a note at the end, so as not to keep the press waiting*.

(18) The proof for the case of the first class and the mode of establishing the correspondence between the partitions of this class of the two kinds is not far to seek. I use as previously $a^{(b)}$ to signify a repeated b times.

Consider then any sequence of consecutive numbers for the cases where the number of terms is odd and where it is even separately, calling s the sum of the first and last terms, and i the number of terms; where i is odd, so that s is even, the sequence may be replaced by $i^{\left(\frac{s}{2}\right)}$, and where i is even (so that s is odd) by $s^{\left(\frac{i}{2}\right)}$. Hence each partition of the first class of the first kind may be transformed into one of the first class of the second kind.

It is necessary to show the converse of this, which may be done as follows: Let λ^{μ} be any partition of the second kind so that λ is necessarily odd. I say that this must be transformable into one or the other (but not into both) of two sequences, namely, one of λ terms of which the sum of the first and last is 2μ, the other of which the sum of the first and last terms is λ and the number of terms 2μ. The former supposition is admissible if 2μ is equal to or greater than $\lambda + 1$, inadmissible if 2μ is less than $\lambda + 1$. The second supposition is admissible if λ is equal to or greater than $2\mu + 1$, inadmissible if λ is less than $2\mu + 1$.

The two conditions of admissibility coexisting would imply that 2μ is equal to or greater than $2\mu + 2$; the two conditions of inadmissibility the one that 2μ is equal to or less than $\lambda - 1$, the other that λ is equal to or less than $2\mu - 1$, that is, $\lambda - 1$ equal to or less than $2\mu - 2$, which are inconsistent. Hence one of the two transformations is always possible and the other impossible to be effected; which proves the correlation that was to be established. A single example will serve to show that this correspondence is entirely different from that offered by the first and (so to say) grosser method; suppose $N = 15$, then $1.2.3.4.5$ will be a partition of the first kind and will be converted by the new rule into $5.5.5$, whereas, by the former rule, it would be inverted into $1.1.1.3.1.1.1.1.5$, that is, into $1^7.3.5$ belonging to the third class instead of to the first.

(19) I will now pass on to the conjugate theorem corresponding to $fx = 1/\phi x$.

* A complete proof of the general theorem will be given in the 3rd Act.

It may be well here to recall that this identity essentially depends upon the identity $1 - x = 1/(1 + x)(1 + x^2)(1 + x^4) \dots$ which, interpreted*, signifies that any number greater than unity may be made up in as many ways with an odd as with an even number of numbers restricted to the geometrical progression 1, 2, 4, 8 .... This may be called, for brevity, a geometric partition. The correspondence to which this points is itself worthy of notice; one mode of establishing it would be to proceed to decompose N into such parts in regular dictionary order—it would easily be seen that each pair of partitions thus deduced would be of contrary *parities*, but it would not be easy, or at all events evident, how to determine at once the conjugate to a given partition by reference to this principle; but if we observe that it is possible to pass from the geometric partitions of n immediately to those of $n + 1$ by the addition of a unit to each of the former, and consequently to those of $n + 2$ from the partitions of $E\frac{n}{2}, E\frac{n-2}{2}, E\frac{n-4}{2}, \dots 2, 1$, by an obvious process of doubling and adding complementary units, another rule or law of correspondence, which proves itself as soon as stated (and is not identical in effect with that supplied by the dictionary-order method), looms into the field of vision, than which nothing can be simpler. Hence we may derive a transcendental equation in differences for u_n, the number of geometric partitions (with radix 2) to n, namely, to find the conjugate of any geometric partition, look at its greatest part—if it is repeated add two of them together: if it is unrepeated split it into two equal parts; these processes are obviously reversible, just as in Dr Franklin's method of correspondence for the pentagonal-series-theorem; and the method is equally open to the remark made thereon by Prof. Cayley; that is to say, there will be four classes, extensile even, extensile odd, contractile even and contractile odd, and the number of partitions in any class will be the same as in the class in which both the characters are reversed.

The application of this transformation to the construction indicated by the equation $fx = 1/\phi x$ will be obvious. Let any partition containing only unrepeated numbers consist of odd numbers $p, q, r, \dots t$, each multiplied by one or more powers of 2; form batches of these terms which have the same greatest odd divisor $(p, q, r, \dots t)$, and arrange those batches in a line according to the order of magnitude of $p, q, r, \dots t$. Then we may agree to proceed either from left to right or from right to left in reading off the batches, and that convention being established once for all, as soon as a batch is reached which does not consist of a single odd term, if it contain one term larger than all the rest that term is to be split into two equal parts, but if it contain two terms not less than any

* Just so the equation $1/(1 - x) = (1 + x)(1 + x^2)(1 + x^4) \dots$ teaches that there is one and only one way of effecting the unrepetitional geometric partition of any number—a theorem which has been applied in the preceding theory.

others in the batch, those two are to be amalgamated into one. In this way the order of a partition consisting of terms not all of them distinct odd numbers, will have its *parity* (quality of being odd or even) reversed, and it is obvious that if A has been under the operation of the rule converted into B, B by the operation of the same rule will be converted back into A. Hence it follows that (making abstraction of the partitions consisting exclusively of unrepeated odd numbers) all the rest will be separable into as many contractile of an odd as into extensile of an even order, and into as many extensile of an odd as into contractile of an even order, so that the difference between the entire number of the partitions of N into an odd and those of an even order of repeatable numbers (odd or even) will be the number of partitions of N into unrepeated odd numbers, and those of an odd or of an even order will be in the majority according as N itself is odd or even*.

It will be convenient to interpolate here Dr F. Franklin's constructive proof of the theorems referred to in p. [4] of what precedes, as there will be frequent occasion to refer to them in what follows. The theory is thus made completely self-contained. I give the proofs in the author's own words, which I think cannot be bettered.

(20) *Constructive Proof of the Formula for Partitions into Repeatable Parts, limited in Number and Magnitude.* The partitions herein spoken of are always partitions into a fixed number, j, of parts, *written in descending order.*

Take any partition of w in which the first excess† is greater than i; subtracting $i+1$ from the first part we get a partition of $w-(i+1)$; and conversely if to the first part in a partition of $w-(i+1)$ we add $i+1$ we get a partition of w in which the first excess is greater than i. Hence the number of partitions of w in which the first excess is greater than i is equal to the whole number of partitions of $w-(i+1)$; so that if the generating

* Dr F. Franklin has remarked that "the theorem admits of the following extensions," which the method employed in the text naturally suggests, and "which are very easily obtained either by the constructive proof or by generating functions":

1. The number of ways in which w can be made up of any number of odd and k distinct even parts is equal to the number of ways in which it can be made up of any number of unrepeated and k distinct repeated parts.

2. The number of ways in which w can be made up of parts not divisible by m is equal to the number of ways in which it can be made up of parts not occurring as many as m times.

3. The number of ways in which w can be made up of an infinite number of parts not divisible by m, together with k parts divisible by m, is equal to the number of ways in which it can be made up of an indefinite number of parts occurring less than m times, together with k parts occurring m or more times. (3) of course comprehends (1) and (2) as special cases.

Dr Franklin adds, "another extension is naturally contained in the mode of proof, which it is perhaps not worth while to state." See *Johns Hopkins Circular* for March, 1883.

† The first excess signifies the excess of the largest part over the next largest; the second excess the excess of the largest over the next part but one, and so on.

function for the partitions of w is $f(x)$, that for those partitions in which the first excess is *not greater* than i is $(1-x^{i+1})f(x)$. Confining ourselves now to this class of partitions, consider any one of them in which the second excess is greater than i; subtracting $i+1$ from the first part and 1 from the next, and putting the reduced first part into the second place we have a partition of $w-(i+2)$ in which the first excess is not greater than i; and conversely if in any partition of $w-(i+2)$ in which the first excess is not greater than i, we add $i+1$ to the second part and 1 to the first part and transfer the augmented second part to the first place, we get a partition of w in which the first excess is not greater than i and the second excess *is* greater than i. Hence the generating function for those partitions in which the second excess is *not* greater than i is $(1-x^{i+1})(1-x^{i+2})f(x)$. Considering now exclusively the partitions last mentioned, any one of them in which the third excess is greater than i may be converted into a partition of $w-(i+3)$ in which the second excess is not greater than i, by subtracting $i+1$ from the first part, 1 from the second part, and 1 from the third part, and removing the reduced first part to the third place, and, as before, by the reverse operation, the latter class of partitions are converted into the former. Hence the generating function for the partitions in which the third excess is not greater than i is

$$(1-x^{i+1})(1-x^{i+2})(1-x^{i+3})f(x).$$

So in like manner, the generating function for the partitions in which the k-th excess is not greater than i is

$$(1-x^{i+1})(1-x^{i+2})(1-x^{i+3})\dots(1-x^{i+k})f(x);$$

and for the partitions in which the j-th or absolute excess is not greater than i, that is in which the greatest part does not exceed i, the generating function is

$$(1-x^{i+1})(1-x^{i+2})(1-x^{i+3})\dots(1-x^{i+j})f(x).$$

(21) *Constructive Proof of the Formula for Partitions into Unrepeated Parts, limited in Number and Magnitude.* All the partitions to be considered consist of a fixed number, j, of *unrepeated* parts, written in *descending order.*

Take any partition of w in which the first excess is greater than $i+1$; subtracting $i+1$ from the first part we get a partition of $w-(i+1)$; conversely, if to the first part in any partition of $w-(i+1)$ we add $i+1$, we get a partition of w in which the first excess is greater than $i+1$; hence the number of partitions of w in which the first excess is greater than $i+1$ is equal to the whole number of partitions of $w-(i+1)$; so that, if the generating function for all the partitions is $\phi(x)$, the generating function for partitions whose first excess is *not* greater than $i+1$ is $(1-x^{i+1})\phi(x)$.

Considering now only partitions subject to this condition, if in any such partition of w the second excess is greater than $i+2$, we obtain by subtracting $i+2$ from the first part and removing the part so diminished to the second place a partition of $w-(i+2)$ subject to the condition; and conversely from any partition of $w-(i+2)$ in which the first excess is not greater than $i+1$, we obtain, by adding $i+2$ to the second part and removing the augmented part to the first place, a partition of w, in which the first excess is not greater than $i+1$ and the second excess *is* greater than $i+2$; hence the generating function for the partitions in which the second excess is *not* greater than $i+2$ (which restriction includes the condition that the first excess is not greater than $i+1$) is

$$(1-x^{i+1})(1-x^{i+2})\phi(x).$$

Confining ourselves now to this class of partitions, and taking any partition of w in which the third excess is greater than $i+3$, we obtain, by subtracting $i+3$ from the first part and removing the diminished part to the third place, a partition of $w-(i+3)$ belonging to the class now under consideration; and reversely. Hence the number of partitions in which the third excess is not greater than $i+3$ is given by the generating function

$$(1-x^{i+1})(1-x^{i+2})(1-x^{i+3})\phi(x).$$

Proceeding in this manner, we have finally that the generating function giving the number of partitions into j unrepeated parts, in which the absolute excess, that is the magnitude of the greatest part, is not greater than $i+j$, is

$$(1-x^{i+1})(1-x^{i+2})(1-x^{i+3})\dots(1-x^{i+j})\phi(x).$$

For example, if $w=18$, $j=3$, $i=4$, the partitions

15, 2, 1   14, 3, 1   13, 4, 1   13, 3, 2   12, 5, 1   12, 4, 2   11, 5, 2   11, 4, 3

in which the first excess is greater than 5, become by subtraction of 5 from their first part,

10, 2, 1   9, 3, 1   8, 4, 1   8 3, 2   7, 5, 1   7, 4, 2   6, 5, 2   6, 4, 3

which are *all* the partitions of 13; the partitions

11, 6, 1   10, 7, 1   10, 6, 2   10, 5, 3   9, 8, 1   9, 7, 2

in which the first excess is not greater than 5, but the second excess is greater than 6 become, by the subtraction of 6 from the first part and its removal to the second place,

6, 5, 1   7, 4, 1   6, 4, 2   5, 4, 3   8, 3, 1   7, 3, 2

which are all the partitions of 12 whose first excess is not greater than 5; the partitions

9, 6, 3   9, 5, 4   8, 7, 3   8, 6, 4

in which the second excess is not greater than 6, but the third excess (the greatest part) is greater than 7, become, by the subtraction of 7 from the first part and its removal to the last place,

6, 3, 2   5, 4, 2   7, 3, 1   6, 4, 1

which are all partitions of 11 whose second excess is not greater than 6. The only remaining partition of 18 is 7, 6, 5.

Interact.

Notes on certain Generating Functions and their Properties.

(22) (A) It may be as well to reproduce here (so as to keep the whole subject together) the entire proof of the well-known expansions of

$$1+ax\,.\,1+ax^2\,.\,1+ax^3\dots1+ax^i,$$

and of the reciprocal of

$$1-a\,.\,1-ax\,.\,1-ax^2\,.\,1-ax^3\dots1-ax^i,$$

which appeared in *part* in the *Johns Hopkins Circular* for February* last. This is, I think, distinguishable from the ordinary proofs as being, so to say, *classical* in form (using the word in an algebraical sense), inasmuch as it establishes the identity of two rational integral functions, one explicitly, the other implicitly given, by comparison of their zeros.

Let the coefficient of a^j in the expansion of

$$(1+ax)(1+ax^2)\dots(1+ax^i),$$

say in the expansion of $F(x, a)$, be called J_x, and

$$\frac{1-x^i\,.\,1-x^{i-1}\dots1-x^{i-j+1}}{1-x\,.\,1-x^2\dots1-x^j}$$

be called X_j.

J_x being the sum of the j-ary combinations of $x, x^2, \dots x^i$ will necessarily contain $x^{1+2+\dots+j}$, that is $x^{\frac{j^2+j}{2}}$, and will be of the degree

$$i+(i-1)+\dots+(i-j+1)$$

in x, and therefore of the same degree as $X_j x^{\frac{j^2+j}{2}}$.

All the linear factors of X_j are obviously of the form $x-\rho$, where $x-\rho$ is a primitive factor of some binomial expression x^r-1: the number of times that any $x-\rho$ occurs in X_j will obviously be equal to $E\frac{i}{r}-E\frac{j}{r}-E\frac{i-j}{r}$ which is either 1 or 0. Now consider $F(\rho, a)$, the value of $F(x, a)$ when x becomes ρ. Let $i=kr+\delta$, where $\delta<r$; then $F(\rho, a)=(1\pm a^r)^k$ multiplied

[* Vol. III. of this Reprint, p. 677.]

by δ linear functions of a, and consequently if $j = k'r + \delta'$, where $\delta' < r$, J_x vanishes when $\delta' > \delta$, in which case

$$E\frac{i}{r} - E\frac{j}{r} - E\frac{i-j}{r} = 1.$$

Hence any linear factor $x - \rho$ of X_j possesses the two-fold property of being unrepeated and of being contained in J_x. Hence J_x must contain $X_j x^{\frac{j^2+j}{2}}$, and being of the same degree as it is in x must bear to it a constant ratio, which, by making $x = 1$, is seen to be that of the coefficient of a^j in $(1+a)^i$, that is of $\frac{i(i-1)(i-2)\dots(i-j+1)}{1.2.3\dots j}$ to the product of the fractions in their vanishing state

$$\frac{1-x^i}{1-x}, \frac{1-x^{i-1}}{1-x^2}, \dots, \frac{1-x^{i-j+1}}{1-x^j},$$

that is, is a ratio of equality, so that $J_x = X_j x^{\frac{j^2+j}{2}}$. Q.E.D.

(23) Again let X_j and J_x now stand respectively for

$$\frac{1-x^{i+1}.1-x^{i+2}\dots 1-x^{i+j}}{1-x.1-x^2\dots 1-x^j}$$

and the coefficient of a^j in the reciprocal of $1-a.1-ax\dots 1-ax^i$ (say $F(x, a)$); this latter is the sum of homogeneous products of the jth order of $1, x, x^2, \dots x^i$, and is therefore of the degree ij which is also the degree (as is obvious) of X_j in x. For like reason as in what precedes $x - \rho$, any linear factor of $x^r - 1$, is contained 1 or 0 times in X_j according as

$$E\frac{i+j}{r} - E\frac{i}{r} - E\frac{j}{r} = 1 \text{ or } 0.$$

Let the minimum negative residue of $i+1$ to modulus r be $-\delta$; $F(\rho, a)$ may be expressed as the product of δ linear functions of a, divided by a power of $1 - a^r$, and the only power of a (say a^θ) which appears in its development will accordingly be those for which the residue of θ in respect to r is $0, 1, 2, \dots \delta$, and consequently if a^θ appears in the development

$$E\frac{i+\theta}{r} - E\frac{i}{r} - E\frac{\theta}{r} = 0,$$

or conversely if $x - \rho$ is a factor of X_j so that

$$E\frac{i+\theta}{r} - E\frac{i}{r} - E\frac{\theta}{r} = 1,$$

J_x vanishes. Hence J_x contains each linear factor of X_j, and these being simple, contains X_j itself, and on account of their degrees in x being the same must bear to it a ratio independent of x, which, by making $x = 1$,

so that the things to be compared are the coefficient of a^j in $\frac{1}{(1-a)^{i+1}}$ and the product of the vanishing fractions $\frac{1-x^{i+1}}{1-x}, \frac{1-x^{i+2}}{1-x^2}, \dots, \frac{1-x^{i+j}}{1-x^j}$, is readily seen to be a ratio of equality, so that $J_x = X_j$. Q.E.D.

(24) (B) *On the General Term in the Generating Function to Partitions into parts limited in number and magnitude*, by Dr F. Franklin.

To prove that the coefficient of a^j in the development of

$$\frac{1}{(1-a)(1-ax)(1-ax^2)\dots(1-ax^i)} \text{ is } \frac{(1-x^{j+1})(1-x^{j+2})\dots(1-x^{j+i})}{(1-x)(1-x^2)\dots(1-x^i)},$$

I showed that the number of partitions of w into i or fewer parts, subject to the condition that the first excess (the excess of the first part over the second) is not greater than j, is the coefficient of x^w in

$$\frac{1-x^{j+1}}{(1-x)(1-x^2)\dots(1-x^i)},$$

and in general that the number of partitions in which the rth excess (the excess of the first part over the $(r-1)$th) is not greater than j, is the coefficient in

$$\frac{(1-x^{j+1})(1-x^{j+2})\dots(1-x^{j+r})}{(1-x)(1-x^2)\dots(1-x^i)}.$$

If we look at the question reversely, namely, the coefficient of a^j in

$$\frac{1}{(1-a)(1-ax)(1-ax^2)\dots(1-ax^i)}$$

being known to be

$$\frac{(1-x^{j+1})(1-x^{j+2})\dots(1-x^{j+i})}{(1-x)(1-x^2)\dots(1-x^i)},$$

if we ask what is the significance of the fractions

$$\frac{1-x^{j+1}}{(1-x)(1-x^2)\dots(1-x^i)}, \dots, \frac{(1-x^{j+1})(1-x^{j+2})\dots(1-x^{j+r})}{(1-x)(1-x^2)\dots(1-x^i)},$$

the answer is immediately given by the generating function itself. For

$$\frac{1-x^{j+1}}{(1-x)(1-x^2)\dots(1-x^i)}$$

$$= \frac{1}{(1-x^2)(1-x^3)\dots(1-x^i)} \cdot \frac{1-x^{j+1}}{1-x}$$

$$= \frac{1}{(1-x^2)(1-x^3)\dots(1-x^i)} \left(\text{co. of } a^j \text{ in } \frac{1}{(1-a)(1-ax)}\right)$$

$$= \text{co. of } a^j \text{ in } \frac{1}{(1-a)(1-ax)(1-x^2)(1-x^3)\dots(1-x^i)}.$$

But the coefficient of $a^j x^w$ in the last written fraction is obviously the number of ways in which w can be composed of the numbers $1, 2, 3, \ldots i$, using not more than j 1's. And the number of 1's in a given partition is equal to the excess of the first part over the second part in its conjugate. In like manner

$$\frac{(1-x^{j+1})(1-x^{j+2})\ldots(1-x^{j+r})}{(1-x)(1-x^2)\ldots(1-x^i)}$$

$$= \text{co. of } a^j \text{ in } \frac{1}{(1-a)(1-ax)\ldots(1-ax^r)(1-x^{r+1})\ldots(1-x^i)};$$

and the coefficient of $a^j x^w$ in the fraction on the right is the number of ways in which w can be composed of the parts $1, 2, 3, \ldots i$, not more than j of the parts being as small as r. But the number of 1's in a given partition is equal to the excess of the first part over the second in its conjugate; the number of 2's to the excess of the second part over the third, and so on. Hence the number of 1's plus the number of 2's ... plus the number of r's in a given partition is equal to the excess of the first part over the rth part in its conjugate; and we have thus proved that the coefficient of x^w in the development of

$$\frac{(1-x^{j+1})(1-x^{j+2})\ldots(1-x^{j+r})}{(1-x)(1-x^2)\ldots(1-x^i)}$$

may be indifferently regarded as the number of partitions of w into parts none greater than i and not more than j of them as small as r or as the number of partitions of w into j or fewer parts, the excess of the first part over the rth part being as small as j. These results may obviously be extended by introducing the a in non-consecutive factors of the product

$$(1-x)(1-x^2)\ldots(1-x^i).$$

(25) (C) *On the theorem of one-to-one and class-to-class correspondence between partitions of n into uneven and its partitions into unequal parts*, by Dr F. FRANKLIN.

The number of partitions of w into k distinct odd numbers, each repeated an indefinite number of times, is evidently the coefficient of $a^k x^w$ in the development of

$$\left(1+a\frac{x}{1-x}\right)\left(1+a\frac{x^3}{1-x^3}\right)\left(1+a\frac{x^5}{1-x^5}\right)\ldots.$$

It is not easy to form the generating function for the number of partitions containing k sequences, but it is plain that the number of partitions of w containing *one* sequence is the coefficient of x^w in

$$S_1 + S_2 + S_3 + \ldots,$$

where

$$S_1 = x + x^2 + x^3 + x^4 + x^5 + \ldots = \frac{x}{1-x}$$

$$S_2 = x^3 + x^5 + x^7 + x^9 + x^{11} + \ldots = \frac{x^3}{1-x^2}$$

$$S_3 = x^6 + x^9 + x^{12} + x^{15} + x^{18} + \ldots = \frac{x^6}{1-x^3}$$

$$S_4 = x^{10} + x^{14} + x^{18} + x^{22} + x^{26} + \ldots = \frac{x^{10}}{1-x^4}$$

$$S_5 = x^{15} + x^{20} + x^{25} + x^{30} + x^{35} + \ldots = \frac{x^{15}}{1-x^5},$$

and in general

$$S_r = x^{1+2+3+\ldots+r} + x^{2+3+4+\ldots+(r+1)} + \ldots = \frac{x^{\frac{1}{2}r(r+1)}}{1-x^r}.$$

So much of Prof. Sylvester's theorem as relates to a single sequence follows from inspection of the above scheme. For $S_1 = \frac{x}{1-x}$; adding to S_3 the first term of S_2, we get $\frac{x^3}{1-x^3}$; adding to S_5 the first term of S_4 and the second term of S_2, we get $\frac{x^5}{1-x^5}$; adding to S_{2m+1} the first term of S_{2m}, the second term of $S_{2(m-1)}$, the third term of $S_{2(m-2)}, \ldots$, and the mth term of S_1, we get $\frac{x^{2m+1}}{1-x^{2m+1}}$; thus the proposition is proved. The fact is made more evident to the eye if we write the scheme as follows:

$$\begin{array}{ll}
S_1 = x + x^2 + x^3 + x^4 + x^5 + \ldots & S_2 = x^3 + x^5 + x^7 + x^9 + x^{11} + \ldots \\
S_3 = x^6 + x^9 + x^{12} + x^{15} + x^{18} + \ldots & S_4 = \quad x^{10} + x^{14} + x^{18} + x^{22} + \ldots \\
S_5 = x^{15} + x^{20} + x^{25} + x^{30} + x^{35} + \ldots & S_6 = \qquad x^{21} + x^{27} + x^{33} + \ldots \\
S_7 = x^{28} + x^{35} + x^{42} + x^{49} + x^{56} + \ldots & S_8 = \qquad\quad x^{36} + x^{44} + \ldots \\
S_9 = x^{45} + x^{54} + x^{63} + x^{72} + x^{81} + \ldots & S_{10} = \qquad\qquad x^{55} + \ldots
\end{array}$$

Here $\frac{x^9}{1-x^9}$, for instance, is obtained by adding the fourth column on the right to the fifth row on the left.

It may be noted that we have thus found that

$$\frac{x}{1-x} + \frac{x^3}{1-x^3} + \frac{x^5}{1-x^5} + \ldots + \frac{x^{2m+1}}{1-x^{2m+1}} + \ldots$$

$$= \frac{x}{1-x} + \frac{x^3}{1-x^2} + \frac{x^6}{1-x^3} + \ldots + \frac{x^{\frac{1}{2}n(n+1)}}{1-x^n} + \ldots.$$

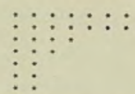
(26) [Compare Jacobi's theorem contained in the last-but-one two lines of the last but one page of the *Fundamenta Nova*, which may be easily reduced to the form

$$\frac{x}{1+x}-\frac{x^3}{1+x^3}+\frac{x^5}{1+x^5}\cdots=\frac{x}{1+x}-\frac{x^3}{1+x^2}+\frac{x^5}{1+x^3}-\cdots.\quad \text{J. J. S.]}$$

Act II. On the Graphical Conversion of Continued Products into Series.

Naturelly, by composiciouns
Of anglis, and slie reflexiouns.
The Squieres Tale.

(27) The method about to be explained of representing the elements of partitions by means of a succession of angles fitting into one another arose out of an investigation (instituted for the purpose of facilitating the arrangement of tables of symmetric functions)* as to the number of partitions of n which are their own conjugates. The ordinary graphs to such partitions must obviously be symmetrical in respect to the two nodal boundaries, as seen below.



Let the above figure be any such graph; it may be dissected into a square (which may contain one or any greater square number) of say i^2 nodes, and two perfectly similar appended graphs, each having the content $\frac{n-i^2}{2}$, and subject to the sole condition that the number of its lines (or columns), that is that the number (or magnitude) of the parts in the partition which it represents, shall be i or less; such number is the coefficient of $x^{\frac{n-i^2}{2}}$ in $\frac{1}{1-x.1-x^2\ldots1-x^i}$, which is the same as that of x^{n-i^2} in

$$\frac{1}{1-x^2.1-x^4\ldots1-x^{2i}}$$

or of x^n in

$$\frac{x^{i^2}}{1-x^2.1-x^4\ldots1-x^{2i}}.$$

* By Mr Durfee, of California (Fellow of the Johns Hopkins University), to whom I suggested the desirability of investigating more completely than had been done the method of arrangement of such tables founded upon the notion of self-conjugate partitions, which M. Faà de Bruno had the honour of initiating. The very valuable results of Mr Durfee's inquiries, embodying, systematising and completing the theory of arrangement originated by Professor Cayley, and further illustrated by the labours of Professors Betti and De Bruno, will probably appear in the next number of the *Journal*.

Hence giving i all possible values we see that the coefficient of x^n in the infinite series

$$1+\frac{x}{1-x^2}+\frac{x^4}{1-x^2.1-x^4}+\frac{x^9}{1-x^2.1-x^4.1-x^6}+\cdots$$

is the number of self-conjugate partitions of n, or which is the same thing of symmetrical groups whose content is n.

(28) But any such graph, in which there is a square of i^2 nodes with its two appendices, may be dissected in another manner into i angles or bends, each containing any continually decreasing odd number of nodes, and *vice versâ*, any set of equilateral angles of nodes continually decreasing in number (which condition is necessary in order that the lower lines and posterior columns may not protrude beyond the upper lines and anterior columns) when fitted into one another in the order of their magnitudes will form a regular graph. Thus the actual figure (where there is a square of 9 nodes) formed by the intersections of the lines and columns may be dissected into 3 angles containing respectively 13, 11, 3 nodes; and so in general the number of ways in which n can be made up of odd and unrepeated parts will be the same as the number of ways in which $\frac{n-j^2}{2}$ can be partitioned into not more than j parts; hence we see that the coefficients of $x^n a^j$ in

$$(1+ax)(1+ax^3)\ldots(1+ax^{2j-1})\ldots$$

and in

$$\frac{x^{j^2}}{1-x^2.1-x^4\ldots1-x^{2j}}$$

are the same, so that the continued product above written is equal to

$$1+\frac{x}{1-x^2}a+\ldots+\frac{x^{j^2}}{1-x^2.1-x^6\ldots1-x^{2j}}a^j+\ldots$$

as is well known.

(29) In like manner if the expansion in a series of ascending powers of a of the finite continued product

$$(1+ax)(1+ax^3)\ldots(1+ax^{2i-1})$$

be required, the coefficient of x^n in the coefficient of a^j will be the number of ways in which n can be made up with j of the unrepeated numbers $1, 3, \ldots 2i-1$, and as $2i-1$ is the number of nodes in an equilateral angle whose sides contain i nodes, it follows that this coefficient will be the number of ways in which $\frac{n-j^2}{2}$ can be composed with parts none *exceeding* $i-j$ in magnitude, and will therefore be the same as the coefficient of $x^{\frac{n-j^2}{2}}$ in

$$\frac{1-x^{i-j+1}.1-x^{i-j+2}\ldots1-x^i}{1-x.1-x^2\ldots1-x^j},$$

and consequently the finite continued product above written is equal to

$$1+\ldots+\frac{1-x^{2i-2j+2}.1-x^{2i-2j+4}\ldots1-x^{2i}}{1-x^2.1-x^4\ldots1-x^{2j}}x^{j^2}a^j+\ldots$$

(30) If it be required to ascertain how many self-conjugate partitions of n there are containing exactly i parts, this may be found by giving j all possible values and making p_j equal to the number of ways in which $\frac{n-j^2}{2}$ can be composed with j or fewer parts the greatest of which is $i-j$, that is $(n-j^2+2j-2i)/2$ with $j-1$ or fewer parts none greater than $i-j$, so that p_j will be the coefficient of $x^{(n-j^2+2j-2i)/2}$ in

$$\frac{1-x^{i-j+1} \,.\, 1-x^{i-j+2} \ldots 1-x^{i-1}}{1-x \,.\, 1-x^2 \ldots 1-x^{j-1}}$$

or of x^n in

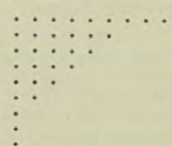
$$\frac{1-x^{2i-2j+2} \,.\, 1-x^{2i-2j+4} \ldots 1-x^{2i-2}}{1-x^2 \,.\, 1-x^4 \ldots 1-x^{2j-2}} \, x^{j^2-2j+2i};$$

the sum of the values of p_j for all values of j will be the number required: this number, therefore, writing ω for $2i-1$, will be the coefficient of x^n in

$$x^{\omega} + \frac{1-x^{\omega-1}}{1-x^2} x^{\omega+1} + \frac{1-x^{\omega-1} \,.\, 1-x^{\omega-3}}{1-x^2 \,.\, 1-x^4} x^{\omega+4} + \text{etc.};$$

the outstanding factor in the qth term in this series being $x^{\omega+(q-1)^2}$ we may suppose q the least integer number not less than $1+\sqrt{(n-\omega)}$ and then the subsequent term to the $(q+1)$th being inoperative may be neglected.

(31) In order to see how any self-conjugate graph may be recovered, so to say, from the corresponding partition consisting of unrepeated odd numbers, consider the diagrammatic case of the partition 17, 9, 5, 1 represented by the angles of the graph below written

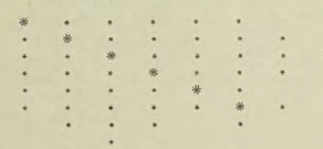


The number of angles is the number of the given parts, that is 4, and the first four lines of the graph will be obtained by adding 0, 1, 2, 3 to the major half (meaning the integer next above the half) of 17, 9, 5, 1, that is will be 9, 6, 5, 4, the total number of lines will be the major half of the highest term (17) and the remaining lines will have the same contents, namely 3, 2, 1, 1, 1, as the columns of the graph found by subtracting 4 (the number of the parts) from the numbers last found, that is will be the lines of the graph which is conjugate to 5, 2, 1. And so in general the self-conjugate graph corresponding to any partition of unrepeated odd numbers $q_1, q_2, \ldots q_j$ will be found by the following rule:

Let P be the system of partitions $k_1, k_2, \ldots k_j$, in which any term k_θ is the major half of q_θ augmented by $\theta-1$, and P' another system of $k_1', k_2', \ldots k_j'$, obtained by subtracting j from each term in P, then P and the conjugate to P' will be the self-conjugate partition corresponding to the given q partition. Thus as an example, 19, 11, 7, 5 being given, P, P' will be 10, 7, 6, 6; 6, 3, 2, 2 respectively, and the self-conjugate system required will be 10, 7, 6, 6, 4, 4, 2, 1, 1, 1. Of course P' might also be obtained by taking the minor halves of the given parts in inverse (ascending) order and subtracting from them the numbers 0, 1, 2, ... respectively.

To pass from a given self-conjugate to the corresponding unrepeated odd numbers-partition is a much simpler process, the rule being to take the numbers in descending order and from their doubles subtract the successive odd numbers in the natural scale until the point is reached at which the difference is about to become negative; thus the partition 6 6 5 4 3 2 is self-conjugate, and the correspondent to it is 11 9 5 1.

(32) The expansion of the reciprocal to $(1-ax)(1-ax^3)\ldots(1-ax^{2i-1})$ may be read off with the same facility as the direct product. In this case we are concerned with partitions of odd numbers capable of being repeated in the same partition; now, therefore, if we use the same method of equilateral angles as before, and fit them into one another in regular order of magnitude, it will no longer be the case that their sum will form a regular graph, for if there be θ parts alike, each line and column which ranges with either side of any (but the first one) of these will jut out one step beyond the anterior line and column (respectively), so that the line joining the extremities of the lines or columns will be parallel to the axis of symmetry. The figure then corresponding to i odd parts can no longer be dissected into a *square* of nodes and two equal regular graphs, but it may be dissected into a *line* of nodes lying in the axis of symmetry, and two regular graphs one of which has for its boundaries one of the original boundaries and a line of nodes parallel to the axis of symmetry, and the other one the other original boundary and a line of nodes parallel to the same axis, as seen in the annexed figure, where the axial points are distinguished by being made larger than the rest.



The graph read off in angles represents the partition 11 11 11 7 3 3. On removing the six diagonal nodes it breaks up into two regular graphs, of which one is 5 5 5 3 1 1, and the other the conjugate thereto; hence the coefficient of x^n in the coefficient of a^j in the expansion of the reciprocal of $1-ax \,.\, 1-ax^3 \ldots .\, 1-ax^{2i-1}$ in ascending powers of a is the number of ways in which $\frac{n-j}{2}$ can be resolved into j parts limited not to exceed $i-1$, which is the coefficient of $x^{\frac{n-j}{2}}$ in

$$\frac{1-x^i \,.\, 1-x^{i+1} \ldots 1-x^{i+j-1}}{1-x \,.\, 1-x^2 \ldots 1-x^j}$$

or of x^n in

$$\frac{1-x^{2i} \,.\, 1-x^{2i+2} \ldots 1-x^{2i+2j-2}}{1-x^2 \,.\, 1-x^4 \ldots 1-x^{2j}} x^j.$$

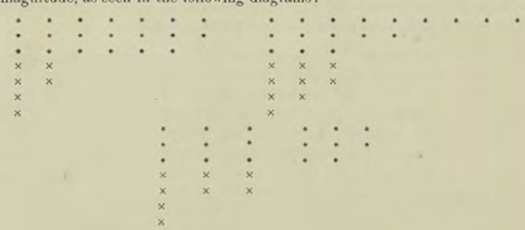
(33) Although I shall not require any intermediate expansion whatever in order to obtain the transcendant $\Theta_1 x$ product in the form of a series, I will give another of those which are sometimes employed together in combination (see Cayley, *Elliptic Functions*, pp. 296—7) to obtain this result: thus to prove that the continued product of the reciprocal of

$$(1-ax)(1-ax^2)(1-ax^3)\ldots$$

is identical with

$$1+\frac{x}{1-x}\cdot\frac{a}{1-xa}+\frac{x^4}{1-x \,.\, 1-x^2}\cdot\frac{a^2}{1-xa \,.\, 1-x^2a}$$
$$+\frac{x^9}{1-x \,.\, 1-x^2 \,.\, 1-x^3}\cdot\frac{a^3}{1-xa \,.\, 1-x^2a \,.\, 1-x^3a}+\text{etc.}$$

if n is partitioned into j parts, the regular graph which represents the result of any such partition must consist either of $1, 2, 3, \ldots j-1$ or of not less than j columns, and its graph may accordingly in these several cases be dissected into a square of $1, 4, 9, \ldots j^2$ nodes; suppose that such square consists of θ parts, then there will be $n-\theta^2$ nodes remaining over subject to distribution into two groups limited by the condition as to one of the groups that it may contain an unlimited number of parts none exceeding θ in magnitude, and as to the other that it must contain $j-\theta$ parts none exceeding θ in magnitude, as seen in the following diagrams:



in all of which the partible number is 26, and j and θ are 7 and 3 respectively. Now the number of such distributions is the coefficient of $x^{n-\theta^2}a^{j-\theta}$ in

$$\frac{1}{1-x \,.\, 1-x^2 \ldots 1-x^\theta}\cdot\frac{1}{1-ax \,.\, 1-ax^2 \ldots 1-ax^\theta},$$

that is of $x^n a^j$ in

$$\frac{x^{\theta^2}}{1-x \,.\, 1-x^2 \ldots 1-x^\theta}\cdot\frac{a^\theta}{1-ax \,.\, 1-ax^2 \ldots 1-ax^\theta},$$

and consequently giving θ all values from 1 to ∞, the proposed equation is verified.

(34) It may be desired to apply the same method to obtain a similar development for the reciprocal of the limited product

$$(1-ax)(1-ax^2)\ldots(1-ax^i);$$

the construction will be the same as in the last case; the distribution into two groups can be made as before; the second group will remain subject to the same condition as in the preceding case (seeing that the number of parts being less than $j-\theta$, will necessarily be less than $i-\theta$, for j cannot exceed i), but the first group will be subject to the condition of being partitioned not now into an unlimited but into $i-\theta$ (or fewer) parts none exceeding θ in magnitude, and the number of such distributions into the two groups will accordingly become the coefficient of $x^{n-\theta^2}a^{j-\theta}$ in

$$\frac{1-x^{i-\theta+1} \,.\, 1-x^{i-\theta+2} \ldots 1-x^i}{1-x \,.\, 1-x^2 \ldots 1-x^\theta}\cdot\frac{1}{1-ax \,.\, 1-ax^2 \ldots 1-ax^\theta}$$

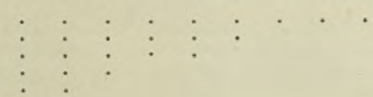
or of $x^n a^j$ in the last written fraction multiplied by $x^{\theta^2} \,.\, a^\theta$, so that the required expansion will be

$$1+\frac{1-x^i}{1-x}\cdot\frac{xa}{1-ax}+\frac{1-x^i \,.\, 1-x^{i-1}}{1-x \,.\, 1-x^2}\cdot\frac{x^4a^2}{1-ax \,.\, 1-ax^2}$$
$$+\frac{1-x^i \,.\, 1-x^{i-1} \,.\, 1-x^{i-2}}{1-x \,.\, 1-x^2 \,.\, 1-x^3}\cdot\frac{x^9a^3}{1-ax \,.\, 1-ax^2 \,.\, 1-ax^3}+\ldots.$$

(35) It is interesting to investigate what will be the form of the mixed development resulting from an application of the same method to the *direct* product

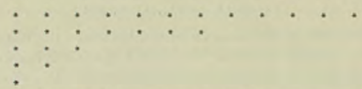
$$1+ax \,.\, 1+ax^2 \ldots 1+ax^i.$$

For greater clearness I shall first suppose i indefinitely great. Consider the diagram:



In the above graph j and θ used in the same sense as *ante* are 5 and 3 respectively, so that there is a square of 9 points; an appendage to the right of and another appendage below the square, which I shall call the lateral and subjacent appendages respectively. The content of the graph being 25, there are 16 points to be distributed between these two appendages. What now are the conditions of the distribution of the $n-\theta^2$ points between them?

I say that there will be two sorts of such distribution—one in which the lateral appendage will consist of θ unrepeated parts, none of them zero, as in the graph above, and the subjacent appendage of $j-\theta$ unrepeated parts, limited not to exceed θ in magnitude, and another sort as in the graph below written,



in which the jth line of the lateral appendage is missing, and consequently the subjacent graph will consist of $j-\theta$ unrepeated parts limited not to exceed $\theta-1$ in magnitude, for there could not be a part so great as θ without the last line of the square having the same content as the first line of the subjacent appendage.

It should be observed that only the *last* admissible line of the lateral appendage can be wanting, for if more than this were wanting, two lines of the square would belong to the graph, and consequently there would be two equal parts θ.

Hence there are two kinds of association of the appendages, one leading to a distribution of $n-\theta^2$ between one group of θ unrepeated but unlimited parts, and another of $j-\theta$ unrepeated parts limited not to exceed θ; the other to a distribution of $n-\theta^2$ between one group of $\theta-1$ unrepeated but unlimited parts, and another of $j-\theta$ unrepeated parts limited not to exceed $\theta-1$.

The number of distributions of the first kind is the coefficient of $x^{n-\theta^2}.\,a^{j-\theta}$ in

$$\frac{x^{\frac{\theta^2+\theta}{2}}}{1-x\,.\,1-x^2\ldots1-x^\theta}\,.\,(1+ax)(1+ax^2)\ldots(1+ax^\theta),$$

the other of $x^{n-\theta^2}.\,a^{j-\theta}$ in

$$\frac{x^{\frac{\theta^2-\theta}{2}}}{1-x\,.\,1-x^2\ldots1-x^{\theta-1}}\,.\,(1+ax)(1+ax^2)\ldots(1+ax^{\theta-1});$$

hence the sum of the distributions of the two kinds is the coefficient of the same argument in

$$\frac{x^{\frac{\theta^2-\theta}{2}}}{1-x\,.\,1-x^2\ldots1-x^\theta}\{x^\theta(1+ax^\theta)+(1-x^\theta)\}\{1+ax\,.\,1+ax^2\ldots1+ax^{\theta-1}\},$$

that is of $x^n a^j$ in

$$x^{\frac{3\theta^2-\theta}{2}}a^\theta\left(\frac{1+ax\,.\,1+ax^2\ldots1+ax^{\theta-1}}{1-x\,.\,1-x^2\ldots1-x^{\theta-1}}\,.\,\frac{1+ax^{2\theta}}{1-x^\theta}\right),$$

and consequently we obtain the equation

$$1+ax\,.\,1+ax^2\,.\,1+ax^3\ldots=1+\frac{1+ax^2}{1-x}xa+\frac{1+ax\,.\,1+ax^4}{1-x\,.\,1-x^2}x^5a^2+\ldots$$

$$+\frac{1+ax\,.\,1+ax^2\ldots1+ax^{j-1}\,.\,1+ax^{2j}}{1-x\,.\,1-x^2\ldots1-x^{j-1}\,.\,1-x^j}x^{\frac{3j^2-j}{2}}a^j+\ldots,$$

and thus by a very unexpected route we arrive at a proof of Euler's celebrated pentagonal-number theorem; for on making $a=-1$ the above equation becomes

$$1-x\,.\,1-x^2\,.\,1-x^3\ldots=1-(1+x)x+(1+x^2)x^5\ldots+(-)^j(1+x^j)x^{\frac{3j^2-j}{2}}+\ldots.$$

Such is one of the fruits among a multitude arising out of Mr Durfee's ever-memorable example of the dissection of a graph (in the case of a symmetrical one) into a square, and two regular graph appendages.

Even the trifling algebraical operation above employed to arrive at the result might have been spared by expressing the continued product as the sum of the two series (which flow immediately from the graphical dissection process), left uncombined, namely,

$$1+\frac{1+ax}{1-x}x^2a+\frac{1+ax\,.\,1+ax^2}{1-x\,.\,1-x^2}x^7a^2+\frac{1+ax\,.\,1+ax^2\,.\,1+ax^3}{1-x\,.\,1-x^2\,.\,1-x^3}x^{15}a^3+\ldots,$$

together with

$$+xa+\frac{1+ax}{1-x}x^5a^2+\frac{1+ax\,.\,1+ax^2}{1-x\,.\,1-x^2}x^{12}a^3+\ldots,$$

which for $a=-1$ unite into the single series

$$1-x-x^2+x^5+x^7-x^{12}-x^{15}\text{ etc.}$$

(36) I will now proceed to find the expression in a mixed series of the limited product

$$1+ax\,.\,1+ax^2\ldots1+ax^i.$$

In each of the two systems of distribution (as shown already in the theory of the reciprocal of such product) the second group will remain unaffected by the new limitation, but the first group will now consist of partitions (limited in number as before), but in magnitude instead of being unlimited, limited not to exceed $(i-\theta)$, so that we will have to take the coefficient of $x^{n-\theta^2}.a^{j-\theta}$ in the sum of

$$x^{\frac{\theta^2+\theta}{2}}\frac{1-x^{i-\theta}.1-x^{i-\theta-1}\ldots 1-x^{i-2\theta+1}}{1-x.1-x^2\ldots 1-x^{\theta}}.(1+ax)(1+ax^2)\ldots(1+ax^{\theta})$$

and

$$x^{\frac{\theta^2-\theta}{2}}\frac{1-x^{i-\theta}.1-x^{i-\theta-1}\ldots 1-x^{i-2\theta+2}}{1-x.1-x^2\ldots 1-x^{\theta-1}}.(1+ax)(1+ax^2)\ldots(1+ax^{\theta-1}).$$

This will be the same as the coefficient of $x^n a^j$ in

$$x^{\frac{3\theta^2-\theta}{2}}a^{\theta}(1+ax)(1+ax^2)\ldots(1+ax^{\theta-1})\frac{1-x^{i-\theta}.1-x^{i-\theta-1}\ldots 1-x^{i-2\theta+2}}{1-x.1-x^2\ldots 1-x^{\theta-1}.1-x^{\theta}}$$
$$\times\{1-x^{\theta}+(1-x^{i-2\theta+1})(x^{\theta}+ax^{2\theta})\},$$

where the quantity within the final bracket is equal to

$$1-x^{i+1}a-x^{i-\theta+1}+x^{2\theta}a.$$

Hence the required series is

$$\left\{1+\frac{1-x^i}{1-x}xa+\frac{1-x^{i-1}.1-x^{i-2}}{1-x.1-x^2}(1+ax)x^5a^2\right.$$
$$\left.+\frac{1-x^{i-2}.1-x^{i-3}.1-x^{i-4}}{1-x.1-x^2.1-x^3}.1+ax.1+ax^2.x^{12}a^3+\ldots\right\}$$
$$+\left\{\frac{1-x^{i-1}}{1-x}x^3a^2+\frac{1-x^{i-2}.1-x^{i-3}}{1-x.1-x^2}(1+ax)x^9a^3\right.$$
$$\left.+\frac{1-x^{i-3}.1-x^{i-4}.1-x^{i-5}}{1-x.1-x^2.1-x^3}.1+ax.1+ax^2.x^{18}a^4+\ldots\right\},$$

the indices in the outstanding powers of x being the pentagonal numbers in the first, and the triangular numbers trebled, in the second of the above series.

In obtaining in the preceding articles mixed series for continued products, it will be noticed that the graphical method has been employed, not to exhibit correspondence, but as an instrument of transformation. The graphs are virtually segregated into classes, and the number of them contained in each class separately determined. (The magnitude of the square in the Durfee-dissection serves as the basis of the classification.)

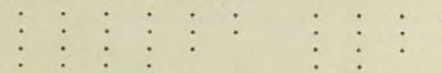
(37) Now let us consider the famous double product of

$$(1+ax)(1+ax^3)(1+ax^5)\ldots$$

by $$(1+a^{-1}x)(1+a^{-1}x^3)(1+a^{-1}x^5)\ldots.$$

Here it will be expedient to introduce a new term and to explain the meaning of a bi-partition and a system of parallel bi-partitions of a number. The former indicates that the elements are to be distributed into two groups, say into a left and right-hand group: the latter that the number of the elements

(on one, say) on the left-hand side of each bi-partition of the system is to be equal to or exceed by a constant difference the number (on the other, say) on the right-hand side of the same bi-partition. If we use dots, regularly spaced, to represent the elements (themselves numbers and not units), we get a figure or pair of figures such as the following:



for which the corresponding lines of the contour are respectively parallel—hence the name. When the numbers of elements on the two sides are identical, I call the system an equi-bi-partition-system—in the general case, a parallel bi-partition-system to a constant difference j, where j is the excess of the number of elements in the left-hand over that in the right-hand part of any of the bi-partitions.

(38) Consider now the given double product—it is obvious that it may be expanded in terms of paired powers a^j+a^{-j} of a, and the coefficient of x^n in the term not involving a will evidently be the number of equi-bi-partitions of n that can be formed with unrepeated odd numbers; and so the coefficient of x^n associated with a^j or a^{-j} will be the number of parallel bi-partitions of n to the constant difference j that can be so formed.

For the equi-bi-partitions; suppose $l_1, l_2\ldots l_i, \lambda_1, \lambda_2\ldots\lambda_i$ is an equi-bi-partition, all the elements being odd and unrepeated; take successive angles whose (say horizontal and vertical) sides are the major halves of l_1, λ_1; $l_2, \lambda_2\ldots$; l_i, λ_i; these angles will fit on to one another so as to form a regular graph by reason of the relations

$$l_1>l_2+1,\quad l_2>l_3+1\ldots l_{i-1}>l_i+1,$$
$$\lambda_1>\lambda_2+1,\quad \lambda_2>\lambda_3+1\ldots\lambda_{i-1}>\lambda_i+1.$$

Conversely any regular graph may be resolved into angles whose horizontal sides shall be the major halves of one set of odd numbers, and their vertical sides the major halves of another set of as many odd numbers, and these two sets of odd numbers will each form a decreasing series; hence there is a one-to-one conjugate correspondence between any bi-partition of n written in regular order, and the totality of regular graphs whose content is $\frac{n}{2}$, so that the number of the equi-bi-partitions of n will be the coefficient of $x^{\frac{n}{2}}$ in

$$\frac{1}{1-x.1-x^2.1-x^3\ldots},$$

that is of x^n in $$\frac{1}{1-x^2.1-x^4.1-x^6\ldots},$$

which fraction is therefore equal to the totality of the terms not involving a.

(39) Next for the coefficient of a^j.

Let $l_1, l_2, \dots l_j, l_{j+1}, l_{j+2}, \dots l_{j+\theta}; \lambda_1, \lambda_2, \dots \lambda_\theta$ be an equi-parallel bi-partition to the difference j (with the elements on each side written in descending order); with the equi-bi-partition $l_{j+1}, l_{j+2}, \dots l_{j+\theta}; \lambda_1, \lambda_2, \dots \lambda_\theta$, form a graph, as in the preceding case; say, for distinctness, with major halves of the l series horizontal and of the λ series vertical; over the highest horizontal line the successive quantities*

$$\frac{l_j-1}{2},\ \frac{l_{j-1}-3}{2},\ \frac{l_{j-2}-5}{2}, \dots \frac{l_1-(2j-1)}{2}$$

may be laid so as to form a regular graph of which the content will be $\frac{n-j^2}{2}$.

Conversely every regular graph whose content is $\frac{n-j^2}{2}$ will correspond to a parallel bi-partition of unrepeated odd numbers to a difference j; to obtain the bi-partition the first j lines of the graph must be abstracted†, and the graph thus diminished resolved into angles; the doubles of the contents of each vertical side of these angles diminished by unity will constitute the right-hand side of the bi-partition, and the doubles of the contents of each horizontal side preceded by the doubles of the lines of the abstracted portion of the graph increased by $1, 3, 5, \dots 2j-1$ respectively, will form the left-hand portion. Hence the number of such bi-partitions will be the number of ways of resolving $\frac{n-j^2}{2}$ into unrestricted parts, that is, will be the coefficient of x^n in

$$\frac{1}{1-x^2 \,.\, 1-x^4 \,.\, 1-x^6 \dots} x^{j^2},$$

and this being true for all values of n and j, we see that the double product in question will be identical with the infinite series

$$\frac{1}{1-x^2 \,.\, 1-x^4 \,.\, 1-x^6 \dots} \{1 + x(a+a^{-1}) + x^4(a^2+a^{-2}) + x^9(a^3+a^{-3}) + \dots\}.$$

(40) To expand the limited double product

$$(1+ax)(1+ax^3)\dots(1+ax^{2i-1})$$

into

$$(1+a^{-1}x)(1+a^{-1}x^3)\dots(1+a^{-1}x^{2i-1})$$

the procedure and reasoning will be precisely the same as in the extreme case of i infinite, the only difference being that the elements of the bi-partition instead of being unlimited odd numbers will be limited not to exceed $2i-1$. In the case of $j=0$ the equi-bi-partition will furnish a series of nodal angles in which neither side can exceed the major half of $2i-1$,

* Any number of these quantities may happen to become zero.

† If the actual number of horizontal lines in the graph is less than j, it must be made to count as j, by understanding lines of zero content to be supplied underneath the graph.

that is i, and the coefficient of x^n in the term not containing any power of a will consequently be the number of ways in which n can be divided into parts limited as well in number as in magnitude not to exceed i, and will therefore be the same as the coefficient of $x^{\frac{1}{2}n}$ in the development of

$$\frac{1-x^{i+1} \,.\, 1-x^{i+2} \dots 1-x^{2i}}{1-x \,.\, 1-x^2 \dots 1-x^i},$$

or, which is the same thing, of x^n in the development of

$$\frac{1-x^{2i+2} \,.\, 1-x^{2i+4} \dots 1-x^{4i}}{1-x^2 \,.\, 1-x^4 \dots 1-x^{2i}},$$

and when the bi-partition system has a constant difference j, the corresponding graph will be of the same form, except that it will be overlaid with j lines, obtained as in the preceding case by subtracting $1, 3, \dots 2j-1$ from the first j left-hand elements, and taking the halves of the remainders; the graphs thus formed will be subject to the condition of having a content $\frac{n-j^2}{2}$, and parts limited not to exceed $i-j$ in magnitude nor $i+j$ in number [$i-j$ in magnitude because the topmost line cannot exceed $\frac{(2i-1)-(2j-1)}{2}$ in content; $i+j$ in number because without reckoning the j superimposed lines the subjacent portion of the graph cannot contain more than i lines]. The converse that out of every regular graph fulfilling these conditions may be spelled out a parallel bi-partition with a difference j, and containing only unrepeated odd numbers limited not to exceed $2i-1$ in magnitude may be shown as in the preceding case. Hence the coefficient of x^n in the coefficient of $a^j + a^{-j}$ in the expansion, is the number of ways of resolving $\frac{n-j^2}{2}$ into parts none exceeding $i-j$ in magnitude nor $i+j$ in number, that is, is the coefficient of x^n in

$$\frac{1-x^{2i+2j+2} \,.\, 1-x^{2i+2j+4} \dots 1-x^{4i}}{1-x^2 \,.\, 1-x^4 \dots 1-x^{2i-2j}} x^{j^2}.$$

Hence by the process of reasoning, which has been so often applied, we see that the finite double product

$$1+ax \,.\, 1+ax^3 \dots 1+ax^{2i-1}$$

into

$$1+a^{-1}x \,.\, 1+a^{-1}x^3 \dots 1+a^{-1}x^{2i-1}$$

$$= \frac{1-x^{2i+2} \,.\, 1-x^{2i+4} \dots 1-x^{4i}}{1-x^2 \,.\, 1-x^4 \dots 1-x^{2i}} \left\{1 + \frac{1-x^{2i}}{1-x^{2i+2}} x + \frac{1-x^{2i} \,.\, 1-x^{2i-2}}{1-x^{2i+2} \,.\, 1-x^{2i+4}} x^4 + \frac{1-x^{2i} \,.\, 1-x^{2i-2} \,.\, 1-x^{2i-4}}{1-x^{2i+2} \,.\, 1-x^{2i+4} \,.\, 1-x^{2i+6}} x^9 + \dots\right\}.$$

Compare Hermite, *Note sur les fonctions elliptiques*, p. 35, where Cauchy's method is given of arriving at this and the preceding identity.

ACT III. ON THE ONE-TO-ONE AND CLASS-TO-CLASS CORRESPONDENCE BETWEEN PARTITIONS INTO UNEVEN AND PARTITIONS INTO UNEQUAL PARTS.

. . . mazes intricate,
Eccentric, intervolved, yet regular
Then most, when most irregular they seem.

Paradise Lost, v. 622.

(41) It has been already shown that any partition of n into unequal parts may be converted into a partition consisting of odd numbers equal or unequal by, first, expressing any even part by its longest odd divisor, say its nucleus and a power of 2, and, second, adding together the powers of 2 belonging to the same nucleus, so that there will result a sum of odd nuclei, each occurring one or more times; a like process is obviously applicable to convert a partition in which any number occurs $1, 2, \ldots$ or $(r-1)$ times into one in which only numbers not divisible by r occur with unrestricted liberty of recurrence. The nuclei will here be numbers not divisible by r multiplied by powers of r, and by adding together the powers of r belonging to the same nucleus there results a series of nuclei, each occurring one or more times. Conversely when the nuclei and the number of occurrences of each are given, there being only one way in which any such number can be expressed in the scale whose radix is r, it follows that there is but one partition of the previous kind in which one of the latter kind can originate, and there is thus a one-to-one correspondence, and consequently equality of content between the two systems of partitions.

(42) To return to the case of $r = 2$, with which alone we shall be here occupied, we see that the number of parts in the unequal partition which corresponds after this fashion with a partition made up of given odd numbers depends on the sum of the places occupied when the number of occurrences of each of the odd numbers is expressed in the notation of dual arithmetic. Such correspondence then is eminently arithmetical and transcendental in its nature, depending as it does on the forms of the numbers of repetitions of each different integer with reference to the number 2.

Very different is the kind of correspondence which we are now about to consider between the self-same two systems, as well in its nature, which is essentially graphical, as in its operation, which is to bring into correspondence the two systems, not as wholes but as separated each of them into distinct classes; and it is a striking fact that the pairs arithmetically and graphically associated will be entirely different, thus evidencing that correspondence is rather a creation of the mind than a property inherent in the things associated*.

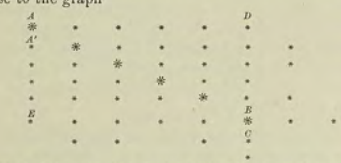
* Just so it is possible for two triangles to stand in a treble perspective relation to each other, as I have had previous occasion to notice in this *Journal*.

(43) I shall call the totality of the partitions of n consisting of odd numbers the U, and that consisting of unequal numbers the V system.

I say that any U may be converted into a V by the following rule: Let each part of the given U be converted into an equilateral bend, and these bends fitted into one another as was done in the problem of converting the reciprocal of

$$(1-ax)(1-ax^3)(1-ax^5)\ldots$$

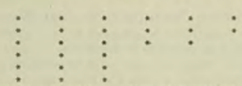
into an infinite series, considered in the preceding section. We thus form what may be called a bent graph. Then, as there shown, such graph may be dissected into a diagonal line of points and two precisely similar regular graphs. The graph compounded of the diagonal and one of these, it is obvious, will also be regular, and I shall call it the major component of the bent graph; the remaining portion may be called the minor component. Each of these graphs will be bounded by lines inclined to each other at an angle one-half of that contained between the original bounding lines, and each may be regarded as made up of bends fitting into one another. The contents of these bends taken in alternate succession, commencing with the major graph, will form a series of continually decreasing numbers, that is to say, a V partition. As an example let 11 11 9 5 5 5 be the given U partition; this gives rise to the graph



Reading off the bends on the major and minor graphs alternately, commencing with BAD, $CA'E$ respectively, there results the regularized partition into unequal numbers

11 10 9 8 6 2.

(44) The application of the rule is facilitated to the eye by at once constructing a graph, the number of points in whose horizontal lines are the major halves of the given parts, and construing this to signify two graphs, one the graph actually written down, the other the same graph with its first column omitted; for instance in the case before us the graph will be*



* This may be regarded as a parallel-ruler form of dislocation of the figure produced by making the portion to the right of the diagonal of larger asterisks revolve about that diagonal

If we call the lines and columns in the directions of the lines and columns of the Durfee-square appurtenant to the graph $a_1 a_2 \ldots a_i$, $\alpha_1 \alpha_2 \ldots \alpha_i$ [i (here 3) being the extent of the side of the square], the partition given by the rule will be

$$a_1+\alpha_1-1,\quad a_1+\alpha_2-2,\quad a_2+\alpha_2-3,\quad a_2+\alpha_3-4,\quad a_3+\alpha_3-5,\ldots$$
$$\ldots[a_{i-1}+\alpha_{i-1}-(2i-3)],\quad [a_{i-1}+\alpha_i-(2i-2)],\quad [a_i+\alpha_i-(2i-1)],\quad [a_i-i],$$

and inasmuch as

$$a_1 = \text{ or } > a_2 = \text{ or } > a_3 \ldots \quad \text{and} \quad \alpha_1 = \text{ or } > \alpha_2 = \text{ or } > \alpha_3 \ldots$$

the above series is necessarily made up of continually decreasing numbers, at all events until the last term is reached. But this term will form no exception, for the fact of i being the content of the side of the square belonging to the transverse graph $\alpha_1, \alpha_2 \ldots, \alpha_i, \alpha_{i+1} \ldots$ implies that $\alpha_i =$ or $> i$, hence

$$[a_i+\alpha_i-(2i-1)]-(a_i-i)=\alpha_i-i+1>0.$$

In the above example the side of the square *nucleus* in the original total graph was supposed to be the same for the major and minor graphs of which it is composed. If we suppose that graph to contain only i nodes in the ith line, then the side of the square to the minor graph which it contains will be $i-1$, and the number of parts given by the angular readings of the two graphs combined will be $2i-1$ instead of $2i$, as for example if the 3rd line in the graph above written be 3 instead of 5, the resulting partition will be 11 10 9 8 2, but we may, if we please, regard this as 11 10 9 8 2 0 and the last term will then still be $a_i - i$, and the general expression will remain unchanged from what it was before.

Next I proceed to the converse of what has been established, namely, that every U may be transformed by the rule into a V, and shall show that any V may be derived from some one (and only one) U.

Whether the number of effective parts in the given V be odd or even, we may always suppose it to be even by supplying a zero part if necessary, and may call the parts $l_1, \lambda_1, l_2, \lambda_2 \ldots, l_i, \lambda_i$. Suppose that it is capable of being derived from a certain U: form with the parts of U a graph expressed in the usual way by equilateral bends or elbows, then the side of the square appurtenant to the regular graph formed by the major half of this, say G, must have for content the given number i.

until it coincides with the portion to the left of the diagonal; the graph thus formed (merely as a matter of convenience to the eye) may be then made to revolve about an axis perpendicular to the plane, so as to bring the diagonal out of its oblique into the more usual horizontal position. All this trouble of description might have been saved by beginning not with a bent graph but with a graph formed with *straight* lines of points written *symmetrically* under each other, which is made possible by the fact of there being an *odd* number of points in each line. The graph so formed then resolves itself naturally into a major and minor regular graph.

