Measuring Financial Time Series Dependence: A Survey

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Measuring Financial Time Series Dependence: A Survey

Anqi Du [†]

Abstract

This paper surveys the most important developments in measuring dependence of multivariate financial time series. It reviews the Asymmetric Dynamic Conditional Correlation model specifications and inference methods, the copula approach and tail dependence coefficients derived from it. We have a discussion about some statistical tests for the existence of asymmetric dependence, and more important, the non-universality of asymmetries in dependence. We highlight the importance of studying asymmetric dependence in risk management.

JEL Classification: C13, C16, C32, C51, G15.

Keywords: A-DCC models, dynamic correlation, Copula, Tail dependence coefficient, Asymmetric dependence.

1 Introduction

Measuring the relationship of two or more financial series plays a key role for risk management, portfolio selection and asset pricing, which have received a lot of attention of researchers in the past. The main challenge lies in constructing and analyzing models that embody time-varying and asymmetric relationships between variables, especially the latter. As shown in Hong et al. (2007), Jiang et al. (2016) etc., the correlations between portfolio returns are higher during bear market periods than bull market periods. This asymmetry is important because the benefits of diversification will decline dramatically when portfolios become highly dependent and prices decline all together in a downside period. Yet some literature (eg., Rodriguez (2005), Chiristoffersen, Errunza et al. (2012)) do not find empirical evidence that such asymmetric dependency exist. Therefore, there are two essential questions: does the asymmetries exist in dependence and how should the asymmetries in dependence be measured? In this paper, we focus on the most important developments in measuring dependence for multivariate financial data particularly the methodologies

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that take asymmetry into account.

One option is to take the covariance and correlation matrices as a measure for representing the relationship among assets. A Sequence of multivariate volatility models have been proposed, for instance, the BEKK model by Engel and Kroner (1995), the Constant Conditional Correlation (CCC) model of Bollerslev (1990) and so on. Worth mentioning, the CCC model gives a multivariate time series model specification with time varying conditional variances and covariances, but constant conditional correlations. As the correlation coefficients should be considered dynamic over the sample period, in light of the CCC model, Engel (2002) and Tse and Tsui (2002) both develop the Dynamic Conditional Correlation (DCC) model which is capable to provide dynamic conditional correlations among assets. More development on this model has focused on the study of asymmetries and regime switching.

The traditional DCC model specifications indicate symmetric time-varying correlations, however, the comovements of assets may not be adequately captured by a symmetric correlation coefficients. Ang and Chen (2002) point out that correlations between US stocks and the aggregate US market are much greater for downside moves than for upside moves. Hong et al. (2007) argue that correlations between portfolio returns and market returns are much higher when both returns are below some criteria, but relatively lower when both returns are above some criteria. Cappiello, Engle, and Sheppard (2006) extend the DCC model into an Asymmetric DCC (A-DCC) model which permitting conditional asymmetries in correlation dynamics, and find that conditional correlation of equity and bond returns show evidence of asymmetry, and equities show a stronger response than bonds to joint bad news[1)].

To develop the multivariate volatility models, besides taking account of the asymmetries in conditional correlations, a regime switching approach is also quite plausible. Ledoit et al. (2003) find that the level of correlation for international stock markets depends on the phase of the business cycle. Kirishnan (2009) argues that the correlations of returns varied substantially over time and investors would pay a premium for securities that perform well in regimes in which the correlation is high. All these findings suggest a state-dependent time-varying correlation for multivariate financial applications. Pelletier (2006) designed the Regime Switching DCC (RS-DCC) model to capture the presence of smooth and abrupt changes in the time-varying correlations. Lee (2010) has extend the model into an Independent Switching DCC (IS-DCC) model and provides a general framework to model multi-state regime switching dynamic correlation for financial assets. Pan et al. (2014) have proposed the regime switching asymmetric DCC (RS-ADCC) model by taking account of both of regime switching and asymmetry in cor- relations. Bauwens and Otranto (2016) proposed the class

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1) The joint bad news is defined as both shocks to returns being negative. Conversely, if both shocks are positive, we call it joint good news.

of Volatility DCC (VDCC) model in the sense that permitting the model including the dependence of the correlations on the market volatility. Bauwens and Otranto (2019) also design a new parameterizations of the DCC model which provide a specific dynamics for each correlation.

Along with the availability of high frequency financial data in recent decades, the multivariate volatility models have also been extended to incorporate intraday returns, such as Bollerslev et al. (2020) and Dhaene and Wu (2020).

On the other hand, besides these multivariate volatility models, copula is another popular way to model the inter dependence among variables. Pearson correlation coefficient, as a traditional means of correlation analysis, is widely used in practice due to its simplicity of calculation. However, Pearson correlation coefficient is a linear correlation coefficient requires that the relationship between variables is linear and the variance is finite, which is often not satisfied in practical applications. Many data in the financial markets exhibit fat-tailed characteristics, so using linear correlation coefficients to portray correlations for those data is problematic. In order to overcome the shortcomings of traditional statistical analysis of correlations, Copula theory, first proposed by Sklar (1973), has shown great superiority and is widely adopted in the financial sector.

Firstly, the copula function does not restrict the choice of marginal distributions and can be used to construct flexible multivariate distributions. Secondly, the marginal distributions of random variables and the correlation structure between them can be studied separately when building the model. Their dependence structure is described by a copula function. In addition, if a nonlinear monotonic incremental transformation of the variables is performed, the values of the linear correlation coefficients will change, but the dependence measures derived by the copula function do not change. Thus, the copula has a wide range of applications and practicality. The copula has also been taken into consideration of conditional copula by Patton (2009) and cooperating with the DCC-family models.

