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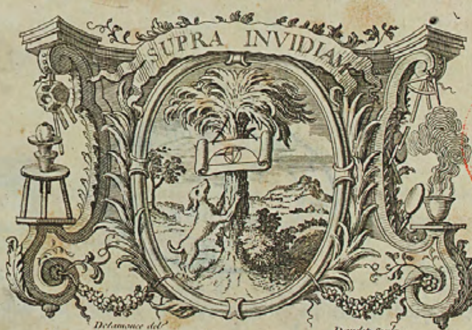

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Sumptibus MARCI-MICHAELIS BOUSQUET & Sociorum.
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JOHANNIS BERNOULLI OPERA ΑΝΕΚΔΟΤΑ.

ANALYTICA.

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N°. CL.

DE SERIEBUS VARIA.

PROPOSITIO I.

PROBLEMA.

S*eriei fractionum, quarum numeratores & denominatores arithmetice sint progressionales, invenire terminum infinitesimum.*

Sit Series $\frac{a}{c}$, $\frac{a+b}{c+e}$, $\frac{a+2b}{c+2e}$, $\frac{a+3b}{c+3e}$, ... $\frac{a+mb}{c+me}$, &c. Si itaque pro numero infinito ponatur n, erit terminus infinitesimus

ſimus $= \frac{a+nb}{c+ne}$. Sit ille $= x$. Ergo $cx + nex = a + nb$, & $cx - a = nb - nex$: Dividendo per $b - ex$, erit $n = \frac{cx-a}{b-ex}$. Quia autem numerator hujus fractionis eſt finitus [nam infinitus eſſe non poteſt, alias x deberet eſſe æqualis infinito, ideoque eſſet $b - ex$ negativa quantitas; ergo etiam $\frac{cx-a}{b-ex}$; quod eſt abſurdum,] erit $b - ex = 0$, proinde $b = ex$, & $x = b:e$. Hic itaque eſt terminus infiniteſimus. *Q. E. I.*

PROPOSITIO II.

PROBLEMA.

Seriei infinitæ fractionum, quarum numeratores ſint arithmeticè, denominatores geometrice progreſſionales, invenire ſummam.

Sit $a:c$ primus terminus, b communis differentia numeratorum, e ratio in qua creſcunt denominatores; fiet itaque hæc progreſſio $\frac{a}{c} + \frac{a+b}{ce} + \frac{a+2b}{cee} + \frac{a+3b}{ce^3}$, &c. hujus progreſſionis ſumma ita invenitur: Dividantur primo numeratores in ſuas partes, ut fiat $\frac{a}{c} + \frac{a+b}{ce} + \frac{a+b+b}{cee} + \frac{a+b+b+b}{ce^3}$ &c. poteſt autem hæc progreſſio in alias infinitas pure geometricas reſolvi, videlicet in has A, B, C, D, &c.

A, $\frac{a}{c} + \frac{a}{ce} + \frac{a}{cee} + \frac{a}{ce^3}$ &c. ſumma eſt $\frac{ae}{ce-c}$

B.... $\frac{b}{ce} + \frac{b}{cee} + \frac{b}{ce^3}$ &c. ſumma $\frac{b}{ce-c}$

C........ $\frac{b}{cee} + \frac{b}{ce^3}$ &c. ſumma $\frac{b}{cee-ce}$

D........... $\frac{b}{ce^3}$ &c. ſumma $\frac{b}{ce^3-cee}$

quarum ſummæ, juxta vulgarem regulam, inveniri poſſunt. Summæ autem Serierum B, C, D, &c. à ſerie B incipientium conſti-

tuunt aliam Seriem infinitam geometricam [ut ex operatione patet], cujus Seriei ſummæ, ſi addatur ſumma Seriei A, habebitur ſumma ſummarum, ſeu ſumma Seriei propoſitæ: Ergo ſummæ Serierum B, C, D, conſtituunt hanc Seriem geometricam $\frac{b}{ce-c} + \frac{b}{cee-ce} + \frac{b}{ce^3-cee}$ &c. cujus ſummæ $\frac{be}{cee-2ce+c}$ addatur ſumma Seriei $A = \frac{ae}{ce-c}$, & habebitur $\frac{aee-ae+be}{cee-2ce+c}$ = ſummæ Seriei propoſitæ. *Q. E. F.*

PROPOSITIO III.

PROBLEMA.

Seriei hujus $\frac{a}{b\times(b+c)} + \frac{a}{(b+c)\times(b+2c)} + \frac{a}{(b+2c)\times(b+3c)}$ &c. *ſummam invenire.*

Hæc Series ex hac $\frac{a}{b} + \frac{a}{b+c} + \frac{a}{b+2c} + \frac{a}{b+3c}$ reſolvi poteſt hoc modo: $\frac{a}{b} = \frac{a}{b+c} + \frac{ac}{b\times(b+c)}$; $\frac{a}{b+c} = \frac{a}{b+2c} + \frac{ac}{(b+c)\times(b+2c)}$; $\frac{a}{b+2c} = \frac{a}{b+3c} + \frac{ac}{(b+2c)\times(b+3c)}$ &c. Ergo $\frac{a}{b} + \frac{a}{b+c} + \frac{a}{b+2c} + \frac{a}{b+3c}$ &c. $= \frac{a}{b+c} + \frac{a}{b+2c} + \frac{a}{b+3c}$ + &c. $+ \frac{ac}{b\times(b+c)} + \frac{ac}{(b+c)\times(b+2c)} + \frac{ac}{(b+2c)\times(b+3c)}$; ideoque $\frac{a}{b} = \frac{ac}{b\times(b+c)} + \frac{ac}{(b+c)\times(b+2c)} + \frac{ac}{(b+2c)\times(b+3c)}$ + &c. & $\frac{a}{bc} = \frac{a}{b\times(b+c)} + \frac{a}{(b+c)\times(b+2c)} + \frac{a}{(b+2c)\times(b+3c)}$ + &c.

COROLLARIUM I.

Hinc ſi Series propoſita ſit finita, & numerus terminorum $= m$, erit omnium ſumma $= \frac{ma}{bb+mbc}$. Hoc patet ex operatione.

COROL-

COROLLARIUM II.

Si proponatur Series fractionum, cujus numeratores sint æquales, denominatores vero numeri trigonales $\frac{1}{1}, \frac{1}{3}, \frac{1}{6}, \frac{1}{10}, \frac{1}{15}, \frac{1}{21}$, &c. summa sic investigabitur. Ponatur $a = b = c = 1$; habebitur ergo per Prop. hanc, summa Seriei $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \frac{1}{30}$ &c. $= 1$. Hæc Series autem est subdupla illius, quia hujus denominatores sunt dupli illius: Ergo illius summa est 2. Summa propositæ Seriei finitæ est $= \frac{2ma}{bb + mbc} = \frac{2m}{1+m}$.

COROLLARIUM III.

Sequitur ex his, progressionis harmonicæ $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$ &c. summam esse infinitam; nam ob

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \text{\&c.} = 1$$
$$\frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \text{\&c.} = \frac{1}{2}$$
$$\frac{1}{12} + \frac{1}{20} + \text{\&c.} = \frac{1}{3}$$
$$\frac{1}{20} + \text{\&c.} = \frac{1}{4}$$

erit $\frac{1}{2} + \frac{2}{6} + \frac{3}{12} + \frac{4}{20} +$ &c. vel, fractionibus ad minimos terminos redactis, $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \text{\&c.} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{\&c.}$ Quod si utriusque progressionis summa esset finita, sequeretur totum æquale esse suæ parti: nam prior incipit a posterioris termino secundo. Ergo summa est infinita. *Q. E. D.*

PROPOSITIO IV.

THEOREMA.

Seriei $\frac{2b+c}{b^2 \times (b+c)^2} + \frac{2b+3c}{(b+c)^2 \times (b+2c)^2} + \frac{2b+5c}{(b+2c)^2 \times (b+3c)^2} + \text{\&c.}$ *summa* $= \frac{1}{bbc}$.

Demonstratio hujus deducitur ut supra, ex progressione

$$\frac{1}{b^2} + \frac{1}{(b+c)^2} + \frac{1}{(b+2c)^2} + \frac{1}{(b+3c)^2} + \text{\&c.}$$

COROLLARIUM.

$$\frac{3}{1^2} + \frac{5}{3^2} + \frac{7}{6^2} + \frac{9}{10^2} + \text{\&c.} = 4;$$

$$\frac{1}{(4-1)^2} + \frac{2}{(16-1)^2} + \frac{3}{(36-1)^2} + \frac{4}{(64-1)^2} + \frac{5}{(100-1)^2} = \frac{1}{8}$$

PROPOSITIO V.

THEOREMA.

Seriei $\frac{1a}{1\times2} + \frac{2a}{1\times2\times3} + \frac{3a}{1\times2\times3\times4} + \frac{4a}{1\times2\times3\times4\times5} + \text{\&c.}$ *summa est* $= a$.

Demonstratio hujus desumitur ab hac Serie $\frac{a}{1} + \frac{a}{1\times2} + \frac{a}{1\times2\times3} + \frac{a}{1\times2\times3\times4} +$ &c. ut in Propositione penultima factum.

COROLLARIUM.

Hinc $\frac{1}{1\times2} + \frac{4}{1\times2\times3} + \frac{9}{1\times2\times3\times4} + \text{\&c.} = \frac{1}{1} + \frac{1}{1\times2} + \frac{1}{1\times2\times3} + \frac{1}{1\times2\times3\times4} + \text{\&c.}$

Nam resolvatur prior hoc modo

$$\frac{1}{1\times2} + \frac{2}{1\times2\times3} + \frac{3}{1\times2\times3\times4} + \text{\&c.} = 1 = \frac{1}{1}$$
$$\frac{2}{1\times2\times3} + \frac{3}{1\times2\times3\times4} + \text{\&c.} = \frac{1}{2} = \frac{1}{1\times2}$$
$$\frac{3}{1\times2\times3\times4} + \text{\&c.} = \frac{1}{6} = \frac{1}{1\times2\times3}$$

&c. &c.

Q. E. D.

PROPOSITIO VI.

PROBLEMA.

Aliarum Serierum ſummam invenire:

$\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+\frac{1}{256}+$ &c. $=2$. ſubtrahatur $\frac{1}{1}+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}$ &c. $=\frac{4}{3}$. Ergo $\frac{1}{4}+\frac{3}{16}+\frac{7}{64}+\frac{15}{256}+$ &c. $=\frac{2}{3}$ ☉. Loco ſubtrahendi addatur hæc Series $\frac{1}{1}+\frac{1}{4}+\frac{1}{16}+$ &c. $=\frac{4}{3}$, proveniet $\frac{2}{1}\div\frac{3}{4}+\frac{5}{16}+\frac{9}{64}+$ &c. $=\frac{10}{3}$ ☽. Ubi notandum quod in Serie ☉ inventa, numeratores ſint in ratione dupla aucta unitate; in ☽ vero inventa in eadem ratione ſed diminuta unitate.

$\frac{1}{1}+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}+\frac{1}{729}+$ &c. $=\frac{3}{2}$. Subtrahatur $\frac{1}{1}+\frac{1}{9}+\frac{1}{81}+\frac{1}{729}+$ &c. $=\frac{9}{8}$ prodibit $\frac{2}{9}+\frac{8}{81}+\frac{26}{729}+$ &c. $=\frac{3}{8}$ ☿; ſi vero addatur $\frac{1}{1}+\frac{1}{9}+\frac{1}{81}+$ &c. $=\frac{9}{8}$, proveniet $\frac{2}{1}+\frac{4}{9}+\frac{10}{81}+\frac{28}{729}+$ &c. $=\frac{21}{8}$ ♀; ubi iterum advertitur, quod numeratores in Serie ☿ ſint in ratione tripla aucta binario, in ♀ vero in eadem ratione ſed eodem numero diminuta.

PROPOSITIO VII.

Poteſt inveniri ratio quam habent ſummæ (etiamſi incognitæ) duarum Serierum.

Ex. gr. hoc modo: Series ſequens $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}$ &c. æquatur ſuis partibus

partibus, videlicet Seriebus $\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+$ &c. $=2$ & $\frac{1}{3}+\frac{1}{6}+\frac{1}{12}+\frac{1}{24}+$ &c. $=\frac{2}{3}$, & $\frac{1}{5}+\frac{1}{10}+\frac{1}{20}+\frac{1}{40}+$ &c. $=\frac{2}{5}$; & $\frac{1}{7}+\frac{1}{14}+\frac{1}{28}+\frac{1}{56}+$ &c. $=\frac{2}{7}$, & ita conſequenter. Erit ergo $\frac{2}{1}+\frac{2}{3}+\frac{2}{5}+\frac{2}{7}+\frac{2}{9}$ &c. $=$ Seriei propoſitæ $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$ &c., ideoque $\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}$ &c., dimidia eſt Seriei propoſitæ. Ex quo rurſum patet Seriem propoſitam eſſe infinitam, nam quia $\frac{1}{1}>\frac{1}{2}$ & $\frac{1}{3}>\frac{1}{4}$ & $\frac{1}{5}>\frac{1}{6}$ &c. erit $\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+$ &c. $>\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}$ &c. Ergo ſi propoſita Series eſſet finita, ſequeretur $\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+$ &c. non eſſe æqualem dimidiæ Seriei propoſitæ. Quod eſt abſurdum.

Eodem modo, ſi proponatur Series $\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+\frac{1}{49}+$ &c. $=$ Seriebus $\frac{1}{1}+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}$ &c. $=\frac{4}{3}$ & $\frac{1}{9}+\frac{1}{36}+\frac{1}{144}$ &c. $=\frac{4}{3\times 9}$ & $\frac{1}{25}+\frac{1}{100}+\frac{1}{400}+$ &c. $=\frac{4}{3\times 25}$ &c. Ergo $\frac{4}{3\times 1}+\frac{4}{3\times 9}+\frac{4}{3\times 25}+\frac{4}{3\times 49}$ &c. $=\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+$ &c. ideoque, multiplicando per 3, $\frac{4}{1}+\frac{4}{9}+\frac{4}{25}+\frac{4}{49}+$ &c. $=\frac{3}{1}+\frac{3}{4}+\frac{3}{9}+\frac{3}{16}+\frac{3}{25}+$ &c. ablatis æqualibus $\frac{1}{1}+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+$ &c. $=\frac{3}{4}+\frac{3}{16}+\frac{3}{36}+$ &c. Ergo ſumma terminorum imparium Seriei propoſitæ eſt ad ſummam terminorum parium ut 3 ad 1.

Si denominatores Seriei propoſitæ eſſent cubi numerorum naturalium; inveniretur ſummam imparium terminorum eſſe ad ſummam parium ut 7 ad 1.

Generaliter : Si denominatores sint quæcunque potestas numerorum naturalium; erit summa terminorum imparium ad summam parium ut eadem potestas diminuta unitate ad unitatem.

PROPOSITIO VIII.

PROBLEMA.

Seriei, cujus numeratores sint in progressione geometrica aucta communi quadam quantitate, denominatores vero in quacunque alia majore progressione geometrica, summam invenire.