Let $a_1, a_2 \ldots a_i$, $\alpha_1, \alpha_2 \ldots \alpha_i$ be the contents of the first i rows and first i columns respectively of G, then the equations to be satisfied are

$$a_1+\alpha_1-1=l_1,\quad a_2+\alpha_2-3=l_2,\quad a_3+\alpha_3-5=l_3\ldots,\quad a_i+\alpha_i-(2i-1)=l_i,$$
$$a_1+\alpha_2-2=\lambda_1,\quad a_2+\alpha_3-4=\lambda_2,\quad a_3+\alpha_4-6=\lambda_3\ldots,\quad a_i \quad -i \quad =\lambda_i.$$

Hence

$$a_1-a_2=\lambda_1-l_2-1 \quad a_2-a_3=\lambda_2-l_3-1\ldots$$
$$a_{i-1}-a_i=\lambda_{i-1}-l_i-1 \quad a_i=\lambda_i+i,$$
$$\alpha_1-\alpha_2=l_1-\lambda_1-1 \quad \alpha_2-\alpha_3=l_2-\lambda_2-1\ldots$$
$$\alpha_{i-1}-\alpha_i=l_{i-1}-\lambda_{i-1}-1 \quad \alpha_i=l_i-\lambda_i+i-1,$$

and for all values of θ,

$$l_\theta > \lambda_\theta > l_{\theta+1}.$$

Hence $a_1, a_2 \ldots a_i$ are all positive, and $\alpha_1, \alpha_2 \ldots \alpha_i$ are all at least equal to i. There will therefore be one and only one graph G satisfying the required conditions, namely a graph the contents of whose lines are

$$a_1, a_2, \ldots a_i, \quad A_1, A_2, \ldots A_{\alpha_i}-i$$

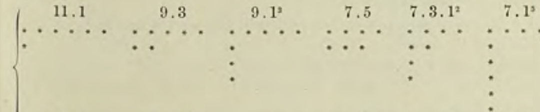
[where $A_1, A_2, \ldots A_{\alpha_i}-i$ is the conjugate partition to $\alpha_1-i, \alpha_2-i, \ldots \alpha_i-i$]; the partition U will be found by subtracting unity from the doubles of each of those parts. Thus then it has been shown that every U will give rise to some one V, and every V be derived from a determinate U; hence there must exist a one-to-one correspondence between the U and V systems. In a certain sense it is a work of supererogation to show that there is a U corresponding to each V; it would have been sufficient to infer from the linear form of the equations that there could not be more than one U transformable into a V; for each U being associated with a distinct V it would follow that there could be no V's not associated with a U, since otherwise there would be more V's than U's, which we know *aliunde* is impossible.

As an example of what precedes let the partible number be 12. The U system computed exhaustively will be

$11.1 \quad 9.3 \quad 9.1^3 \quad 7.5 \quad 7.3.1^2 \quad 7.1^5 \quad 5^2.1^2 \quad 5.3.1^4$
$5.3^2.1 \quad 5.1^7 \quad 3^4 \quad 3^3.1^3 \quad 3^2.1^6 \quad 3.1^9 \quad 1^{12}$

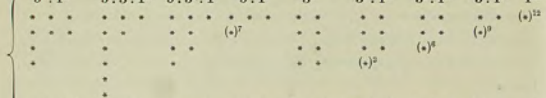
Underneath of these partitions I will write the major component graph, and underneath this again the corresponding V; we shall thus have the table

11.1	9.3	9.1^3	7.5	$7.3.1^2$	7.1^5



7.5	6.5.1	8.4	5.4.2.1	7.4.1	9.3

$5^2.1^2$ $5.3.1^4$ $5.3^2.1$ 5.1^7 3^4 $3^3.1^3$ $3^2.1^6$ 3.1^9 1^{12}



6.3.2.1 8.3.1 6.4.2 10.2 5.4.3 7.3.2 9.2.1 11.1 12

Thus we obtain for the *V* system:

7.5 6.5.1 8.4 5.4.2.1 7.4.1 9.3 6.3.2.1 8.3.1
6.4.2 10.2 5.4.3 7.3.2 9.2.1 11.1 12

which are all the ways in which 12 can be broken up into unequal parts*.

The *U*'s corresponding to those given by the arithmetical method of effecting correspondence would be:

7.5 $1.3^2.5$ 1^{12} $1^7.5$ $1^5.7$ 3.9 $1^3.3^3$ $1^9.3$ $1^6.3^2$
$1^2.5^2$ $3.1^4.5$ $1^2.3.7$ $1^3.3^3$ 11.1 3^4

instead of

11.1 9.3 9.1^3 7.5 $7.3.1^2$ 7.1^5 $5^2.1^2$ $5.3.1^4$
$5.3^2.1$ 5.1^7 3^4 $3^3.1^3$ $3^2.1^6$ 3.1^9 1^{12}

so that there is absolutely not a single pair the same in the two methods of conjugation.

(45) The object, however, of instituting the graphical correspondence is not to exhibit this variation, however interesting to contemplate, but to find a correspondence between the two systems which shall resolve itself into correspondences between the classes into which each may be subdivided.

Thus we may call U_i that class of *U*'s in which there are i distinct odd numbers, and V_i that class of *V*'s in which there are i sequences with a gap between each two successive ones: the theorem now to be established is that the *V* corresponding to any U_i is a V_i, so that class corresponds with class, and as a corollary, that the number of ways in which n can be made up by a series of ascending numbers constituting i distinct sequences is the same as the number of ways in which it can be composed with any i distinct odd numbers each occurring any number of times. This part of the investigation which I will presently enter upon is purely graphical. A few remarks and illustrations may usefully precede.

In the example above worked out it will be observed that there are three classes of *U*'s, namely,

1^{12} 3^4: 11.1 9.3 9.1^3 7.5 7.1^5 $5^2.1^2$
$3^3.1^3$ $3^2.1^6$ 3.1^9: $7.3.1^2$ $5.3.1^4$ $5.3^2.1$

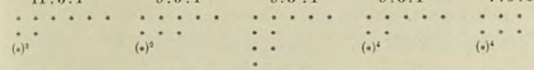
* In Note D, *Interact*, Part 2, I show how this transformation can be accomplished by the continual doubling of a string on itself.

and three classes of *V*'s agreeing with those above in the number of partitions in each, namely,

12 3.4.5: 11.1 9.3 10.2 8.4 7.5 9.2.1
7.3.2 6.5.1 5.4.2.1: 8.3.1 7.4.1 6.4.2.

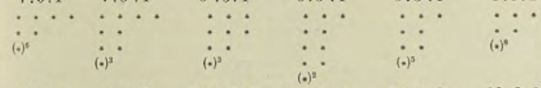
So again for $n=16$ there will be found to be eleven partitions into odd parts of the third class, which, with their quasi-graphs and corresponding partitions into unequal parts are exhibited below:

$11.3.1^2$ $9.5.1^2$ $9.3^2.1$ $9.3.1^4$ $7.5.1^4$



9.6.1 8.5.2.1 8.6.2 10.5.1 9.4.2.1

$7.3.1^6$ $7.3^2.1^3$ $5^2.3.1^3$ $5.3^3.1^2$ $5.3^2.1^5$ $5.3.1^8$

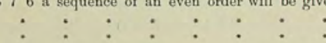


11.4.1 9.5.2 8.4.3.1 8.5.3 10.4.2 12.3.1

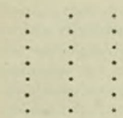
The transformed partitions above written are all of them of the third class (that is consist of three distinct sequences) and comprise all that exist of that class. 16 will correspond to 1^{16} and 1.3.5.7 to itself. All the other partitions of each of the two systems will be of the second class, and will necessarily have a one-to-one graphical correspondence inasmuch as the entire systems have been proved to have such correspondence.

It is worthy of preliminary remark that the association of the first classes of *U*'s and *V*'s given in the previous section will be identical with the association furnished by the graphical method—but whereas in converting *V* into *U* by the antecedent process, the two cases of the sequence being of an odd or even order had to be separately considered, the graphical method is uniform in its operation.

Thus 9 8 7 6 a sequence of an even order will be given graphically by



corresponding to 15^2, and 9 8 7 6 5 a sequence of an odd order will be given graphically by



corresponding to 5^7, whereas it will be observed that $15^2 = (9+6)^{\frac{4}{2}}$ and $5^7 = 5^{\frac{9+5}{2}}$.

It may be noticed that when the major component is an oblate rectangle it gives rise to a sequence of an even order, and when a quadrate or prolate rectangle to one of an odd order.

I subjoin an example of the algorithm by means of which a given V can be transformed into its corresponding U, taking as a first example $V = 10\ 9\ 8\ 5\ 4\ 1$.

The process of finding U is exhibited below:

3	3	5	5	(9)
2	2	3	3	(8)
	4	4	2	(7)
	1	3	3	(6)
	10	8	4	(1)
	9	5	1	(2)
	1	1	1	(3)
	4	4	4	(4)
	7	7	7	(5)

$3^2 . 5^2 . 7^3$ will be the U required.

As a second example let $V = 12\ 10\ 9\ 8\ 5\ 4\ 1$; the algorithm will be as shown below:

			1	(9)
			1	(8)
1	0	0	0	(7)
2	1	1	1	(6)
12	9	5	1	(1)
10	8	4	0	(2)
1	3	3	0	(3)
8	8	6	4	(4)
15	15	11	7	(5)

1 7 11 15 15 will be the U required. Lines (1) and (2) are the parts of the given V written alternately in the upper and lower line; lines (3) and (6) are obtained by oblique and direct subtraction performed between (1) and (2); line (4) is obtained from (3) by adding the number of terms in (1) to the last term in (3) which gives the last term in (4) and then adding in successively the other terms in (3) each diminished by one unit; (7) is derived from (6) by diminishing each term in the latter by a unit and taking the continued sum of the terms thus diminished; (8) is found by the usual

rule of "calling"* from its conjugate (7); and finally (5) and (9) are obtained by subtracting a unit from the doubles of the several terms in (4) and (8).

It thus becomes apparent that the passage back from a V to a U is a much more complicated operation than that of making the passage from a U to a V, so much more so that it would seemingly have been labour in vain to have attacked the problem of transformation by beginning from the V end.

(46) I now proceed to the main business, which is to show that any U containing i distinct odd numbers will, by the method described, be graphically converted into a V containing i distinct sequences.

Let G be any regular graph; H what G becomes when the first column of G is removed; a, b, c, d ... the contents of the angles of G, H taken in succession.

Also let i be the number of lines of unequal content in G, j the number of distinct sequences in $a, b, c, d, e, \ldots$.

The two first lines of G, say L, L', and also the two first columns, say K, K', may be equal or unequal†.

If $L = L'$ and $K = K'$, $a - 1 = b$, $b - 1 = c$.

If $L = L'$ and $K > K'$, $a - 1 = b$, $b - 1 > c$.

If $L > L'$ and $K = K'$, $a - 1 > b$, $b - 1 = c$.

If $L > L'$ and $K > K'$, $a - 1 > b$, $b - 1 > c$.

Let G', H' represent what G, H become on removing the first bend, that is the first line and the first column, and let i', j' be the values of i, j for G', H', so that j' is the number of sequences in $c, d, e \ldots$.

It is obvious from what precedes that in the four cases considered $j' = j$, $j' = j - 1$, $j' = j - 1$, $j' = j - 2$ respectively. But in these four cases $i' = i$, $i' = i - 1$, $i' = i - 1$, $i' = i - 2$ respectively.

Hence on each supposition $i - j = i' - j'$, and continuing the process by removing each bend in succession, $i - j$ must for any number of bends have the same values as it has for *one* bend; but in that case if h and k are the contents of the line and column of the bend, the reading of the corresponding G, G' will be $h + k - 1$, $h - 1$, so that for that case j will be 1 or 2 according as h and k are not or are both greater than 1, that is according as i is 1 or 2‡.

* I borrow this term from the vernacular of the American Stock Exchange.

† For brevity I use line and column to signify the extent of (that is, the number of nodes in) either.

‡ The final graph after denudation pushed as far as it will go must be either a single bend, a column, a line or a single node. In the first case $i = 2, j = 2$, in each of the remaining three cases $i = 1, j = 1$.

Hence $i-j$ is always equal to zero, consequently a U of the ith class will be transformed by the graphical process into a V of the ith class, as was to be proved.

(47) I have previously noticed [p. 25 above] that the simplest case of $i=j=1$ leads to the formula

$$\frac{q}{1-q}+\frac{q^3}{1-q^3}+\frac{q^5}{1-q^5}+\frac{q^7}{1-q^7}+\ldots=\frac{q}{1-q}+\frac{q^3}{1-q^2}+\frac{q^6}{1-q^3}+\frac{q^{10}}{1-q^4}+\ldots,$$

which is a sort of pendant to Jacobi's formula

$$\frac{q}{1+q}-\frac{q^3}{1+q^3}+\frac{q^5}{1+q^5}-\frac{q^7}{1+q^7}+\ldots=\frac{q}{1+q}-\frac{q^3}{1+q^2}+\frac{q^6}{1+q^3}-\frac{q^{10}}{1+q^4}+\ldots^{*}.$$

These formulae may be derived from one another or both obtained simultaneously as follows: From addition of the left-hand sides of the two equations there results the double of

$$\frac{q}{1-q^2}+\frac{q^6}{1-q^6}+\frac{q^5}{1-q^{10}}+\frac{q^{14}}{1-q^{14}}+\ldots \text{ or of } \sum_{i=1}^{i=\infty}\left(\frac{q^{4i-3}}{1-q^{8i-6}}+\frac{q^{8i-2}}{1-q^{8i-2}}\right),$$

and from addition of the right-hand sides of the same there results the double of

$$\frac{q}{1-q^2}+\frac{q^5}{1-q^4}+\frac{q^6}{1-q^6}+\frac{q^{14}}{1-q^8}+\ldots \text{ or of } \sum_{i=1}^{i=\infty}\left(\frac{q^{i(2i-1)}}{1-q^{4i-2}}+\frac{q^{i(2i+3)}}{1-q^{4i}}\right).$$

Consequently in order by the operation of addition of the two equations to deduce one from the other we must be able to show that these expressions are identical: observing then that $4i-3$ and $8i-2$ are odd and even respectively for all values of i, but $i(2i-1)$ and $i(2i+3)$ odd or even, according as for i, $2i-1$ or $2i$ be written, it has to be shown that

$$\sum_1^\infty \frac{q^{4i-3}}{1-q^{8i-6}}=\sum_1^\infty\left(\frac{q^{2i-1.4i-3}}{1-q^{8i-6}}+\frac{q^{2i-1.4i+1}}{1-q^{8i-4}}\right) \qquad (A)$$

and

$$\sum_1^\infty \frac{q^{8i-2}}{1-q^{8i-2}}=\sum_1^\infty\left(\frac{q^{i(8i-2)}}{1-q^{8i-2}}+\frac{q^{i(8i+6)}}{1-q^{8i}}\right). \qquad (B)$$

(A) is equivalent to

$$\sum_1^\infty q^{4i-3}\,\frac{1-q^{i-1.8i-6}}{1-q^{8i-6}}=\sum_1^\infty\frac{q^{2i-1.4i+1}}{1-q^{8i-4}}$$

or

$$\sum_1^\infty q^{4i+1}\,\frac{1-q^{i(8i+2)}}{1-q^{8i+2}}=\sum_1^\infty\frac{q^{2i-1.4i+1}}{1-q^{8i-4}}.$$

Hence if i signify any number from 1 to ∞ and k signify any number from 0 to $i-1$, it has to be shown that $(4i+1)(2k+1)$ contains the same integers and each taken the same number of times as $(2m-1)(4m+1+4n)$, where m is any number from 1 to ∞ and n is any number from 0 to ∞. But the $(4i+1)(2k+1)$ is the same as $(2k+1)\{4(k+l+1)+1\}$ where k and l

* My formula is what Jacobi's becomes when every middle *minus* sign in it is changed into *plus* and every inferior *plus* sign into *minus*.

each extend from 0 to ∞, and the $(2m-1)(4m+4n+1)$ is the same as $(2m+1)\{4(m+n+1)+1\}$ where m and n each extend from 0 to ∞, and the two latter expressions on writing $k=m$, $l=n$ become identical.

Again (B) is equivalent to

$$\sum_1^\infty q^{8i-2}\,\frac{1-q^{i-1.8i-2}}{1-q^{8i-2}}=\sum_1^\infty\frac{q^{i(8i+6)}}{1-q^{8i}}.$$

Hence we have to show that $(8i-2)(1+j)$ when $i=2, 3, \ldots \infty$ and $j=0, 1, 2, \ldots, (i-2)$, or say $(8i+6)(1+j)$, where $i=1, 2, \ldots \infty$ and $j=0, 1, 2, \ldots (i-1)$ is identical with $l(8l+6+8m)$, where $l=1, 2, \ldots \infty$ and $m=0, 1, 2, \ldots \infty$; the former of these is identical with

$$(1+j)\{8(j+k+1)+6\},$$

where $j=0, 1, \ldots \infty$; $k=0, 1, \ldots \infty$, and the latter is identical with

$$(1+l)\{8(l+m+1)+6\},$$

where $l=0, 1, \ldots \infty$; $m=0, 1, \ldots \infty$, consequently the two expressions are coextensive, which proves (B), and (A) has been already proved. Hence we see that either of the two original equations can be deduced from the other from the fact that their sum leads to an identity.

In like manner subtraction performed between the two allied equations leads to the fissiparous equation

$$\sum_0^\infty\left\{\frac{x^{8i+2}}{1-x^{8i+2}}+\frac{x^{4i+3}}{1-x^{8i+6}}\right\}=\sum_0^\infty\left\{\frac{x^{(i+2)(2i+1)}}{1-x^{4i+2}}+\frac{x^{i+1.2i+3}}{1-x^{4i+4}}\right\},$$

which gives birth to the pair

$$\sum_0^\infty \frac{x^{4i+3}}{1-x^{8i+6}}=\sum_0^\infty\left\{\frac{x^{2i+3.4i+3}}{1-x^{8i+6}}+\frac{x^{2i+1.4i+3}}{1-x^{8i+4}}\right\} \qquad (C)$$

and

$$\sum_0^\infty \frac{x^{8i+2}}{1-x^{8i+2}}=\sum_0^\infty\left\{\frac{x^{2i+2.4i+1}}{1-x^{8i+2}}+\frac{x^{2i+2.4i+5}}{1-x^{8i+8}}\right\}. \qquad (D)$$

(C) is equivalent to

$$\sum_0^\infty \frac{x^{4i+3}(1-x^{i+1.8i+6})}{1-x^{8i+6}}=\sum_0^\infty\frac{x^{2i+1.4i+3}}{1-x^{8i+4}},$$

which is an identity by virtue of the equivalence of

$(4i+3)[1+2\{j<(i+1)\}]$ that is $(4j+4k+3)(1+2j)$ to $(2\lambda+1)(4\lambda+3+4\mu)$ where j, k, λ, μ each extend from zero to infinity, and

(D) is equivalent to

$$\sum_0^\infty \frac{x^{8i+2}(1-x^{i(8i+2)})}{1-x^{8i+2}}=\sum_0^\infty\frac{x^{2i+2.4i+5}}{1-x^{8i+8}},$$

which is an identity by virtue of the equivalence of

$(8i+2)\{1+(j<i)\}$ that is $\{8(j+k+1)+2\}(1+j)$ to $(2\lambda+2)(4\lambda+5+4\mu)$, each symbol j, k, μ having as before the same range, namely from zero to infinity. Thus then the difference of the two allied equations (as previously their sum) is reduced to an identity which establishes the validity of each of them.

INTERACT, PART 2.

With notes of many a wandering bout,
Of linkèd sweetness long drawn out.
L'Allegro.

(48) D. *Transformation of Partitions by the Cord Rule.*—The figures below are designed to show how it is possible by means of the continuous doubling of a string upon itself to pass from an arrangement of groups of repetitions of r distinct odd integers to the corresponding one with like sum, made up of r distinct sequences. Each of the two figures duplicated by rotation about its upper horizontal boundary of nodes through two right angles will represent an arrangement of repeated odd numbers, the parts being represented by the contents of the *vertical* lines in the figures so duplicated.

Fig. 1.

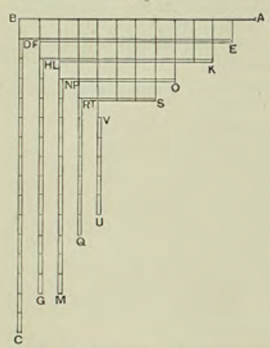
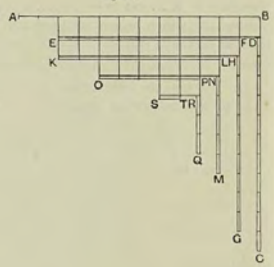


Fig. 2.



The first duplicated figure represents the arrangement 33, 29^2, 23, 21, 9^3, 7, 5^2, 3, 1 whose sum is 183; its correspondent will be the contents of the lengths of * *ABC, CDE, EFG, GHK, KLM, MNO, OPQ, QRS, STU, UV*, namely the arrangement 29, 27, 24 (22, 21,), 18, 14, 12, 10, 6 which is the same number 183 partitioned into (ten parts but) nine sequences: the second duplicated figure represents the arrangement 25, 23, 17, 15, 9^2, 7^3, 5^2, 1^2, whose sum is 130; its correspondent is represented by the lengths of *ABC, CDE, DEF, FGH, HKL, LMN, NOP, PQR, RST, TU*, which is the same number 130 partitioned into the (nine parts but) eight sequences 25, 22 (20, 19,), 15, 12, 10, 6, 1.

* A line containing i units of length represents $(i+1)$ nodes.

(49) E. *On Graphical Dissection.*—It may be not unworthy of notice that there is a sort of potential anticipation of Mr Durfee's dissection of a symmetrical graph, in a method which, whether it is generally known or not I cannot say, but is substantially identical with Dirichlet's for finding approximately $\sum_1^n \left[\frac{n}{i}\right]$ and other such like series (a bracketed quantity being used to signify that quantity's integer part). Constructing the hyperbola $xy = n$, drawing its ordinates to the abscissas 1, 2, 3, ... n, and in each of them planting nodes to mark the distances 1, 2, 3, ... from its foot, there results a *symmetrical graph* included between one branch of the curve, its two asymptotes, and lines parallel to and cutting each of them at the distance n from the original. Its content will be the sum in question. The Durfee-square to it will be limited by the square whose side is $[\sqrt{n}]$, and this added to the original area gives twice over the area in which the number of nodes is $\sum_1^{\sqrt{n}} \left[\frac{n}{i}\right]$, and consequently neglecting magnitudes of the order $\sqrt{n}$,

$$\sum_1^n \left[\frac{n}{i}\right] = 2n \sum_1^{\sqrt{n}} \frac{1}{i} - i^2 = n(\log n + 2C - 1)$$

and as a corollary

$$\sum_1^{\sqrt{n}} \left\{\frac{n}{i} - \left[\frac{n}{i}\right]\right\} = n(C - 2C + 1) = (1 - C)n,$$

where C is Euler's number ·57721, so that $1 - C$ for large values of n will be the average value of the fractional part of n divided by an inferior number. Furthermore a similar graph, but with $xy = 2n$ diminished by the portion contained between a branch of the new curve, one of its asymptotes and two parallel ordinates cutting that asymptote at distances n and $2n$ from the origin (which portion obviously contains $(2n - n)$ that is n nodes) will represent $\sum_1^n \left[\frac{2n}{i}\right]$, and consequently the sum $\sum_1^n \left\{\left[\frac{2n}{i}\right] - 2\Sigma\left[\frac{n}{i}\right]\right\}$, that is (see *Berl. Abhand.* 1849, p. 75) the number of times that $\frac{n}{i} - \left[\frac{n}{i}\right]$ equals or exceeds $\frac{1}{2}$, as i progresses from 1 to n (within the same limits of precision as previously) $= 2n(\log 2n + 2C - 1) - n$ less $2n(\log n + 2C - 1)$, that is $= (\log 4 - 1)n$, so that the probability of the fractional part of n divided by an inferior number not falling under $\frac{1}{2}$ is $\log 4 - 1$*.

* What precedes I recall as having been orally communicated to me many years ago by the late ever to be regretted Prof. Henry Smith, so untimely snatched away when in the very zenith of his powers, and so to say, in the hour of victory, at the moment when his intellectual eminence was just beginning to be appreciated at its true value, by the outside world. I was under the impression until lately that he was quoting literally from Dirichlet when so communicating with me, but as the geometrical presentation given in the text is not to be found in the

(50) F. *Mr Ely's method of finding the asymptotic value of the number of improper fractions with a very large given numerator which are nearer to the integer below than to the integer above**.

"Let a number n be divided by all the numbers from 1 to n; then a value is required for the number of residues which are equal to or greater than $\frac{1}{2}$. An example will make evident a method by which we may obtain limits to the value sought. If n be 100 the residues $= > \frac{1}{2}$ are

(1)	$\frac{49}{51}$	$\frac{48}{52}$	$\frac{47}{53}$	$\frac{46}{54}$	$\frac{45}{55}$	$\frac{44}{56}$	$\frac{43}{57}$	$\frac{42}{58}$	$\frac{41}{59}$	$\frac{40}{60}$	$\frac{39}{61}$	$\frac{38}{62}$	$\frac{37}{63}$	$\frac{36}{64}$	$\frac{35}{65}$	$\frac{34}{66}$
(2)	$\frac{32}{34}$	$\frac{30}{35}$	$\frac{28}{36}$	$\frac{26}{37}$	$\frac{24}{38}$	$\frac{22}{39}$	$\frac{20}{40}$									
(3)	$\frac{22}{26}$	$\frac{19}{27}$	$\frac{16}{28}$													
(4)	$\frac{16}{21}$	$\frac{12}{22}$														
(5)	$\frac{15}{17}$	$\frac{10}{18}$														
(6)	$\frac{10}{15}$															
(a)	$\frac{4}{6}$	$\frac{4}{8}$	$\frac{9}{13}$													

memoir cited from the *Berlin Transactions*, I infer that it originated with himself. In comparing Mertens' memoir, *Crelle*, 1874, with Dirichlet's (1849), upon which it is a decided step in advance, one cannot fail to be struck with surprise that the point to which the closer drawing of the limits to the values of certain transcendental arithmetical functions achieved by the former is owing, should have escaped the notice of so profound and keen an intellect as Dirichlet's, and those who came after him in the following quarter of a century. The point I refer to is the almost self-evident fact that if in the cases under consideration

$$\Sigma\phi(Fi\,.\,x)=\psi x \text{ then } \phi x=\Sigma\mu(i)\,\psi(Fi\,.\,x)$$

where $\mu(i)$ means 0, if i contains any repeated prime factors, but otherwise 1 or $\bar{1}$ according as the number of prime factors in i is even or odd. Dirichlet works with a function given implicitly by an equation, Mertens with the same function expressed in a series, wherein exclusively lies the secret of his success.

* It is proper to state that what follows in the text was handed in to me by Mr Ely on the morning after I had proposed to my class to think of some "common sense method" to explain the somewhat startling fact brought to light by Dirichlet, of more than three-fifths of the residues of n in regard to $i=1, 2, 3, \ldots n$ being less than $\frac{i}{2}$. Mr Ely's method shows at once, in a very common sense manner, why the proportion must be considerably greater than the half, inasmuch as whilst the terms in the first few harmonic ranges are approximately $\frac{n}{1\,.\,2}, \frac{n}{2\,.\,3}, \frac{n}{3\,.\,4}$, etc., in number, the number of them which employed as denominators to n give fractional parts greater than $\frac{1}{2}$, instead of being the halves of these are only $\frac{n}{2\,.\,3}, \frac{n}{3\,.\,5}, \frac{n}{4\,.\,7}$, etc. The mean value in both methods to quantities of the order of $\sqrt{n}$ *inclusive*, turns out to be the same, whichever method is employed, but the margin of unascertained error by the use of Mr Ely's method (as compared with Dirichlet's) is reduced in the proportion of $1 : 1+\sqrt{2}$, that is, nearly $2 : 5$.

In which it will be observed that the residues $= > \frac{1}{2}$ occur in batches. Let X be the whole number, and x_i the number in batch i. In batch i the numerators decrease by i and the denominators increase by 1. (Those marked (a) of which the denominators are less than $\sqrt{200}$ are left out of account for the present.) It is evident for the general case we have approximately

$$\frac{\left[\frac{n}{i+1}\right]-ix_i}{\left[\frac{n}{i+1}\right]+x_i}=\frac{1}{2}$$

or accurately

$$x_i=\left[\frac{n}{(i+1)(2i+1)}\right] \text{ or } \left[\frac{n}{(i+1)(2i+1)}\right]+1\text{*."}$$

Mr Ely is then able to show that by limiting the calculation of x_i to the values of i which do not exceed $[\sqrt{n}/2]$, so that roughly speaking the character of $\sqrt{2n}$ of the remainders is left undetermined (and no account taken of them in finding the value of X), and giving to x_i its approximate value $\frac{n}{(i+1)(2i+1)}$, and then extending the series $\frac{n}{2\,.\,3}+\frac{n}{3\,.\,5}+\frac{n}{4\,.\,7}$ beyond the $[\sqrt{n}/2]$th term, where it ought to stop, to infinity, the errors arising from each of these three sources† and therefore their combined effect will be of the order $\sqrt{n}$, so that the asymptotic value of X will be

$$\left(\frac{1}{2\,.\,3}+\frac{1}{3\,.\,5}+\frac{1}{4\,.\,7}+\ldots\right)n,$$

which is $(2\log 2-1)\,n$, with an uncertainty of the order $\sqrt{n}$, as was to be shown.

(51) It may be seen that Mr Ely's method consists in distributing the n numbers from n to 1 into what I have elsewhere termed *harmonic ranges* and determining what portions of the several ranges employed as denominators to n give fractional parts, greater or less than $\frac{1}{2}$. It may assist in forming a more vivid idea of this kind of distribution, if the reader takes a definite case, say of $n=121$, the first (10) harmonic ranges will then comprise

* I find by an exact calculation that if R is the remainder of n in regard to $(i+1)(2i+1)$ and $R=\lambda(i+1)+\mu$, where $\lambda<2i+1$ and $\mu<i+1$, then for $\lambda=2\theta-1$ or 2θ, $x_i=\left[\frac{n}{(i+1)(2i+1)}\right]+1$ if $\mu=i-1$ or $i-2\ldots$ or $i-\theta$, and $x_i=\left[\frac{n}{(i+1)(2i+1)}\right]$ for all other values of μ. Hence it follows that out of $(2i^2+3i+1)$ successive values of n, (i^2+i) and (i^2+2i+1) will be the respective numbers of the cases for which the one or the other of these two values of x_i is employed, so that for larger values of i the chances for the two values are nearly the same, but with a slight preponderance in favour of the smaller value. See p. [54].

† The error from the first cause makes the determination of X too small by an unknown amount, that from the third cause too large by a known amount, and that from the second too large or too small (as it may happen) by an unknown amount.

all the numbers from 121 to 12 inclusive, and the remaining 111 harmonic ranges will comprise the remaining 11 numbers from 11 to 1; that is to say 11 of them will contain a single number, and the remaining 100 ranges be vacant of content.

So again if $n=20$ the first four ranges will contain all the numbers from 20 to 5 inclusive; the 5th, 6th, 9th and 20th range will consist of the sole numbers 4, 3, 2, 1, and the remaining 12 ranges will be vacant. I shall proceed to compare the precision of Mr Ely's result with that of Dirichlet's—for this purpose it will be enough to determine the asymptotic value of the uncertainty and to take no account of quantities of a lower order than $\sqrt{n}$.

Let us then suppose that $\sqrt{(kn)}$ ranges are preserved, and consequently $\sqrt{\left(\frac{n}{k}\right)}$ fractions left out (k being an arbitrary constant which will eventually be determined so as to make the uncertainty a minimum).

The first cause of error necessitates a correction of which the limits are 0 and $\sqrt{\left(\frac{n}{k}\right)}$; the second cause a correction of which the limits are $\sqrt{(kn)}$ and $-\sqrt{(kn)}$; and the third, namely the overreckoning of

$$\frac{n}{(j+1)(2j+1)}+\frac{n}{(j+2)(2j+3)}+\dots$$

where $j=\sqrt{(kn)}$, a correction of which the value is $-\frac{n}{2j}$ or $-\frac{1}{2}\sqrt{\left(\frac{n}{k}\right)}$.

Hence making $(\log 4-1)\,n=U$, the superior limit of X is

$$U+\frac{1}{2}\sqrt{\left(\frac{n}{k}\right)}+\sqrt{(kn)},$$

and the inferior limit $U-\frac{1}{2}\sqrt{\left(\frac{n}{k}\right)}\sqrt{(kn)}$. Consequently $X=U+\rho n^{\frac{1}{2}}$ where $\rho<\sqrt{k}+\frac{1}{2}\sqrt{\left(\frac{1}{k}\right)}$, of which the minimum value is found by making $k=\frac{1}{2}$, so that $\rho<\sqrt{2}$ and the *uncertainty* is $\sqrt{2}\,.\,n^{\frac{1}{2}}$. Adopting Mertens' asymptotic value of the *uncertainty* of $\sum_{1}^{n}\left[\frac{n}{i}\right]$, namely $\sqrt{n}$, and using Dirichlet's formula, $\sum_{n}^{1}\left[\frac{2n}{i}\right]-2\sum_{n}^{1}\left[\frac{n}{i}\right]$, X has the same mean value as above, but the uncertainty becomes $(\sqrt{2}+2)\,n^{\frac{1}{2}}$ which is nearly two and a half times as great as that given by the direct method employed by Mr Ely.

I use the word *uncertainty*, it will be noticed, in a different sense from *error*; the latter is objective, referring to fact, the former subjective, referring to knowledge. Both methods in the case here presented give the same mean value, and therefore the *error* is the same, but the uncertainty is widely

different according to the method made use of. Of two formulae referring to the same fact one might very well give the smaller error and the other the smaller uncertainty.

I have shown above that for considerable values of i, the average value of x_i is $\frac{n}{(i+1)(2i+1)}+\frac{1}{2}$; if then it may be assumed (and there seems no reason for suspecting the contrary) that for $i=1, 2, \dots, \sqrt{2n}$, the mean value of $\frac{n}{i}-\left[\frac{n}{i}\right]$ is $\frac{1}{2}$, U will not only be the mean value of the known limits of X but also the mean value of X itself. The value found for k shows that the most advantageous mode of employing Mr Ely's method is to make the series $\frac{n}{2\,.\,3}+\frac{n}{3\,.\,5}+\dots\frac{n}{(i+1)(2i+1)}+\dots$ stop at one of the terms which is approximately equal to unity.

(52) It is not without interest to consider the exact law for the extent of a harmonic range of a given denomination, say i: this it is easily seen will be always equal to $\left[\frac{n}{i^2+i}\right]$ or $\left[\frac{n}{i^2+i}\right]+1$.

I shall regard i as given and determine the values of n which correspond to the one or the other of the two formulae: this will depend not on the absolute value of n but on its remainder in respect to the modulus i^2+i. To fix the ideas, let $i=4$ so that $i^2+i=20$, and let n take in successively all values from 40 to 59 inclusive.

Then corresponding to n equal to

40	44	48	52	56
41	45	49	53	57
42	46	50	54	58
43	47	51	55	59

the fourth range will be

10, 9	11, 10, 9	12, 11, 10	13, 12, 11	14, 13, 12
10, 9	11, 10	12, 11, 10	13, 12, 11	14, 13, 12
10, 9	11, 10	12, 11	13, 12, 11	14, 13, 12
10, 9	11, 10	12, 11	13, 12	14, 13, 12

that is in half the terms of the period $\left[\frac{n}{i^2+i}\right]$ and in the other half $\left[\frac{n}{i^2+i}\right]+1$ gives the extent of the range.

So in general, if $n=k(i^2+i)+\lambda i+\mu$, where $\lambda=0, 1, 2, \dots i$, and $\mu\equiv 0, 1, 2, \dots(i-1)$, when the remainder of n to modulus (i^2+i) is of the form

$\lambda(i^2+i)+\{0, 1, 2, \dots (\lambda-1)\}$ that is in $\frac{i^2+i}{2}$ cases the extent of the ith harmonic range to n is $\left[\frac{n}{i^2+i}\right]+1$, and when of the form

$$\lambda(i^2+i)+\{\lambda, \lambda+1, \dots (i-1)\},$$

that is in the remaining $\frac{i^2+i}{2}$ cases it is $\left[\frac{n}{i^2+i}\right]$.

As the sum of the harmonic ranges to n is n itself, and

$$\frac{n}{1.2}+\frac{n}{2.3}+\dots+\frac{n}{n(n+1)}=n-\frac{n}{n+1},$$

it follows that if we separate all the numbers from 1 to n into two classes, say i's and j's, i being any number for which n is of the form

$$k(i^2+i)+\lambda i+0, 1, 2, \dots (\lambda-1),$$

and j any other number within the prescribed limits, then

$$\sum_1^n \frac{n}{t}-\sum_1^n\left[\frac{n}{t}\right]=\text{number of } i\text{'s}-\frac{n}{n+1},$$

and consequently the number of the i terms has $(1-C)\,n$ for its asymptotic value.

(53) In like manner the law previously stated in a footnote, p. [51], for giving the extent of that portion of the ith range for which $\frac{n}{t}$ contains a fractional part not less than $\frac{1}{2}$ may be verified. Thus let $i=3$ then $(i+1)(2i+1)=28$, let $n=56, 57, \dots 83$. Then for the values of n

28	32	36	40	44	48	52
29	33	37	41	45	49	53
30	34	38	42	46	50	54
31	35	39	43	47	51	55

the portion of the third range having the required character will contain the numbers

8	9	10	11	12	13	14
8	9	10	11	12	14, 13	15, 14
8	9	10	12, 11	12, 11	14, 13	15, 14
8	10, 9	11, 10	12, 11	13, 12	14, 13	15, 14

so that there are $2(1+2+3)$, that is 3.4 forms of n out of 7.4 for which the formula $\left[\frac{n}{4.7}\right]+1$ has to be employed, and so in general if R is the residue of n in respect to $(i+1)(2i+1)$, there are i^2+i cases where the formula $\left[\frac{n}{(i+1)(2i+1)}\right]+1$ and $(i+1)^2$ where the formula $\left[\frac{n}{(i+1)(2i+1)}\right]$ has to be employed.

G. *On Farey Series.*

(54) This note is a natural sequel to and has grown out of the two which precede; it has also a collateral affinity with the subject-matter of the Acts, inasmuch as a graph affords the most simple mode of viewing and stating the fundamental property of an ordinary Farey series, and any series *ejusdem generis*. For instance, let A, B, C be a reticulation in the form of an equilateral triangle, where B is a right angle, and n the number of nodes in the base or height of the triangle; if the hypothenuse be made to revolve in the plane of the triangle about (either end say about) A, the triangle formed by joining A with any two consecutive nodes of greatest proximity to the centre of rotation traversed by the rotating line will be equal in area to the minimum triangle which has any three nodes for its apices, that is its double will be equal to unity. This law of uniform description of areas (say of *equal areas in equal jerks*) is identical with the characteristic law of an ordinary Farey series which deals with terms whose number is the sum-totient τn: but it will also hold good if the triangle be scalene instead of equilateral, which corresponds to Glaisher's extension of a Farey series, to the case where the numerator and denominator of each term has its own separate limit (*Phil. Mag.* 1879), or again, when the rotation takes place about the right angle B as centre, which gives rise to a Farey series of a totally different species, defined by the inequality $ax+by<n$, or again when the hypothenuse is replaced by the quadrant of a circle or ellipse, and in an infinite variety of other cases, as for example when the graph is contained between a branch of an equilateral hyperbola and the asymptotes, which case corresponds to the subject-matter of the theory of Dirichlet (*Berl. Abhand.* 1844) concerning the sum of the number of ways in which all integers up to n can be resolved into the product of two relative primes, which is the same thing as the half of the number of divisors (containing no repeated prime factors) which enter into the several integers up to n, or as the entire number of solutions in relative primes of the inequality $xy=$ or $<n$. The law of equal description of areas ($pq'-p'q=\pm 1$), Mr Glaisher has shown very acutely, is an immediate inference (by an obvious induction) from the well-known fact that between a fraction and its two nearest convergents (namely the one ordinarily so called and that which is obtained by substituting $\delta-1$ and 1 for the last partial quotient), no other fraction can be interposed whose denominator is not greater than that of the one first named.

From the areal-law obviously follows the equation $\frac{p''}{q''}=\frac{xp'-p}{xq'-q}$ (where $\frac{p}{q}$, $\frac{p'}{q'}$, $\frac{p''}{q''}$ are any three consecutive terms of the series), so that in order to construct explicitly such a series from the two first terms, all we have to do is to give to x at each step the highest value it can assume, consistent with the imposed limit or limits. Thus for example I have found by this method when the limiting inequality is $x+y=$ or <15, the series

$$\frac{0}{1}\ \frac{1}{15}\ \frac{1}{14}\ \frac{1}{13}\ \frac{1}{12}\ \frac{1}{11}\ \frac{1}{10}\ \frac{1}{9}\ \frac{1}{8}\ \frac{1}{7}\ \frac{1}{6}\ \frac{2}{11}\ \frac{1}{5}$$
$$\frac{2}{9}\ \frac{1}{4}\ \frac{3}{11}\ \frac{2}{7}\ \frac{3}{10}\ \frac{1}{3}\ \frac{4}{11}\ \frac{3}{8}\ \frac{2}{5}\ \frac{3}{7}\ \frac{4}{9}\ \frac{1}{2}\ \frac{1}{1}^{*}$$

and the complements in respect to unity of the several terms which precede $\frac{1}{2}$ taken in reverse order, and again for $xy=$ or <15 the series (which might be called the Dirichlet-Farey series)

$$\frac{0}{1}\ \frac{1}{15}\ \frac{1}{14}\ \frac{1}{13}\ \frac{1}{12}\ \frac{1}{11}\ \frac{1}{10}\ \frac{1}{9}\ \frac{1}{8}\ \frac{1}{7}\ \frac{1}{6}\ \frac{1}{5}$$
$$\frac{1}{4}\ \frac{2}{7}\ \frac{1}{3}\ \frac{2}{5}\ \frac{1}{2}\ \frac{3}{5}\ \frac{2}{3}\ \frac{3}{4}\ \frac{1}{1}.$$

In general if we agree to understand respectively by the *decernent* and the *secernent* to x, the number of divisors without restriction, and the number of divisors restricted to contain no square number, that go into x, and denote the sum-secernent and sum-decernent of n by Sn and Dn respectively, Dirichlet's mode of looking at the question leads immediately to the equation $\sum_1^n S\frac{n}{i^2}=Dn$. Mertens' equation $\left[Sn=\sum_1^n \mu i D\frac{n}{i^2}\right]$ obtained by a longer and somewhat more difficult process is in point of fact merely that equation *reverted*. On pointing out to Mr F. Franklin this elegant passage in Dirichlet's memoir, he remarked to me to the effect that it was an example, which might admit of wide generalization, of a concept resembling that inherent in the subject-matter of the ordinary Farey series; which excellent and keen-witted observation led me to look into the subject from the point of view herein explained. The present theory diverges from the ordinary one in quite another and more natural direction (I imagine) than that pursued by M. Darboux, whose article on the subject of quasi-Farey series (*Bulletin de la Société Mathématique de France*, tome VI.) I have not been able to obtain sight of, and can only conjecture its purport through the reference made to it in a subsequent article which I *have* been able to procure in the same journal by M. Edouard Lucas.

* It is advisable for the purpose of securing generality in reasoning upon Farey series not to omit the initial and final terms $\frac{0}{1}$, $\frac{1}{1}$ which seem generally to have been lost sight of by previous writers on the subject. Even then the series is only half complete, for after $\frac{1}{1}$ should follow the reciprocals of the preceding terms until $\frac{1}{0}$ is reached. Thus a complete *ordinary* Farey series beginning with $\frac{0}{1}$ and ending with $\frac{1}{0}$ consists of two symmetrical branches with $\frac{1}{1}$ as their point of junction, each made up of two symmetrical sub-branches meeting respectively in the terms $\frac{1}{2}$ and $\frac{2}{1}$, and such that the sum of a corresponding pair of fractions on the one side of $\frac{1}{1}$ and of their reciprocals on the other side is equal to unity; whereas in the two complete branches the *product* of each corresponding pair is unity.

(55) I prove the persistency of the fundamental property of ordinary Farey series for such series generalized in the manner supposed above, as follows.

Let us use $O.\ F.\ S_i$ to denote an ordinary Farey series for which the limit is i, and $G.\ F.\ S.$ a Farey series in which, calling the numerator and denominator of any term x, y, $\phi(x, y) <= i$, $\phi(x, y)$ meaning a rational function which increases when either x or y increases. If in an $O.\ F.\ S_i$ any two consecutive terms be $\frac{a}{b}$, $\frac{c}{d}$, and in an $O.\ F.\ S_{i+1}$ $\frac{p}{q}$ intervenes between $\frac{a}{b}$, $\frac{c}{d}$ we know, p being greater than b and d, the two nearest convergents to $\frac{p}{q}$ must be contained in $O.\ F.\ S_i$, and consequently must be $\frac{a}{b}$, $\frac{c}{d}$ themselves, so that $p=a+c$, $q=b+d$, and as a corollary if $\frac{a}{b}$, $\frac{c}{d}$ be consecutive terms in any $O.\ F.\ S.$, and $\frac{p}{q}$ be any one of the terms which subsequently intervene between $\frac{a}{b}$, $\frac{c}{d}$, we must have $p=$ or $>a+c$, $q=$ or $>b+d$. In order to fix the ideas let us suppose $\phi(x, y)$ to represent $x+y$, so that $x+y<=n$.

For the values 2, 3, 4, 5, 6, 7, 8, 9 ... of n, the $G.\ F.\ S.$ will be

$$\frac{0}{1}\ \frac{1}{1};\ \ \frac{0}{1}\left(\frac{1}{2}\right)\frac{1}{1};\ \ \frac{0}{1}\left(\frac{1}{3}\right)\frac{1}{2}\ \frac{1}{1};\ \ \frac{0}{1}\left(\frac{1}{4}\right)\frac{1}{3}\ \frac{1}{2}\left(\frac{2}{3}\right)\frac{1}{1};\ \ \frac{0}{1}\left(\frac{1}{5}\right)\frac{1}{4}\ \frac{1}{3}\ \frac{1}{2}\ \frac{2}{3}\ \frac{1}{1};$$
$$\frac{0}{1}\left(\frac{1}{6}\right)\frac{1}{5}\ \frac{1}{4}\ \frac{1}{3}\left(\frac{2}{5}\right)\frac{1}{2}\ \frac{2}{3}\left(\frac{3}{4}\right)\frac{1}{1};\ \ \frac{0}{1}\left(\frac{1}{7}\right)\frac{1}{6}\ \frac{1}{5}\ \frac{1}{4}\ \frac{1}{3}\ \frac{2}{5}\ \frac{1}{2}\left(\frac{3}{5}\right)\frac{2}{3}\ \frac{3}{4}\ \frac{1}{1};$$
$$\frac{0}{1}\left(\frac{1}{8}\right)\frac{1}{7}\ \frac{1}{6}\ \frac{1}{5}\ \frac{1}{4}\left(\frac{2}{7}\right)\frac{1}{3}\ \frac{2}{5}\ \frac{1}{2}\ \frac{3}{5}\ \frac{2}{3}\ \frac{3}{4}\left(\frac{4}{5}\right)\frac{1}{1};\ldots$$

where the terms in parenthesis are the new terms which intervene as n increases from any value to the next following integer, and where it will be noticed that if $\frac{p}{q}$ be any such parenthesised fraction lying between $\frac{a}{b}$ and $\frac{c}{d}$, $p=a+c$ and $q=b+d$, just as in the successive form of an $O.\ F.\ S.$ The theorem to be proved may be made to depend on the following lemma.

If for any given value of n every two consecutive terms in a $G.\ F.\ S.$ appear as consecutive terms in an $O.\ F.\ S.$ for the same or any smaller value of n; this will continue to be true for all superior values of n.

The proof is immediate, for let $\frac{a}{b}$, $\frac{c}{d}$ be any two consecutive terms in the $G.\ F.\ S_j$ which are also consecutive terms in $O.\ F.\ S_i$ where $i=$ or $<j$.

If a term $\frac{p}{q}$ intervene between $\frac{a}{b}$, $\frac{c}{d}$ in *G. F.* S_{j+1}, $p=$ or $> a+c$, $q=$ or $> b+d$, by virtue of the remark made. But if $p > a+c$ and $q > b+d$,

$$\phi(a+c, b+d) < \phi(p, q) < j+1,$$

but $\frac{a+c}{b+d}$ is intermediate in value between $\frac{a}{b}$, $\frac{c}{d}$, hence $\frac{a+c}{b+d}$ must have appeared in a *G. F.* $S_{j'}$, where $j' < j$, which is contrary to hypothesis.

Hence $\frac{a}{b}$, $\frac{p}{q}$, $\frac{c}{d}$ will have been consecutive terms in some *O. F. S.*, and in like manner any two consecutive terms in *G. F. S.* either remain consecutive in *G. F.* S_{j+1} or admit a new term between them which is consecutive to each of them in some *O. F. S.*, so that the supposed relation if it holds good for j is true for all superior values of j; but $\frac{0}{1}$, $\frac{1}{1}$ will in any of the supposed cases be a *G. F. S.*; consequently in all these cases no two terms are consecutive in any *G. F. S.* which are not so in some *O. F. S.*, and therefore the law of equal description of areas will apply to them equally as to the *O. F. S.*, as was to be proved.

The theory may be extended to *G. F. S.*, defined by several concurrent limiting equations. Thus for example Mr Glaisher has proved this for the case of $x <= m$, $y <= n$: I have not had time as yet to consider what are the restrictions to which the limiting functions may be subject, but the theorem is obviously an extremely elastic one, and the above proof suffices for all the special cases which I have enumerated*.

(56) I am indebted to Mr Ely for the following additional examples of Farey series, in the enlarged sense, which may interest some of my readers.

Ex. (1). $x+y=$ or <20

$$\frac{1}{19}\cdots\frac{1}{9}\ \frac{2}{17}\ \frac{1}{8}\ \frac{2}{15}\ \frac{1}{7}\ \frac{2}{13}\ \frac{1}{6}\ \frac{3}{17}\ \frac{2}{11}\ \frac{3}{16}$$

$$\frac{1}{5}\ \frac{3}{14}\ \frac{2}{9}\ \frac{3}{13}\ \frac{1}{4}\ \frac{4}{15}\ \frac{3}{11}\ \frac{2}{7}\ \frac{3}{10}\ \frac{4}{13}\ \frac{1}{3}\ \frac{5}{14}\ \frac{4}{11}\ \frac{3}{8}\ \frac{5}{13}\ \frac{2}{5}$$

$$\frac{5}{12}\ \frac{3}{7}\ \frac{4}{9}\ \frac{5}{11}\ \frac{6}{13}\ \frac{1}{2}\ \frac{7}{13}\ \frac{6}{11}\ \frac{5}{9}\ \frac{1}{7}\ \frac{7}{12}\ \frac{3}{5}\ \frac{5}{8}\ \frac{7}{11}\ \frac{2}{3}\ \frac{7}{10}\ \frac{5}{7}$$

$$\frac{8}{11}\ \frac{3}{4}\ \frac{7}{9}\ \frac{4}{5}\ \frac{9}{11}\ \frac{5}{6}\ \frac{6}{7}\ \frac{7}{8}\ \frac{8}{9}\ \frac{9}{10}.$$

Ex. (2). $x^2+y=$ or <20

$$\frac{1}{19}\cdots\frac{1}{9}\ \frac{1}{8}\ \frac{2}{15}\ \frac{1}{7}\ \frac{2}{13}\ \frac{1}{6}\ \frac{2}{11}\ \frac{1}{5}\ \frac{2}{9}\ \frac{1}{4}\ \frac{3}{11}\ \frac{2}{7}\ \frac{3}{10}\ \frac{1}{3}\ \frac{3}{8}\ \frac{2}{5}\ \frac{3}{7}\ \frac{1}{2}\ \frac{2}{3}\ \frac{3}{4}.$$

* Since the above was in type I have discovered the true principle of Farey series, for which see Note H following the Exodion.

Ex. (3). $y-\sqrt{x}=$ or <15

$$\frac{1}{16}\cdots\frac{1}{8}\ \frac{2}{15}\ \frac{1}{7}\ \frac{2}{13}\ \frac{1}{6}\ \frac{2}{11}\ \frac{3}{16}\ \frac{1}{5}\ \frac{3}{14}\ \frac{2}{9}\ \frac{3}{13}\ \frac{4}{17}\ \frac{1}{4}\ \frac{4}{15}\ \frac{3}{11}\ \frac{2}{7}$$

$$\frac{5}{17}\ \frac{3}{10}\ \frac{4}{13}\ \frac{5}{17}\ \frac{1}{3}\ \frac{6}{17}\ \frac{5}{14}\ \frac{4}{11}\ \frac{3}{8}\ \frac{5}{13}\ \frac{2}{5}\ \frac{7}{17}\ \frac{5}{12}\ \frac{3}{7}\ \frac{7}{17}\ \frac{4}{0}\ \frac{5}{11}\ \frac{6}{13}$$

$$\frac{7}{15}\ \frac{8}{17}\ \frac{1}{2}\ \frac{9}{17}\ \frac{8}{15}\ \frac{7}{13}\ \frac{6}{11}\ \frac{5}{9}\ \frac{9}{16}\ \frac{4}{7}\ \frac{7}{12}\ \frac{10}{17}\ \frac{3}{5}\ \frac{11}{18}\ \frac{8}{13}\ \frac{5}{8}\ \frac{7}{11}$$

$$\frac{9}{14}\ \frac{11}{17}\ \frac{2}{3}\ \frac{11}{16}\ \frac{9}{13}\ \frac{7}{10}\ \frac{12}{17}\ \frac{5}{7}\ \frac{13}{18}\ \frac{8}{11}\ \frac{11}{15}\ \frac{3}{4}\ \frac{13}{17}\ \frac{10}{13}\ \frac{7}{9}\ \frac{11}{14}$$

$$\frac{4}{5}\ \frac{13}{16}\ \frac{9}{11}\ \frac{14}{17}\ \frac{5}{6}\ \frac{16}{19}\ \frac{11}{13}\ \frac{6}{7}\ \frac{13}{15}\ \frac{7}{8}\ \frac{15}{17}\ \frac{8}{9}\ \frac{17}{19}\ \frac{9}{10}\ \frac{10}{11}\ \frac{11}{12}\ \frac{12}{13}\ \frac{13}{14}$$

$$\frac{14}{15}\ \frac{15}{16}\ \frac{16}{17}\ \frac{17}{18}\ \frac{18}{19}\ \frac{1}{1}.$$

EXODION. *On the Correspondence between certain Arrangements of Complex Numbers.*

At which he wondred much and gan enquere
What stately building durst so high extend
Her lofty towres, unto the starry sphere.
Faerie Queene I. x. 56.

(57) Starting from the expansion in a series of $\Theta_1 x$, multiplying in the usual notation both sides of the equation by

$$(1-q^2)(1-q^4)(1-q^6)\ldots,$$

and intercalating the factors of this product between those of

$$(1-qz)(1-q^3z)\ldots(1-qz^{-1})(1-q^3z^{-1})\ldots$$

taken in alternate order, there results the equation

$$(1-qz^{-1})(1-qz)(1-q^2)(1-q^3z^{-1})(1-q^3z)(1-q^4)\ldots=\sum_{i=-\infty}^{i=+\infty}(-)^i q^{i^2}z^i,$$

and writing q^n in place of q and making $z=\mp q^m$, Jacobi (*Crelle*, Vol. XXXII. p. 166) derives the identity

$$(1\pm q^{n-m})(1\pm q^{n+m})(1-q^{2n})(1\pm q^{3n-m})(1-q^{3n+m})(1-q^{4n})\ldots=\sum_{-\infty}^{+\infty}(\pm)^i q^{ni^2+mi}.$$

From this equation, using the lower sign and making $n=\frac{3}{2}$, $m=\frac{1}{2}$, he observes, may be deduced Euler's expression in a series for

$$(1-q)(1-q^2)(1-q^3)\ldots,$$

and using the *upper* sign and making $n=\frac{1}{2}$, $m=\frac{1}{2}$, another known series "given by Gauss in the first volume of the *Göttingen Commentaries* for the years 1808–11."

It is not without interest, I think, to observe that by making $n=\frac{1}{2}$, $m=\frac{1}{2}+\epsilon$ (where ϵ is an infinitesimal), and using the *lower* sign we may immediately deduce Jacobi's own celebrated postscript (so to say) to Euler's equation, namely,

$$(1-q)^3(1-q^2)^3(1-q^3)^3\ldots=\sum_{-\infty}^{+\infty}(-)^i q^{\frac{i^2+i}{2}+i\epsilon}\div(1-q^{-\epsilon})$$
$$=1-3q+5q^3-7q^6\ldots,$$

the general term being

$$\sum_0^\infty(-)^i\left\{(q^{\frac{i^2+i}{2}+i\epsilon}-q^{\frac{i^2+i}{2}-(i+1)\epsilon})\div\frac{1}{1-q^{-\epsilon}}\right\},$$

which is
$$(-)^i(2i+1)q^{\frac{i^2+i}{2}}.$$

(58) It is obvious, that by the same right and within the same limits of legitimacy as the equation involving q, n, m (or if we please to say so in q, m) has been derived from the equation in (q, z), the equation in q, z may be recovered from the equation in q and m, if this latter can be shown to be true, morphologically interpreted for general values of m. I shall show that regarding m and n as absolutely general symbols, such as $\sqrt{(-1)}$ or $\sqrt{2}$ or ρ or the quaternion units, or any other heterogeneous or homogeneous units we please, the equation in question which I shall write under the equivalent form

$$(1\mp q^a)(1\mp q^b)(1-q^c)(1\mp q^{a+c})(1\mp q^{b+c})(1-q^{2c})\ldots=\sum_{i=-\infty}^{i=+\infty}(\mp)^i q^{\frac{i^2}{2}c+\frac{i}{2}(a-b)}$$

[where $c=a+b$, and a, b are absolutely general symbols or species of units entirely independent of one another] does hold good as a morphological identity*. Thus interpreted, it amounts to a theorem in complex quantities, dealing with arrangements of three sorts of elements which I shall call C's, B's, A's respectively, meaning by a C any non-negative integer (that is zero or any positive integer) multiple of c, by a B such multiple augmented by a single b, and by an A such multiple augmented by a single a.

The C's, the B's and the A's in any such arrangement will be regarded as three separate series, the terms in each of which flow from left to right in descending order, that is the multiples of c which represent totally or with the exception of a single b or a single a, the terms in each such series taken in severalty are to form a continually decreasing series.

* This theorem is less transcendental than Newton's binomial theorem when the same latitude is given to the meaning of the symbols in either case: for $(1+x)^m=1+mx+\frac{m^2-m}{2}x^2+\ldots$ does not admit of *direct* interpretation when m is a general symbol. The passage from numerical proximate equality to absolute identity, prepared but not perfected nor capable of being explained by infinitesimal gradation, brings to mind the analogous transfiguration of sensibility into sensation, or of sensation into consciousness, or of consciousness into thought.

The total number of elements and the number of C's will be called the major and minor parameters respectively—the relation to the modulus 2 (that is the parity or imparity) of either one of them its character: and for brevity, the terms major and minor character will be used to signify the character of the major or minor parameter. The totality of all arrangements whatever of A's, B's, C's in which *no element is repeated*, will constitute the sphere of the investigation, limited only by the absence of what I term the exceptional or isolated arrangements, consisting exclusively of a series of *consecutive* B's ending in b, or of consecutive A's ending in a. Within the prescribed sphere I shall prove that a process may be instituted for transforming any arrangement which shall satisfy the five following conditions:

(1) That it shall be capable of acting on every licit and unexceptional arrangement.

(2) That it shall transform it into another such arrangement.

(3) That operating once upon an arrangement, and then again upon the operate, it brings back the original arrangement.

(4) That it leaves the sum of the elements in the arrangement unaltered.

(5) That it reverses each of its two characters*.

From (3) it will follow that all the arrangements within the prescribed sphere are associated in pairs, and from (1) that the sum of the elements in each such pair is the same. This being so, it is obvious from the fact of the parity of the total number of elements being opposite for any pair of associated arrangements, that in the development in a series of

$$(1-q^a)(1-q^b)(1-q^c)(1-q^{a+c})\ldots,$$

no term will appear in which the index of q is other than the sum of the terms in one of the exceptional (we may now call them unconjugated or unconjugable) arrangements, and from the fact of the parity of the number of the C's being opposite in any pair, the same will be true of the development in a series of

$$(1+q^a)(1+q^b)(1-q^c)(1+q^{a+c})\ldots.$$

As regards the coefficient in this latter series of any term whose index is

* It will presently be seen that all the licit and unexceptional arrangements will be divided into 3 classes and a specific operator be found for each class capable of acting on each arrangement of that class and converting it into another of the same class, and which will satisfy also the 3rd, 4th and 5th of the enumerated conditions. The total operator contemplated in the text may then be regarded as the sum of these specific ones, each of which, within its own sphere, will have to fulfil the five conditions of Catholicity, Homoeogenesis, Mutuality, Inertia and Enantiotropy (the last a word used in the school of Heraclitus to signify "the conversion of the primeval being into its opposite"). See Kant's *Critique of Pure Reason* by Max Müller, Vol. I., p. 18.

the sum of the elements in an unconjugate arrangement it will manifestly be the number of ways in which the same complex number can be thrown under the form of a sum of the arithmetical series

$$a,\ a+c, \ldots,\ a+(i-1)c,$$

which is

$$\frac{i^2-i}{2}c+ia,$$

that is

$$\frac{i^2}{2}c+\frac{i}{2}(a-b),$$

or of

$$b,\ b+c, \ldots,\ b+(i-1)c,$$

which is

$$\frac{i^2}{2}c-\frac{i}{2}(a-b).$$

If

$$\frac{i^2}{2}c+\frac{i}{2}(a-b)=\frac{j^2}{2}c+\frac{j}{2}(a-b),$$

then

$$\frac{i^2+i}{2}a+\frac{i^2-i}{2}b=\frac{j^2+j}{2}a+\frac{j^2-j}{2},$$

which necessitates $i=j$, and if

$$\frac{i^2}{2}c+\frac{i}{2}(a-b)=\frac{j^2}{2}c-\frac{j}{2}(a-b),$$

then

$$\frac{i^2+i}{2}a+\frac{i^2-i}{2}b=\frac{j^2-j}{2}a+\frac{j^2+j}{2}b,$$

so that

$$i^2+i-(i^2-i)=(j^2-j)-(j^2+j) \text{ or } i=-j.$$

Hence the general term is $q^{\frac{i^2}{2}c\pm\frac{i}{2}(a-b)}$, where i is an integer stretching from zero to infinity, and in like manner, and for the same reason, the general term in the former series will be $(-)^i q^{\frac{i^2}{2}c\pm\frac{i}{2}(a-b)}$ with the like interpretation: or which is the same thing, comprising both cases in one and interpreting i to be integer stretching from $-\infty$ to $+\infty$, the general term will be $(\mp)^i q^{\frac{i^2}{2}c+\frac{i}{2}(a-b)}$.

(59) The task before us then is to show the *possibility* of instituting, by *actually* instituting, a law of operation which shall satisfy the five preliminary conditions of catholicity, homoeogenesis, reciprocity, reversal of characters and conservation of sum.

The following notation will be found greatly to conduce to clearness in effecting the needful separation into classes or species. A capital letter with a point above, as $\dot{X}$, will be used to signify the greatest value, and with a point below, as $\underset{\cdot}{X}$, the least value of any term in a series which that letter is used to denote. $X=0$, $X>0$, $X+Y=0$, $X+Y>0$ will signify respectively that there are no X's, that there are X's, that there are no X's and no Y's, that there are either X's or Y's or both in any arrangement under consideration. B's will be separated into $'B$ and B''s, or as we may write it $B='BB'$, where $'B$ is the general name for all the B's, which beginning with the highest term $\dot{B}$ form an arithmetical series of which c is the common difference. If there is a gap of more than one c between $\dot{B}$ and the next lowest B, $'B$ is of course the single term $\dot{B}$: B' is any B which is not a $'B$.