There are ample methodologies to measuring the correlation and dependence between financial time series. We intend to introduce the main models and statistics of measuring the associations among financial time series, in order to discuss: Does asymmetry in dependency widespread? What will be the factor that influence the existence of asymmetry in correlation dynamics? And are the differences in dependence statistically significant? The rest parts of this survey are generated as follows. In Section 2, we review the specification of DCC model and A-DCC model. In Section 3, we introduce the definition and estimation of Copulas. In Section 4, we have the discussion about some empirical evidence, potential factors that may influence the asymmetric phenomenon, tests for asymmetric dependence between two variables, and what kind of role does correlation play in risk management.

2 The Dynamic Conditional Correlations

Understanding and measuring the relationship between movements in different assets play key roles in designing a portfolio. As a constant coefficient cannot describe the relationship between series completely, we consider the correlation dynamics. One essential and commonly applied class of models is the dynamic conditional correlation (DCC) models. Notice that, the volatility (conditional variance) of an individual asset series, is usually calculated by a univariate GARCH-type model, and then with the standardized marginal innovations we fit the DCC model to obtain the conditional correlation dynamics.

2.1 The Dynamic Conditional Correlation model

Consider a k-dimensional asset return series x_t with innovation term ϵ_t, given F_{t-1}, the information available at time $t-1$. The conditional covariance matrix of ϵ_t is heterogeneous, and denoting it as H_t. The covariance matrix can be decomposed as:

$$H_t = D_t R_t D_t, \tag{1}$$

where $D_t = \mathrm{diag}\{h_{11,t}^{1/2}, \cdots, h_{kk,t}^{1/2}\}$ is the diagonal matrix of the k marginal volatilities at time t. R_t is the correlation matrix, contains $k(k-1)/2$ elements. The time-varying correlation matrix R_t must be inverted at every point in time, which may make the calculation much slower. In the meantime, it is constrained to be positive definite.

The marginal volatility $h_{ii,t}^{1/2}$ can be calculated by any uni-variate volatility model, such as the GARCH model. Then for each series x_i, $i = \{1, 2, \cdots, k\}$, we can get the marginal standardized series by following equation:

$$\eta_{i,t} = \epsilon_{i,t} / \sqrt{h_{ii,t}}, \tag{2}$$

and the vector $(\eta_{i,1}, \cdots, \eta_{i,t}, \cdots, \eta_{i,T})$ follows an independently identically distribution which mean equals 0 and variance equals 1. We denote $\eta_t = (\eta_{1,t}, \cdots, \eta_{k,t})'$ be the marginally standardized innovation vector. Notice that the $\eta_{i,t}$ and $\eta_{j,t}$ may not be independent. Let Q_t be the conditional covariance matrix of η_t. Engle (2002) has proposed one type of the DCC model to get a proper estimation of Q_t and it is defined as

$$Q_t = (1-a-b)\bar{Q} + aQ_{t-1} + b\eta_{t-1}\eta'_{t-1}. \tag{3}$$

It is actually the form of a DCC(1, 1) model. More generally, a DCC(m, n)model is specified as follows:

$$Q_t = \left(1 - \sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j\right)\bar{Q} + \sum_{i=1}^{m} a_i Q_{t-i} + \sum_{j=1}^{n} b_j \eta_{t-j}\eta'_{t-j}, \tag{4}$$

$$R_t = (diag(Q_t))^{-1/2} Q_t (diag(Q_t))^{-1/2}, \tag{5}$$

where $\bar{Q} = E[\eta_t \eta'_t]$, is the unconditional covariance matrix of η_t, a_i and b_j are non-negative real numbers satisfying $0<\sum_{i=1}^{m}a_i+\sum_{j=1}^{n}b_j<1$. After the normalization we can finally obtain the correlation matrix R_t. More specifically, the element of R_t is:

$$\rho_{ij,t}=\frac{q_{ij,t}}{\sqrt{q_{ii,t}q_{jj,t}}}. \tag{6}$$

The log-likelihood function is given by:

$$\log L(\theta,\ \phi)=\left[-\frac{1}{2}\sum_{t=1}^{T}(k\log(2\pi)+\log|D_1|^2+\epsilon'_t D_t^{-2}\epsilon_t)\right]+\left[-\frac{1}{2}\sum_{t=1}^{T}(\log|R_t|+\eta_t R_t^{-1}\eta_t-\eta_t\eta'_t)\right], \tag{7}$$

where θ is the parameter set characterizes univariate volatility model, and ϕ is the parameter set of DCC model. The first term in eq.(7) is the sum of the likelihoods of each uni-variate volatility model. When we maximize the first term, we can get the maximum likelihood estimator of θ, which is the parameter set in the individual model. Then we can move on to the next stage, maximizing the second term in eq.(7) to obtain the correlation coefficients ϕ.

Practically, to estimate DCC models, univariate GARCH models are typically used to obtain the standardized marginal series $\{u_t\}$ and the parameter set θ. In general, the procedure to build DCC models contains 3 steps as follows:

1. Obtain estimates of the condition mean and residual series via a VAR(p) model.
2. Estimate the volatility series via a univariate volatility model (eg. GARCH family models). Then calculate the marginal standardized innovation series.
3. By the standardized innovation series obtained in Step 2, fit a DCC model.

2.2 The Asymmetric Dynamic Conditional Correlation model

The conditional correlations of DCC models are symmetrically distributed respect to shocks, which may not be able to detect the asymmetries and does not allow for asset-specific news and smoothing parameters. Cappiello et.al (2006) proposed the Asymmetric Dynamic Conditional Correlation (A-DCC) model, which allows for asymmetries in correlations.