Sit primus terminus $\frac{a}{e}$, ratio in qua ascendunt numeratores & denominatores sit $\frac{c}{m}$, communis quantitas b; procreabitur itaque hæc Series $\frac{a}{e}+\frac{ac+b}{em}+\frac{acc+bc+b}{emm}+\frac{ac^3+bcc+bc+b}{em^3}$ &c. cujus summa invenitur resolvendo hanc Seriem in alias

$$\frac{a}{e}+\frac{ac}{em}+\frac{acc}{emm}+\frac{ac^3}{em^3}\ \&c. = \frac{am}{em-ec}$$
$$\frac{b}{em}+\frac{bc}{emm}+\frac{bcc}{em^3}\ \&c. = \frac{b}{em-ec}$$
$$\frac{b}{emm}+\frac{bc}{em^3}\ \&c. = \frac{b}{emm-emc}$$
$$\frac{b}{em^3}\ \&c. = \frac{b}{em^3-ecmm}$$
$$\&c. \qquad \&c.$$

quarum summæ, omissa tantisper prima, constituunt hanc Seriem geometricam $\frac{b}{em-ec}+\frac{b}{emm-ecm}+\frac{b}{em^3-ecmm}$ &c. quæ est æqualis $\frac{bm}{emm-ecm-em-ec}$. Addatur prima $\frac{am}{em-ec}$; provenit $\frac{amm-am+bm}{emm-ecm-em+ec}$ = summæ Seriei propositæ.

PRO-

PROPOSITIO IX.

THEOREMA.

Summa Seriei $\sqrt{(2+\sqrt{(2+\sqrt{(2+\sqrt{(2+\sqrt{(2\ \&c.)})})})})}$ *est* $=2$.

Sit si potest $=2+a$: Ergo eorum quadrata erunt æqualia, id est, $4+4a+aa=2+\sqrt{(2+\sqrt{(2+\sqrt{(2+\sqrt{(2\ \&c.)})})})}$: Ergo [subtractis utrinque æqualibus] $2+4a+aa=\sqrt{(2+\sqrt{(2+\sqrt{(2+\sqrt{(2\ \&c.)})})})}=2+a$. *Quod est absurdum.*

Si nunc dicatur esse æqualem $2-a$; erit ergo $4-4a+aa=2+\sqrt{(2+\sqrt{(2+\sqrt{(2+\sqrt{(2\ \&c.)})})})}$ & [subtractis utrinque æqualibus] $2-4a+aa=\sqrt{(2+\sqrt{(2+\sqrt{(2+\sqrt{(2\ \&c.)})})})}=2-a$, ideoque transponendo & dividendo per a, erit $a=3$, & supra $2>a$, quæ repugnant; quin potius summa Seriei propositæ est $=2$. *Q. E. D.*

COROLLARIUM.

Eodem modo demonstratur $\sqrt{(6+\sqrt{(6+\sqrt{(6+\sqrt{(6\ \&c.)})})})}=3$, & $\sqrt{(12+\sqrt{(12+\sqrt{(12+\sqrt{(12\ \&c.)})})})}=4$, & $\sqrt{(20+\sqrt{(20+\sqrt{(20+\sqrt{(20\ \&c.)})})})}=5$ &c. Et generaliter $\sqrt{(a+\sqrt{(a+\sqrt{(a+\sqrt{(a+\sqrt{(a+\&c.)})})})})}$ summa ita invenitur. Ponatur summa æqualis x: Ergo ipsorum quadrata æquantur $xx=a+\sqrt{(a+\sqrt{(a+\sqrt{(a+\sqrt{(a\ \&c.)})})})}$: aufer utrinque æqualia, remanebit $xx-a=\sqrt{(a+\sqrt{(a+\sqrt{(a+\sqrt{(a\ \&c.)})})})}=x$, proinde $xx=x+a$ & $x=\frac{1}{2}+\sqrt{(\frac{1}{4}+a)}$.

Hinc si series ita proponatur $\sqrt[3]{(a+\sqrt[3]{(a+\sqrt[3]{(a+\sqrt[3]{(a\ \&c.)})})})}$; summa ita invenitur. Ponatur $=x$: ergo $x^3-a=\sqrt[3]{(a+\sqrt[3]{(a+\sqrt[3]{(a+\sqrt[3]{(a\ \&c.)})})})}=x$; proinde $x^3-x-a=0$ & universaliter $\sqrt[n]{(a+\sqrt[n]{(a+\sqrt[n]{(a+\sqrt[n]{(a\ \&c.)})})})}$ pro æquatione habebitur $x^n-x-a=0$.

 PRO-

PROPOSITIO X.

THEOREMA.

Si sint duæ Series figuratæ immediate sibi subsequentes, & sit ubique summa terminorum primæ ad totidem maximo æqualium ut 1 *ad* r; *erit & ubique summa terminorum secundæ ad totidem maximo æqualium ut* 1 *ad* r + 1.

Est enim per naturam Serierum figuratarum, & per hypothesin:

$$n\left\{\begin{array}{ll} a \ldots o \\ \vdots \quad \vdots \\ b \ldots f \\ c \ldots g \\ d \ldots h \\ e \ldots i \\ \quad\;\; k \end{array}\right. \quad k+i+h+g+f \ldots +o = \frac{ne}{r} + \frac{(n-1)d}{r} + \frac{(n-2)c}{r} + \frac{(n-3)b}{r} \ldots + \frac{(n-n+1)a}{r}$$

$= \text{(connexis homologis)} \; \frac{n.(e+d+c+b \ldots +a)}{r}$ $\frac{-1d-2c-3b \ldots -n(+1)a}{r} =$ (per naturam Serierum) $\frac{nk}{r} - \frac{i-h-g-f \ldots -o}{r}$, ideoque $rk + r(i+h+g+f \ldots +o) = nk - i - h - g - f \ldots - o$. & reducta æquatione invenitur $nk - rk = (r+1)(i+h+g+f \ldots +o)$; dividatur utrumque per $r+1$, addaturque dein utrique k, habebitur $\frac{(n+1).k}{r+1} = k+i+h+g+f \ldots +o$. &c. *Q. E. D.*

COROLLARIUM.

Patet hinc cujuslibet Seriei figuratæ quamlibet summam terminorum habere ad totidem maximo æqualium rationem constantem. Series enim unitatum, quæ est prima omnium figuratarum, habet hanc proprietatem requisitam; sequitur itaque ex Propositione secundam Seriem eandem proprietatem habere; & ex secunda demonstratur, tertiam; ex tertia, quartam; ex quarta, quintam &c; & ita consequenter omnes in infinitum usque. *Q. E. D.*

PROPOSITIO XI.

THEOREMA.

Esto numerus quilibet n datus, & fractio qualibet assumta $\frac{a}{b}$, *ex qua formetur hæc altera* $\frac{b}{(n-1)a+2b}$, *& ex hac formetur eadem lege tertia* $\frac{(n-1)a+2b}{(n-1)b+2(n-1)a+4b}$, *ex tertia quarta, & sic deinceps, considerato scilicet semper numeratore ut a & denominatore ut b; erit terminus infinitesimus* $\frac{1}{1+\sqrt{n}} = \frac{-1+\sqrt{n}}{-1+n}$.

Cum enim terminus infinitesimus suo proximo termino seu sequenti censeatur æqualis, esto terminus infinitesimus $\frac{x}{y}$; ergo sequens ex natura Seriei erit $\frac{y}{(n-1)x+2y}$; adeoque $\frac{x}{y} = \frac{y}{(n-1)x+2y}$; reducatur æquatio & habebitur $y = x + x\sqrt{n}$, unde $\frac{x}{y} = \frac{x}{x+x\sqrt{n}} = \frac{1}{1+\sqrt{n}} = \frac{-1+\sqrt{n}}{-1+n}$. *Q. E. D.*

COROLLARIUM.

Hinc patet ratio extrahendi commodissime radicem quadratam numeri propositi n, per approximationem continuam: Incipiatur namque a quacunque fractione $\frac{a}{b}$, & continuetur Series, servata lege in Propositione exhibita, donec termini se mutuo insequentes sensibiliter non differant, vocetur unus ex illis V, erit ergo $V =$ (quam proxime) $\frac{1}{1+\sqrt{n}} = \frac{-1+\sqrt{n}}{-1+n}$, unde $\sqrt{n} = \frac{1-V}{V} = (n-1)V+1$.

ALITER.

Ponatur $\frac{a}{b} = \frac{(n+1)ab}{aa+bb}$, & reliqua fiant ut ante; dabit terminus infiniteſimus hujus Seriei $\frac{a}{b}$, $\frac{(n+1)ab}{aa+bb}$, &c. immediate radicem quæſitam $\sqrt{n}$.

Simili modo extrahitur radix cubica ex numero dato n: ponendo ſcil. $\frac{a}{b} = \frac{(n+1)abb}{a^3+b^3}$; imo & cujuſcunque poteſtatis p, hoc modo $\frac{a}{b} = \frac{(n+1)ab^{p-1}}{a^p+b^p}$.

ADHUC ALITER ET FACILIUS.

Fiat $\frac{a}{b} = \frac{a^{p-1}+nb^{p-1}}{(a+b)a^{p-2}}$; erit ultima fractio radix poteſtatis p ex numero n.

N°. CLI.

METHODUS

Exhibendi ſummas progreſſionum finitarum per numerorum naturalium quamcunque potentiam datam procedentium; imo cujuſvis alterius progreſſionis finitæ conſtantis terminis utcunque complicatis, modo contineant quantitates rationales & integras.

Sit ex. gr. progreſſio quadratorum $1+4+9+16 \ldots\ldots + mm$, cujus ſumma quæritur. Hæc ponatur eſſe $am^3+bmm+cm$ (per a, b, c, intelligo coëfficientes incognitos ipſarum m^3, mm, m). Aucto nunc numero terminorum unitate, habebitur, ponendo ubique $m+1$ loco m, in aſſumta quantitate $am^3+bmm+cm$, hæc

$+cm$, hæc altera

$$\begin{array}{r} am^3+3amm+3am+a \\ +bmm+2bm+b \\ +cm+c \end{array}$$

quæ per conſequens æqualis eſſe debet expoſitæ progreſſioni, una cum termino poſt mm ſequenti, qui eſt $mm+2m+1$; ideoque erit æquatio inter quantitatem hanc ex aſſumta generatam & inter ipſam aſſumtam, auctam termino poſt mm ſequenti, $mm+2m+1$. Hinc ſi æquatio ad o redigatur; obſervata terminorum homogeneitate habebitur

$$\left.\begin{array}{r} +3amm+3am+a \\ -1mm+2bm+b \\ -2m+c \\ -1 \end{array}\right\} = 0$$

Singulos ergo terminos æquando o, reperitur $a=\frac{1}{3}$, $b=\frac{1}{2}$, $c=\frac{1}{6}$; & per conſequens $am^3+bmm+cm$, id eſt $1+4+9+16 \ldots\ldots\ldots + mm$ eſt $= \frac{1}{3}m^3+\frac{1}{2}mm+\frac{1}{6}m = \frac{2m^3+3mm+m}{6} = \frac{2m^3+3mm+m}{1.2.3}$. Sit nunc generaliter potentia numerorum naturalium n, & quæratur ſumma progreſſionis finitæ $1^n+2^n+3^n+4^n+5^n \ldots\ldots + m^n$. Aſſumatur ergo itidem hæc quantitas [pro quæſita ſumma] $am^{n+1}+bm^n+cm^{n-1}+dm^{n-2}+$ &c. & augeatur unitate numerus terminorum m; unde provenit

$$\begin{array}{llll} am^{n+1}+(n+1)am^n & +\frac{(n+1).n}{1.2}am^{n-1} & +\frac{(n+1).n.(n-1)}{1.2.3.}am^{n-2}, & \text{\&c.} \\ \quad + \quad bm^n & + \; n.bm^{n-1} & +\frac{n.(n-1)}{1.2}bm^{n-2}, & \text{\&c.} \\ & + \quad cm^{n-1} & +(n-1)cm^{n-2}, & \text{\&c.} \\ & & + \quad dm^{n-2}, & \text{\&c.} \end{array}$$

$$= am^{n+1} + bm^{n} + cm^{n-1} + dm^{n-2} + \&c.$$
$$+ m^{n} + nm^{n-1} + \frac{n(n-1)}{1.2.} m^{n-2}$$
$$+ \frac{n.(n-1).(n-2)}{1\ 2.\ 3} m^{n-3} + \&c.$$

Æquatione reducta ad o, & obſervata homogeneitate terminorum, oritur

$$\left.\begin{array}{l} +(n+1).am^{n} + \frac{(n+1).n}{1.2} am^{n-1} + \frac{(n+1).n.(n-1)}{1.2.3} am^{n-2} + \&c. \\ -1m^{n} \quad + nbm^{n-1} + \frac{n.(n-1)}{1.2} bm^{n-2} + \&c. \\ \quad - nm^{n-1} + (n-1)\ cm^{n-2} + \&c. \\ \quad - \frac{n.(n-1)}{1.2}\ m^{n-2} + \&c. \end{array}\right\} = 0$$

Singulis itaque terminis cum o æquatis, innoteſcent coëfficientes

$$a = \frac{1}{n+1}$$
$$b = 1 - \frac{n+1}{1.2} a$$
$$c = \frac{n}{1.2} - \frac{(n+1).n}{1.2.3} a - \frac{n}{1.2} b$$
$$d = \frac{n.(n-1)}{1.2.3} - \frac{(n+1).n.(n-1)}{1.2.3.4} a - \frac{n.(n-1)}{1.2.3} b - \frac{n-1}{1.2} c$$
$$e = \frac{n.(n-1).(n-2)}{1.2.3.4} - \frac{(n+1).n.(n-1).(n-2)}{1.2.3.4.5} a - \frac{n.(n-1).(n-2)}{1.2.3.4} b$$
$$- \frac{(n-1).(n-2)}{1.2.3} c - \frac{n-2}{1.2} d,$$

&c.

quibus explicitis, erit

$$a = \frac{1}{n+1}$$
$$b = \tfrac{1}{2}$$
$$c = \frac{n}{3.4}$$
$$d = 0$$

e=

$$e = - \frac{n.(n-1).(n-2)}{1.2.3.4.5.6}$$
$$f = 0$$
$$g = \frac{n.(n-1).(n-2).(n-3).(n-4)}{1.2.3.4.5.6.7.6}$$
$$h = 0$$
$$i = - \frac{3n.(n-1).(n-2).(n-3).(n-4).(n-5).(n-6)}{1.2.3.4.5.6.7.8.9.10}$$
$$k = 0$$
$$l = \frac{5n.(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)(n-7)(n-8)}{1.2.3.4.5.6.7.8.9.10.11.6.}$$
$$m = 0$$

&c.

EXEMPLUM.

Quæritur ſumma omnium numerorum integrorum ad decimam poteſtatem elevatorum, quorum radices in millenario continentur; id eſt: quæritur ſumma hujus progreſſionis $1^{10} + 2^{10} + 3^{10} + 4^{10} + 5^{10} \ldots\ldots + 1000^{10}$. Hoc in caſu, cum ſit $n = 10$, & $m = 1000$, erunt $a = \frac{1}{11}$, $b = \frac{1}{2}$, $c = \frac{5}{6}$, $d = 0$, $e = -1$, $f = 0$, $g = 1$, $h = 0$, $i = -\frac{1}{2}$, $k = 0$, $l = \frac{5}{66}$, adoque ſumma totius progreſſionis erit $= \ldots$ $= 91,409,924,241,424,243,424,241,924,242,500$.

COROLLARIUM.

Si n eſt numerus negativus vel fractio; ſumma acquiret terminos infinitos. Hinc licet numeros irrationales plures ex. gr. $\sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5} \ldots + \sqrt{1000}$; per unicam Seriem conſtantem terminis mere rationalibus exprimere; cum alioquin unuſquiſque numerus irrationalis peculiarem poſcat legem.

N°. CLII.

Summatio Seriei $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}$ &c.

seu $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} +$ &c.

I.

