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Backward Stochastic Differential Equations and Solutions

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Abstract

In this paper, we consider backward stochastic differential equations (BSDEs for short). We are interested in two topics: The existence of L^p solutions to BSDEs with non-Lipschitz generators, and the higher order differentiability of solutions in the sense of the Malliavin calculus.

First, we deal with BSDEs with linear growth generators and show directly the existence of L^p solutions by constructing a Cauchy sequence of solutions to BSDEs approximationg the original one. Second, we will argue the differentiability of solutions in the sense of the Malliavin calculus. It is known that a solution is differentiable and the derivative is also a solution to a linear BSDE. Under additional conditions, we will show that the higher order differentiability of a solution to a BSDE and that it also becomes a solution to a linear BSDE.

Contents

Chapter 1

Introduction

1.1 Backward Stochastic Differential Equations

Backward stochastic differential equations containing general nonlinear cases are first introduced by Pardoux and Peng [23]. Since then, these equations have been studied by a lot of researchers and known to have various applications on pricing and hedging financial derivatives, stochastic optimal control, connection with partial differential equations and so on.

General BSDEs are formulated as follows:

$$\begin{cases} -dY_t = f(t, Y_t, Z_t)dt - Z_t^* dW_t, \quad 0 \le t \le T, \\ Y_T = \xi, \end{cases}$$

or, equivalently,

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s^* dW_s, \quad 0 \le t \le T, \tag{1.1}$$

where T is a positive constant, ξ is a d-dimensional random variable, $(W_t)_{0 \le t \le T}$ is an n-dimensional standard Brownian motion, the generator f is a d-dimensional random function defined on $[0, T] \times \mathbb{R}^d \times \mathbb{R}^{n \times d}$ and the notation " $*$ " represents the transpose of a matrix. We say a pair of $\mathbb{R}^d$- and $\mathbb{R}^n$-valued adapted processes (Y, Z) is a solution if (Y, Z) satisfies the equation.

Stochastic differential equations (SDEs) are kinds of differential equations with randomness given by stochastic integration, and their solutions are stochastic processes represented by one symbol generally. SDEs are often called forward SDEs in the context of contrasting to BSDEs since they are usually discussed under initial conditions are given. On the other hand, it is characteristic that BSDEs have terminal conditions and each of solutions consists of two processes symbolized by Y and Z such that Z enjoys the role of controlling Y to achieve the terminal conditions. Furthermore, it is important that the solutions are adapted. The solutions to the equations (1.1) could be considered formally by the framework of usual forward SDEs. Such solutions, however, might fail to be adapted, that is, they might contain future information. This would impact significantly on applications.

In mathematical finance, for instance, pricing or hedging contingent claims is an important problem. The writers of claims have to replicate them on the maturity by determining asset allocation, at which time only market information obtained up to that time can be used. In this situation, for a solution to a BSDE (Y, Z), Y and Z correspond to a wealth process and a hedging strategy, respectively. Therefore, we see that adaptedness of (Y, Z) is necessary for consideration of real typical problems. In addition, studying properties of solutions is quite important not only for mathematical interest but also for applying to the problems.

1.2 Literature on BSDE and Our Work

Nonlinear BSDEs are first introduced by Pardoux and Peng [23] and they proved the existence and uniqueness of solution under Lipschitz condition on f. Since then, BSDEs have been studied by a lot of researchers in connection with mathematical finance. It is known that ξ, Y and Z correspond respectively to a contingent claim, a value of a replicating portfolio and a replicating strategy. And in connection with stochastic optimal control, BSDEs with values in Hilbert spaces are studied by [3, 9, 10, 11, 12].

In this paper, we are interested in two themes on the equations: First, the existence of L^p solutions under lack of Lipschitz condition on generators f, and second, smoothness of solutions in the sense of the Malliavin calculus.

We begin with the explanation of the first one. As for L^p $(p > 1)$ solutions to the BSDE, El Karoui et al. [7] proved an existence and uniqueness result when f is Lipschitz continuous and ξ is in L^p by using a fixed-point theorem. A natural question then arises whether the Lipschitz condition can be relaxed. On account of the standard forward SDEs, the linear growth condition of the generator seems to be a candidate for a weaker condition to guarantee the existence and the L^p-integrability of solutions. When f is continuous and of linear growth order and ξ is in L^p, the existence results were shown by Lepeltier and San Martin [17] for $p = 2$, by Chen [6] for $1 < p \leq 2$ and after them by Fan and Jiang [8] for general $p > 1$. In these papers, a key role is played by an approximation sequence. When $1 < p \leq 2$, the existence was obtained by proving that the sequence is a Cauchy one. When $p > 2$, an L^p solution was constructed by taking advantage of a stopping time argument. And, it remained open to prove for the sequence to be a Cauchy one when $p > 2$.