2.2.1 Model Specification

Based on the DCC specification of dynamic correlation, Cappiello et.al (2006) insert a leverage term into eq.(4) in order to model the asymmetric correlation, and the structure of an A-DCC model is given by:

$$Q_t=\left(1-\sum_{i=1}^{m}a_i-\sum_{j=1}^{n}b_j\right)\bar{Q}-\tau\bar{N}+\tau(n_{t-1}n'_{t-1})+\sum_{i=1}^{m}a_iQ_{t-i}+\sum_{j=i}^{n}b_j(\eta_{t-j}\eta'_{t-j}), \tag{8}$$

where $\bar{Q}_t$ and u_t are exactly same as in the DCC case. $n_t=I_{[\eta_t<0]}\circ\eta_t$, the $\circ$ denotes the Hadamard product, and $I_{[\eta_t<0]}$ is a $k\times1$ indicator function, which takes value 1 when $\eta_t<0$, and 0 otherwise. $\bar{N}=E[n_t n'_t]$ is the unconditional covariance matrix of n_t.

Consider the elements $n_{i,t}$ and $n_{j,t}$ in n_t, when we have joint bad news, the indicators of both $n_{i,t}$ and

$n_{j,t}$ are 1. Thus, this specification implies a higher correlation when there is a joint bad news. In order to ensure the positive definiteness of Q_t, we have the restrictions that $\sum_{i=1}^{m} a_i + \sum_{j=1}^{n} b_j + \tau < 1$, and a_i, b_j, and τ are all non-negative.

The A-DCC model is also estimated via maximum likelihood as eq.(7), with τ in the parameter set ϕ, assuming conditional multivariate normality. Although the conditional distribution is often misspecified, the quasi-maximum likelihood estimator exists, which is consistent and asymptotically normal (Engle and Sheppard (2001)).

Cappiello et al. (2006) also provide a more general form of the correlation model called the Asymmetric Generalized Dynamic Conditional Correlation (AG-DCC) model, which we demonstrate in Appendix A.

2.2.2 The news impact surface

To better illustrate the asymmetry in the dynamic correlations, we can create the news impact surface for A-DCC introduced by Cappiello et al (2006) to indicate that respect to a specific shock, how much the correlation will change.

The news impact surface shows the relation between shocks and correlation for asset i and j, and it is given by:

$$f(\eta_i, \eta_j) = \frac{\tilde{c}_{ij} + (a_i a_j + \tau I_{[\eta_i<0]} I_{[\eta_j<0]}) \eta_i \eta_j + b_i b_j \rho_{ij,t}}{\sqrt{[\tilde{c}_{ii} + a_i^2 \eta_i^2 + b_i^2][\tilde{c}_{jj} + a_j^2 \eta_j^2 + b_j^2]}} \quad \text{for} \quad \eta_i, \eta_j < 0, \tag{9}$$

$$f(\eta_i, \eta_j) = \frac{\tilde{c}_{ij} + a_i a_j \eta_i \eta_j + b_i b_j \rho_{ij,t}}{\sqrt{[\tilde{c}_{ii} + a_i^2 \eta_i^2 + b_i^2][\tilde{c}_{jj} + (a_j^2 + \tau I_{[\eta_j<0]}) \eta_j^2 + b_j^2]}} \quad \text{for} \quad \eta_i > 0, \eta_j < 0, \tag{10}$$

$$f(\eta_i, \eta_j) = \frac{\tilde{c}_{ij} + a_i a_j \eta_i \eta_j + b_i b_j \rho_{ij,t}}{\sqrt{[\tilde{c}_{ii} + (a_i^2 + \tau I_{[\eta_i<0]}) \eta_i^2 + b_i^2][\tilde{c}_{jj} + a_j^2 \eta_j^2 + b_j^2]}} \quad \text{for} \quad \eta_i < 0, \eta_j > 0, \tag{11}$$

$$f(\eta_i, \eta_j) = \frac{\tilde{c}_{ij} + a_i a_j \eta_i \eta_j + b_i b_j \rho_{ij,t}}{\sqrt{[\tilde{c}_{ii} + a_i^2 \eta_i^2 + b_i^2][\tilde{c}_{jj} + a_j^2 \eta_j^2 + b_j^2]}} \quad \text{for} \quad \eta_i, \eta_j > 0, \tag{12}$$

where η_i and η_j are the standardized shocks on asset i and j respectively. By figures of news impact surface, we can easily check that in each quadrant, the relationship between the joint shock and the correlation, and the slope of the surface will distinguish the asymmetry.

Noteworthiness, Gjika and Horvath (2013) give the news impact surface of the asymmetric correlation of the pair BUX(stock index of Hungary)-WIG(stock index of Poland) as the Figure 1. From the upper panel of Figure 1, we can clearly observe that for a joint negative shock, the correlation varies more than a joint positive shock.

3 The Copulas

Asset returns might not be normally distributed, and the marginal distribution (the distribution of a