LEIBNITIUS ostendit per methodum, quam dedit in *Actis Lipſ.* 1693, p. 180, dato arcu circuli x, fore ſinum y [ſumto radio vel ſinu toto $= 1$] expreſſum per hanc Seriem $y = x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \frac{x^7}{2.3\ldots7} +$ &c. Quam eandem Seriem anno ſequenti 1694, p. 438, & 439 *, inveni per aliam methodum univerſalem; Et anno 1722, p. 398, eorund. *Actor.*† adhuc per aliam methodum, & quidem ſine adminiculo Calculi differentialis.

II.

Quod ſi jam detur y, palam eſt, infinitos eſſe valores ipſius x eidem y reſpondentes, ob infinitos infinitarum dimenſionum terminos ipſius x; quot nimirum ſunt radices in hac æquatione dimenſionis infinitæ $x - \frac{1}{2.3}x^3 + \frac{1}{2.3.4.5}x^5 - \frac{1}{2.3\ldots7}x^7 +$ &c. $- y = 0$. Cujus ſingulæ radices, totidem arcus exhibentes, eidem ſinui y reſpondent.

III.

Sit nunc ſinus y infinite parvus, ſeu $= 0$, erit utique primus arcus x ſinui y æqualis, ſeu etiam $= 0$, ipſaque æquatio, diviſa per x, abibit in hanc $1 - \frac{1}{2.3}xx + \frac{1}{2.3.4.5}x^4 - \frac{1}{2.3\ldots7}x^6 +$ &c. $= 0$. Liquet autem, præter arcum illum initialem infinite parvum ſeu 0, qui, diviſa æquatione per x, jam eſt ſequeſtra-

* N°. XXI. pag. 127. *Tom. I.* † N°. CXXVII. pag. 533. *Tom. II.*

queſtratus, reliquos omnes eſſe vel ſemicircumferentiam, vel duplam ſemicircumferentiam, vel triplam, vel quadruplam, vel quintuplam, & ita in infinitum; quippe qui arcus ſinguli habent ſinum ſuum $= 0$. Liquet etiam, præter hos arcus nullum alium dari, qui habeat ſuum ſinum $= 0$.

IV.

Nominando itaque ſemicircumferentiam circuli $= c$; continebuntur radices omnes prædictæ æquationis in hac progreſſione, c, $2c$, $3c$, $4c$, $5c$, &c. in infinitum continuata. Ut autem æquatio iſta mutetur in aliam, quæ incipiat a maxima dimenſione litteræ incognitæ; ponamus $x = 1 : z$; adeoque $xx = 1 : zz$, $x^4 = 1 : z^4$ &c. poſitoque exponente infiniteſimo $2n$, prodibit hæc æquatio $1 - \frac{1}{2.3.} \times \frac{1}{zz} + \frac{1}{2.3.4.5} \times \frac{1}{z^4} - \frac{1}{2.3\ldots7} \times \frac{1}{z^6} \ldots + \frac{1}{2.3.4\ldots2n+1} \times \frac{1}{z^{2n}} = 0$; quæ multiplicata per z^{2n} dat hanc $z^{2n} - \frac{1}{2.3}z^{2n-2} + \frac{1}{2.3.4.5}z^{2n-4} - \frac{1}{2.3\ldots7}z^{2n-6} \ldots \frac{1}{2.3.4\ldots2n+1} = 0$.

V.

Hujus autem æquationis (quæ omnes ſuos terminos habet parium dimenſionum) radix $zz = 1 : xx$, hoc eſt $=$ vel $1 : cc$, vel $1 : 4cc$, vel $1 : 9cc$, vel $1 : 16cc$, vel &c. Adeoque, cum ex natura æquationum algebraïcarum coëficiens ſecundi termini mutato ſigno ſit æqualis ſummæ omnium radicum, erit ſane $\frac{1}{2.3}$ ſeu $\frac{1}{6} = \frac{1}{cc} + \frac{1}{4cc} + \frac{1}{9cc} + \frac{1}{16cc} +$ &c. proinde $\frac{cc}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} +$ &c. $= \frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} +$ &c. Unde patet ſummam Seriei fractionum, quarum numeratores ſunt unitates, denominatores vero quadrata numerorum naturalium esse subsextuplum quadrati semicircumferentiæ cujus radius $= 1$; vel (quod eodem recidit) subsextuplum quadrati totius circumferentiæ cujus diameter $= 1$.

VI.

SCHOLIUM I.

Atque ita satisfactum est ardenti desiderio Fratris mei, qui agnoscens summæ hujus pervestigationem difficiliorem esse quam quis putaverit, ingenue fassus est, omnem suam industriam fuisse elusam: Si quis *inveniat*, inquit, *nobisque communicet*, *quod industriam nostram elusit hactenus*, *magnas de nobis gratias feret*. Vid. Tractat. *de Seriebus infinitis*. p. 254. Utinam Frater superstes esset!

VII.

Simili modo inveniri potest summa Seriei fractionum, cujus, existentibus numeratoribus inter se æqualibus, denominatores procedunt ut quadrata, vel cubi, vel biquadrata, vel qualescunque potentiæ denominatorum 1, 4, 9, 16, 25, &c. hoc est, ut 1, 4^2, 9^2, 16^2, 25^2 &c. vel ut 1, 4^3, 9^3, 16^3, 25^3, &c. vel in genere ut 1, 4^m, 9^m, 16^m, 25^m, &c. aut quia ipsi numeri 1, 4, 9, 16, 25 &c. sunt quadrati numerorum naturalium 1, 2, 3, 4, 5, &c. poterit inveniri summa Seriei fractionum cujus denominatores sunt quælibet potentiæ pares numerorum naturalium, nempe hujus Seriei fractionum $1 + \frac{1}{2^{2m}} + \frac{1}{3^{2m}} + \frac{1}{4^{2m}} + \frac{1}{5^{2m}} +$ &c.

VIII.

Hujus investigationis fundamentum petitur ex elegantissimo Theoremate *Newtoniano*, quod sine demonstratione extat in illius *Algebra* p. 251, Edit. Lond. anni 1707: cujus autem demonstrationem ego inveni: ubi traditur modus quo ex coëfficienti-

ficientibus terminorum datæ alicujus æquationis determinantur summa non tantum radicum, sed & ex radicibus summa quadratorum, cuborum, quadrato-quadratorum &c.

IX.

Ita ejus regulæ ductu, æquationis nostræ §. 4 expressæ $z^{2n} - \frac{1}{2.3} z^{2n-2} + \frac{1}{2.3.4.5.} z^{2n-4} \ldots \frac{1}{2.3\ldots 2n+1} = 0$, radices $\frac{1}{cc}$, $\frac{1}{4cc}$, $\frac{1}{9cc}$, &c. habebunt pro summa quadratorum $(\frac{1}{c^4} + \frac{1}{4^2c^4} + \frac{1}{9^2c^4} + \frac{1}{16^2c^4} +$ &c.$)$ hanc quantitatem ex primo & secundo coëfficiente deductam $(\frac{1}{2.3})^2 - \frac{2}{2.3.4.5} = \frac{1}{36} - \frac{1}{60} = \frac{1}{90}$. Est itaque $1 + \frac{1}{4^2} + \frac{1}{9^2} + \frac{1}{16^2} +$ &c. seu $1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} +$ &c. $= \frac{c^4}{90}$.

X.

Sic pariter radices $\frac{1}{cc}$, $\frac{1}{4cc}$, $\frac{1}{9cc}$, $\frac{1}{16cc}$ &c. obtinebunt pro summa cuborum $(\frac{1}{c^6} + \frac{1}{4^3c^6} + \frac{1}{9^3c^6} + \frac{1}{16^3c^6} +$ &c.$)$ hanc quantitatem ex primo, secundo & tertio coëfficiente, seu ex coëfficientibus termini secundi, tertii & quarti elicitam $(\frac{1}{2.3}) \times \frac{1}{90} - (\frac{1}{2.3.4.5}) \times (\frac{1}{2.3}) + \frac{3}{2.3\ldots 7} = \frac{1}{940}$; Proinde erit $1 + \frac{1}{4^3} + \frac{1}{9^3} + \frac{1}{16^3} +$ &c. seu $1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{2}{4^6} +$ &c. $= \frac{c^6}{940}$.

XI.

Ex istis eliciemus summam hujus Seriei $1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} +$ &c.

atque

atque ita successive progredi licebit ad altiores dimensiones numerorum naturalium, modo exponentes sint numeri pares; quomodo vero Series tractandæ sint, si denominatores terminorum sunt numeri naturales ad dimensiones impares elevati, ex. gr. si hæc simplicissima proponatur Series $1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \&c.$ cujus utique summa est finita, nondum constat per hanc nostram methodum. Invitantur Analystæ, ut defectui succurrant.

XII.

SCHOLIUM II.

Supponitur cæterum, æquationem nostram §. 4 inventam $z^{2n} - \frac{1}{2.3} z^{2n-2} + \frac{1}{2.3.4.5.} z^{2n-4} - \ldots \frac{1}{2.3.4\ldots 2n+1} = 0$, nullas continere radices impossibiles, seu imaginarias, nullas quoque admixtas habere (ut in solutione Problematum subinde accidit) radices peregrinas, præter illas genuinas $\frac{1}{cc}$, $\frac{1}{4cc}$, $\frac{1}{9cc}$, $\frac{1}{16cc}$, &c. Alias vacillaret fundamentum desumtum ex notissima illa proprietate coëfficientium. Neutrum autem hic timendum esse, satis est probabile vel ex ipsa Seriei generatione. Cum præsertim per regulam NEWTONI traditam p. 242, ad nostram æquationem infinitam probe applicatam, fere omnino pateat nullas in illa contineri radices impossibiles.

APPENDIX.

Esto circuli radius $= 1$, tangens alicujus arcus $= t$, erit arcus cui respondet t, hoc est $\int \frac{dt}{1+tt} = t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7 + \&c.$ Hoc jam invenit LEIBNITIUS. Sed sine alternatione signorum inveni ego hanc seriem $\int \frac{dt}{1+tt} = \frac{t}{1.(1+tt)^1} + \frac{2t^3}{1.3.(1+tt)^2} + \frac{2.4.t^5}{1.3.5(1+tt)^3} + \frac{2.4.6.t^7}{1.3.5.7.(1+tt)^4} + \&c.$

Posito

Posito sinu $= x$, erit, ut dudum constat, arcus $= x + \frac{1}{2.3}x^3 + \frac{1.3}{2.4.5}x^5 + \frac{1.3.5}{2.4.6.7.}x^7 + \&c.$ Sed sine successione signorum reperi ego hanc seriem $\frac{x}{1(1-xx)^{1:2}}, - \frac{x^3}{3(1-xx)^{3:2}}, + \frac{x^5}{5(1-xx)^{5:2}} - \frac{x^7}{7(1-xx)^{7:2}} + \&c. = \int \frac{dx}{\sqrt{(1-xx)}} = \int \frac{dx}{(1-xx)^{1:2}}$. Hinc itaque $\int \left(\frac{dx}{\sqrt{(1-xx)}} \int \frac{dx}{\sqrt{(1-xx)}}\right) = \int \frac{x\,dx}{1-xx} - \frac{1}{3}\int \frac{x^3 dx}{(1-xx)^2} + \frac{1}{5}\int \frac{x^5 dx}{(1-xx)^3} - \frac{1}{7}\int \frac{x^7 dx}{(1-xx)^4} + \&c.$

Ex serie secunda fit $\int \left(\frac{dt}{1+tt} \int \frac{dt}{1+tt}\right) = \int \frac{t\,dt}{1.(1+tt)^2} + \frac{2}{1.3} \int \frac{t^3 dt}{(1+tt)^3} + \frac{2.4}{1.3.5} \int \frac{t^5 dt}{(1+tt)^4} + \frac{2.4.6}{1.3.5.7.} \int \frac{t^7 dt}{(1+tt)^5} + \&c.$ cujus singuli termini sunt integrabiles. Si vero, post integrationem peractam & more solito rectificatam, ponatur $t = \infty$; prodibit ista Series $\frac{1}{1.2} + \frac{2}{1.3.4} + \frac{2.4}{1.3.5.6} + \frac{2.4.6}{1.3.5.7.8} + \&c.$ cujus adeo summa (posita semicircumferentia $= c$) erit $= \frac{1}{8}cc$.

N°. CLIII.

PROBLEMA.

Maximum terminum binomii ad quamcunque dimensionem elevati invenire.

SIT $a + b$ binomium, & c numerus dimensionis; erit $(a+b)^c = a^c + \frac{c}{1} a^{c-1} b + \frac{c.(c-1)}{1.2.} a^{c-2} bb + \frac{c.(c-1).(c-2)}{1.2.3.} a^{c-3} b^3$; Quæritur maximus hujus Seriei terminus?

Hic ibi erit, ubi duo subsequentes sunt æquales. Sit igitur numerus quotus termini quæsiti $= x$, & quia subsequens

 huic

huic debet esse æqualis, dividatur subsequens per antecedentem, & proveniet ex natura Seriei $\frac{c - x + 1}{x} \times \frac{b}{a}$; quod erit (ob æquale divisum per æquale) æquale unitati, id, quod hanc format æqualitatem $\frac{cb - bx + b}{ax} = 1$, vel $cb - bx + b = ax$; Ergo $x = \frac{cb + b}{a + b}$ = numero quoto termini maximi quæsiti. *Q. E. I.*

Hinc si $\frac{cb + b}{a + b}$ est numerus integer, duo erunt termini maximi; nam subsequens invento est æqualis. Sin autem $\frac{cb + b}{a + b}$ non est numerus integer; erunt duo termini subsequentes quidem æquales; sed, ob numerum quotum non integrum, neuter in Serie existet, ideoque unicus erit maximus terminus; nempe ille qui inter utrumque imaginarium includitur: proinde numerus fractus $\frac{cb + b}{a + b}$ ad proximam unitatem augendus est, ad habendum verum numerum quoti termini maximi quæsiti.

EXEMPLUM.

Sit $a = 2$, $b = 6$, $c = 7$, erit $\frac{cb + b}{a + b} = 6$ = numero integro, quod indicio est 6^{tum} & 7^{mum} terminum esse maximos, utpote æquales. Si vero $a = 3$, $b = 4$, $c = 5$, erit $\frac{cb + b}{a + b} = \frac{24}{7} = 3\frac{3}{7}$ = numero non integro; quod denotat quartum terminum, eumque solum, esse maximum. Sic sit $a = 2$, $b = 1$, $c = 6$, erit $\frac{cb + b}{a + b} = \frac{7}{3} = 2\frac{1}{3}$; ideoque maximus terminus erit tertius.

PROBLEMA.

Data progressione arithmetica, ab unitate incipiente, & eo modo dispositа quo hic vides; invenire Seriem transversalem, cujus summa omnium sit maxima.

1 2 3 4 5 6
7 8 9 10 11
12 13 14 15
16 17 18
19 20
21

Sit numerus terminorum primæ Seriei transversalis $= a$, numerus quotus Seriei quæsitæ $= x$. Ex natura progressionis arithmeticæ, primi & ultimi Serierum transversalium, sic se habebunt.

$$\begin{array}{lcl} 1,\ 2,\ 3,\ 4 & \ldots\ldots\ldots\ldots & a \\ a + 1, & \ldots\ldots\ldots\ldots & 2a - 1 \\ 2a. & \ldots\ldots\ldots\ldots & 3a - 3 \\ 2a - 2. & \ldots\ldots\ldots\ldots & 4a - 6 \\ 4a - 5. & \ldots\ldots\ldots\ldots & 5a - 10 \\ \vdots & & \\ ax - a - \frac{xx + 3x}{2} & \ldots\ldots & ax - \frac{xx + x}{2} \end{array}$$

Est autem numerus terminorum Seriei quæsitæ $= a - x + 1$; ergo summa Seriei quæsitæ erit $= (x^3 - 3axx - 3xx + 2aax + 5ax + 2x - aa - a) : 2$, & posito $x + 1$ loco x, proveniet summa Seriei subsequentis $= (x^3 - 3axx + 2aax - ax - x + aa + a) : 2$. Si itaque, ad modum præcedentem, æquatio instituatur inter has duas quantitates, invenietur $x = a + \frac{1}{2} - \sqrt{(\frac{1}{3}aa + \frac{1}{3}a + \frac{1}{4})}$. Sic si hæc quantitas est numerus integer, erunt duæ Series transversales maximæ: Sin minus, quantitas inventa ad proximam unitatem augenda est, & unica erit Series transversalis maxima.