As for BSDEs and the Malliavin calculus, our second interest, Pardoux and

Peng [24] and El Karoui, Peng and Quenez [7] studied the differentiability in the sense of the Malliavin calculus. They showed that the solution is differentiable under some conditions and the Malliavin derivative of the solution is also a solution of a linear BSDE. In addition, they found the relation between Y and Z; $D_tY_t = Z_t$. For the notation "D_t", see Section 2. Since then, BSDEs have been studied via the Malliavin calculus from some viewpoints such as numerical simulations [5, 13] and densities [1, 2, 20]. Mastrolia et. al. [21] studied new conditions, which can be applicable also to quadratic growth BSDEs, to enable solutions to be differentiable. On the higher order differentiability of solutions, Lin [18] studied the second order differentiability under similar conditions in [7, 24]. Then arises a natural question if solutions have higher than the second order differentiability and the similar property holds between Y and Z under the same kind of assumptions as [7, 18, 24].

In this paper, we will discuss the two themes mentioned earlier; the existence of L^p solutions under continuity and linear growth condition on f, and higher order differentiability of solutions in the sense of the Malliavin calculus. On the first one, one-dimensional BSDEs are dealt with: The comparison theorem, Theorem 2.1.2, plays an important role to show the convergence of the sequence of solutions to the approximating BSDEs, which yields the existence of L^p solution. Next, the higher order differentiability of solutions in the sense of the Malliavin calculus is discussed. Under our notation, the derivative of a solution takes values on the Cameron-Martin space. Then in order to deal with the higher order Malliavin derivatives, we introduce BSDEs which take values on Hilbert spaces. After showing the differentiability of solutions on Hilbert spaces, we will discuss the infinite differentiability of solutions on $\mathbb{R}^d$. Showing the third or higher differentiability of solutions needs additional conditions, under which the result on the differentiability of solutions on Hilbert spaces can be used. In comparison with the results [7, 18, 24], our result is new in the point that we establish a framework to deal with any order differentiability of solutions simultaneously via the differentiability of solutions to BSDEs with values on Hilbert spaces as well as showing just higher than the second differentiability of solutions.

This paper is organized as follows. In the rest of this chapter, we introduce some notations on the Malliavin calculus as well as definitions on BSDEs and solution spaces. The paper is separated into two parts corresponding to our two interest, Chapter 2 and 3, respectively. In Chapter 2, the existence of L^p solutions to BSDEs with linear growth generators is shown. We construct an approximation sequence of solutions and see it is a Cauchy one by a priori estimates, Proposition 2.2.1. In Chapter 3, we discuss the differentiability of solutions in the sense of the Malliavin calculus. In the chapter, we introduce the first order differentiability result [7]. Under our notation, the derivative of a solution to a BSDE is also a solution to a linear BSDE on a Hilbert space. In order to show the higher order differentiability, we introduce a linear BSDE on a Hilbert space and consider the differentiability of a solution to the BSDE. Then, the second order differentiability of a solution is shown. The rest of the chapter is devoted to showing the higher order differentiability of a solution under a condition on f and boundedness assumption of the Malliavin derivative of Z. Moreover, some examples are discussed.

1.3 Notations

In this section, we introduce some preliminaries of notations used hereafter.

1.3.1 BSDE and solution spaces

Let $T > 0$ be fixed throughout this paper. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $(W_t)_{0\le t\le T}$ be an n-dimensional Brownian motion defined on the probability space and the filtration $(\mathcal{F}_t)_{0\le t\le T}$ be the Brownian filtration augmented by all P-negligible sets. We consider the following BSDEs on a real separable Hilbert space $\mathcal{K}$;

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s \cdot dW_s, \quad 0 \le t \le T, \tag{1.2}$$

where ξ is a $\mathcal{K}$-valued $\mathcal{F}_T$-measurable random variable, $f(t, \omega, y, z)$ is a function, which is called a generator, defined on $[0, T] \times \Omega \times \mathcal{K} \times \mathcal{K}^n$ with values in $\mathcal{K}$ which is $(\mathcal{F}_t)$-progressively measurable for each y, z, and "$\cdot$" represents the Euclidean type product, that is, $z \cdot w = \sum_{j=1}^n z^j w^j$ for $z \in \mathcal{K}^n, w \in \mathbb{R}^n$.