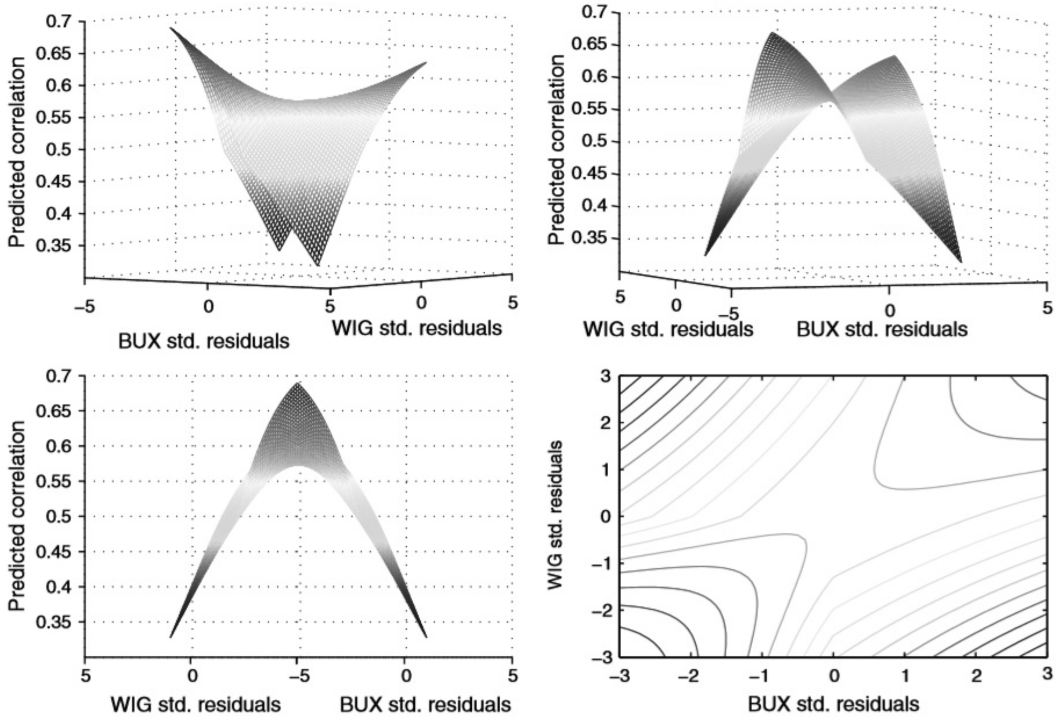

Fig. 1: News impact surface of BUX-WIG. (Source: Gjika and Horvath (2013))

single return series) would have a heavy tail. Thus the probability of two asset returns exhibiting movements in the same direction may be much higher than it would be with a bivariate Gaussian. When two series are not Gaussian, correlation may not be the best way of capturing the relationship between them. Copula offers a more radical and flexible way of modelling dependence structure between variables with no restriction on marginal distributions.

3.1 What is copula?

A k-dimensional copula is a joint distribution function on $[0, 1]^k$ with standard uniform marginal distributions. Denoting an element in $[0, 1]^k$ by $\mathbf{u}=(u_1, \cdots, u_k)'$, which u_i follows a standard uniform distribution. As the copula is a distribution function, its range of values is $[0, 1]$. Thus a k-dimensional copula $C(\mathbf{u})=C(u_1, \cdots, u_k)$, is actually a mapping from $[0, 1]^k$ to $[0, 1]$.

The importance of copulas in multivariate analysis is established by the Sklar Theorem (1973), which shows that all multivariate distribution functions contain copulas and copulas can be used in conjunction with univariate distribution function to construct multivariate distribution function.

Theorem. *Sklar Theorem*

Let $(X_1, \cdots, X_k)'$ *be a random vector with marginal distributions* $F_1(x_1), \cdots, F_k(x_k)$. *Then there* *exists a copula C such that*

$$F(x_1,\ \cdots,\ x_k)=C(F_1(x_1),\ \cdots,\ F_k(x_k)), \tag{13}$$

where $F(\cdot)$ is the joint distribution function. If the marginal distributions are continuous, then C is uniquely determined. Conversely, with a copula C, and marginal distributions $F_1,\ \cdots,\ F_k$, the function in eq.(13) is the joint distribution function with these margins.

A proof of the Sklar theorem can be found in Nelsen (1999) or McNeil et al. (2005).

One subtle properties of copula is that, any monotonic incremental transformation of the variables, no matter it is linear or nonlinear, the dependence measures derived by copula keep the same. We demonstrate this unchanged property of copula as following proposition, and give the detailed proof of it.

Proposition. *(Invariance of the copula under monotonic transformations)*

Let $(X_1,\ \cdots,\ X_k)'$ be a random vector with continuous marginal distributions $F_1(x_1),\ \cdots,\ F_k(x_k)$ and copula C. Denoting the transformation functions of variables as $T_1(X_1), \cdots, T_k(X_k)$. As long as $T_i(X_i)$, $i=1,\ \cdots,\ k$, are continuous and monotonically increasing functions, $(T_1(X_1),\ \cdots,\ T_k(X_k))'$ also has copula C.

Proof. Let $\tilde{F}_i(x_i)$ be the marginal distribution function of $T_i(X_i)$. As $T_i^{-1}(\cdot)$ is also strictly increasing, then

$$u=\tilde{F}_i(y)=P(T_i(X_i)\leq y)=P(X_j\leq T_i^{-1}(y))=F_i(T_i^{-1}(Y)).$$

So we have

$$y=\tilde{F}_i^{-1}(u)$$

and

$$F_i^{-1}(u)=T_i^{-1}(y),\quad \text{thus}\quad T_i(F_i^{-1}(u))=y.$$

Above all, we have

$$T_i(F_i^{-1}(u))=\tilde{F}_i^{-1}(u)\quad \text{for}\quad 0\leq u\leq 1.$$

By Sklar Theorem,

$$\begin{aligned} C(u_1,\ \cdots,\ u_k)&=P[F_1(X_1)\leq u_1,\ \cdots,\ F_k(X_k)\leq u_k]\\ &=P[X_1\leq F_1^{-1}(u_1),\ \cdots,\ X_k\leq F_k^{-1}(u_k)]\\ &=P[T_1(X_1)\leq T_1(F_1^{-1}(u_1)),\ \cdots,\ T_k(X_k)\leq T_k(F_k^{-1}(u_k))] \qquad (14)\\ &=P[T_1(X_1)\leq \tilde{F}_1^{-1}(u_1),\ \cdots,\ T_k(X_k)\leq \tilde{F}_k^{-1}(u_k)]\\ &=P[\tilde{F}_1(T_1(X_1))\leq u_1,\ \cdots,\ \tilde{F}_k(T_k(X_k))\leq u_k]. \end{aligned}$$

Consequently, this C is also the Copula of $((T_1(X_1)),\ \cdots,\ T_k(X_k))'$. A similar proof can be found in Tsay (2014).