N°. CLIV.

DE ALEA, SIVE ARTE CONJECTANDI;

PROBLEMATA QUÆDAM.

PROBLEMA I.

Aliquot Colluſores teſſera ludunt, quorum primus certum numerum jactuum habet; quæritur quot jactus ſecundo, tertio, quarto, quinto adjudicandi ſint ante ludum, ita ut ſors ſingulorum fiat æqualis?

Sit numerus facierum teſſeræ quibus obtinetur $= a$, & numerus facierum quibus perditur $= b$, numerus jactuum primi Colluſoris $= c$, numerus Colluſoris quoti $= n$, & numerus jactuum Colluſoris quoti, una cum jactibus præcedentium $= x$. In uno jactu ſunt a caſus ad obtinendum, & b ad perdendum; ergo ſors unius jactus $= \frac{a}{a+b}$. In primo duorum jactuum ſunt a caſus ad obtinendum, & b ad perveniendum ad ſortem $\frac{a}{a+b}$; ergo ſors duorum jactuum $= \frac{ab}{(a+b)^2} + \frac{a}{a+b}$. In primo trium jactuum ſunt a caſus ad 1, & b ad $\frac{ab}{(a+b)^2} + \frac{a}{a+b}$; ergo ſors trium jactuum $= \frac{abb}{(a+b)^3} + \frac{ab}{(a+b)^2} + \frac{a}{a+b}$. Eodem modo habetur ſors quatuor jactuum $= \frac{ab^3}{(a+b)^4} + \frac{abb}{(a+b)^3} + \frac{ab}{(a+b)^2} + \frac{a}{a+b}$. Et ſors c jactuum $= \frac{ab^{c-1}}{(a+b)^c} + \frac{ab^{c-2}}{(a+b)^{c-1}} \ldots\ldots \frac{a}{a+b} =$ (quia eſt progreſſio geometrica) $1 - b^c : (a+b)^c =$ ſorti primi Colluſoris. Pari ratione erit ſors x jactuum $= 1 - b^x : (a+b)^x$; quia autem ſingulorum Colluſorum ſors eadem ponitur, erit $1 - b^x : (a+b)^x = n - nb^c : (a+b)^c$, aut $1 - n + nb^c : (a+b)^c = b^x : (a+b)^x$; ideoque Log. $(1 - n + nb^c : (a+b)^c) = x\,lb - x\,l(a+b)$; ideoque $x = l(1 - n + nb^c : (a+b)^c) : (lb - l(a+b))$, a quo ſi auferatur numerus jactuum præcedentium [qui habetur ponendo $n - 1$ pro n] reſtabit

$$\frac{l(2 - n + (n-1)b^c : (a+b)^c) - l(1 - n + nb^c : (a+b)^c)}{l(a+b) - lb}$$

$=$ numero jactuum quæſito.

ALITER.

Quoniam tantundem eſt, & eadem expectatio habetur, ſi unica teſſera aliquot jactus inſtituendi ſint, quam cum totidem teſſeris unicus jactus eſt faciendus; ponatur loco numeri jactuum primi Colluſoris numerus teſſerarum $= c$, & loco numeri jactuum Colluſoris quoti una cum jactibus præcedentium, numerus teſſerarum $= x$. Patet ex Arte combinandi, quod c teſſeræ (ob $a + b$ facies unius teſſeræ) variari poſſint $(a+b)^c$ caſibus, & b^c caſibus quibus nulla facies ipſarum a cadit, id eſt, quibus perditur; ideoque ſunt $(a+b)^c - b^c$ caſus quibus obtinetur: Invenitur ergo ſors Colluſoris primi $= ((a+b)^c - b^c) : (a+b)^c = 1 - b^c : (a+b)^c$, ut ante. Pariter erit ſors x jactuum $= 1 - b^x : (a+b)^x$; reliqua peraguntur ut prius.

PROBLEMA II.

Datis faciebus a + b *unius tesseræ; quæritur quot vicibus cum unica tessera, seu, quod tantundem est, quot tesseris unica vice aliquis suscipere possit ut jaciat unam, duas,* 3, 4, &c. n, *ex faciebus a.*

Esto numerus tesserarum $= x$, erunt $(a+b)^x$ casus quibus x tesseræ variari possunt, b^x casus quibus nulla facies ipsarum a cadit, $\frac{x}{1} b^{x-1} a^1$ casus quibus una, $\frac{x.(x-1).}{1.2} b^{x-2} a^2$ casus quibus duæ, $\frac{x.(x-1).(x-2)}{1.2.3} b^{x-3} a^3$ quibus tres &c. Si itaque unam ex faciebus a jacere debeat; erunt $(a+b)^x - b^x$ casus quibus lucratur: si duas, $(a+b)^x - b^x - \frac{x}{1} b^{x-1} a$ casus: si tres, $(a+b)^x - b^x - \frac{x}{1} b^{x-1} a - \frac{x.(x-1)}{1.2} b^{x-2} a^2$ casus; si n facies suscipiantur; erunt $(a+b)^x - b^x - \frac{x}{1} b^{x-1} a - \frac{x.(x-1)}{1.2} b^{x-2} a^2 \ldots\ldots\ldots - \frac{x(x-1)\ldots(x-n+2)}{1.2.3\ldots n-1} b^{x-n+1} a^{n+1}$ casus; ideoque si quæratur sors, erit illa $= \frac{1}{2}$; quod hanc dabit æquationem $-\frac{1}{2} + (a+b)^x - b^x - \frac{x}{1} b^{x-1} a^1 - \frac{x.(x-1)}{1.2} b^{x-2} a^2$ &c. $= 0$. *Q. E. I.*

PROBLEMA III.

Petrus & Paulus, *quorum dexteritates sint æquales inter se, datis globis numero* p & q *certent; jam post ludos aliquot peractos desint Petro ludi* f *quominus victor evadat*, Paulo *vero desint ludi* g. *Quæritur ratio inter ipsorum sortes?*

Solutio

Solutio ex sequenti tabella habetur. Esto $p+q=m$

$$\left.\begin{array}{l}
\frac{p.(p-1).(p-2)\ldots(p-f+1)}{m.(m-1)\ldots(m-f+1)}\,1 + \frac{q.(q-1).(q-2)\ldots(q-g+1)}{m.(m-1)\ldots(m-g+1)}\,0 \\
\frac{pq.(p-1).(p-2)\ldots(p-f+2)}{m.(m-1)\ldots(m-f+1)}\,A + \frac{pq.(q-1).(q-2)\ldots(q-g+2)}{m.(m-1)\ldots(m-g+1)}\,\alpha \\
\frac{pq.(p-1).(p-2)\ldots(p-f+3)}{m.(m-1)\ldots(m-f+2)}\,B + \frac{pq.(q-1).(q-2)\ldots(q-g+3)}{m.(m-1)\ldots(m-g+2)}\,\beta \\
\frac{pq.(p-1).(p-2)\ldots(p-f+4)}{m.(m-1)\ldots(m-f+3)}\,C + \frac{pq.(q-1).(q-2)\ldots(q-g+4)}{m.(m-1)\ldots(m-g+3)}\,\gamma \\
\quad\vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\
\frac{pq}{m.(m-1)}\,X \qquad\qquad\qquad\qquad \frac{pq}{m.(m-1)}\,\xi
\end{array}\right\} = \text{Expectationi } Petri$$

NB. Intelligo per *A*, *B*, *C*, *D*, &c. expectationes *Petri* cum ipsi desunt ludi 1, 2, 3, 4 &c. & *Paulo* ludi g; & per $\alpha, \beta, \gamma, \delta$ &c. expectationes *Petri*, cum ipsi desunt ludi f, & *Paulo* ludi 1, 2, 3, 4 &c.

PROBLEMA IV.

Petrus *ludit cum* Paulo *tesseris, hac conditione, ut si medium arithmeticum proportionalem jaciat inter maximum & minimum jactum, vel si plura puncta jaciat quam iste medius proportionalis, ille lucretur; sin minorem jactum faciat, lucretur* Paulus. *Quæritur ratio sortium?*

SOLUTIO.

Si numerus tesserarum est impar quicunque, liquet sortes inter se esse æquales. Si vero sit par, sit ille $= 2n$, ita ut numerus casuum, quibus jactus omnes variare possunt, sit 6^{2n}; inter hos erunt $\frac{(7n-1).(7n-2).(7n-3)\ldots\ldots(5n+1)}{1.2.3.4\ldots\ldots(2n-1)}$

G 2 casus,

casus, quibus medius arithmeticus inter extremos cadere potest; adeoque sors *Petri* est ad sortem *Pauli* ut $6^{2n} + \frac{(7n-1).(7n-2)....(5n+1)}{1.2.3....(2n-1)}$ ad $6^{2n} - \frac{(7n-1).(7n-2)...(5n+1)}{1.2.3...(2n-1)}$;

PROBLEMA V.

Duo lusores A & B ludunt una tessera, hac conditione, ut A tres jactus faciat continuos, numeretque puncta quæ tribus istis jactibus jecerit in unam summam; B vero tot faciat jactus quot puncta A primo jactu jecerit, omnia jacta puncta pariter collecturus in unum: Qui autem majorem punctorum summam habuerit, ille lucrabitur; Quod si vero utriusque punctorum numerus sit æqualis, tunc depositum bipartientur. Quæritur ratio sortis utriusque?

R. Sors A est ad sortem B ut 4200563 ad 5877133.

PROBLEMA VI.

Cæteris positis ut prius, sit æqualem uterque habuerint punctorum summam, tunc etiam A vincet. Quæritur sortium ratio?

R. Sors A est ad sortem B ut 282571 ad 347285.

PROBLEMA VII.

Dato numero Electorum, qui sit multiplex ternarii, non tamen minor senario; Duo autem ex Electoribus A & B se declaraverint faventes alicui ex Candidatis C. Quæritur quantam spem habeat C, seu quænam sit probabilitas, ut A & B in unam eandemque trium Electorum classem per sortem collocentur.

SOLUTIO.

Sit numerus Electorum $= 6 + 3n$. Dico probabilitatem quæsitam fore $= \frac{1+n}{5+3n}$; Hoc est probabilitas ut A & B in eadem classe conjungantur, est ad probabilitatem eventus contrarii ut $1 + n$ ad $4 + 2n$.

COROL-

COROLLARIUM I.

Si 6 sint Electores, erit ratio expectationis ad metum sinistri successus, ut 1 ad 4.

COROLLARIUM II.

Si numerus Electorum esset infinitus, foret illa ratio ut 1 ad 2.

COROLLARIUM III.

Hinc quo major est Electorum numerus, eo favorabiliorem expectationem habet *C*. Quod paradoxum esse videtur.

N°. CLV.

GEOMETRICA.

PROPOSITIO I.

THEOREMA.

Si per quodvis punctum A *in Triangulo quovis rectilineo* BCD *ex singulis angulis ducantur rectæ ad latera opposita; erunt solida ex tribus laterum segmentis, non contiguis, facta inter se æqualia; nempe* CF×DE×BG = DF×CG×BE. TAB. LXXVII. *Fig.* 1.

DEMONSTRATIO.

Producta BF, agantur DH & CI parallelæ ipsis AC & AD. Jam, ob similia Triangula ACI, HDA, ut & AFC, HFD, erit CF : FD = AI : AH = AI : AB [CG : GB] + AB : AH [BE : ED]; id est, CF : FD = CG×BE : BG×DE. Ergo CF×BG×DE = DF×CG×BE. *Q. E. D.*

PROPOSITIO II.

PROBLEMA.

Datis in peripheria circuli duobus punctis A *&* B, *invenire tertium quoddam in illa, ut* C, *ita ut ductis* AC, BC *& perpendiculari* BD, *lineæ* AD *&* BD *simul sumtæ sint æquales lineis* BC *&* DC *simul sumtis.*

SOLUTIO.

TAB. LXXVII. Fig. 2.

Ducatur subtensa AB, bisceturque illa in H, & per punctum H agatur diameter HI, voceturque AH, vel HB $= a$, GH $= c$, radius FG $= b$, HE $= x$; erit AE $= \surd(aa+xx)$. Nunc, ob similia Triangula AHE & ABD, est AE $[\surd(aa+xx)]$: AH $[a]$ = AB $[2a]$: AD $[2aa : \surd(aa+xx)]$, & AH $[a]$: HE $[x]$ = AD $[2aa : \surd(aa+xx)]$: BD $[2ax : \surd(aa+xx)]$. Quia GE $= c+x$, & IE $= 2b - c - x$, erit rectangulum GEI $= 2bc + 2bx - cc - 2cx - xx$ = rectangulo AEC, proinde $\frac{2bc+2bx-cc-2cx-xx}{\surd(aa+xx)}$ = EC, & $\frac{2bc+2bx-cc-2cx-xx}{\surd(aa+xx)} + \surd(aa+xx) = \frac{aa+2bc+2bx-cc-2cx}{\surd(aa+xx)}$ = AC. Ergo AC − AD [DC] $= \frac{-aa+2bc+2bx-cc-2cx}{\surd(aa+xx)}$; & BD^2 $\left[\frac{4aaxx}{aa+xx}\right]$ + DC^2 $[(a^4+4bbcc+4bbxx+c^4+4ccxx-4aabc-4aabx+2aacc+4aacx+8bbcx-4bc^3-12bccx-8bcxx+4c^3x):(aa+xx)]$ = BC^2. Jam autem quia AD + BD debet esse = BC + DC, erit etiam AD + BD − DC = BC, id est, $(3aa+2ax-2bc-2bx+cc+2cx):\surd(aa+xx) = \surd(4aaxx+a^4+4bbcc+4bbxx+c^4+4ccxx-4aabc-4aabx+2aacc+4aacx+8bbcx-4bc^3-12bccx-8bcxx+4c^3x):\surd(aa+xx)$.

Abjecto

Abjecto communi denominatore, sumantur numeratorum quadrata, erit,

$$\begin{array}{lll} +4aaxx & -4aabx & +a^4 \\ +4bb & +4aac & +c^4 \\ +4cc & +8bbc & +4bbcc \\ -8bc & -12bcc & -4aabc \\ & +4c^3 & +2aacc \\ & & -4bc^3 \end{array} = \begin{array}{lll} 4aaxx & +12a^3x & +9a^4 \\ +4bb & -12aab & +c^4 \\ +4cc & +12aac & +4bbcc \\ -8ab & -8abc & -12aabc \\ +8ac & +4acc & +6aacc \\ -8bc & +8bbc & -4bc^3 \\ & -12bcc & \\ & +4c^3 & \end{array}$$

Reducta hac æquatione, provenit $xx = \left\{\begin{array}{ll} -3a^2x & -2a^3 \\ +2ab & +2abc \\ -2ac & -acc \\ +2bc & \\ -cc & \end{array}\right\} : (2c-2b).$

Ut autem hæc æquatio paucioribus literis & terminis habeatur, quæratur valor ipsius b, & quidem sic: IHG rectangulum = AH^2, id est $2bc - cc = aa$, proinde $b = (aa+cc):2c$; si itaque loco b, substituatur valor hic inventus, habebitur $xx = ((a^3-2aac-acc)x - a^3c):(cc-aa)$ & $x = (\frac{1}{2}a^3 - aac - \frac{1}{2}acc + \surd(\frac{1}{4}a^6 + \frac{1}{2}a^4cc + \frac{1}{4}aac^4)):(cc-aa)$; aut, quia ex posteriori quantitate radix potest extrahi, fit $x = (a^3 - aac):(cc-aa)$ & $x = (-acc-aac):(cc-aa)$, vel dividatur numerator & denominator prioris per $a-c$, & posterioris per $-a-c$, proveniet $x = aa:(-a-c)$ & $x = ac:(a-c)$. Ubi notandum est quod prior radix inventa valeat, si punctum C quæratur in minori arcu AGB, posterior autem, si idem illud in majori AIB.