So again, A_1 is any A which belongs to a series of A's forming an arithmetical series whose constant difference is c and lowest term a, so that unless $\underset{\cdot}{A}=a$, $A_1=0$: any other A will be designated by ${}_1A$. The signs of accent and point may of course be separate or combined: thus for example $\underset{\cdot}{C}$ will mean the smallest C in any given arrangement, $\dot{B}$ will mean the greatest B, $\underset{\cdot}{A}$ will mean the lowest A, ${}_1\underset{\cdot}{A}$ will mean the lowest of the ${}_1A$'s and $\dot{A}_1$ the highest of the A_1's. Every $'B$ is necessarily greater than any B', and every ${}_1A$ than any A_1. If $'\underset{\cdot}{B}-b=0$, this will indicate that all the B''s will form a consecutive series of terms (that is having a constant difference c) and ending in b, so that here $B'=0$, that is there are no B's except those that belong to the regular arithmetical progression ending in b. If ${}_1A=0$, all the A's will form an arithmetical progression ending in a. Thus we see that the arrangements belonging to the 1st terms (those that I have called exceptional) will consist of two species denoted respectively by

$${}_1A+B+C=0 \text{ and } ('\underset{\cdot}{B}-b)+A+C=0.$$

It may sometimes be found convenient to use a point to the left centre of a quantitative letter to signify that the quantity denoted is to be increased, and a point to the right centre to signify that the quantity denoted is to be diminished, by c. Thus $\dot{B}\cdot$ will mean $\dot{B}-c$, and $\cdot\underset{\cdot}{A}_1$ will mean $\underset{\cdot}{A}_1+c$, the first signifying the greatest B diminished by and the second the smallest A_1 increased by c. When any general letter, say X, is wanting as indicated by the equation $X=0$, $\underset{\cdot}{X}$ must be understood to mean zero. So for instance if $A=0$, and consequently ${}_1A=0$ and $A_1=0$, ${}_1\underset{\cdot}{A}=0$. Again, when there is a gap between the highest B and the one that follows it in any arrangement, the arithmetical progression of $'B$'s reduces as above remarked to a single term and there results $'\dot{B}='\underset{\cdot}{B}$. It may be noticed also that always $'\dot{B}=\dot{B}$, and $\underset{\cdot}{A}_1=\underset{\cdot}{A}$.

The arrangements which are comprised under the forms

$$(\alpha)\quad A,\ A-c,\ A-2c, \ldots,\ a,$$
$$(\beta)\quad B,\ B-c,\ B-2c, \ldots,\ b,$$

may be regarded as belonging to what I shall term the first genus.

The second genus, namely that consisting of unexceptional combinations of unrepeated A's, B's, C's, may then be divided into the following three species, the conditions by which they are severally distinguished being attached to each in its proper place.

1st Species. Conditions (γ) $'\underset{\cdot}{B} - b > 0$,
or (γ') $'\underset{\cdot}{B} - b = 0,\ C > 0,\ \underset{\cdot}{C} - c <= '\dot{B} - b$.

2nd Species. (δ) $'\underset{\cdot}{B} - b = 0,\ A + C > 0,\ C = 0$ or $\underset{\cdot}{C} - c > '\dot{B} - b$,
or (δ') $B = 0,\ C > 0,\ A = 0$, or ${}_1\underset{\cdot}{A} - a = > C$.

3rd Species. (ϵ) $B = 0,\ A > 0,\ {}_1A + C > 0,\ C = 0$, or $\underset{\cdot}{C} > {}_1\underset{\cdot}{A} - a$.

Where it is to be understood that the conditions set out in the same line are simultaneous conditions. Thus for example the conditions of an arrangement being of the second species are when all the conditions of the upper or else all the conditions of the lower of the two lines written under that species are fulfilled: the conditions of the upper line (be it noticed) are that $'\underset{\cdot}{B}$ is b, and that there are either some A's or some C's, and that if there are some C's, $\underset{\cdot}{C} - c > '\dot{B} - b$, and of the lower line, that there are no B's and some C's, and that if there are A's, $\underset{\cdot}{A} - a = > \underset{\cdot}{C}$, and so for the interpretation of the conditions of the existence of each of the other two species.

To these (7) systems of conditions $\alpha, \beta, \gamma, \gamma', \delta, \delta', \epsilon$ may be joined the trivial system (ω) $A = 0,\ B = 0,\ C = 0$*; the (8) systems thus constituted will easily be seen to be mutually exclusive and between them to comprehend the entire sphere of possibility, leaving no space vacant to be occupied by any other hypothesis. I will now proceed to assign the operators ϕ, ψ, ϑ appropriate to the three species of the second genus.

Office of the Operator ϕ. $\phi = '\phi + \phi'$.

When in Genus 2, Species 1, $C = 0$ or $\underset{\cdot}{C} - c > '\dot{B} - '\underset{\cdot}{B}$, $'\phi$ is to be performed, meaning that for each $'B$, $'B\cdot$ is to be substituted, and the inertia kept constant by forming a new $\underset{\cdot}{C}$ with the sum of the c's thus abstracted. In the contrary case ϕ' is to be performed, meaning that $\underset{\cdot}{C}$ is to be resolved into simple c's and as many of the $'B$'s, commencing with $'\dot{B}$ and taken in regular order to be converted into $'\cdot B$ as are required to maintain the inertia constant, that is c is to be added to each B in succession, until all the c's which together make up $\underset{\cdot}{C}$ are absorbed.

Office of the Operator ψ. $\psi = '\psi + \psi'$.

When in Genus 2, Species 2, $C = 0$ or $\underset{\cdot}{C} > '\dot{B} + \underset{\cdot}{A}$, $'\psi$ is to be performed, meaning that for $'\dot{B}$ and A their sum is to be substituted, producing a C [which, on the second hypothesis, will be a new $\underset{\cdot}{C}$]. In the contrary case ψ' is to be performed, meaning that for $\underset{\cdot}{C}$ is to be substituted $'\cdot\dot{B}$ (which will form a new $'\dot{B}$) and $\underset{\cdot}{C} - '\cdot\dot{B}$ which will form a new A_1.

* It would be perfectly logical, and indeed is necessary to regard the trivial case as belonging to the cases of exception, and then we might say that there are two genera, each containing three species, those of the first genus solitary, and those of the second, each of them comprising two sub-species, namely the sub-species subject to the action of the left-accented and that subject to the operation of the right-accented operators. The *trivial* species of the first genus consists of a single individual.

Office of the Operator ϑ. $\vartheta = '\vartheta + \vartheta'$.

When $C > 0$ and $\underset{\cdot}{C} + \dot{A}_1 < {}_1\underset{\cdot}{A}$, $'\vartheta$ is to be performed, meaning that for $\underset{\cdot}{C}$ and $\dot{A}_1$ their sum is to be substituted, producing a new ${}_1\underset{\cdot}{A}$. In the contrary case ϑ' is to be performed, meaning that for ${}_1\underset{\cdot}{A}$, $\cdot\dot{A}_1$ forming a new $\dot{A}_1$ and ${}_1\underset{\cdot}{A} - \dot{A}_1$ forming a new C are to be substituted.

(60) It will be seen that every species of the second genus consists of two contrary sub-species having opposite characters, and it will presently appear that any arrangement belonging to one of these sub-species under the effect of its appropriate operator passes over into the other, which operated upon in its turn by its appropriate operator becomes identical with the original one, so that any two contrary sub-species may be said to be of equal extent: in fact if the sum of the parts is supposed to be given there will be as many arrangements in any sub-species as in its opposite, for each one will be conjugated with some one of the others.

It may not be amiss to call attention here to the fact that the scheme of classification adopted is, in a certain sense, artificial. Thus, for instance, it proceeds upon an arbitrary choice between which shall be regarded as the A and which as the B series, so that by an interchange of these letters a totally different correspondence would be brought about between the arrangements of the second genus, those of the first genus remaining unaltered. Nor is there any reason for supposing that these are the only two correspondences capable of being instituted between the arrangements of the second genus—in particular there is great reason to suspect that a symmetrical mode of procedure might be adopted, remaining unaffected by the interchange between A and B. As a simple example of the effect of interchange, applying the method here given, suppose $A = 0$, $B = 0$, a case belonging to the second species and that sub-species thereof to which ψ' is applicable, and imagine further that the C series is monomial. Then C will be associated according to the scheme here given with $b, C - b$, but in the correlative scheme it would be associated with $a, C - a$.

(61) I need hardly say that so highly organized a scheme, although for the sake of brevity presented in a synthetical form, has not issued from the mind of its composer in a single gush, but is the result of an analytical process of continued residuation or successive heaping of exception upon exception in a manner dictated at each point in its development by the nature of the process and the resistance, so to say, of its subject-matter. The initial step (that applicable to species γ) is akin to the procedure applied by Mr F. Franklin to the pentagonal-number theorem of Euler, of which I shall have more to say presently. It will facilitate the comprehension of the scheme to take as an example the particular case where a and b represent actual and real quantities, say, to fix the ideas, $b = 1$, $a = 2$. Nothing, it will

be noticed, turns upon the fact of this specialization, which is adopted solely for the purpose of greater concision and to afford more ready insight into the *modus operandi*.

To illustrate the classes and laws of transformation consider (with $b=1$, $a=2$*, $c=a+b=3$) all the arrangements, the sum of whose parts is 12, namely 12, 11.1, 10.2, 9.2.1, 8.4, 8.3.1, 7.5, 7.4.1, 7.3.2, 6.5.1, 6.4.2, 5.4.3, 5.4.2.1.

One of these, 7.4.1, belongs to the exceptional genus. The rest will be conjugated and fall into species in the manner shown below, where the first species means where the conditions (γ) or (γ'), the second that where (δ) or (δ'), and the third where the conditions (ϵ) are satisfied. The C's, B's, A's are now numbers whose residues are 0, 1 or 2 in respect to the modulus 3. For greater clearness in each arrangement, numbers belonging to the same series are kept together, the law of descent only applying in this theory to elements belonging to the same series.

Species 1. 10.2 3.7.2; 4.8 3.1.8; 7.5; 3.4.5; 6.4.2 6.3.1.2; 5.7 3.2.7.

Species 2. 9.1.2 9.3; 6.1.5 4.1.5.2;

Species 3. Caret.

Or again let the collection of arrangements be one in which the sum is 18. The partitions of 18 are 18 17.1 16.2 15.3 15.2.1 14.4 14.3.1 13.5 13.4.1 13.3.2 12.6 12.5.1 12.4.2 12.3.2.1 11.7 11.6.1 11.5.2 11.4.3 11.4.2.1 10.8 10.7.1 10.6.2 10.5.3 10.5.2.1 10.4.3.1 9.8.1 9.7.2 9.6.3 9.6.2.1 9.5.4 9.5.3.1 9.4.3.2 8.7.3 8.7.2.1 8.6.4 8.6.3.1 8.5.4.1 8.5.3.2 8.4.3.2.1 7.6.5 7.6.4.1 7.6.3.2 7.5.4.2 7.5.3.2.1 6.5.4.3 6.5.4.2.1. In this case there are no exceptional arrangements.

1st Species. 16.2 3.13.2; 4.14 3.1.14; 13.5 3.10.5; 13.4.1 3.10.4.1; 7.11 3.4.11; 10.8 3.7.8; 12.4.2 12.3.1.2; 10.7.1 6.7.4.1; 6.10.2 6.3.7.2; 10.1.5.2 3.7.1.5.2; 9.4.5 9.3.1.5; 6.7.5 6.3.4.5; 7.1.8.2 3.4.1.8.2; 6.4.8 6.3.1.8; 7.4.5.2 6.4.1.5.2;

2nd Species. 18 17.1; 15.3 15.1.2; 12.6 12.5.1; 6.1.11 4.1.11.2; 9.1.8 4.1.8.5; 9.7.2 9.3.4.2; 9.6.3 9.6.1.2; 11.5.2 3.8.5.2.

3rd Species. Caret.

If the partible number is 11, of which the partitions are 11 10.1 9.2 8.3 8.2.1 7.4 7.3.1 6.5 6.4.1 6.3.2 5.4.2 5.3.2.1, there will be no exceptional arrangements and the pairs of unexceptional ones will be as below.

* No use it will be seen is made of the *accidental* relation $a=b+b$.

1st Species. 10.1 3.7.1; 7.4 6.4.1; 4.5.2 3.1.5.2.

2nd Species. 3.8 1.8.2.

3rd Species. 11 9.2; 6.5 6.3.2.

By interchanging a and b, that is making $a=1$, $b=2$, the correspondence changes into the following:

1st Species. 11, 3.8; 6.3.2, 6.5; 8.2.1, 3.5.2.1; 7.4, 6.4.1.

2nd Species. Caret.

3rd Species. 10.1, 6.4.1; 7.4, 3.7.1.

According to Mr Franklin's process the correspondence takes a form quite distinct from either of the above, namely 11, 10.1; 9.2, 8.2.1; 8.3, 7.3.1; 7.4, 6.4.1; 6.5, 5.4.2; 6.3.2, 5.3.2.1, all these arrangements constituting one single species.

A careful study of the preceding examples will sufficiently explain to the reader the ground of the divisions into species with their appropriate rules of transformation, and might almost supersede the necessity of a formal proof of the operator supplying the conditions of catholicity, homoeogenesis and mutuality; from their very definition they are seen to comply with the other two essential conditions of inertia and enantiotropy.

Signifying by Ω the total operator $\phi+\psi+\vartheta$, it has been already remarked that Ω will in the general case have two values which only come together when $a=b$, or which is the same thing, each of them is 1; a special case of the special case when the complex reduces to simple numbers, namely, it is the case indicated in the well-known equation

$$(1-q)^2(1-q^3)^2(1-q^5)^2\ldots=\frac{1}{(1-q^2)(1-q^4)\ldots}\sum_{i=-\infty}^{i=\infty} q^{i^2}.$$

But besides the two correspondences given by the two values of Ω, if we take the actual (no longer a diagrammatic case) $b=2$, $a=1$, we revert to Euler's theorem concerning the partitions of all pentagonal and non-pentagonal numbers, and can obtain by Dr Franklin's process, given in Art. (12), a totally different distribution into genera and species, namely the first genus instead of containing arrangements of the species

$$1, 4, 7, \ldots 3i-2;\quad 2, 5, 8, \ldots 3i-1$$

will, as previously shown, consist of the very different arrangements (giving the same infinite series of numbers as those for other sums)

$$i, i+1, i+2, \ldots 2i-1;\quad i+1, i+2, i+3 \ldots;\ 2i.$$

The character of each arrangement in the new solution depends in part on the relation to the modulus 2 of the whole number of parts and of the number of parts which are divisible by 3, so that we may divide the conjugate arrangements into four groups* designated respectively by *Oo*, *Oe*; *Eo*, *Ee*, using the capital letters to signify the oddness or evenness of the whole set of parts, and the small letters the same for the parts divisible by 3. There will thus be a cross classification of the arrangements of the second genus into groups over and above that into species, each species in fact consisting of four groups, which may be denoted as above, and of which *Oo* and *Ee* are one associative couple, and *Oe*, *Eo* the other†.

(62) The following elegant investigation has been handed in to me by Arthur S. Hathaway, fellow and one of my hearers at the Johns Hopkins University, to which, although it does not exactly strike at the object of the constructive theory here expounded, I gladly give hospitality in these pages.

"The theorem to be proved is as follows:

$$\begin{aligned} & 1+\epsilon x^{a}\,.\,1+\epsilon x^{a+h}\,.\,1+\epsilon x^{a+2h}\ldots \\ \times\, & 1+\epsilon x^{b}\,.\,1+\epsilon x^{b+h}\,.\,1+\epsilon x^{b+2h}\ldots \\ \times\, & 1-x^{h}\,.\,1-x^{2h}\,.\,1-x^{3h}\ldots=\sum_{\delta=-\infty}^{\delta=+\infty}\epsilon^{\delta}\,.\,x^{\frac{a+b}{2}\delta^2+\frac{a-b}{2}\delta}, \end{aligned}$$

where $\epsilon^2=1$ and $h=a+b$, a and b being any quantities whatever.

"The general term contains, say i exponents of x selected from the first line, j from the second line, and k from the third line, namely

$$\begin{gathered} a+\alpha_0 h,\ldots a+\alpha_{i-1}h, \\ b+\beta_0 h,\ldots b+\beta_{j-1}h, \\ \gamma_1 h,\ldots \gamma_k h, \end{gathered}$$

where $\alpha_0\ldots\alpha_{i-1}$, $\beta_0\ldots\beta_{j-1}$, $\gamma_1\ldots\gamma_k$ are respectively sets of i, j, k unequal integers arranged in ascending order, none representing a less integer than its subscript. This term is (remembering that $h=a+b$)

$$\epsilon^{i+j}(-)^k x^{ma+nb},$$

where

$$m=[(\alpha_0+1)+\ldots(\alpha_{i-1}+1)]+[\beta_0+\ldots\beta_{j-1}]+[\gamma_1+\ldots\gamma_k] \qquad (1)$$

$$n=[\alpha_0+\ldots\alpha_{i-1}]+[(\beta_0+1)+\ldots(\beta_{j-1}+1)]+[\gamma_1+\ldots\gamma_k] \qquad (2)$$

* It will be seen later on that there is a division into sixteen groups analogous to the division into four groups first noticed by Prof. Cayley arising under the Franklin process.

† The *Oe* and *Eo* conjugation has a very striking analogue in nature (as I am informed) in the existence of dissimilar hermaphrodite characters in two sorts of the wild English *primrose* and the American flower *Spring-beauty* or *Quaker-lady*—it being the law of nature that only those of different sorts can fertilize one another. Possibly the double symbolic character of *Oo* and *Ee* will justify or suggest the inquiry whether there may not be a latent duality in the unisexual specimens of such flowers as those just mentioned, where male and female are found codomiciled with the bisexual florets. There is also, it seems, a trace of analogy to the sparsely distributed unconjugate individuals of my first genus in Darwin's "complemental males."

In addition to these we obtain by subtraction

$$m-n=i-j\equiv i+j \bmod 2. \qquad (3)$$

Whence (since $\epsilon^2=1$) $\epsilon^{i+j}=\epsilon^{m-n}$.

"Thus all the above general terms having the same m and the same n divide themselves into positive and negative groups (corresponding to even and odd values of k), a term from one group cancelling a term from the other group. I propose to prove that the number of terms in each of these groups are equal, except when a certain relation exists between m and n, namely

$$m-\frac{(m-n)(m-n+1)}{2}=0,\ (\text{or } m=0 \text{ if } m=n),$$

corresponding to which there is but one general term having the same m and the same n which falls into the positive group ($k=0$). This establishes the theorem in question, as we see by putting $m-n=\delta$.

"It is sufficient to consider (1) in connection with (3). In the first place the first two partitions in (1) may be converted by a (1 : 1) correspondence into an indefinite partition (bearing in mind (3)) with a decrease ($m-n>0$) in the sum or content of the integers by $\frac{1}{2}(m-n)(m-n+1)$, as follows: extend α_0+1 in a horizontal line of dots, and *under* the first dot extend β_0 in a vertical line of dots, thus forming an *elbow*; in a similar manner form elbows out of α_1+1, β_1 &c. until one of the partitions is exhausted; this will be according to (3), the first or the second, according as $m<$ or $>n$, leaving in the inexhausted partition $m-n$ integers; place these elbows successively one without the other, and place on top ($m-n>0$) horizontal lines of dots corresponding to the successive unmatched integers decreased respectively by $0, 1, \ldots (n-m-1)$ or $1, 2, \ldots (m-n)$, according as $m<$ or $>n$; in either case the total decrease is $\frac{1}{2}(m-n)(m-n+1)$. In other words, the above tri-partition of m has a (1 : 1) correspondence with a bi-partition of

$$m-\frac{(m-n)(m-n+1)}{2},\ (\text{or } m \text{ if } m=n),$$

consisting of an indefinite partition on one side and a partition of unrepeated integers on the other ($\gamma_1,\ldots\gamma_k$). Such a bi-partition (on removing the line of demarcation) is an indefinite partition; and, conversely, every indefinite partition involving θ different integers gives rise as follows to $(1+1)^\theta$ such bi-partitions, the number of those involving even and odd values of k being respectively the positive and negative parts of the expansion of $(1-1)^\theta$, which are equal: namely, first, the indefinite partition itself ($k=0$); second, the θ bi-partitions obtained by placing each of the θ integers successively on the k side ($k=1$); third, the $\frac{1}{2}\theta(\theta-1)$ bi-partitions obtained by placing the $\frac{1}{2}\theta(\theta-1)$ pairs of the θ integers successively on the k side ($k=2$), and so on.

The only exception to this equality of the number of partitions for even and odd values of k is when the partible number,

$$m - \frac{(m-n)(m-n+1)}{2} \text{ or } m,$$

is zero, for which case there is but one bi-partition $[0]+[0]\,(k=0)$. Q.E.D. The tri-partition of m corresponding to the celibate case reduces to the natural sequence above subtracted whose content is

$$\frac{(m-n)(m-n+1)}{2} \text{ (or 0)},$$

which is the second or the first partition (according as $m <$ or $> n$), the others being wanting."

(63) The same infinitesimal method which applied to the expansion of $\Theta_1 x$ gives rise as was shown to the expression for the cubes of the successive rational binomial functions may be applied to the development of

$$(1+ax)(1+ax^2)(1+ax^3)\ldots$$

given in Art. (35), but will not lead to any new result. Making $a = -x^{-1-\epsilon}$, where ϵ is infinitesimal, we obtain from the general theorem

$$(1-x^{\epsilon})(1-x)(1-x^2)(1-x^3)\ldots$$

$$= 1 - \frac{1-x^{\epsilon}}{1-x}x + \frac{1-x^{\epsilon}.1-x}{1-x.1-x^2}x^5 - \frac{1-x^{\epsilon}.1-x.1-x^2}{1-x.1-x^2.1-x^3}x^{12}\ldots$$

$$-x^{\epsilon} + \frac{1-x^{\epsilon}}{1-x}x^3 - \frac{1-x^{\epsilon}.1-x}{1-x.1-x^2}x^9\ldots,$$

or $$(1-x)(1-x^2)(1-x^3)\ldots = 1 - \frac{x-x^3}{1-x} + \frac{x^5-x^9}{1-x^2}\ldots$$

$$= 1 - x(1+x) + x^5(1+x^2)\ldots,$$

the same equation as results from writing $a=-1$.

To arrive at any new result it would be necessary to have recourse to processes of differentiation; the above calculation serves, however, as a verification if any were needed of the accuracy of the theorem to which it refers.

(64) Since sending what precedes to press I have thought it would be desirable in the interest of sound logic to set out the marks or conditions of the several species of the arrangements of unrepeated A, B, C's, somewhat more fully and explicitly than before. And first, I may observe that since it has been convenient to understand that when there are no X terms $\underset{.}{X}$ shall signify zero, the quantitative equation $\underset{.}{X}=0$ dispenses with the necessity of

using the symbolical one $X=0$, and in like manner $\underset{.}{X}>0$ supersedes the symbolical inequality $X>0$, and, of course, the same remark extends to the equality or inequality $X+Y=$ or >0.

We have then for what I shall term the first, second and third species of genus 1, the conditions

$$\underset{.}{C}+\underset{.}{B}+\underset{.}{A}=0,\quad \underset{.}{C}+'\underset{.}{B}+\underset{.}{A}=b,\quad \underset{.}{C}+\underset{.}{B}+\underset{.}{A}_1=0$$

respectively—the first, the trivial case of vacuous content; the second, of only a complete natural B progression, that is, one ending with b (the minimum value of B), and the third, the same for A similarly ending with the minimum a. In what follows the conditions in each separate *line* are to be understood to be not disjunctive but simultaneous or accumulative; they of course refer to the species of the second genus.

Marks of species (1) (α) $\underset{.}{B}-b>0$,
or (β) $\underset{.}{B}-b=0,\ '\dot{B}-'\underset{.}{B}=>\underset{.}{C}-c,\ \underset{.}{C}>0$.
" " (2) (α) $\underset{.}{B}-b=0,\ \underset{.}{C}-c>'\dot{B}-'\underset{.}{B}$,
or (β) $\underset{.}{B}-b=0,\ \underset{.}{C}=0\ [\underset{.}{A}>0]$,
or (γ) $\underset{.}{B}=0,\ \underset{.}{A}-a=>\underset{.}{C},\ \underset{.}{C}>0$,
or (δ) $\underset{.}{B}=0,\ \underset{.}{A}=0\ [\underset{.}{C}>0]$.
" " (3) (α) $\underset{.}{B}=0,\ \underset{.}{C}>\underset{.}{A}-a,\ \underset{.}{A}>0$,
or (β) $\underset{.}{B}=0,\ \underset{.}{C}=0\ [{}_1\underset{.}{A}-a>0]$.

The three inequalities included in brackets are only required in order to exclude arrangements belonging to the first genus. Leaving these out of account for the moment, merely for the sake of greater concision of statement, it is easy to see by mere inspection of the above table that the three species are mutually exclusive and share between them the total sphere of possibility, for (1) α exhausts the hypothesis of there being other B's besides those forming a complete natural progression, (1) β and (2) α of the B's forming such progression when there are existent C's, and (2) β when there are not. Also ((2) γ, (2) δ), (3) α exhaust between them the hypothesis of there being no B's when there are some existent C's, and (3) β of neither B's nor C's appearing in an arrangement.

Thus all unexceptional arrangements must bear the marks occurring in one or the other of the first four lines of the table, and all those where no B's occur, either of the last line when there are neither B's nor C's, and of the three preceding ones when there are no B's but some C's, and the total sum of these hypotheses plus the hypothesis of the first genus together make up necessity, as was to be shown.

The convention $\underset{.}{X}=0$ when an arrangement contains no X with the consequent reduction of the conditions to a purely quantitative form has lent itself very advantageously to the above bird's-eye view of the completeness of the scheme (as covering the whole ground of possibility); it also will be found to simplify the expression of the proof. I did not employ it until the necessity for so doing forced itself upon my notice, for a very obvious reason, namely that X is a B (or an A), which is defined to be congruous to b (or a) [mod c], which zero is not: there is thus an apparent paralogism in admitting that any X of these two *where there is a* B (or when there is an A) is congruent to b (or to a), but that *when there is no* B (or no A) then the conventional least B (or A) is zero. It will be seen, however, *ex post facto*, that no inconvenience in working the scheme results from this extended definition which constitutes an important gain to the perfect evolution of the method. It is usually in the form of some apparent contradiction or paradox that a scientific advance makes its first appearance.

(65) Aided by this clearer and fuller expression of the definitions of the genera and species, I will now set out a logical proof that the respective operators fulfil the three additional necessary conditions. I may observe preliminarily that the Greek letterings α, β; $\alpha, \beta, \gamma, \delta$; α, β, do not express sub-species, for one distinguishing mark of species (or sub-species) may be taken to be that conjugation cannot take place except between individuals of the same species or sub-species, but it will be presently seen that individuals belonging to the differently lettered divisions of the above species are susceptible of mutual conjugation—and are therefore in conformity with biological precedent to be regarded as mere varieties. Besides these varieties of each of the species there is another entirely different principle of cross classification applicable to each of them, namely in general an arrangement must belong to one of sixteen groups designated by combining together one out of each of the four pairs of opposite symbols X, C; x, c; O, E; o, e, where the large O, E refer to the oddness or evenness of the major, and the small o, e to the same for the minor parameter; and in like manner the large X and large C to the result of the operation appropriate to any arrangement, being to extend or contract the major, and x, c to extend or contract the minor parameter. There are thus eight pairs of groups, and conjugation can only take place between individuals belonging to the same pair.

The pairs are as follows:

$$\binom{XxOo}{CcEe},\ \binom{XxOe}{CcEo},\ \binom{XxEo}{CcOe},\ \binom{XxEe}{CcOo};$$

and

$$\binom{XcOo}{CxEe},\ \binom{XcOe}{CxEe},\ \binom{XcEo}{CxOe},\ \binom{XcEe}{CxOo}.$$

Species (1) and species (3) it will be seen may each be separately divided into four sub-species denoted by the upper four, and species (2) into the four sub-species denoted by the lower four pairs of combined characters, so that there will be in all twelve (and not as might at first be supposed twenty-four)

sub-species of conjugable arrangements. The different sub-species of the same species do not admit of cross-conjugation; it is the property which they have in common of being subject to the same law of transformation when passage is made from an individual to its conjugate, which binds them together into a single species. In the arrangements peculiar to Euler's problem, we see that there was no division of the second genus at the outset, but that a separation would be made of it into two pairs of groups with conjugation possible only between individuals belonging to the same pair, and consequently there may be said in this case to be two species of the second genus, analogous, however, not to the species but the sub-species in the more general theory. The final separation of a pair of groups into its component elements has nothing to do with the concept of species, sub-species or variety, but may be regarded as similar to the separation of the sexes.

In what follows, a bracket enclosing a letter will be used to denote that it belongs to an arrangement after it has been operated upon by its appropriate operator, or what may be called its operate.

Species (1). When $\underset{\cdot}{B}-b>0$, if $\underset{,}{C}-c>{}'\dot{B}-{}'\underset{\cdot}{B}$ or $\underset{,}{C}=0$, ${}'\phi$ may be performed, giving $[\underset{,}{C}]={}'\underset{\cdot}{B}-{}'B+C<\underset{,}{C}$ so that the law of descending magnitude is maintained; we have then $[\dot{B}]-[{}'\underset{\cdot}{B}]=$ or $>\dot{B}-{}'\underset{\cdot}{B}=>[\underset{,}{C}]-c$; hence ϕ' has to be performed and will obviously restore the original arrangement. Again if in the original arrangement $\dot{B}-{}'\underset{\cdot}{B}=>\underset{,}{C}-c$ and $\underset{,}{C}>0$, ϕ' has to be applied; a resolution of $\underset{,}{C}$ can take place into c's and the $\underset{,}{C}/c$ first ${}'B$'s, and will each be increased by c and $[\dot{B}]-{}'[\underset{\cdot}{B}]=\underset{,}{C}-c$, so that either $[\underset{,}{C}]=0$ or $[\underset{,}{C}]-c<\underset{,}{C}-c<[\dot{B}]-{}'[\underset{\cdot}{B}]$, and ${}'\phi$ being applicable to the new arrangement will convert it back to the original one.

First Species (β). When $\underset{\cdot}{B}-b=0$ and $\dot{B}-{}'\underset{\cdot}{B}=>\underset{,}{C}-c$ and $\underset{,}{C}>0$, ϕ' can be performed, and the new arrangement as before may be operated upon by ϕ' and so brought back to its original value. If $\underset{,}{C}=0$ or $\underset{,}{C}-c>\dot{B}-{}'B$, ${}'\phi$ could not be performed, for then $\underset{\cdot}{B}=b$ and has no c to part with to help make up $[\underset{,}{C}]$.

These two hypotheses belong to Species (2), which we will now proceed to consider throughout its full extent. When $\underset{\cdot}{B}-b=0$, then ${}'\underset{\cdot}{B}=b$, and I shall first suppose [(α) and (β)] that $\underset{,}{C}=0$ or $\underset{,}{C}-c>B-b$. When $\underset{,}{C}=0$ or $\dot{B}+\underset{\cdot}{A}>\underset{,}{C}$, then ${}'\psi$ will be applicable, making $[\underset{,}{C}]=\dot{B}+\underset{\cdot}{A}$; if now $[B]>0$ and $[\underset{\cdot}{A}]>0$, $[B]+[\underset{\cdot}{A}]=>(\dot{B}-c)+(\underset{\cdot}{A}+c)=>\dot{B}+\underset{\cdot}{A}=>[\underset{,}{C}]$, and

$$[\underset{,}{C}]-c=\dot{B}+\underset{\cdot}{A}-c=[B]+\underset{\cdot}{A}>[\underset{\cdot}{B}]-b.$$

Hence we are still within Species 2 and have fallen upon the case to which the reversing operator ψ' has to be applied. If $[\underset{\cdot}{B}]=0$, $[\underset{\cdot}{A}]=0$ we must have $B[\underset{,}{C}]>0$, inasmuch as the original content (or inertia) is originally greater than zero and is kept constant, and this is a case which still belongs to Species 2 and falls under the operation of ψ'.

If $[\underset{\cdot}{B}]=0$ so that $\dot{B}=\underset{\cdot}{B}=b$ and $[\underset{\cdot}{A}]>0$, then

$$[\underset{\cdot}{A}]-a=>\underset{\cdot}{A}+c-a=>\underset{\cdot}{A}+\dot{B}=>\underset{\cdot}{C},$$

which also falls within the second species and is amenable to the reversing operator ψ'.

Finally, if $[\underset{\cdot}{B}]>0$, that is $\underset{\cdot}{B}-b=0$ and $[\underset{\cdot}{A}]=0$,

$$[\underset{\cdot}{C}]-c=\dot{B}+\underset{\cdot}{A}-c=>[\dot{B}]-b,$$

that is $=>[\dot{B}]-{}'\underset{\cdot}{B}$, and we are still within Species (2) and in the case amenable to the reversing operator ψ'.

If now on the other hand we begin with an arrangement of the second species in the case amenable to ψ' we must suppose either $\underset{\cdot}{B}=0$ or $\underset{\cdot}{A}=0$, or else $\underset{\cdot}{C}>0$ and $\underset{\cdot}{C}<=\dot{B}+A$.

Take first this last supposition. The operation of ψ' gives $[\underset{\cdot}{C}]=>\underset{\cdot}{C}+c$,

$$[\dot{B}]=\dot{B}+c \text{ and } [\underset{\cdot}{A}]=\dot{C}-c-\dot{B}>\dot{B}-b-\dot{B}>-b=>c-b=>a.$$

And $\quad [\dot{B}]+[\underset{\cdot}{A}]=\dot{B}+\underset{\cdot}{C}-\dot{B}=\underset{\cdot}{C}<[\underset{\cdot}{C}].$

$$[\underset{\cdot}{C}]-c=>(\underset{\cdot}{C}-c)+c=>B-b+c=>[B]-[\underset{\cdot}{B}].$$

Hence the operate is licit, belongs to the second species and is amenable to the reversing operator $'\psi$.

If $\underset{\cdot}{B}=0$ and $\underset{\cdot}{A}=0$, $[\dot{B}]=[\underset{\cdot}{B}]=b$ and $[\underset{\cdot}{A}]=\underset{\cdot}{C}-b$ and $[\underset{\cdot}{C}]=0$ or $>\underset{\cdot}{C}$.

If $[\underset{\cdot}{C}]=0$ since $[\underset{\cdot}{A}]>0$, the operate is included in variety (β) of the second species and amenable to the reversing operator $'\psi$, and if

$$[\underset{\cdot}{C}]>\underset{\cdot}{C}\,[\underset{\cdot}{C}-c]>\underset{\cdot}{C}-c>0,$$

that is $>[\dot{B}]-\underset{\cdot}{B}$ which belongs to variety (α) of the second species; and since $[\underset{\cdot}{C}]>\underset{\cdot}{C}>[\dot{B}]+[\underset{\cdot}{A}]$ is amenable to the reversing operator $'\psi$.

If $\underset{\cdot}{B}>0$ and $\underset{\cdot}{A}=0$, then $\underset{\cdot}{C}>0$ [otherwise it would be an arrangement in Genus 1, Species 2] $[\underset{\cdot}{C}]=0$ or $>\underset{\cdot}{C}$, $[\dot{B}]=\dot{B}+c$,

$$[\underset{\cdot}{A}]=\underset{\cdot}{C}-[\dot{B}]>(c+\dot{B}-b)-(c+\dot{B})=>a,$$

and either $[\underset{\cdot}{C}]=0$ and $[\underset{\cdot}{A}]>0$ or

$$[\underset{\cdot}{C}]-c>(\underset{\cdot}{C}-c)+c>\dot{B}+c-b>[\dot{B}]-{}'\underset{\cdot}{B}$$

and $[\underset{\cdot}{A}]+[\dot{B}]=\underset{\cdot}{C}>[\underset{\cdot}{C}]$. Hence in either hypothesis the operate is still in Species (2) and amenable to the reversing operator $'\psi$.

Lastly, if $\underset{\cdot}{B}=0$, $\underset{\cdot}{A}-a=>\underset{\cdot}{C}$ and $\underset{\cdot}{C}>0$, the arrangement is amenable to the operator ψ', which will make $[\underset{\cdot}{B}]=b$, $[\underset{\cdot}{A}]=\underset{\cdot}{C}-b<\underset{\cdot}{C}+a<\underset{\cdot}{A}$. We have then $[\underset{\cdot}{B}]-b=0$ and $[\underset{\cdot}{C}]=0$, and consequently also $\underset{\cdot}{A}>0$ or

$$[\underset{\cdot}{C}]-c>\underset{\cdot}{C}-c>0,$$

that is $>[\dot{B}]-{}'[B]$, and the result is still contained within Species (2) and is amenable to the reversing operator $'\psi$.

(66) The following are examples of paired arrangements belonging to the first species, adapted to the case of $a=2$, $b=1$. The C and B terms are

expressed; the A line is the same for each of any pair of this species, and may be filled in at will.

$$\phi'\begin{Bmatrix} X.9. \\ 16.13.10.Y \end{Bmatrix}=\begin{Bmatrix} X. \\ 19.16.13.Y \end{Bmatrix}$$

where X, Y represent any licit series of C's and B's respectively.

$$'\phi\begin{Bmatrix} X.9 \\ 16.13.7.Y \end{Bmatrix}=\begin{Bmatrix} X.9.6. \\ 13.10.7.Y \end{Bmatrix} \quad \phi'\begin{Bmatrix} X.9 \\ 16.13.10.4 \end{Bmatrix}=\begin{Bmatrix} X. \\ 19.16.13.4 \end{Bmatrix}$$

$$\phi'\begin{Bmatrix} X.9 \\ 7.4.1 \end{Bmatrix}=\begin{Bmatrix} X. \\ 10.7.4 \end{Bmatrix} \quad '\phi\begin{Bmatrix} 10.7.4 \end{Bmatrix}=\begin{Bmatrix} 9. \\ 7.4.1 \end{Bmatrix}$$

$$\phi'\begin{Bmatrix} 3. \\ 13.7.4.1 \end{Bmatrix}=\begin{Bmatrix} 16.7.4.1 \end{Bmatrix}.$$

The following are examples of paired arrangements of the second species with $a=2$ and $b=1$ as usual.

$$'\psi\begin{Bmatrix} X.12. \\ 7.4.1. \\ Y.2 \end{Bmatrix}=\begin{Bmatrix} X.12.9. \\ 4.1 \\ Y \end{Bmatrix} \quad \psi'\begin{Bmatrix} X.12. \\ 7.4.1. \\ Y.5 \end{Bmatrix}=\begin{Bmatrix} X \\ 10.7.4.1. \\ Y.5.2 \end{Bmatrix}$$

$$'\psi\begin{Bmatrix} 7.4.1. \\ Y.5. \end{Bmatrix}=\begin{Bmatrix} 12. \\ 4.1. \\ Y \end{Bmatrix} \quad \psi'\begin{Bmatrix} X.15 \\ 7.4.1 \\ Y.8 \end{Bmatrix}=\begin{Bmatrix} X. \\ 10.7.4.1 \\ Y.8.5 \end{Bmatrix}$$

$$\psi'\begin{Bmatrix} X.9. \\ \dots. \\ \dots. \end{Bmatrix}=\begin{Bmatrix} X. \\ 1. \\ 8 \end{Bmatrix} \quad \psi'\begin{Bmatrix} 6. \\ 1. \\ 8 \end{Bmatrix}=\begin{Bmatrix} \dots \\ 4.1. \\ 8.2 \end{Bmatrix}$$

$$\psi'\begin{Bmatrix} X.9. \\ \dots \\ Y.11 \end{Bmatrix}=\begin{Bmatrix} X. \\ 1. \\ Y.11.8 \end{Bmatrix}.$$

We come now to the third species. Here, I think, the reader will find it a great relief to the strain upon his attention if I invite him before attacking the demonstration to consider the annexed diagrammatic cases accommodated to the supposition $a=2$, $b=1$. The B's it will be remembered in this species do not exist, and the action neither of $'\vartheta$ nor ϑ' introduces any B into the transformed arrangement. In the examples given below the C and A terms occupy the higher and lower lines respectively—the comma is used in the latter to mark off the ${}_1A$'s from the A_1's.

$$'\vartheta\begin{Bmatrix} 9.6. \\ 14.11.8.5, \end{Bmatrix}=\begin{matrix} 9.6.3. \\ 14.11.8,\ 2 \end{matrix} \quad \vartheta'\begin{Bmatrix} 6.3. \\ 14.11.8,\ 2 \end{Bmatrix}=\begin{matrix} 6. \\ 14.11.8.5, \end{matrix}$$

$$'\vartheta\left(17.8.5\right)=\begin{matrix} 3. \\ 17.8,\ 2 \end{matrix} \quad '\vartheta\left(17.8.5,\right)=\begin{matrix} 3. \\ 17.8,\ 2 \end{matrix}$$

$$'\vartheta\left(17,\ 8.5.2\right)=\begin{matrix} 6. \\ ,\ 11.8.5.2 \end{matrix} \quad '\vartheta\left(17.14,\ 8.5.2\right)=\begin{matrix} 3. \\ 17,\ 11.8.5.2 \end{matrix}$$

$$'\vartheta\,11,=\begin{matrix} 9. \\ ,\ 2 \end{matrix} \quad \vartheta'\begin{Bmatrix} 12.9.3. \\ ,\ 11.8.5.2 \end{Bmatrix}=\begin{matrix} 12.9. \\ 14,\ 8.5.2 \end{matrix}$$

$$\vartheta'\begin{Bmatrix} 9.6.3. \\ ,\ 11.8.5.2 \end{Bmatrix}=\begin{matrix} 9.6 \\ 14,\ 8.5.2 \end{matrix}.$$

The left-hand accent is used here as elsewhere to signify that phase of the operator which brings about an increase and the right-hand one a decrease in the number of C's. It will readily be seen that the action of the operator in each of the above examples prepares the arrangement for the action of the contrary one which will restore it to its original value. It is worthy of notice that in any two associated arrangements above, an a (here 2) *may* appear in each and *must* appear in one of them. I will now proceed to the general demonstration.

(67) Let us first suppose $\underset{\cdot}{A}_1 = 0$, then ${}_1\underset{\cdot}{A} > 0$, otherwise we shall be dealing with the antecedent species and ${}'\vartheta$ will be applicable, making $[\underset{\cdot}{A}] = [\dot{A}_1] = a\,[\underset{\cdot}{C}] = \underset{\cdot}{A} - a < \underset{\cdot}{C}$ and $> (\underset{\cdot}{A} - a)$. Thus the generated arrangement is licit and belongs still to the third species; but now $[\underset{\cdot}{C}] + [\dot{A}_1] = \dot{A}$ and $[{}_1\underset{\cdot}{A}] = 0 > \underset{\cdot}{A}$. Hence the reversing operator ϑ' is applicable to the new arrangement; the remaining cases to consider (in which $\underset{\cdot}{A} = a$ for the arrangement as well before as after being operated upon) may be separated into those where $\underset{\cdot}{C} > 0$, and at the same time either $\underset{\cdot}{C} + \dot{A}_1 < {}_1\dot{A}$ or ${}_1\dot{A} = 0$, which are amenable to the operator ϑ' and the complementary cases which are amenable to ${}'\vartheta$.

In the cases first considered $[\dot{A}_1] = \dot{A}_1 - c$, $[{}_1\dot{A}] = \underset{\cdot}{C} = \underset{\cdot}{A}_1\,\vartheta\,[\underset{\cdot}{C}] + 0$ or $> \underset{\cdot}{C}$ (and *à fortiori* > 0), consequently the new arrangement is licit and still belongs to the third species, and since either $[\underset{\cdot}{C}] = 0$ or else

$$[\underset{\cdot}{C}] + [\dot{A}_1] > \underset{\cdot}{C} + \dot{A}_1 - \underset{\cdot}{C} = > [{}_1\dot{A}]$$

and $[{}_1\dot{A}] > 0$, it is one of the complementary cases and is subject to the reversing operator ${}'\vartheta$.

Again, any arrangement for which $\underset{\cdot}{A} = a$ belonging to the complementary cases is defined by the conditions ${}_1\dot{A} > 0$ and $\underset{\cdot}{C} + \dot{A}_1 = > {}_1\underset{\cdot}{A}$ and is by hypothesis to be subjected to the operator ${}'\vartheta$ which will make $[\dot{A}_1] = \dot{A}_1 + c$, $[{}_1\underset{\cdot}{A}] = 0$ or $> {}_1\underset{\cdot}{A}\,[\underset{\cdot}{C}] = {}_1\underset{\cdot}{A} - \dot{A}_1 - c$, and since $\underset{\cdot}{C} = > {}_1\underset{\cdot}{A} - \dot{A}_1$, $[\underset{\cdot}{C}] < \underset{\cdot}{C}$, so that the operation leads to a licit new arrangement.

Also $[\underset{\cdot}{C}] + [\dot{A}_1] = {}_1\underset{\cdot}{A}$, and consequently either $[{}_1\underset{\cdot}{A}] = 0$ or $[\underset{\cdot}{C} + \dot{A}_1] < [{}_1\underset{\cdot}{A}]$, which is a condition belonging to the first considered class of cases, subject to the reversing operator ϑ', and thus for the third as for both the antecedent species of the second genus, it has been proved that each designated operator prior to any arrangement being performed does not take away its licit character nor carry it out of the species to which it belongs, and on being repeated brings it back to its original form, and that the effect of any single operation is to maintain the content (or inertia) of the arrangement constant but to reverse each of its characters. This is the thing that was to be proved and brings my wearisome but indispensable task to an end.

(68) Another and perhaps somewhat clearer image of the classification of the numbers of the second Genus may be presented as follows: The combinations of the characters *XCOExcoe* give rise to eight pairs of groups, say eight classes. Of these classes four belong to Species 2, and may be represented by four indefinite vertical parallelograms, set side to side, and subdivided each of them into four, (say) black, white, grey and tawny stripes, corresponding to the four varieties of the second species. The other four classes may be similarly represented by four such parallelograms as before, but separated by a transverse horizontal line into eight sub-classes, four corresponding to the first species and four to the second. The upper parallelograms may then be each divided into blue and green, the lower into yellow and red stripes to represent the respective couples of varieties of the first and third species. There will thus be in all thirty-two stripes, namely four blue, green, yellow and red, and four black, white, grey and tawny, each of which is bifid, representing two groups of opposite sexual characters, which may be fittingly represented by the upper and under sides of the sixteen unlimited single-coloured stripes of the first and the eight unlimited double-coloured stripes of the second set of parallelograms.

The above logical scheme is not intended to convey any notion of the relative frequency of the three species. The general case is that of the first species. The second is conditioned by ${}'B = b$ or $B = 0$, and the third by $B = 0$. When ${}'B = b$ it is about an even chance whether the arrangement is of the second or first species, and when $B = 0$ of the second or third. Either equality is a particularization of the B series, the latter signifying that there are no B's in the arrangement, the former that there are B's descending in rational progression down to b: this supposition is apparently infinitely more general than the former, because there is no limit to the number of terms in the progression, and the case of a natural progression of B's of the kind mentioned with any given number of terms as regards the probability of its occurring in an arrangement seems to be on a par with the case of the B's being all wanting. Hence the first species is infinitely more frequent than the second, and the second than the third. According to Prof. Max Müller's theory of the relation of thought to language (if I interpret it rightly) I ought to have thought out my divisions and schemes of operation in language, but I certainly had formed in my mind a dim abstract of them before I had found the language that was competent to give them expression.

In conclusion, I may remark that whilst the experience of the past indicated the probability that there did exist (if one could find it) a method of distributing the arrangements of the second genus into pairs, in such a way that in each pair the total or partial character should be reversed in passing from the one to the other, there was nothing to induce a reasonable degree of assurance that both those characters should be found simultaneously reversed in one and the same distribution; for aught that could have been foreseen to the contrary, it might very well have happened that one mode of distribution might have been needed to prove Jacobi's theorem for the case of only negative signs appearing in the factors on the left-hand side of the equation, and a different one for the other case where only every third factor contains such sign—indeed upon the principle of *divide et impera* or doing one thing at a time (as invaluable a maxim to the algebraist as to the politician) I had completed the proof for the former case without thinking of the latter, and only when on the point of attacking it was agreeably surprised to find that there was nothing left to be done, for that the proof found for the one extended to the other—in familiar phrase, I had hit two birds with one stone. We may now ask whether this was a happily found chance solution or was predestined by the nature of things, and that *simple* necessarily implies *double* enantiotropy of conjugation. Probably I think not, and if so, a question arises as to the number of solutions for each of the two sorts of enantiotropy and whether the number of each kind of simply-enantiotropic conjugations is the same.

Viewed merely as a question of direct multiplication, I think it must be allowed that what I have here called Jacobi's theorem (including Euler's marvellous one, as the ocean a drop of water) is the most surprising revelation that has been made in elementary algebra since the discovery of the general binomial theorem, and that the space devoted to its independent, and so to say, materialistic proof in these pages, although considerable, is not out of proportion to its intrinsic importance.

H. *Intuitional Exegesis of Generalized Farey Series**.

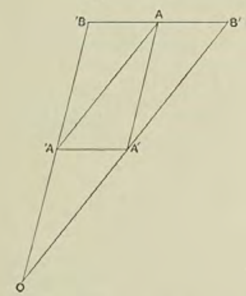
(69) The demands of the press will only admit of a rapid sketch of what appears to me to be the true underlying principles of the theory initiated by Farey, honoured by the notice of Cauchy, and to a certain extent generalized by Mr Glaisher, whose inductive method in the cases treated by him finds its full development in the method of continuous change of boundary, explained in the course of what follows. Let us start from the conception of an infinite cross-grating formed by two orthogonal systems of parallel lines in a plane, the distance between any two parallels being made equal to unity. The intersections of any two lines of the grating may, as heretofore, be termed nodes. A triangle which has nodes at its apices and at no other point on or within its periphery, may be termed an elementary triangle, and the double of the area of any such triangle will be unity. If any finite aggregate of nodes be given it must be possible to pick out a certain number of them which may be formed together by right lines so as to form a sort of ring-fence, within which all the rest are included: the area thus formed, if it

* Continued from note G, Interact, Part 2.

admits of being mapped out into elementary triangles, may be termed a *complete* nodal aggregate. Any other contour consisting of lines of any form (curved or straight) drawn outside of this ring-fence in such a manner that no nodes occur between the two, may be termed a regular contour.

If any node O be taken as origin and any nodal lines through O as axes of coordinates, and if $'A$, A' are the nearest nodes to O in the radial lines on which they lie, and if no nodes of the given aggregate are passed over as an indefinite line rotating round O, passes from one of these radial lines to the other, $'AOA$ is an elementary triangle, and if $'p, 'q$; p, q be the coordinates of $'A$, A respectively, $'pq - p'q = \epsilon$ where ϵ is $+1$ or -1 but is fixed in sign when the direction of the rotation is given.

When the aggregate is *complete*, if the values of the coordinates of the successive points passed over by the rotating line be called ... $''p, ''q$; $'p, 'q$; p, q; p', q', p'', q''; ..., we shall have a Farey series formed by the successive couples p, q, that is $p''{}'q - p'q'' = \epsilon$; $p'q - pq' = \epsilon$; $pq' - p'q = \epsilon$.... Thus we see that the Farey property is invariantive in the sense of being independent of the position of the origin.



Next I say, that if any contour to a given aggregate is regular, every contour similar thereto in respect to any node of the aggregate regarded as the centre of similitude is also regular, provided the boundary is simple; meaning that there are no interior limiting lines giving rise to holes or perforations in aggregate, and no loops formed by the boundary cutting itself.

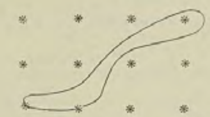
In the above figure $'BOB'$ is any triangle whose sides are bisected in $'A$, A, A'. Suppose O to be the origin, $'A$, A' two nodes of greatest proximity to O successively passed over by the rotating line for a given contour. As this contour expands uniformly in all directions through O, the line $'AA'$ remains parallel to itself. Since $'AOA'$ is an elementary triangle so also must the similar triangles $'AAA'$, $A'AB'$, $'AA'B$ be all elementary, consequently A will be the first new node intervening between $'A$, A' brought into the enlarged aggregate as $'AA'$ moves continuously parallel to itself, and $'AOA$, AOA' will be elementary triangles; it may be noticed in order to bring this method into relation with that indicated by Mr Glaisher, that the coordinates of this new node A are the sums of the coordinates of its neighbours $'A$, A'. If the contour were not supposed to be simple, this condition could not be drawn; for if there were a hole round the middle point of $'AA'$ the node A would be missing in the enlarged aggregate, and if the first node to intervene as the contour went on enlarging be called (A), $'AO(A)$ or $(A)OA'$ or each of them would be a multiple of the elementary triangle, so that the constancy of the value of the successive determinants would no longer hold. In like manner it will be seen that on the same supposition as above made, if in consequence of the contour contracting about O as the centre of similitude, two points $'A$, A' which originally are non-contiguous, at any moment become contiguous, at the moment previous to this taking place A (and no other point) must have intervened, and after A has disappeared from the reduced aggregate, no other point can make its appearance between $'A$, A'.

(70) Hence we may contract at pleasure the given contour about any node as origin, and if the contour so contracted contains at least one node besides the origin, it will suffice to determine whether the given contour is or is not regular.

Thus for example in the case of a triangle limited by the axes and by the right line $x + y = n$, we may make $n = 1$ and the trial series will then become $\frac{0}{1}\ \frac{1}{1}\ \frac{1}{0}$ which possesses the Farey property. Hence this will hold good for a triangular boundary of any size and wherever the origin is situated: this includes the case of the ordinary Farey series when the origin is taken at either extremity of the hypothenuse. So again for the area contained within the axes and the hyperbola $xy = n$, we may take $xy = 1$ and the trial series is the same as before.

(71) It is easy to form *unperforated* areas of any magnitude which shall not satisfy the Farey law: for example we may as in the annexed figure draw a curve passing through the origin, the point (0, 1), and the point (2, 3). $\frac{0}{1}$, $\frac{2}{3}$ does not satisfy the Farey law, and consequently no similar contour obtained by treating any one of the three nodes which it contains as a centre of similitude will be a "complete contour," and the successive values of (p, q)

obtained by the rotation of a line round the origin in such contour will not constitute a Farey series.



The theory will, I believe, admit of being extended to solid reticulations, formed by the intersections of three systems of equidistant parallel planes, determinants of the third order between the three coordinates of successive points, replacing the $pq' - p'q$ of the plane theory. The chief difference will consist in the introduction of a new element in the multiplicity of the "normal orders" in which a given set (of points in a plane or) of radii *in solido* may be taken. (Points in a plane arranged in any order of sequence, such that the successive determinants formed by their trilinear coordinates are of uniform sign, are said to be in a normal order. Rays of a conical pencil arranged in any order of sequence, such that their intersections by a plane satisfy the above condition, are also said to be in a normal order: see privately printed syllabus* of my lectures on Partitions, 1859, or M. Halphen's theory of *Aspects*.) But as far as I can see this will in no way militate against the existence of the laws of invariance and similitude established for the case of a plane reticulation, but will only introduce a further principle of invariance, namely that the law of unit-determinants if satisfied by one normal arrangement of the points of the solid reticulation will be satisfied by every other.

APPENDIX†.

LIST OF CORRECTIONS SUGGESTED BY M. JENKINS TO PROFESSOR SYLVESTER'S CONSTRUCTIVE THEORY OF PARTITIONS.

Page 5, 5 lines from end, $2n - (i + 3)$ should be $n - (i + 3)$.
„ 6, between 2nd and 3rd rows of sinister table insert 13 . 2 . 0.
„ „ „ 7th and 8th „ „ „ „ 11 . 2 . 2.
„ „ in 6th row of dexter table, for 8 . 4 . 3 (2) write 8 . 4 . 3 (1).
„ 11, line 8 from the end, interchange protraction and contraction so as to read "*contraction* could not now be applied to A' and B' nor *protraction* to C'."
„ 13, line 25. If $f(x) = (1 - x)(1 - x^3)(1 - x^3)(1 - x^7)(1 - x^9)$, for the second x^3 read x^5.

[* Vol. II. of this Reprint, p. 119.]

[† These corrections have been included in those made in the text preceding.]

Page 13, line 29, for "latter" read "former."

" 15, line 11 from end, for l^r read l^λ.

" 20, line 4, for $1+2$ read $i+2$.

" " line 5, for $1+2$ read $i+2$.

" 22, line 11, for $X_j x^{\frac{i^2+i}{2}}$ read $X_j x^{\frac{j^2+j}{2}}$.

" " line 20, for "the minimum negative residue of $i-1$" read $i+1$.

" 25, line 7, for $\frac{x^{\frac{1}{2}n(n+1)}}{1-x^n}$ read $\frac{x^{\frac{1}{2}r(r+1)}}{1-x^r}$.

" " line 4 from the end, for "to the 5th now" read "to the 5th row now."

" 27, line 15, for 15, 7, 3 read 13, 11, 3.

" " line 19, for $(1+ax)(1-ax^3)(1-ax^j)\dots$ read $(1+ax)(1+ax^3)\dots(1+ax^{2j-1})$.

" " line 22, for $\frac{x}{1-x}\alpha$ read $\frac{x}{1-x^2}\alpha$.

" " line 30, for "angle whose *nodes* contain i nodes" read whose *sides*.

" 28, line 5, for "with $j-i$ or fewer parts" read $j-1$.

" " line 12, for $1+\frac{1-x^{\omega+1}}{1-x^2}x^{\omega}+\frac{1-x^{\omega+1}\,.\,1-x^{\omega+3}}{1-x\,.\,1-x^4}x^{\omega+1}$ etc.

read $x^{\omega}+\frac{1-x^{\omega-1}}{1-x^2}x^{\omega+1}+\frac{1-x^{\omega-1}\,.\,1-x^{\omega-3}}{1-x^2\,.\,1-x^4}x^{\omega+4}+$ etc.

If in the expression in line 9, namely in

$$\frac{1-x^{2i-2j+2}\,.\,1-x^{2i-2j+4}\dots1-x^{2i-2}}{1-x^2\,.\,1-x^4\dots1-x^{2j-2}}x^{j^2-2j+2i},$$

we put $j=3$ we obtain

$$\frac{1-x^{2i-4}\,.\,1-x^{2i-2}}{1-x^2\,.\,1-x^4}\,.\,x^{9-6+2i}=\frac{1-x^{2i-2}\,.\,1-x^{2i-4}}{1-x^2\,.\,1-x^4}\,.\,x^{2i+3}$$

$$=\frac{1-x^{\omega-1}\,.\,1-x^{\omega-3}}{1-x^2\,.\,1-x^4}\,.\,x^{\omega+4},$$

since $\omega=2i-1$, and similarly for other terms when we put $j=2$ and $j=1$.

The correction which I offer seems to me to be right, and the expression in the paper to give a wrong result in the case when n happens to be equal to $\omega+2$; for then the number of parts being supposed to be exactly i, the first bend contains $2i-1$ or ω nodes, and there is then no way of placing the remaining 2 nodes so as to make the partition a conjugate partition—supposing I have not misunderstood the article.

Page 29, line 8, for 19, 7, 6, 6 read 10, 7, 6, 6.

" " figure, either insert a node at junction of 5th column and 7th row or remove a node from junction of 7th column and 5th row.

" " lines 7 and 8 from the bottom, if we remove a node from the figure no change is required in these two lines; but if we

insert a node in the figure, then 11 11 11 7 3 3 should be 11 11 11 7 5 3 and 5 5 5 3 1 1 should be 5 5 5 3 2 1.

Page 31, line 15 from end, after $\frac{1}{1-ax\,.\,1-ax^2\dots1-ax^\theta}$ insert "or of $x^n a^j$."

" 34, line 7, for a^j read a^θ.

" " line 8, for $(x^\theta+ax^{1\theta})\}$ read $(x^\theta+x^{2\theta})$.

" 36, line 8, for $\frac{l_1(2-j-1)}{2}$ read $\frac{l_1-(2j-1)}{2}$.

" 37, line 4, for x^n read $x^{\frac{n}{2}}$.

" " line 7, for x^{2i+1} read x^{2i+2}.

" 40, line 6, a_i-i is, I believe, the right final term; but it appears as if it were the first of a pair instead of the last of a pair, a_i-i being a quantity which may vanish.

If the pair of expressions which in the text precede a_i-i, if definitely expressed and not left to be understood, should be

$$[a_{i-1}+\alpha_{i-1}-(2i-3)],\quad[a_{i-1}+\alpha_{i-2}-(2i-2)],$$

and not as in the text

$$[a_{i-1}+\alpha_{i-1}-(2i-1)],\quad[a_{i-1}+\alpha_i-2i],$$

the factor which should precede a_i-i is $[a_i+\alpha_i-(2i-1)]$.

I do not quite follow lines 9—13 of p. 40, possibly from the oversight in the subscripts I do not see what is intended. But it seems to me the following proof would be right:

The expressions of the same form succeeding $a_1+\alpha_1-1$ and $a_1+\alpha_2-2$ must be continued so long as they are positive, and must be rejected when they become negative.

Now from the fact of i being the content of the side of the square belonging to the transverse graph $a_i=$ or $>i$, $\alpha_i=$ or $>i$, therefore $a_i+\alpha_i-(2i-1)$ is positive and is therefore one of the terms of the series. Also $a_{i+1}=$ or $<i$ and $\alpha_{i+1}=$ or $<i$, therefore $a_{i+1}+\alpha_{i+1}-(2i+1)$ is negative and must consequently be rejected.

The intermediate expression is $a_i+\alpha_{i+1}-2i$; and for this we may in all cases put a_i-i as the last term of the series for the following reason:

If the extreme inside bend have more than one node in the row, then $\alpha_{i+1}=i$ and $a_i+\alpha_{i+1}-2i$ is $=a_i-i$, which is not negative since $a_i=$ or $>i$. If the extreme inside bend degenerate, so that it consists only of a vertical line or of a single point, then $a_i=i$; and since $\alpha_{i+1}<i$ in this case, therefore $a_i+\alpha_{i+1}-2i$ is negative and inadmissible as a term in the series; but since $a_i-i=0$ there is no harm in putting it as the final term in the series.

Page 601, *Vol. III. of this Reprint*, line 6 from the end, for 3100 read 3110.

2.

SUR LES NOMBRES DE FRACTIONS ORDINAIRES INÉGALES QU'ON PEUT EXPRIMER EN SE SERVANT DE CHIFFRES QUI N'EXCÈDENT PAS UN NOMBRE DONNÉ.

[*Comptes Rendus*, XCVI. (1883), pp. 409—413.]

DANS le *Philosophical Magazine*, 1881, p. 175, M. Airy, associé étranger de l'Institut, annonce qu'il a calculé, pour l'usage de l'Institution of civil Engineers, à Londres, les valeurs logarithmiques de toutes les fractions ordinaires $\frac{m}{n}$, dans lesquelles m et n ne contiennent nul facteur commun et n'excèdent pas 100, arrangées dans l'ordre de leurs grandeurs, et que le nombre de ces fractions est 3043.

Je vais montrer qu'on peut appliquer la méthode dont M. Tchebycheff s'est servi dans sa théorie célèbre sur les nombres premiers, avec l'addition que j'y ai faite*, pour trouver des limites supérieures et inférieures au nombre d'un système pareil de fonctions quand la limite des valeurs de m et de n est un nombre quelconque donné.

1. Je dis que si Ti signifie le nombre de nombres inférieurs et premiers à i, nombre entier (ce que nous nommons, à Baltimore, le *totient* de i), on aura l'identité

$$\sum_{r=x}^{r=1}\left(E\frac{i}{r}\,Tr\right)=\frac{i^2+i}{2}.$$

C'est une conséquence du théorème plus général que "si $a_1, a_2, \ldots, a_i$ sont des nombres entiers quelconques, et si l'on nomme le nombre des a qui contiennent r la fréquence de r par rapport au système des a, et qu'on prenne le produit de la fréquence de r par son totient, la somme de ces produits (quand r prend toutes les valeurs de 1 jusqu'à l'infini) sera la somme des a."

* Voir *American Journal of Mathematics*. [Vol. III. of this Reprint, pp. 530, 605, 672.]