□

In population terms, the joint probability that observations at the same time from X_i is less than the u_i-quantile, $F_i^{-1}(u_i)$[2), is

$$Pr(X_1 \leq F_1^{-1}(u_1), \ \cdots, \ X_k \leq F_k^{-1}(u_k)) = F(F_1^{-1}(u_1), \ \cdots, \ F_k^{-1}(u_k)). \tag{15}$$

where the F_i^{-1} are the quantile functions of the marginal series, and $0 \leq u_1, \ \cdots, \ u_k \leq 1$. Such probabilities are given by the copula $C(u_1, \ \cdots, \ u_k)$, and it can be written as

$$C(u) = C(u_1, \ \cdots, \ u_k) = F(F_1^{-1}(u_1), \ \cdots, \ F_k^{-1}(u_k)). \tag{16}$$

Then we are able to obtain the joint distribution function with known marginal distributions with eq. (16).

There are several kinds of copulas which can model different dependent structures among multiple series. In next section, resulting from the fact that we often observe heavy tails in financial returns data, and the t copula can better capture the phenomenon of dependent extreme values (see for example, Literatures such as Mashal Zeevi (2002) and Breymann et al. (2003)), we concentrate on the t copula to explain its definition and how does it work with multivariate volatility modeling[3]. Rest copulas are introduced in Appendix B.

3.2 The definition of *t* copula

The t copula (see, for example, Embrechts, McNeil Straumann (2001) or Fang (2002)) can be thought of as representing the dependence structure implicit in a multivariate t distribution.

Let $X = (X_1, \ \cdots, \ X_k)'$ be a k-dimensional multivariate Student-t random vector. The pdf of X is a multivariate t distribution with v degrees of freedom, mean vector μ and positive-definite covariance matrix Σ, denoted by $X \sim t_k(v, \mu, \Sigma)$, which can be expressed as:

$$f(x) = \frac{\Gamma\left(\frac{v+k}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\sqrt{(\pi v)^k |\Sigma|}} \left(1 + \frac{(x-\mu)'\Sigma^{-1}(x-\mu)}{v}\right)^{-\frac{v+k}{2}}. \tag{17}$$

As we discuss in the Proposition, the copula remains unchanged under any monotonically increasing transformations of a random vector. Consider the covariance matrix Σ being the random vector[4], and denoting R as the correlation matrix implied by the covariance matrix Σ. We have

$$R = \frac{\Sigma}{\sqrt{Var(X_1) \cdots Var(X_k)}}.$$

The denominator is positive, thus we can regard R as a monotonic transformation of Σ. This indicates that the copula of a $t_k(v, \mu, \Sigma)$ is identical to that of a $t_k(v, 0, R)$ distribution. We take $F_i^{-1}(u_i) = t_i^{-1}(u_i)$ in eq. (16), where t_i^{-1} denotes the quantile function of a standard univariate t distribution for ith variable. The unique copula is thus given by

2) More specifically, considering a binary case, an observation from the first series X_1 is less than the u_1-quantile $F_1^{-1}(u_1)$, at the same time as the corresponding observation from the second series X_2 is below the u_2-quantile $F_2^{-1}(u_2)$, the joint probability of these happen is $P(X_1 \leq F_1^{-1}(u_1), \ X_2 \leq F_2^{-1}(u_2)) = F(F_1^{-1}(u_1), \ F_2^{-1}(u_2))$.

3) Copulas have many other properties, see for example McNeil et al.(2005), Joe and Kurowicka(2010). Here we only discuss the properties that are useful in multivariate volatility modeling.

4) More specifically, $\Sigma = (\Sigma_1, \ \cdots, \ \Sigma_k)'$, where Σ_i is the covariance between ith variable and others.

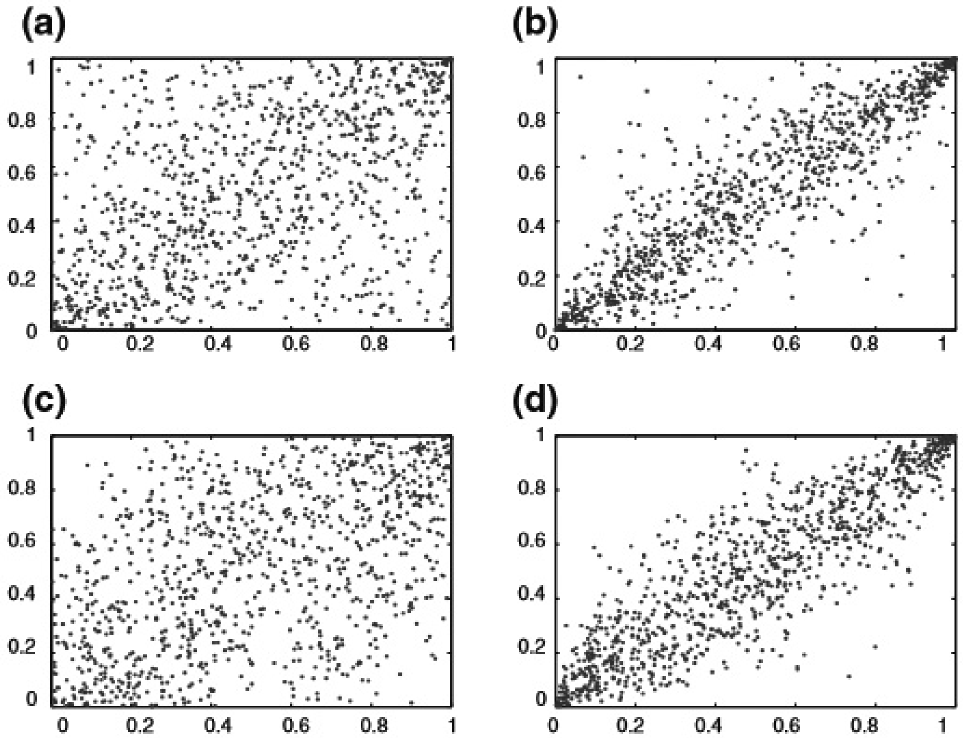

Fig. 2: Scattered points of t-copulas. **a.** $\rho=0.5$, $v=3$, **b.** $\rho=0.9$, $v=3$, **c.** $\rho=0.5$, $v=30$, **d.** $\rho=0.9$, $v=30$. (Source: Cherubini et al. (2016))

$$C^t_{v,R}(u)=\int_{-\infty}^{t_1^{-1}(u_1)}\cdots\int_{-\infty}^{t_k^{-1}(u_k)}\frac{\Gamma\left(\frac{v+k}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\sqrt{(\pi v)^k|R|}}\left(1+\frac{v'R^{-1}x}{v}\right)^{-\frac{v+k}{2}}dx. \tag{18}$$

Simulation of the t copula is particularly easy: we generate a multivariate t-distributed random vector $X\sim t_k\,(v,\,0,\,R)$ using eq.(17) and then return a vector $U=(t_v(X_1),\ \cdots,\ t_v(X_k))'$, where t_v denotes the distribution function of a standard univariate t with v degrees of freedom.