CONSTRUCTIO.

Sumatur HK = HG ad utramque partem, & agatur KL parallela ipsi HG, & quidem in priori casu sursum & æqualis ipsi AH, in posteriori deorsum & æqualis HG; perque puncta L ducantur lineæ AC; designabunt hæ lineæ in periphe-

ria

ria Circuli duo puncta C, adeo ut, ductis BC & perpendicularibus BD, sint $AD + BD = BC + DC$.

PROPOSITIO III.

PROBLEMA.

TAB. LXXVII. Fig. 3. *In data linea* BF *ad diametrum Circuli* GH *perpendiculari; invenire punctum* F *ita comparatum, ut ducta* FC *tangens Circulum sit æqualis lineæ* CD *perpendiculari ad eandem diametrum.*

SOLUTIO.

Intelligatur ductam esse AC, & CE parallelam diametro GH; voceturque radius circuli $AC = a$, $AB = b$, & $AD = x$, erit $DC = \sqrt{(aa - xx)}$. Quia nunc angulus ACF est rectus, ob tangentem CF, & angulus DCE etiam rectus, ob CE parallelam GH; erit, ablato communi ACE, reliquus ACD = reliquo FCE: est autem CDA = CEF, utpote uterque rectus; erunt ideo Triangula ADC & CEF similia; proinde $AC[a] : DC[\sqrt{(aa - xx)}] = CF[\sqrt{(aa - xx)}] : CE$ seu $DB[(aa - xx) : a] = b + x$; erit reducendo æquationem $xx = -ax - ab + aa$, & $x = -\frac{1}{2}a \pm \sqrt{(\frac{5}{4}aa - ab)}$.

PROPOSITIO IV.

PROBLEMA.

TAB. LXXVII. Fig. 4. *Si a puncto quodam peripheriæ Circuli, ut* B, *infinitæ ducantur lineæ* BF, *arcusque* BF, *quos subtendunt, bisecentur in* D, *& a punctis bisectionis agantur lineæ* DE, *parallelæ diametro* BA, *donec occurrant lineis* BF *in* E; *invenire naturam curvæ, quam puncta* E *describunt?*

SOLU-

SOLUTIO.

Ducatur AD ex centro A, & EC perpendicularis ad BA: Sit radius $AB = a$, $BC = x$, & $CE = y$. Quia nunc arcus BF per hypothesin bisectus est in D, erit AD perpendicularis ad BF; proinde Triangula AGB & ACH sunt similia: est autem Triangulum AGB etiam simile Triangulo BCE; ergo Triangulum ACH est simile Triangulo BCE, ideoque $EC[y] : CB[x] = AC[a - x] : CH[(ax - xx) : y]$. Quia vero Triang. ACH & DHE etiam sunt similia, erit $EH : HC = DH : HA$ & *componendo* $EC[y] : HC[(ax - xx) : y]$. $DA[a] : HA[(aax - axx) : yy]$. Nunc iterum $CH[(ax - xx) : y] : HA[(aax - axx) : yy] = 1 : \frac{a}{y} = BC[x] : BE[ax : y]$, cujus quadratum $BE^2[aaxx : yy] = BC^2[xx] + CE^2[yy]$ & $aaxx = xxyy + y^4$, tandemque $y^4 + xxyy - aaxx = 0$.

PROPOSITIO V.

PROBLEMA.

Si a puncto quodam peripheriæ Circuli ut B, *infinitæ ducantur lineæ* BF, *arcusque* BF *quos subtendunt bisecentur in* D, *& a punctis bisectionis demittantur lineæ* DC *normales ad diametrum* BA, *quæ lineas* BF *secent in* E; *invenire naturam Curvæ quam puncta* E *describunt?* TAB. LXXVII. Fig. 5.

SOLUTIO.

Ducatur AD ex centro A, & DC perpendicularis ad BA (ut prius): Sit radius $AB = a$, $BC = x$, & $CE = y$. Quia nunc arcus BF per hypothesin bisectus est in D, erit AD perpendicularis ad BF, Triangula BCE & AHB sunt similia; est autem Triangulum AHB simile Triangulo ACD; Ergo Triangulum BCE est simile Triangulo ACD, ideo-

que

que CE [y]: BC [x] = AC [$a - x$]: CD [$(ax - xx):y$] cujus quadratum CD^2 [$(aaxx - 2ax^3 + x^4):yy$] = rectang. BCG [$2ax - xx$] & $aaxx - 2ax^3 + x^4 = 2axyy - xxyy$. Diviſum utrumque per x, erit $aax - 2axx + x^3 = 2ayy - xyy$; tandemque $x^3 - 2axx + aax + yyx - 2ayy = 0$.

PROPOSITIO VI.

PROBLEMA.

TAB. LXXVII. Fig. 6. *Data in dato Circulo* ABC *ſubtenſa* AC; *invenire in majoris ſegmenti peripheria punctum* B, *ita ut ducta per centrum* D *linea* BDH, *terminata a ſubtenſa* AC, *ſit media proportionalis inter ductas* BA & BC.

SOLUTIO.

Sit BD vel DG $= a$, AF vel FC $= b$, DF $= \sqrt{(aa - bb)} = c$, DH $= x$, FH $= \sqrt{(xx - cc)}$. Quia nunc HD : DB = HF : FE, erit FE $= \sqrt{(xx - cc)}\,a : x$ & AE $= (bx - a\sqrt{(xx - cc)}) : x$ & EC $= (bx + a\sqrt{(xx - cc)}) : x$, quia etiam HD : DF = HB : BE, erit BE $= (cx + ac) : x$, eſt autem $AE^2 + EB^2 = AB^2$, & $EC^2 + BE^2 = BC^2$; erit ergo AB $= \sqrt{((aax + bbx + ccx + 2acc - 2ab\sqrt{(xx - cc)}) : x)}$ & BC $= \sqrt{((aax + bbx + ccx + 2acc + 2ab\sqrt{(xx - cc)}) : x)}$. Subſtituatur utrobique valor ipſius $cc = aa - bb$, & habebitur AB $= \sqrt{((2aax + 2a^3 - 2abb - 2ab\sqrt{(xx - aa + bb)}) : x)}$ & BC $= \sqrt{((2aax + 2a^3 - 2abb + 2ab\sqrt{(xx - aa + bb)}) : x)}$. Quoniam itaque debet eſſe AB : BH = BH : BC, erit rectangulum ABC $= BH^2$, id eſt $\sqrt{(4a^4xx - 4aabbxx + 8a^5x - 8a^3bbx + 4a^6 - 4a^4bb)} : x = aa + 2ax + xx$. Multiplica utrumque per x, & divide per $x + a$, & invenies $xx + ax = \sqrt{(4a^4 - 4aabb)}$ & $xx = -ax + \sqrt{(4a^4 - 4aabb)}$; ergo erit $x = -\frac{1}{2}a + \sqrt{(\frac{1}{4}aa + \sqrt{(4a^4 - 4aabb)})}$.

Idem PROBLEMA *faciliori modo ſolvere.*

Serventur eædem litteræ quæ prius, videlicet DB $= a$, DF $= c$, DH $= x$, & ducatur CG; ideoque erit angulus BCG = recto = BEA & BGC = BAE, quia inſiſtunt eidem ſegmento BC, proinde Triangula ABE & BGC ſimilia ſunt, idcirco AB : BE = BG : BC; hincque rectangulum ABC = EBG; ſed eſt HD : DF = HB : BE, id eſt $x : c = x + a : \frac{ac + cx}{x} =$ BE; ideoque rectangulum EBG $= (2aac + 2acx) : x =$ ABC. Hoc autem æquale debet eſſe BH^2, erit ergo $(2aac + 2acx) : x = aa + 2ax + xx$; multiplica utrumque per x, & divide per $a + x$, habebis $xx + ax = 2ac$, proinde $x = -\frac{1}{2}a + \sqrt{(\frac{1}{4}aa + 2ac)}$. TAB. LXXVII. Fig. 7.

CONSTRUCTIO.

Producatur diameter OI ad K, ita ut IK ſit = DF, & diametro OK deſcribatur circulus KLO, cujus peripheriæ occurrat recta IL tangens circulum ICO in I, & a medio ipſius DI puncto M applicetur MN = ductæ ML, radioque DN deſcribatur circulus HNH ſecans ſubtenſam AC in punctis H, H; per quæ & per centrum D, ſi ducantur rectæ HDB, deſignabunt in peripheria puncta B quæſita, ut videlicet BH ſit media inter ductas BA, BC.

DEMONSTRATIO.

Quia NM = ML, erit NM^2 [$ND^2 + DM^2 + NDI$] $= ML^2$ [$MI^2 + IL^2$], & ablatis æqualibus, erit $ND^2 + NDI = IL^2 =$ KIO; eſt autem $ND^2 +$ NDI = IND = BHD; ergo BHD = KIO, ideoque DH : KI vel FD [BH : BE] = IO : BH; proinde $BH^2 = BE \times IO =$ EBG = ABC [ut in analyſi demonſtratur]; ergo AB : BH = BH : BC. *Q. E. D.*

PROPOSITIO VII.

PROBLEMA.

TAB. LXXVII. Fig. 8. *Quæritur natura curvæ* ABCG, *ut ducta a puncto dato* D *utcunque recta* DBC, *intercepta inter puncta intersectionis* BC *sit semper æqualis constanti.*

Sit $BC = 1$, $DB = x$, $BE = y = a + bx + cx^2 + ex^3 + fx^4 + gx^5 + hx^6$ &c. Quoniam igitur $DB : DC = BE : CF$ erit

$$CF = a : x \begin{array}{l} + a \\ + b + bx \\ \quad + cx + cxx \\ \quad\quad + exx + ex^3 \\ \quad\quad\quad + fx^3 + fx^4 \\ \quad\quad\quad\quad + gx^4 + gx^5 \\ \quad\quad\quad\quad\quad + hx^5 + hx^6 \text{ \&c. \&;} \end{array}$$

ob identitatem relationis duorum punctorum in curva ad puncta in axe erit etiam

$$CF = \begin{array}{l} + a \\ + b + bx \\ + c + 2cx + cxx \\ + e + 3ex + 3exx + ex^3 \\ + f + 4fx + 6fxx + 4fx^3 + fx^4 \\ + g + 5gx + 10gxx + 10gx^3 + 5gx^4 + gx^5 \\ + h + 6hx + 15hxx + 20hx^3 + 15hx^4 + 6hx^5 + hx^6. \\ \text{\&c.} \end{array}$$

Ergo, comparando dimensiones æquales, invenietur pro coëfficientibus $a = 0$, $c + e + f + g + h = 0$, $c + 3e + 4f + 5g + 6h = 0$, $2e + 6f + 10g + 15h = 0$, $3f + 10g + 20h = 0$, $4g + 15h = 0$, $5h = 0$. Quoniam autem provenit $h = 0$, erunt pariter omnes aliæ litteræ $= 0$, quod indicio est, hoc modo Problema non posse solvi. Ponatur ergo punctum D inter A & G; erit, cæteris positis ut prius, ob $DB : DC = BE : CF$

$$CF = a : x \begin{array}{l} - a \\ + b - bx \\ \quad + cx - cxx \\ \quad\quad + exx - ex^3 \\ \quad\quad\quad + fx^3 - fx^4 \\ \quad\quad\quad\quad + gx^4 - gx^5 \\ \quad\quad\quad\quad\quad + hx^5 - hx^6 \\ \quad\quad\quad\quad\quad\quad \text{\&c.} \end{array}$$

$=$, ob identitatem relationis punctorum in curva,

$$\begin{array}{l} + a \\ + b - bx \\ + c - 2cx + cxx \\ + e - 3ex + 3exx - ex^3 \\ + f - 4fx + 6fxx - 4fx^3 + fx^4 \\ + g - 5gx + 10gxx - 10gx^3 + 5gx^4 - gx^5 \\ + h - 6hx + 15hxx - 20hx^3 + 15hx^4 - 6hx^5 + hx^6 \\ \text{\&c.} \end{array}$$

unde comparando dimensionum æqualium coëfficientes, reperientur $a = 0$, $c + e + f + g + h = 0$, $-3c - 3e - 4f - 5g - 6h = 0$, $+2c + 2e + 6f + 10g + 15h = 0$, $-5f - 10g - 20h = 0$, $+2f + 4g + 15h = 0$, $-7h = 0$, $-2h = 0$; Ex quo concluditur litteras b, c, e, esse arbitrarias, ceteræ vero erunt $a = 0$, $f = -2c - 2e$, $g = c + e$, $h = 0$; ideoque substitutis hisce valoribus, habebitur æquatio quæsita pro natura curvæ, talis $y = bx + cxx + ex^3 - (2c - 2e)x^4 + (c + e)x^5$. Rursum inde patet, quod hæc Series continuari possit quantum placuerit, ita ut infinita genera curvarum, præter Circulum, e vestigio exhiberi possint, quæ lineas rectas per punctum quoddam datum ductas capiant æquales.

Nota, quod punctum D possit etiam, imo nonnunquam debeat esse extra A, G, sed tunc $CD + DB$ statuitur $=$ unitati.

Conferatur N^rus^. XXX, pag. 158, *Tom. I.* ubi Solutio hujus Problematis exstat, sed sine Demonstratione.

DE ANALYSI INFINITORUM VARIA.

****

N°. CLVI.

Modus resolvendi æquationem $(ax + by)\,dx + (cx + ey)\,dy = 0$, *per logarithmos circulares, sine præcedente separatione indeterminatarum.*

Conferatur Nrus. CXXXVI. *Tom. III*, pag. 108.