2. Nommons Jx la somme-totient de x, c'est-à-dire la somme des totients de tous les nombres qui n'excèdent pas la valeur de Ex (la partie entière de x).

Je me servirai désormais de $\left(\frac{p}{q}\right)$ pour signifier la partie entière de $\frac{p}{q}$.

Or écrivons les suites successives

$$\begin{array}{llllllll} x, & x-1, & \ldots, & \left(\frac{x}{2}\right)+1; & \left(\frac{x}{2}\right), & \left(\frac{x}{2}\right)-1, & \ldots, & \left(\frac{x}{3}\right)+1; \\ \left(\frac{x}{3}\right), & \left(\frac{x}{3}\right)-1, & \ldots, & \left(\frac{x}{4}\right)+1; & \left(\frac{x}{4}\right), & \left(\frac{x}{4}\right)-1, & \ldots, & \left(\frac{x}{5}\right)+1; \\ \ldots, & \ldots\ldots, & \ldots, & \ldots\ldots; & \ldots, & \ldots\ldots, & \ldots, & \ldots\ldots; \\ \left(\frac{x}{2q-1}\right), & \left(\frac{x}{2q-1}\right)-1, & \ldots, & \left(\frac{x}{2q}\right)+1; & \left(\frac{x}{2q}\right), & \left(\frac{x}{2q}\right)-1, & \ldots, & \left(\frac{x}{2q+1}\right)+1; \\ \ldots, & \ldots\ldots, & \ldots, & \ldots\ldots; & \ldots, & \ldots\ldots, & \ldots, & \ldots\ldots; \end{array}$$

q augmentant *ad libitum*.

Je dis que, "si r est un nombre entier quelconque qui se trouve dans les suites d'ordre impair, c'est-à-dire commençant avec x, $\left(\frac{x}{3}\right)$, $\left(\frac{x}{5}\right)$, ..., et si $j=2i$ ou $2i+1$, on aura

$$E\left(\frac{j}{r}\right)-2E\left(\frac{i}{r}\right)=1,$$

et que, si r appartient à une suite quelconque d'ordre pair, on aura

$$E\left(\frac{j}{r}\right)-2E\left(\frac{i}{r}\right)=0."$$

Conséquemment, en appliquant le théorème précédent, on aura

$$\frac{j(j+1)}{2}-2\frac{i(i+1)}{2}=S_1+S_3+\ldots+S_{2q-1}+\ldots,$$

où S_{2q-1} est la somme des totients des nombres qui sont en même temps égaux ou inférieurs à $E\frac{j}{2q-1}$ et plus grands que $E\frac{j}{2q}$, c'est-à-dire

$$S_{2q-1}=J\left(\frac{j}{2q-1}\right)-J\left(\frac{j}{2q}\right).$$

Si donc on écrit

$$\theta x=Jx-J\frac{x}{2}+J\frac{x}{3}-J\frac{x}{4}+J\frac{x}{5}-J\frac{x}{6}+\ldots,$$

on aura, quand $x=$ un nombre entier pair (soit $2i$),

$$\theta x=(2i^2+i)-(i^2+i)=i^2=\frac{x^2}{4},$$

et, quand $x=$ un nombre entier impair (soit $2i+1$),

$$\theta x=(i+1)(2i+1)-(i^2+i)=\frac{(x+1)^2}{4}.$$

Avec l'aide de ces égalités, si x est un nombre positif quelconque entier ou fractionnel, on obtient facilement les inégalités

$$\theta x = \text{ou} > \frac{x^2 - 2x}{4}$$

$$\theta x = \text{ou} < \frac{x^2 + 2x + 1}{4}.$$

En appliquant à ces deux inégalités la méthode d'approximation successive que j'ai appliquée, dans* le Mémoire cité, aux inégalités auxquelles est assujettie la fonction $\psi(x)$ (*voir* Serret, *Algèbre supérieure*, édition de 1879, t. II. p. 233), je parviens facilement et rigoureusement à démontrer que, étant donnée une quantité ϵ aussi petite qu'on veut, on peut trouver une limite supérieure L et une limite inférieure Λ à Jx, où

$$L = \left(\frac{3}{\pi^2} + \eta\right) x^2 - Ax + R(\log x)$$

$$\Lambda = \left(\frac{3}{\pi^2} - \eta'\right) x^2 - A'x + R'(\log x),$$

où $R(\log x)$, $R'(\log x)$ sont tous les deux fonctions rationnelles et entières de $\log x$ d'un degré fini, dont les coefficients aussi bien que A et A' restent toujours finis et où η, η' sont tous les deux plus petits que ϵ.

Il s'ensuit que la fraction $\frac{J(x)}{x^2}$ possède une valeur asymptotique $\frac{3}{\pi^2}$ (ce qui n'est pas démontré pour la fraction analogue $\frac{\psi x}{x}$, dans la théorie parallèle de M. Tchebycheff) et que la valeur de $\frac{Jx}{x^2}$ approche indéfiniment près quand x est pris suffisamment grand de $\frac{3}{\pi^2}$, c'est-à-dire de ·30396....

Il est facile de voir que la quantité Jx diminuée de l'unité n'est autre chose que le nombre des fractions dans les Tables pareilles à celles de M. Airy. Ainsi, pour le cas de $x = 100$ selon M. Airy, $Jx = 3044$. Pour ce cas $\frac{3}{\pi^2}x^2 = 3039{\cdot}6$.

Avec l'aide de ces limites on peut calculer la probabilité que deux nombres dont la limite supérieure est très grande soient premiers entre eux. Car si cette limite est x, le nombre total des cas qui peuvent arriver est x^2, et le nombre des cas pour lesquels les nombres choisis sont premiers entre eux sera $2Jx - 1$. Conséquemment, la probabilité en question sera $\frac{6}{\pi^2}$.

M. Franklin, l'auteur de la belle démonstration, insérée dans les *Comptes rendus*, du théorème d'Euler sur le produit $(1-x)(1-x^2)(1-x^3)\ldots$, a bien

[* Vol. III. of this Reprint, p. 532.]

voulu m'adresser la remarque que cette conclusion peut être au moins confirmée, peut-être même absolument démontrée, de la manière suivante :

x étant pris très grand, la probabilité que deux nombres inférieurs à x, pris au hasard, ne contiennent pas tous les deux le nombre premier p, sera $1 - \frac{1}{p^2}$. Donc, la probabilité cherchée sera

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)\left(1 - \frac{1}{7^2}\right)\ldots,$$

qui est la réciproque de

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \ldots,$$

c'est-à-dire est égal à $\frac{6}{\pi^2}$.

Il y a une suite doublement infinie d'équations fonctionnelles exactes qu'on peut former avec les $J(x)$. En particulier, il y a une série simplement infinie de telles fonctions où les signes sont alternativement positifs et négatifs, et conséquemment peuvent servir chacun à donner une suite infinie de limites à Jx.

Ainsi, si l'on écrit

$$\theta x = Jx - J\frac{x}{2} + J\frac{x}{3} - J\frac{x}{4} + J\frac{x}{5} - J\frac{x}{6} + \ldots \ldots + \ldots \ldots$$

$$\theta_2 x = 2J\frac{x}{2} - 3J\frac{x}{3} + 2J\frac{x}{4} - J\frac{x}{6} + 2J\frac{x}{8} - 3J\frac{x}{9} + 2J\frac{x}{10} - J\frac{x}{12} + 2J\frac{x}{14} - 3J\frac{x}{15} + 2J\frac{x}{16} - J\frac{x}{18} + \ldots \ldots + \ldots \ldots$$

$$\theta_3 x = 3J\frac{x}{3} - 4J\frac{x}{4} + 3J\frac{x}{6} - 4J\frac{x}{8} + 3J\frac{x}{9} - J\frac{x}{12} + 3J\frac{x}{15} - 4J\frac{x}{16} + 3J\frac{x}{18} - 4J\frac{x}{20} + 3J\frac{x}{21} - J\frac{x}{24} + 3J\frac{x}{27} - 4J\frac{x}{28} + 3J\frac{x}{30} - 4J\frac{x}{32} + 3J\frac{x}{33} - J\frac{x}{34} + \ldots \ldots + \ldots \ldots$$

on aura toujours, quand

$$x = (k^2 + k)\,i, \quad \theta_k x = \frac{x^2}{2(k^2 + k)},$$

et quand

$$x = (k^2 + k)\,i - 1, \quad \theta_k x = \frac{(x+1)^2}{2(k^2 + k)},$$

et, quel que soit le résidu de x par rapport au module $k^2 + k$, on peut calculer la valeur de $\theta_k x$. Enfin, si x est une quantité positive quelconque, on trouvera

$$\theta_k x = \text{ou} > \frac{x^2 - x}{2(k^2 + k)}, \quad \theta_k x = \text{ou} < \frac{x^2 + 2x + 1}{2(k^2 + k)}.$$

3.

NOTE SUR LE THÉORÈME DE LEGENDRE CITÉ DANS UNE NOTE INSÉRÉE DANS LES *COMPTES RENDUS.*

[*Comptes Rendus*, XCVI. (1883), pp. 463—465.]

Le théorème de Legendre, cité par MM. de Jonquières et Lipschitz, est une conséquence immédiate d'un théorème logique bien connu, lequel, *mis sous forme sensible,* équivaut à dire que, si $A, B, C, \ldots$ sont des corps avec la faculté de s'entrecouper, contenus dans un vase d'eau, et si $a, ab, abc, \ldots$ représentent symboliquement les volumes de A, de la partie commune à A et à B, de la partie commune à $A, B, C, \ldots$, alors le volume du liquide déplacé par la totalité des corps sera

$$\Sigma a - \Sigma ab + \Sigma abc - \ldots.$$

Conséquemment, ce théorème admet une généralisation infinie dont je donnerai un seul exemple.

Nommons les nombres premiers qui n'excèdent pas n, nombres premiers subordonnés à n, et distinguons entre eux ceux qui sont plus grands que $\sqrt{n}$ comme supérieurs.

Le théorème de Legendre équivaut à dire que, si $p_1, p_2, \ldots, p_i$ sont les nombres premiers subordonnés à $\sqrt{n}$, le nombre des nombres premiers subordonnés à n du genre supérieur augmenté de l'unité est égal à

$$n - \Sigma\left(\frac{n}{p_1}\right) + \Sigma\left(\frac{n}{p_1 p_2}\right) - \Sigma\left(\frac{n}{p_1 p_2 p_3}\right) + \ldots.$$

Or, représentons la fonction $\frac{1}{2}x(x+1)$ par Δx; alors on aura le théorème que la *somme* des nombres premiers subordonnés à n du genre supérieur augmenté de l'unité sera égale à

$$\Delta n - \Sigma p_1 \Delta\left(\frac{n}{p_1}\right) + \Sigma p_1 p_2 \Delta\left(\frac{n}{p_1 p_2}\right) - \ldots.$$

Par exemple, si $n = 11$, les nombres premiers subordonnés à 11 du genre supérieur seront 5, 7, 11, et les nombres premiers subordonnés à $\sqrt{n}$ sont 2, 3.

On doit donc trouver, et en effet on trouve

$$(11 \,.\, 12) - 2\,(5 \,.\, 6) - 3\,(3 \,.\, 4) + 6\,(1 \,.\, 2) = 2\,(1 + 5 + 7 + 11).$$

Je saisis cette occasion pour dire que j'ai fait calculer la valeur de $J(n)$, "somme-totient de n," pour toutes les valeurs entières de n jusqu'à 500, et je trouve que sans aucune exception $J(n)$ est toujours plus grand que $\frac{3}{\pi^2}(n^2)$ et plus petit que $\frac{3}{\pi^2}(n+1)^2$.

Il reste à démontrer que ces limites sont d'application universelle pour un nombre entier quelconque n.

On peut faire une extension illimitée du théorème donné dans le numéro précédent des *Comptes rendus* sur les *sommes-totients,* tout à fait analogue à l'extension ci-dessus donnée au théorème de Legendre sur les nombres premiers. Nommons, par exemple, $u(j)$ la somme de tous les nombres premiers et inférieurs à j, et Uj la somme

$$u(1) + u(2) + \ldots + u(j).$$

On établit facilement* l'identité

$$\sum_{r=\infty}^{r=1} \Delta\left(E\frac{j}{r}\right) u\left(\frac{j}{r}\right) = \tfrac{1}{6}j(j+1)(j+2),$$

où Δx signifie le nombre triangulaire $\frac{1}{2}x(x+1)$, et avec ce théorème, en se servant, comme dans la théorie des sommes-totients, du principe† de la division harmonique et en écrivant

$$Vj = Uj - 2U\frac{j}{2} + 3U\frac{j}{3} - 4U\frac{j}{4} + 5U\frac{j}{5} - \ldots,$$

on en déduit facilement $Vj = \frac{j^2}{12} - \frac{j}{3}$ quand j est pair,

$$Vj = \frac{(j+1)^2}{12} + \frac{j+1}{6} \text{ quand } j \text{ est impair, etc.}$$

Dans ma Note‡ *Sur le nombre des fractions ordinaires inégales*, etc., j'ai omis de dire que l'équation

$$\sum_r E\frac{j}{r}\, Tr = \frac{j^2+j}{2}$$

peut être écrite sous la forme

$$Jj + J\frac{j}{2} + J\frac{j}{3} + J\frac{j}{4} + \ldots = \frac{j^2+j}{2}. \qquad (1)$$

[* With $u(r) = \frac{1}{2}rT(r)$, $u(1) = \frac{1}{2}$, $T(r)$ being the totient of r, we have

$$2\sum_{r=1} \Delta\left(E\frac{i}{r}\right) u(r) = \tfrac{1}{6}i(i+1)(2i+1).]$$

[† Vol. III. of this Reprint, p. 673.] [‡ p. 84 above.]

De même, l'équation

$$\Sigma\Delta E\frac{j}{r}\,u\frac{j}{r}=\frac{j(j+1)(j+2)}{6}$$

équivaut à l'équation *

$$Uj+2U\frac{j}{2}+3U\frac{j}{3}+4U\frac{j}{4}+\ldots=\frac{j(j+1)(j+2)}{6}. \qquad (2)$$

Il est facile de démontrer, avec l'aide des équations (1) et (2), que les valeurs asymptotiques de $\frac{Jj}{j^2}$ et $\frac{Uj}{j^3}$ pour j indéfiniment grand sont $\frac{3}{\pi^2}$ et $\frac{1}{\pi^2}$ respectivement.

Cauchy, MM. Halphen et Lucas ont écrit sur *les suites de Farey.* Il est donc bon de faire remarquer que Jj est le nombre des fractions et Uj la somme des numérateurs des fractions dans une telle suite pour laquelle la limite donnée est j.

[* For $\frac{1}{6}j(j+1)(j+2)$ *read* $\frac{1}{12}j(j+1)(2j+1)$.]

4.

SUR LE PRODUIT INDÉFINI $1-x.1-x^2.1-x^3\ldots$.

[*Comptes Rendus*, XCVI. (1883), p. 674.]

DANS le *Johns Hopkins Circular,* numéro de février*, on trouvera l'explication d'une méthode *graphique* pour convertir les produits continus en séries. J'ai appliqué cette méthode pour obtenir la formule connue (Cayley, *Elliptic Functions*, p. 296)

$$\frac{1}{1-ax.1-ax^2.1-ax^3\ldots}$$
$$=1+\frac{xa}{1-x.1-ax}+\frac{x^4a^2}{1-x.1-x^2.1-ax.1-ax^2}$$
$$+\frac{x^9a^3}{1-x.1-x^2.1-x^3.1-ax.1-ax^2.1-ax^3}+\ldots.$$

Je me suis demandé quelle serait l'expression obtenue en appliquant la même construction (ou dissection) graphique (qui fournit la formule citée en haut), au produit $1+ax.1+ax^2.1+ax^3\ldots$, et j'ai trouvé sans aucune difficulté l'expression suivante:

$$1+xa\frac{1+ax^2}{1-x}+x^5a^2\frac{1+ax.1+ax^4}{1-x.1-x^2}+\ldots$$
$$+x^{\frac{3j^2-j}{2}}a^j\frac{1+ax.1+ax^2\ldots1+ax^{j-1}}{1-x.1-x^2\ldots1-x^{j-1}}.\frac{1+ax^{2j}}{1-x^j}+\ldots.$$

En faisant $a=-1$, on obtient

$$1-x.1-x^2.1-x^3\ldots$$
$$=1-x(1+x)+x^5(1+x^2)+\ldots+(-)^jx^{\frac{3j^2-j}{2}}(1+x^j)+\ldots.$$

C'est le théorème bien connu d'Euler, lequel, sous ce point de vue, n'est qu'un corollaire d'un théorème plus général.

Par la même méthode, j'obtiens la série pour les *théta* fonctions et d'autres séries beaucoup plus générales, sans calcul algébrique aucun.

[* Vol. III. of this Reprint, pp. 669, 686; and above pp. 30, 33.]

5.

SUR UN THÉORÈME DE PARTITIONS.

[*Comptes Rendus*, XCVI. (1883), pp. 674, 675.]

SOIENT $s_1, s_2, \ldots, s_i$ des suites de nombres consécutifs, telles que le plus petit terme dans aucune d'elles n'excède de plus de l'unité le plus grand terme dans la suite qui précède ; bien entendu que i peut devenir l'unité et qu'une suite quelconque peut se réduire à un seul terme. On peut envisager ce système de suites comme une partition de la somme des nombres contenus dans leur totalité : alors on aura le théorème suivant :

Le nombre de systèmes de i suites de nombres consécutifs dont la somme est N est le même que le nombre de partitions de N qu'on peut former avec les repétitions de i nombres impairs. Comme exemple, en faisant $N = 10$ et $i = 1, 2, 3$ successivement, on aura d'un côté les divers groupes de partitions

10	9, 1	1, 2, 7	1, 3, 6
1, 2, 3, 4	8, 2	2, 3, 5	
	7, 3	1, 4, 5	
	6, 4		

et de l'autre (en se servant d'un indice supérieur pour signifier le nombre des réflexions de sa base),

5^2	9, 1	3^3, 1	1^2, 3, 5
1^{10}	7, 3	3^2, 1^4	
	7, 1^3	3, 1^7	
	5, 1^5		

En ajoutant ensemble les équations qui, pour la même valeur de N, répondent à toutes les valeurs possibles de i, on retombe sur le théorème bien connu d'Euler que *le nombre des partitions de N, en excluant seulement les répétitions, est le même que le nombre de ses partitions en excluant seulement les nombres pairs.* Ainsi, on peut envisager ce dernier théorème comme un corollaire d'un théorème bien autrement profond et qui n'est pas du tout facile à démontrer, sinon pour le cas le plus simple, c'est-à-dire quand il n'y a qu'une seule suite. Pour ce cas, le théorème peut s'exprimer en disant que *le nombre de suites de nombres consécutifs dont la somme est N est égal au nombre de diviseurs impairs de N.*

6.

PREUVE GRAPHIQUE* DU THÉORÈME D'EULER SUR LA PARTITION DES NOMBRES PENTAGONAUX.

[*Comptes Rendus*, XCVI. (1883), pp. 743—745.]

UNE partition quelconque de n peut être représentée par un assemblage de points uniformément distribués sur un plan et limités par deux lignes droites. Ainsi, par exemple, l'arrangement suivant :

```
.  .  .  .  .  .  .
.  .  .  .  .
.  .  .  .  .
.  .  .
.  .
```

sera la représentation graphique de la partition du nombre 22 dans les parties

7, 5, 5, 3, 2.

Mais, de plus, un tel arrangement de points peut être distribué dans un carré et deux groupes que je nommerai *latéral* et *inférieur*. Ainsi, l'arrangement écrit ci-dessus peut être décomposé dans un carré de neuf points, dans un groupe latéral de huit et dans un groupe inférieur de cinq points.

Considérons les partitions de n dans j parties *inégales*. Tous les arrangements de points qui correspondent à ces partitions peuvent être classifiés selon la valeur du côté du carré qui y correspond et que je nommerai θ. Alors, pour une valeur donnée de θ, le groupe latéral contiendra nécessairement ou θ ou $\theta - 1$ lignes de points, car autrement il y aurait des parties égales dans l'arrangement. Dans le premier cas, le nombre de colonnes dans ce groupe inférieur peut être un nombre quelconque, mais pas plus grand que θ ; dans le second cas, pas plus grand que $\theta - 1$. Donc, en se rappelant que le nombre de partitions de ν en θ parties inégales est le coefficient de x dans le développement de

$$\frac{x^{\frac{\theta^2+\theta}{2}}}{1 - x \,.\, 1 - x^2 \ldots 1 - x^\theta}$$

et que le nombre de partitions de ν dans $j - \theta$ parties inégales et pas plus grandes que θ est le coefficient de $x^\nu a^{j-\theta}$ dans le développement de

$$(1 + ax)(1 + ax^2) \ldots (1 + ax^\theta),$$

on voit que, quand le nombre de lignes dans le groupe latéral est θ, le nombre

[* See p. 32 above.]

total d'arrangements de n dans j parties inégales qui correspondent à cette espèce de distribution sera le coefficient de $x^{n-\theta^2}a^{j-\theta}$ dans le développement de

$$\frac{1+ax\,.\,1+ax^2\ldots 1+ax^\theta}{1-x\,.\,1-x^2\ldots 1-x^\theta}\,x^{\frac{\theta^2+\theta}{2}}.$$

De même, le nombre des partitions qui correspondent à la seconde hypothèse sera le coefficient de $x^{n-\theta^2}a^{j-\theta}$ dans le développement de

$$\frac{1+ax\,.\,1+ax^2\ldots 1+ax^{\theta-1}}{1-x\,.\,1-x^2\ldots 1-x^{\theta-1}}\,x^{\frac{\theta^2-\theta}{2}}.$$

En donnant à θ toutes les valeurs depuis 1 jusqu'à l'infini, on obtiendra toutes les partitions de n dans j parties inégales. Les cas où θ excède j n'offrent rien d'exceptionnel, car, pour ces cas, le coefficient de $a^{j-\theta}$ dans les deux fonctions génératrices sera nul.

Or le coefficient de $x^{n-\theta^2}a^{j-\theta}$ dans chacune de ces deux fonctions est le même que le coefficient de $x^n a^j$ dans les produits qui résultent de leur multiplication par $x^{\theta^2}a^\theta$.

En comparant les coefficients de $x^n a^j$ pour toute valeur de n et i, on trouve donc

$$\begin{aligned}(1+xa)(1+x^2a)(1+x^3a)+\ldots\\ =1+\frac{1+ax}{1-x}\,x^2a+\frac{1+ax\,.\,1+ax^2}{1-x\,.\,1-x^2}\,x^7a^2+\ldots\\ +\frac{1+ax\,.\,1+ax^2\ldots 1+ax^\theta}{1-x\,.\,1-x^2\ldots 1-x^\theta}\,x^{\frac{3\theta^2+\theta}{2}}a^\theta+\ldots\\ +xa+\frac{1+ax}{1-x}\,x^5a^2+\ldots\\ +\frac{1+ax\,.\,1+ax^2\ldots 1+ax^{\theta-1}}{1-x\,.\,1-x^2\ldots 1-x^{\theta-1}}\,x^{\frac{3\theta^2-\theta}{2}}a^\theta+\ldots.\end{aligned}$$

En mettant $a=-1$, on obtient ainsi

$$1-x\,.\,1-x^2\,.\,1-x^3-\ldots=1-x-x^2-\ldots+(-)^\theta\left(x^{\frac{3\theta^2-\theta}{2}}+x^{\frac{3\theta^2+\theta}{2}}\right)+\ldots,$$

ce qui est le théorème d'Euler.

En réunissant les deux séries dans une seule, on obtient, pour le cas général,

$$\begin{aligned}(1+xa)(1+x^2a)(1+x^3a)+\ldots\\ =1+\frac{1+ax^2}{1-x}\,xa+\frac{1+ax\,.\,1+ax^4}{1-x\,.\,1-x^2}\,x^5a^2+\frac{1+ax\,.\,1+ax^2\,.\,1+ax^6}{1-x\,.\,1-x^2\,.\,1-x^3}\,x^{12}a^3+\ldots,\end{aligned}$$

c'est-à-dire l'équation que j'ai donnée dans la Note précédente [p. 91].

Je dois dire que c'est M. Durfee, étudiant à Baltimore, qui, le premier (dans un tout autre problème), a fait usage du genre de décomposition d'une *assemblée régulière* de points dans un carré et deux groupes supplémentaires dont j'ai profité dans l'analyse précédente (voir *Johns Hopkins Circular*, [Vol. III. of this Reprint, pp. 661 ff.]).

7.

DÉMONSTRATION GRAPHIQUE* D'UN THÉORÈME D'EULER CONCERNANT LES PARTITIONS DES NOMBRES.

[*Comptes Rendus*, XCVI. (1883), pp. 1110—1112.]

COMME confirmation de la puissance de la méthode graphique appliquée à la théorie des partitions, la preuve suivante d'un théorème que je crois être nouveau ne sera pas, je l'espère, tout à fait dépourvue d'intérêt pour les géomètres; car il serait, il me semble, assez difficile d'en trouver une preuve directe analytique au moyen de la comparaison de fonctions génératrices, comme on le fait ordinairement pour des théorèmes de ce genre.

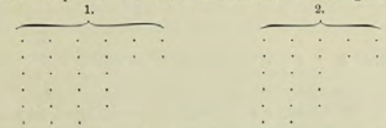
Euler a trouvé facilement, par une comparaison de telles fonctions, que le nombre de partitions de n en nombres impairs est le même que le nombre de partitions de n en nombres inégaux; je précise ce théorème en ajoutant que le nombre de partitions de n en nombres impairs, qui se divisent en i groupes de nombres distincts, est égal au nombre de partitions de n en i suites tout à fait distinctes de nombres consécutifs.

Nommons U une partition en nombres impairs et V une partition en nombres inégaux.

Je dis qu'on peut passer de U à V par la méthode suivante. Supposons, par exemple, que U soit la partition 11 . 11 . 7 . 7 . 7 . 5.

Je forme deux assemblages réguliers de points en prenant dans l'un d'eux, sur chaque ligne, un nombre de points égal à $\frac{11+1}{2}$, $\frac{11+1}{2}$, $\frac{7+1}{2}$, $\frac{7+1}{2}$, $\frac{7+1}{2}$, $\frac{5+1}{2}$, et l'autre assemblage en diminuant de l'unité chacun de ces nombres de points. On forme ainsi ces deux assemblages :

1. 2.



et, en comptant le nombre de points dans les *angles* successifs de chaque figure, on obtient, dans l'un, 11, 9, 5, 2, et, dans l'autre, 10, 8, 3; en les réunissant, on obtient la partition

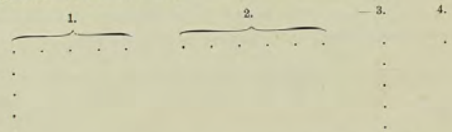
11 . 10 . 9 . 8 . 5 . 3 . 2,

qui est un V.

[* See p. 39 above.]

Or il est facile de voir que dans cette méthode de transformation U devient V, et l'on démontre (en construisant un certain système d'équations linéaires) que, pour un V quelconque donné, on peut trouver un et un seul U qui se transformera dans ce V, de sorte qu'il y a correspondance un à un entre la totalité des U et la totalité des V, ce qui sert à démontrer le théorème original d'Euler. Mais si tel était le but de cette recherche, cette méthode de transformation serait peine perdue, car il existe une tout autre méthode, infiniment plus simple, d'établir une telle correspondance: on la trouvera expliquée dans le cahier de l'*American Journal of Mathematics* qui va paraître. L'utilité de cette méthode spéciale de créer la correspondance consiste en ceci: que le V ainsi conjugué avec un U contiendra le même nombre de suites distinctes de nombres consécutifs que le U contient de nombres impairs distincts: cela veut dire que le nombre des lignes inégales (disons i) dans l'un ou l'autre assemblage de points est toujours égal à j, nombre de suites distinctes obtenu en opérant de la manière expliquée ci-dessus. La preuve en est facile; car si l'on enlève l'angle extérieur à l'un et à l'autre des assemblages, on verra facilement que quatre cas se présenteront: pour un de ces cas, j ne change pas de valeur, à cause du changement opéré dans les deux assemblages; dans un autre cas, j subira une diminution de deux unités, et dans les deux cas intermédiaires d'une seule unité. Ces cas correspondent aux quatre suppositions qui résultent de la combinaison des hypothèses que les deux premières lignes ou les deux premières colonnes dans l'un ou l'autre des assemblages sont ou ne sont pas égales entre elles: de sorte qu'on verra facilement que le j et le i seront toujours diminués de la même quantité, ou 0, ou 1 ou 2, et conséquemment on aura $i-j$ *constant*; si l'on enlève l'un après l'autre les angles des deux assemblages jusqu'à ce qu'on arrive à un assemblage qui sera de l'une ou l'autre des quatre formes suivantes:

1. 2. — 3. 4.



pour lesquels cas $i=2$, $j=2$; $i=1$, $j=1$; $i=1$, $j=1$; $i=1$, $j=1$; respectivement on aura toujours ainsi $i=j$, de sorte qu'il y a correspondance une à une entre les partitions du même nombre n qui contiennent justement i nombres impairs répétés (ou non) à volonté, et celles qui contiennent justement i suites distinctes de nombres consécutifs, et conséquemment il y aura le même nombre des unes et des autres: ce qui est le théorème que j'ai voulu démontrer.

8.

SUR UN THÉORÈME DE PARTITIONS* DE NOMBRES COMPLEXES CONTENU DANS UN THÉORÈME DE JACOBI.

[*Comptes Rendus*, XCVI. (1883), pp. 1276—1280.]

DANS le *Journal de Crelle*, t. XXXII. p. 166, Jacobi fait la remarque que le développement en série de $\Theta_1 x$ donne lieu à un théorème que j'exprime de la manière suivante.

Soient a et b deux quantités $c=a+b$; alors le produit infini

$$(1\mp q^a)(1\mp q^b)(1-q^c)(1\mp q^{a+c})(1\mp q^{b+c})(1-q^{2c})\ldots=\sum_{-\infty}^{+\infty}(\mp)^i q^{\frac{i^2c+i(a-b)}{2}}.$$

Ce théorème étant vrai pour un nombre infini de valeurs de $\frac{a}{b}$ sera, par sa forme même, nécessairement vrai quand a et b sont de symboles absolument arbitraires, et l'on voit facilement que, pour le montrer dans ce sens universel, il suffira d'énoncer un certain théorème sur les nombres complexes dont voici l'énoncé:

Désignons par C, B, A des nombres complexes de la forme fc, $fc+b$, $fc+a$, où f est ou zéro ou un nombre entier et positif quelconque.

Considérons un arrangement composé avec des C, des B et des A non répétés ou avec des C, B, A pris seuls ou combinés deux à deux, en excluant les arrangements (que je nomme exceptionnels) qui ne contiennent que des B formant une série arithmétique dont b est le dernier terme et c la différence constante, ou des A formant une série semblable dont a est le dernier terme.

Par le caractère majeur et le caractère mineur d'un tel arrangement, je désigne la parité ou l'imparité du nombre total *des termes et du nombre des C qu'il contient. Je dis qu'à chaque arrangement (non exceptionnel) on peut en associer un autre pareil dont la somme totale des éléments (les A, B, C) sera la même, mais dont les caractères seront tous les deux opposés.*

La démonstration deviendra plus claire en se servant de la notation suivante. En désignant par X un symbole d'une série de termes, je me servirai de X et de X pour signifier le terme le plus haut et le terme le plus

[* See above, p. 59 ff.]

bas de la série, et en me servant de Y ou Z pour signifier un symbole ou simple ou affecté de marques quelconques, j'emploie les notations

$$Y=0, \quad Y+Z=0, \quad Y>0, \quad Y+Z>0,$$

pour signifier que les Y manquent, que les Y et les Z manquent tous les deux, que les Y ne manquent pas, que les Y et les Z ne manquent pas tous les deux.

Je divise les B (d'un arrangement quelconque) en deux espèces, ${}'B$ et B', dont ${}'B$ représente un B appartenant à la série arithmétique (la plus grande qu'on puisse former) commençant avec le plus grand B, et B' les autres B qui se trouvent dans l'arrangement.

Ainsi je divise les A en ${}_{,}A$ et en $A_{,}$; $A_{,}$ signifie un A appartenant à la série arithmétique la plus grande qu'on puisse former, dont a est le terme minimum (de sorte que, si l'arrangement ne contient pas un a, $A_{,}$ manque) et ${}_{,}A$ signifie les autres A de l'arrangement.

Finalement un point au centre d'un symbole à droite ou à gauche signifiera ce symbole diminué ou augmenté respectivement de c.

On voit que dans cette notation les arrangements exceptionnels seront exprimés ainsi : ceux qui appartiennent à l'une des deux classes par les conditions ${}'B-b=0$ avec $A+C=0$, et les autres par les conditions $B=0$ avec ${}_{,}A+C=0$.

Je divise les arrangements non exceptionnels en trois classes, dont les conditions seront respectivement les suivantes :

Première classe :

$${}'\underset{\cdot}{B}-b>0 \quad \text{ou} \quad ({}'\underset{\cdot}{B}-b=0 \quad \text{avec} \quad \underset{\cdot}{C}-c \leq {}'\dot{B}-b).$$

Deuxième classe :

$${}'B-b=0 \quad \text{avec} \quad (\underset{\cdot}{C}-c>{}'\dot{B}-b \quad \text{ou} \quad C=0, \quad \text{mais} \quad A+C>0),$$

ou

$$B=0 \quad \text{avec} \quad (A=0 \quad \text{ou} \quad \underset{\cdot}{A}-a \geqq C).$$

Troisième classe :

$$B=0 \quad \text{avec} \quad A>0 \quad \text{et} \quad \underset{\cdot}{A}-a<\underset{\cdot}{C} \quad \text{et} \quad {}_{,}A+C>0.$$

Toutes les hypothèses possibles se trouvent comprises dans ces tableaux des arrangements exceptionnels et non exceptionnels.

A chacune des trois classes des derniers je vais assigner un opérateur qui peut être appliqué à chaque arrangement de cette classe et qui le transformera dans un autre arrangement appartenant à la même classe ; cette disposition, appliquée deux fois successivement, reproduira l'arrangement sur lequel on opère, lequel ne changera pas la somme des éléments, mais changera chacun des deux caractères en sens opposé : c'est-à-dire que chacun des trois opérateurs que je vais définir, et que je nommerai ϕ, ψ, ϑ, doit

satisfaire à cinq conditions qu'on peut nommer *catholicité, homœogénèse, mutualité, inertie* et *énantiotropie.*

1. ϕ signifie que, si $C=0$ ou $\underset{\cdot}{C}-c>{}'\dot{B}-{}'\underset{\cdot}{B}$, on doit former un nouveau C, en substituant, pour chaque ${}'B$, ${}'B\cdot$ (c'est-à-dire sa valeur diminuée de c), et reconstituer l'inertie originale en ajoutant ensemble les c ainsi soustraits pour former un nouveau C, et que, dans le cas contraire, $\underset{\cdot}{C}$ doit être décomposé en simples c, dont on ajoutera un au premier ${}'B$ (le B le plus grand), un au second ${}'B$, etc., jusqu'à ce que tous les c dont on a à disposer soient épuisés.

2. ψ signifie que, si $B>0$ ou $C=0$, ou $\underset{\cdot}{C}>{}'\dot{B}+A$, on doit former un nouveau C en substituant à ${}'\dot{B}$ et A leur somme et que, dans le cas contraire, $\underset{\cdot}{C}$ doit être décomposé en ${}'{\cdot}\dot{B}$ et A si $B>0$ et en b et A si $B=0$.

3. ϑ signifie que, si $C=0$ ou $\underset{\cdot}{C}+\dot{A}_{,}=>\underset{\cdot}{A}$, il faut décomposer ${}_{,}A$ en ${\cdot}\dot{A}_{,}$ et C ou en a et C, selon que $A_{,}=$ ou >0, et que, dans le cas contraire, pour $\underset{\cdot}{C}$ et $\dot{A}_{,}$, il faut substituer leur somme. On sera satisfait en étudiant les conditions des trois classes que les ϕ, ψ, ϑ possèdent tous les trois cinq attributs voulus : la preuve en est facilitée en supposant que, dans chaque série des C, des B et des A, prise séparément, on suit un ordre régulier de grandeur dans l'arrangement de ces termes respectivement au multiple de c qui entre dans chacun d'eux.

Si l'on donne à a et à b des valeurs quantitatives (ce qui est toujours permis), et en particulier les valeurs 1 et 2 respectivement, on retombe sur le théorème d'Euler, mais (chose à noter) la correspondance donnée par le procédé général appliqué à ce cas ne sera nullement identique à la correspondance donnée par le procédé de Franklin. En effet, les arrangements exceptionnels ne seront pas les mêmes dans les deux méthodes : selon le procédé de Franklin, les arrangements non conjugables sont de la forme

$$i, \; i+1, \; \ldots, \; 2i-1 \quad \text{ou} \quad i+1, \; i+2, \; \ldots, \; 2i,$$

tandis que la méthode actuelle donnera, comme non conjugués, les arrangements de la forme

$$1, \; 4, \; \ldots, \; 3i-2 \quad \text{ou} \quad 2, \; 5, \; \ldots, \; 3i-1.$$

La méthode employée ici fournira elle-même toujours deux systèmes de correspondance absolument distincts, dont on obtient l'un, qui n'est pas exprimé, en échangeant entre eux les a, A et les b, B, car la méthode n'est pas symétrique dans son opération sur ces deux systèmes de lettres.

Ce cas est analogue à celui de la correspondance perspective entre deux triangles, laquelle peut être simple ou triple, comme je l'ai montré ailleurs. Jacobi, dans l'endroit cité, a fait la remarque que, pour $a=1$, $b=2$, en se servant du signe supérieur $(\mp)$ dans son théorème, on retombe sur le théorème d'Euler et que, pour le cas de $a=1$, $b=1$, en se servant du signe inférieur, sur un théorème donné (il y a longtemps par Gauss). On peut ajouter que, si avec cette supposition on se sert du signe supérieur, on obtient $0=0$, mais si l'on écrit $a=1-\epsilon$, $b=1$, en faisant ϵ infinitésimal, on tombe (chose singulière) sur l'équation de Jacobi elle-même,

$$(1-q)^3(1-q^2)^3(1-q^3)^3+\ldots=1-3q+5q^3-7q^6+\ldots.$$

Puisque j'ai introduit le nom de l'auteur des *Fundamenta nova*, qu'on me permette la remarque que, dans les deux avant-dernières lignes de l'avant-dernière page de cet immortel Ouvrage, on trouve un théorème qui équivaut à l'équation

$$\frac{q}{1+q}-\frac{q^3}{1+q^3}+\frac{q^5}{1+q^5}-\ldots=\frac{q}{1+q}-\frac{q^{1+2}}{1+q^2}+\frac{q^{1+2+3}}{1+q^3}-\ldots;$$

or, le premier cas du théorème intitulé: *Sur un théorème d'Euler*, contenu dans une Note précédente des *Comptes rendus**, affirme que le nombre des séries arithmétiques avec lesquelles on peut exprimer n est égal au nombre des diviseurs impairs de n, laquelle considération mène immédiatement à une conséquence qu'on ne pourrait manquer d'observer (mais que M. Franklin, effectivement, a remarquée le premier) et qui s'exprime par l'équation

$$\frac{q}{1-q}+\frac{q^3}{1-q^3}+\frac{q^5}{1-q^5}+\ldots=\frac{q}{1-q}+\frac{q^{1+2}}{1-q^2}+\frac{q^{1+2+3}}{1-q^3}+\ldots,$$

équation très ressemblante à l'autre et qui peut être combinée avec elle de manière à donner naissance à quatre autres équations de la même espèce.

On n'a pas besoin de dire que le théorème qui constitue la matière principale de cette Note, en faisant $a=1$ et en considérant b comme une quantité arbitraire, contient ou au moins conduit immédiatement au développement de $\Theta_1 x$ dont Jacobi l'a traité comme conséquence.

[* p. 95 above. Cf. p. 25 above.]

9.

ON THE NUMBER OF FRACTIONS CONTAINED IN ANY "FAREY SERIES" OF WHICH THE LIMITING NUMBER IS GIVEN.

[*Philosophical Magazine*, XV. (1883), pp. 251—257; XVI. (1883), pp. 230—233.]

A *Farey series* ("suite de Farey") is a system of all the unequal vulgar fractions arranged in order of magnitude, the numerator and denominator of which do not exceed a given number.

The first scientific notice of these series appeared in the *Philosophical Magazine*, Vol. XLVII. (1816), pp. 385, 386. In 1879 Mr Glaisher published in the *Philosophical Magazine* (pp. 321—336) a paper on the same subject containing a proof of their known properties, an important extension of the subject to series in which the numerators and denominators are subject to distinct limits, and a bibliography of Mr Goodwyn's tables of such series. Finally, in 1881 Sir George Airy contributed a paper also to the *Philosophical Magazine* of that year, in which he refers to a table calculated by him "some years ago," and printed in the Selected Papers of the *Transactions* of the Institution of Civil Engineers, which is in fact a Farey table with the logarithms of the fractions appended to each of them. Previous tables had only given the decimal values of such fractions. The drift of this paper is to point out a caution which it is necessary to observe in the use of such tables, and which limits their practical utility: this arises from the fact of the differences receiving a very large augmentation in the immediate neighbourhood of the fractions which are a small aliquot part of unity—a fact which may be inferred *à priori* from the well-known law discovered by Farey applicable to those differences, but to which the author of the paper makes no allusion.

In addition to the tables of Farey series by Goodwyn, Wucherer, an anonymous author mentioned in the Babbage Catalogue, and Gauss, referred to by Mr Glaisher in his Report to the Bradford Meeting of the British Association (1873), may be mentioned one contained in Herzer's *Tabellen*

(Basle, 1864) with the limit 57, and another in Hrabak's *Tabellen-Werk* (Leipsic, 1876), in which the limit is taken at 50.

The writers on the theory are:—Cauchy (as mentioned by Mr Glaisher), who inserted a communication relating to it in the *Bulletin des Sciences par la Société Philomathique de Paris*, republished in his *Exercices de Mathématiques*; Mr Glaisher himself (*loc. cit.*); M. Halphen, in a recent volume of the Proceedings of the Mathematical Society of France; and M. Lucas, in the next following volume of the same collection. I am indebted to my friend and associate Dr Story for these later references.

For theoretical purposes it is desirable to count $\frac{1}{1}$ as one of the fractions in a Farey series. The number of such fractions for the limit j then becomes identical with the sum of the *totients* of all the natural numbers up to j inclusive—a totient to x (which I denote by τx) meaning the number of numbers less than x and prime to it. Such sum, that is, $\sum_{x=j}^{x=1} \tau x$, I denote by Tj. My attention was called to the subject by this number Tj expressing the number of terms in a function whose residue (in Cauchy's sense) is the generating function to any given simple denumerant (see *American Journal of Mathematics*, [Vol. III. of this Reprint, p. 605]); and I became curious to know something about the value of Tj. I had no difficulty in finding a functional equation which serves to determine its limits (see *Johns Hopkins University Circular*, Jan. and Feb. 1883*). The most simple form of that equation (omitted to be given in the *Circular*) is

$$Tj + T\frac{j}{2} + T\frac{j}{3} + T\frac{j}{4} + T\frac{j}{5} + \ldots = \frac{j^2+j}{2},$$

(where, when x is a fraction, Tx is to be understood to mean Tj, j being the integer next below x); and from this it is not difficult to deduce by strict demonstration that Tj/j^2, when j increases indefinitely, approximates indefinitely near to $3/\pi^2$.

I have subsequently found that if ux be used to denote the sum of all the numbers inferior and prime to x, and $Uj = \sum_{x=j}^{x=1} ux$, then†

$$Uj + 2U\frac{j}{2} + 3U\frac{j}{3} + 4U\frac{j}{4} + \ldots = \frac{j(j+1)(j+2)}{3}$$

(where Ux, when x is a fraction, means the U of the integer next inferior to x). From this equation it is also possible to prove that Uj/j^3, when j becomes indefinitely great, approximates to $1/\pi^2$. Uj, it may be well to notice, is the sum of all the numerators of the fractions in a Farey series whose limit is j, just as Tj is the number of these fractions.

In the annexed Table the value of τx (the totient), of Tx (the sum-totient), and of $3/\pi^2 . x^2$ is calculated for all the values of x from 1 to 1000; and the

[* See pp. 84, 89 above.] [† The right side should be $\frac{1}{12}j(j+1)(2j+1)$.]

remarkable fact is brought to light that Tx is always greater than $3/\pi^2 . x^2$ (the number opposite to it), and less than $3/\pi^2 . (x+1)^2$, the number which comes after the following one in the same table.

I have calculated in my head the first few values of Ux, and find (if I have made no mistake) that it obeys an analogous law, namely is always intermediate between $1/\pi^2 . x^3$ and $1/\pi^2 . (x+1)^3$.

It may also be noticed that when n is a prime number, Tn is always nearer, and usually very much nearer, to the superior than to the inferior limit—as might have been anticipated from the circumstance that, when this is the case, in passing from $n-1$ to n the T receives an augmentation of $n-1$, whereas its average augmentation is only $\frac{3}{\pi^2}(2n-1)$.

In like manner and for a similar reason, when n contains several small factors Tn is nearer to the inferior than to the superior limit. For instance, when $n = 210$, $Tn = 13414$ and $3/\pi^2 . n^2 = 13404{\cdot}79$.

TABLE *of Totients, of Sum-totients, and of* $3/\pi^2$ *into the Squares of all the Numbers from* 1 *to* 1000 *inclusive.*

$$\left[\frac{3}{\pi^2} = {\cdot}30396355\right].$$

n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2}n^2$	n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2}n^2$	n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2}n^2$
1	1	1	·30	27	18	230	221·59	53	52	882	853·83
2	1	2	1·22	28	12	242	238·31	54	18	900	886·36
3	2	4	2·74	29	28	270	255·63	55	40	940	919·49
4	2	6	4·86	30	8	278	273·56	56	24	964	953·23
5	4	10	7·60	31	30	308	292·11	57	36	1000	987·58
6	2	12	10·94	32	16	324	311·26	58	28	1028	1022·54
7	6	18	14·90	33	20	344	331·01	59	58	1086	1058·10
8	4	22	19·46	34	16	360	351·38	60	16	1102	1094·27
9	6	28	24·62	35	24	384	372·35	61	60	1162	1131·05
10	4	32	30·40	36	12	396	393·93	62	30	1192	1168·44
11	10	42	36·78	37	36	432	416·12	63	36	1228	1206·43
12	4	46	43·77	38	18	450	438·92	64	32	1260	1245·03
13	12	58	51·37	39	24	474	462·32	65	48	1308	1284·25
14	6	64	59·58	40	16	490	486·34	66	20	1328	1324·07
15	8	72	68·39	41	40	530	510·96	67	66	1394	1364·49
16	8	80	77·81	42	12	542	536·19	68	32	1426	1405·53
17	16	96	87·84	43	42	584	562·02	69	44	1470	1447·17
18	6	102	98·48	44	20	604	588·47	70	24	1494	1489·42
19	18	120	109·73	45	24	628	615·52	71	70	1564	1532·28
20	8	128	121·58	46	22	650	643·19	72	24	1588	1575·75
21	12	140	134·05	47	46	696	671·45	73	72	1660	1619·82
22	10	150	147·12	48	16	712	700·33	74	36	1696	1664·51
23	22	172	160·79	49	42	754	729·82	75	40	1736	1709·80
24	8	180	175·08	50	20	774	759·91	76	36	1772	1755·69
25	20	200	189·98	51	32	806	790·61	77	60	1832	1802·20
26	12	212	205·48	52	24	830	821·92	78	24	1856	1849·31

TABLE (*continued*).

n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$	n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$	n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$
79	78	1934	1897·04	134	66	5498	5457·97	189	108	10904	10857·88
80	32	1966	1945·37	135	72	5570	5539·74	190	72	10976	10973·09
81	54	2020	1994·31	136	64	5634	5622·11	191	190	11166	11088·90
82	40	2060	2043·85	137	136	5770	5705·09	192	64	11230	11205·31
83	82	2142	2094·01	138	44	5814	5788·68	193	192	11422	11322·34
84	24	2166	2144·77	139	138	5952	5872·88	194	96	11518	11439·97
85	64	2230	2196·14	140	48	6000	5957·69	195	96	11614	11558·21
86	42	2272	2248·12	141	92	6092	6043·10	196	84	11698	11677·06
87	56	2328	2300·70	142	70	6162	6129·12	197	196	11894	11796·52
88	40	2368	2353·90	143	120	6282	6215·75	198	60	11954	11916·59
89	88	2456	2407·70	144	48	6330	6302·99	199	198	12152	12037·26
90	24	2480	2462·10	145	112	6442	6390·83	200	80	12232	12158·54
91	72	2552	2517·12	146	72	6514	6479·29	201	132	12364	12280·43
92	44	2596	2572·75	147	84	6598	6568·35	202	100	12464	12402·93
93	60	2656	2628·98	148	72	6670	6658·02	203	168	12632	12526·03
94	46	2702	2685·82	149	148	6818	6748·29	204	64	12696	12649·75
95	72	2774	2743·27	150	40	6858	6839·18	205	160	12856	12774·07
96	32	2806	2801·33	151	150	7008	6930·67	206	102	12958	12899·00
97	96	2902	2860·00	152	72	7080	7022·77	207	132	13090	13024·54
98	42	2944	2919·27	153	96	7176	7115·48	208	96	13186	13150·68
99	60	3004	2979·15	154	60	7236	7208·80	209	180	13366	13277·43
100	40	3044	3039·64	155	120	7356	7302·72	210	48	13414	13404·79
101	100	3144	3100·73	156	48	7404	7397·26	211	210	13624	13532·76
102	32	3176	3162·44	157	156	7560	7492·40	212	104	13728	13661·34
103	102	3278	3224·75	158	78	7638	7588·15	213	140	13868	13790·52
104	48	3326	3287·67	159	104	7742	7684·51	214	106	13974	13920·32
105	48	3374	3351·20	160	64	7806	7781·47	215	168	14142	14050·72
106	52	3426	3415·34	161	132	7938	7879·04	216	72	14214	14181·73
107	106	3532	3480·08	162	54	7992	7977·22	217	180	14394	14313·34
108	36	3568	3545·44	163	162	8154	8076·01	218	108	14502	14445·57
109	108	3676	3611·40	164	80	8234	8175·41	219	144	14646	14578·40
110	40	3716	3677·96	165	80	8314	8275·41	220	80	14726	14711·84
111	72	3788	3745·14	166	82	8396	8376·02	221	192	14918	14845·89
112	48	3836	3812·92	167	166	8562	8477·24	222	72	14990	14980·54
113	112	3948	3881·31	168	48	8610	8579·07	223	222	15212	15115·81
114	36	3984	3950·31	169	156	8766	8681·50	224	96	15308	15251·68
115	88	4072	4019·92	170	64	8830	8784·55	225	120	15428	15388·16
116	56	4128	4090·14	171	108	8938	8888·20	226	112	15540	15525·25
117	72	4200	4160·96	172	84	9022	8992·46	227	226	15766	15662·94
118	58	4258	4232·39	173	172	9194	9097·33	228	72	15838	15801·24
119	96	4354	4304·43	174	56	9250	9202·80	229	228	16066	15940·15
120	32	4386	4377·08	175	120	9370	9308·88	230	88	16154	16079·67
121	110	4496	4450·33	176	80	9450	9415·57	231	120	16274	16219·80
122	60	4556	4524·19	177	116	9566	9522·87	232	112	16386	16360·53
123	80	4636	4598·66	178	88	9654	9630·78	233	232	16618	16501·87
124	60	4696	4673·74	179	178	9832	9739·29	234	72	16690	16643·82
125	100	4796	4794·43	180	48	9880	9848·42	235	184	16874	16786·38
126	36	4832	4825·72	181	180	10060	9958·15	236	116	16990	16929·55
127	126	4958	4902·63	182	72	10132	10068·49	237	156	17146	17073·32
128	64	5022	4980·14	183	120	10252	10179·44	238	96	17242	17217·70
129	84	5106	5058·26	184	88	10340	10290·99	239	238	17480	17362·70
130	48	5154	5136·98	185	144	10484	10403·15	240	64	17544	17508·30
131	130	5284	5216·32	186	60	10544	10515·92	241	240	17784	17654·51
132	40	5324	5296·26	187	160	10704	10629·30	242	110	17894	17801·32
133	108	5432	5376·81	188	92	10796	10743·29	243	162	18056	17948·74

TABLE (*continued*).

n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$	n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$	n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$
244	120	18176	18096·77	299	264	27318	27174·65	354	116	38174	38091·50
245	168	18344	18245·41	300	80	27398	27356·72	355	280	38454	38307·01
246	80	18424	18394·66	301	252	27650	27539·40	356	176	38630	38523·12
247	216	18640	18544·51	302	150	27800	27722·69	357	192	38822	38739·85
248	120	18760	18694·97	303	200	28000	27906·59	358	178	39000	38957·18
249	164	18924	18846·04	304	144	28144	28091·10	359	358	39358	39175·13
250	100	19024	18997·72	305	240	28384	28276·21	360	96	39454	39393·68
251	250	19274	19150·01	306	96	28480	28461·93	361	342	39796	39612·83
252	72	19346	19302·90	307	306	28786	28648·26	362	180	39976	39832·60
253	220	19566	19456·40	308	120	28906	28835·20	363	220	40196	40052·97
254	126	19692	19610·51	309	204	29110	29022·75	364	144	40340	40273·95
255	128	19820	19765·23	310	120	29230	29210·90	365	288	40628	40495·54
256	128	19948	19920·56	311	310	29540	29399·66	366	120	40748	40717·74
257	256	20204	20076·49	312	96	29636	29589·03	367	366	41114	40940·55
258	84	20288	20233·03	313	312	29948	29779·01	368	176	41290	41163·96
259	216	20504	20390·18	314	156	30104	29969·59	369	240	41530	41387·98
260	96	20600	20547·94	315	144	30248	30160·79	370	144	41674	41612·61
261	168	20768	20706·30	316	156	30404	30352·59	371	312	41986	41837·85
262	130	20898	20865·28	317	316	30720	30545·00	372	120	42106	42063·69
263	262	21160	21024·86	318	104	30824	30738·01	373	372	42478	42290·15
264	80	21240	21185·05	319	280	31104	30931·64	374	160	42638	42517·21
265	208	21448	21345·84	320	128	31232	31125·87	375	200	42838	42744·87
266	108	21556	21507·25	321	212	31444	31320·71	376	184	43022	42973·15
267	176	21732	21669·26	322	132	31576	31516·16	377	336	43358	43202·04
268	132	21864	21831·88	323	288	31864	31712·22	378	108	43466	43431·53
269	268	22132	21995·11	324	108	31972	31908·88	379	378	43844	43661·63
270	72	22204	22158·95	325	240	32212	32106·15	380	144	43988	43892·34
271	270	22474	22323·39	326	162	32374	32304·03	381	252	44240	44123·65
272	128	22602	22488·44	327	216	32590	32502·52	382	190	44430	44355·58
273	144	22746	22654·10	328	160	32750	32701·62	383	382	44812	44588·11
274	136	22882	22820·37	329	276	33026	32901·32	384	128	44940	44821·25
275	200	23082	22987·25	330	80	33106	33101·63	385	240	45180	45055·00
276	88	23170	23154·73	331	330	33436	33302·55	386	192	45372	45289·35
277	276	23446	23322·82	332	164	33600	33504·08	387	252	45624	45524·32
278	138	23584	23491·52	333	216	33816	33706·22	388	192	45816	45759·89
279	180	23764	23660·83	334	166	33982	33908·96	389	388	46204	45996·07
280	96	23860	23830·75	335	264	34246	34112·31	390	96	46300	46232·86
281	280	24140	24001·27	336	96	34342	34316·27	391	352	46652	46470·25
282	92	24232	24172·40	337	336	34678	34520·84	392	168	46820	46708·25
283	282	24514	24344·14	338	156	34834	34726·01	393	260	47080	46946·87
284	140	24654	24516·49	339	224	35058	34931·80	394	196	47276	47186·09
285	144	24798	24689·44	340	128	35186	35138·19	395	312	47588	47425·91
286	120	24918	24863·00	341	300	35486	35345·19	396	120	47708	47666·35
287	240	25158	25037·18	342	108	35594	35552·80	397	396	48104	47907·39
288	96	25254	25211·96	343	294	35888	35761·01	398	198	48302	48149·04
289	272	25526	25387·34	344	168	36056	35969·83	399	216	48518	48391·30
290	112	25638	25563·34	345	176	36232	36179·26	400	160	48678	48634·17
291	192	25830	25739·94	346	172	36404	36389·30	401	400	49078	48877·64
292	144	25974	25917·15	347	346	36750	36599·95	402	132	49210	49121·73
293	292	26266	26094·97	348	112	36862	36811·21	403	360	49570	49366·42
294	84	26350	26273·40	349	348	37210	37023·07	404	200	49770	49611·72
295	232	26582	26452·43	350	120	37330	37235·54	405	216	49986	49857·62
296	144	26726	26632·07	351	216	37546	37448·61	406	168	50154	50104·14
297	180	26906	26812·32	352	160	37706	37662·30	407	360	50514	50351·26
298	148	27054	26993·18	353	352	38058	37876·59	408	128	50642	50598·99

TABLE (*continued*).

n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2}n^2$	n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2}n^2$	n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2}n^2$
409	408	51050	50847·33	464	224	65630	65442·14	519	344	82028	81875·93
410	160	51210	51096·27	465	240	65870	65724·52	520	192	82220	82191·75
411	272	51482	51345·83	466	232	66102	66007·51	521	520	82740	82508·18
412	204	51686	51595·99	467	466	66568	66291·11	522	168	82908	82825·21
413	348	52034	51846·76	468	144	66712	66575·31	523	522	83430	83142·85
414	132	52166	52098·14	469	396	67108	66860·13	524	260	83690	83461·10
415	328	52494	52350·12	470	184	67292	67145·55	525	240	83930	83779·95
416	192	52686	52602·72	471	312	67604	67431·58	526	262	84192	84099·42
417	276	52962	52855·92	472	232	67836	67718·22	527	480	84672	84419·49
418	180	53142	53109·73	473	420	68256	68005·46	528	160	84832	84740·17
419	418	53560	53364·15	474	156	68412	68293·32	529	506	85338	85061·46
420	96	53656	53619·17	475	360	68772	68581·78	530	208	85546	85383·36
421	420	54076	53874·80	476	192	68964	68870·85	531	348	85894	85705·87
422	210	54286	54131·04	477	312	69276	69160·52	532	216	86110	86028·98
423	276	54562	54387·89	478	238	69514	69450·81	533	480	86590	86352·70
424	208	54770	54645·35	479	478	69992	69741·70	534	176	86766	86677·03
425	320	55090	54903·42	480	128	70120	70033·20	535	424	87190	87001·97
426	140	55230	55162·09	481	432	70552	70325·31	536	264	87454	87327·51
427	360	55590	55421·39	482	240	70792	70618·03	537	356	87810	87653·66
428	212	55802	55681·26	483	264	71056	70911·35	538	268	88078	87980·42
429	240	56042	55941·76	484	220	71276	71205·29	539	420	88498	88307·79
430	168	56210	56202·86	485	384	71660	71499·83	540	144	88642	88635·77
431	430	56640	56464·57	486	162	71822	71794·98	541	540	89182	88964·35
432	144	56784	56726·89	487	486	72308	72090·73	542	270	89452	89293·54
433	432	57216	56989·82	488	240	72548	72387·10	543	360	89812	89623·34
434	180	57396	57253·36	489	324	72872	72684·07	544	256	90068	89953·75
435	224	57620	57517·50	490	168	73040	72981·65	545	432	90500	90284·77
436	216	57836	57782·26	491	490	73530	73279·84	546	144	90644	90616·39
437	396	58232	58047·62	492	160	73690	73578·63	547	546	91190	90948·62
438	144	58376	58313·58	493	448	74138	73878·04	548	272	91462	91281·46
439	438	58814	58580·16	494	216	74354	74178·05	549	360	91822	91614·91
440	160	58974	58847·34	495	240	74594	74478·67	550	200	92022	91948·97
441	252	59226	59115·14	496	240	74834	74779·90	551	504	92526	92283·64
442	192	59418	59383·54	497	420	75254	75081·73	552	176	92702	92618·91
443	442	59860	59652·54	498	164	75418	75384·18	553	468	93170	92954·79
444	144	60004	59922·16	499	498	75916	75687·23	554	276	93446	93291·28
445	352	60356	60192·38	500	200	76116	75990·89	555	288	93734	93628·38
446	222	60578	60463·22	501	332	76448	76295·15	556	276	94010	93966·08
447	296	60874	60734·66	502	250	76698	76600·03	557	556	94566	94304·39
448	192	61066	61006·70	503	502	77200	76905·52	558	180	94746	94643·31
449	448	61514	61279·36	504	144	77344	77211·61	559	504	95250	94982·84
450	120	61634	61552·62	505	400	77744	77518·31	560	192	95442	95322·98
451	400	62034	61826·49	506	220	77964	77825·62	561	320	95762	95663·72
452	224	62258	62100·97	507	312	78276	78133·54	562	280	96042	96005·07
453	300	62558	62376·06	508	252	78528	78442·06	563	562	96604	96347·03
454	226	62784	62651·75	509	508	79036	78751·19	564	184	96788	96689·60
455	288	63072	62928·05	510	128	79164	79060·93	565	448	97236	97032·77
456	144	63216	63204·97	511	432	79596	79371·28	566	282	97518	97376·55
457	456	63672	63482·48	512	256	79852	79682·23	567	324	97842	97720·94
458	228	63900	63760·61	513	324	80176	79993·79	568	280	98122	98065·94
459	288	64188	64039·35	514	256	80432	80305·96	569	568	98690	98411·55
460	176	64364	64318·69	515	408	80840	80618·74	570	144	98834	98757·76
461	460	64824	64598·64	516	168	81008	80932·13	571	570	99404	99104·58
462	120	64944	64879·20	517	460	81468	81246·12	572	240	99644	99452·01
463	462	65406	65160·36	518	216	81684	81560·72	573	380	100024	99800·05

TABLE (*continued*).