When we consider a binary case, the correlation R is a (2×2) matrix. The off-diagonal element is a scalar, denoted by ρ. Figure 2 (source from Cherubini et al. (2016)) shows the scatter plots of four bivariate t copulas $(k=2)$ with **a.** $\rho=0.5$, $v=3$, **b.** $\rho=0.9$, $v=3$, **c.** $\rho=0.5$, $v=30$, **d.** $\rho=0.9$, $v=30$, respectively. These plots are based on 1000 random draws from the corresponding t copulas. As the correlation coefficient ρ goes closer to 1 (right panels), the scattered points are lying tighter, and as the degree of freedom goes higher (bottom panels), the scattered points are more dispersed. Both parameters ρ and v in the copula function are measures of tail dependence. We denote the parameter set of copula function as ϕ (for t copula, $\phi=(\rho,\ v)$), which includes the parameters of multivariate dependence modeling.

For estimation purposes, it is useful to note that the density of the t copula may be easily calculated from eq.(16) and has the form

$$c^t_{v,R}(u)=\frac{f_{v,R}(t_1^{-1}(u_1), \ \cdots, \ t_k^{-1}(u_k))}{\prod_{i=1}^k f_i(t_i^{-1}(u_i))}, \quad u\in(0, \ 1)^k, \tag{19}$$

where $f_{v,R}$ is the joint density of a k-dimensional $t(v, 0, R)$-distributed random vector and f_i is the pdf of the univariate standard t-distribution for ith series.

3.3 Estimation of the Copula

In this section, we discuss the estimation of Copulas with financial data. Broadly speaking the estimation procedure is similar to DCC model estimation mentioned in previous sections. Firstly, obtain the standardized marginal series via any univariate model for each series. Then we transform the standardized margins η_t with the quantile function t^{-1} for each series to fit the copula function of eq.(18) or eq.(19). The estimation technique is the maximum likelihood, and the log likelihood function of the copula model is given by

$$\begin{aligned}\log L(\theta, \ \phi)=&\sum_{t=i}^{T}\log f_1(\eta_{1,i} \ ; \ \theta_1)+\cdots+\sum_{t=1}^{T}\log f_k(\eta_{k,t} \ ; \ \theta_k)\\&+\sum_{t=1}^{T}\log c(F_1(\eta_{1,t} \ ; \ \theta_1)), \ \cdots, \ F_k((\eta_{k,t} \ ; \ \theta_k) \ ; \ \phi).\end{aligned} \tag{20}$$

This log-likelihood has to be maximized with respect to all parameters $(\theta_1, \ \cdots, \ \theta_k, \ \phi)$, where $\theta=(\theta_1, \ \cdots, \ \theta_k)$ is the parameter set for margins and ϕ is the parameter set for the copula. However, this maximization is very computationally intensive especially in the case of a high dimension.

To overcome this problem, Joe and Xu (1996) propose the estimation technique called Inference for the Margins (IFM) which separate eq.(20) into two parts: **1.** the first k terms related to the margins and their parameters; **2.** the last term involving the copula density and its parameters. Thus to obtain the IFM estimator contains two steps:

- maximize the first k terms of eq.(20) to obtain the $\hat{\theta}$.
- then with $\hat{\theta}$ maximize the last term related to the copula to obtain estimator $\hat{\phi}$.

Under some regularity conditions, the IFM estimator is asymptotically Gaussian (see Joe (1997)).

3.4 Tail Dependence Coefficients

For this section it suffices to consider a bivariate random vector (X_1, X_2) with continuous and strictly increasing marginal distribution functions and unique copula C. As the Copula models the dependence structure between two random variables, the tail dependence coefficients can be easily derived from a Copula. Notice that, in this section, the tail dependence coefficients are including but not limited to the t copula.

The tail dependence coefficient provides an asymptotic measure of the tail dependence of the bivariate distribution of (X, Y). Denote the upper tail dependence parameter as λ_U, then

$$\begin{aligned}\lambda_U &= \lim_{u \to 1^-} P(Y > F_Y^{-1}(u) | X > F_X^{-1}(u)) \\ &= \lim_{u \to 1^-} \frac{1 - P(X \leq F_X^{-1}(u)) - P(Y \leq F_Y^{-1}(u)) + P(X \leq F_X^{-1}(u),\ Y \leq F_Y^{-1}(u))}{1 - P(X \leq F_X^{-1}(u))}, \\ &= \lim_{u \to 1^-} \frac{(1 - 2u + C(u,\ u))}{1 - u}\end{aligned} \tag{21}$$

which indicate the probability when X takes extreme large value and Y also abnormally large. If $\lambda_U \neq 0$, X and Y are asymptotically dependent in the upper tail; if $\lambda_U = 0$, X and Y are said to be asymptotically independent in the upper tail.