Ponatur $\pi\int\frac{(\alpha^2 ydx - \alpha^2 xdy).(\beta-\varepsilon)}{\alpha\alpha(x+\beta y)^2+(x+\varepsilon y)^2} + l(\alpha\alpha(x+\beta y)^2+(x+\varepsilon y)^2) = C.$
Differentietur hæc æquatio ut habeatur

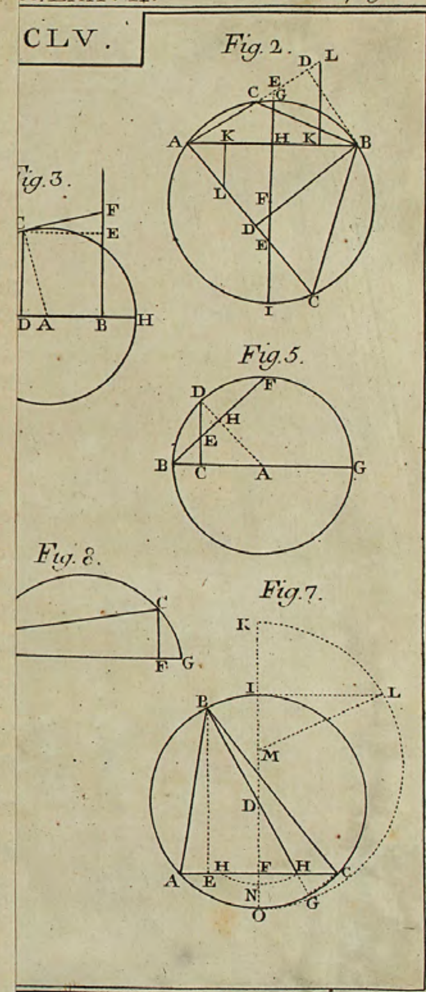
$$\begin{array}{llll} + 2\alpha\alpha xdx & + 2\alpha\alpha\beta ydx & + 2\alpha\alpha\beta xdy & + 2\alpha\alpha\beta\beta ydy = 0 \\ + 2xdx & + 2\varepsilon ydx & + 2\varepsilon xdy & + 2\varepsilon\varepsilon ydy \\ + (\beta\pi - \varepsilon\pi)\alpha^2 ydx & & + \alpha^2 xdy.(-\beta\pi + \varepsilon\pi) & \end{array}$$

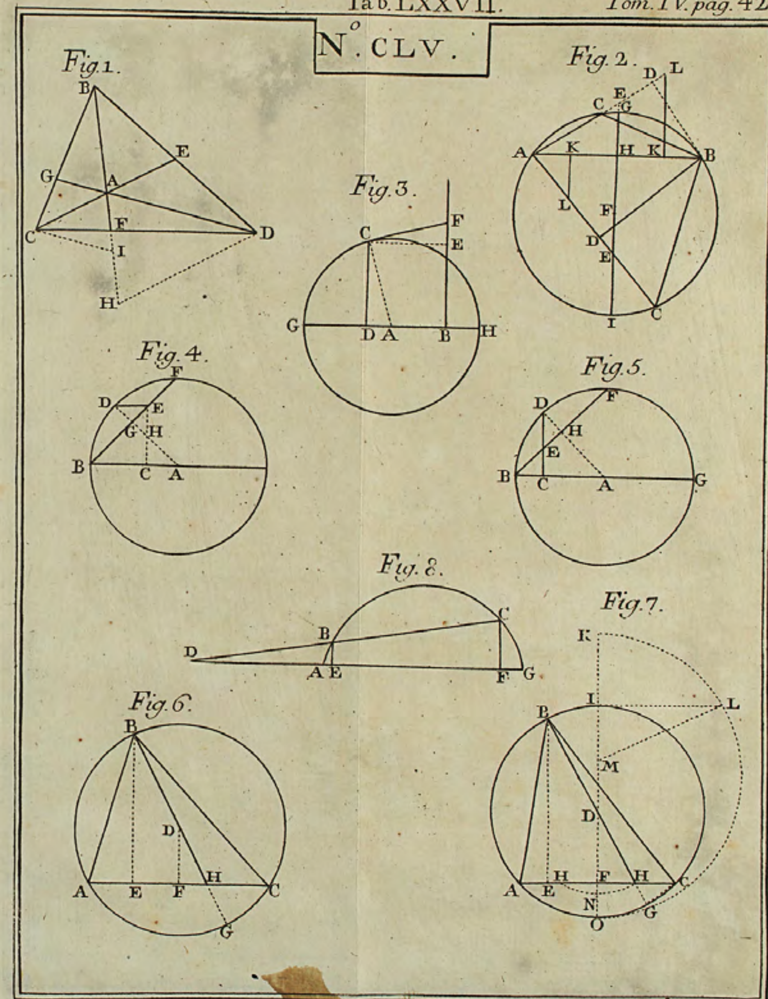
Hujus termini comparentur cum terminis homogeneis æquationis propositæ; prodibunt valores litterarum assumtarum $\alpha, \beta, \varepsilon, \pi$; Tot enim occurrunt comparationes instituendæ, quot sunt assumtæ litteræ. Earum itaque valores, substituti in æquatione supposita, formabunt eam convenientem æquationi propositæ &c.
Superest ut ostendam quantitatem $\pi\int\frac{(\alpha\alpha ydx - \alpha\alpha xdy).(\beta-\varepsilon)}{\alpha\alpha.(x+\beta y)^2+(x+\varepsilon y)^2}$ exprimere arcum circuli; quod sic facio:

$$\frac{(\alpha\alpha ydx - \alpha\alpha xdy).(\beta-\varepsilon)}{\alpha\alpha.(x+\beta y)^2+(x+\varepsilon y)^2} = \frac{\frac{(\alpha\alpha ydx - \alpha\alpha xdy)}{(x+\beta y)^2}\times(\beta-\varepsilon)}{\alpha\alpha+\left(\frac{x+\varepsilon y}{x+\beta y}\right)^2}.$$

Est autem numerator hujus fractionis differentiale ipsius $\frac{x+\varepsilon y}{x+\beta y}$ multiplicatum per α^2: Quare descriptus arcus circuli radio α, cujus tangens $= \frac{x+\varepsilon y}{x+\beta y}$, erit $= \int\frac{(\alpha^2 ydx - \alpha^2 xdy).(\beta-\varepsilon)}{\alpha\alpha.(x+\beta y)^2+(x+\varepsilon y)^2}$, qui arcus π vicibus sumtus, & additus logarithmo $l(\alpha\alpha(x+\beta y)^2 + (x+\varepsilon y)^2)$ dabit æquationem integralem propositæ canonicæ differentialis primi gradus $(ax + by)\,dx + (cx + ey)\,dy = 0.$

SCHO-




N°. CLV.
Fig. 1.
Fig. 2.
Fig. 3.
Fig. 4.
Fig. 5.
Fig. 6.
Fig. 7.
Fig. 8.

D

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$\beta = (b +$
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$+ l\frac{1}{2}a.(x$

assumta for

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cus circuli

$= (y\sqrt{}$
sub signo
$+ \frac{(b+c)}{2}$

erit æquati

$+ l(\frac{1}{2}axx$

SCHOLION.

Poteſt, brevitatis gratia, in $(x+\varepsilon y)^2$, negligi x, & poni ſimpliciter $\pi\int\frac{\alpha^2\varepsilon xdy-\alpha^2\varepsilon ydx}{\alpha\alpha(x+\beta y)^2+\varepsilon\varepsilon yy}+l(\alpha\alpha(x+\beta y)^2+\varepsilon\varepsilon yy)=C.$ Quæ differentiata dat hanc $2\alpha\alpha xdx+2\alpha\alpha\beta ydx+2\alpha\alpha\beta xdy+2\alpha\alpha\beta\beta ydy=0$
$-\pi\alpha^2\varepsilon ydx+\pi\alpha^2\varepsilon xdy+2\varepsilon\varepsilon ydy$, comparandam terminotenus cum $axdx+bydx+cxdy+eydy=0$; faciendo ſcilicet $2\alpha\alpha=a$, $2\alpha\alpha\beta-\pi\alpha^2\varepsilon=b$, $2\alpha\alpha\beta+\pi\alpha^2\varepsilon=c$, $2\alpha\alpha\beta\beta+2\varepsilon\varepsilon=e$. Ex prima habetur $\alpha=\surd\frac{1}{2}a$, ex ſecunda & tertia $4\alpha\alpha\beta=b+c$, & $2\pi\alpha^2\varepsilon=c-b$, unde porro fluit $\beta=(b+c):2a$, $2\alpha\alpha\beta\beta=(b+c)^2:4a$, adeoque ex quarta $(b+c)^2:4a+2\varepsilon\varepsilon=e$, hinc $\varepsilon^2=(4ae-(b+c)^2):8a$. Quoniam vero $2\pi\alpha^2\varepsilon=c-b$, inde $\pi=(c-b):2\alpha^2\varepsilon=(2c-2b):\surd(2aae-\frac{1}{2}a(b+c)^2)$. Inveniuntur itaque litterarum aſſumtarum hi quatuor valores $\alpha=\surd\frac{1}{2}a$, $\beta=(b+c):2a$, $\varepsilon=\surd((4ae-(b+c)^2):8a)$, $\pi=(2c-2b):\surd(2aae-\frac{1}{2}a(b+c)^2)$. His ſubſtitutis emergit quæſita æquatio in terminis finitis hæc

$$\frac{2c-2b}{\surd(2aae-\frac{1}{2}a.(b+c)^2)}\int\frac{\frac{1}{4}\surd(2aae-\frac{1}{2}a.(b+c)^2).(xdy-ydx)}{\frac{1}{2}a.(x+\frac{b+c}{2a}y)^2+\frac{4ae-(b+c)^2}{8a}yy}$$

$+l\frac{1}{2}a.(x+\frac{b+c}{2a}y)^2+\frac{4ae-(b+c)^2}{8a}yy=C.$ Eſt enim in aſſumta formula $\frac{\alpha^2\varepsilon xdy-\alpha^2\varepsilon ydx}{\alpha\alpha.(x+\beta y)^2+\varepsilon\varepsilon yy}$ idem quod $(\frac{\alpha^2\varepsilon xdy-\alpha^2\varepsilon ydx}{(x+\beta y)^2})$: $(\alpha\alpha+\frac{\varepsilon\varepsilon yy}{(x+\beta y)^2})$ id quod nihil aliud eſt, quam elementum arcus circuli, cujus radius $=\alpha=\surd\frac{1}{2}a$, & tangens $=\frac{\varepsilon y}{x+\beta y}=(y\surd\frac{(4ae-(b+c)^2}{8a}):(x+\frac{b+c}{2a}y)$. Porro quantitas ſub ſigno logarithmicali rite ordinata abit in hanc $l(\frac{1}{2}axx+\frac{(b+c)^2}{2}xy+\frac{1}{2}eyy)$; Ergo ſi prædictus arcus dicatur $=A$, erit æquatio finita ſatisfaciens canonicæ hæc $\frac{2c-2b}{\surd(2aae-\frac{1}{2}a.(b+c)^2)}\times A$ $+l(\frac{1}{2}axx+\frac{b+c}{2}xy+\frac{1}{2}eyy)=C.$

COROLLARIUM.

Hinc liquet $4ae$ debere esse $> (b+c)^2$ ut hæc methodus succedat. Altera vero, quæ per meros logarithmos peragitur requirit contrarium, ut nimirum $4ae$ sit $< (b+c)^2$: adeoque quodlibet particulare exemplum per alterutram methodum solvi potest, excepto unico casu quo $4ae = (b+c)^2$. Hunc autem in scripto *Petropolin* misso §. 19 *, peculiari modo solutum dedi, monstrans eum lineæ rectæ convenire.

Applicatio ad Exemplum de curva velocitatum in Cycloide. Vid. N^us. CLXXXIII, infra.

Æquatio hujus curvæ est $sds - gvds + vdv = 0$. Supponendo igitur $x = s$ & $y = v$, erit hic $a = 1, b = -g, c = 0, e = 1$. Quibus substitutis in æquatione finita generali, prodit $\frac{2g}{\sqrt{(2 - \frac{1}{2}gg)}} \times A + l(\frac{1}{2}ss - \frac{1}{2}gsv + \frac{1}{2}vv) = C$, quæ evanescente v, dat $\frac{2g}{\sqrt{(2 - \frac{1}{2}gg)}} \times 0 + 2\,l\alpha - l2 = C$, quia tangens arcus A etiam evanescit, & s evadit = longitudini totius descensus, quam vocavi α, sicuti longit. ascensus $= \beta$. Unde vera æquatio correcta hæc est $\frac{2g}{\sqrt{(2 - \frac{1}{2}gg)}} \times A + l(ss - gsv + vv) - 2\,l\alpha = 0$, quæ evanescente v & ipsa tota evanescet. Sed existente $s = 0$, tangens arcus A fit $= \frac{\sqrt{(2 - \frac{1}{2}gg)}}{-g}$, ubi notandum etiamsi hic tangens sit negativa, ideo arcum ipsum non esse negativum, sed potius quadrante majorem, & affirmativum, seu complementum ad duos quadrantes ejus qui quadrante minor pro tangente habet $\frac{\sqrt{(2 - \frac{1}{2}gg)}}{+g}$. Vocetur igitur hic arcus quadrante major $= Q + B$. Et erit in casu $s = 0$, $\frac{2g}{\sqrt{(2 - \frac{1}{2}gg)}} \times (Q + B) + 2\,lv - 2\,l\alpha = 0$,

* N°. CXXXVI, *Tom. III*, pag. 121.

unde $lv = l\alpha - \frac{g}{\sqrt{(2 - \frac{1}{2}gg)}} \times (Q + B)$. Haud absimili ratiocinio concluditur, pro ascensu fore $lv = l\beta + \frac{g}{\sqrt{(2 - \frac{1}{2}gg)}} \times (Q - B)$; Qui duo valores consentire deprehenduntur cum illis, quos per separationem indeterminatarum invenimus *; sicuti patebit attentione facta ad id quod in uno modo assumtus sit radius $= 1$, in altero vero $= \sqrt{\frac{1}{2}}$; & quod in uno est B, id in altero sit complementum ad quadrantem.

NOTA. Poterat pro multiplo arcus simpliciter poni .... $\int \frac{\pi x dy - \pi y dx}{\alpha\alpha.(x + \beta y)^2 + \varepsilon\varepsilon yy}$; hic enim pariter habentur quatuor assumtæ litteræ $\alpha, \beta, \varepsilon, \pi$, adeoque sufficientes ad quatuor comparationes instituendas. Et ita semper invenitur tangens arcus simpli, qui aliquoties sumtus dat $\int \frac{\pi x dy - \pi y dx}{\alpha\alpha.(x + \beta y)^2 + \varepsilon\varepsilon yy}$. Est enim $\frac{\pi x dy - \pi y dx}{\alpha\alpha.(x + \beta y)^2 + \varepsilon\varepsilon yy} = \left(\frac{\pi x dy - \pi y dx}{(x + \beta y)^2}\right) : \left(\alpha\alpha + \frac{\varepsilon\varepsilon yy}{(x + \beta y)^2}\right)$ $\frac{\pi}{\alpha\alpha\varepsilon}\left(\frac{\alpha\alpha\varepsilon x dy - \alpha\alpha\varepsilon y dx}{(x + \beta y)^2}\right) : \left(\alpha\alpha + \frac{\varepsilon\varepsilon yy}{(x + \beta y)^2}\right)$. Est vero numerator hujus fractionis differentiale ipsius $\frac{\varepsilon y}{x + \beta y}$ multiplicatum per $\alpha\alpha$, sumtum $\frac{\pi}{\alpha\alpha\varepsilon}$ vicibus. Adeoque $\int \frac{\pi x dy - \pi y dx}{\alpha\alpha.(x + \beta y)^2 + \varepsilon\varepsilon yy}$ designabit arcum circuli $\frac{\pi}{\alpha\alpha\varepsilon}$ vicibus sumti, cujus radius $= \alpha$, & tangens $= \frac{\varepsilon y}{x + \beta y}$. Quare, si statuatur $\pi \int \frac{x dy - y dx}{\alpha\alpha.(x + \beta y)^2 + \varepsilon\varepsilon yy} + l(\alpha\alpha.(x + \beta y)^2 + \varepsilon\varepsilon yy) = C$, dabit ea differentiata,

$$\begin{array}{l} 2\alpha\alpha x dx + 2\alpha\alpha\beta y dx + 2\alpha\alpha\beta x dy + 2\alpha\alpha\beta^2 y dy = 0 \\ \qquad - \pi y dx \quad + \pi x dy \quad + 2\varepsilon\varepsilon y dy \end{array}$$

comparandam cum proposita $axdx + bydx + cxdy + eydy = 0$. Faciendo ergo $2\alpha\alpha = a, 2\alpha\alpha\beta - \pi = b, 2\alpha\alpha\beta + \pi = c, 2\alpha\alpha\beta\beta + 2\varepsilon\varepsilon = e$, & quærendo valores litterarum, invenietur nunc $\alpha = \sqrt{\frac{1}{2}a}$, $\beta = (b+c) : 2a, \pi = \frac{1}{2}c - \frac{1}{2}b; \varepsilon = \sqrt{(4ae - (b+c)^2)} : \sqrt{8a}$.