n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2}n^2$	n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2}n^2$	n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2}n^2$
574	240	100264	100148·70	629	576	120544	120260·45	684	216	142380	142211·17
575	440	100704	100497·95	630	144	120688	120643·14	685	544	142924	142627·30
576	192	100896	100847·81	631	630	121318	121026·44	686	294	143218	143044·03
577	576	101472	101198·28	632	312	121630	121410·35	687	456	143674	143461·37
578	272	101744	101549·36	633	420	122050	121794·86	688	536	144010	143879·32
579	384	102128	101901·05	634	316	122366	122179·98	689	624	144634	144297·88
580	224	102352	102253·34	635	504	122870	122565·71	690	176	144810	144717·05
581	492	102844	102606·24	636	208	123078	122952·05	691	690	145500	145136·82
582	192	103036	102959·75	637	504	123582	123338·00	692	344	145844	145557·20
583	520	103556	103313·87	638	280	123862	123726·55	693	360	146204	145978·19
584	288	103844	103668·60	639	420	124282	124114·71	694	346	146550	146399·79
585	288	104132	104023·93	640	256	124538	124503·48	695	552	147102	146821·99
586	292	104424	104379·87	641	640	125178	124892·86	696	224	147326	147244·80
587	586	105010	104736·42	642	212	125390	125282·85	697	640	147966	147668·22
588	168	105178	105093·58	643	642	126032	125673·44	698	348	148314	148092·25
589	540	105718	105451·35	644	264	126296	126064·64	699	464	148778	148516·89
590	232	105950	105809·72	645	336	126632	126456·45	700	240	149018	148942·14
591	392	106342	106168·70	646	288	126920	126848·87	701	700	149718	149367·99
592	288	106630	106528·29	647	646	127566	127241·89	702	216	149934	149794·45
593	592	107222	106888·49	648	216	127782	127635·52	703	648	150582	150221·52
594	180	107402	107249·29	649	580	128362	128029·76	704	320	150902	150649·20
595	384	107786	107610·70	650	240	128602	128424·60	705	368	151270	151077·48
596	296	108082	107972·72	651	360	128962	128820·06	706	352	151622	151506·37
597	396	108478	108335·35	652	324	129286	129216·12	707	600	152222	151935·87
598	264	108742	108698·59	653	652	129938	129612·79	708	232	152454	152365·98
599	598	109340	109062·43	654	216	130154	130010·07	709	708	153162	152796·70
600	160	109500	109426·88	655	520	130674	130407·96	710	280	153442	153228·02
601	600	110100	109791·94	656	320	130994	130806·46	711	468	153910	153659·95
602	252	110352	110157·61	657	432	131426	131205·56	712	352	154262	154092·49
603	396	110748	110523·89	658	276	131702	131605·27	713	660	154922	154525·64
604	300	111048	110890·77	659	658	132360	132005·59	714	192	155114	154959·40
605	440	111488	111258·26	660	160	132520	132406·52	715	480	155594	155393·76
606	200	111688	111626·36	661	660	133180	132808·06	716	356	155950	155828·73
607	606	112294	111995·07	662	330	133510	133210·20	717	476	156426	156264·31
608	288	112582	112364·39	663	384	133894	133612·95	718	358	156784	156700·50
609	336	112918	112734·31	664	328	134222	134016·31	719	718	157502	157137·30
610	240	113158	113104·84	665	432	134654	134420·28	720	192	157694	157574·70
611	552	113710	113475·98	666	216	134870	134824·86	721	612	158306	158012·71
612	192	113902	113847·73	667	616	135486	135230·04	722	342	158648	158451·33
613	612	114514	114220·09	668	332	135818	135635·83	723	480	159128	158890·56
614	306	114820	114593·05	669	444	136262	136042·23	724	360	159488	159330·40
615	320	115140	114966·62	670	264	136526	136449·24	725	560	160048	159770·84
616	240	115380	115340·80	671	600	137126	136856·86	726	220	160268	160211·89
617	616	115996	115715·59	672	192	137318	137265·08	727	726	160994	160653·55
618	204	116200	116090·99	673	672	137990	137673·91	728	288	161282	161095·82
619	618	116818	116466·99	674	336	138326	138083·35	729	486	161768	161538·69
620	240	117058	116843·60	675	360	138686	138493·40	730	288	162056	161982·17
621	396	117454	117220·82	676	312	138998	138904·05	731	672	162728	162426·26
622	310	117764	117598·65	677	676	139674	139315·31	732	240	162968	162870·96
623	528	118292	117977·08	678	224	139898	139727·18	733	732	163700	163316·27
624	192	118484	118356·12	679	576	140474	140139·66	734	366	164066	163762·18
625	500	118984	118735·77	680	256	140730	140552·75	735	336	164402	164208·70
626	312	119296	119116·03	681	452	141182	140966·44	736	352	164754	164655·83
627	360	119656	119496·90	682	300	141482	141380·74	737	660	165414	165103·57
628	312	119968	119878·37	683	682	142164	141795·65	738	240	165654	165551·92

TABLE (*continued*).

n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2}n^2$	n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2}n^2$	n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2}n^2$
739	738	166392	166000·87	794	396	191870	191629·56	849	564	219340	219097·23
740	288	166680	166450·43	795	416	192286	192112·56	850	320	219660	219613·66
741	432	167112	166900·60	796	396	192682	192596·17	851	792	220452	220130·71
742	312	167424	167351·38	797	796	193478	193080·39	852	280	220732	220648·36
743	742	168166	167802·77	798	216	193694	193565·21	853	852	221584	221166·62
744	240	168406	168254·76	799	736	194430	194050·64	854	360	221944	221685·48
745	592	168998	168707·36	800	320	194750	194536·67	855	432	222376	222204·96
746	372	169370	169160·57	801	528	195278	195023·32	856	424	222800	222725·04
747	492	169862	169614·39	802	400	195678	195510·57	857	856	223656	223245·73
748	320	170182	170068·82	803	720	196398	195998·43	858	240	223896	223767·03
749	636	170818	170523·85	804	264	196662	196486·90	859	858	224754	224288·93
750	200	171018	170979·50	805	528	197190	196975·98	860	336	225090	224811·44
751	750	171768	171435·75	806	360	197550	197465·66	861	480	225570	225334·56
752	368	172136	171892·61	807	536	198086	197955·96	862	430	226000	225858·29
753	500	172636	172350·07	808	400	198486	198446·86	863	862	226862	226382·62
754	336	172972	172808·14	809	808	199294	198938·37	864	288	227150	226907·57
755	600	173572	173266·82	810	216	199510	199430·48	865	688	227838	227433·12
756	216	173788	173726·11	811	810	200320	199923·21	866	432	228270	227959·28
757	756	174544	174186·01	812	336	200656	200416·54	867	544	228814	228486·05
758	378	174922	174646·52	813	540	201196	200910·48	868	360	229174	229012·43
759	440	175362	175107·63	814	360	201556	201405·03	869	780	229954	229541·41
760	288	175650	175569·35	815	648	202204	201900·19	870	224	230178	230070·01
761	760	176410	176031·68	816	256	202460	202395·95	871	792	230970	230599·21
762	252	176662	176494·62	817	756	203216	202892·32	872	432	231402	231129·02
763	648	177310	176958·16	818	408	203624	203389·30	873	576	231978	231659·43
764	380	177690	177422·31	819	432	204056	203886·89	874	396	232374	232190·46
765	384	178074	177887·07	820	320	204376	204385·09	875	600	232974	232722·09
766	382	178456	178352·44	821	820	205196	204883·89	876	288	233262	233254·33
767	696	179152	178818·42	822	272	205468	205383·30	877	876	234138	232787·18
768	256	179408	179285·00	823	822	206290	205883·32	878	438	234576	234320·64
769	768	180176	179752·19	824	408	206698	206383·95	879	584	235160	234854·70
770	240	180416	180219·99	825	400	207098	206885·19	880	320	235480	235389·37
771	512	180928	180688·40	826	348	207446	207387·03	881	880	236360	235924·65
772	384	181312	181157·42	827	826	208272	207889·48	882	252	236612	236460·54
773	772	182084	181627·04	828	264	208536	208392·54	883	882	237494	236997·04
774	252	182336	182097·27	829	828	209364	208896·21	884	384	237878	237534·14
775	600	182936	182568·11	830	328	209692	206400·49	885	464	238342	238071·85
776	384	183320	183039·56	831	552	210244	209905·37	886	442	238784	238610·17
777	432	183752	183511·61	832	384	210628	210410·86	887	886	239670	239149·10
778	388	184140	183984·28	833	672	211300	210916·96	888	288	239958	239688·64
779	720	184860	184457·55	834	276	211576	211423·67	889	756	240714	240228·78
780	192	185052	184931·43	835	664	212240	211930·98	890	352	241066	240769·53
781	700	185752	185405·92	836	360	212600	212438·91	891	540	241606	241310·89
782	352	186104	185881·01	837	540	213140	212947·44	892	444	242050	241852·86
783	504	186608	186356·71	838	418	213558	213456·58	893	828	242878	242395·43
784	336	186944	186833·02	839	838	214396	213966·32	894	296	243174	242938·62
785	624	187568	187309·94	840	192	214588	214476·68	895	712	243886	243482·41
786	260	187828	187787·47	841	812	215400	214987·64	896	384	244270	244026·81
787	786	188614	188265·60	842	420	215820	215499·21	897	528	244798	244571·81
788	392	189006	188744·34	843	560	216380	216011·39	898	448	245246	245117·43
789	524	189530	189223·69	844	420	216800	216524·18	899	840	246086	245663·65
790	312	189842	189703·65	845	624	217424	217037·57	900	240	246326	246210·48
791	672	190514	190184·22	846	276	217700	217551·58	901	832	247158	246757·91
792	240	190754	190665·39	847	660	218360	218066·19	902	400	247558	247305·96
793	720	191474	191147·17	848	416	218776	218581·40	903	504	248062	247854·61

TABLE* (*continued*).

n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2}n^2$	n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2}n^2$	n	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2}n^2$
904	448	248510	248403·88	937	936	267256	266870·57	970	384	286076	285999·30
905	720	249230	248953·75	938	396	267652	267440·51	971	970	287046	286589·30
906	300	249530	249504·22	939	624	268276	268011·05	972	324	287370	287179·90
907	906	250436	250055·31	940	368	268644	268582·19	973	828	288198	287771·11
908	452	250888	250607·00	941	940	269584	269153·95	974	486	288684	288362·92
909	600	251488	251159·31	942	312	269896	269726·31	975	480	289164	288955·35
910	288	251776	251712·22	943	880	270776	270299·28	976	480	289644	289548·39
911	910	252686	252265·73	944	464	271240	270872·86	977	976	290620	290142·03
912	288	252974	252819·86	945	432	271672	271447·05	978	324	290944	290736·28
913	820	253794	253374·59	946	420	272092	272021·84	979	880	291824	291331·13
914	456	254250	253929·93	947	946	273038	272597·25	980	336	292160	291926·60
915	480	254730	254485·88	948	312	273350	273173·26	981	648	292808	292522·67
916	456	255186	255042·44	949	864	274214	273749·88	982	490	293298	293119·35
917	780	255966	255599·61	950	360	274574	274327·10	983	982	294280	293716·64
918	288	256254	256157·38	951	632	275206	274905·94	984	320	294600	294314·54
919	918	257172	256715·76	952	384	275590	275483·38	985	784	295384	294913·04
920	352	257524	257274·75	953	952	276542	276062·43	986	448	295832	295512·15
921	612	258136	257834·34	954	312	276854	276642·09	987	552	296384	296111·87
922	460	258596	258394·55	955	760	277614	277222·36	988	432	296816	296712·20
923	840	259436	258955·36	956	476	278090	277803·23	989	924	297740	297313·14
924	240	259676	259516·78	957	560	278650	278384·71	990	240	297980	297914·68
925	720	260396	260078·81	958	478	279128	278966·80	991	990	298970	298516·83
926	462	260858	260641·45	959	816	279944	279549·50	992	480	299450	299119·59
927	612	261470	261204·69	960	256	280200	280132·81	993	660	300110	299722·96
928	448	261918	261768·55	961	930	281130	280716·72	994	420	300530	300326·94
929	928	262846	262333·01	962	432	281562	281301·24	995	792	301322	300931·52
930	240	263086	262898·07	963	636	282198	281886·37	996	328	301650	301536·71
931	756	263842	263463·75	964	480	282678	282472·11	997	996	302646	302142·51
932	464	264306	264030·03	965	768	283446	283058·46	998	498	303144	302748·92
933	620	264926	264596·93	966	264	283710	283645·41	999	648	303792	303355·93
934	466	265392	265164·43	967	966	284676	284232·97	1000	400	304192	303963·55
935	640	266032	265732·53	968	440	285116	284821·14				
936	288	266320	266301·25	969	576	285692	285409·92				

* In the extended as well as in the original Table it will be seen that the sum-totient is always intermediate between $3/\pi^2 . n^2$ and $3/\pi^2 . (n+1)^2$.

The formula of verification applied at every tenth step to the T column precludes the possibility of the existence of other than typographical errors or errors of transcription. Accumulative errors are rendered impossible.

10.

ON THE EQUATION TO THE SECULAR INEQUALITIES IN THE PLANETARY THEORY.

[*Philosophical Magazine*, XVI. (1883), pp. 267—269.]

A VERY long time ago I gave, in this *Magazine**, a proof of the reality of the roots in the above equation, in which I employed a certain property of the square of a symmetrical matrix which was left without demonstration. I will now state a more general theorem concerning the *product* of *any* two matrices of which that theorem is a particular case. In what follows it is of course to be understood that the product of two matrices means the matrix corresponding to the *combination* of two *substitutions* which those matrices represent.

It will be convenient to introduce here a notion (which plays a conspicuous part in my new theory of multiple algebra), namely that of the *latent roots* of a matrix—latent in a somewhat similar sense as vapour may be said to be latent in water or smoke in a tobacco-leaf. If from each term in the diagonal of a given matrix, λ be subtracted, the determinant to the matrix so modified will be a rational integer function of λ; the roots of that function are the latent roots of the matrix; and there results the important theorem that the latent roots of any function of a matrix are respectively the same functions of the latent roots of the matrix itself: for example, the latent roots of the square of a matrix are the squares of its latent roots.

The latent roots of the product of two matrices, it may be added, are the same in whichever order the factors be taken. If, now, m and n be any two matrices, and $M = mn$ or nm, I am able to show that the sum of the products of the latent roots of M taken i together in every possible way is equal to the sum of the products obtained by multiplying every minor determinant of the ith order in one of the two matrices m, n by its *altruistic opposite* in the other: the reflected image of any such determinant, in respect to the principal diagonal of the matrix to which it belongs, is its *proper* opposite, and the corresponding determinant to this in the other matrix is its *altruistic opposite*.

[* Vol. I. of this Reprint, p. 378.]

The proof of this theorem will be given in my large forthcoming memoir on Multiple Algebra designed for the *American Journal of Mathematics*.

Suppose, now, that m and n are transverse to one another, that is, that the lines in the one are identical with the columns in the other, and *vice versâ*, then any determinant in m becomes identical with its altruistic opposite in n; and furthermore, if m be a symmetrical matrix, it is its own transverse. Consequently we have the theorem (the one referred to at the outset of this paper) that the sum of the i-ary products of the latent roots of the square of a symmetrical matrix (that is, of the squares of the roots of the matrix itself) is equal to the sum of the squares of all the minor determinants of the order i in the matrix; whence it follows, from Descartes's theorem, that when all the terms of a symmetrical matrix are real, none of its latent roots can be *pure* imaginaries, and, as an easy inference, cannot be *any kind* of imaginaries; or, in other words, all the latent roots of a symmetrical matrix are real, which is Laplace's theorem.

I may take this opportunity of stating the important theorem that if $\lambda_1, \lambda_2, \ldots \lambda_i$ are the latent roots of any matrix m, then

$$\phi m = \Sigma \frac{(m-\lambda_2)(m-\lambda_3)\ldots(m-\lambda_i)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\ldots(\lambda_1-\lambda_i)} \phi\lambda.$$

This theorem of course presupposes the rule first stated by Prof. Cayley (*Phil. Trans.* 1857) for the addition of matrices.

When any of the latent roots are equal, the formula must be replaced by another obtained from it by the usual method of infinitesimal variation. If $\phi m = m^{\frac{1}{\omega}}$, it gives the expression for the ωth root of the matrix; and we see that the number of such roots is ω^i, where i is the order of the matrix. When, however, the matrix is *unitary*, that is, all its terms except the diagonal ones are *zeros*, or *zeroidal*, that is, when all its terms are *zeros*, this conclusion is no longer applicable, and a certain definite number of arbitrary quantities enter into the general expressions for the roots.

The case of the extraction of any root of a unitary matrix of the second order was first considered and successfully treated by the late Mr Babbage; it reappears in M. Serret's *Cours d'Algèbre supérieure*. This problem is of course the same as that of finding a function $\frac{ax+b}{cx+d}$ of any given order of periodicity. My memoir will give the solution of the corresponding problem for a matrix of any order. Of the many unexpected results which I have obtained by my new method, not the least striking is the *rapprochement* which it establishes between the theory of Matrices and that of Invariants. The theory of invariance relative to associated Matrices includes and transcends that relative to algebraical functions.

11.

ON THE INVOLUTION AND EVOLUTION OF QUATERNIONS.

[*Philosophical Magazine*, XVI. (1883), pp. 394—396.]

THE subject-matter of quaternions is really nothing more nor less than that of substitutions of the second order, such as occur in the familiar theory of quadratic forms. A linear substitution of the second order is in essence identical with a square matrix of the second order, the law of multiplication between one such matrix and another being understood to be the same as that of the composition of one substitution with another, and therefore depending on the order of the factors; but as regards the multiplication of three or more matrices, subject to the same associative law as in ordinary algebraical multiplication.

Every matrix of the second order may be regarded as representing a quaternion, and *vice versâ*; in fact if, using i to denote $\sqrt{(-1)}$, we write a matrix m of the second order under the form

$$\begin{matrix} a+bi, & c+di, \\ -c+di, & a-bi, \end{matrix}$$

we have by definition,

$$m = a\alpha + b\beta + c\gamma + d\delta,$$

where $$\alpha = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}, \quad \beta = \begin{matrix} i & 0 \\ 0 & -i \end{matrix}, \quad \gamma = \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}, \quad \delta = \begin{matrix} 0 & i \\ i & 0 \end{matrix}.$$

Now $$\alpha^2 = \alpha, \quad \beta^2 = \gamma^2 = \delta^2 = -\alpha,$$
$$\alpha\beta = \beta\alpha = \beta, \quad \alpha\gamma = \gamma\alpha = \gamma, \quad \alpha\delta = \delta\alpha = \delta,$$
$$\beta\gamma = -\gamma\beta = \alpha, \quad \gamma\delta = -\delta\gamma = \beta, \quad \delta\beta = -\beta\delta = \gamma;$$

so that we may for α, β, γ, δ, substitute 1, h, k, l, four symbols subject to the same laws of self-operation and mutual interaction as unity and the three Hamiltonian symbols. Now I have given the universal formula for expressing any given function of a matrix of *any* order as a rational function of that matrix and its latent roots; and consequently the qth power or root of any

quadratic matrix, and therefore of any quaternion, is known. As far as I am informed, only the square root of a quaternion has been given in the text-books on quaternions, notably by Hamilton in his Lectures on Quaternions.

The latent roots of m are the roots of the quadratic equation

$$\lambda^2 - 2a\lambda + a^2 + b^2 + c^2 + d^2 = 0.$$

The general formula

$$\phi m = \Sigma\phi\lambda_1 \frac{(m-\lambda_2)(m-\lambda_3)\dots(m-\lambda_i)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\dots(\lambda_1-\lambda_i)};$$

where i is the order of the matrix m, when $i = 2$ and $\phi m = m^{\frac{1}{q}}$, becomes

$$m^{\frac{1}{q}} = \frac{\lambda_1^{\frac{1}{q}} - \lambda_2^{\frac{1}{q}}}{\lambda_1 - \lambda_2} m - \frac{\lambda_2\lambda_1^{\frac{1}{q}} - \lambda_1\lambda_2^{\frac{1}{q}}}{\lambda_1 - \lambda_2},$$

where λ_1, λ_2 are the roots of the above equation. If μ is the modulus of the quaternion, namely is $\sqrt{(a^2+b^2+c^2+d^2)}$, and $\mu\cos\theta = a$, the latent roots λ_1, λ_2 assume the form

$$\mu(\cos\theta \pm i\sin\theta).$$

When the modulus is zero the two latent roots are equal to one another, and to a, the scalar of the quaternion; so that in this case the ordinary theory of vanishing fractions shows that

$$m^{\frac{1}{q}} = a^{\frac{1}{q}}\left(\frac{m}{a} + \frac{q-1}{q}\right).$$

In the general case there are q^2 roots of the qth order to a quaternion. Calling $\frac{\pi}{q} = \omega$, and writing $m^{\frac{1}{q}} = Am + B$,

$$A = \frac{\mu^{\frac{1}{q}}}{\mu} \frac{\cos\left(\frac{\theta}{q}+2k\omega\right) + i\sin\left(\frac{\theta}{q}+2k\omega\right) - \cos\left(\frac{\theta}{q}+2k'\omega\right) + i\sin\left(\frac{\theta}{q}+2k'\omega\right)}{2i\sin\theta},$$

$$B = -\mu^{\frac{1}{q}} \frac{\cos\left(\frac{q-1}{q}\theta+2k\omega\right) + i\sin\left(\frac{q-1}{q}\theta+2k\omega\right) - \cos\left(\frac{q-1}{q}\theta+2k'\omega\right) + i\sin\left(\frac{q-1}{q}\theta+2k'\omega\right)}{2i\sin\theta}.$$

For the q system of values $k = k' = 1, 2, 3 \dots q$, the coefficients A and B will be real, for the other $q^2 - q$ systems of values imaginary; so that there are q quaternion-proper qth roots of a quaternion-proper in Hamilton's sense, and $q^2 - q$ of the sort which, by a most regrettable piece of nomenclature, he terms bi-quaternions. The real or proper-quaternion values of $m^{\frac{1}{q}}$ are

$$\frac{\mu^{\frac{1}{q}}}{\sin\theta}\left\{\sin\left(\frac{\theta}{q}+2k\omega\right)\frac{m}{\mu} + \sin\left(\frac{q-1}{q}\theta+2k\omega\right)\right\},$$

$\mu^{\frac{1}{q}}$ meaning *the* or (when there is an alternative) *either* real value of the qth root of the modulus.

In the qth root (or power) of a quaternion m, the form $Am + B$ shows that the vector-part remains constant to an ordinary algebraical factor *près*; and we know *à priori* from the geometrical point of view that this ought to be the case. When the vector disappears a porism starts into being; and besides the values of the roots given by the general formula, there are others involving arbitrary parameters. Babbage's famous investigation of the form of the homographic function of $\frac{px+q}{rx+s}$ of x, which has a periodicity of any given degree q, is in fact (surprising as such a statement would have appeared to Babbage and Hamilton) one and the same thing as to find the qth root of unity under the form of a quaternion!

It is but justice to the eminent President of the British Association to draw attention to the fact that the substance of the results here set forth (although arrived at from an independent and more elevated order of ideas) may be regarded as a statement (reduced to the explicit and most simple form) of results capable of being extracted from his memoir on the Theory of Matrices, *Phil. Trans.* Vol. CXLVIII. (1858) (*vide* pp. 32—34, arts. 44—49).

12.

ON THE INVOLUTION OF TWO MATRICES OF THE SECOND ORDER.

[*British Association Report, Southport* (1883), pp. 430—432.]

IF m, n be two matrices of any order i, then, taking the determinant of the matrix $z + yn + xm$, there results a ternary quantic in the variables x, y, z, which may be termed the quantic of the corpus m, n.

In what follows I confine myself almost exclusively to the case of a corpus of the second order; the quantic may be written

$$z^2 + 2bzx + 2cyz + dx^2 + 2exy + fy^2:$$

it is then easy to establish the identical relations

$$m^2 - 2bm + d = 0,$$
$$mn + nm - 2bn - 2cm + 2e = 0,$$
$$n^2 - 2cn + f = 0.$$

It hence easily appears that any given function of m, n can, by aid of the five parameters b, c, d, e, f, be expressed in the form $A + Bm + Cn + Dmn$.

This form containing four arbitrary constants, it follows that in general any given matrix of the second order can be expressed as a function of m and n; for there will be four linear equations between A, B, C, D and the four elements of the given matrix. But this statement is subject to two cases of exception.

The first of these is when n and m are functions of one another: for in this case $A + Bm + Cn + Dmn$ is reducible to the form $P + Qm$, and there will be only two disposable constants wherewith to satisfy the four linear equations.

The second case is when the determinant of the fourth order formed by the elements of the four matrices $\begin{vmatrix} 1, & m \\ n, & mn \end{vmatrix}$ vanishes; writing

$$m, n = \begin{vmatrix} t_1, & t_2 \\ t_3, & t_4 \end{vmatrix}, \quad \begin{vmatrix} \tau_1, & \tau_2 \\ \tau_3, & \tau_4 \end{vmatrix}$$

respectively, it is not difficult to show that the value of this determinant is

$$(t_2\tau_3 - \tau_2 t_3)^2 + \{(t_1 - t_4)\,\tau_2 - (\tau_1 - \tau_4)\,t_2\}\,\{(t_1 - t_4)\,\tau_3 - (\tau_1 - \tau_4)\,t_3\}.$$

This expression is a function of the five parameters b, c, d, e, f, as may be shown in a variety of ways.

Thus it is susceptible of easy proof that if μ_1, μ_2 are the roots of the equation $\mu^2 - 2b\mu + d = 0$, and ν_1, ν_2 the roots of the equation $\nu^2 - 2c\nu + f = 0$, then, the two matrices being related as above, we must have

$$(m - \mu_1)\,(n - \nu_1) = 0,$$
$$(n - \nu_2)\,(m - \mu_2) = 0,$$

and consequently, by virtue of the middle one of the three identities,

$$\mu_1\nu_1 + \mu_2\nu_2 - 2e = 0.$$

Writing this in the form

$$(\mu_1\nu_1 + \mu_2\nu_2 - 2e)\,(\mu_1\nu_2 + \mu_2\nu_1 - 2e) = 0,$$

this is $$4e^2 - 2e\,.\,4bc + (\mu_1^2 + \mu_2^2)\,\nu_1\nu_2 + (\nu_1^2 + \nu_2^2)\,\mu_1\mu_2 = 0,$$

which gives $$e^2 - 2bce + b^2f + c^2d - df = 0\,;$$

the function on the left hand is the invariant (discriminant) of the ternary quantic appurtenant to the corpus, and we have this invariant $= 0$ as the necessary and sufficient condition of the involution of the elements of the corpus; the invariant in question is for this reason called the involutant.

Expressing the values of the coefficients in terms of the elements of the two matrices, namely

$$2b = t_1 + t_4, \quad 2c = \tau_1 + \tau_4,$$
$$d = t_1t_4 - t_2t_3, \quad 2e = t_1\tau_4 + \tau_1t_4 - t_2\tau_3 - t_3\tau_2, \quad f = \tau_1\tau_4 - \tau_2\tau_3,$$

it at once appears that the two expressions for the involutant are, to a numerical factor *près*, identical.

It can be shown *à priori* that the involutant of a corpus of the second order must be expressible in terms of the coefficients of the function; and therefore, being obviously invariantive in regard to linear substitutions impressed on m, n, it must be also invariantive for linear substitutions impressed on z, x, y, and must therefore be the invariant of the function. The corresponding theorem is not true, it should be observed, for the involutant of a corpus beyond the second order; for such involutant cannot in general be expressed in terms of the coefficients of the function.

The expression for the involutant in terms of the t's and τ's may also be obtained directly from the equation $(m - \mu_1)\,(n - \nu_1) = 0$. To this end it is only necessary to single out any term of the matrix represented by the left-hand side of the equation and equate it to zero: the resulting equation rationalised will be found to reproduce the expression in question.

I have thus indicated four methods of obtaining the involutant to a matrix-corpus of the second order; but there is yet a fifth, the simplest of all, and the most suggestive of the course to be pursued in investigating the higher order of involutants.

I observe that for a corpus of any order the function $mn - nm$ is invariantive for any linear substitution impressed on m and n. Its determinant will therefore be an invariant for any substitution impressed on m and n. When m and n are of the second order, reducing each term of $(mn - nm)^2$, that is $mnmn - mn^2m - nm^2n + nmnm$, and of $mn - nm$, by means of the three identical equations, to the form of a linear function of $mn, m, n, 1$, it will be found without difficulty that there results the identical equation

$$(mn - nm)^2 + I = 0,$$

the coefficient of $mn - nm$ vanishing. Consequently the determinant of the matrix $mn - nm$ is equal to I, which on calculation will be found to be identical with the invariant of the ternary quadric function.

It is obvious from the three identical equations that if m, n are in involution—that is, if their involutant is zero—every rational and integral function of m, n will be in involution with every other rational and integral function of m, n. Hence follows this new and striking theorem concerning matrices of the second order: If $f(m, n)$ and $\phi\,(m, n)$ are any rational functions whatever of m, n, the determinant to the matrix $mn - nm$ is contained as a factor in the determinant to the matrix $f\phi - \phi f$.

It may be noticed that f, ϕ need not be integer functions *by stipulation*, because any linear function of $mn, m, n, 1$, divided anteriorly or posteriorly by a second like function, can itself be expressed as a linear function of the same four terms.

As a very simple example of the theorem, observe that the determinant of $m^2n - mnm$ will contain as a factor the determinant of $mn - nm$.

13.

SUR LES QUANTITÉS FORMANT UN GROUPE DE NONIONS ANALOGUES AUX QUATERNIONS DE HAMILTON.

[*Comptes Rendus*, XCVII. (1883), pp. 1336—1340.]

ON sait qu'on peut tout à fait (et très avantageusement) changer la base de la théorie des quaternions en considérant les trois symboles i, j, k de Hamilton comme des matrices binaires.

Si h, j sont des matrices binaires qui satisfont à l'équation $hj = -jh$, on démontre facilement que, en écartant le cas où $hj = jh = 0$, h^2 et k^2 seront de la forme

$$\begin{matrix} c & 0 \\ 0 & c \end{matrix}, \quad \begin{matrix} \gamma & 0 \\ 0 & \gamma \end{matrix}$$

c'est-à-dire cu, γu, où u est l'unité binaire

$$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$$

On peut ajouter, si l'on veut, les deux conditions $c^2 = \bar{1}$, $\gamma^2 = \bar{1}$; alors, en supprimant, pour plus de brièveté, le u, qui jouit de propriétés tout à fait analogues à celles de l'unité ordinaire, on obtient facilement les équations connues

$$h^2 = \bar{1}, \quad j^2 = \bar{1}, \quad k^2 = \bar{1},$$

$$hj = -jh = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

De plus, en supposant que (i, j) soit un système particulier qui satisfait à l'équation $ij = -ji$, on peut déduire les valeurs universelles de I, J qui satisfont à l'équation $IJ = -JI$ en termes de i, j. En effet, on démontre rigoureusement que, en écartant toujours la solution $mn = nm = 0$, on aura

$$I = ai + bj + cij,$$

$$J = \alpha i + \beta j + \gamma ij,$$

avec la seule condition $a\alpha + b\beta + c\gamma = 0$. De plus, si l'on suppose $i^2 = j^2 = \bar{u}$ et aussi $I^2 = J^2 = \bar{u}$, on aura

$$a^2 + b^2 + c^2 = 1, \quad \alpha^2 + \beta^2 + \gamma^2 = 1,$$

de sorte que, en écrivant $ij = k$, $IJ = K$ et $K = Ai + Bj + Ck$, la matrice

$$\begin{matrix} a & b & c \\ \alpha & \beta & \gamma \\ A & B & C \end{matrix}$$

formera une matrice orthogonale. Une solution, parmi les plus simples, des équations $ij = -ji$, $i^2 = \bar{u}$, $j^2 = \bar{u}$, est la suivante:

$$i = \begin{vmatrix} \theta & 0 \\ 0 & -\theta \end{vmatrix}, \quad j = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}$$

et conséquemment

$$k = ij = \begin{vmatrix} 0 & -\theta \\ -\theta & 0 \end{vmatrix},$$

où $\theta = \sqrt{(-1)}$.

En écrivant une quantité binormale quelconque (c'est-à-dire une matrice binaire) sous la forme

$$\begin{matrix} a + b\theta, & -c - d\theta, \\ c - d\theta, & a - b\theta, \end{matrix}$$

on voit qu'elle peut être mise sous la forme $au + bi + cj + dk$, où il est souvent commode de supprimer (c'est-à-dire de sous-entendre) sans écrire l'unité binaire u.

On peut construire d'une manière tout à fait analogue un système de nonions en considérant l'équation $m = \rho n$, où m, n sont des matrices ternaires et ρ une racine cubique primitive de l'unité (voir* la *Circular* du *Johns Hopkins University* qui va prochainement paraître), en prenant pour les nonions fondamentaux u (l'unité ternaire)

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

et les huit matrices m, m^2; n, n^2; m^2n, mn^2; mn, m^2n^2 construites avec les valeurs les plus simples de m, n qui satisfont aux équations

$$nm = \rho mn, \quad m^3 = u, \quad n^3 = u.$$

Les valeurs

$$m = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^2 \end{vmatrix} \quad \text{et} \quad n = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & \rho \\ \rho^2 & 0 & 0 \end{vmatrix}$$

peuvent être prises pour les valeurs basiques du système de nonions.

Une quantité ternaire (c'est-à-dire une matrice) quelconque s'exprime alors sous la forme

$$a + bm + \beta m^2 + cn + \gamma n^2 + dm^2n + \delta mn^2 + emn + \epsilon m^2n^2;$$

[* Vol. III. of this Reprint, p. 647. Also below, p. 122.]

mais, quand cette matrice M est capable de s'associer avec une autre N dans l'équation $NM = \rho MN$, alors il devient nécessaire que

$$a = 0, \quad b\beta + c\gamma + d\delta + e\epsilon = 0.$$

Je n'entrerai pas ici dans les détails de la méthode d'associer la solution générale de l'équation $NM = \rho MN$ avec une solution quelconque particulière de cette équation, mais je me bornerai à expliquer quelles sont les conditions auxquelles les éléments de M et de N doivent satisfaire afin que cette équation ait lieu.

M. Cayley a résolu la question analogue pour les matrices binaires dans le beau Mémoire, qu'il a publié dans les *Transactions of the Royal Society* de 1858. En supposant que m et n sont les matrices

$$\begin{matrix} a & b \\ c & d \end{matrix} \qquad \begin{matrix} a' & b' \\ c' & d' \end{matrix}$$

il trouve que, afin que $nm = -mn$, il faut avoir

$$a + d = 0, \quad a' + d' = 0, \quad aa' + bc' + cb' + dd' = 0.$$

Au lieu de cette troisième équation (en la combinant avec les deux précédentes), on peut écrire

$$ad' + a'd - bc' - b'c = 0.$$

Alors ces trois conditions équivalent à dire que le déterminant de la matrice $xu + my + nz$ (u étant l'unité binaire), qui, en général, est de la forme

$$x^2 + 2Bxy + 2Cxz + Dy^2 + 2Eyz + Fz^2,$$

se réduira à la forme

$$x^2 + Dy^2 + Fz^2,$$

car, dans le déterminant de $xu + my + nz$, c'est-à-dire de

$$\begin{vmatrix} x + ay + a'z & by + b'z \\ cy + c'z & x + dy + d'z \end{vmatrix},$$

les coefficients de xy, xz, yz seront évidemment

$$a + d, \quad a' + d', \quad ad' + a'd - bc' - b'c$$

respectivement.

Passons au cas de m et n, matrices ternaires qui satisfont à l'équation

$$nm = \rho mn.$$

Formons le déterminant de $xu + ym + zn$, où u représente l'unité ternaire

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

Ce déterminant sera de la forme

$$x^3 + 3Bx^2y + 3Cx^2z + 3Dxy^2 + 6Exyz + 3Fxz^2 + Gy^3 + 3Hy^2z + 3Kyz^2 + Lz^3,$$

et je trouve que, dans le cas supposé, il faut que les sept conditions souscrites soient satisfaites; $B = 0$, $C = 0$, $D = 0$, $E = 0$, $F = 0$, $H = 0$, $K = 0$, de sorte que la fonction en x, y, z devient une somme de trois cubes, mais ces sept conditions, qu'on pourrait nommer *conditions paramétriques*, quoique nécessaires, ne sont pas suffisantes; il faut y ajouter une huitième condition que je nommerai $Q = 0$.

Pour former Q, voici la manière de procéder:

En supposant que

$$m = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} \quad \text{et} \quad n = \begin{vmatrix} a' & b' & c' \\ d' & e' & f' \\ g' & h' & k' \end{vmatrix},$$

on écrit, au lieu de m, son transversal

$$\begin{vmatrix} a' & d' & g' \\ b' & e' & h' \\ c' & f' & k' \end{vmatrix},$$

et l'on forme neuf produits en multipliant chaque déterminant mineur du second ordre contenu dans m avec le déterminant mineur semblablement posé dans le transversal de n: la somme de ces neuf produits est Q.

Ces huit conditions que je démontre sont suffisantes et nécessaires (en écartant comme auparavant le cas où $nm = mn = 0$) pour que $nm = \rho mn$.

On pourrait très bien se demander ce qui arrive dans le cas où les sept conditions paramétriques sont satisfaites, mais non pas la huitième condition supplémentaire.

Dans ce cas, je trouve* que mn et nm restent fonctions l'une et l'autre et qu'on aura

$$nm = \quad A + B_1 mn + C(mn)^2,$$
$$mn = -A + B_2 nm + C(nm)^2,$$

où B_1, B_2 sont les racines de l'équation algébrique

$$B^2 + B + 1 = 0,$$

A, C étant deux quantités arbitraires et indépendantes, sauf que l'une d'elles ne peut pas s'évanouir sans l'autre, les deux s'évanouissant ensemble pour le cas (et seulement pour le cas) où Q (qui fournit la condition supplémentaire) s'évanouit.

[* See footnote [†], p. 154 below.]

14.

ON QUATERNIONS, NONIONS, SEDENIONS, ETC.

[*Johns Hopkins University Circulars*, III. (1884), pp. 7—9.]

(1) SUPPOSE that m and n are two matrices of the second order.

Then if we call the determinant of the matrix $x + my + nz$,

$$x^2 + 2bxy + 2cxz + dy^2 + 2eyz + fz^2,$$

the necessary and sufficient conditions for the subsistence of the equation $nm = -mn$ is that $b = 0$, $c = 0$, $e = 0$, and if we superadd the equations $m^2 + 1 = 0$, $n^2 + 1 = 0$, then $d = 1$ and $f = 1$, or in other words in order to satisfy the equations $mn = -nm$, $m^2 = -1$, $n^2 = -1$, where it will of course be understood that in these (as in the equations $m^2 + 1 = 0$, $n^2 + 1 = 0$) 1 is the abbreviated form of the matrix $\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}$ and $\bar{1}$ of* the form $\begin{smallmatrix} \bar{1} & 0 \\ 0 & \bar{1} \end{smallmatrix}$, the necessary and sufficient condition is that the determinant of $x + my + nz$ shall be equal to $x^2 + y^2 + z^2$.

The simplest mode of satisfying this condition is to write $m = \begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}$, $n = \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}$, i meaning $\sqrt{(-1)}$, which gives $mn = \begin{smallmatrix} 0 & -i \\ -i & 0 \end{smallmatrix}$ and $nm = \begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}$.

It is easy to express any matrix of the second order as a linear function of 1 $\left(\text{meaning } \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ m, n, p, where p stands for mn.

For if $\begin{smallmatrix} a, & b \\ c, & d \end{smallmatrix}$ be any such matrix it is only necessary to write

$$a = f + ig, \quad b = -h - ki,$$
$$d = f - ig, \quad c = -h + ki,$$

and then $\begin{smallmatrix} a, & b \\ c, & d \end{smallmatrix} = f + gm + hn + kp$.

The most general solution of the equations $MN = -NM$, $M^2 = N^2 = -1$, must contain three arbitrary constants, namely, the difference between the number of terms in m and n, and the number of conditions $b = 0$, $c = 0$, $e = 0$, $d = 1$, $f = 1$, which are to be satisfied.

[* $\bar{1}$ denotes -1.]

Suppose M, N to be the most general solution fulfilling these conditions; we may write

$$M = f + gm + hn + kp,$$
$$N = f' + g'm + h'n + k'p,$$

where m, n is any particular solution and $p = mn$, and we shall have inasmuch as $M^2 = \bar{1}$,

$$f^2 - g^2 - h^2 - k^2 + 2fgm + 2fhn + 2fkp = \text{the matrix } \bar{1},$$

and consequently $$g^2 + h^2 + k^2 = 1 + f^2,$$

$$fg = 0, \quad fh = 0, \quad fk = 0.$$

Hence $f = 0$ and $$g^2 + h^2 + k^2 = 1.$$

Similarly $f' = 0$ and $$g'^2 + h'^2 + k'^2 = 1,$$

and also inasmuch as $MN = -NM$,

$$gg' + hh' + kk' = 0,$$

and since the equations $M^2 = \bar{1}$, $N^2 = \bar{1}$, $MN = -NM$ imply if we make $MN = P$ that $P^2 = -1$, and $MP = -PM$, and $NP = -PN$, it follows that M, N, P, are connected with m, n, p, in the same way as the coordinates of a point referred to one set of rectangular coordinates in space are connected with the coordinates of the same point referred to any other set of the same*.

Herein lies the ground of the geometrical interpretation to which quaternions lend themselves and it is hardly necessary to do more than advert to the fact that the theory of Quaternions is one and the same thing as that of Matrices of the second order viewed under a particular aspect†.

(2) Let m, n now denote matrices of the third order.

We might propose to solve the equation $mn = -nm$.

The result of the investigation is that we must have $m^2 = n^2$, $m^3 = 0$, $n^3 = 0$, and writing $mn = p$, $m^2 = n^2 = q$, there results a set of *quinions*, $1, m, n, p, q$, for which the multiplication is that marked (a_5) p. 144 of the late Prof. Peirce's invaluable memoir in Vol. IV. of the *American Journal of Mathematics*.

* There is another solution possible, obtained by writing

$$-\frac{f}{f'} = \frac{g}{g'} = \frac{h}{h'} = \frac{k}{k'}, \quad f^2 + g^2 + h^2 + k^2 = 0$$

but this leads to a linear relation between m and n, so that $mn = nm$ and consequently $mn = nm = 0$ which is not the kind of solution proposed in the question.

† See my article in the *Lond. and Edin. Phil. Mag.* on "Involution and Evolution of Quaternions," November, 1883. [Above, p. 112.]

But instead of this let us propose the equation $mn = \rho nm$, where ρ is one of the imaginary roots of unity; if now we write the determinant of $x + my + nz$ under the form

$$x^3 + 3bx^2y + 3cx^2z + 3dxz^2 + 6exyz + 3fyz^2 + gy^3 + 3hy^2z + 3kyz^2 + lz^3,$$

it may be shown [cf. p. 126, below] that we must have

$$b = 0, \quad c = 0, \quad d = 0, \quad e = 0, \quad f = 0, \quad h = 0, \quad k = 0,$$

and if we superadd the conditions $m^3 = 1$, $n^3 = 1$, we must also have $g = 1$, $l = 1$, or in other words the determinant to $x + my + nz$ must take the form $x^3 + y^3 + z^3$; but this condition (or system of conditions) although necessary *is not sufficient* (a point which I omitted to notice in my article entitled "A Word on Nonions" inserted* in a previous *Circular*).

It is obviously necessary that we must have $(mn)^3 = 1$.

Now if the *identical equation* to mn be written under the form

$$(mn)^3 - 3B(mn)^2 + 3Dmn - E = 0,$$

B may be shown to be a linear homogeneous function of b, c, and e; also $E = gl = 1$; but D is not a function of $b, c, d, e, f, g, h, k, l$, and will not in general vanish (as it is here required to do) when b, c, d, e, f, h, k vanish. Its value is the sum of the products obtained on multiplying each quadratic minor of m by its *altruistic* opposite in n: (the *proper* opposite to a minor of m means the minor which is the reflected image of such minor viewed in the Principal Diagonal of m regarded as a mirror; and the *altruistic* opposite is the minor which occupies in n a position precisely similar to that of the proper opposite in m). There are, therefore, 10 equations in all to be satisfied between the coefficients of m and n when $m^3 = n^3 = 1$ and $nm = \rho mn$.

These *ten* conditions I have demonstrated are *sufficient* as well as necessary. There remains then $18 - 10$ or 8 arbitrary constants in the general solution. If m, n is a particular solution we may take for M, N (the matrices of the general solution),

$$M = \alpha + \beta m + \gamma m^2 + \alpha' n + \beta' mn + \gamma' m^2 n + \alpha'' n^2 + \beta'' mn^2 + \gamma'' m^2 n^2,$$

$$N = \alpha_1 + \beta_1 m + \gamma_1 m^2 + \alpha_1' n + \beta_1' mn + \gamma_1' m^2 n + \alpha_1'' n^2 + \beta_1'' mn^2 + \gamma_1'' m^2 n^2,$$

and 10 relations between the 18 coefficients *must* be sufficient to enable to be satisfied the equations $M^3 = N^3 = 1$, $NM = \rho MN$: but what these relations are and how they may most simply be expressed I am not at present in a condition to state†.

[* Vol. III. of this Reprint, p. 647.]

† The solution of this problem would seem to involve some unknown expansion of the idea of orthogonalism. Unless $MN = NM = 0$, a solution to be neglected, it may be proved that $\alpha = 0$, $\alpha_1 = 0$.

I showed in "A Word on Nonions" that the 9 first conditions are satisfied by taking

$$m = \begin{matrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^2, \end{matrix} \qquad n = \begin{matrix} 0 & 0 & 1 \\ \rho & 0 & 0 \\ 0 & \rho^2 & 0. \end{matrix}$$

The 10th condition is also satisfied; for the only quadratic minors (not having a zero determinant) in m are $\begin{matrix} 1 & 0 \\ 0 & \rho \end{matrix}$, $\begin{matrix} \rho & 0 \\ 0 & \rho^2 \end{matrix}$, $\begin{matrix} 1 & 0 \\ 0 & \rho^2 \end{matrix}$; the *altruistic opposites* to which in n are $\begin{matrix} 0 & \rho \\ 0 & 0 \end{matrix}$, $\begin{matrix} 0 & \rho^2 \\ 0 & 0 \end{matrix}$, $\begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix}$, the determinants to each of which are zeros, and accordingly we find

$$m^3 = n^3 = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1, \end{matrix}$$

$$nm = \begin{matrix} 0 & 0 & \rho^2 \\ \rho & 0 & 0 \\ 0 & 1 & 0, \end{matrix} \qquad mn = \begin{matrix} 0 & 0 & 1 \\ \rho^2 & 0 & 0 \\ 0 & \rho & 0, \end{matrix}$$

so that $mn = \rho nm$ and $m^3 = n^3 = 1$ as required.

I subjoin an outline proof of the fundamental portion of the theory of Quaternions and Nonions above stated as it will serve to throw much light upon the nature of the processes employed in that new world of thought to which I gave the name of Universal Algebra or the Algebra of multiple quantity: a fuller explanation will be found in the long memoir which I am preparing on the entire subject for the *American Journal of Mathematics*.

(1) As regards the equation $nm = -mn$, where m, n are matrices of the second order.

As before let the determinant of $(x + ym + zn)$ be

$$x^2 + 2bxy + 2cxz + dy^2 + 2eyz + fz^2.$$

I may observe here parenthetically that the Invariant of the above Quantic is equal to the determinant of $mn - nm$, and that when it vanishes $1, m, n, mn$, as also $1, n, m, nm$ are linearly related—or, as I express it, this Invariant is the Involutant of the system m, n or n, m. When m, n are of higher than the second order, the Involutant of m, n, say I, is that function whose vanishing implies that the 9 matrices $(1, m, m^2 \between 1, n, n^2)$ are linearly related, and the Involutant of n, m, say J, that function whose vanishing implies that the 9 quantities $(1, n, n^2 \between 1, m, m^2)$ are so related (I, J being two distinct functions), and so for matrices of any order higher than the second.

By virtue of a general theorem for any two matrices m, n of the second order, the following identities are satisfied:

$$m^2 - 2bm + d = 0,$$
$$mn + nm - 2bn - 2cm + 2e = 0,$$
$$n^2 - 2cn + f = 0.$$

If then $mn + nm = 0$, since m and n cannot be functions of one another (for then $mn = nm$), the second equation shows that $b = 0$, $c = 0$, $e = 0$, and conversely if $b = 0$, $c = 0$, $e = 0$, $mn + nm = 0$, and $m^2 + d = 0$, $n^2 + f = 0$, where, if we please, we may make $d = 1$, $f = 1$.

(2) Let m, n be matrices of the third order, and write as before,

$$\text{Det.}\,(x + ym + zn) = x^3 + 3bx^2y + 3cx^2z + 3dxy^2 + 6exyz + 3fxz^2 + gy^3 + 3hy^2z + 3kyz^2 + lz^3.$$

Then by virtue of the general theorem last referred to* there exist the identical equations

$$m^3 - 3bm^2 + 3dm - g = 0,$$
$$m^2n + mnm + nm^2 - 3b(mn + nm) - 3cm^2 + 3dn + 6em - 3h = 0,$$
$$mn^2 + nmn + n^2m - 3c(mn + nm) - 3bn^2 + 3fm + 6en - 3k = 0,$$
$$n^3 - 3cn^2 + 3fn - l = 0.$$

Let now $nm = \rho mn$, where ρ is either imaginary cube root of unity, then

(1) $m^2n + mnm + nm^2 = 0$ and (2) $mn^2 + nmn + n^2m = 0$;

for greater simplicity suppose also that $m^3 = n^3 = 1$, where 1 means the matrix

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0. \\ 0 & 0 & 1 \end{matrix}$$

From the 1st and 2nd of the four identical equations combined it may be proved that $b = 0$, $d = 0$; I do not produce the proof here because to make it *rigorous*, the theory of Nullity would have to be gone into which would occupy too much space; and in like manner from the 3rd and 4th it may be shown that $c = 0$, $f = 0$†. Hence returning to the two middle equations it follows that $e = 0$, $h = 0$, $k = 0$, and from the two extremes that $g = 1$, $l = 1$.

If then $nm = \rho mn$, $m^3 = 1$, and $n^3 = 1$, it is necessary that

$$b = 0,\quad c = 0,\quad d = 0,\quad e = 0,\quad f = 0,\quad g = 1,\quad h = 0,\quad k = 0,\quad l = 1.$$

But these equations although necessary are manifestly insufficient; for they lead to the equations $m^3 - 1 = 0$, $n^3 - 1 = 0$, and

(1) $m^2n + mnm + nm^2 = 0$; (2) $mn^2 + nmn + n^2m = 0$,

[* By Cayley's theorem, if in Det. $(x + ym + zn)$ we replace x by $-ym - zn$, the result vanishes identically in regard to y and z.]

† Except when m, n are functions of one another, so that mn and nm are identical and consequently are each of them zero.

but not necessarily to $nm = \rho mn$. In fact the supposed equations between m and n involve as a consequence the equation $(mn)^3 = 1$. Now the general identical equation to (mn) is

$$(mn)^3 - 3B(mn)^2 + 3D(mn) - F = 0,$$

where B is the sum of each term in m by its altruistic opposite in n $= 3bc - 2e = 0$, $F = gl = 1$, and D is the sum of each first minor in m by its altruistic opposite in n which sum does not necessarily vanish when b, c, d, e, f, h, k, all vanish. Hence there is a 10th condition necessary not involved in the other 9, namely, $D = 0$. These 10 conditions I shall show are sufficient as well as necessary. For when they are satisfied since $(mn)^3 = 1$, $mn\,.\,mn = n^2m^2$.

Hence from (1) $\qquad m^2n^2 + n^2m^2 + nm^2n = 0,$

and from (2) $\qquad m^2n^2 + n^2m^2 + mn^2m = 0.$

Hence $nm\,.\,mn = mn\,.\,nm$*, and consequently nm is a function of mn [cf. p. 149, below]. Hence we may write

$$nm = A + Bmn + C(mn)^2.$$

But the *latent roots* of mn and nm (which are always identical) are $1, \rho, \rho^2$, hence

$$A + B + C,\quad A + B\rho + C\rho^2,\quad A + B\rho^2 + C\rho,$$

must be equal to $1, \rho, \rho^2$, each to each taken in some one of the 6 orders in which these quantities can be written†.

Solving these 6 systems of linear equations there results:

$$A = 0, \qquad B = 0, \qquad C = 1,\ \rho \text{ or } \rho^2$$

or

$$A = 0, \qquad B = 1,\ \rho \text{ or } \rho^2, \qquad C = 0.$$

Hence $nm = \theta mn$, or $\theta(mn)^2$ where $\theta = 1, \rho, \rho^2$.

If $\qquad nm = \theta(mn)^2, \quad nmmn = \theta(mn)^3 = \theta.$

Hence $\qquad m^2 = \theta n^2\,.\,\theta n^2 = \theta^2 n;$

and $\qquad m^2n + mnm + nm^2 = 3\theta m^4 = 3\theta m = 0,$

so that $m = 0$, and $m^3 = 0 = 1$; and again if $nm = mn$,

$$m^2n + mnm + nm^2 = 2m^2n + mnm = 3m^2n = 0,$$

* This equation is independent of the equation $(mn)^3 = 1$; for

$$nm^2n - mn^2m = (m^2n + mnm + nm^2)\,n - m\,(mn^2 + nmn + n^2m) = 0$$

by virtue of equations (1) and (2) above: accordingly these equations taken alone imply the equations

$$nm = A + B_1 mn + C(mn)^2, \quad mn = -A + B_2 nm - C(nm)^2$$

where B_1, B_2 are the roots of $B^2 + B + 1 - \frac{AC}{2} = 0$; A, C being arbitrary and independent except that each vanishes when and only when the cube of mn and (as a consequence) of nm, is a scalar matrix. [See below, p. 154. Footnote [†].]

† By virtue of the general theorem that the latent roots of any function of a matrix are the like functions of the latent roots of the original matrix.

so that $m^2n = 0$, $n = 0$, and $n^3 = 0 = 1$ as before, where it should be noticed that $0 = 1$ means that $\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$ is identical with $\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$.

Hence the only available hypothesis remaining is the equation $nm = v \, . \, mn$, where v is one of the imaginary cube-roots of unity as was to be proved.

(3) It remains to say a few words on the general equation $nm = kmn$, where m, n are matrices of any order ω. To avoid prolixity I shall confine my remarks to the general case, which is, that where the determinants (or as I am used to say the contents) of m and n are each of them finite; with this restriction, the proposed equation is impossible for general values of k as will be at once obvious from the fact that the totalities of the latent roots of mn and of nm are always identical, but the individual latent roots are by virtue of the proposed equation in the ratio to one another of $1 : k$, which, since by hypothesis no root is zero, is only possible when $k^\omega = 1$.

When the above equation is satisfied the ω^2 equations arising from the identification of nm with kmn cease to be incompatible and (as is necessary or at all events usual in such a contingency) become mutually involved. Thus, for example, when $\omega = 1$ and $k = 1$, the number of independent equations is 0, that is, $1 - 1$, when $\omega = 2$ and $k = -1$ the number is 3, that is, $4 - 1$, when $\omega = 3$ and $k = \rho$ or ρ^2 the number is 8, that is, $9 - 1$; it is fair therefore to presume (although the assertion requires proof) that for any value of ω when k is a primitive ωth root of unity the number of conditions to be satisfied when $nm = kmn$ is $\omega^2 - 1$. Of these the condition that the content of $x + my + nz$ shall be of the form $x^\omega + cy^\omega + c'z^\omega$ will supply

$$\frac{(\omega+1)(\omega+2)}{2} - 3, \text{ that is, } \frac{\omega^2 + 3\omega}{2} - 2,$$

and there will therefore be

$$\frac{\omega^2 - 3\omega}{2} + 1 \text{ or } \frac{(\omega-1)(\omega-2)}{2}$$

to be supplied from some other source.

When k is a non-primitive ωth root of unity, the number of equations of condition is no longer the same. Thus when $k = 1$ we know that n may be of the form

$$A + Bm + Cm^2 + \ldots + Lm^{\omega-1},$$

where $A, B, \ldots L$, and all the ω^2 terms in m are arbitrary, and consequently the number of conditions for that case is $2\omega^2 - (\omega^2 + \omega)$ or $\omega^2 - \omega$. It seems then very probable that if k is a qth power of a primitive ωth root of unity the number of conditions required to satisfy $nm = kmn$ is $\omega^2 - \delta$ where δ is

the greatest common measure of q and ω: but, of course, this assertion awaits confirmation.

When $\omega = 4$ besides the case of $nm = mn$, that is, of n being a function of m of which the solution is known, there will be two other cases to be considered, namely, $nm = -mn$ and $nm = imn$: the former probably requiring 14 and the latter 15 conditions to be satisfied between the coefficients of m, the coefficients of n and the two sets of coefficients combined.

It is worthy of notice that the conditions resulting from the content of $x + my + nz$ becoming a sum of 3 powers are incompatible with the equation $nm = vmn$ when v is other than a primitive ωth root of unity (ω being of course the order of m or n).

Thus suppose $\omega = 4$; the conditions in question applied to the middle one of the 5 identical equations give

$$m^2n^2 + n^2m^2 + mn^2m + nm^2n + mnmn + nmnm = 0;$$

when $nm = imn$ the left-hand side of this equation becomes

$$(1 + i^4 + i^2 + i^2 + i + i^3)\, m^2n^2,$$

that is, is zero, but when $nm = -mn$, the value is

$$(1 + 1 - 1 - 1 - 1 - 1)\, m^2n^2$$

which is not zero, and so in general. Thus the pure power form of the content of $x + my + nz$ is a condition applicable to the case of $\frac{nm}{mn}$ being a primitive root of unity and to no other.

The case of nm being a primitive root of ordinary unity is therefore the one which it is most interesting to thrash out.

There are in this case, we have seen, $\frac{1}{2}(\omega^2 + 3\omega - 4)$ simple conditions expressible by the vanishing of that number of coefficients in the content of $x + my + nz$ and $\frac{1}{2}(\omega - 1)(\omega - 2)$ supplemental ones. What are these last? I think their constitution may be guessed at with a high degree of probability. For revert to the case of $\omega = 3$ in which there is one such found by equating to zero the second coefficient in the identical equation

$$(mn)^3 - 3B(mn)^2 + 3Dmn - G = 0.$$

Suppose now $(m^2n^2)^3 - 3B'(m^2n^2)^2 + 3D'm^2n^2 - G' = 0$

is the identical equation to m^2n^2. By virtue of the 8 conditions supposed to be satisfied we *know* that $nm = \rho mn$ as well as $m^3 = 1$, $n^3 = 1$, and consequently that $(m^2n^2)^3 = 1$. Hence $B' = 0$, $D' = 0$, by virtue of the 7 parameters in the oft-quoted content and of D being all zero, and thus the evanescence of B' or D' imports no new condition.

Now suppose $\omega = 4$, and that

$$(mn)^4 - 4B(mn)^3 + 6D(mn)^2 - 4Gmn + M = 0,$$
$$(m^2n^2)^4 - 4B'(m^2n^2)^3 + 6D'(m^2n^2)^2 - 4G'm^2n^2 + M' = 0.$$

Here we know that B vanishes by virtue of b, c and e vanishing, but $D = 0$, $G = 0$, which must be satisfied if $nm = imn$, will be two new conditions not implied in those which precede. It seems then, although not certain, highly probable that $B' = 0$, $D' = 0$, will be implied in the satisfaction of the antecedent conditions but that $G' = 0$ will be an independent condition, so that $D = 0$, $G = 0$, $G' = 0$, will be the three supplemental conditions: and again when $\omega = 5$ forming the identical equations to mn, m^2n^2, m^3n^3, and using an analogous litteration to what precedes, the supplemental conditions will be

$$\begin{array}{lll} D = 0, & G = 0, & M = 0, \\ & G' = 0, & M' = 0, \\ & & M'' = 0, \end{array}$$

and so in general for any value of ω.

The functions D, G, M, etc., above equated to zero are known from the following theorem of which the proof will be given in the forthcoming memoir*.

If $$(\overline{mn})^\omega + k_1(\overline{mn})^{\omega-1} + \ldots + k_i(\overline{mn})^{\omega-i} + \ldots = 0$$

is the *identical equation* to mn, then k_i is equal to the sum of the product of each minor of order i in m multiplied by its *altruistic* opposite in n.

The annexed example will serve to illustrate in the case of $\omega = 3$ that unless the supplemental condition is satisfied we cannot have $nm = \rho mn$.

Write
$$m = \begin{matrix} 1 & 0 & 0, \\ 0 & \rho & 0, \\ 0 & 0 & \rho^2, \end{matrix} \qquad n = \begin{matrix} 0 & c & k, \\ k & 0 & c\rho, \\ c\rho^2 & k & 0, \end{matrix}$$

then the determinant to $x + my + nz$ will be easily found to be

$$x^3 + y^3 + (c^3 + k^3)z^3;$$

but D becomes $-3\rho ck$, and does not vanish unless $c = 0$ or $k = 0$, and accordingly we find

$$nm = \begin{matrix} 0 & \rho c & \rho^2 k, \\ k & 0 & c, \\ \rho^2 c & \rho k & 0, \end{matrix} \qquad mn = \begin{matrix} 0 & c & k, \\ \rho k & 0 & \rho^2 c, \\ \rho c & \rho^2 k & 0. \end{matrix}$$

When $k = 0$ $mn = \rho^2 nm$, when $c = 0$ $nm = \rho^2 mn$, but on no other supposition will $\frac{nm}{mn}$ be a primitive cube root of unity.

* This theorem furnishes as a Corollary the principle employed to prove the stability of the Solar System. (See *Lond. and Edin. Phil. Mag.*, October, 1883.) [Above, p. 110.]

ADDENDUM.

Referring to the equation $MN = -NM$, and to the eight equations expressing M and N in terms of the combinations of the powers of m with those of n, in which it is to be understood that M and N are *non-vacuous*, we know that the sums of the latent roots of M and of N must each vanish and consequently, as may be proved, that $a = 0$, $a' = 0$, leaving $8 - 2$ or 6 conditions to be satisfied. If we further stipulate that $M^3 = 1$, $N^3 = 1$, there will be 8 relations connecting the coefficients $b, c, \ldots k$ and $b', c', \ldots k'$, so that the 64 coefficients in the 8 equations connecting M, M^2; N, N^2; MN, M^2N^2; M^2N, MN^2, or say rather M, M^2; N, N^2; $\rho^2 MN, \rho^2 M^2N^2$; $\rho M^2N, \rho MN^2$, with like combinations or multiples of combinations of powers of m, n* will be connected together by 56 equations; the coefficients in the expression for any one of the above 8 terms may then be arranged in pairs f_i, f_i'; g_i, g_i'; h_i, h_i'; k_i, k_i'; and in the expression for its fellow by F_i, F_i'; G_i, G_i'; H_i, H_i'; K_i, K_i'; so that the Matrix is resolved as it were into 4 sets of paired columns and 4 sets of paired lines; the 4 different sets of paired lines being found by writing successively $i = 1, 2, 3, 4$.

It is then easy to see that there will be 4 equations of the form

$$\Sigma(f_\alpha G_\alpha' + f_\alpha G_\alpha') = 1,$$

and 6 quaternary groups (that is, 24 equations) of the form

$$\Sigma(f_\alpha G_\beta' + f_\alpha G_\beta') = 0,$$

with liberty to change f into F or G into g or each into each: together then the above are 28 of the 56 conditions required. But inasmuch as the 8 $[m, n]$ arguments may be interchanged with the 8 $[M, N]$ ones, we may transform the above equations by substituting for each letter f its conjugate $\frac{d \log \Delta}{df}$ (where Δ is the content of the Matrix) and thus obtain 28 others, giving in all (if the two sets as presumably is the case are independent) the required 56 conditions: the latter 28, however, may be replaced by others of much simpler form†.

* It is easy to see that the sum of the latent roots of M^iN^j must be zero for all values of i, j so that it is a homogeneous linear function of the 8 quantities $m, m^2, \ldots, mn, m^2n^2$.

† I am still engaged in studying this matrix, which possesses remarkable properties. Is it orthogonal? I rather think not, but that it is allied to a system of 4 pairs of somethings drawn in four mutually perpendicular hyperplanes in space of 4 dimensions. In the general case of $MN = \rho NM$ where ρ is a primitive ωth root of unity, there will be an analogous matrix of the order $\omega^2 - 1$ where each line and each column will consist of $\omega + 1$ groups of $\omega - 1$ associated terms.

The value of the cube of any one of the 8 matrices $M, M^2; \ldots; MN, M^2N^2$ may be expressed as follows: It is P into ternary unity. Such a quantity may be termed by analogy a Scalar. To find $P_{i,j}$ I imagine the 8 letters corresponding to M^iN^j (but without powers of ρ attached) to be set over 8 of the 9 points of inflexion to any cubic curve, the paired letters being made suitably

To me it seems that this vast new science of multiple quantity soars as high above ordinary or quaternion Algebra as the Mécanique Céleste above the "Dynamics of a Particle" or a pair of particles, (if a new Tait and Steele should arise to write on the Dynamics of such pair,) and is as well entitled to the name of Universal Algebra as the Algebra of the past to the name of Universal Arithmetic.

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collinear with the missing 9th point. Then among themselves the 8 letters may be taken in 8 different ways to form collinear triads and the product of the letters in each triad may be called a collinear product; $P_{i,j}$ (which is identical with the Determinant to M^iN^j) will be the sum of the cubes of the 8 letters less 3 times the sum of their 8 collinear products, and its 8 values will be analogous to the 3 values of the sum of 3 squares in the Quaternion Theory. Each of these 8 values is assumed equal to unity.