In the similar way, the lower tail dependence parameter denoting as λ_L is defined as

$$\lambda_L = \lim_{u \to 0^+} P(Y \leq F_Y^{-1}(u) | X \leq F_X^{-1}(u)) = \lim_{u \to 0^+} \frac{C(u,\ u)}{u}. \tag{22}$$

The derivation of the expression of λ_L can be found in Embrechts et al. (2001). And if $\lambda_L \neq 0$, X and Y are asymptotically dependent in the lower tail; if $\lambda_L = 0$, X and Y are asymptotically independent in the lower tail.

As we discuss in Section 3.1, u is the probability that X_i less than the u-quantile ($u = P(X_i \leq F_i^{-1}(u))$, where F_i is the distribution function of X_i). Once we have estimated the copula function C, by artificially selecting the value of u[5], it will be possible to obtain λ_U and λ_L via eq.(21) and eq.(22).[6]

By tail dependence coefficients, Nikoloulopoulos et al. (2012) studies daily log-return data for five European market indexes (CAC40 France, DAX Germany, OSEAX Norway, SMI Switzerland and FTSE England) through an upturn period (2003-2006) and a downturn period (2007-2009). Their empirical results suggest that tail dependence exists and there is a slight tendency to have more tail dependence in the lower tail than upper tail, but this might not be significant as it is not true for both periods.

4　Discussions

4.1　Non-universality of asymmetric dependence in financial data

Through a copula approach, measuring the correlation coefficient, Rodriguez (2005) has found that, during the major financial crises in Asian and Latin American of 1990s[7], there are increased tail

5) For example, for upper tail dependence we may select $u \in \{0.9, 0.95, 0.98, 0.99\}$; and for lower tail dependence $u \in \{0.1, 0.05, 0.02, 0.01\}$.

6) Notice that, the tail dependence exists for some of the families of copulas, while some families do not allow tail dependence, for example, the Gaussian copula (Nikoloulopoulos et al. (2012)).

7) ERM attacks (1992), the Mexican devaluation (1994), the East Asian crisis (1997), the Russian default (1998), and the devaluation of the Brazilian real (1999).

dependence and asymmetry in Asian markets, but tail independence and symmetry in Latin American countries.

This kind of result is not unique. For instance, Chiristoffersen, Errunza et al. (2012) examine the nonlinear dependence and asymmetries in international equity markets as well. By proposing dynamic asymmetric copula model, they find that correlations increase dramatically in developed markets but relatively less increase in emerging markets. Also for the tail dependence, the increased level is far less in the emerging markets than in developed markets. Therefore, they conclude that in the emerging markets, diversification still has significant benefits particularly during the "downside" period.

Gjika and Horvath (2013), by measuring the correlation among Central Europe stock markets via A-DCC model, they find that asymmetric volatility is common in these stock markets, whereas the asymmetric correlations only be found in one pair (Hungary (BUX) and Poland (WIG)). The relationship between conditional correlations and conditional variances (the volatilities) are examined as well, and their results suggest that the conditional correlations are positively related to the conditional variances, in another words, correlation between two markets goes higher at a turmoil period.

On the other hand, some literature find asymmetries in the correlations. For instance, through an A-DCC approach, Tamakoshi and Hamori (2014) examine the dynamic linkages among three major European currencies (EUR, GBP, and CHF) exchange rates against USD for a sample period from January 1999 to December 2010. Their empirical analysis finds evidence of an asymmetric response in the correlation among the three exchange returns, and higher degree of interdependence during periods of appreciation than depreciation. This finding is interesting and may be explained by two reasons: 1. the increased economic and financial convergence lead to high dependency in correlations before the financial crisis (2007); 2. the different vulnerability of the three European currencies after the financial crisis.

Ji et al. (2019) analyse the dynamic dependence between WTI crude oil and the exchange rates of USD/CNY via a time-varying copula model. Their empirical results indicate that the dependence is significantly asymmetry in Chinese exchange rate market in response to rising and falling oil returns, though the asymmetry for the oil returns and US dollar index is not significant. These results might be attributed to the Chinese government's exchange rate control policy.

All these empirical results show that it seems the asymmetries in the correlations are not as common as in volatilities. Some markets show the existence of asymmetry while some markets not. Why the asymmetries in the correlations are not as widespread as asymmetries in volatilities? What factors will influence the existence of asymmetric correlations?

Many recent studies work on these issues and Chung and Kim (2017) argue that the assets with asymmetric correlation tend to cause portfolios to have negative skewness, even if every asset in the portfolio has positive skewness. And the asymmetric correlation is better measured with the skewness of smaller portfolios but is not well captured with larger portfolios.

Based on all these research, in our opinion, the non-universal asymmetry in dependence might be result from both following reasons: the measure we apply, and the features of the data.

The method we use to capture the asymmetric dependence may affect the result whether we can detect asymmetries. We principally introduce the dynamic conditional correlations and the tail dependence coefficients λ_U and λ_L via Copulas. Although in the A-DCC model specification, the value of coefficient τ can be interpreted as the degree of asymmetry among marginal series, the significance of τ can solely tell whether the model is correctly specified or not. Noteworthy, the news impact surfaces illustrate the asymmetries in conditional correlations respect to the shocks on returns, and it much depends on the specification of the model, as we may obtain a symmetric diagram through a standard DCC model.

Numerous other measurements of asymmetric dependence have been proposed, such as exeedance correlation measurements (Ang and Chen (2002)), local Gaussian correlation (Lacal and Tjostheim, (2019)) and so on. Some of them capture the global dependence while some are especially for local dependence structure.

More likely, the financial data itself may affect the result whether we can detect asymmetries. The market-wide liquidity, volatility (here the volatility may refer to the realized volatility), investor sentiment, price-earnings ratio, and etc., all may potentially influence the existence of asymmetry. However, it still remains inconclusive and further studies are required.