 His

* N°. CLXXXIII, infra.

His ergo valoribus ſubſtitutis in formula habetur $\frac{c-b}{2}$

$$\int \frac{x\,dy - y\,dx}{\frac{1}{2}a(x+\frac{b+c}{2a}y)^2 + \frac{4ae-(b+c)^2}{8a}yy} + l(\tfrac{1}{2}a(x+\frac{(b+c)}{2a}y)^2$$

$+\frac{4ae-(b+c)^2}{a}yy = C.$ Reliqua fiunt ut ſupra.

Propoſita jam ſit æquatio canonica ſecundi ordinis, cujus reſolutio pendeat a logarithmis partim, partim a rectificatione arcus circularis, $(axx+bxy+cyy)dx+(exx+fxy+gyy)dy=0$.

Hic jam pono hanc integralem ſatisfacere $\pi\int\frac{x\,dy-y\,dx}{\alpha\alpha.(x+\beta y)^2+\varepsilon\varepsilon yy}$ $+\lambda l(x+\mu y)+l((x+\mu y)\times\alpha\alpha.(x+\beta y)^2+\varepsilon\varepsilon yy)=C.$ Differentiata enim dat æquationem etiam ſecundi ordinis, & adſunt ſex litteræ aſſumtæ pro totidem comparationibus inſtituendis. Adeoque reſolvetur propoſita.

Propoſita æquatio canonica tertii ordinis poſtulat, ut integralis hoc pacto ponatur: $\pi\int\frac{x\,dy-y\,dx}{\alpha\alpha.(x+\beta y)^2+\varepsilon\varepsilon yy}+\phi\int\frac{x\,dy-y\,dx}{\gamma\gamma.(x+\sigma y)^2+\theta\theta yy}$ $+l(\alpha\alpha.(x+\beta y)^2+\varepsilon\varepsilon yy)\times(\gamma\gamma.(x+\delta y)^2+\theta\theta yy))=C.$ Et ita habentur octo aſſumtæ litteræ pro totidem comparationibus faciendis.

Pro canonica quarti ordinis, addenda eſt ad aſſumtam adhuc $\lambda l(x+\mu y)$. Et ita procedendum in infinitum. Hac methodo poteſt æquatio canonica $(ax+by)\,dx+(cx+ey)\,dy=0$, reſolvi per meros logarithmos, ubi rei natura hoc permittit, ut faciliori forſan modo perveniatur ad æquationem exponentialem, quam edocui in *Commentariis Petropolit.* * In hunc finem ita procedo. Fiat $\pi\int\frac{x\,dy-y\,dx}{\alpha\alpha.(x+\beta y)^2-\varepsilon\varepsilon yy}+l(\alpha\alpha.(x+\beta y)^2$ $-\varepsilon\varepsilon yy)=C.$ Quod ſi differentietur, habebitur

$$\begin{matrix}2\alpha\alpha x\,dx+2\alpha\alpha\beta y\,dx+2\alpha\alpha\beta x\,dy+2\alpha\alpha\beta^2 y\,dy=0,\\ -\ \pi y\,dx\ +\ \pi x\,dy\ -\ 2\varepsilon\varepsilon y\,dy\end{matrix}$$

compa-

* N°. CXXXV, *Tom. III*, pag. 108.

comparanda cum propoſita $ax\,dx+by\,dx+cx\,dy+ey\,dy=0$, ubi nulla eſt alia differentia, quam quod hic ſemper habeatur $-\varepsilon\varepsilon$ pro $+\varepsilon\varepsilon$. Faciendo ergo $2\alpha\alpha=a$, $2\alpha\alpha\beta-\pi=b$, $2\alpha\alpha\beta+\pi=c$; nunc vero $2\alpha\alpha\beta\beta-2\varepsilon\varepsilon=e$, & quærendo valores litterarum invenietur $\alpha=\sqrt{\tfrac{1}{2}a}$, $\beta=(b+c):2a$, $\pi=\tfrac{1}{2}c-\tfrac{1}{2}b$; ſed $\varepsilon=\sqrt{((b+c)^2-4ae)}:\sqrt{8a}$. His ergo valoribus ſubſtitutis in formula habetur

$$\frac{c-b}{2}\int\frac{x\,dy-y\,dx}{\frac{1}{2}a.(x+\frac{b+c}{2a}y)^2-(\frac{(b+c)^2-4ae}{8a})yy}+$$

$l(\tfrac{1}{2}a(x+\frac{b+c}{2a}y)^2-\frac{(b+c)^2-4ae}{8a}yy)=C.$ Hinc ergo, & ex præcedentibus, patet formulam noſtram pendere ex logarithmis quotieſcunque $(b+c)^2 > 4ae$, ſed ex rectificatione arcus circularis quando $(b+c)^2 < 4ae$. In poſteriori caſu rem jam vidimus; In priori caſu vero res ita peragitur: In aſſumta expreſſione habetur $\frac{x\,dy-y\,dx}{\alpha\alpha.(x+\beta y)^2-\varepsilon\varepsilon yy}=(\frac{x\,dy-y\,dx}{(x+\beta y)^2}):(\alpha\alpha-\frac{\varepsilon\varepsilon yy}{(x+\beta y)^2})$ $=$ [poſito $\frac{\varepsilon y}{x+\beta y}=z$] $\frac{\frac{1}{\varepsilon}dz}{\alpha\alpha-zz}=\frac{1}{2\alpha\alpha\varepsilon}\times\frac{2\alpha\alpha\,dz}{\alpha\alpha-zz}=\frac{1}{2\alpha\alpha\varepsilon}$ $(\frac{\alpha\,dz}{\alpha+z}+\frac{\alpha\,dz}{\alpha-z})$. Ergo integrando & multiplicando per π, prodibit $\pi\int\frac{x\,dy-y\,dx}{\alpha\alpha.(x+\beta y)^2-\varepsilon\varepsilon yy}=\frac{\pi}{2\alpha\varepsilon}l\frac{\alpha+z}{\alpha-z}$, & reponendo pro z ejus valorem $\frac{\varepsilon y}{x+\beta y}$, prodibit $\pi\int\frac{x\,dy-y\,dx}{\alpha\alpha.(x+\beta y)^2-\varepsilon\varepsilon yy}=\frac{\pi}{2\alpha\varepsilon}$ $l((\alpha+\frac{\varepsilon y}{x+\beta y}):(\alpha-\frac{\varepsilon y}{x+\beta y}))=\frac{\pi}{2\alpha\varepsilon}l\frac{\alpha x+\alpha\beta y+\varepsilon y}{\alpha x+\alpha\beta y-\varepsilon y}$ Adeoque ſubſtitutis pro $\alpha,\beta,\varepsilon,\pi$ eorum valoribus, habetur æquatio logarithmica hæc $\frac{c-b}{\sqrt{((b+c)^2-4ae)}}\times$

$$l\frac{x\sqrt{\tfrac{1}{2}a}+\frac{b+c}{2\sqrt{2a}}y+y\sqrt{((b+c)^2-4ae)}:\sqrt{8a}}{x\sqrt{\tfrac{1}{2}a}+\frac{b+c}{2\sqrt{2a}}y-y\sqrt{((b+c)^2-4ae)}:\sqrt{8a}}+l(\tfrac{1}{2}a(x+\frac{b+c}{2a}y)^2$$

$-\frac{(b+c)^2-4ae}{8a}yy)=C.$

Quia

Quia autem $\frac{1}{2}a.\left(x+\frac{b+c}{2a}y\right)^2-\frac{(b+c)^2-4ae}{8a}yy=\left(x\sqrt{\tfrac{1}{2}a}+\frac{b+c}{2\sqrt{2a}}y+\frac{y\sqrt{((b+c)^2-4ae)}}{2\sqrt{2a}}\right)\times\left(x\sqrt{\tfrac{1}{2}a}+\frac{b+c}{2\sqrt{2a}}y-\frac{y\sqrt{((b+c)^2-4ae)}}{2\sqrt{2a}}\right)$; poterit æquatio inventa, per $\sqrt{((b+c)^2-4ae)}$ prius multiplicata ita exprimi $(c-b)\,l\left(x\sqrt{\tfrac{1}{2}a}+\frac{b+c}{2\sqrt{2a}}y+\frac{y\sqrt{((b+c)^2-4ae)}}{2\sqrt{2a}}\right)-(c-b)\,l\left(x\sqrt{\tfrac{1}{2}a}+\frac{b+c}{2\sqrt{2a}}y-\frac{y\sqrt{((b+c)^2-4ae)}}{2\sqrt{2a}}\right)+\sqrt{((b+c)^2-4ae)}\,l\left(x\sqrt{\tfrac{1}{2}a}+\frac{b+c}{2\sqrt{2a}}y+\frac{y\sqrt{((b+c)^2-4ae)}}{2\sqrt{2a}}\right)+\sqrt{((b+c)^2-4ae)}\,l\left(x\sqrt{\tfrac{1}{2}a}+\frac{b+c}{2\sqrt{2a}}y-\frac{y\sqrt{((b+c)^2-4ae)}}{2\sqrt{2a}}\right)=C$. Habebitur transeundo a logarithmis ad numeros, $\left(x\sqrt{\tfrac{1}{2}a}+\frac{b+c}{2\sqrt{2a}}y+\frac{y\sqrt{((b+c)^2-4ae)}}{2\sqrt{2a}}\right)^{c-b+\sqrt{((b+c)^2-4ae)}}\times\left(x\sqrt{\tfrac{1}{2}a}+\frac{b+c}{2\sqrt{2a}}y-\frac{y\sqrt{((b+c)^2-4ae)}}{2\sqrt{2a}}\right)^{-c+b+\sqrt{((c+b)^2-4ae)}}=C$. Et omnia sub parenthesibus multiplicando per $2\sqrt{2a}$; erit $\left(2ax+(b+c)y+y\sqrt{((b+c)^2-4ae)}\right)^{c-b+\sqrt{((b+c)^2-4ae)}}\times\left(2ax+(b+c)y-y\sqrt{((b+c)^2-4ae)}\right)^{-c+b+\sqrt{((c+b)^2-4ae)}}=C$: vel, brevitatis gratia, scribendo m pro $\sqrt{((b+c)^2-4ae)}$, prodit tandem hæc æquatio $(2ax+(b+c+m)y)^{c-b+m}$ $(2ax+(b+c-m)y)^{-c+b+m}=C$; eodem prorsus modo ut inveni in priori Schediasmate edito in *Comment. Petropolitanis* *.

* N°. CXXXV, *Tom. III*, pag. 120.

PROBLE-

N°. CLVII.

PROBLEMA.

Invenire conditiones separabilitatis differentialium in æquationibus hujus formæ $as^m ds+bu^q s^p ds=du$.

SOLUTIO.

Cas. 1. Si $m=p$, patet separari posse dividendo per $a+bu^q$.

Cas. 2. Fiat $u=z^\alpha$, adeoque $du=\alpha z^{\alpha-1}dz$; quibus substitutis prodit $as^m ds+bz^{\alpha q}s^p ds=\alpha z^{\alpha-1}dz$; ubi si ponatur $m=\alpha q+p=\alpha-1$ [ut nimirum indeterminatæ s & z eundem in omnibus terminis dimensionis numerum efficiant, adeoque per regulam generalem separabiles evadant] habebitur $m=(p+q):(1-q)$.

Cas. 3. Fiat $u=xy$, adeoque $du=xdy+ydx$; per substitutionem provenit $as^m ds+bx^q y^q s^p ds=xdy+ydx$. Ponatur $bx^q y^q s^p ds=xdy$, & $as^m ds=ydx$. Ut vero ubique æquationi satisfiat, sit $q=1$; mutabitur prior in hanc $bys^p ds=dy$, ac proinde $bs^p ds=y^{-1}dy$; datur itaque y per s: sit ergo $y=S$; quo substituto in altera $as^m ds=ydx$, oritur $as^m ds=Sdx$, adeoque $\frac{as^m}{S}ds=dx$; dabitur ergo etiam x per S, sit igitur $x=\Sigma$: unde [existente $q=1$] erit u $[xy]=S\Sigma$. Hac methodo jam usus sum in *Actis Lips.* A. 1697, *p.* 115 *, pro solutione æquationis similis a Fratre meo mihi propositi.

* N°. XXXV, *Tom. I*, pag. 175.

Caf. 4. Fiat $s = z^{\alpha}$, adeoque $ds = \alpha z^{\alpha - 1} dz$; quæ ſubſtituantur, & orietur $\alpha a z^{\alpha m + \alpha - 1} dz + \alpha b u^{q} z^{\alpha p + \alpha - 1} dz = du$; quare ut indeterminatæ ubique eaſdem dimenſiones obtineant, ponendum eſt $\alpha m + \alpha - 1 = 0 = q + \alpha p + \alpha - 1$; ex priori habetur $m = (1 - \alpha) : \alpha$; ex altera vero $q = 1 - \alpha - \alpha p$; ubi notandum numerum α eſſe arbitrarium. Ex. gr. aſſumta $\alpha = \frac{1}{2}$, erit $m = 1$, & $q = (1 - p) : 2$. Hic caſus ex ſecundo etiam deduci poteſt, atque ſi eliminetur α, ad eandem conditionem pervenietur.

Caf. 5. Dantur etiam caſus, in quibus æquatio propoſita ad algebraicam omnino reduci poteſt. Hoc modo: $as^{m} ds + bu^{q} s^{p} ds = du = \alpha du + (1 - \alpha) du$. Ponatur $as^{m} ds = \alpha du$, & $bu^{q} s^{p} ds = (1 - \alpha) du$, ſeu $bs^{p} ds = (1 - \alpha) u^{-q} du$; per utriuſque integrationem provenit $\frac{a}{m+1} s^{m+1} = \alpha u$, & $\frac{b}{p+1} s^{p+1} = \frac{1 - \alpha}{-q+1} u^{-q+1}$; prior dat $u = \frac{a}{\alpha(1+m)} s^{m+1}$, altera vero $u = \left(\frac{b(1-q)}{(1-\alpha)(1+p)}\right)^{1:(1-q)} s^{(1+p):(1-q)}$. Ut igitur hi duo valores ipſius u identificentur, coæquandæ ſunt tam dimenſiones, quam coëfficientes; h. e. fiat $m + 1 = (1 + p) : (1 - q)$ & $a : \alpha(1 + m) = \left(\frac{b(1-q)}{(1-\alpha)(1+p)}\right)^{1:(1-q)}$; unde primo habetur $m = (p + q) : (1 - q)$; quæ eſt eadem conditio jam *Caf.* 2, & 4, inventa pro ſeparabilitate tantum. Secundo erit $\left(\frac{a}{\alpha.(1+m)}\right)^{1-q} = \frac{b.(1-q)}{(1-\alpha).(1+p)}$, vel ſubſtituto valore ipſius $1 + m$, $\left(\frac{a.(1-q)}{\alpha.(1+p)}\right)^{1-q} = \frac{b.(1-q)}{(1-\alpha).(1+p)}$, hoc eſt $\left(\frac{a}{\alpha}\right)^{1-q} \times \left(\frac{1-q}{1+p}\right)^{1-q} = \frac{b}{1-\alpha} \times \frac{1-q}{1+p}$, adeoque $\left(\frac{a}{\alpha}\right)^{1-q} \times \left(\frac{1-q}{1+p}\right)^{-q} = \frac{b}{1-\alpha}$; reducta æquatione habetur

$b\alpha^{1-q} = (1 - \alpha) \times a^{1-q} \times \left(\frac{1-q}{1+p}\right)^{-q} = (1 - \alpha) \times a^{1-q} \times (1-q)^{-q} \times (1+p)^{q}$. Cujus radix α determinat quæſitum.