It may be not amiss to add that the product of four squares by four is representable rationally as a sum of four squares, so if we place (not now 8 specially related but) nine perfectly arbitrary letters over the nine points of inflexion of a cubic curve the sum of their 9 cubes less three times their 12 collinear products multiplied by a similar function of 9 other letters may be expressed by a similar function of 9 quantities lineo-linear functions of the two preceding sets of 9 terms.

By the 8 letters of any set as, for example, $b, \ldots, h'$ being "specialized," I mean that they are subject to the condition $bb'+dd'+ff'+hh'=0$. When this equation is satisfied, and not otherwise, M^3 will be a Scalar, and it *must* be satisfied when $MN=\rho NM$.

15.

ON INVOLUTANTS AND OTHER ALLIED SPECIES OF INVARIANTS TO MATRIX SYSTEMS.

[*Johns Hopkins University Circulars*, III. (1884), pp. 9—12, 34, 35.]

To make what follows intelligible I must premise the meaning and laws of vacuity and nullity.

A matrix is said to be vacuous when its content (the determinant of the matrix) is zero, but it may have various degrees of vacuity from 0 up to ω the order of the matrix.

If from each term in the principal diagonal of a matrix λ be subtracted, the content of the resulting matrix is a function of degree ω in λ; the ω values of λ which make this content vanish are called its latent roots, and if i of these roots are zero, the vacuity (treated as a number) is said to be i. This comes to the same thing as saying that the vacuity is i when the determinant, and the sums of the determinants of the principal minors of the orders $\omega-1$, $\omega-2$, ... $(\omega-i+1)$ are each zero. A principal minor of course means one which is divided into 2 [equal] triangles by the principal diagonal of the parent matrix.

Again the nullity is said to be i when *every* minor of the order $(\omega-i+1)$, and consequently of each superior order, is zero. It follows therefore that it means the same thing to predicate a vacuity 1 and a nullity 1 of any matrix, but for any value of i greater than 1, a nullity i implies a vacuity i but not *vice versâ*; the vacuity may be i, whilst the nullity may have any value from 1 up to i inclusive.

The law of nullity which I am about to enunciate is one of paramount importance in the theory of matrices*.

* The three cardinal laws or landmarks in the science of multiple quantity are (1) the law of *nullity*, (2) the law of *latency*, namely, that if $\lambda_1, \lambda_2, \ldots \lambda_\omega$ are the latent roots of m, then $f\lambda_1, f\lambda_2, \ldots f\lambda_\omega$ are those of fm, including as a consequence that

$$fm=\Sigma f\lambda_1 \frac{(m-\lambda_2)(m-\lambda_3)\ldots(m-\lambda_\omega)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\ldots(\lambda_1-\lambda_\omega)},$$

and (3) the law of *identity*, namely, that the powers and combinations of powers of two matrices m, n of the order ω are connected together by $(\omega+1)$ equations whose coefficients are all included among the coefficients of the determinant to the Matrix

$$x+ym+zn.$$

The law is that the nullity of the product of *two* (and therefore of any number of) matrices cannot be less than the nullity of any factor nor greater than the sum of the nullities of the several factors which make up the product.

Suppose now that $\lambda_1, \lambda_2, \ldots \lambda_\omega$ are the latent roots of any matrix with unequal latent roots of the order ω. It is obvious that any such term as $m - \lambda_1$ will have the nullity 1, for its latent roots will be $0, \lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \ldots \lambda_\omega - \lambda_1$, and consequently its *vacuity* is 1.

Moreover we know from Cayley's famous identical equation that the nullity of the product of all the ω factors is ω.

Hence it follows that if M_i contains i, and M_j the remaining $\omega - i$ of these factors (so that $i + j = \omega$), the nullity of M_i must be exactly i and of M_j exactly j.

For the theorem above stated shows that M_i cannot have a nullity greater than i, nor M_j a nullity greater than j.

Hence if the nullity of the one were less than i or of the other less than j, the nullity of M_iM_j would be less than $i + j$, that is, less than ω, whereas its nullity is ω; hence the two nullities are respectively i and j as was to be shown.

Furthermore we know that the latent roots of $(m - \lambda_1)^a$ are $(\lambda_1 - \lambda_1)^a$; $(\lambda_2 - \lambda_1)^a$; ... $(\lambda_\omega - \lambda_1)^a$.

Hence if the latent roots of m are all distinct, the nullity of $(m - \lambda_1)^a$ is unity and consequently by the same reasoning as that above employed it follows that the nullity of

$$(m - \lambda_1)^{a_1} . (m - \lambda_2)^{a_2} \ldots (m - \lambda_i)^{a_i}$$

is exactly i.

I will now explain what is meant by the Involutant or Involutants of a system of two matrices of like order.

It will be convenient here to introduce the term "topical resultant" of a system of ω^2 matrices each of order ω.

We may denote any matrix say

$$\begin{matrix} a_{1,1} & a_{1,2} & \cdots & a_{1,\omega} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,\omega} \\ \cdots & \cdots & \cdots & \cdots \\ a_{\omega,1} & a_{\omega,2} & \cdots & a_{\omega,\omega} \end{matrix}$$

by the linear form

$$\begin{aligned} & a_{1,1}\, t_{1,1} + a_{1,2}\, t_{1,2} + \ldots + a_{1,\omega}\, t_{1,\omega} \\ + & a_{2,1}\, t_{2,1} + a_{2,2}\, t_{2,2} + \ldots + a_{2,\omega}\, t_{2,\omega} \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ + & a_{\omega,1}\, t_{\omega,1} + a_{\omega,2}\, t_{\omega,2} + \ldots + a_{\omega,\omega}\, t_{\omega,\omega} \end{aligned}$$

where the t system is the same for all matrices of the order ω. If, then, we have ω^2 such matrices, their topical resultant is the Resultant in the ordinary sense of the ω^2 linear forms above written, proper to each of them respectively.

Suppose now that m, n are two independent matrices of the order ω, we may form ω^2 matrices by taking each power of m from 0 to $\omega - 1$ as an antecedent factor, and can combine it with similar powers of n as a consequent factor, and in this way obtain ω^2 matrices, of which the first will be the ω-ary unity, that is, a matrix of the order ω in which the principal diagonal terms are all units and the other terms all zero. The topical resultant of these ω^2 matrices I shall for brevity denote as the Involutant to m, n.

In like manner, inverting the position of the powers of m and of n so as to make the latter precede instead of following the former in the ω^2 products above referred to, we shall obtain another topical resultant which may be termed the Involutant to n, m.

The reason why I speak of these topical resultants as involutants to m, n or n, m is the following:

In general if m, n are two independent matrices, any other matrix p, by means of solving ω^2 linear equations, may obviously be expressed as a linear function of the ω^2 products

$$(1, m, m^2, \ldots, m^{\omega-1})\,(1, n, n^2, \ldots, n^{\omega-1}).$$

There are, however, exceptions to this fact.

The most obvious exception is that which takes place when n is a function of m; for then any ω of the ω^2 products will be linearly related, and there will be substantially only ω disposable quantities to solve ω^2 equations.

Another exception is when the m, n Involutant, that is, the topical resultant of the ω^2 matrices, is zero; in which case the general values of the ω^2 disposable quantities each becomes infinite. So that m, n may be said to be in a kind of mutual involution with one another. So, again, p may in general be expressed as a linear function of the ω^2 matrices

$$(1, n, n^2, \ldots, n^{\omega-1})\,(1, m, m^2, \ldots, m^{\omega-1}),$$

but when the n, m Involutant vanishes this is no longer possible.

When $\omega = 2$ the two involutants, considered as definite determinants, are absolutely equal in magnitude and in Algebraical sign, but when ω exceeds 2 this is no longer the case; the two Involutants are then entirely distinct functions of the elements of m and n.

Thus to take a simple example: if $m = \begin{matrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^2 \end{matrix}$ and $n = \begin{matrix} 0 & \rho & k \\ k & 0 & \rho^2 \\ 1 & k & 0 \end{matrix}$ it will be found by direct calculation of two topical resultants of the 9th order, that the two involutants will be

$$81\,(\rho - \rho^2)\,(k^3 - \rho)^3 \text{ and } 81\,(\rho^2 - \rho)\,(k^3 - \rho^2)^3$$

respectively. The reason why the two involutants coincide in the case of $\omega = 2$ is not far to seek. It depends upon the fact of the existence of the mixed identical equation

$$mn + nm - 2bn - 2cm + 2e = 0\,;$$

from which it is obvious that the topical resultant of 1, m, n, mn is the negative of that of 1, m, n, nm or identical with that of 1, n, m, nm.

By direct calculation it will be found that the Involutant m, n, or n, m, where $m = \begin{matrix} f & g \\ h & k \end{matrix}$ $n = \begin{matrix} f' & g' \\ h' & k' \end{matrix}$ is

$$-(gh' - g'k)^2 + \{(f-k)\,g' - (f'-k')\,g\}\,\{(f-k)\,h' - (f'-k')\,h\},$$

which is the same thing as the content of the matrix $(mn - nm)$. It may also be shown *à priori* or by direct comparison to be identical (to a numerical factor *près*) with the Discriminant of the Determinant to the matrix $(x + ym + zn)$ which is a ternary quantic of the second order. Its actual value is 4 times that discriminant.

Let us consider the analogous case of Mechanical Involution of lines in a plane or in space. There are two questions to be solved. The one is to find the condition that the Involution may exist, that is, that a set of equilibrating forces admit of being found to act along the lines; the second, to determine the relative magnitudes of the forces when the involution exists, and this is the simpler question of the two.

In like manner we may consider two questions in the case of m, n being in either of the two kinds of involution; the one being to find what the condition is of such involution existing, the other what are the coefficients of the ω^2 coefficients in the equation which connects the ω^2 products, when the involution exists.

This latter part of the question (surprising as the assertion may appear and is) admits of a very simple and absolutely general direct and almost instantaneous solution by means of the Law of Nullity, above referred to, as I will proceed to show.

The determination of the Involutants, or at all events of their product, will then be seen to follow as an immediate consequence from this prior determination of the form of the equations which express the involutions of the two kinds respectively.

But first it may be well to explain why and in what sense I refer in the title to Involutants as belonging to a class of invariants. I say, then, that universally involutants are invariants in this sense, that if for m and for n, any function of m, or any function of n be substituted, the ratio of the two Involutants, say I and J, remains unaltered. By virtue of the Identical Equation $(m)^i$ will be of the form of

$$A_i + B_i + C_i m^2 + \ldots + L_i m^{\omega-1}$$

and as a consequence it is easy to see that when m^i is substituted for m, I and J will become respectively PI, PJ where P is the ωth power of the determinant to the matrix formed by writing under one another the $(\omega - 1)$ lines of terms, of which the line $B_i, C_i, \ldots; L_i$ is the general expression.

Moreover, in the particular case where $\omega = 2$ and $I = J$*, besides being an Invariant in this modified sense, I will be an invariant in a sense including but transcending the more ordinary conception of an Invariant; for if when, for m and n, $f(m, n)$ and $\phi(m, n)$ are substituted, I becomes I', then I' will contain I as a factor; this is a consequence of the fact that when m and n are in involution $f(m, n)$ and $\phi(m, n)$ will also be in involution, for in consequence of the identical equation

$$mn + nm - 2bn - 2cm + 2e = 0$$

f and ϕ and $f\phi$ will each be reducible to the form

$$A + Bm + Cn + Dmn$$

and it is obvious from the ordinary theory of the determinants that the topical resultant of 1, $\left(\text{meaning } \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right)$, and three linear functions of 1, m, n, nm, will contain as a factor the topical resultant of 1, m, n, mn.

Nor must it be supposed that Involutants are the only species of invariants in the modified sense first described which appertain to the

* I for some time had imagined, and indeed thought I had proved, that the two involutants were always identical. When crossing the Atlantic last month on board the "Arizona," having hit upon a pair of matrices of the third order, for which the two topical resultants admitted of easy calculation, I found, to my surprise, that they were perfectly distinct. The cause of the failure of the supposed proof constitutes a paradox which will form the subject of a communication to a future meeting of the Johns Hopkins Mathematical Society.

I will here only premise that the seeming contradiction between the logical conclusion and the facts of the case takes its rise in a sort of mirage with which invariantists are familiar, namely: the apparent *à priori* establishment of algebraical forms as the result of perfectly valid processes, which forms have no more real existence in nature than the Corona of the Sun under our Dr Hastings' scrutinizing gaze: the contradiction between the logical inference and the truth being accounted for by the circumstance that any such supposed form on actual performance of the operations indicated, turns out to be a congeries of terms, each affected with a null coefficient; we are thus taught the lesson that all *à priori* reasoning until submitted to the test of experience, is liable to be fallacious, and it is impossible to prove that a proof may not be erroneous by any other method than that of actual trial of the results which it is supposed to yield.

system m and n. Thus, for example, when $\omega = 2$ it is not only true that the determinant of the matrix $mn - nm$ is such a kind of Invariant (which for greater clearness it may be desirable to denote by the term Perpetuitant*), but each element of that matrix will also be a perpetuitant, and these 4 perpetuitants, when for m, n $\rho m, \phi n$ are substituted, will be in an invariable ratio to one another and to either square root of the Involutant.

In like manner it will eventually be seen that for two matrices m, n of any order ω, it is possible to form a matrix of the order ω analogous to $mn - nm$ $\left(\text{which be it observed may be regarded as the Determinant of the matrix } \begin{matrix} m & n \\ m & n \end{matrix}\right)$ each of whose ω^2 terms will be in a constant ratio to each other and to any ωth root of I and of J.

I will now return to the problem of finding what is the form of the equation which connects the ω^2 matrices denoted by

$$(1, m, m^2, \ldots m^{\omega-1})(1, n, n^2, \ldots n^{\omega-1})$$

when such an equation admits of being formed, that is, $I = 0$.

To fix the ideas let us suppose that m, n are matrices of the 3rd order of perfectly general form so that the m, n involution necessitates the satisfaction of one single condition, $I = 0$.

Let $A + Bn + Cn^2 = 0$ be the equation whose form is to be determined where A, B, C, are each of them quadratic functions of m. I say that neither A, B, nor C, can contain a non-vacuous linear factor. For suppose that any one of them as A should contain the non-vacuous factor $m + q$, and that

$$A = (m + q)(am + p).$$

Then we may multiply the equation by $(m + q)^{-1}$ and thus obtain the equation

$$(am + p) + B'n + C'n^2 = 0,$$

that is, we have an equation in which not all 9 but only 8 of the terms signified by $(1, m, m^2)(1, n, n^2) = 0$ are linearly related. But this obviously implies, contrary to the hypothesis, the existence of two equations of condition instead of one.

Hence then A must be of the form $c(m - \lambda)(m - \lambda')$ where λ, λ' are each of them a latent root of m; whether the same or different remains to be determined.

In like manner it may be shown that B is of the form $c_1(m - \lambda_1)(m - \lambda_1')$ and C of the form $c_2(m - \lambda_2)(m - \lambda_2')$. But now I say further that

$$(m - \lambda)(m - \lambda'), \quad (m - \lambda_1)(m - \lambda_1'), \quad (m - \lambda_2)(m - \lambda_2')$$

must be identical.

* *Perpetuitant* formed from *perpetuity* by analogy to Annuitant from Annuity. Perpetuant would have been better, but that it has already been applied by myself in the theory of Invariants in a sense recognized and adopted by Cayley, Hammond, and MacMahon.

For, firstly, suppose that any one pair of the λ's, say λ, λ', are distinct. If any other pair, say λ_2, λ_2', is not identical with this pair, on multiplying the equation by $m - \lambda''$, where λ'' is the 3rd latent root of M, the term containing the term $A(\lambda \ldots \lambda'')$ will vanish, but $B(\lambda \ldots \lambda'')$ will not vanish and consequently there will be an equation, if $C(\lambda \ldots \lambda'')$ does not vanish, between 6 only, and if $C(\lambda \ldots \lambda'')$ does vanish, between 3 only of the 9 terms denoted by $(1, m, m^2)(1, n, n^2)$, contrary to hypothesis.

The only remaining supposition is that A, B, C are each perfect squares. Suppose, then, that any one of them as A is a multiple of $(m - \lambda)^2$; unless B, C are each of them also multiples of the same, on multiplying the equation by $(m - \lambda')(m - \lambda'')$, one of the three coefficients of $1, n, n^2$ will vanish but one at least of the other two will not vanish, which is impossible for the same reason as before. Hence the left-hand side of the equation of involution must contain $(m - \lambda)(m - \lambda')$ as a sinister factor where λ, λ' (whether the same or different) are latent roots of λ. And in like manner precisely, by arranging the equation of involution under the form $A' + mB' + m^2C'$ where A', B', C' are quadratic functions of n, it may be found that the same function must contain $(n - \mu)(n - \mu')$ where μ, μ' are latent roots of n as a dexter factor.

Hence the form of the equation must be

$$(m - \lambda)(m - \lambda')(n - \mu)(n - \mu') = 0.$$

It is easy to see that we cannot have λ and λ' the same latent root of m and at the same time μ, μ' the same latent root of n, for then the above product would have at most the nullity 2 whereas it is an absolute null, that is, has the nullity 3.

But I will now show that λ, λ' and μ, μ' must each consist of unlike roots. Let t be any term of the matrix

$$(m - \lambda)(m - \lambda')(n - \mu)(n - \mu'),$$

where t will be a known function of the elements of m, n, of λ, λ' entering symmetrically, and of μ, μ' also entering symmetrically: this is the same thing as saying that t will be a function of the elements of m and n, of λ'', μ'', and of the coefficients of the equations which contain the 3 latent roots of λ and μ respectively.

Consequently the product of the 9 values of t found by writing $\lambda'', \lambda', \lambda$ for λ'', and μ'', μ', μ for μ'', will be a rational integer function of the elements of m, n which vanishes when the Involutant I vanishes and must consequently contain I as a factor. If then, *in any single instance*, the matrix

$$(m - \lambda)^2(n - \mu')(n - \mu'')$$

does not vanish for some one value of λ and μ when I vanishes, it cannot be the form, or *one of two* conceivably possible coexisting forms, of the left-hand side of the general equation of involution. A similar remark of course applies to

$$(m-\lambda_1)(m-\lambda_2)(n-\mu_1)^2.$$

Let now

$$m = \begin{array}{ccc} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^2 \end{array}, \quad n = \begin{array}{ccc} 0 & \rho & k \\ k & 0 & \rho^2 \\ 1 & k & 0 \end{array}.$$

The latent roots of m are $1, \rho, \rho^2$, and of n are $\theta, \rho\theta, \rho^2\theta$, where $\theta = \sqrt[3]{(1+k^3)}$; we have also

$$m^2 = \begin{array}{ccc} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho \end{array}, \quad n^2 = \begin{array}{ccc} -\rho^2 k & k^2 & 1 \\ \rho^2 & -k & k^2 \\ k^2 & \rho & -\rho k \end{array}.$$

The three values of $(m-\lambda')(m-\lambda'')$ are

$$\begin{array}{ccc} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}, \quad \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 3\rho^2 & 0 \\ 0 & 0 & 3\rho \end{array}, \quad \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 3\rho & 0 \\ 0 & 0 & 3\rho^2 \end{array}$$

and the three values of $(n-\mu_1)(n-\mu_2)$ are

$$\begin{vmatrix} -\rho^2 k + \theta^2 & k^2 + \rho\theta & 1 + \theta k \\ \rho^2 + \theta k & -k + \theta^2 & k^2 + \rho^2\theta \\ k^2 + \theta & \rho + \theta k & -\rho k + \theta^2 \end{vmatrix} \quad \begin{vmatrix} -\rho^2 k + \rho^2\theta^2 & k^2 + \rho^2\theta & 1 + \rho\theta k \\ \rho^2 + \rho\theta k & -k + \rho^2\theta^2 & k^2 + \theta \\ k^2 + \rho\theta & \rho + \rho\theta k & -\rho k + \rho^2\theta^2 \end{vmatrix}$$

$$\begin{vmatrix} -\rho^2 k + \rho^2\theta & k^2 + \theta & 1 + \rho^2\theta k \\ \rho^2 + \rho^2\theta k & -k + \rho\theta^2 & k^2 + \rho\theta \\ k^2 + \rho^2\theta & \rho + \rho^2\theta k & -\rho k + \rho\theta^2 \end{vmatrix}.$$

The general value of

$$(m-\lambda_1)(m-\lambda_2)(n-\mu_1)(n-\mu_2)$$

will (to a numerical factor *près*) be a matrix consisting of a single column accompanied by two columns of zeros, the non-zero column being some one of the 9 columns found in the above 3 matrices.

Now by direct calculation we know that the n, m Involutant in this case is a numerical multiple of $(k^3-\rho^2)^3$ and vanishes when $k^3 = \rho^2$, which gives $\theta = \sqrt[3]{(1+\rho^2)}$, that is, $-\rho = \theta^3$, and if we please $k = \theta^2$.

Hence not merely one but three of the products of

$$(m-\lambda')(m-\lambda'')(n-\mu')(n-\mu'')$$

will in this case vanish, for the above equations will cause the 2nd, 4th and 9th columns all to become columns of nulls.

If now instead of the factor $(m-\lambda')(m-\lambda'')$ we substitute the factor $(m-\lambda)^2$, the three values of $(m-\lambda)^2$ will become

$$\begin{array}{ccc} 0 & 0 & 0 \\ 0 & -3\rho & 0 \\ 0 & 0 & -3\rho^2 \end{array} \quad \begin{array}{ccc} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3\rho^2 \end{array} \quad \begin{array}{ccc} -3 & 0 & 0 \\ 0 & -3\rho & 0 \\ 0 & 0 & 0 \end{array}$$

so that if

$$(m-\lambda)^2(n-\mu')(n-\mu'')$$

is to vanish, it will readily be seen that each of two columns of one or the other of the two matrices representing $(n-\mu')(n-\mu'')$ will have to vanish simultaneously, and that this cannot be brought to pass when $\theta^3 = -\rho$ and $k^3 = \rho^2 = \theta^6$ whether we make $k = \theta^2$ or $-\theta^5$ or θ^8.

Hence

$$(m-\lambda)^2(n-\mu')(n-\mu'') = 0$$

is not an admissible general involution form of equation. Similarly by interchanging the above special values assigned to m and n, it may be shown that

$$(m-\lambda')(m-\lambda'')(n-\mu)^2 = 0$$

is not an admissible form, and consequently that the one universal form of the involution equation is expressed by saying that

$$(m-\lambda')(m-\lambda'')(n-\mu')(n-\mu'')$$

is an absolute null. If no connexion exists between the elements of m and n, we know from the law of nullity that the above matrix has a nullity 2, that is, that all its minors except the elements themselves have zero contents. The effect of the vanishing of I is to make the elements themselves one and all vanish when the two sets of latent roots are duly selected.

So in general if

$$F = \lambda^\omega - A_1\lambda^{\omega-1} + A_2\lambda^{\omega-2} - A_3\lambda^{\omega-3} \ldots = 0,$$

and

$$G = \mu^\omega - B_1\mu^{\omega-1} + B_2\mu^{\omega-2} - B_3\mu^{\omega-3} \ldots = 0,$$

are the two equations to the latent roots of m, n matrices of order ω, and if

$$M = m^{\omega-1} - (A_1-\lambda)m^{\omega-2} + (A_2 - A_1\lambda + \lambda^2)m^{\omega-3} \ldots$$

and

$$N = n^{\omega-1} - (B_1-\mu)n^{\omega-2} + (B_2 - B_1\lambda + \lambda^2)n^{\omega-3} \ldots,$$

$MN = 0$ for some value of λ and of μ is the one equation of involution, and $NM = 0$ for some value of λ and some value of μ is the other such equation.

I will now show how to deduce from the above statement the following marvellous theorem.

Let H represent the sum of the product of each term in the matrix M by its *altruistic opposite* in N (so that H is a function of λ and μ and of degree $\omega - 1$ in each of them) then will the ordinary Algebraical Resultant of F, G, H* be exactly equal (in magnitude as well as form) to the product of the two involutants to the *corpus* m, n†.

* The system of equations whose resultant expresses the undifferentiated condition of involution, may be written under the form $(x, y)^\omega = 0$; $(z, t)^\omega = 0$; $(x, y)^{\omega-1} = 0$. *Quære* whether such a resultant may not be written under the form of a determinant by an application of the Dialytic Method?

† If I and J be the two involutants, $I=0$ will be the condition of left-handed involution of m, n or right-handed of n, m, and $J=0$ of right-handed involution of m, n or left-handed of n, m, for Involution, like light, "has sides." But $IJ=0$ will be the condition of *one* or *the other* kind, or so to say of undifferentiated Involution.

By the theorem proved at the beginning of this note, the nullity of M and that of N are each $\omega - 1$, hence the nullity of MN and consequently *à fortiori* its vacuity cannot be less than $\omega - 1$, and accordingly the identical equation to MN may be written under the form

$$(MN)^{\omega} - H(MN)^{\omega-1} = 0,$$

where H is the sum of the product of each element in the Matrix M or the Matrix N multiplied by its altruistic opposite in the other. Suppose now that $I = 0$ then for some one system of λ, μ out of the ω^2 systems given by the equations $F = 0$, $G = 0$, H must vanish (for the nullity and *à fortiori* the vacuity of MN in that case becomes ω); hence the *double norm* of H, that is, the product of the ω^2 values of H, or, which comes to the same thing, the resultant of F, G, H, must vanish when I vanishes and must therefore contain I; in like manner because the nullity of NM and *à fortiori* its vacuity is ω when $J = 0$, it follows that the same resultant, say R, must contain also J; R will therefore contain IJ, from which it may readily be concluded that it can differ from IJ, if it differ at all, only by a numerical factor.

I need hardly pause to defend the assumption that I, J have no common factor, and that it is the first and not necessarily any higher power of R which contains IJ; the single instance, when

$$m = \begin{matrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^2 \end{matrix}, \quad n = \begin{matrix} 0 & \rho & k \\ k & 0 & \rho^2 \\ 1 & k & 0 \end{matrix},$$

of I, J being respectively (to a numerical factor *près*) the cubes of $k^3 - \rho$ and $k^3 - \rho^2$ which have no common factor, settles the first part of this assumption at all events for the case of $\omega = 3$, and as regards the second, it is only necessary to show that neither I nor J is equal to, or contains a square or higher power of a function of the letters in m and n as may be done easily enough when $\omega = 3$ by another simple instance*. We may then at once proceed to compare the dimensions of R with those of I and J.

* Limiting ourselves to the case of matrices of the third order, if we take for m, n the matrices $\begin{matrix} 0 & b & 0 \\ d & 0 & f \\ 0 & h & 0 \end{matrix}$, $\begin{matrix} 0 & B & 0 \\ D & 0 & F \\ 0 & H & 0 \end{matrix}$, it may be shown by direct computation that one of the Involutants becomes

$$(bH - hB)^2 (fD - dF)^2 (bd + fh)(BD - FH)(dB - fH) . \{(hF + bD)^2 - (bd + fh)(BD + FH)\},$$

and consequently if there were any square factor in either involutant such factor would contain the elements belonging to the two sets indecomposably blended, but on the other hand, if we take for m, n the matrices $\begin{matrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^2 \end{matrix}$, $\begin{matrix} 0 & f & F \\ G & 0 & g \\ h & H & 0 \end{matrix}$, either involutant to m, n may easily be shown (also by direct computation) to be made up of three factors, each of which is an indecomposable cubic function of f, g, h, F, G, H. Hence it follows that neither involutant can in its general

R being the product of ω^2 values of $\lambda^{\omega-1}\mu^{\omega-1}$ + etc., where λ, μ are codimensional with the elements in m and n respectively, is obviously of the degree $\omega^2 . (\omega - 1)$ in regard to each set of elements, that is, of the degree $2\omega^2(\omega - 1)$ in regard to the two sets taken together.

Consider now the degree of I; this is the topical resultant of ω^2 matrices of the form $m^i . n^j$, where

$$i = 0, 1, 2, \dots \omega - 1, \quad j = 0, 1, 2, \dots \omega - 1,$$

so that each term in I will consist of a combination of ω^2 elements selected respectively from these ω^2 matrices. If ω is even, there will be $\frac{\omega^2}{2}$ pairs of matrices, one of any such pair of the form $m^i n^j$, the other of form $m^{\omega-1-i} . n^{\omega-1-j}$, and the combination of elements taken from any such pair will be of the collective degree $2(\omega - 1)$ in the two sets of elements, so that the total degree of the Involutant will be $\frac{\omega^2}{2} . 2(\omega - 1)$ or $\omega^2(\omega - 1)$. If again ω is odd, there will be $\frac{1}{2}(\omega^2 + 1)$ such pairs, and one factor (unpaired) belonging to the matrix $m^{\frac{\omega-1}{2}} . n^{\frac{\omega-1}{2}}$ of the collective degree $(\omega - 1)$. Hence the degree of the involutant will be

$$(\omega^2 - 1)(\omega - 1) + (\omega - 1) \text{ or } \omega^2(\omega - 1)$$

as before.

Hence the product of IJ is of the degree $2\omega^2(\omega - 1)$, or the same as R, and consequently (at all events to a numerical factor *près*) R and IJ coincide, which is the essential thing to be proved.

N.B. As regards $\omega = 3$, the above proof is exact; for higher values of ω to make it valid, it must be demonstrated as a Lemma that the two general twin involutants (even were they decomposable forms, which they undoubtedly are not) could not have any common factor, nor either of them contain any square factor. The Resultant of F, G, H may be compared to a cradle just large enough to contain the twin forms in question, so as to give assurance that no other form is mixed up with them; and the proof given above shows that this must be the case if neither twin is doubled

form contain any square factor. As a matter of fact, not only for ternary matrices but for matrices of any order, there can be no reasonable doubt whatever in any sane mind that every Involutant is *absolutely* indecomposable. One must try, however, to obtain a strict proof of this upon the general principle of crushing every logical difficulty regarded as a challenge to the human reason, which falls in our way; it is in overcoming the difficulties attendant upon the proof of negative propositions that the mind acquires new strength and accumulates the materials for future and more significant conquests. To prove that involutants in their general form are indecomposable may possibly, I imagine, prove to be a hard nut to crack, or it may be exceedingly easy.

up upon itself, and if the two do not grow into one another, but like such creatures each possesses a perfectly distinct organization.

A single instance will serve to establish the fact that the Resultant of F, G, H is the very product IJ itself, without any numerical multiplier. I have made this verification for binary and ternary matrices, and as the point is not one of an essential importance need not dwell here further upon it.

To pass to a much more important subject, I am inclined to anticipate as the result of a long and interesting investigation into the relations of the involutants of a certain particular *corpus* of the third order that the *sum* of the two involutants of any *corpus* admits of being represented by means of invariants similar in kind to that which expresses the single involutant to a binary *corpus* (m, n), namely, the content of (that is, the determinant to) the matrix $mn - nm$, which itself (as previously observed) may be written as the determinant to the matrix $\begin{Bmatrix} m & n \\ m & n \end{Bmatrix}$, or say $(m, n)_2$; and in some similar way it is, I think, not unlikely that the *product* also of the two involutants (the resultant of F, G, H) is capable of being expressed; but I must for the present content myself with exhibiting the bare fact of the existence of invariants of the kind referred to for matrices of any order.

Suppose then that m, n is a *corpus* of the third order. Form the determinant

$$\left.\begin{matrix} m & n & m^2 & n^2 \\ m & n & m^2 & n^2 \\ m & n & m^2 & n^2 \\ m & n & m^2 & n^2 \end{matrix}\right\}, \text{ say } (m, n, m^2, n^2)_4.$$

The number of terms, half of them positive and half of them negative, in such determinant is 24; but of these, all but 8 will obviously appear as pairs of equal terms affected with opposite signs and so cancel one another: the 8 excepted ones are those in which no m and n come together, *to wit*:

$$mnm^2n^2 + nmn^2m^2 + m^2n^2mn + n^2m^2nm$$
$$- m^2nmn^2 - nm^2n^2m - mn^2m^2n - n^2mnm^2.$$

The determinant to this matrix will be of the total degree 18 in the two sets of elements belonging to m and n respectively, that is, of the degree 9 in respect to each set of elements *per se*. And so in general if m, n be of the order ω the determinant

$$(m, m^2, \dots m^{\omega-1}, n, n^2, \dots n^{\omega-1})_{2\omega}$$

will contain only $2(\pi\omega)^2$ effective terms, of which half will bear the positive and the others the negative sign.

The determinant to this matrix will be of the order

$$\omega[2\{1 + 2 + \dots + (\omega - 1)\}], \text{ that is, } (\omega - 1)\omega^2,$$

in regard to the combined elements in m and n, that is, equi-dimensional with either involutant to the *corpus* m, n.

Whatever else may be its properties (on which I do not dare yet to pronounce), it is certain that such determinant (and over and above that, every term in the matrix of which it is the content) will be an Invariant to the *corpus* in the same sense in which either Involutant has been previously shown to be entitled to bear that name. And here for the present it becomes necessary for me to break off, bidding *au revoir* to any reader who may peruse this sketch, and trusting to meet him again in the broader field of the *American Journal of Mathematics*, where I hope to be spared to set out this portion of the theory with more certainty, and the whole doctrine of multiple quantity with much greater completeness and in more ample detail than is possible within the limits of the *Circulars* and in the short interval remaining between the present time and the date of my intended departure for Europe.

16.

ON THE THREE LAWS OF MOTION IN THE WORLD OF UNIVERSAL ALGEBRA.

[*Johns Hopkins University Circulars*, III. (1884), pp. 33, 34, 57.]

IN the preceding *Circular* allusion was made to the three cardinal principles or conspicuous landmarks in Universal Algebra; these may be called, it seems to me (without impropriety), its Laws of Motion, on the ground that as motion is operation in the world of pure space, so operation is motion in the world of pure order, and without claiming any exact analogy between these and Newton's laws, it will be seen that there is an element in each of the former which matches with a similar element in the latter, so that there is no difficulty in pairing off the two sets of laws and determining which in one set is to be regarded as related by affinity with which in the other. They may be termed the law of *concomitance* or *congruity*, the law of *consentaneity* and the law of *mutuality* or *community*.

The law of congruity is that which affirms that the latent roots of a matrix follow the march of any functional operation performed upon the matrix, not involving the action of any foreign matrix; it is the law which asserts that any function of a latent root to a matrix is a latent root to that same function of the matrix; in so far as it regards a matrix *per se*, or with reference solely to its environment, it obviously pairs off with Newton's first law.

The law of *consentaneity*, which is an immediate inference from the rule for combining or multiplying substitutions or matrices, is that which affirms that a given line (or parallel of latitude) can be followed out in the matrices resulting from the continued action of a matrix upon a fixed matrix of the same order, that is, in the series M, mM, m^2M, m^3M, ... (which may be regarded as so many modified states of the original matrix) without reference to any other of the lines or parallels of latitude in the series, or again any column or parallel of longitude in the correlated series M, Mm, Mm^2, ... without reference to any other such column or parallel of longitude.

An immediate consequence of this obvious fact (a direct consequence for the rule of multiplication) obtained by dealing at will with either of the systems of parallels referred to, is that a system of simultaneous linear equations in differences may be formed for finding each term in any given line or in any given column at any point in the series, and the integration of these equations leads at once to the conclusion that any term of given latitude and longitude in the ith term of either series is a syzygetic function of the ith powers of the latent roots of m.

If, then, M be made equal to multinomial unity, this at once shows that supposing ω to be the order of m, on substituting m for the *carrier* (or latent variable) in the latent function to m, and multiplying the last term by the proper multinomial unit, the matrix so formed is an absolute null, which proves the proposition concerning the "identical equation" first enunciated by Professor Cayley in his great paper on Matrices in the *Philosophical Transactions* for 1858.

This proposition admits of augmentation, (1), from within, as shown in a former note, by applying to it the limiting law of the nullity of a product (a branch of the 3rd law), which leads to the very important conclusion that the nullity of any factor of the function of a matrix which is an absolute null, or more generally of any product of powers of its linear factors, is exactly equal to the number of distinct linear factors which such factor or product contains, at all events, in the general case where the latent roots are all unequal; and (2), from without, by substituting for m, $m+\epsilon n$ where n is any second matrix whatever and ϵ is an infinitesimal. This leads to the *catena* of identities, to which allusion has been made in the preceding *Circular*. Then, again, the *endogenous* growth of the theorem (that which determines the exact nullity of any factor of the left-hand side of the identical equation) in its turn seems to lead to a remarkable theorem concerning the form of the general term of any power of m into M.

Observe that every such term is expressed as a syzygetic function of powers of the ω latent roots, and contains, therefore, ω constants, so that the total number of syzygetic multipliers is ω^3; but the number of variables in m and M together is $2\omega^2$; and, consequently, apart from the ω arbitrary latent roots the number of independent constants in m^iM should be $2\omega^2-\omega$. The ω^3 syzygetic multipliers ought then to contain only $\omega(2\omega-1)$ arbitrary constants, and such will be found to be the case by virtue of the following hypothetical theorem: Calling λ any one of the latent roots, the multipliers of λ^i in m^iM will form a square of ω^2 quantities; the theorem in question* is that every minor of the second order in such square is zero, so that the ω^2 terms in the square is given when the bounding angle containing

* I have not had leisure of mind, being much occupied in preparing for my departure, to reduce this theorem to apodictic certainty. I state it therefore with all due reserve.

$2\omega - 1$ terms is given; and the same being true for the multipliers of each latent root (which resolve themselves into ω squares) the number of arbitrary quantities in all is $\omega(2\omega - 1)$ as has to be shown.

The law of *consentaneity* in so far as it relates to the decomposition of the motion of a matrix into a set of parallel motions, has an evident affinity with Newton's second law*.

Remains the law of *mutuality*, which is concerned with the effect of the mutual action upon one another of two matrices, and so claims kindred with Newton's third law.

This law branches off into two, one of which may be termed the law of reversibility, the other that of co-occupancy or permeability.

The law of reversibility affirms that the latent function of the product of two matrices is independent of the sense in which either of them operates upon the other, that is, is the same for mn as for nm, just as the kinetic energy developed by the mutual action of two bodies is not affected by their being supposed to change places.

As regards the second branch of the third law, the word co-occupancy refers to the fact that although the space occupied by two similarly shaped figures (say two spheres) is not absolutely determined (in the absence of other data) by the spaces occupied by them each separately (for they may intersect or one of them coincide with or contain the other), a superior as well as an inferior limit to such joint occupation is so determined; the inferior limit being the space occupied by either such figure, that is, the *dominant* of these two given spaces, and the superior limit their arithmetical sum. So the nullity resulting from the action in either sense of two matrices upon one another is not given when their separate nullities are assigned, but has for an inferior limit the dominant of these two nullities and for a superior limit their sum; the nullities of the two component matrices may also be conceived under the figure of two gases or other fluids which are mutually *permeable* and capable of occupying each other's pores.

Although the limits spoken of are independent of the sense in which the two matrices act on one another, it must not however be supposed that the actual resultant nullity is unaffected by that circumstance; thus, for example, if the latent roots of a ternary matrix m are $\lambda, \lambda', \lambda''$, the nullity resulting from $(m - \lambda)(m - \lambda')$ acting sinistrally upon $(m - \lambda'')n$, that is, of $(m - \lambda)(m - \lambda')(m - \lambda'')n$ is 3, but from the same acting dextrally upon the same, that is, of $(m - \lambda'')n(m - \lambda)(m - \lambda')$, need not necessarily exceed 2.

* For another and closer bond of affinity between the two laws see concluding paragraph of this note.

Such then are the three primary Laws of Algebraical Motion; but as Conservation of areas, *Vis viva*, D'Alembert's Principle, the principle of Synchronous Vibrations, of Least action, and various other general laws may be deduced from Newton's three ground laws, so, of course, various subordinate but very general laws may be deduced from the interaction of the above stated three ground laws, namely, the law of Congruity, the law of Consentaneity, and the law of Mutuality.

The deduction of the catena of identical equations connecting two matrices m and n from the second and third laws combined, affords an instance of such derivative general laws. Another instance of the same is the theorem that when the product resulting from the action upon one another of two matrices, is the same in whichever of the two senses the action takes place, the matrices must be functionally related, unless one of them is a scalar, that is, a multiple of multinomial unity, at all events when neither m nor n possesses a pair of equal latent roots.

This very important and almost fundamental law (seemingly so simple and yet so hard to prove) may be obtained as an immediate inference from that identical equation in the catena of such equations connecting the matrices m and n, in which one of the two enters only singly at most in any term. As for example if m and n are of the 3rd order, beside the identical equation $m^3 - 3bm^2 + 3dm - g = 0$ we have* the identity

$$m^2n + mnm + nm^2 - 3b(mn + nm) - 3cm^2 + 3dn + 6em - 3h = 0.$$

But if $nm = mn$ then $mnm = m^2n, nm^2 = mnm = m^2n$, so that this equation becomes

$$m^2n - 2bmn + dn = m^2c - 2em + h, \quad \text{or} \quad n = \frac{cm^2 - 2em + h}{m^2 - 2bm + d}\dagger,$$

unless $m^2 - 2bm + d$ is vacuous.

The first branch of the third law, namely, the law of *reversibility*, is an almost immediate inference from the rule for the multiplication of matrices, and becomes intuitively evident when the process of multiplication in each of the two senses between m and n is actually set out. The second branch, namely, the law of co-occupancy or permeability, as it is the most far-reaching so it is the most deep seated (the most *caché*) of all the primary laws of

[* See p. 126 above.]

† Whence it follows that n must be a function of m convertible into an integral polynomial form, unless the numerator and denominator of the fraction to which n is equated vanish simultaneously, which is what happens when m is scalar. If the numerator exactly contains the denominator n becomes a scalar. Seeing that a constant c is a specialized case of a function of a variable x although the converse is not true, we may say that whenever $nm = mn$, *one at least* of the two matrices m and n is a function of the other, and that *each* is a function of the other unless that other is a scalar. Compare Clifford's "Fragment on Matrices" in the posthumous edition of his collected works.

motion. I found my proof of it upon the fact that the value of any minor determinant, say of the ith order, in either product of m and n (two matrices of the order ω) may be expressed as the quantitative product of a certain couple of rectangular matrices (in Cauchy's sense of the term), of which one is formed by i columns and the other by i lines in the two given matrices respectively. Such rectangle as shown by Cauchy (and as may be intuitively demonstrated by the simplest of my umbral theorems on compound determinants) is the sum of the

$$\frac{\pi(\omega)}{\pi(\omega-i)\,\pi i}$$

complete determinants of the one rectangle multiplied respectively by the corresponding complete determinants of the other rectangle.

This shows at once the truth of the proposition in so far as relates to the lower limit, that is, that if $mn=p$, and m, n have the nullities ϵ, ζ, and p the nullity θ, then θ must be at least as great as ϵ and at least as great as ζ. As regards the superior limit the proof is also founded on the theorem in determinants already cited, and the form of it is as follows. If ϵ be any number r, it may be shown that ζ must be at least as great as $\theta-r$; hence giving r all values successively from 0 to $\zeta-1$, it follows that $\epsilon+\zeta$ cannot be less than θ, that is, that θ cannot be greater than $\epsilon+\zeta$.

The proof of the first law, that of concomitance or congruity, I ought to have stated antecedently, is a deduction from the theory of resultants and the well-known fact that the determinant of a product of matrices is the product of their determinants. Thus each of the three laws of motion is deduced independently of the two others.

As another example of a derivative law of motion, I may quote the very notable one which results from the interaction of the first and second fundamental laws upon one another, and which gives the general expression for any function whatever of a matrix in the form of a rational polynomial function of the same and of its latent roots, to wit, the magnificent theorem that whatever the form of the functional symbol ϕ, and whether it be a single or many valued function, if $\lambda_1, \lambda_2, \ldots \lambda_\omega$ be the latent roots of m,

$$\phi m = \Sigma\phi\lambda_1 \frac{(m-\lambda_2)(m-\lambda_3)\ldots(m-\lambda_\omega)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\ldots(\lambda_1-\lambda_\omega)}.$$

As for example if $\phi m = m^{\frac{p}{q}}$, $m^{\frac{p}{q}}$ will have q^ω roots which are completely determined by the above formula.

The first law, as already stated, regards a single body or matrix, uninfluenced by the action of any external force. The second law regards the effect upon a single matrix, subject to external impulses, taking their rise in an external source; whilst the third law has regard to the mutual

action or joint effect of two bodies or matrices simultaneously operating upon one another.

Note. Making [in p. 149] $m^3-3bm^2+3dm-g=F(m)$, we found

$$(F'm)\,n = cm^2-2em+g.$$

When two of the latent roots of m are equal, it is easy to prove that $F'm$ is vacuous, and conversely, that when $F'm$ is vacuous, two of the latent roots of m are equal; but when $F'm$ is vacuous it is no longer permissible to drive it out of the equation, and accordingly the true statement of the theorem in question is that when m. n are two matrices of (any) the same order, such that $mn=nm$, n must *in general* be a function of m, but that this ceases to be true, when and only when m has two equal roots. The theorem requires further investigation in order to make out what happens when, or how it can happen that, two of the latent roots of one and only one of the two convertible matrices are equal; for supposing this to happen it would seem to lead to the conclusion that n may be a function of m, but m not a function of n; which, however, is not quite so paradoxical as it looks, inasmuch as in ordinary algebra a constant may be regarded as a specialized function of a variable, whilst a variable in no sense can be regarded as a function of a constant. The following example of two matrices not functions of one another, but forming commutable products, has recently occurred to me in practice, and led to the discovery of the oversight I had committed in stating the theorem in question in too absolute terms.

If $x = \begin{matrix} 0 & \rho & \rho^2 \\ 1 & 0 & 1 \\ \rho^2 & \rho & 0 \end{matrix}$, $y = \begin{matrix} 0 & 1 & 1 \\ \rho & 0 & \rho^2 \\ \rho & \rho^2 & 0 \end{matrix}$ where $\rho^2+\rho+1=0$, it will be found that $xy=yx$, but that neither x nor y is a function of the other; this may easily be deduced from the fact that $x^2-\rho^2x-2\rho=0$, so that if y were any function of x, it would be reducible to the form of a linear function thereof, and consequently (on account of the zeros in the two matrices) y must be a multiple of x, which is absurd.

In like manner it will be found that $y^2-\rho^2y-2\rho=0$, and that consequently x cannot be a function of y.

17.

EQUATIONS IN MATRICES.

[*Johns Hopkins University Circulars*, III. (1884), p. 122.]

I HAVE been lately considering the subject of equations in matrices. Sir William Hamilton in his *Lectures on Quaternions* has treated the case of what I call unilateral equations of the form $x^2+px+q=0$, or $x^2+xp+q=0$, where we may, if we please, regard x, p, q as general matrices of the second order. He has found there are six solutions, which may be obtained by the solution of an ordinary cubic equation. In a paper now in print and which will probably appear in the May number of the *Philosophical Magazine*, I have discussed by my own methods the general *unilateral* equation, say

$$x^{\omega}+px^{\omega-1}+qx^{\omega-2}+\dots+l=0,$$

where $x, p, q \dots l$, are quaternions or matrices of the second order, and have shown, by a method satisfactory if not absolutely rigorous, that the number of solutions is $\omega^3-\omega^2+\omega$, that is to say, the nearest superior integer to the general maximum number of roots (ω^4) divided by the augmented degree $(\omega+1)$.

But after I had done this it occurred to me that there were multitudinous failing cases of which neither Hamilton nor myself had taken account, as for example $x^2+px=0$, besides the solutions $x=0$, $x=-p$, will admit of a solution containing an arbitrary constant, I think; but that is a matter which I shall have to look further into before committing myself to a positive assertion about it. I have only had time to pass in review the more elementary case of a unilateral simple equation, say $px=q$, where p, q are matrices of any order ω.

If p is non-vacuous there is one solution, namely, $x=p^{-1}q$; but suppose p is vacuous: what is the condition that the equation may be soluble?

(1) Suppose $q=0$, p being vacuous has for its identical equation $pP=0$, and consequently we may make $x=\lambda P$ where λ is an arbitrary constant.

(2) Suppose q is finite and that $x=r$ is one solution, then obviously the general solution is $x=r+\lambda P$.

We have now to inquire what is the condition that r may exist. I find from the mere fact of x being indeterminate (and confirm the result by another order of considerations) that the determinant of $q+\lambda p$ must vanish identically; so that for instance when p, q are of the second order and $\begin{matrix} b'c \\ def \end{matrix}$ are the *parameters to the corpus* (p, q), we must have when $d=0$, which is implied in the vacuity of p, $f=0$ and $e=0$. The first of these conditions is known *à priori* immediately from my third law of motion; but not so, without introducing a slight intervening step, the intermediate one (I mean the connective to d and f, namely) $e=0$.

So in general in order that $px+q=0$ may be soluble, that is, in order that $p^{-1}q$ where p is simply vacuous may be *Actual* and not Ideal, q must satisfy as many conditions as there are units in the order of p or q, all implied in the fact that the determinant to $p+\lambda q$, where λ is an arbitrary constant, vanishes identically. When these conditions are satisfied $p^{-1}q$ becomes actual but indeterminate. (This, by the way, shows the disadvantage of calling a *vacuous* matrix *indeterminate*, as was done in the infancy of the theory by Cayley and Clifford—for we want this word as you see to signify a combination of the inverse of a vacuous matrix with another which takes the combination out of the ideal sphere and makes it actual.)

So in general in order that $p^{-1}q$ where p is a null of the ith order (that is where all the $(i+1)$th but not all the ith minors of p are zero) shall be an actual (although indeterminate) matrix, it is necessary and sufficient that $p+\lambda q$, where λ is arbitrary, shall be a null of the same (ith) order. What will be the degree of indeterminateness in $p^{-1}q$, that is, how many arbitrary constants are contained in the value of x which satisfies the equation $px=0$ remains to be considered.

The law as to the conditions is an immediate *corollary* to my third law of motion, for if $px=q$ then $p+\lambda q=p(1+\lambda x)$; consequently $p+\lambda q$, whatever λ may be, must have at least as high a degree of nullity as p. Q.E.D.

18.

SUR LES QUANTITÉS FORMANT UN GROUPE DE NONIONS ANALOGUES AUX QUATERNIONS DE HAMILTON.

[*Comptes Rendus*, XCVIII. (1884), pp. 273—276, 471—475.]

Dans une Note précédente*, j'ai fait allusion au cas où le déterminant de $x+ym+zn$ devient une fonction linéaire de x^3, y^3, z^3 sans que la quantité nommée Q s'évanouisse. Dans ce cas, on aura

$$(mn)^3+Q(mn)-R=0, \qquad (1)$$

R étant le déterminant de mn. C'est bien la peine, comme on va le voir, de donner plus de précision aux équations qui lient ensemble mn et nm pour ce cas.

En suivant la même marche que pour le cas particulier où $Q=0$, on trouvera sans difficulté les résultats suivants:

$$nm=-\frac{3Q}{\zeta}(mn)^2-\frac{\zeta+9R}{2\zeta}mn-\frac{2Q^2}{\zeta}, \qquad (2)$$

$$mn=\frac{3Q}{\zeta}(nm)^2-\frac{\zeta-9R}{2\zeta}nm+\frac{2Q^2}{\zeta}, \qquad (3)$$

ζ étant le produit des différences des racines de la fonction $\lambda^3+Q\lambda-R$, de sorte que $\zeta^2=-(4Q^3+27R^2)$.

Conséquemment on peut écrire

$$nm=A(mn)^2+Bmn+C, \qquad (4)$$

$$mn=-A(nm)^2+B'nm-C, \qquad (5)$$

où A et C peuvent être tous les deux zéro, ou tous les deux des quantités finies quelconques, mais non pas l'un d'entre eux une quantité finie et l'autre zéro, et B, B' les deux racines par rapport à B de l'équation

$$B^2+B+1+\frac{AC}{2}=0\dagger. \qquad (6)$$

* *Comptes rendus*, t. XCVII. p. 1336.

[† It follows from $n(mn+\theta)=(nm+\theta)n$ that $M, =mn$ and $N, =nm$ both satisfy equation (1); further $MN=NM$ (footnote * p. 127 above), so that (p. 149 above) there exists an equation $N=pM^2+qM+r$; from (1), if $|M-N|\neq 0$, follows $M^2+MN+N^2+Q=0$. Hence (2), (3) can be deduced.]

On peut vérifier, comme je l'ai fait, par un calcul algébrique direct, que les équations (4) et (5), en vertu des équations (1) et (6), sont compatibles.

Or une chose digne de remarque, c'est ce qui arrive quand $\zeta=0$, car cela servira à révéler un phénomène d'Algèbre universelle d'un genre que personne n'avait encore même soupçonné.

Dans ce cas, les deux équations (4) et (5) changent leur caractère et deviennent

$$Q(mn)^2+3Rmn+\tfrac{2}{3}Q^2=0,$$

$$Q(nm)^2+3Rnm+\tfrac{2}{3}Q^2=0,$$

de sorte que mn et nm cessent d'être fonctions l'un de l'autre.

Nommons, pour le moment, $mn=u$, $nm=v$; on aura, comme auparavant, $uv=vu$, sans que v et u soient fonctionnellement liés ensemble. Dans le *Johns Hopkins Circular* de janvier 1884 (dans l'article intitulé *On the three laws of motion in the world of universal Algebra*, [above p. 146]), on trouvera le moyen d'établir qu'en général cette équation amène à la conclusion que ou u doit être un *scalar*, c'est-à-dire de la forme $\begin{matrix} C & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & C \end{matrix}$, ou bien v un scalar, ou sinon que nm, mn doivent être fonctions l'un de l'autre; mais on remarquera (ce qui m'avait alors échappé) que, si $Fu=0$ est l'équation identique en u et que la dérivée fonctionnelle $F'u$ est une matrice *vide* (*vacuous*), c'est-à-dire dont le déterminant est zéro, le raisonnement est en défaut; cette vacuité a lieu dans le cas, et seulement dans le cas, où deux des racines latentes (lambdaïques) de m sont égales. On peut généraliser cette conclusion et l'étendre à deux matrices u et v d'un ordre quelconque au-dessus du deuxième; c'est-à-dire quand les racines latentes de u (ou bien de v) ne sont pas toutes inégales, *il est des cas* où $uv=vu$, sans que u ou v soient des scalars et sans que v et u soient fonctions l'un de l'autre. Par exemple, si l'on fait

$$u=\begin{vmatrix} 0 & \rho & \rho^2 \\ 1 & 0 & 1 \\ \rho^2 & \rho & 0 \end{vmatrix}, \qquad v=\begin{vmatrix} 0 & 1 & 1 \\ \rho & 0 & \rho^2 \\ \rho & \rho^2 & 0 \end{vmatrix},$$

on trouvera

$$uv=\begin{vmatrix} -\rho & \rho & 1 \\ \rho & -\rho & 1 \\ \rho^2 & \rho^2 & -\rho \end{vmatrix}=vu.$$

Mais on démontrera sans difficulté que v ne peut pas s'exprimer comme somme de puissances de u, ni *vice versa* v comme somme de puissances de u.

On n'a pas besoin de remarquer que la seule condition de l'existence de racines latentes égales en u ou en v ne peut pas suffire en elle-même pour assurer que $uv = vu$, mais il faut réserver pour une autre occasion la pleine discussion de la totalité des solutions de cette équation importante.

J'ajouterai seulement cette remarque, qui est essentielle. En supposant l'existence des équations

$$\begin{aligned} m^2n + mnm + nm^2 &= 0, \\ n^2m + nmn + mn^2 &= 0, \\ (mn)^3 + Qmn - R &= 0, \\ (nm)^3 + Qnm - R &= 0, \end{aligned}$$

qui ont lieu nécessairement quand le déterminant de $x + ym + zn$ devient une fonction linéaire de x^3, y^3, z^3, et en regardant nm comme fonction de mn (en vertu de l'équation $mn \,.\, nm = nm \,.\, mn$), alors, en additionnant aux deux valeurs de nm (exprimé comme fonction de mn) données ci-dessus, qui correspondent aux deux valeurs de ζ, c'est-à-dire $\sqrt{-(4Q^3 + 27R^2)}$, on a à considérer quatre autres valeurs, le nombre total en étant six. Car si l'on suppose $nm = A(mn)^2 + Bmn + C$ et si $\lambda_1, \lambda_2, \lambda_3$ sont les trois racines de $\lambda^3 + Q\lambda - R = 0$, les valeurs de A, B, C sont déterminées en mettant

$$\begin{aligned} A\lambda_1^2 + B\lambda_1 + C &= \lambda_i, \\ A\lambda_2^2 + B\lambda_2 + C &= \lambda_j, \\ A\lambda_3^2 + B\lambda_3 + C &= \lambda_k, \end{aligned}$$

où i, j, k sont respectivement

$$1 \quad 2 \quad 3 \qquad \text{ou} \qquad \begin{matrix} 2 & 3 & 1 \\ 3 & 1 & 2 \end{matrix} \qquad \text{ou bien} \qquad \begin{matrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{matrix}$$

Les valeurs de A, B, C données ci-dessus correspondent au deuxième de ces groupes de valeurs de i, j, k.

Si l'on écrit $i = 1$, $j = 2$, $k = 3$, on trouvera $nm = mn$.

Si l'on écrit $i = 1$, $j = 3$, $k = 2$, en faisant $\lambda_1 = \Lambda$, on trouvera

$$nm = \frac{3\Lambda(mn)^2 - Qmn + 2\Lambda Q}{3\Lambda^2 + Q}.$$

Dans le cas critique où $\zeta = 0$, de sorte que $3\Lambda^2 + Q = 0$, l'équation devient $(mn)^2 + \Lambda mn - 2\Lambda^2 = 0$, comme dans le cas déjà traité. Quand on suppose Q égal à zéro et R (c'est-à-dire le déterminant de mn) fini, les seules solutions possibles avec ces conditions sont celles fournies en écrivant $i, j, k = 2, 3, 1$, ou $3, 1, 2$; mais, pour le cas général, il n'y a pas de raison (au moins *très évidente*) pour exclure aucune des trois classes de solution. Si l'on admet la légitimité des solutions de la troisième classe, en écrivant

$$nm = A(mn)^2 + Bmn + C,$$

on trouvera

$$B^2 + B + \frac{AC}{2} = 0$$

au lieu de l'équation

$$B^2 + B + 1 + \frac{AC}{2} = 0,$$

qui est applicable aux solutions de la deuxième classe.

Avant de considérer l'équation $xy = yx$, il importe d'avoir une idée nette d'une certaine classe de matrices que je nomme *privilégiées* ou *dérogatoires*, en tant qu'elles dérogent à la loi générale que toute matrice est assujettie à satisfaire à une équation identique dont le degré ne peut pas être moindre que l'ordre de la matrice.

Les matrices dérogatoires sont justement celles qui satisfont à une équation d'un ordre inférieur à leur ordre propre; on peut les nommer *simplement, doublement, triplement, … dérogatoires*, selon que le degré de l'équation identique à laquelle elles satisfont diffère par une, deux, trois, … unités du degré minimum ordinaire.

Pour le cas des matrices du deuxième ordre, il n'y a que les *scalars* $\begin{matrix} a & 0 \\ 0 & a \end{matrix}$ qui soient dérogatoires.

Pour le cas des matrices du troisième ordre, en écartant les *scalars* de la forme $\begin{matrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{matrix}$, toute matrice x dérogatoire peut être ramenée ou à la forme

$$a + b(\epsilon + \epsilon^2),$$

où ϵ est une matrice qui satisfait à l'équation $\epsilon^3 = 1$, c'est-à-dire une matrice dont les racines latentes sont $1, \rho, \rho^2$, ou à la forme

$$a + b(1 + \epsilon + \epsilon^2)\zeta,$$

où

$$\epsilon^3 = 1, \quad \zeta^3 = 1 \quad \text{et} \quad \zeta\epsilon = \rho\epsilon\zeta,$$

ρ signifiant une racine cubique primitive de l'unité. Dans le premier cas,

$$x^2 - (2a + b)x + (a^2 + ab - 2b^2) = 0,$$

et dans le second

$$x^2 - 2ax + a^2 = 0,$$

car on trouvera facilement que

$$(1 + \epsilon + \epsilon^2)\zeta(1 + \epsilon + \epsilon^2)\zeta = 0.$$

Pour le cas du quatrième ordre, en écartant les scalars et en se bornant au cas où l'équation identique dérogée (vue pour le moment comme une équation ordinaire en x) ne contient pas des racines égales, toute matrice x peut être ramenée à l'une ou à l'autre des deux formes suivantes :

$$a + b(U + U^3) \quad \text{ou bien} \quad a + b\left(U + \frac{1 + ki}{1 + i}U^2 + kU^3\right).$$

où U est une matrice du quatrième ordre telle que $U^4+1=0$; a, b, k sont des scalars arbitraires et i est une racine primitive biquadratique de l'unité; quand, pour la seconde forme $k=1$, on trouvera qu'il y aura une dérogation double de l'ordre de l'équation satisfaite par x, l'équation identique pour x ne sera que du deuxième degré.

En réservant les détails du calcul, voici le résultat général que j'ai démontré rigoureusement (en m'aidant de la notation des nonions) pour les matrices du troisième degré qui satisfont à l'équation $xy=yx$.

A moins que x ne soit une matrice privilégiée ou dérogatoire, y sera toujours une fonction rationnelle et entière quadratique de x, et de même, à moins que y ne soit privilégiée, x sera une fonction pareille de y.

Il est bien entendu que le caractère dérogatoire d'une seule des deux matrices n'empêche pas qu'elle ne soit une fonction entière et rationnelle quadratique de l'autre. Dans le cas où x et y sont tous les deux dérogatoires, ni l'un ni l'autre ne peut être exprimé comme fonction explicite l'un de l'autre, mais ils seront liés ensemble par une équation linéo-linéaire.

Il paraît peu douteux qu'une règle semblable doive être applicable à l'équation $xy=yx$, quel que soit l'ordre des matrices x et y, sauf quand l'équation qui lie ensemble x et y pourra être d'un degré moindre que l'ordre de chacune d'elles.

Il est bon de remarquer que nulle matrice ne peut être dérogatoire, sauf pour le cas où il existe des égalités entre ses racines latentes; mais ces égalités peuvent parfaitement subsister sans que la matrice à laquelle elles appartiennent soit dérogatoire. En général, si $x=a+by+cy^2$, on peut, par une formule générale que j'ai déjà donnée, exprimer y sous la forme

$$\alpha+\beta x+\gamma x^2;$$

avec l'aide des racines latentes de x, cette formule ne cesse pas en général d'être valable, même pour le cas où x contient des racines égales, en regardant leur différence comme une quantité infinitésimale; seulement le nombre des racines finies subira dans ce cas une diminution; mais, dans le cas où l'équation $xy=yx$ (x étant dérogatoire) mènerait à l'équation

$$x=a+by+cy^2,$$

on trouverait que nulle fonction explicite de x avec des coefficients finis ne peut exprimer le y cherché.

Il est à peine nécessaire d'ajouter que rien n'empêche, dans le cas où l'un ou l'autre de x et y ou tous les deux sont dérogatoires, qu'on puisse satisfaire à $xy=yx$, en supposant que x et y soient des fonctions explicites chacune l'une de l'autre: tout ce qu'on affirme, c'est que, dans le cas admis, cette supposition cesse d'être obligatoire; c'est un cas très semblable à ce qui arrive dans le cas de défaut (*failing case*) du théorème de Maclaurin: c'est

celui où une variable est une fonction sans pouvoir être développée dans une série de puissances d'une autre variable.

Dans ce qui précède, on a vu un exemple du fait général que, m étant une matrice donnée, l'équation $\phi(x, m)=0$, pour certaines valeurs de m, cesse d'admettre la solution ordinaire $x=Fm$.

Mais il existe encore une classe assez étendue d'équations entre x et m pour lesquelles, quand m prend certaines valeurs, x n'a aucune existence actuelle; par exemple, m étant une matrice *vide* d'un ordre quelconque, si $mx=1$, la matrice x devient inexprimable et n'a, pour ainsi dire, qu'une existence idéale.