4.2 Test of asymmetric dependence

Besides the most popular methodology we introduced in previous sections, several other measurements for asymmetric dependence and statistical tests based on them have been proposed by literature. To our knowledge, Ang and Chen (2002) firstly provide a novel test for the null hypothesis of symmetric correlations with developed the so-called exceedance correlation measurement, which explains the degree of correlation's asymmetry in the data. But their test is model dependent, testing the joint hypothesis of both validity of a given model and symmetry, so that a rejection of null may be solely due to a rejection of the model. Hong et al. (2007) provide a model-free test for asymmetric correlations where stocks move more often with the market when the market goes down than when it goes up. Jiang et al. (2018) propose a modified mutual information measure to capture general asymmetric dependence between two random variables and a test based on this measure. Tjostheim et al. (2013, 2019) develop a nonlinear local measure of dependence called local Gaussian correlation and develop a test based on this measure of dependence as well.

4.3 Risk Management

The Capital Asset Pricing Model (CAPM) and the Arbitrage Pricing Theory (APT) (Campbell, Lo, and MacKinlay 1997) use correlation as a measure of dependence between different financial instruments, which is essentially founded on an assumption of multivariate normally distributed returns, in order to arrive at an optimal portfolio selection. Unfortunately, the multivariate normal distribution is not usually suitable for construct a portfolio that gives minimum variance for a given expected return.

Asset returns are not normally distributed, and their comovements are not exactly captured by correlation coefficients. Heavy tails are a feature of distributions of financial time series, and the probability of two asset prices exhibiting big movements in the same direction may be much higher than it would be with a bivariate Gaussian. These two observations suggest a relatively high probability that two markets experience large falls at the same time. The implications for many asset allocation strategies are quite serious. Furthermore, the magnitude of comovements when the markets turn down and up may be different.

The study of asymmetric dependence is important for two reasons. First, hedging relies crucially on the correlations between the assets hedged and the financial instruments used. The presence of asymmetric correlations can cause problems in hedging effectiveness. Second, though standard investment theory advises portfolio diversification, the value of this advice might be questionable if all stocks tend to fall as the market falls.

One essential problem is that most existing tests which designed to detect the existence of asymmetric dependence can hardly further facilitate future investment or risk management. Because no matter the asymmetric correlations or the asymmetric dependence coefficient, they are time-varying, and difficult to predict. It is unarguable that out-of-sample prediction of asymmetry in correlations is important because it affects both returns and risks in the future. For example, Gupta and Donleavy (2009) and Virbickaite et al. (2016) both showed that asymmetric correlations are vital in out-of-sample asset allocation. Whether financial markets become more interdependent, or whether the portfolios become more related to each other during a turmoil period, is a critical task of asset allocation. We suggest that the risk management must be aware of those pitfalls and the sophisticated measuring of dependence is needed to model the risks of the real world.

5 Conclusion

Asymmetric dependency in financial data have been investigated by many studies. The multivariate volatility models (the DCC-family models) and the Copulas are two most common and most essential methodology to detect the dependency among financial time series. We introduce them in detail and give some empirical results derived from them.

However, there still remains two major problems. First, by these methods, the empirical evidence of asymmetric dependence is not widespread as expected. Researchers may have given explains for a specific data that why it does not show the asymmetric phenomenon, but the generally conclusive causes of asymmetric dependency are unclear. Second, the asymmetric correlations and the dependent coefficients are time-varying and difficult to predict. The existing asymmetry measurements can only tell the dependence structure in the past, thus cannot be directly applied for future investment. The cause and predictability of asymmetric dependence in financial time series may be the focal points of future research.

A　Asymmetric Generalized Dynamic Conditional Correlation (AG-DCC)models

The model specification of AG-DCC model is defined as:

$$Q_t=(\bar{Q}-A'\bar{Q}A-B'\bar{Q}B-G'\bar{Q}^{-}G)+A'u_{t-1}u_{t-1}A+B'Q_{t-1}B+G'u_t^{-}u'^{-}_tG,$$

where A, B and G are $k\times k$ matrices, u_t^{-} are the zero-threshold standardized error which are equal to u_t when less than zero else zero otherwise. $\bar{Q}$ and $\bar{Q}^{-}$ are the unconditional covariance matrices of u_t and u_t^{-} respectively. Resulting from the fact that this specification has high dimensionality, restricted models have been used, including the scalar, diagonal and symmetric versions:

- DCC：$G=[0]$, $A=\sqrt{a}$, $B=\sqrt{b}$
- A-DCC：$G=\sqrt{g}$, $A=\sqrt{a}$, $B=\sqrt{b}$
- G-DCC：$G=[0]$.

B　Copula Families

B.1　The Frechet Family

Through a convex combination we can obtain the Frechet family copula. It has the representation as:

$$C(u)=\phi\prod_d(u)+(1-\phi)M_d(u),\quad u\in[0,\ 1]^d$$

where $\phi\in[0,\ 1]$.

B.2　The Archimedean Family

According to McNeil (2009), the copulas through a generator satisfying some suitable assumptions are belong to this family. An Archimedean generator is any decreasing and continuous function $\psi:[0,\infty)\to[0,1]$.

A d-dimensional copula C is called Archimedean if it satisfies that:

$$C_\psi(u)=\psi(\psi^{-1}(u_1)+\cdots+\psi^{-1}(u_d)),$$

Gumbel Copula The generator $\psi(t)=\exp(-t^{1/\theta})$, for $\theta\geq1$, and the copula representation is:

$$C_\phi(u)=\exp\left(-\left(\sum_{i=1}^{d}(-\log u_i)^\phi\right)^{1/\phi}\right),\quad \phi\geq1.$$

Clayton Copula The generator $\psi_\phi(t)=(max\{1+\phi t,\ 0\})^{-1/\phi}$, for $\phi\in\left[\frac{-1}{d-1},0\right)\cap(0,\infty)$, and the copula representation is:

$$C_\phi(u)=\left(\max\left\{\sum_{i=1}^{d}u_i^{-\phi}-(d-1),\ 0\right\}\right)^{-1/\phi},\ \ \phi\geq\frac{-1}{d-1},\ \ \phi\neq 0.$$

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