Ex quo patet æquationem $as^{m} ds + bu^{q} s^{p} ds = du$ [quando nempe $m = (p+q) : (1 - q)$,] non tantum ad ſeparabilitatem differentialium reduci, ſed omnino algebraice exprimi poſſe, ei quippe reſpondet hæc $\frac{a}{\alpha.(1+m)} s^{1+m} = u$; adeo ut curva, cujus natura per dictam æquationem differentialem determinatur, eſſe poſſit ex Parabolarum genere.

ALITER ET FACILIUS.

Ponatur æquationem propoſitæ reſpondentem eſſe $\pi s^{\varsigma} = u$, erit $\varsigma \pi s^{\varsigma - 1} ds = du$. Subſtituendo in propoſita loco u^{q} & du ipſarum valores, mutabitur in hanc $as^{m} ds + b\pi^{q} s^{\varsigma q + p} ds = \varsigma \pi s^{\varsigma - 1} ds$, h. e. $as^{m} + b\pi^{q} s^{\varsigma q + p} = \varsigma \pi s^{\varsigma - 1}$; quæ ut fiant perfecte æqualia, faciendum eſt $m = \varsigma q + p = \varsigma - 1$, & $a + b\pi^{q} = \varsigma \pi$; ex priori venit $m = (p + q) : (1 - q)$ & $\varsigma = (1 + p) : (1 - q)$, ut ante; altera vero $b\pi^{q} - \varsigma\pi + a = 0$, dat $b\pi^{q} + \frac{-1-p}{1-q}\pi + a = 0$, cujus radix π erit parameter Parabolæ quæſitæ: adeoque in caſu quo $m = (p + q) : (1 - q)$, æquationi propoſitæ $as^{m} ds + bu^{q} s^{p} ds = du$ ſatisfacit hæc algebraïca $\pi s^{(1+p):(1-q)} = u$.

N°. CLVIII.

FORMULÆ REDUCTIONUM.

Videantur N^ri. CXIV & CXV, *Tom. II*, pag. 402.

I. $\int \frac{dx}{(e+fx^q)^n} = \int \frac{1}{(e+fx^q)^n} \times dx = \frac{x}{(e+fx^q)^n} + nq\int \frac{fx^q\,dx}{(e+fx^q)^{n+1}}$ $= \frac{x}{(e+fx^q)^n} + nq\int \frac{e+fx^q}{(e+fx^q)^{n+1}}\,dx - nq\int \frac{e\,dx}{(e+fx^q)^{n+1}} =$ $\frac{x}{(e+fx^q)^n} + nq\int \frac{dx}{(e+fx^q)^n} - nq\int \frac{e\,dx}{(e+fx^q)^{n+1}}$: Unde nqe $\int \frac{dx}{(e+fx^q)^{n+1}} = \frac{x}{(e+fx^q)^n} + (nq-1)\int \frac{dx}{(e+fx^q)^n}$.

Coroll. I. Existente $nq = 1$, erit $\frac{dx}{(e+fx^q)^{n+1}}$ absolute integrabile, utpote cujus integrale $= \frac{1}{e} \times \frac{x}{(e+fx^q)^n}$.

Coroll. II. $\int \frac{dx}{(e+fx^q)^{n+1}}$ dependet a $\int \frac{dx}{(e+fx^q)^n}$, & per eandem rationem $\int \frac{dx}{(e+fx^q)^n}$ a $\int \frac{dx}{(e+fx^q)^{n-1}}$, & ita porro; usquedum, supposito n numero integro & affirmativo, perveniatur ad $\int \frac{dx}{e+fx^q}$; a quo per consequens dependet $\int \frac{dx}{(e+fx^q)^n}$ nisi fortassis $(n-1).\,q = 1$; quo casu esset hoc absolute integrabile, ut & quicquid ab hoc dependet, nempe $\int \frac{dx}{(e+fx^q)^{n+m}}$.

II. $\int \frac{e\,dx}{e+fx^q} = \int \frac{e+fx^q}{e+fx^q}\,dx - \int \frac{fx^q\,dx}{e+fx^q} = x - \int \frac{fx^q\,dx}{e+fx^q}$;

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adeoque $f\int \frac{x^q\,dx}{e+fx^q} = x - e\int \frac{dx}{e+fx^q}$; hinc ergo $\int \frac{x^q\,dx}{e+fx^q}$ dependet a $\int \frac{dx}{e+fx^q}$.

III. $\int \frac{e\,dx}{(e+fx^q)^2} = \int \frac{e+fx^q}{(e+fx^q)^2}\,dx - \int \frac{fx^q\,dx}{(e+fx^q)^2} =$ $\int \frac{dx}{e+fx^q} - \int \frac{fx^q\,dx}{(e+fx^q)^2}$; unde $f\int \frac{x^q\,dx}{(e+fx^q)^2} = \int \frac{dx}{e+fx^q} -$ $e\int \frac{dx}{(e+fx^q)^2}$; = [per primam] $\frac{1}{q}\int \frac{dx}{e+fx^q} - \frac{1}{q} \times \frac{x}{e+fx^q}$; adeoque $fq\int \frac{x^q\,dx}{(e+fx^q)^2} = \frac{-x}{e+fx^q} + \int \frac{dx}{e+fx^q}$. Hinc igitur $\int \frac{x^q\,dx}{(e+fx^q)^2}$ dependet a $\int \frac{dx}{e+fx^q}$.

IV. $\int \frac{dx}{(e+fx^q)^{n+1}} = \frac{1}{e}\int \frac{e+fx^q}{(e+fx^q)^{n+1}}\,dx - \frac{1}{e}\int \frac{fx^q\,dx}{(e+fx^q)^{n+1}}$ $= \frac{1}{e}\int \frac{dx}{(e+fx^q)^n} - \frac{f}{e}\int \frac{x^q\,dx}{(e+fx^q)^{n+1}}$. Itaque $\frac{f}{e}\int \frac{x^q\,dx}{(e+fx^q)^{n+1}}$ $= \frac{1}{e}\int \frac{dx}{(e+fx^q)^n} - \int \frac{dx}{(e+fx^q)^{n+1}}$ = [per primam] $\frac{1}{e}\int \frac{dx}{(e+fx^q)^n} - \frac{1}{nqe} \times \frac{x}{(e+fx^q)^n} - \frac{nq-1}{nqe} \times \int \frac{dx}{(e+fx^q)^n} =$ $\frac{1}{nqe}\int \frac{dx}{(e+fx^q)^n} - \frac{1}{nqe} \times \frac{x}{(e+fx^q)^n}$; adeoque $nfq \times \int \frac{x^q\,dx}{(e+fx^q)^{n+1}}$ $= \int \frac{dx}{(e+fx^q)^n} - \frac{x}{(e+fx^q)^n}$. Hinc ergo $\int \frac{x^q\,dx}{(e+fx^q)^{n+1}}$ dependet a $\int \frac{dx}{(e+fx^q)^n}$.

V. Quia

V. Quia $\int \frac{x^q\,dx}{(e+fx^q)^{n-1}} = \int \frac{ex^q + fx^{2q}}{(e+fx^q)^n}\,dx = e\int \frac{x^q\,dx}{(e+fx^q)^n} + f\int \frac{x^{2q}\,dx}{(e+fx^q)^n}$; patet $\int \frac{x^{2q}\,dx}{(e+fx^q)^n} = \frac{1}{f}\int \frac{x^q\,dx}{(e+fx^q)^{n-1}} - \frac{e}{f}\int \frac{x^q\,dx}{(e+fx^q)^n}$, hoc eſt per præced. a $\int \frac{dx}{e+fx^q}$ dependere.

VI. Eodem modo demonſtratur $\int \frac{x^{3q}\,dx}{(e+fx^q)^n}$, $\int \frac{x^{4q}\,dx}{(e+fx^q)^n}$ & [ſumto p pro numero integro & affirmativo] generaliter $\int \frac{x^{pq}\,dx}{(e+fx^q)^n}$ dependere a $\int \frac{dx}{e+fx^q}$. Quod eſt THEOR. V *.

VII. Quia, per Coroll. I, Prop. I, exiſtente $nq = 1$; erit $\frac{dx}{(e+fx^q)^{n+1}}$ abſolute integrabile, & quia per ipſam Prop. I, $\int \frac{dx}{(e+fx^q)^{n+2}}$ dependet a $\int \frac{dx}{(e+fx^q)^{n+1}}$, atque $\int \frac{dx}{(e+fx^q)^{n+3}}$ a $\int \frac{dx}{(e+fx^q)^{n+2}}$, & ita deinceps, ſequitur [aſſumto k pro quolibet numero integro affirmativo, excepta cyphra] fore $\int \frac{dx}{(e+fx^q)^{1:q+k}}$ abſolute quadrabilem. Quod eſt THEOR. I. †

VIII. Quia etiam, per Propoſitionem IV, $nfq\int \frac{x^q dx}{(e+fx^q)^{n+1}} = \int \frac{dx}{(e+fx^q)^n} - \frac{x}{(e+fx^q)^n}$, exiſtente vero $nq - q = 1$; per Coroll. II, eſt $\int \frac{dx}{(e+fx^q)^n}$ abſolute quadrabilis; erit per con-

* N°. CXIV., *Tom. II*, pag. 418. † Ibid. pag. 417.

ſequens etiam $\int \frac{x^q\,dx}{(e+fx^q)^{(2q+1):q}}$ abſolute quadrabilis.

IX. Quia $\int \frac{x^{2q}\,dx}{(e+fx^q)^n} = \frac{1}{f}\int \frac{x^q\,dx}{(e+fx^q)^{n-1}} - \frac{e}{f}\int \frac{x^q\,dx}{(e+fx^q)^n}$, per V; hæ vero partes ſimul non poſſunt eſſe quadrabiles; ſequitur $\int \frac{x^{2q}\,dx}{(e+fx^q)^n}$ non eſſe quadrabilem. Idem etiam valet de $\int \frac{x^{3q}\,dx}{(e+fx^q)^n}$, $\int \frac{x^{4q}\,dx}{(e+fx^q)^n}$, & in genere de $\int \frac{x^{pq}\,dx}{(e+fx^q)^n}$; quamvis interim ſingulæ dependeant a $\int \frac{dx}{e+fx^q}$.

X. $\int \frac{dx}{(e+fx^q)^{1:q+k}} = \int \frac{(e+fx^q)\,dx}{(e+fx^q)^{1:q+k+1}}$; quia itaque, per VII, totum eſt quadrabile, & per eandem etiam pars $\int \frac{e\,dx}{(e+fx^q)^{1:q+k+1}}$; erit quoque reliquum $f\int \frac{x^q\,dx}{(e+fx^q)^{1:q+k+1}}$ quadrabile.

XI. Pariter $\int \frac{x^q\,dx}{(e+fx^q)^{1:q+k+1}} = \int \frac{(ex^q+fx^{2q})\,dx}{(e+fx^q)^{1:q+k+2}}$. Ergo quia, per præced. totum eſt quadrabile; ut & ablatum; erit etiam quadrabile reliquum $f\int \frac{x^{2q}\,dx}{(e+fx^q)^{1:q+k+2}}$.

XII. Per proceſſus continuationem probabitur generaliter, quadrabilem eſſe hanc Formulam $\int \frac{x^{pq}\,dx}{(e+fx^q)^{1:q+k+p}}$. Quod eſt THEOR. II. *

* Ibid. pag. 417.

XIII. Quia itaque $\int \frac{x^{pq}\,dx}{(e+fx^q)^{1:q+k+p}} =$ [dividendo numeratorem & denominatorem per $x^{1+kq+pq}$] $\int \frac{x^{-kq-1}\,dx}{(ex^{-q}+f)^{1:q+k+p}}$; Ergo nominando e per f, f per e, $-q$ per $+q$, erit $\int \frac{x^{kq-1}\,dx}{(e+fx^q)^{-1:q+k+p}}$ quadrabilis. Quod eſt THEOR. III. & IV. *

XIV. Porro quia $\int \frac{x^{pq}dx}{(e+fx^q)^n} =$ [dividendo numeratorem & denominatorem per x^{qn}] $\int \frac{x^{pq-nq}dx}{(ex^{-q}+f)^n}$; ergo etiam nominando e per f, & f per e, & $-q$ per $+q$, prodibit $\int \frac{x^{nq-pq}\,dx}{(e+fx^q)^n}$; vel, vocando $n-p$ per $-p$, provenit $\int \frac{x^{-pq}\,dx}{(e+fx^q)^n}$ dependens a $\int \frac{dx}{e+fx^q}$. Quod eſt THEOR. VI. † Hæc dudum inventa habui.

NB. *Theorema* VII facile reducitur ad *Theor.* V, ponendo tantum $x^{l+1} = z$. Et Theor. VIII eſt tantum Corollarium utriuſque.

Alia Methodus pro reductione Formulæ $dx : \sqrt[n]{(x^n + a)}$ Petropoli *nuper miſſæ.*

Ponatur $\sqrt[n]{(x^n+a)} = xz$, unde $x^n + a = x^n z^n$, ſeu $(z^n - 1)x^n = a$, vel $x^n = a : (-1 + z^n)$. Hinc $nlx = la - l(-1+z^n)$; differentiando $dx : x = -z^{n-1} dz : (z^n - 1)$. Per ſubſtitutionem aſſumptæ quantitatis xz pro $\sqrt[n]{(x^n + a)}$, provenit $dx : \sqrt[n]{(x^n + a)} = dx : xz$, ubi pro $dx : x$ ſubſtituatur valor inventus $- z^{n-1} dz : (z^n - 1)$; emergit $dx : \sqrt{(x^n + a)} = -z^{n-2} dz : (z^n - 1)$.

* Ibid. pag. 417, 418. † Ibid. pag. 418.

Pro EULERI noſtri, vel Filii mei *Danielis* expreſſione $dx : \sqrt[n]{(a + bx^n)}$ ſic pariter procedo. Pono $\sqrt[n]{(a+bx^n)} = xz$, $a + bx^n = x^n z^n$, $(z^n - b)x^n = a$, $x^n = a : (z^n - b)$; $nlx = la - l(z^n - b)$, $dx : x = -z^{n-1} dz : (z^n - b)$. Per ſubſtitutionem xz pro $\sqrt[n]{(a + bx^n)}$ habetur $dx : \sqrt[n]{(a+bx^n)} = dx : xz = -z^{n-2} dz : (z^n - b)$. Quæ fractio cum ſit rationalis, patet reduci poſſe ad quadraturam Circuli, vel ad Logarithmos; ſed non video dari caſus abſolute quadrabiles.

GENERALIUS.

Sit formula reducenda $dx : \sqrt[n]{(ax^p + bx^n)}$. Pono, ut ante, $ax^p + bx^n = x^n z^n$, $(z^n - b)x^n = ax^p$, $z^n - b = ax^{p-n}$, $l(z^n - b) = la + (p - n)lx$, $nz^{n-1} dz : (z^n - b) = (p - n)dx : x$. Hinc $dx : \sqrt[n]{(ax^p + bx^n)} = dx : xz = \frac{n}{p-n} z^{n-2} dz : (z^n - b)$.

COROLL. Vocetur $p = q + n$, & erit $dx : \sqrt{(ax^{q+n} + bx^n)} = dx : x\sqrt[n]{(ax^q + b)}$; quæ nova eſt formula reducibilis.