Je citerai encore l'exemple $x^2=m$, m étant une matrice du deuxième ordre; si les racines latentes de m sont inégales, on trouvera, par la formule générale, quatre valeurs de x. Si les deux racines latentes sont égales et finies, ces quatre valeurs se réduisent à deux; mais, si les deux racines sont toutes les deux égales à zéro, il n'y aura aucune valeur de x qui satisfasse à l'équation donnée, c'est-à-dire si $m=\begin{matrix} a & -\frac{a}{k} \\ ka & -a \end{matrix}$; l'équation devient absolument insoluble, ou, si l'on peut s'exprimer ainsi, les quatre racines carrées de m sont toutes idéales.

Dans le cas supposé, on vérifiera aisément que $m^2=0$ et, *vice versa*, toute racine carrée du zéro binomial est de la forme $\begin{matrix} a & -\frac{a}{k} \\ ka & -a \end{matrix}$, de sorte que l'on peut dire qu'une racine carrée quelconque du zéro binomial ne possède pas elle-même des racines algébriques quelconques, ou, en d'autres termes, une racine algébrique quelconque du quaternion $i+\sqrt{(-1)}\,j$ est purement idéale et n'admet pas d'être représentée sous la forme d'un quaternion. Finalement je remarque que toute matrice est d'un certain ordre et d'une certaine classe; l'ordre, c'est le nombre total de ses racines latentes; la classe, c'est le degré minimum de l'équation latente (c'est-à-dire de l'équation identique à laquelle la matrice satisfait), lequel ne peut être plus petit que le nombre des racines latentes inégales.

Je dois ajouter (ce que j'aurais dû dire auparavant) que, quand x est une matrice ternaire dérogatoire dont *toutes les racines latentes sont égales*, l'équation $xy=yx$ peut subsister sans que ni x ni y ne soit une fonction explicite l'un de l'autre, même quand y n'est pas une matrice privilégiée; c'est le cas où, ϵ et ζ faisant partie d'un groupe de nonions élémentaires, on a $x=a+b(1+\epsilon+\epsilon^2)\zeta$. Les calculs sont un peu compliqués pour ce cas spécial, mais je crois ne pas me tromper en faisant cette correction. Le champ de la théorie de la quantité multiple est tellement nouveau et inexploité que, sans les plus grandes précautions, on est toujours en danger de se heurter contre quelque cause imprévue d'incertitude ou même d'erreur.

19.

SUR UNE NOTE RÉCENTE DE M. D. ANDRÉ*.

[*Comptes Rendus*, XCVIII. (1884), pp. 550, 551.]

LE théorème de M. André est une conséquence immédiate de la généralisation que j'ai donnée du théorème de Newton (*Arithmétique universelle*, 2ᵉ Partie, Ch. II.) sur les racines imaginaires des équations.

On verra, en consultant mon travail† sur ce sujet (*Proceedings of the London Mathematical Society*, No. 2), que si $u_0, u_1, u_2, \dots u_m$ sont les coefficients d'une équation du degré m et si

$$G_r' = ru_r^2 - (r+1)\,\gamma_r\, u_{r-1}\, u_{r+1}$$

ou

$$\gamma_r = \frac{v+r-1}{v+r},$$

γ_r étant une quantité réelle quelconque qui n'est pas intermédiaire entre 0 et $-m$, l'équation aura nécessairement au moins autant de racines imaginaires qu'il y a de variations de signes dans la série $G_0, G_1, G_2, \dots, G_m$.

En faisant $v = -m$, on a le théorème de Newton; en faisant $v = 1$, on voit qu'on peut prendre $G_r = u_r^2 - u_{r-1}\,u_{r+1}$. Conséquemment le théorème de M. André subsiste, quel que soit le signe de la quantité qu'il nomme α et quels que soient les signes des quantités qu'il nomme $u_0, u_1, \dots, u_m$.

De plus, le théorème subsistera encore quand, outre ces modifications, au lieu de l'équation

$$u_n = \alpha u_{n-1} + \beta u_{n-2},$$

on écrit

$$v_n = \alpha v_{n-1} + \beta v_{n-2}$$

ou

$$v_0, v_1, v_2, \dots, v_m,$$

identiques avec

$$u_0, \quad \frac{u_1}{m}, \quad \frac{u_2}{\frac{1}{2}(m\,.\,m-1)}, \quad \frac{u_3}{\frac{1}{1.2.3}\,m\,(m-1)\,(m-2)}, \dots.$$

* *Comptes rendus*, séance du 18 février 1884.

[† Vol. II. of this Reprint, pp. 501, 507.]

Il y a encore une autre extension importante à ajouter, en considérant l'équation

$$u_{n-1}\,u_{n+1} - u_n^2 = A\alpha^x + B\beta^x + C\gamma^x,$$

dont j'ai donné une solution particulière dans l'*American Mathematical Journal*, Vol. IV. [Vol. III. of this Reprint, pp. 546, 633.]

Il est peut-être digne de remarqué que si, dans la formule établie pour γ_r, on fait v infini, la règle calquée sur celle de Newton (mais plus générale) enseigne que, quels que soient a, b, c ou m, l'équation

$$a\left(1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \dots + \frac{x^m}{1.2\dots m}\right) + b\left(1 - x + \frac{x^2}{2} - \frac{x^3}{1.2.3} + \dots \pm \frac{x^m}{1.2\dots m}\right) + c = 0$$

ne peut jamais avoir plus de deux racines réelles.

20.

SUR LA SOLUTION D'UNE CLASSE TRÈS ÉTENDUE D'ÉQUATIONS EN QUATERNIONS.

[*Comptes Rendus*, XCVIII. (1884), pp. 651, 652.]

L'ÉQUATION parfaitement générale du deuxième degré en quaternions sera de la forme

$$\Sigma(axbxc + dxe) + f = 0$$

et admettra seize solutions, qu'on pourrait obtenir d'une manière directe au moyen de quatre équations, chacune du deuxième degré, contenant les quatre éléments de x comme inconnus. De même, l'équation en quaternions ou en matrices du deuxième ordre du degré ω admettra ω^4 solutions. Parmi ces formes générales, on peut distinguer celles dans lesquelles tous les quaternions donnés se trouvent du même côté du quaternion cherché, par exemple $ax^2 + bx + c = 0$. On peut nommer de telles équations *équations unilatérales*. Hamilton a considéré le seul cas de l'équation quadratique (voir *Lectures on Quaternions*, art. 636, pp. 631—2), et a déterminé le nombre (6) des racines.

Or, je trouve que ma méthode générale de traiter les matrices amène directement à la solution d'une équation unilatérale d'un ordre quelconque ω (c'est-à-dire la fait dépendre de la solution d'une équation algébrique ordinaire) et donne sans la moindre difficulté et sans aucun effort d'invention le nombre des racines. Ce nombre est exprimé par la fonction $\omega^3 - \omega^2 + \omega$, de sorte que le nombre des racines, pour ainsi dire évanouies par suite de l'unilatéralisme de la forme, est $\omega^4 - \omega^3 + \omega^2 - \omega$, c'est-à-dire $(\omega^2 - \omega)(\omega^2 + 1)$. On comprend bien qu'en certains cas le nombre des racines subit une réduction; par exemple, le nombre des racines de $x^\omega + l = 0$ est ω^2 et celui de $x^\omega + kx + l = 0$ est $2\omega^2 - \omega$. Il semble que le nombre, pour l'équation

$$x^\omega + p_\theta x^\theta + p_{\theta-1} x^{\theta-1} + \ldots + p_0 = 0,$$

doit être $(\theta + 1)\omega^2 - \theta\omega$, lequel, quand $\theta = \omega - 1$, devient le nombre général $\omega^3 - \omega^2 + \omega$. Les détails de ce petit travail seront donnés dans un prochain numéro du *London and Edinburgh Philosophical Magazine*.

21.

SUR LA CORRESPONDANCE ENTRE DEUX ESPÈCES DIFFÉRENTES DE FONCTIONS DE DEUX SYSTÈMES DE QUANTITÉS, CORRÉLATIFS ET ÉGALEMENT NOMBREUX.

[*Comptes Rendus*, XCVIII. (1884), pp. 779—781.]

VOICI le théorème à démontrer, dans lequel, par *somme-puissance*, on sous-entend une somme de puissances de quantités données:

A i quantités on peut en associer i autres telles, que chaque fonction symétrique (qui est une fonction des différences) des premières sera une fonction des sommes-puissances du 2*e*, *du* 3*e*, ..., *du* $i^{\text{ième}}$ *ordre des dernières.*

Faisons, pour plus de clarté, $i = 3$.

Soient r_1, r_2, r_3 les racines de l'équation

$$fr = ar^3 + br^2 + cr + d = 0.$$

En prenant b, c, d; r_1, r_2, r_3 comme deux systèmes corrélatifs de variables indépendants, on trouve

$$\delta_b = -\Sigma \frac{r^2}{f'r}\delta_r, \quad \delta_c = -\Sigma \frac{r}{f'r}\delta_r, \quad \delta_d = -\Sigma \frac{1}{f'r}\delta_r.$$

Donc

$$3a\delta_b + 2b\delta_c + c\delta_d = -\Sigma\delta_r,$$

$$a\delta_b + b\delta_c + c\delta_d = d\,\Sigma \frac{1}{rf'r}\delta_r.$$

Soient $a = \alpha$, $b = 3\beta$, $c = 3.2.\gamma$, $d = 3.2.1.\delta$, et soient ρ_1, ρ_2, ρ_3 les racines de l'équation

$$\alpha\rho^3 + \beta\rho^2 + \gamma\rho + \delta = 0.$$

Alors, si $\Sigma\delta_r\phi = 0$, on aura $(\alpha\delta_\beta + \beta\delta_\gamma + \gamma\delta_\delta)\phi = 0$. C.Q.F.D.

L'intégrale générale de la première équation est

$$\phi = \mathfrak{F}(r_1 - r_2, r_1 - r_3),$$

et celle de la dernière est

$$\phi = \mathfrak{F}_1(\rho_1^2 + \rho_2^2 + \rho_3^2, \rho_1^3 + \rho_2^3 + \rho_3^3).$$

Ces deux intégrales sont donc identiques, et, le raisonnement étant général pour une valeur quelconque de i, on voit que chaque fonction des différences des r doit pouvoir s'exprimer comme une fonction de $i-1$ sommes-puissances consécutives des ρ (commençant avec la seconde), les r et les ρ étant liés ensemble par les équations

$$ar^i + br^{i-1} + cr^{i-2} + dr^{i-3} + \ldots = 0,$$

$$a\rho^i + \frac{b}{i}\,\rho^{i-1} + \frac{c}{i(i-1)}\,\rho^{i-2} + \frac{d}{i(i-1)(i-2)}\,\rho^{i-3} + \ldots = 0,$$

et conséquemment une fonction *symétrique* des différences des r sera une fonction rationnelle et entière des $i-1$ puissances consécutives (dont on a déjà fait mention) des ρ.

En prenant $i = \infty$, on voit que le théorème équivaut à dire que tous les *sous-invariants*, sources des covariants de $(a, b, c \between x, y)^2$, $(a, b, c, d \between x, y)^3, \ldots$ (à l'infini), seront des fonctions des sommes-puissances prises à l'infini, avec la seule exception de la somme linéaire, des racines de l'équation

$$a + bx + \frac{c}{1\,.\,2}\,x^2 + \frac{d}{1\,.\,2\,.\,3}\,x^3 + \ldots \text{ (à l'infini).}$$

Tel est le théorème capital découvert par M. le capitaine Mac-Mahon, de l'Artillerie royale anglaise, dont il a fait le plus heureux usage en développant la théorie des perpétuants (voir *American Journal of Mathematics*). Il est évident que le même principe peut être appliqué aux invariants de toute espèce, de sorte que, grâce à la belle découverte de M. Mac-Mahon, avec la généralisation (qui en sort presque intuitivement) que j'ai donnée, on est aujourd'hui en état de traiter les parties les plus difficiles et les plus essentielles de la théorie des formes algébriques, comme M. Schubert l'a fait avec sa *Zahl-Geometrie* pour les figures dans l'espace, en faisant abstraction, pour ainsi dire, de toute question de substance (de matière contenue dans les formes), et en se bornant à un calcul purement arithmétique.

Je dois avertir que le théorème de correspondance, tel que M. Mac-Mahon l'a donné, a paru dans l'*American Journal of Mathematics* (Vol. VI. p. 131). M. Mac-Mahon affirme (mais sans aucune preuve) que, si ($\alpha, \beta, \gamma, \ldots$ étant des nombres entiers plus grands chacun que l'unité) ϕ est de la forme $\Sigma r^\alpha s^\beta t^\gamma, \ldots$, où $r, s, t, \ldots$ sont les racines de l'équation

$$\left(a_0, a_1, \frac{a_2}{1\,.\,2}, \frac{a_3}{1\,.\,2\,.\,3}, \ldots\right)(x, 1)^n = 0,$$

alors $$(a_0\delta_{a_1} + a_1\delta_{a_2} + a_2\delta_{a_3} + \ldots)\,\phi = 0,$$

et il donne à ϕ le nom de *fonction symétrique non unitaire* des racines. Ce théorème est vrai seulement pour le cas où n est infini (ce que M. Mac-

Mahon a oublié de dire), et dans ce cas il conduit à la conséquence que les *différentiants* (c'est-à-dire les sous-invariants) de

$$(a_0, a_1, a_2, \ldots)(x, 1)^\infty$$

sont des *fonctions symétriques non unitaires* des racines de l'équation

$$a_0 + a_1 x^{-1} + \frac{a_2}{1\,.\,2}\,x^{-2} + \frac{a_3}{1\,.\,2\,.\,3}\,x^{-3} + \ldots = 0$$

et *vice versâ*. Or il est évident que chaque fonction *symétrique non unitaire* d'un nombre *infini* de quantités n'est autre chose qu'une fonction des sommes de toutes les puissances de ces quantités au delà de la première. Voilà pourquoi j'ai attribué à M. Mac-Mahon, dans ce qui précède (pour le cas d'une équation dont le degré est infini), la connaissance du théorème que j'ai démontré dans toute sa généralité.

22.

SUR LE THÉORÈME DE M. BRIOSCHI, RELATIF AUX FONCTIONS SYMÉTRIQUES.

[*Comptes Rendus*, XCVIII. (1884), pp. 858—862.]

DANS la démonstration du théorème sur une correspondance algébrique, inséré dans les *Comptes rendus* de la semaine dernière [p. 163 above], j'ai eu occasion de considérer l'intégrale de l'équation

$$\left(a_0 \frac{d}{da_1} + a_1 \frac{d}{da_2} + \dots + a_{n-1} \frac{d}{da_n}\right) \phi = 0.$$

Je me suis aperçu depuis que cette intégrale peut se déduire immédiatement du beau théorème de M. Brioschi, *sur les fonctions symétriques*, à savoir que :

$$r \frac{d\phi}{ds_r} + a_0 \frac{d\phi}{da_r} + a_1 \frac{d\phi}{da_{r+1}} + \dots + a_{n-r} \frac{d\phi}{da_n} = 0.$$

On en tire cette conséquence immédiate que, si ϕ est une fonction des n premières sommes-puissances des racines de l'équation

$$a_0 x^n + a_1 x^{n-1} + \dots = 0,$$

avec exclusion de la puissance $r^{\text{ième}}$, on aura

$$a_0 \frac{d\phi}{da_r} + \dots + a_{n-r} \frac{d\phi}{da_n} = 0,$$

et conséquemment $F(s_1, s_2, \dots, s_{r-1}, s_{r+1}, \dots, s_n)$ sera l'équivalent complet de l'expression

$$\left(a_0 \frac{d}{da_r} + a_1 \frac{d}{da_{r+1}} + \dots + a_{n-r} \frac{d}{da_n}\right)^{-1} . \, 0.$$

Dans le cas que j'ai considéré, $r = 1$, et nous avons trouvé

$$\left(a_0 \frac{d}{da_1} + a_1 \frac{d}{da_2} + \dots + a_{n-1} \frac{d}{da_n}\right)^{-1} . \, 0 = F(s_2, s_3, \dots, s_n).$$

On peut trouver aussi facilement l'intégrale complète de l'équation

$$\left(a_0 \frac{d}{da_1} + a_1 \frac{d}{da_2} + \dots + a_{n-1} \frac{d}{da_n}\right)^{*i} \phi = 0,$$

où l'astérisque signifie qu'on doit prendre le *produit complet* de l'action de la forme linéaire agissant $i - 1$ fois sur elle-même. Ainsi, par exemple,

$$\left(a \frac{d}{db} + b \frac{d}{dc}\right)^{*2} \text{signifie } a^2 \left(\frac{d}{db}\right)^2 + 2ab \frac{d}{db} \frac{d}{dc} + b^2 \left(\frac{d}{dc}\right)^2 + a \frac{d}{dc}.$$

On trouvera sans difficulté que la valeur de cette intégrale est

$$F + s_1 F_1 + s_1^2 F_2 + \dots + s_1^{i-1} F_{i-1},$$

où chaque F est une fonction exclusivement de $s_2, s_3, \dots, s_n$.

Conséquemment le $i^{\text{ième}}$ coefficient d'un covariant quelconque de

$$(\alpha_0, \alpha_1, \dots, \alpha_n)(x, y)^n$$

peut être mis sous cette forme, si l'on se sert de s_ω pour exprimer la somme des $\omega^{\text{ièmes}}$ puissances des racines de

$$x^n + \alpha_1 x^{n-1} + \frac{\alpha_2}{1 \,.\, 2} x^{n-2} + \frac{\alpha_3}{1 \,.\, 2 \,.\, 3} x^{n-3} + \dots = 0.$$

En effet, en écrivant $\frac{s_1}{n} = s$, tout covariant de degré arbitraire ν appartenant à ce quantic sera de la forme

$$[u_0, \ (u_0, u_1 \between s, 1), \ (u_0, u_1, u_2 \between s, 1)^2, \ (u_0, u_1, u_2, u_3 \between s, 1)^3, \dots] (x, y)^\nu,$$

où, en général,

$$u_{\theta+1} = \frac{du_\theta}{ds_2} v_2 + \frac{du_\theta}{ds_3} v_3 + \dots + \frac{du_\theta}{ds_n} v_n,$$

v_ω étant une fonction exclusivement de ω, n; $s_2, s_3, \dots, s_n$ du poids $\omega + 1$. J'ajoute encore cette observation que tout différentiant (c'est-à-dire *sous-invariant* ou *seminvariant*) d'un système de i quantics des degrés $m, \mu, \dots, M$ sera fonction exclusivement de $s_2, s_3, \dots, s_m$; $\sigma_2, \sigma_3, \dots, \sigma_\mu, \dots, S_2, S_3, \dots, S_M$ et de $i - 1$ fonctions linéaires indépendantes de la forme

$$l s_1 + \lambda \sigma_1 + \dots + L S_1,$$

soumises à la condition que $l + \lambda + \dots + L = 0$.

Je ne sais s'il vaut la peine de dire, comme conclusion, qu'en combinant le théorème de M. Brioschi avec le mien sur les puissances (*avec astérisque*) on trouve, pour l'équation

$$\left(a_0 \frac{d}{da_1} + a_1 \frac{d}{da_2} + a_2 \frac{d}{da_3} + \dots\right)^i \phi = 0$$

(où le i est *sans astérisque*), l'intégrale partielle

$$\phi = F + F_1 s_1 + F_2 s_1^2 + \dots + F_{i-1} s_1^{i-1},$$

où chaque F est une fonction arbitraire de $s_{i+1}, s_{i+2}, \dots, s_n$.

En effet, cette expression est l'intégrale complète du système formé par l'équation supposée conjointe avec les équations

$$\left(a_0 \frac{d}{da_2} + \dots\right) \phi = 0, \quad \left(a_0 \frac{d}{da_3} + \dots\right) \phi = 0, \quad \dots, \quad \left(a_0 \frac{d}{da_r} + \dots\right) \phi = 0.$$

On voit aussi facilement que l'intégrale de

$$\left(a_0 \frac{d}{da_r} + a_1 \frac{d}{da_{r+1}} + \dots\right)^{*i} \phi = 0$$

est

$$\phi = U_0 + U_1 s_r + U_2 s_r^2 + \dots + U_{r-1} s_r^{i-1},$$

où chaque U est une fonction arbitraire de $s_1, s_2, \dots, s_{r-1}, s_{r+1}, \dots s_n$.

On peut former un nombre infini de systèmes construits au moyen des opérateurs $\left(a_0 \frac{d}{da_r} + \dots\right)$ dont on connaîtra d'avance les intégrales ; ainsi, par exemple, le système de r équations

$$\left(a_0 \frac{d}{da_2} + \dots\right)^i \phi = 0, \quad \left(a_0 \frac{d}{da_r} + \dots\right) \phi = 0, \quad \dots, \quad \left(a_0 \frac{d}{da_{2r}} + \dots\right) \phi = 0$$

aura pour intégrale complète

$$\phi = U_0 + s_2 U_1 + s_2^2 U_2 + \dots + s_2^{i-1} U_{i-1},$$

où chaque U représente une fonction arbitraire de $(s_1 s_3 s_5 \dots s_{2i-1} s_{2i} \dots s_n)$, en omettant celles des quantités $s_1, s_3, \dots, s_{2i-1}$ dont les sous-indices excèdent n.

Pour indiquer le moyen de justifier ces énoncés, prenons comme exemple le cas des équations simultanées

$$(a_0 \delta a_1 + \dots + a_{n-1} \delta a_n)^3 \phi = 0, \quad \text{ou} \quad E_1^3 \phi = 0,$$
$$(a_0 \delta a_2 + \dots + a_{n-2} \delta a_n) \ \phi = 0, \quad \text{ou} \quad E_2 \phi = 0,$$
$$(a_0 \delta a_3 + \dots + a_{n-3} \delta a_n) \ \phi = 0, \quad \text{ou} \quad E_3 \phi = 0.$$

On trouvera facilement qu'en général $E_1^3 = E^*{}_1^3 - 2E^*{}_1^1 E_2 + E_3$, de sorte que le système donné équivaut au système

$$E^*{}_1^3 \phi = 0, \quad E_2 \phi = 0, \quad E_3 \phi = 0.$$

Pour que ces équations soient satisfaites séparément, il faut et il suffit que ϕ soit respectivement de la forme

$$F(s_2 s_3 s_4 \dots s_n) + s_1 F_1 (s_2 s_3 s_4 \dots s_n) + s_1^2 F_2 (s_2 s_3 s_4 \dots s_n),$$
$$G(s_1 s_3 s_4 \dots s_n), \quad H(s_1 s_2 s_4 \dots s_n).$$

Conséquemment, afin que les trois équations soient toutes satisfaites simultanément, la condition suffisante et nécessaire sera que ϕ soit de la forme

$$F(s_4 \dots s_n) + s_1 F_1 (s_4 \dots s_n) + s_1^2 F_2 (s_4 \dots s_n),$$

laquelle est conséquemment l'intégrale complète du système donné. De même, on démontre facilement que l'intégrale complète des équations

$$(a_0 \delta a_1 + \dots + a_{n-1} \delta a_n)^2 \phi = 0,$$
$$(a_0 \delta a_2 + \dots + a_{n-2} \delta a_n) \ \phi = 0,$$
$$(a_0 \delta a_2 + \dots + a_{n-2} \delta a_n)^2 \phi = 0$$

sera

$$\phi = F(s_3 s_5 s_6 \dots s_n) + s_1 F_1 (s_3 s_5 s_6 \dots s_n).$$

23.

SUR UNE EXTENSION DE LA LOI DE HARRIOT RELATIVE AUX ÉQUATIONS ALGÉBRIQUES.

[*Comptes Rendus*, XCVIII. (1884), pp. 1026—1030.]

On peut envisager la loi de Harriot comme une loi qui affirme la possibilité de décomposer d'une seule manière un polynôme en x dans un produit de facteurs linéaires composés avec les différences entre x et les racines du polynôme. En réfléchissant sur la cause de cette possibilité et la manière de la démontrer, on voit facilement que le même principe doit, avec une certaine modification, s'appliquer à toute équation en matrices d'un ordre quelconque dont les coefficients sont transitifs entre eux-mêmes, c'est-à-dire qui agissent les uns sur les autres exactement comme les quantités de l'Algèbre ordinaire, si chaque coefficient, par exemple, est une fonction rationnelle de la même matrice. On peut nommer les équations dont les coefficients satisfont à cette condition équations *monothétiques :* on remarquera que de telles équations forment une classe spéciale des équations que j'ai nommées *unilatérales* dans une Note précédente.

Pour fixer les idées, prenons comme exemple une équation monothétique du second degré en matrices binaires, laquelle peut toujours être ramenée à la forme

$$x^2 - 2px + Ap + B = 0.$$

En supposant que $p^2 - (\alpha + \beta) p + \alpha\beta = 0$ soit l'*équation identique* de p, on aura

$$x = \frac{p - \beta}{\alpha - \beta} \{\alpha \pm \sqrt{(\alpha^2 - A\alpha - B)}\} + \frac{p - \alpha}{\beta - \alpha} \{\beta \pm \sqrt{(\beta^2 - A\beta - B)}\}.$$

Faisons $\frac{p - \beta}{\alpha - \beta} \sqrt{(\alpha^2 - A\alpha - B)} = u$, $\frac{p - \alpha}{\beta - \alpha} \sqrt{(\beta^2 - A\beta - B)} = v$.

Alors les quatre racines de p seront

$$p + u + v, \quad p - u - v; \quad p + u - v, \quad p - u + v.$$

Disons r_1, r_2, r_3, r_4.

On trouve

$$(p-\beta)^2=(p-\beta)(p-\alpha)+(\alpha-\beta)(p-\beta)=(\alpha-\beta)(p-\beta),$$

et de même $$(p-\alpha)^2=(\beta-\alpha)(p-\alpha),$$

de sorte que

$$u^2+v^2=\frac{p-\beta}{\alpha-\beta}(\alpha^2-A\alpha-B)+\frac{p-\alpha}{\beta-\alpha}(\beta^2-A\beta-B)$$
$$=(\alpha+\beta)p-\alpha\beta-Ap-B=p^2-Ap-B.$$

On a aussi $uv=0$ et conséquemment $(u+v)^2=u^2+v^2=(u-v)^2$. Donc

$$(x-r_1)(x-r_2)=(x-p)^2-(u+v)^2=x^2-2px+Ap+B,$$
$$(x-r_3)(x-r_4)=(x-p)^2-(u-v)^2=x^2-2px+Ap+B.$$

Or considérons le cas général d'une équation monothétique du degré n en matrices de l'ordre ω.

Cette équation (que j'écrirai $fx=0$), en vertu de ce que j'ai nommé la seconde loi de mouvement algébrique (c'est-à-dire la formule

$$\phi m=\Sigma\frac{(m-b)(m-c)\dots(m-l)}{(a-b)(a-c)\dots(a-l)}\phi a,$$

où $a, b, c, \dots, l$ sont les *racines latentes* de la matrice m), aura n^ω racines qu'on peut représenter par les symboles composés

$$r_1,\quad r_2,\quad \dots,\quad r_\omega,$$

où chaque r parcourt les valeurs $1, 2, 3, \dots, n$.

En réfléchissant sur la manière de démontrer le principe de Harriot, on arrivera facilement à la conclusion suivante: en prenant une combinaison quelconque de n symboles $r_1, r_2, \dots, r_\omega$, de telle manière que chaque r parcoure toutes ses n valeurs, $R_1, R_2, \dots, R_n$, on aura

$$fx=(x-R_1)(x-R_2)\dots(x-R_n).$$

Ainsi on arrive au théorème suivant:

Toute fonction monothétique rationnelle et entière de x du degré n en matrices de l'ordre ω peut être représentée de $(1.2.3\dots n)^{\omega-1}$ manières différentes comme un produit de n facteurs linéaires dont chacun sera la différence entre x et une des racines de la fonction donnée.

Telle est la loi de Harriot, étendue au cas des quantités multiirrationnelles.

Dans le cas de l'Algèbre ordinaire, $\omega=1$, et le nombre des décompositions de fx en facteurs, selon la formule, devient unique, comme il doit être.

De même, pour les quaternions, le nombre des décompositions d'une fonction monothétique du degré n en facteurs linéaires sera πn. Par

exemple, si $n=3$, les racines de fx peuvent être exprimées par les neuf symboles

$$\begin{matrix}0.0 & 0.1 & 0.2\\ 1.0 & 1.1 & 1.2\\ 2.0 & 2.1 & 2.2\end{matrix}$$

La fonction (comme on le démontrera facilement) peut être mise sous la forme $x-0.0$ multipliée par une fonction quadratique dont les racines seront des racines de fx, et conséquemment, par raison de symétrie, seront les quatre racines

$$\begin{matrix}1.1 & 1.2,\\ 2.1 & 2.2;\end{matrix}$$

donc la fonction quadratique dont j'ai parlé sera égale à

$$(x-1.1)(x-2.2)$$

et à $$(x-1.2)(x-2.1).$$

Ainsi il y aura deux décompositions de fx qui correspondent aux deux diagonales 0.0, 1.1, 2.2; 0.0, 1.2, 2.1, et de même il y aura des décompositions qui répondent aux diagonales 0.1, 1.2, 2.0; 0.1, 1.0, 2.2; 0.2, 1.0, 2.1; 0.2, 1.1, 2.0, de sorte que le nombre total est égal à 1.2.3.

De même, quand fx est monothétique et matrice du troisième ordre, on peut prendre les diagonales d'un cube. Par exemple, les racines de l'équation monothétique du second degré en matrices du troisième ordre peuvent être représentées par

$$\begin{matrix}0.0.0 & 0.0.1 & 0.1.0 & 0.1.1\\ 1.1.1 & 1.1.0 & 1.0.1 & 1.0.0\end{matrix}$$

et l'on aura les quatre décompositions

$$(x-0.0.0)(x-1.1.1);\quad (x-0.0.1)(x-1.1.0);$$
$$(x-0.1.0)(x-1.0.1);\quad (x-0.1.1)(x-1.0.0);$$

et de même, en général, pour le degré n, le nombre des diagonales (en se servant de ce mot dans le sens analytique, bien entendu) sera

$$(1.2.3\dots n)^2.$$

C'est ainsi qu'on trouve l'expression générale que j'ai donnée $(\pi n)^{\omega-1}$ pour le nombre des décompositions quand le degré est n et que l'ordre des matrices est ω.

En multipliant ensemble toutes les équations de décomposition, et en nommant v chacune des n^ω racines, on parvient à l'équation

$$\pi(x-v)^{\pi(n-1)^{\omega-1}}=(fx)^{\pi n^{\omega-1}};$$

donc, quoiqu'on ne puisse pas en général conclure que, si $X^j=Y^j$ (X et Y

étant des matrices), X est nécessairement égal à Y, il y a toute raison de croire qu'on pourra démontrer que, dans le cas actuel, on aura

$$\pi(x-v) = (fx)^{n^{\omega-1}}.$$

Ainsi la règle de Harriot se reproduira de nouveau sous la forme très peu modifiée qu'un polynôme (monothétique) en x (élevé à une puissance convenable) est égal au produit des différences entre x et toutes les racines en succession de ce polynôme.

On aura remarqué, dans ce qui précède, qu'en appliquant la seconde des trois lois du mouvement algébrique aux équations monothétiques, on a trouvé que le nombre des racines est n^{ω}, et conséquemment est n^2 dans le cas des quaternions, tandis que le nombre des racines pour la classe des équations en quaternions unilatérales (à laquelle les formes monothétiques appartiennent) est en général $n^3 - n^2 + n$ (*voir* le numéro d'avril 1884 du *London and Edinburgh Phil. Mag.*), de sorte qu'il y a une élimination $n(n-1)^2$ de racines en passant du cas général au cas particulier.

Il reste à examiner s'il n'est pas possible d'étendre la loi de Harriot aux équations unilatérales polythétiques. C'est ce que je vais étudier, mais sans cela, et en me bornant au cas monothétique, il me semble qu'en attribuant aux éléments des matrices des valeurs entières (simples ou complexes), comme le fait M. le professeur Lipschitz pour les quaternions, on voit s'ouvrir un nouveau champ immense de recherches arithmétiques fondées sur la loi fondamentale de Harriot généralisée de la manière indiquée dans ce qui précède.

24.

SUR LES ÉQUATIONS MONOTHÉTIQUES.

[*Comptes Rendus*, XCIX. (1884), pp. 13—15.]

DANS une Note précédente sur une extension de la loi de Harriot, j'ai eu occasion de considérer les équations dites *monothétiques* dont tous les coefficients sont des fonctions d'une seule matrice. Or il y a une circonstance très intéressante et importante relative aux équations de cette forme qu'il est essentiel de faire connaître; car, à défaut d'une telle explication, le lecteur de la Note citée pourrait facilement être induit dans une erreur très grave. Voici en quoi consiste l'addition à faire.

Supposons que tous les coefficients d'une équation donnée soient des fonctions d'une seule matrice m. En appelant x l'inconnue, on peut résoudre l'équation en regardant x comme fonction de m, et l'on trouvera ainsi n^{ω} racines, en supposant que n soit le degré de l'équation et ω l'ordre de m. Ces racines seront parfaitement déterminées: mais on n'a nullement le droit de supposer qu'il n'y a pas d'autres racines qui ne sont pas des fonctions de m, qu'on peut nommer racines *aberrantes*, et un exemple, des plus simples qu'on puisse imaginer, suffira à démontrer que de telles racines, en effet, existent; je me servirai, pour cet objet, de l'équation en quaternions (ou matrices binaires) $x^2 - px = 0$.

En effet, on connaît déjà, *a priori*, la possibilité de l'existence des racines aberrantes, car l'équation en matrices $x^n + q = 0$, quand q est une matrice *scalar* $\left(\text{comme si, par exemple, } q = \begin{Bmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q \end{Bmatrix}\right)$, possède, on le sait, bien des racines qui ne sont pas scalars et conséquemment ne sont pas des fonctions de q, et, de plus, ces racines contiennent des constantes *arbitraires*. Comme on va le voir, c'est aussi le cas pour l'équation $x^2 - px = 0$, qui possède une seule constante.

Si l'on veut trouver ses racines normales (ou non aberrantes), on n'a qu'à résoudre cette équation comme une équation ordinaire, et l'on trouve ainsi

$$x = \tfrac{1}{2}\{p + \sqrt{(p^2)}\}.$$

En nommant r et s les racines latentes de p, on obtient par ma formule d'interpolation (pour ainsi dire), récemment citée par M. Weyr,

$$x=\frac{1}{2}\left(p\pm\frac{p-s}{r-s}r\pm\frac{p-r}{s-r}s\right),$$

c'est-à-dire $x=0$, p, $\frac{r(p-s)}{r-s}$, $\frac{s(p-r)}{s-r}$, et il n'y a pas d'autres racines de ce caractère. Mais sortons de cette restriction arbitraire (produit de la paresse de l'esprit humain, qui se fatigue enfin en voyant sans cesse se reproduire des horizons nouveaux et inattendus), et posons hardiment

$$x=\begin{matrix}\alpha & \beta\\ \gamma & \delta\end{matrix},\quad p=\begin{matrix}a & b\\ c & d\end{matrix},$$

où α, β, γ, δ sont les quantités à déterminer.

Puisqu'on fait abstraction des solutions $x=0$, $x=p$, on sent, en vertu de la *troisième loi du mouvement algébrique*, que x et $x-p$ auront chacun un degré de nullité (car leur produit possède deux degrés); ainsi, si $\alpha+\delta=0$, on aura

$$x^2=0,$$

donc aussi

$$px=0,$$

et p sera aussi une matrice vide, c'est-à-dire qu'on aura

$$ad-bc=0.$$

La solution pour ce cas (dont, dans ce qui suit, je veux faire abstraction) sera

$$x=\lambda\begin{Bmatrix}ac & -a^2\\ a^2 & -ac\end{Bmatrix},$$

λ étant arbitraire.

Dans tout autre cas, en égalant la raison du second au troisième membre de x^2 avec la même pour px, on trouve sans difficulté que x sera de la forme

$$\begin{matrix}-\lambda(d-r) & \lambda b\\ \mu c & -\mu(a-r)\end{matrix}$$

où r et s sont les racines latentes de p, c'est-à-dire les racines de l'équation

$$r^2-(a+d)r+ad-bc=0.$$

Alors, en calculant x^2 et px, et en les égalant terme à terme, on obtient les quatre équations suivantes :

$$\begin{aligned}
&\lambda(d-r)^2 \quad +\mu bc &&= bc-a(d-r),\\
&b[\lambda(d-r)+\mu(a-r)] &&= -br,\\
&c[\lambda(d-r)+\mu(a-r)] &&= -cr,\\
&\lambda bc \quad +\mu(a-r)^2 &&= bc-d(a-r).
\end{aligned}$$

En écartant le cas spécial pour lequel $b=0$ et $c=0$, on voit (et c'est M. Franklin, de Baltimore, qui le premier s'est aperçu de cette conclusion capitale) que toutes ces équations seront satisfaites avec la seule supposition

$$\lambda(d-r)+\mu(a-r)+r=0,$$

de sorte qu'une constante reste parfaitement libre dans la solution aberrante de l'équation $x^2-px=0$.

Dans le cas où $p=\begin{matrix}a & 0\\ 0 & d\end{matrix}$ on trouvera facilement les deux solutions déterminées

$$x=\begin{matrix}a & 0\\ 0 & 0\end{matrix}\quad\text{et}\quad x=\begin{matrix}0 & 0\\ 0 & d\end{matrix}.$$

Dans ses *Lectures sur les quaternions*, Hamilton n'a pas mis le doigt sur les cas véritablement singuliers des équations quadratiques unilatérales. La condition de singularité, c'est-à-dire de la présence de l'un ou de l'autre des cas où une ou plusieurs des trois paires de racines de l'équation $px^2+qx+r=0$ disparaissent ou deviennent indéterminées (c'est-à-dire affectées de constantes arbitraires), peut se résumer dans la seule équation $I=0$, où I est l'invariant quartique ternaire quadratique (en u, v, w) qui exprime le déterminant d'une matrice $up+vq+wr$.

25.

SUR L'ÉQUATION EN MATRICES $px = xq$.

[*Comptes Rendus*, XCIX. (1884), pp. 67—71; 115, 116.]

SOIENT p et q deux matrices de l'ordre ω.

Pour résoudre l'équation $px = xq$, on obtiendra ω^2 équations homogènes linéaires entre les ω^2 éléments de l'inconnue x et les éléments de p et de q, de sorte que, afin que l'équation donnée soit résoluble, les éléments de p et de q doivent être liés ensemble par une et une seule équation.

Mais, si l'*équation identique* en p est écrite sous la forme

$$p^\omega + Bp^{\omega-1} + Cp^{\omega-2} + \ldots + L = 0,$$

on aura apparemment, en vertu de l'équation $p = xqx^{-1}$,

$$xq^\omega x^{-1} + Bxq^{\omega-1}x^{-1} + Cxq^{\omega-2}x^{-1} + \ldots + L = 0$$

ou bien
$$q^\omega + Bq^{\omega-1} + Cq^{\omega-2} + \ldots + L = 0;$$

donc les ω racines de q seront identiques avec celles de p et, au lieu d'une seule équation, on aura en apparence (*au moins*) ω équations entre les éléments de p et de q.

Pour faire disparaître ce paradoxe, il n'y a qu'une seule supposition à faire : c'est que x, sous les suppositions faites, devient une matrice vide, car alors x^{-1} n'a plus une existence actuelle, et l'équation $p = xqx^{-1}$ n'aura pas lieu ; c'est ce qu'on va voir arriver *dans le cas général*, où $px = xq$.

Pour fixer les idées, supposons $\omega = 1$ et faisons

$$p = \begin{vmatrix} a & b \\ c & d \end{vmatrix}, \quad q = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}, \quad x = \begin{vmatrix} \lambda & \mu \\ \nu & \pi \end{vmatrix}.$$

En égalant px à xq, on obtient les quatre équations simultanées et homogènes entre λ, μ, ν, π suivantes :

$$(a-\alpha)\lambda + c\mu - \beta\nu + 0\pi = 0,$$
$$b\lambda + (d-\alpha)\mu + 0\nu - \beta\pi = 0,$$
$$-\gamma\lambda + 0\mu + (a-\delta)\nu + c\pi = 0,$$
$$0\lambda + \gamma\mu + b\nu + (d-\delta)\pi = 0,$$

et conséquemment on aura*

$$b^2c^2 + \beta^2\gamma^2 - 2bc\beta\gamma - 2abcd - 2\alpha\beta\gamma\delta + (bc + \beta\gamma)(a+d)(\alpha+\delta)$$
$$- bc(\alpha^2 + \delta^2) - \beta\gamma(a^2 + d^2) + \alpha\delta(a^2 + d^2) + ad(\alpha^2 + \delta^2)$$
$$+ 2ad\alpha\delta + a^2d^2 + \alpha^2\delta^2 - (a+d)(\alpha+\delta)(ad + \alpha\delta) = 0,$$

ou, en écrivant $a + d = B,\ ad - bc = D,\ \alpha + \delta = C,\ \alpha\delta - \beta\gamma = F$,

$$(D-F)^2 + (B-C)(BF - CD) = 0;$$

c'est-à-dire, si R est le résultant de $X^2 - Bx + D$, $X^2 - Cx + F$, $R = 0$ sera la condition générale de la possibilité de satisfaire à l'équation $px = xq$.

Il est facile de faire voir que ce résultat peut être étendu au cas général où p et q sont des matrices de l'ordre ω : on n'a qu'à démontrer que si une des racines latentes de p est égale à une de q, l'équation $px = xq$ est résoluble; et de plus, sans que cette condition soit satisfaite, l'équation est irrésoluble. Soient donc $\lambda_1, \lambda_2, \ldots, \lambda_\omega$ les racines latentes de p et $\mu_1, \mu_2, \ldots, \mu_\omega$ de q et supposons que $\lambda_1 = \mu_1$, alors

$$(p - \lambda_1)x = x(q - \mu_1),$$

et l'on peut satisfaire à cette équation en écrivant

$$x = (p-\lambda_2)(p-\lambda_3)\ldots(p-\lambda_\omega)(q-\mu_2)(q-\mu_3)\ldots(q-\mu_\omega).$$

Conséquemment, si les racines latentes de p et de q sont les racines des deux formes algébriques $X^\omega + BX^{\omega-1} + \ldots + L$, $X^\omega + CX^{\omega-1} + \ldots + M$, quand R (le résultant de ces deux formes) s'évanouit, le résultant des ω^2 équations homogènes linéaires obtenues en égalant $px = xq$ s'évanouira ; mais R est indécomposable et du même degré (ω^2) que ce dernier résultant dans les éléments de p et q. Conséquemment les deux résultants (à un facteur numérique près) sont identiques : ce qui démontre que la condition $R = 0$ est non pas seulement nécessaire, mais de plus suffisante afin que $px = xq$ soit résoluble.

Pour ce qui regarde la valeur de x, posons $x = UV$, où

$$U = (p-\lambda_2)(p-\lambda_3)\ldots(p-\lambda_\omega);\quad V = (q-\mu_2)(q-\mu_3)\ldots(q-\mu_\omega),$$

le seul fait que x contient U comme facteur ou que x contient V comme facteur suffit à constater que x n'est pas seulement vide, mais de plus possède au moins $\omega - 1$ degrés de nullité, c'est-à-dire que tous ses déterminants mineurs du second ordre sont des zéros.

Cela est la conséquence d'un théorème que j'ai démontré dans le *Johns Hopkins Circular*† relatif au degré de nullité des combinaisons des *facteurs latents* d'une matrice, dont le théorème relatif à l'*équation* dite *identique* de Cayley ou de Hamilton n'est qu'un cas particulier, ou pour mieux dire le cas extrême; seulement il faut y ajouter un théorème qui fait partie de ma troisième loi de mouvement algébrique, c'est-à-dire que le degré de nullité d'un facteur ne peut jamais excéder le degré de nullité du produit auquel il appartient.

[* The expressions for p, q in line 7 from the bottom of p. 176 should be interchanged; in the last line of p. 176, for $+\gamma\mu$ *read* $-\gamma\mu$.]

[† p. 134 above.]

Nous avons donc complètement résolu le paradoxe qui était à expliquer. Mais, sur-le-champ, une nouvelle contradiction surgit, car il semble que nous avons démontré que, dans *tout* cas sans exception, si $px=xq$, x est nécessairement une matrice vide, ce qui est évidemment faux, car on sait bien que, si, ω étant de l'ordre de p et de q, $q=\sqrt[\omega]{(1)}\,p$, alors, afin que l'équation $px=xq$ soit résoluble, il n'est jamais nécessaire que x soit vide. Ainsi, par exemple, pour les matrices binaires, l'équation $qx=xq$ est satisfaite quand x est une fonction quelconque de q, et l'équation $qx=-xq$ est résoluble, pourvu que q^2 soit *scalar*, en imposant deux conditions (dont une que son carré soit *scalar*) sur x. Pour lever cette contradiction, revenons au cas où $\omega=2$ et aux équations fondamentales

$$(a-\alpha)\,\lambda+c\mu-\beta\nu=0,$$
$$b\lambda+(d-\alpha)\,\mu-\beta\pi=0,$$
$$-\gamma\lambda+(a-\delta)\,\nu+c\pi=0,$$
$$-\gamma\mu+b\nu+(d-\delta)\,\pi=0.$$

Certes, si ces équations donnent des valeurs *déterminées* aux rapports λ, μ, ν, π, le raisonnement précédent rend certain que x doit être vide, c'est-à-dire que $\lambda\pi-\mu\nu=0$, mais cette conclusion devient fausse aussitôt que p et q sont pris tels que ces rapports deviennent indéterminés, ce qui arrive quand tous les premiers déterminants mineurs de la matrice

$$\begin{vmatrix} (a-\alpha) & c & -\beta & 0 \\ b & (d-\alpha) & 0 & -\beta \\ -\gamma & 0 & (a-\delta) & c \\ 0 & -\gamma & b & (d-\delta) \end{vmatrix}$$

s'évanouissent simultanément.

Dans ce cas, quoique la solution générale qui donne *x vide* tienne bon, rien n'empêche qu'il n'existe d'autres valeurs de x, c'est-à-dire de $\begin{matrix}\lambda & \mu \\ \nu & \pi\end{matrix}$, pour lequels cela n'est pas vrai.

La matrice écrite en haut doit posséder et possède, en effet, la propriété remarquable que, en supprimant une ligne horizontale quelconque et en nommant A, B, C, D les quatre déterminants mineurs de la matrice rectangulaire qui survient, affectés de signes convenables, la quantité $AD-BC$ contiendra le déterminant complet comme facteur. Il sera peut-être utile, avant de conclure, de donner un exemple d'un genre nouveau de subsistance de l'équation $px=xq$ avec une valeur finie du déterminant de x. Faisons donc

$$a-\delta=0,\quad d-\alpha=0,\quad bc-\beta\gamma=0,$$

on aura

$$(a-d)\,\lambda+c\mu-\beta\nu=0,$$
$$b\lambda-\beta\pi=0,\quad -\gamma\lambda+c\pi=0,$$
$$-\gamma\mu+b\nu+(d-a)\,\pi=0,$$

équations qui n'équivalent qu'à deux,

$$b\lambda-\beta\pi=0,\quad (a-d)\,\lambda+(c\mu-\beta\nu)=0,$$

et le déterminant de x, c'est-à-dire $\lambda\pi-\mu\nu$, aura en général une valeur finie.

Dans la dernière Note (insérée dans les *Comptes rendus**) qui roule sur l'équation en matrices binaires $x^2-px=0$, j'ai remarqué qu'en addition aux solutions normales

$$x=0,\quad x=p,\quad x=r\frac{p-s}{r-s},\quad x=s\frac{p-r}{s-r}$$

(où r, s sont les racines latentes de p), on a la solution indéterminée (due en grande partie à la sagacité de M. Franklin)

$$x=\begin{Bmatrix} -\lambda\,(d-r) & \lambda b \\ \mu c & -\mu\,(a-r)\end{Bmatrix}$$

avec la condition $\lambda\,(d-r)+\mu\,(a-r)+r=0$. Évidemment on a aussi la solution tout à fait distincte

$$x=\begin{Bmatrix} -\lambda\,(d-s) & \lambda b \\ \mu c & -\mu\,(a-s)\end{Bmatrix}$$

avec la condition $\lambda\,(d-s)+\mu\,(a-s)+s=0$; mais on doit noter que, quand on prend $\lambda=\mu$, on reprend les deux valeurs normales $x=r\dfrac{p-s}{r-s}$, $x=s\dfrac{p-r}{s-r}$; le fait curieux que, quand $b=0$ et $c=0$, les deux solutions aberrantes forment un troisième couple tout à fait déterminé a été déjà noté, et l'on peut y ajouter la remarque que si, en addition à $b=0$, $c=0$, on a aussi

$$a-d=0,$$

alors l'indétermination reparaît à pas redoublé, la solution entière étant dans ce cas extra-spécialement constituée par une paire de solutions dont l'une et l'autre contiennent *deux* constantes arbitraires au lieu d'une seule.

Je dois ajouter que, dans le cas où i racines de p $(\lambda_1, \lambda_2, \ldots, \lambda_i)$ sont identiques avec i de q $(\mu_1, \mu_2, \ldots, \mu_i)$, l'équation

$$px=xq,$$

qui amène à

$$p^2x=xq^2,\ \ldots,\ p^ix=xq^i$$

et, par conséquent, à

$$(p-\lambda_1)\ldots(p-\lambda_i)\,x=x\,(q-\mu_1)\ldots(q-\mu_i),$$

sera satisfaite si l'on fait $x=UV$, où

$$U=(p-\lambda_{i+1})\ldots(p-\lambda_\omega),\quad V=(q-\mu_{i+1})\ldots(q-\mu_\omega),$$

[* p. 174 above.]

de sorte que x (en vertu du théorème déjà cité) aura au moins $\omega-\theta$ degrés de nullité, c'est-à-dire tous ses déterminants mineurs de l'ordre $\theta+1$ s'évanouiront. Mais on sait, pour le cas où $\theta=\omega$ (et l'on a toute raison de croire pour le cas où θ a une valeur quelconque au-dessus de l'unité), qu'il existe pour des valeurs spéciales de p et de q des solutions singulières de l'équation $px=xq$, lesquelles (comme dans le cas de l'equation de Riccati) sont bien autrement intéressantes et beaucoup plus importantes que la solution générale.

On remarquera que, quand $\theta=\omega$, la solution générale disparaît, tandis que les solutions singulières pour des valeurs particulières de p et de q, ayant toutes les racines latentes de l'un identiques avec celles de l'autre, forment la base de la présentation des matrices sous la forme de quaternions, nonions, etc.

26.

SUR LA SOLUTION DU CAS LE PLUS GÉNÉRAL DES ÉQUATIONS LINÉAIRES EN QUANTITÉS BINAIRES, C'EST-À-DIRE EN QUATERNIONS OU EN MATRICES DU SECOND ORDRE.

[*Comptes Rendus*, XCIX. (1884), pp. 117, 118.]

SOIENT p, q deux matrices d'un ordre donné et servons-nous du symbole $p(\ )q$ pour signifier l'opérateur, lequel, appliqué à une autre matrice x du même ordre, donne pxq.

Alors, si l'on pose

$$p_1(\ )q_1+p_2(\ )q_2+\ldots+p_n(\ )q_n=\phi,$$

ϕx sera une matrice dont chaque élément sera une fonction linéaire des éléments de x; conséquemment, en supposant que les matrices p, q sont de l'ordre ω, on parvient ainsi à une matrice de l'ordre ω^2, et conséquemment ϕ sera assujetti à une équation identique de l'ordre ω^2; disons $F=0$.

Je vais donner la valeur de F pour le cas où $\omega=2$, c'est-à-dire où F sera une fonction du quatrième degré. Supposons que P et P' sont deux quantics du second ordre dans les deux systèmes de variables $x_1, x_2, \ldots, x_n$; $\xi_1, \xi_2, \ldots, \xi_n$ contragredients. Alors, si l'on représente par $\dot{P}'$ ce que devient P' quand on écrit $\delta_{x_1}, \delta_{x_2}, \ldots, \delta_{x_n}$ au lieu de $\xi_1, \xi_2, \ldots, \xi_n$, $(\dot{P}')^i \,.\, P^i$ sera un invariant du système donné pour toute valeur de i.

Considérons le cas où $P=ax^2+bxy+cy^2$ et $P'=\alpha\xi^2+\beta\xi\eta+\gamma\eta^2$. Dans ce cas, on trouvera que $\frac{1}{8}[(\dot{P}')^2P^2-4(\dot{P}'\,.\,P)^2]$ sera identique avec le résultat de $ax^2+bxy+cy^2$, $\gamma x^2-\beta xy+\alpha y^2$, de sorte qu'on peut le nommer le *contra-résultant* des formes (a, b, c), (α, β, γ). Je nommerai donc, en général, l'invariant $\frac{1}{8}[(\dot{P}')^2P^2-4(\dot{P}'\,.\,P)^2]$ le *quasi contra-résultant* des deux formes P, P' quand elles contiennent un nombre quelconque de variables.

Or, en revenant à l'expression ϕ, nommons P le déterminant de

$$u_1p_1+u_2p_2+\ldots+u_np_n+\phi.v$$

et Q le déterminant de

$$u_1q_1+u_2q_2+\ldots+u_nq_n-v,$$

où ϕ, pour le moment, est traité comme une quantité ordinaire. J'ai trouvé que le quasi contra-résultant de P, Q, quand ϕ appartient à des matrices du second ordre (lequel sera une fonction biquadratique de ϕ), égalé à zéro, est l'équation identique cherchée en ϕ.

Il est probable, mais je n'en suis pas encore absolument convaincu, qu'une méthode analogue donnera l'équation identique de ϕ pour des matrices d'un ordre quelconque.

Si l'on suppose que les p et les q sont des quaternions, rien ne change avec l'exception que P et Q seront définis comme étant les modules (les *tensors* carrés) au lieu d'être les déterminants de $\phi v + \Sigma pu$, $-v + \Sigma qu$ respectivement.

Connaissant ainsi l'équation identique de ϕ, on peut résoudre immédiatement l'équation

$$\Sigma(pxq) = T,$$

car, en écrivant $p(\ \ )q = \phi$, on a l'équation connue

$$\phi^4 + B\phi^3 + C\phi^2 + D\phi + E = 0,$$

et, conséquemment, en exceptant toujours le cas où $E = 0$ (dans lequel cas l'équation devient ou impossible ou indéterminée), on trouve

$$x = \phi^{-1}T = -\frac{D + C\phi + B\phi^2 + \phi^3}{E}T.$$

Par exemple, si l'équation donnée est $pxq + rxs = T$,

$$\begin{aligned}\phi T &= pTq + rTs,\\ \phi^2 T &= p^2Tq^2 + prTsq + rpTqs + r^2Ts^2,\\ \phi^3 T &= p^3Tq^3 + p^2rTsq^2 + prpTqsq\\ &\quad + rp^2Tq^2s + pr^2Ts^2q + rprTsqs + r^2pTqs^2 + r^3Ts^3,\end{aligned}$$

et, éventuellement, en ne se servant que des coefficients qui entrent dans les fonctions P et Q par le moyen de formules connues, on réduit x à une somme de multiples de termes de la forme

$$pT,\ rT,\ prT;\ pTq,\ rTq,\ prTq;\ pTqs,\ rTqs,\ prTqs,$$

et ainsi en général. Donc le problème de la résolution des équations linéaires est complètement résolu ; seulement il reste à traiter en détail le cas singulier où la matrice appartenant à ϕ est *vide*.

27.

SUR LES DEUX MÉTHODES, CELLE DE HAMILTON ET CELLE DE L'AUTEUR, POUR RÉSOUDRE L'ÉQUATION LINÉAIRE EN QUATERNIONS.

[*Comptes Rendus*, XCIX. (1884), pp. 473—476, 502—505.]

UN célèbre quaternioniste m'ayant demandé de lui expliquer la portée de ma solution de l'équation linéaire en matrices sur la solution du même problème en quaternions, il me semble désirable de donner explicitement le moyen de passer d'une solution à l'autre. Préalablement, il sera bon cependant de remarquer que, faute d'un examen suffisamment attentif de la forme du résultat obtenu ou plutôt indiqué par Hamilton (*Lectures on Quaternions*, pp. 559—561), on pourrait attribuer à sa solution une propriété qu'elle ne possède pas, celle de fournir le moyen de trouver la solution de l'équation linéaire en quaternions *sous une forme réduite* semblable à celle que fournit ma méthode : mais, en effet, l'examen d'un seul terme de m (voir au bas de la page 561), par exemple $SrJr^2$, suffit à montrer que le dénominateur m de Hamilton est du douzième degré dans les éléments des quaternions (b et a) de son équation $\Sigma bqa = c$ (p. 559), tandis que le degré pour la forme réduite n'est que huit. Il s'ensuit que le numérateur (si l'on avait la patience de le déduire des formules de Hamilton), aussi bien que le dénominateur obtenu par ce moyen, serait affecté d'un facteur étranger à la question, du quatrième degré, dans les éléments nommés.

J'ajoute qu'il est parfaitement possible de donner la valeur de x dans l'équation $\Sigma pxp' = T$ comme fonction seulement des p et p' et des coefficients des *deux formes associées* sans aucune irrationnalité. Car le déterminant du nivellateur $\Sigma p(\ )p'$, disons N, étant obtenu sous la forme $\Omega_2 + \sqrt{(\Omega_4)}$, le déterminant du nivellateur

$$\Sigma p(\ )p' + \begin{matrix}-1 & 0\\ 0 & -1\end{matrix}\ (\ )\ \begin{matrix}N & 0\\ 0 & N\end{matrix}$$

(disons FN) sera aussi exprimé sous une forme semblable à celle-là, disons $\Phi_2 + \sqrt{(\Phi_4)}$.

Or, au lieu de l'équation identique $FN = 0$, on peut se servir d'un multiple quelconque de cette équation pour obtenir l'inverse de N comme fonction de puissances positives de N. Ainsi l'on peut, dans ce but, se servir de l'équation $\Phi_2^2 - \Phi_4^4 = 0$, au lieu de $FN = 0$, et, avec l'aide de cette équation, on obtiendra x exprimé en fonction des p et p' et de fonctions rationnelles des coefficients des deux formes associées ; mais alors, au lieu d'être obtenu sous sa forme la plus simple, son numérateur et son dénominateur contiendront un facteur commun qui sera une fonction du huitième degré des éléments des p et des p'.

Je passe à la règle pour traduire ma solution de l'équation en matrices $\Sigma pxp' = T$ en solution de cette même équation quand les p, les p' et le T, au lieu d'être matrices, sont donnés comme quaternions. Évidemment tout ce qui est nécessaire, c'est de connaître l'équation qui serait identique pour $\Sigma p(\ )p'$; je vais donner la règle pour l'obtenir.

Sous le signe Σ, je suppose compris $p, q, r, \ldots, p', q', r', \ldots$.

Écrivons la forme symbolique $[Nx + (p)y + (q)z + \ldots]^2$, disons X ; les coefficients de $xy, xz, \ldots$, symboliquement écrits, sont

$$2(p)N,\ 2(q)N,\ \ldots;$$

à $(p), (q), \ldots$ il *faut substituer* $Sp, Sq, \ldots$; le coefficient de y^2 est $(p)^2$ auquel il faut substituer Tp^2 ; finalement le coefficient de yz est $(p)(q)$, auquel il faut substituer $S(Vp.Vq)$*.

De même, on construit et l'on interprète la forme

$$[-x' + (p')y' + (q')z' + \ldots]^2$$

(disons X').

On calcule† la valeur de $\dot{X}'^2X^2 - 4(\dot{X}'X)^2$. Ce résultat (une fonction du quatrième degré en N) (disons ΩN) sera une partie de la fonction qui doit être identiquement zéro. Le reste de cette fonction (disons $64\Omega_1 N$) sera

$$[\Sigma S(VpVqVr)\,S(Vp'Vq'Vr')]\,N - \Sigma SpSp'\,S(VpVqVr)\,S(Vp'Vq'Vr'),$$

et je dis que

$$\Omega N + 64\Omega_1 N = 0$$

sera l'équation identique en N, et servira pour trouver la valeur de x, c'est-à-dire $N^{-1}T$ comme fonction du quaternion T, des quaternions $p, q, \ldots, p', q', \ldots$ et des symboles S, V, T ; de plus la valeur ainsi obtenue sera x *sous sa forme réduite*.

Il y a encore une petite observation à ajouter à mes remarques sur la solution de Hamilton de l'équation $\Sigma bqa = c$ (*Lectures*, p. 559). Il divise q en deux parties, le scalar w et le vecteur ρ.

C'est cette dernière quantité (ρ) qu'il exprime sous la forme $\frac{R}{m}$; alors $w = \frac{S(c) - S\eta'\rho}{\Sigma S(ab)}$, de sorte que, à défaut d'avoir recours à des réductions

[* See first note on p. 191 below.]

[† See p. 181 above and p. 202 below.]

ultérieures, le dénominateur de q contiendra, non seulement le facteur étranger du quatrième degré dans les éléments des a et des b dont j'ai déjà parlé, mais encore le facteur étranger $\Sigma S(ab)$.

On remarquera que, dans cette solution, on aura des combinaisons des b avec des a et des fonctions quaternionistiques de ces combinaisons, tandis que, dans la solution infiniment plus simple que je donne du problème, il ne se trouve nulle part des mélanges de cette nature, mais seulement des fonctions quaternionistiques de combinaisons des a entre eux-mêmes et des b entre eux-mêmes. Le vice fondamental de la méthode de Hamilton, c'est la réduction du problème donné à un autre, où, au lieu de q, il n'entre que sa partie vectorielle. Néanmoins le travail de Hamilton (quoique sa raison d'être ne subsiste plus) méritera toujours d'être regardé comme un monument du génie de son grand et admirable auteur.

C'est là, pour la première fois dans l'histoire des Mathématiques, qu'on rencontre la conception de l'équation identique (voir *Lectures*, pp. 566, 567) qui est la base de tout ce qu'on a fait depuis et de tout ce qui reste à faire dans l'évolution de la Science vivante et remuante de la quantité multiple, c'est-à-dire l'*Algèbre universelle*, née à peu près 250 ans après l'organisation définitive de sa sœur aînée l'*Arithmétique universelle*, dans le Mémoire de M. Cayley sur les matrices, dans les *Philosophical Transactions*, vol. 148.

Dans une Note précédente, on a vu que dans la nouvelle et seule *bonne* méthode pour résoudre, par rapport à x, l'équation en quaternions

$$pxp' + qxq' + rxr' + sxs' + \ldots = \Gamma,$$

on fait trois opérations. La première, à laquelle on peut donner le nom de *nivellation*, consiste à trouver le nivellant, c'est-à-dire le déterminant de la matrice du quatrième ordre appartenant à un nivellateur donné du second ordre. La seconde, qu'on peut appeler *déduction*, consiste à obtenir l'équation identique, à laquelle un nivellateur correspond au moyen d'un autre nivellateur qu'on obtient du nivellateur donné en y adjoignant un couple de plus de la forme $-N(\ )\delta s$, ou, ce qui revient au même, le couple $\sqrt{(-N)}(\ )\sqrt{(-N)}$, où N est considéré comme un *scalar*. Finalement, on arrive à la dernière opération, que je nommerai *substitution et réduction*, et qui consiste à substituer à l'inverse du nivellateur sa valeur en fonction rationnelle du troisième ordre de lui-même, puis à faire des réductions dont je parlerai tout à l'heure.

Au moyen de ces opérations, on arrive à la valeur de l'inconnue de l'équation sous sa forme réduite la plus simple qu'elle puisse prendre.

Pour obtenir la forme de l'équation identique, voici ce que j'ai trouvé en appliquant la méthode indiquée dans la Note précédente.

Pour plus de simplicité, je me sers de la notation suivante, qui s'applique à des lettres quelconques, accentuées ou non, représentant des quaternions.

Je pose

$$Sp=(p),\quad Tp^2=p_2,\quad S(VpVq)=(pq),\quad S(VpVqVr)=(pqr).$$

Alors, en écrivant

$$p(\ )p'+q(\ )q'+r(\ )r'+s(\ )s'+\ldots=N,$$

on aura

$$\begin{aligned}
&N^4-4\Sigma(p)(p')N^3+\Sigma[4(p)^2p'_2+4(p')^2p_2-2p_2p'_2]N^2\\
&\quad-\Sigma\{4(p)(p')p_2p'_2\\
&\qquad+8[(p)(q')(pq).p'_2+(p')(q)p'q'.p_2]-4(p)(p')q_2q'_2\\
&\qquad+4[(q)(p')p_2q'_2+(p)(q')p'_2q_2]-8pp'(qr)(q'r')\\
&\qquad+8[(p)(q')(qr)p'r'+(p')(q)(q'r')(pr)]+8\Sigma(pqr)(p'q'r')\}N\\
&\quad+\Sigma\{p_2^2p_2'^2-2p_2p'_2.q_2q'_2\\
&\qquad+4[p_2q_2(p'q')^2+p'_2q'_2(pq)^2]-4p_2p'_2pq.p'q'\\
&\qquad+4p_2p'_2qr.q'r'+8[p_2(qr)(p'q')(p'r')+p'_2(q'r')(pq)(pr)]\\
&\qquad+8[pq.rs.p'r'.q's'+p'q'.r's'.pr.qs]-8(p)(p')(qrs)(q'r's')\}=0,
\end{aligned}$$

où le dernier terme de la partie fonctionnelle de l'équation est le nivellant de N.

Quant à la substitution, si, dans l'équation précédente

$$N^4-AN^3+BN^2+CN-D=0\ *,$$

on remplace $N^{-1}\Gamma$ par la fraction

$$\frac{N^3\Gamma-AN^2\Gamma+BN\Gamma-C\Gamma}{D},$$

tous les termes du numérateur de cette fraction seront des multiples connus de la forme $P\Gamma P'$, où P est de l'une des formes suivantes : p^3; p^2q, pqp, qp^2; p^2, pq; p; ..., et où de même P' a des types semblables avec des lettres accentuées. Il ne reste plus qu'à réduire chaque P à sa forme la plus simple, c'est-à-dire à l'exprimer comme fonction linéaire de $1, p, q, pq-qp$, et de même pour P'. Alors le numérateur de x ne contiendra plus que des termes dont les arguments seront tous d'un des types suivants (je remplace la moitié de $pq-qp$ par $[pq]$) :

$$\Gamma,\ p\Gamma,\ \Gamma p',\ p\Gamma p',\ p\Gamma q',$$
$$[pq]\Gamma,\ \Gamma[p'q'],\ p\Gamma[p'q'],\ [pq]\Gamma p',\ [pq]\Gamma[p'q'];$$

il faut y ajouter le type $pqr\Gamma r'q'p'$, qui est déjà sous sa forme la plus simple et n'exige aucune formule de réduction.

* D est le déterminant de la matrice qui appartient au nivellement N. Quand $D=0$, la solution de l'équation $Nx=\Gamma$ devient ou *idéale* (ce qui a lieu en général), ou (ce qui a lieu pour des cas particuliers) *actuelle*, mais indéterminée.

Je n'entreprendrai pas pour le moment de calculer les coefficients de ces arguments, mais j'indiquerai du moins les formules de réduction qui seules sont nécessaires pour effectuer ce calcul. Ce travail, bien digne d'attirer l'attention de quelque jeune géomètre, peut très probablement amener à des résultats qui, à l'aide d'une notation symbolique, pourront être présentés sous une forme d'une simplicité tout à fait inattendue et pour ainsi dire providentielle. J'en ai eu l'expérience pareille dans d'autres recherches du même genre, dans la solution de certains cas d'équations quaternionistiques du second degré.

Voici toutes les formules de réduction dont on aura besoin :

$$\begin{aligned}
&p^2=2(p)p-p_2,\quad p^3=[4(p)^2-p_2]p-2(p)p_2,\\
&pq=[pq]+(p)q+(q)p-(pq),\\
&qp=-[pq]+(p)q+(q)p-(pq),\\
&p^2q=2(p)[pq]+2(p)(q)p+(2p^2-p_2)q-2(p)(pq),\\
&pqp=4(p)[pq]+[8(p)(q)-2(pq)]p\\
&\qquad-[4(p)^2+p_2]q-[2(q)p_2+4(p)(pq)];
\end{aligned}$$

dans les formules on peut, au lieu de $[pq]$, écrire $V(VpVq)$.

Remarque.—Quand un nivellateur devient *symétrique*, c'est-à-dire quand $p=p'$, $q=q'$, ..., alors les *deux formes associées* coïncident en une seule dont le nivellant devient un *invariant orthogonal.*

Qu'il me soit permis, avant de conclure, d'ajouter encore une petite réflexion sur l'importance de la question traitée ici. Elle constitue, pour ainsi dire, un canal qui, comme celui de Panama, sert à unir deux grands océans, celui de la théorie des invariants et celui des quantités complexes ou multiples : dans l'une de ces théories, en effet, on considère l'action des substitutions sur elles-mêmes, et dans l'autre, leur action sur les formes; de plus, on voit que la théorie *analytique* des quaternions, étant un cas particulier de celle des matrices, cesse d'exister comme une science indépendante; ainsi, de trois branches d'analyse autrefois regardées comme étant indépendantes, en voilà une abolie ou absorbée, et les deux autres réunies en une seule de substitution algébrique.

28.

SUR LA SOLUTION EXPLICITE DE L'ÉQUATION QUADRATIQUE DE HAMILTON EN QUATERNIONS OU EN MATRICES DU SECOND ORDRE.

[*Comptes Rendus*, XCIX. (1884), pp. 555—558, 621—631.]

HAMILTON, dans ses *Lectures on quaternions* (p. 632), a fourni un moyen de résoudre l'équation (en quaternions ou en matrices binaires) de la forme

$$x^2 - 2px + q = 0;$$

mais les circonstances les plus intéressantes de la solution ne se font pas voir dans sa méthode de traiter la question. Voici la manière analytique directe que nous employons pour obtenir x sous sa forme explicite.

On suppose $\qquad x^2 - 2Bx + D = 0$

l'équation identique pour x, où B et D sont des *scalars* à trouver.

En combinant ces deux équations en x, on obtient

$$2x = (p - B)^{-1}(q - D),$$

et, en supposant que la *forme associée* à [1], p, q, c'est-à-dire le déterminant de $\lambda + \mu p + \nu q$, soit

$$\lambda^2 + 2b\lambda\mu + 2c\lambda\nu + d\mu^2 + 2e\mu\nu + f\nu^2,$$

on aura*

$$4(d - 2bB + B^2)x_0^2 - 4(e - bD - cB + BD)x_0 + f - 2cD + D^2 = 0.$$

Conséquemment, en écrivant $u = B - b$, $v = D - c$,

$$d - b^2 = \alpha, \quad e - bc = \beta, \quad f - c^2 = \gamma,$$

et, en comparant cette équation avec l'équation donnée, on voit qu'on peut écrire

$$u^2 + \alpha = \lambda, \quad uv + \beta = 2\lambda(u + b), \quad v^2 + \gamma = 4\lambda(v + c).$$

De plus, puisque $p^2 - 2bp + d = 0$, on aura

$$x = \frac{(p - b + u)(q - c - v)}{2(b^2 - d - u^2)} = -\frac{(p - b + u)(q - c - v)}{2\lambda}.$$

[* The determinant of $2Bx_0 - D - 2x_0p + q$ being zero, if x_0 is a latent root of x.]

En éliminant u, v entre les trois équations qui les lient avec b, c, α, β, γ, on trouvera l'équation bien remarquable

$$e^{\lambda(2\delta_e - \delta_d)}.I = 0,$$

où I est le discriminant de la forme associée donnée plus haut, c'est-à-dire

$$I = \begin{vmatrix} 1 & b & c \\ b & d & e \\ c & e & f \end{vmatrix} = df + 2bce - dc^2 - e^2 - fb^2,$$

de sorte que la quantité exponentielle symbolique représente une fonction cubique et donne lieu à une équation cubique en λ.

A chaque valeur de λ correspondent les deux valeurs $\pm\sqrt{(\lambda - \alpha)}$ de u et à chaque valeur de u (autre que $u = 0$) correspondra la seule valeur $2\lambda + \frac{(2\lambda + c)b - e}{u}$ de v.

Quand $u = 0$, $\lambda = \alpha = d - b^2$, et l'équation

$$v^2 - 4\lambda v + \gamma - 4\lambda c = 0$$

a ses deux racines finies. Donc, quand $u = 0$, il faut que $\frac{(2\lambda + c)b - e}{u}$ prenne la forme $\frac{0}{0}$, et à cette valeur de u (qu'on peut envisager comme deux valeurs de u réunies en une) correspondront pour v les deux valeurs données par l'équation quadratique ci-dessus.

Ainsi l'on voit qu'en général x a trois paires de valeurs déterminées et qu'aucune de ces valeurs ne cesse d'être *actuelle* et *déterminée* que pour le seul cas où l'une des trois valeurs de λ est égale à zéro, c'est-à-dire où I, l'invariant de la *pleine** forme associée à (p, q), s'évanouit.

Cela revient à dire que I est le critérium de la normalité de l'équation donnée.

Si l'on regarde p et q comme des quaternions, on aura

$$b = Vp, \quad c = Vq, \quad d = Tp^2, \quad e = SpSq - S(VpVq), \quad f = Tq^2.$$

Il est bien digne de remarque que $4I$ est identique avec $(pq - qp)^2$.

On peut démontrer que, si p et q sont des matrices d'un ordre quelconque, les racines de l'équation $x^2 - 2px + q = 0$ seront toujours (comme ici) associées en paires; car, si l'on écrit $x + x_1 = 2p$, on aura

$$x_1^2 - 2x_1p + q = 0,$$

et conséquemment, si $p^\omega - \omega bp^{\omega-1} + \ldots = 0$ est l'équation identique connue en p et $x^\omega - \omega Bx^{\omega-1} + \ldots = 0$ l'équation identique à trouver en x, à chaque valeur

* Nous avons déjà défini la *forme associée* au *corps* $p, q, r, \ldots$. Par la *pleine* forme, on peut sous-entendre ce que devient la forme associée quand on adjoint au corps une matrice unitaire.

de $B - b$ correspondra une valeur égale de $b - B$, c'est-à-dire que l'équation pour trouver B sera de la forme $F(B-b)^2 = 0$.

En se servant de l'équation conjuguée (c'est-à-dire en x_1) dont la somme des racines sera évidemment la même que pour l'équation en x, on obtient immédiatement, dans le cas où p et q sont du second ordre, par le moyen de la formule

$$x = -\frac{(p+b-u)(q-c-v)}{2\lambda}$$

et de l'équation en λ, la valeur de Σx*.

Cette valeur sera $6[p + (2\delta_e - \delta_d) I^{\frac{1}{2}}]$, de sorte que la valeur moyenne d'une racine de l'équation $x^2 - 2px + q = 0$ est p (la valeur moyenne pour le cas où p et q sont *scalars*), augmentée de $(2\delta_e - \delta_d) I^{\frac{1}{2}}$, où $I^{\frac{1}{2}}$ doit avoir le signe qui le rend égal à $\frac{1}{2}(pq - qp)$. De même on trouve

$$\Sigma x^2 = 2p\Sigma x - 6q,$$

et ainsi la valeur moyenne de x^2 sera

$$2p^2 - q + (4\delta_e - 2\delta_d) I^{\frac{1}{2}} p,$$

et l'on peut trouver successivement, par la même méthode, la valeur moyenne d'une puissance quelconque de x. Les détails du calcul précédent, et encore d'autres propriétés de l'équation en x, seront donnés prochainement dans le *Quarterly mathematical Journal* ou quelque autre recueil mathématique. Ici on n'a voulu que produire les résultats principaux obtenus par notre méthode.

L'équation de Hamilton en quaternions ou en matrices binaires est celle que nous avons traitée dans une Note précédente. C'est l'équation

$$x^2 + 2qx + r = 0.$$

Nous avons trouvé que la solution de cette équation dépend d'une équation cubique ordinaire en λ, à chaque valeur de laquelle correspondent deux valeurs de x, et qu'elle est normale ou régulière quand le dernier terme de cette équation diffère de zéro. L'équation est dite *régulière* ou *normale* quand sa solution dépend du nombre maximum de racines déterminées, c'est-à-dire de trois paires de racines déterminées; chaque paire est alors connue comme fonction de λ, q, r et des paramètres b, c, d, e, f qui dépendent de q

* On aura

$$\Sigma x = -\Sigma \frac{(p-b+u)(q-c-v)}{2\lambda},$$

$$\Sigma(2p - x) = -\Sigma \frac{(q-c-v)(p-b+u)}{2\lambda}.$$

On retranche une équation de l'autre, on substitue pour $\Sigma \frac{1}{\lambda}$ sa valeur tirée de l'équation cubique en λ, et on écrit $pq - qp = 2I^{\frac{1}{2}}$.

et r et sont définis au moyen du déterminant de $u + vq + wr$* qu'on a supposé être mis sous la forme

$$u^2 + 2buv + 2cuw + dv^2 + 2evw + fw^2,$$

d'où

$$b = Sq, \quad c = Sr, \quad d = Tq^2, \quad f = Tr^2e = SqSr - S(Vq \,.\, Vr)^*.$$

Dans ce cas, on peut dire que la solution elle-même est régulière.

En nommant I l'invariant de la forme ternaire, écrite plus haut, c'est-à-dire en posant

$$I = df + 2bce - b^2f - c^2d - e^2,$$

nous avons trouvé que l'équation en λ peut être mise sous la forme

$$e^{\lambda\Omega} I = 0,$$

où

$$\Omega = 2\delta_e - \delta_d,$$

c'est-à-dire qu'on aura

$$4\lambda^3 + (4c - 4d)\lambda^2 + (4be - 4cd + c^2 - f)\lambda + I = 0.$$

Ainsi, afin que la solution soit régulière, il faut et il suffit que I diffère de zéro†.

De là il suit que, dans le cas d'une équation régulière, deux x ne peuvent être égaux, à moins qu'ils n'appartiennent à la même paire ou bien que deux λ ne deviennent égaux; car x peut être exprimé comme une fonction linéaire de qr, q, r, 1, dans laquelle le coefficient de qr est $-\frac{1}{2\lambda}$.

Donc, si deux des x sont égaux sans que deux λ le soient, une équation linéaire subsistera entre pq, p, q, 1, mais dans ce cas nous avons trouvé ailleurs que $I = 0$, et la solution cesse d'être régulière.

Nous allons pour le moment nous borner au cas où l'équation est régulière, et conséquemment nous n'aurons qu'à considérer les cas où il y a égalité ou entre deux racines de λ ou bien entre deux valeurs de x qui correspondent à la même valeur de λ.

Si l'on suppose que deux valeurs de λ soient égales, il en résultera que deux des paires de valeurs de x deviendront identiques, de sorte qu'une seule condition suffira à réduire le nombre des racines distinctes de 6 à 4, c'est-à-

* Par un oubli très regrettable nous avons pris, dans une Note précédente, pour le coefficient de $2xy$ dans la forme associée à

$$(up + vq + wr + \ldots),$$

$S(VpVq)$ au lieu de sa vraie valeur, $\quad SpSq - S(VpVq)$,

et de même pour les autres coefficients des termes mixtes, de sorte que le calcul du déterminant du *nivellateur* $\Sigma p(\;\;)p'$ dans la Note sur l'achèvement de la solution de l'équation linéaire en quaternions est erroné et a besoin d'être fait de nouveau.

† Conséquemment, quand l'équation est régulière, ni q ni v ne peut devenir zéro; car, dans l'un et l'autre de ces deux cas, $I = 0$; aussi, pour la même raison, r ne peut pas être une fonction de q.

dire que les valeurs de x, qui, en général, sont de la forme m, m'; n, n'; p, p', deviendront de la forme m, m'; n, n'; n, n'.

Au lieu de calculer directement le discriminant de l'équation en λ, qui donnera un résultat très compliqué, nous allons montrer qu'on peut substituer le discriminant de la forme très simple biquadratique

$$\left(1,\ b,\ \frac{c+2d}{3},\ e,\ f\right)(r,\ s)^4.$$

Mais préalablement il sera utile d'opérer une transformation linéaire sur l'équation en λ.

Écrivons $\lambda = \mu + d$; l'équation en μ sera

$$4\mu^3 + 4(c+2d)\mu^2 + [(c+2d)^2 + 4be - f]\mu + 2b(c+2d)e - b^2 f - e^2 = 0.$$

On voit donc que le discriminant qu'on veut calculer est une fonction complète de b, $c+2d$, e, f.

Nous avons trouvé $u^2 = \lambda - d + b^2$, c'est-à-dire $\mu + b^2$. On aura donc

$$\begin{aligned}4u^6 &+ 4(c+2d-3b^2)u^4\\ &+ [12b^4 - 8(c+2d)b^2 + (c+2d)^2 + (4be-f)]u^2\\ &- [2b^3 - b(c+2d) + e]^2 *.\end{aligned}$$

Dans l'équation donnée, substituons $x + \epsilon$, où ϵ est un infinitésimal (*scalar* si l'on parle de quaternions ou représentant la matrice $\begin{matrix}\epsilon & 0\\ 0 & \epsilon\end{matrix}$ si l'on parle de matrices); alors p sera augmenté par ϵ et q par $2\epsilon p$, et ainsi $(\lambda + \mu p + \nu q)$ deviendra $(\lambda + \epsilon\mu) + (\mu + 2\epsilon\nu)p + \nu q$, de sorte qu'en désignant le discriminant cherché par D, l'accroissement de D est nul quand λ et μ deviennent $\lambda + \epsilon\mu$, $\mu + \epsilon\nu$ simultanément, c'est-à-dire quand la forme ternaire en u, v, w devient

$$\begin{aligned}u^2 + 2(b+\epsilon)uv + 2(c+2\epsilon b)uw + (d+2\epsilon b)v^2&\\ + (2e + 2\epsilon c + 4\epsilon d)vw + (f + 4\epsilon e)w^2&.\end{aligned}$$

Donc $$[a\delta_b + 2b\delta_c + 2c\delta_d + (c+2d)\delta_e + 4e\delta_f]D = 0.$$

Écrivons $c + 2d = 3m$. On sait que D est une fonction complète de b, m, e, f, de sorte que, par rapport à D (comme opérande), $\delta_c + \delta_d = \delta_m$; ainsi, en écrivant $1 = a$, on aura

$$(a\delta_b + 2b\delta_m + 3m\delta_e + 4e\delta_f)D = 0.$$

D sera donc ou un invariant ou un sous-invariant de la forme biquadratique (a, b, m, e, f).

* u sera la partie scalar de x si l'équation est donnée sous la forme quaternionique, ou bien la moitié de la somme du premier et du quatrième élément de x si l'équation est donnée entre des matrices. Hamilton a trouvé l'équation équivalente à celle donnée pour u dans le texte; mais, dans sa formule, les coefficients sont exprimés sous une forme compliquée et assez difficile à débrouiller.

Mais, en faisant attention à l'équation en μ, on voit que D sera de l'ordre 6 dans les coefficients et du poids 12; il est donc un invariant et une fonction linéaire de s^3 et t^2 (où s et t sont les deux invariants irréductibles) de la forme biquadratique.

En nommant Δ le discriminant de cette forme, on a

$$\Delta = s^3 - 27t^2,$$

dont une partie sera $$f^3 - 27b^4 f^2;$$

mais on voit, par l'examen de l'équation en μ, qu'une partie de D sera

$$16b^4 f^2 - \frac{16 f^3}{27}$$

et, conséquemment, $$D = -\tfrac{16}{27}\Delta.$$

Il s'ensuit que la condition nécessaire et suffisante pour l'égalité de deux des racines de l'équation donnée avec deux autres est tout simplement $\Delta = 0$, comme nous l'avons déjà énoncé.

Cherchons la condition pour laquelle les trois paires coïncideront toutes dans une seule paire; alors les trois racines de μ deviennent toutes égales, et l'on a non seulement

$$\Delta = 0,$$

mais encore $$(12m^2) - (9m^2 + 4be - f) = 0,$$

c'est-à-dire $$f - 4be + 3m^2 = 0 \quad \text{ou} \quad s = 0.$$

Donc les conditions nécessaires et suffisantes, pour qu'il n'y ait que deux racines distinctes chacune, prises trois fois dans la solution de l'équation donnée, seront

$$s = 0, \quad t = 0.$$

On peut aussi demander quelle est la condition ou plutôt quelles sont les équations de condition pour que deux racines de la même paire soient égales.

Dans ce cas, nous avons trouvé que $u = 0$; cela exige que le dernier terme dans l'équation à u^2 devienne zéro. On aura donc, en vertu de l'équation en u^2,

$$ae - 3bm + 2b^3 = 0,$$

c'est-à-dire que le sous-invariant gauche ou bien le premier coefficient du Hessien à la forme biquadratique s'évanouit. Mais cela ne suffit pas pour que les deux x d'une paire deviennent parfaitement identiques. Il faut aussi que les deux valeurs de v, qui correspondent à la valeur zéro de u, ou que les deux racines de l'équation

$$v^2 - 4\lambda(v + c) + \gamma = 0,$$

où $$\lambda = \alpha = d - b^2,$$

deviennent égales, c'est-à-dire que

$$\gamma + c^2 - (2\alpha + c^2) = 0,$$

ou bien, puisque $\gamma = f - c^2$, que

$$f - (3m - 2b^2)^2 = 0\,;$$

à cette équation il faut joindre l'équation déjà trouvée

$$ae - 3bm + 2b^3 = 0\,;$$

le système de ces deux équations exprime la condition de la coïncidence des deux x d'une paire. Quoique $f - (3m - 2b^2)^2 = 0$ ne soit pas en elle-même un sous-invariant, les deux équations ci-dessus constituent (comme elles doivent le faire) un *plexus* sous-invariantif; car on trouvera

$$(a\delta_b + 2b\delta_m + 3m\delta_e + 4e\delta_f)\,[af - (3am - 2b^2)^2] = 4\,(ae - 3bm + 2b^3) = 0.$$

En effet, puisque $f - (3m - 2b^2)^2$ ne diffère de $f - 9m^2 + 2abe + 6b^2m$ (le second coefficient du Hessien) que par $-\,2b\,(ae - 3bm + 2b^3)$, on peut substituer, pour le plexus écrit plus haut, le plexus $H_1 = 0$, $H_2 = 0$, où H_1, H_2 sont le premier et le second coefficient du Hessien de la forme quadratique.

Or il est facile de démontrer que, quand dans la forme $(a, b, m, e, f)(x, y)$ a n'est pas zéro, mais que les deux premiers coefficients du covariant irréductible gauche le sont, le covariant s'évanouit complètement*, et la forme biquadratique a deux paires de racines égales.

On sait aussi que, quand les deux invariants irréductibles s'évanouissent, il y a trois racines égales, et, quand en même temps les deux invariants et le covariant gauche s'évanouissent, toutes les racines de la biquadratique sont égales.

Ainsi on voit que les seuls cas d'égalité possibles entre les racines de l'équation quadratique donnée, quand sa solution est régulière, correspondent aux quatre cas d'égalité entre les racines de la biquadratique ordinaire qui s'y est associée.

En prenant les quatre cas: 1° ou la quadratique a deux racines égales; 2° ou elle a deux paires de racines égales; 3° trois racines égales; 4° toutes ses racines égales; alors la quadratique donnée aura, dans le premier cas, deux paires de racines égales; dans le deuxième, quatre racines égales; dans le troisième, trois paires de racines égales, et dans le dernier cas toutes ses racines seront égales.

Quant au rapport de la biquadratique binaire à la forme ternaire quadratique, on passe de la seconde à la première, en se servant de la substitution dont s'est servi notre très honoré collègue, M. Darboux, dans sa belle Note sur la résolution de l'équation biquadratique (*Journal de Liouville*, t. XVIII. p. 220). On n'a qu'à faire $x = u^2$, $y = 2uv$, $z = v^2$, et la forme ternaire passe dans la forme binaire biquadratique. On voit ainsi que les genres de solutions régulières de l'équation en quaternions donnée dépendent ex-

* Quand les deux premiers coefficients du covariant irréductible gauche d'une biquadratique binaire s'évanouissent, le discriminant s'évanouit nécessairement: nous avons trouvé que ce discriminant pris négativement égale 16 fois le produit des coefficients extrêmes, moins le produit du second et l'avant-dernier coefficient du covariant gauche.

clusivement de la relation entre la conique qui s'y est associée avec la conique absolue $y^2 - 4xz$. Dans le cas le plus général, les deux courbes se coupent en quatre points; dans les quatre autres cas, il y aura l'une ou l'autre des quatre espèces de contact entre les deux coniques.

Mais, de plus, on voit évidemment que cette idée des deux coniques peut être étendue à l'équation de Hamilton, même pour le cas où la solution devient irrégulière.

Dans ce cas, la forme ternaire, associée à l'équation $x^2 + qx + r$, perdra sa forme de conique et deviendra un système de deux lignes droites qui se croisent ou de deux lignes coïncidentes. Dans la première supposition, il y aura le cas où les deux droites toutes les deux coupent et les cas où l'une ou toutes les deux touchent la conique fixe; il y aura aussi les cas où la conique fixe passe par le point d'intersection des deux droites en les coupant toutes les deux ou en touchant une. Dans la seconde supposition, il y aura les deux cas où les droites coïncidentes coupent ou touchent la conique fixe.

Ainsi donc il nous paraît qu'on peut affirmer avec pleine confiance que, dans l'équation de Hamilton*, il y a exactement douze cas, ou au moins douze cas principaux, à considérer†. Nous devons cette méthode si simple

* Quant à l'équation plus générale $px^2 + qx + r = 0$, dans le cas où le discriminant ou le tenseur de p devient zéro et que, par conséquent, la forme ne rentre pas dans celle de Hamilton (puisqu'on ne peut plus diviser l'équation par p), il peut se présenter encore un grand nombre de cas singuliers que nous n'avons pas encore étudiés à fond.

† Cela donne lieu à une réflexion curieuse. Si l'on considère tous les genres de rapports qui peuvent avoir lieu entre une vraie conique et une conique variable et capable de dégénérer en n'excluant pas les deux cas où la conique variable coïncide avec l'autre ou s'évanouit tout à fait, le nombre de ces genres sera 14, qui est le nombre de doubles décompositions du nombre 4, savoir:

4: 3, 1: 2, 2: 2, 1, 1: 1, 1, 1, 1: 3 : 1 2, 1 : 1 1, 1, 1 : 1 2 : 2 1, 1 : 2 1, 1 : 1, 1
2 : 1 : 1 1, 1 : 1 : 1 1 : 1 : 1 : 1.

De même on trouvera facilement que, pour le cas de formes binaires, le nombre de genres semblables sera 6, car, ayant sur une ligne droite deux points fixes et deux points variables, ces derniers peuvent être distincts entre eux-mêmes en coïncidant avec un ou tous les deux ou avec ni l'un ni l'autre des deux premiers, ou bien ils peuvent être réunis dans un seul point qui peut coïncider ou ne pas coïncider avec un des points fixes, et finalement ils peuvent disparaître; or le nombre de décompositions doubles du nombre 3, c'est-à-dire

3: 2, 1: 1, 1, 1: 2 : 1 1, 1 : 1 1 : 1 : 1,

est aussi 6.

Mais nous avons démontré autrefois, dans le *Philosophical Magazine*, que pour le cas de deux formes quadratiques de n variables dont chacune reste générale, c'est-à-dire n'a pas le discriminant zéro, le nombre des genres de rapport est exactement le nombre de doubles décompositions du nombre n. C'est une question qui mérite d'être examinée, si cette identité entre le nombre de genres pour n variables dans le second cas avec celui pour le nombre $n - 1$ dans le premier, reste vraie pour toute valeur de n. Une considération qui s'y oppose, c'est que, dans le premier cas, quand $(n - 1 = 1)$ le nombre de genres, au lieu d'être 3 (le nombre de décompositions doubles de 2), n'est que 2, mais il peut arriver que pour ce cas (le cas d'une seule variable), la forme générale étant la même que la forme de coïncidence parfaite, ce genre doit compter pour deux, et ainsi la loi se maintiendra.

de dénombrement à la connaissance que nous avons acquise du Mémoire ci-dessus cité de M. Darboux*.

Mais ce qui plus est, on peut beaucoup simplifier, comme on va voir, la solution de l'équation quadratique $fx = px^2 + qx + r = 0$.

En regardant pour le moment x comme une quantité ordinaire, soient Fx le déterminant de la matrice $x^2p + xq + r$ et ϕx un quelconque des six facteurs quadratiques de Fx; alors $\phi x = 0$ sera l'équation identique d'une des racines de $fx = 0$, et ces deux équations, en éliminant x^2, donneront la valeur précise de cette racine†. De même nous ferons voir qu'en général, quel que soit le degré (n) de fx (fonction rationnelle entière et unilatérale de x), lequel, comme aussi chaque coefficient, est une matrice d'un ordre donné (ω) quelconque, en prenant le déterminant Fx de fx (où pour le moment on regarde x comme une quantité ordinaire), chaque facteur du degré ω de Fx sera la fonction identiquement zéro d'une des racines (prise négativement) de l'équation $fx = 0$, et réciproquement.

Ce beau théorème‡, *pulcherrima regula*, repose sur les considérations suivantes:

Soit $\phi\lambda$ le déterminant de $\lambda + x$; alors on peut démontrer facilement que $\phi x = 0$ sera l'équation identique de x.

Or soit $fx = 0$, alors $f(-\lambda) = f(-\lambda) - f(x)$ et conséquemment contiendra le facteur $x + \lambda$. Donc le déterminant de $f(-\lambda)$ contiendra le déterminant de $(\lambda + x)$, c'est-à-dire contiendra $\phi\lambda$, où $\phi x = 0$ est l'équation identique.

Ainsi ϕx (la fonction de x qui est identiquement zéro) ne peut qu'être un facteur du déterminant de $f(-x)$ pris comme si x était une quantité ordinaire. De plus, puisqu'en général ce déterminant sera une fonction irréductible de x, de sorte qu'on ne peut plus distinguer une racine d'avec une autre, tout facteur qu'il contient dont le degré est égal à l'ordre de x sera la fonction identiquement nulle d'une des racines de l'équation $fx = 0$.

* On doit remarquer que le discriminant de l'équation en λ ou μ ou u^2 est le même que celui de la biquadratique associée à l'équation donnée; en effet, l'équation en μ a pour racines $\frac{(\alpha+\beta)(\gamma+\delta)}{4}$, $\frac{(\alpha+\gamma)(\beta+\delta)}{4}$, $\frac{(\alpha+\delta)(\beta+\gamma)}{4}$, où α, β, γ, δ sont les racines de cette biquadratique; ainsi on peut dire que les six racines cherchées sont associées respectivement aux six côtés du quadrangle complet formé par les quatre points d'intersection de la conique appartenant aux coefficients de l'équation donnée avec la conique absolue $y^2 - 4xz$.

On comprend que la forme appartenant à p, q, r veut dire le déterminant de la matrice $xp + yq + zr$ qui est une courbe dont l'ordre sera toujours celui des matrices p, q, r.

† Ainsi on possède une méthode immédiate, et qui s'applique à tous les cas qui peuvent se présenter pour résoudre l'équation de Hamilton. L'analyse précédente suffit pour en donner une démonstration qui a été passée dans le texte.

‡ On peut donner à cet énoncé une autre forme, à savoir: *Toute racine latente de chaque racine de fx* (fonction rationnelle entière et unilatérale par rapport à x) *est une racine* (prise négativement) *du déterminant de fx* (où x est traité comme une quantité ordinaire) *et réciproquement chaque racine ainsi prise de ce déterminant est une racine latente d'une des racines de fx.*

Il paraît donc (s'il n'y a aucune erreur dans ce dernier raisonnement) que le nombre des racines de fx sera le nombre exact de combinaisons de $n\omega$ choses prises ω à ω ensemble, où n est le degré de fx en x et ω l'ordre des matrices qui paraissent là-dedans; conséquemment le nombre des racines sera

$$\frac{\pi n\omega}{\pi(n-1)\omega \, . \, \pi\omega}\ ^{*};$$

ainsi, par exemple, le nombre des racines dans le cas d'une équation du degré n en quaternions sera $2n^2 - n$†.

Pour trouver ces racines, on n'a qu'à combiner les deux équations $fx = 0$ qui ne change pas, avec $\phi x = 0$, qui varie avec chaque combinaison des racines de Fx [c'est-à-dire le déterminant de $f(-x)$], et, en éliminant les puissances supérieures de x, on trouvera une équation linéaire qui sert à donner x sous la forme d'une fraction: par des procédés qui ne présentent nulle difficulté, cette fraction peut être ramenée (au moins pour le cas des matrices binaires) à la forme d'une autre fraction dont le dénominateur sera une fonction exclusivement des coefficients de la *forme* associée à l'ensemble des coefficients de l'équation donnée dont nous nous proposons d'essayer de trouver la valeur générale. Ce dénominateur sera toujours (comme dans le cas que nous avons traité en détail dans ce qui précède) le *criterium* de la *régularité* de l'équation donnée. Quand ce *criterium* s'évanouit (et pas autrement), quelques-unes des racines vont à l'infini, c'est-à-dire cessent d'être actuelles et deviennent purement conceptuelles.

En général, pour résoudre l'équation unilatérale du degré n et l'ordre ω, on n'aura besoin que de résoudre une équation ordinaire du degré $n\omega$. Si une racine de l'équation donnée est connue, on n'aura qu'à résoudre deux équations ordinaires des degrés ω et $(n-1)\omega$ respectivement. Dans le cas d'une équation quadratique, quand une racine est donnée, on peut trouver immédiatement l'équation identique d'une seule autre qui y est associée, et conséquemment en déterminer la valeur sans résoudre une équation d'un degré supérieur au premier. Quand deux racines de l'équation résolvante (celle du degré $n\omega$) sont égales, on a $\frac{\pi(n\omega - 2)}{\pi(\omega-1) \, . \, \pi[(n-1)\omega - 1]}$ paires de racines égales dans l'équation du degré n qui est à résoudre.

* Dans le cas le plus général d'une équation en x du degré n et de l'ordre ω par rapport aux matrices, on peut supposer un nombre indéfini de termes dans l'équation. Chacun de ces termes sera composé d'un nombre pas plus grand que n des x dont chacun sera suivi et précédé par une matrice multiplicatrice. En appliquant la méthode algébrique directe pour résoudre cette équation, on sera amené à un système de ω^2 équations du degré n chacune. Ainsi le nombre des racines sera en général n^{ω^2}.

† Cela démontre que le nombre 21 que nous avions trouvé pour le cas de $n=3$ dans le *Philosophical Magazine* (mai 1884) [p. 229 below] et la formule générale que nous avons basée là-dessus sont erronés; la raison en est évidemment que l'ordre *apparent* du système d'équations qui nous a fourni ce résultat surpasse l'ordre *actuel* de 6 unités.

Nous n'avions pas discuté en détail ces équations, et ainsi cet abaissement du degré nous a échappé. C'est un point curieux qui reste à discuter.

Prenons comme exemple de l'application de la méthode l'équation en quaternions

$$q_3x^3 + q_2x^2 + q_1x + q_0 = 0.$$

La fonction résolvante sera

$$(3.3)x^6 + (3.2)x^5 + (3.1 + 2.2)x^4 (3.0 + 2.1)x^3$$
$$+ (2.0 + 1.1)x^2 (1.0)x + (0.0) = 0,$$

où en général $i.i$ et $i.j$ signifient

$$Tq_i^2, \quad 2[Sq_iq_j - S(Vq_iVq_j)]$$

respectivement.

Les quinze facteurs quadratiques de cette fonction égalés à zéro donneront chacun une équation quadratique à laquelle doit satisfaire une des quinze racines de l'équation donnée, et, en combinant séparément chacune de ces équations avec la cubique donnée, on peut éliminer x^3 et x^2 et obtenir ainsi quinze équations linéaires pour déterminer les quinze racines voulues.

29.

SUR LA RÉSOLUTION GÉNÉRALE DE L'ÉQUATION LINÉAIRE EN MATRICES D'UN ORDRE QUELCONQUE.

[*Comptes Rendus*, XCIX. (1884), pp. 409—412, 432—436.]

Ce qui intéresse le plus dans les résultats nouvellement acquis que j'ai l'honneur de présenter à l'Académie, c'est l'union ou bien l'anastomose dont ils offrent un exemple frappant et tout à fait inattendu entre les deux grandes théories de l'*Algèbre moderne* et de l'*Algèbre nouvelle*, dont l'une s'occupe des transformations linéaires, et l'autre de la quantité généralisée, de sorte qu'au même titre que Newton définit l'Algèbre ordinaire comme étant l'Arithmétique universelle, on pourrait très bien caractériser cette Algèbre-ci comme étant l'Algèbre universelle, ou au moins une de ses branches les plus importantes.

En général, un invariant de deux formes signifie une fonction de deux systèmes de coefficients qui reste invariable, à un facteur près, quand les deux systèmes des variables sont ou identiques ou assujettis à des substitutions semblables; mais rien n'empêche qu'on n'applique ce même mot au cas où les substitutions sont réciproques: ainsi, sans parler du cas de deux formes mixtes, on aura des invariants de deux formes données à mouvement semblable et des invariants à mouvement contraire; on peut très bien nommer ces derniers (comme titre distinctif) *contrariants*. C'est à une classe spéciale de contrariants que nous aurons affaire dans la solution de l'équation générale linéaire en matrices d'un ordre quelconque.

En supposant que chaque p et p' soit une matrice de l'ordre ω, l'opérateur qui contient i couples

$$p_1(\ )p'_1 + p_2(\ )p'_2 + \dots + p_i(\ )p'_i$$

peut être nommé provisoirement un *nivellateur* de l'ordre ω et de l'étendue i, et on peut le caractériser par le symbole $\Omega_{\omega,i}$. Servons-nous toujours du symbole 0 pour signifier une matrice dont tous les éléments sont des zéros, et désignons par 1 (ou bien par υ indifféremment) une matrice dont tous les éléments sont zéro, à l'exception des éléments de la diagonale qui seront des unités: ce sont les matrices nommées *matrice nulle* et *matrice unitaire* respectivement.

J'ai déjà expliqué comment un nivellateur général, de l'ordre ω, donne naissance à une matrice de l'ordre ω^2: je nomme le déterminant de cette matrice le *déterminant du nivellateur**. Ces déterminants possèdent des propriétés tout à fait analogues à celles des déterminants des matrices simples; ainsi, par exemple, je démontre la propriété dont je me suis servi avec grand avantage dans les recherches actuelles, que le déterminant du produit de deux *nivellateurs* est égal au produit de leurs déterminants séparés, et que le déterminant d'une fonction rationnelle d'un nivellateur, disons $F\Omega$, est égal au résultant (par rapport à Ω regardé comme une quantité ordinaire) de $F\Omega$ et $I\Omega$, où $I\Omega = 0$ représente l'équation identique du degré ω^2 à laquelle Ω est assujetti.

En général, à un système ou *corps* de matrices $p_1, p_2, \ldots, p_i$ de l'ordre ω correspond un quantic de l'ordre ω, c'est-a-dire le déterminant de

$$x_1 p_1 + x_2 p_2 + \ldots + x_i p_i.$$

Je nomme les coefficients de ce quantic les *paramètres du corps*. Ces paramètres doivent être regardés comme des quantités connues. Ainsi, par exemple, si au *corps* p, q (deux matrices binaires) on adjoint la matrice unitaire υ, et qu'on forme le déterminant de la matrice $x + yp + zq$, on obtiendra un quantic

$$x^2 + Bxy + Cxz + Dy^2 + Eyz + Fz^2,$$

où, si l'on regarde p, q comme des *quaternions*, on aura, dans le langage du grand Hamilton,

$$B = Sp, \quad C = Sq, \quad D = T^2p, \quad F = T^2q, \quad E = S(Vp.Vq).$$

Il résulte de cette définition qu'à chaque nivellateur $\Omega_{\omega,i}$ appartiennent deux quantics de l'ordre ω et avec i variables, dont l'un appartient au corps $p_1, p_2, \ldots, p_i$ et l'autre au corps $p'_1, p'_2, \ldots, p'_i$.

Si l'on connaît l'équation identique $I\Omega = 0$ à laquelle le nivellateur Ω obéit, on peut immédiatement, comme je l'ai déjà montré, résoudre l'équation $\Omega x = T$.

Mais il est très facile de voir que $I\Omega$ n'est autre chose que le déterminant du nivellateur $\Omega - \lambda\upsilon(\ )\upsilon$, quand dans ce résultat on substitue Ω à λ. Donc la question de la solution linéaire la plus générale est ramenée à ce seul problème:

Exprimer le déterminant d'un nivellateur en termes de quantités connues.

Or la première conclusion et la plus difficile à établir dans cette recherche, mais que j'ai enfin réussi à démontrer, c'est que ce déterminant est toujours

* Quelquefois ce déterminant sera nommé un *nivellant*.

une *fonction* entière, mais pas nécessairement rationnelle, des *coefficients des deux quantics* qui sont associés au nivellateur.

Cela étant convenu, on démontre avec une extrême facilité que ce déterminant est un *contrariant* du degré ω dans chaque système de coefficients des deux quantics associés.

Cela ne suffit pas ou peut ne pas suffire en soi-même à définir complètement le contrariant cherché; nommons, en général, ce contrariant le *nivellant* des deux quantics.

Supposons que $N_{x,y,\ldots z,t}$ soit le nivellant pour deux quantics d'un ordre donné ω, et représentons par $N_{x,y,\ldots z,0}$ ce que ce nivellant devient quand on réduit à zéro tous les coefficients qui appartiennent aux termes dans les deux quantics qui contiennent t; alors il est facile de voir que

$$N_{x,y,\ldots,z,0} = N_{x,y,\ldots z}.$$

Cette propriété seule est suffisante (avec l'aide d'un quelconque des opérateurs différentiels qui servent pour annuler un contrariant) pour préciser le contrariant (nivellant) dans le cas de deux quantics du second ordre, et c'est ainsi que j'ai obtenu la solution de l'équation linéaire pour le cas des matrices binaires donné dans la Note précédente. Or il est bien concevable que cette loi ne peut pas suffire à déterminer les paramètres arbitraires qui entrent dans le contrariant d'ordre (ω, ω) appartenant à deux quantics de l'ordre ω.

Mais il y a encore une autre loi (constituant par elle-même un très beau théorème) qui doit suffire surabondamment à cette fin.

C'est une loi qui établit une liaison entre les nivellants de deux systèmes de quantics contenant chacun le même nombre de variables, mais dont l'un est d'un ordre plus grand par unité que l'ordre de l'autre.

Supposons que N soit le nivellant de deux quantics de l'ordre ω,

$$F(x, y, \ldots, z) \text{ et } G(x, y, \ldots, z);$$

soit N' ce que devient N quand

$$F(x, y, \ldots, z) = (lx + my + \ldots + nz) F_1(x, y, \ldots, z)$$

et

$$G(x, y, \ldots, z) = (\lambda x + \mu y + \ldots + \nu z) G_1(x, y, \ldots, z);$$

alors je dis que, quand

$$l\lambda + m\mu + \ldots + n\nu = 0,$$

le nivellant de (F_1, G_1) sera contenu comme facteur dans le nivellant modifié N'.

A l'aide de ces principes, je me propose de calculer les nivellants pour les degrés supérieurs au second. On voit par ce qui précède que la solution de l'équation linéaire $\Sigma pxp' = T$ sera alors connue en termes des p, des p', de T et des paramètres des deux corps $p_1, p_2, \ldots, p_i, p'_1, p'_2, \ldots, p'_i$, augmentés l'un et l'autre d'une matrice unitaire.

C'est dans les *Lectures*, publiées en 1844, que pour la première fois a paru la belle conception de l'équation identique appliquée aux matrices du troisième ordre, enveloppée dans un langage propre à Hamilton, après lui mise à nu par M. Cayley dans un très important Mémoire sur les matrices dans les *Philosophical Transactions* pour 1857 ou 1858, et étendue par lui aux matrices d'un ordre quelconque, mais sans démonstration; cette démonstration a été donnée plus tard per feu M. Clifford (*voir* ses œuvres posthumes), par M. Buchheim dans le *Mathematical Messenger* (marchant, comme il l'avoue, sur les traces de M. Tait, d'Édimbourg), par M. Ed. Weyr, par nous-même, et probablement par d'autres; mais les quatre méthodes citées plus haut paraissent être tout à fait distinctes l'une de l'autre.

Par le moyen d'une chaîne de matrices couplées (disons N), opérant non pas sur une matrice générale, mais sur une matrice x (disons du degré ω) d'une forme spéciale suivie par un autre opérateur V qui aura l'effet de réduire la matrice du degré ω de Nx (dont les éléments sont des fonctions linéaires des éléments de x) à une forme identique à celle de x, il est facile de voir qu'à l'opérateur composé VN on peut faire correspondre une matrice d'un *ordre quelconque* non supérieur à ω^2, et c'est ainsi virtuellement que Hamilton, à cause d'une transformation qu'il effectue sur l'équation linéaire générale, est tombé dans ses *Lectures* sur la matrice du troisième ordre, et ce n'est que dans les *Éléments* publiés en 1866 (après sa mort) qu'on trouve quelque allusion à l'équation identique pour les matrices du quatrième ordre.

On pourrait nommer l'opérateur composé VN, pour lequel l'équation identique est d'un degré moindre que ω^2, *nivellateur qualifié*, mais il est essentiel de remarquer que ces opérateurs ne posséderont pas les propriétés analogues à celles des matrices que possèdent ces nivellateurs purs dont il est question dans ma méthode. Comme exemple d'un nivellateur qualifié, on pourrait admettre que le x (matrice du deuxième ordre), sur lequel opère le N, aura son quatrième élément zéro, et que l'effet du V sera d'abolir le quatrième élément dans Nx, où l'on peut supposer (et cette supposition est, dans son essence, à peu près identique à la méthode des vecteurs de Hamilton) que le premier et le quatrième élément de x sont égaux, mais de signes contraires, et que l'effet de V est de substituer dans la matrice du second ordre $N(x)$ la moitié de la différence entre le premier et le quatrième élément au lieu du premier et, au lieu du quatrième, cette même quantité avec le signe algébrique contraire.

Évidemment un tel opérateur donnera naissance à une matrice et sera assujetti à une équation identique du troisième ordre. Avant de conclure, pour convaincre de la justesse de la formule importante

$$\tfrac{1}{8}[(\dot{P}')^2 P^2 - 4(\dot{P}'.P)^2] - \tfrac{1}{2}\sqrt{(I.I')}\text{*},$$

* Pour rendre intelligible cette formule, il est nécessaire de dire que l'expression

$$\tfrac{1}{8}[(\dot{P}')^2 P^2 - 4(\dot{P}'.P)^2],$$

applicable au cas d'un nivellateur du second ordre à quatre couples de matrices, il sera bon d'en donner une démonstration parfaite *a posteriori*, ce qu'une transformation légitime rend très facile à faire. Remarquons que le déterminant du nivellateur du second ordre $\Sigma \begin{matrix} a & b \\ c & d \end{matrix} (\ ) \begin{matrix} \alpha & \beta \\ \gamma & \delta \end{matrix}$ est le déterminant de la matrice suivante:

$$\begin{matrix} \Sigma a\alpha & \Sigma c\alpha & \Sigma a\beta & \Sigma c\beta \\ \Sigma b\alpha & \Sigma d\alpha & \Sigma b\beta & \Sigma d\beta \\ \Sigma a\gamma & \Sigma c\gamma & \Sigma a\delta & \Sigma c\delta \\ \Sigma b\gamma & \Sigma d\gamma & \Sigma b\delta & \Sigma d\delta \end{matrix}$$

laquelle contiendra dans le cas supposé 144 termes, puisque chaque Σ comprend 4 produits: mais, sans perdre en généralité, on peut prendre une forme de nivellateur dont le déterminant ne comprendra pas plus de 24 termes; car il est facile de démontrer que, si aux 4 matrices de gauche on substitue 4 fonctions linéaires quelconques, pourvu que sur les 4 de droite on opère une substitution contragrédiente à la substitution précédente, la valeur du déterminant ne subira nul changement. On peut donc supposer que les 4 matrices de gauche sont

$$\begin{matrix} 10 & 01 & 00 & 00 \\ 00 & 00 & 10 & 01 \end{matrix}$$

respectivement, et, si la formule est vérifiée dans cette supposition (vu que les *contravariants* des deux quantics associés ne sont pas affectés par les substitutions contragrédientes opérées sur les deux systèmes de matrices), elle

donnée dans la Note du 21 juillet [pp. 181, 184 above], a besoin d'une correction (dont je pensais avoir fait mention dans le texte): il fait lui ajouter la *racine carrée* d'un contrariant connue du quatrième degré (appartenant aux deux *formes associées*), laquelle sera une fonction rationnelle des éléments des matrices du nivellateur. Pour le cas d'un nivellateur à quatre couples de matrices, c'est la racine carrée du produit de I et I', les discriminants des deux formes associées prises séparément; en nommant les quatre matrices à gauche

$$\begin{matrix} a & b \\ c & d \end{matrix} \quad \begin{matrix} a' & b' \\ c' & d' \end{matrix} \quad \begin{matrix} a'' & b'' \\ c'' & d'' \end{matrix} \quad \begin{matrix} a''' & b''' \\ c''' & d''' \end{matrix},$$

la racine carrée de I sera égale au déterminant

$$\begin{vmatrix} a & b & c & d \\ a' & b' & c' & d' \\ a'' & b'' & c'' & d'' \\ a''' & b''' & c''' & d''' \end{vmatrix},$$

qu'on peut nommer le développant de ces quatre matrices; de même la racine carrée de I' sera égale au développant des quatre matrices correspondantes à droite, de sorte que le terme irrationnel dans la formule pour le nivellant à quatre couples de matrices est égal au produit de ces deux développants; dans le cas général, la partie *relativement* irrationnelle de la formule pour un nivellant sera égale à la somme de tous les produits de développants accouplés qu'on peut former en combinant quatre à quatre, ensemble, les couples de matrices qui en dépendent. Dans le cas où le nivellateur contient moins de quatre couples, la racine carrée disparaît entièrement de la formule pour le nivellant. Je nommerai $\dot{P}'.P$ et $(\dot{P}')^2 P^2$, ϑ_1 et ϑ_2 respectivement.

sera non pas seulement *vérifiée*, mais absolument *démontrée* pour les valeurs parfaitement générales des deux systèmes.

Avec ces valeurs des matrices gauches, la matrice écrite plus haut, en prenant

$$\begin{matrix} \alpha & \beta \\ \gamma & \delta \end{matrix} \quad \begin{matrix} \alpha' & \beta' \\ \gamma' & \delta' \end{matrix} \quad \begin{matrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{matrix} \quad \begin{matrix} \bar\alpha & \bar\beta \\ \bar\gamma & \bar\delta \end{matrix}$$

pour les matrices à droite, devient

$$\begin{matrix} \alpha & \alpha_1 & \beta & \beta_1 \\ \alpha' & \bar\alpha & \beta' & \bar\beta \\ \gamma & \gamma_1 & \delta & \delta_1 \\ \gamma' & \bar\gamma & \delta' & \bar\delta \end{matrix}$$

dont je nommerai le déterminant Q.

De plus, le quantic à gauche deviendra $xt - yz$, et le quantic à droite

$$(\alpha\delta - \beta\gamma)x^2 + (\bar\alpha\bar\delta - \bar\beta\bar\gamma)t^2 + (\alpha'\delta' - \beta'\gamma')y^2 + (\alpha_1\delta_1 - \beta_1\gamma_1)z^2$$
$$+ (1.2)xy + (3.4)zt + (1.3)xz + (2.4)yt + (1.4)xt + (2.3)yz,$$

où $\quad (1.2) = \alpha\delta' + \delta\alpha' - \beta\gamma' - \beta'\gamma, \quad (3.4) = \alpha_1\bar\delta + \delta_1\bar\alpha - \beta_1\bar\gamma - \gamma_1\bar\beta,$

....

Donc
$$\vartheta_1 = (\alpha\bar\delta + \bar\alpha\delta - \beta\bar\gamma - \bar\beta\gamma) - (\alpha'\delta_1 + \alpha_1\delta' - \beta'\gamma_1 - \beta_1\gamma'),$$
$$\tfrac{1}{4}\vartheta_2 = (\alpha\bar\delta + \bar\alpha\delta - \beta\bar\gamma - \bar\beta\gamma)^2 + (\alpha'\delta_1 + \alpha_1\delta' - \beta'\gamma_1 - \beta_1\gamma')^2$$
$$+ 2(\alpha\delta - \beta\gamma)(\bar\alpha\bar\delta - \bar\beta\bar\gamma) + 2(\alpha'\delta' - \beta'\gamma')(\alpha_1\delta_1 - \beta_1\gamma_1)$$
$$- (\alpha\delta' + \delta\alpha' - \beta\gamma' - \beta'\gamma)(\alpha_1\bar\delta + \delta_1\bar\alpha - \beta_1\bar\gamma - \bar\beta\gamma_1)$$
$$- (\alpha\delta_1 + \delta\alpha_1 - \beta\gamma_1 - \beta_1\gamma)(\alpha'\bar\delta + \delta'\bar\alpha - \beta'\bar\gamma - \bar\beta\gamma')$$
$$- (\alpha\bar\delta + \bar\alpha\delta - \beta\bar\gamma - \bar\beta\gamma)(\alpha'\delta_1 + \alpha_1\delta' - \beta'\gamma_1 - \beta_1\gamma'),$$

et $\sqrt{(I.I')}$ (pris avec le signe convenable) sera le déterminant de la matrice

$$\begin{matrix} \alpha & \beta & \gamma & \delta \\ \alpha' & \beta' & \gamma' & \delta' \\ \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \bar\alpha & \bar\beta & \bar\gamma & \bar\delta. \end{matrix}$$

En faisant les multiplications nécessaires, on trouvera que

$$\tfrac{1}{4}\vartheta_2 - \vartheta_1^2 - \sqrt{(I.I')} = 2Q,$$

ce qui démontre l'exactitude de la formule donnée pour un nivellateur du deuxième ordre à quatre couples de matrices.

D'ici à peu de temps, j'espère avoir l'honneur de soumettre à l'Académie la valeur du déterminant du nivellateur du troisième ordre à trois couples de matrices. Pour présenter l'expression générale de ce déterminant pour une matrice d'un ordre et d'une étendue quelconques*, il faudrait avoir une connaissance des propriétés des formes qui va beaucoup au delà

* C'est-à-dire pour résoudre l'équation linéaire en matrices dans toute sa généralité.

des limites des facultés humaines, telles qu'elles ne sont manifestées jusqu'au temps actuel et qui, dans mon jugement, ne peut appartenir qu'à l'intelligence suprême.

Post-scriptum.—Qu'on me permette d'ajouter une petite observation qui fournit, il me semble, une raison suffisante *a priori* pour le signe ambigu du terme $\sqrt{(I.I')}$ qui entre dans la formule donnée pour un nivellant (c'est-à-dire déterminant d'un nivellateur) du deuxième ordre.

Les déterminants d'un nivellateur et de son *conjugué* étant *identiques* en signe algébrique tout autant qu'en grandeur, ce n'est pas dans cette direction qu'on peut chercher l'origine de l'ambiguïté.

Mais, si, en se bornant aux matrices correspondantes d'un nivellateur *de la même espèce*, c'est-à-dire à main droite ou à main gauche du symbole ( ), on échange entre eux, dans chacune de ces matrices, le premier terme avec le quatrième et le deuxième avec le troisième, on verra facilement que le nivellant et en même temps les deux *quantics associés* restent absolument sans altération; mais, si l'on exécute l'une ou l'autre de ces substitutions séparément, alors, tandis que les deux quantics associés restent constants, le nivellant (quand son nivellateur possède plus de trois couples) subira un changement de valeur (et, pour l'une et l'autre substitution, le *même* changement), de sorte que pour les quatre positions qu'on peut assigner simultanément aux éléments des matrices de la même espèce sans changer en rien les quantics associés, le nivellant aura deux valeurs distinctes. Voilà, il me semble, l'explication suffisante et la véritable origine de l'ambiguïté dont il est question.

A peine est-il nécessaire de remarquer qu'on peut faire 4 autres dispositions semblables et simultanées des matrices à l'un ou l'autre côté du symbole ( ), dispositions qui donneront naissance à des nivellants identiques en valeur avec les deux dont j'ai parlé (c'est-à-dire deux à une valeur et deux à l'autre), et pour lesquelles les deux quantics associés seront sans autre changement que celui du signe algébrique.

En combinant les 24 dispositions semblables des matrices d'un côté d'un nivellateur donné avec les 24 de l'autré côté, on obtiendra un système de 576 nivellateurs corrélatifs dont les déterminants ne prendront que 3 paires de valeurs; de plus, les deux valeurs d'une quelconque de ces paires seront les racines d'une équation quadratique dont les coefficients seront des contrariants rationnels et entiers d'une des trois paires de formes quadratiques; mais le discriminant de ces trois équations sera le même *certainement* quand les nivellateurs du système seront formés avec quatre couples de matrices et *probablement* quel que soit le nombre de ces couples. Quand ce nombre est moindre que 4, le discriminant de ces trois quadratiques devient nul pour toutes les trois.

30.

SUR L'ÉQUATION LINÉAIRE TRINÔME EN MATRICES D'UN ORDRE QUELCONQUE.

[*Comptes Rendus*, XCIX. (1884), pp. 527—529.]

POUR résoudre l'équation trinôme $pxp' + qxq' + r = 0$ (où toutes les lettres désignent des matrices du même ordre ω) sous sa forme symétrique, on a besoin de connaître l'équation identique à un nivellateur de cet ordre à deux couples de matrices, ce qui équivaut virtuellement à connaître le déterminant d'un nivellateur à trois de ces couples. Mais, sans avoir recours à cette méthode générale, il existe, comme on va le voir, un moyen plus court et plus direct pour résoudre l'équation et exprimer x sous la forme essentiellement bonne d'une fraction réduite, si l'on est d'accord à se dispenser de la condition que le numérateur soit symétrique.

A cet effet, on peut multiplier l'équation, à volonté, ou par $q^{-1}(\ )p'^{-1}$ ou par $p^{-1}(\ )q'^{-1}$. Choisissons le premier de ces deux multiplicateurs et écrivons $q^{-1}p = \phi$, $q'p'^{-1} = -\psi$, $-q^{-1}rp'^{-1} = \mu$; alors on obtient l'équation $\phi x - x\psi = \mu$ (mais déjà avec une brèche de symétrie, par la raison du choix d'une entre deux choses pareilles). En multipliant cette équation par le nivellateur $\phi^i(\ ) + \phi^{i-1}(\ )\psi + \phi^{i-2}(\ )\psi^2 + \ldots + (\ )\psi^i$ (disons U_i) et en écrivant $U_i\mu = \mu_{i+1}$, on obtient la suite d'équations

$$\phi x - x\psi = \mu,\quad \phi^2 x - x\psi^2 = \mu_2,\quad \phi^3 x - x\psi^3 = \mu_3,\ \ldots,\quad \phi^\omega x - x\psi^\omega = \mu_\omega.$$

Soient $B_0, B_1, \ldots, B_\omega$ et $C_0, C_1, \ldots, C_\omega$ les coefficients des deux formes associées aux deux systèmes p, q et p', q' respectivement; alors, en vertu d'un théorème général en matrices*, on aura

$$C_\omega\psi^\omega + C_{\omega-1}\psi^{\omega-1} + \ldots + C_0 = 0,\quad B_0 - B_1\phi + \ldots + (-)^\omega B_\omega\phi^\omega = 0.$$

Avec l'aide de ces deux équations et de la suite précédente, on peut déduire une équation de l'une ou de l'autre des deux formes $Mx = N$ ou $xM = N$. Faisons le choix (qui amène encore une fois une brèche de symétrie) de la première.

On aura $(C_\omega\phi^\omega + C_{\omega-1}\phi^{\omega-1} + \ldots + C_1\phi + C_0)x = C_\omega\mu_\omega + C_{\omega-1}\mu_{\omega-1} + \ldots + C_1\mu$.
Or, selon la théorie ordinaire d'élimination, on peut déterminer ϑ et H deux fonctions chacune du degré $(\omega - 1)$ en ϕ (traité comme une quantité ordinaire), telles que

$$\vartheta[B_0 - B_1\phi + \ldots + (-)^\omega B_\omega\phi^\omega] + H(C_\omega\phi^\omega + C_{\omega-1}\phi^{\omega-1} + \ldots + C_0)$$

* Ainsi, par exemple, si p, q sont des quaternions, on a

$$Tp^2(p^{-1}q)^2 - 2S(VpVq)(p^{-1}q) + Tq^2 = 0.$$

sera égal à R, le contre-résultant des deux formes associées à (p, q) et (p', q')* respectivement, et l'on aura

$$x = \frac{C_1H\mu + C_2H\mu_2 + \ldots + C_\omega H\mu_\omega}{R},$$

et ainsi x sera déterminé.

Si μ est zéro, alors, afin que x ne soit pas zéro, le R doit devenir zéro, comme nous avons déjà trouvé dans une Note précédente. En général, si R (le contre-résultant des deux formes adjointes à p, q et p', q' dans l'équation $pxp' + qxq' + r = 0$) s'évanouit, l'équation ne peut pas admettre une solution en même temps actuelle et déterminée; sans autres conditions, la solution deviendra *idéale*; avec conditions convenables, elle peut redevenir *actuelle*, mais contiendra (selon les circonstances) une ou plusieurs constantes arbitraires.

Hamilton, dans ses *Lectures*, a considéré l'équation trinôme pour les quaternions, mais il n'en a pas poussé la solution, c'est-à-dire la valeur de l'inconnue, à sa forme finale dans laquelle le dénominateur doit être un scalar (je dis *doit* être), parce que, ici comme dans toutes les équations en matrices, c'est le dénominateur de l'inconnue convenablement exprimé dont l'évanouissement est le *critérium* pour distinguer le cas où la solution est actuelle et déterminée d'avec les cas où elle devient ou idéale ou indéterminée.

En combinant le résultat ici obtenu avec celui de notre Note précédente, on voit qu'on est entré en pleine possession de la solution de l'équation $Nx = \Gamma$ dans les deux cas où le nivellateur N est de l'ordre 2 et d'une étendue quelconque ou bien de l'étendue 2 et d'un ordre quelconque.

Remarque.—On peut objecter que le numérateur de l'expression trouvée pour x dans l'équation trinôme contient des combinaisons de $q^{-1}p$, $q'p'^{-1}$, $q^{-1}rp'^{-1}$ et que, conséquemment, x pourrait devenir idéal à cause de l'évanouissement du déterminant de p' ou de q sans que le contre-résultant R s'évanouisse. Pour répondre à cette objection, soient D', Δ les déterminants de p' et de q^{-1}; alors, en se servant des équations identiques à p' et à q, on peut substituer pour leurs inverses des fonctions rationnelles de l'un et de l'autre divisées respectivement par D' et Δ, et alors le numérateur de x sera une quantité incapable de devenir infinie, tandis que son dénominateur sera R multiplié par des puissances de D' et de Δ; mais, vu qu'on peut représenter x tout aussi bien par une autre fraction dont le numérateur sera aussi incapable de devenir infini et dont le dénominateur sera R multiplié par des puissances de D' et de Δ (les déterminants de p et de q'), il est évident que ces deux fractions doivent toutes les deux admettre d'être simplifiées et que dans leurs formes réduites le dénominateur sera tout simplement R et qu'ainsi ce contre-résultant est le seul critérium pour distinguer le cas de l'actuel et déterminé d'avec le cas de l'idéal ou indéterminé.

* C'est-à-dire le *résultant* des fonctions multipliées par ϑ et H ci-dessus.