Let $p > 1$. Denote by $\mathscr{S}^p_{rc}(\mathcal{K})$ the space of all $\mathcal{K}$-valued adapted processes $\eta = (\eta_t)_{0\le t\le T}$ whose sample paths are right continuous with left-hand limits (RCLL for short). We note that the space is complete under the norm

$$\|\eta\|_{\mathscr{S}^p_{rc}(\mathcal{K})} := \left\{ E\left[\sup_{0\le t\le T} \|\eta_t\|^p_{\mathcal{K}} \right] \right\}^{\frac{1}{p}} < \infty.$$

The closed subspace in $\mathscr{S}^p_{rc}(\mathcal{K})$ of all $\mathcal{K}$-valued adapted processes with continuous sample paths is denoted by $\mathscr{S}^p(\mathcal{K})$. And $\mathscr{H}^p(\mathcal{K})$ represents the Banach space of all $\mathcal{K}$-valued progressively measurable processes $\zeta = (\zeta_t)_{0\le t\le T}$ endowed with the norm

$$\|\zeta\|_{\mathscr{H}^p(\mathcal{K})} := \left\{ E\left[\left(\int_0^T \|\zeta_s\|^2_{\mathcal{K}}\, ds \right)^{\frac{p}{2}} \right] \right\}^{\frac{1}{p}} < \infty.$$

Definition 1.3.1. *A pair (Y, Z) which consists of $\mathcal{K}$-valued continuous adapted process and $\mathcal{K}^n$-valued progressively measurable process is said to be a solution to* *the BSDE* (1.2) *with respect to the pair* (f,ξ) *if* Y *and* Z *satisfy*

$$\int_0^T \{\|f(s,Y_s,Z_s)\|_{\mathcal{K}} + \|Z_s\|^2_{\mathcal{K}^n}\}ds < \infty \quad a.s.$$

and (1.2). *In addition, we call* (Y,Z) *an* L^p *solution if* $(Y,Z) \in \mathscr{S}^p(\mathcal{K})\times\mathscr{H}^p(\mathcal{K}^n)$.

In what follows, we omit to specify the space $\mathcal{K}$ if $\mathcal{K} = \mathbb{R}$ and write $\mathscr{S}^p(\mathbb{R})$ and $\mathscr{H}^p(\mathbb{R})$ as $\mathscr{S}^p$ and $\mathscr{H}^p$.

1.3.2 The Wiener space and the Malliavin derivatives

Hereafter, we introduce Wiener space and differentiation on Wiener space briefly. For more detail, see [25, 14, 22].

In Chapter 3, we assume $(\Omega,\mathcal{F},P)$ is the n-dimensional Wiener space, i.e., Ω is the space of $\mathbb{R}^n$-valued continuous functions defined on $[0,T]$ starting at the origin endowed with the uniform convergence norm, $\mathcal{F} = \mathcal{F}_T$, $\mathcal{F}_t = \sigma(\omega_s; s \leq t, \omega \in \Omega) \vee \mathcal{N}$ and P is the Wiener measure, that is the measure under which coordinate mapping process becomes Brownian motion, where ω_s is the value of $\omega \in \Omega$ at time $s \in [0,T]$, $\mathcal{N}$ represents the collection of all P-negligible sets. The Cameron-Martin subspace H is the subspace of absolutely continuous functions whose Radon-Nikodym derivatives are square integrable on $[0,T]$. H is a real separable Hilbert space under the inner product;

$$\langle h_1,h_2\rangle_H = \int_0^T \dot{h}_1(t)\cdot\dot{h}_2(t)dt, \quad h_1,h_2 \in H,$$

where we write the Radon-Nikodym derivative of $h \in H$ as $\dot{h}$ and "$\cdot$" represents the Euclidean inner product.

Let $\mathcal{P}$ be the set of all functionals of the form $\phi = p(l_1,\ldots,l_m)$ with $m = 0,1,2,\ldots,$ polynomials p defined on $\mathbb{R}^m$ and continuous linear functionals l_j on Ω, and $\mathcal{P}(\mathcal{K})$ be the set of $\phi = \sum_{j=1}^{m'}\phi_j e_j$ for all $m' = 1,2,\ldots$, $\phi_j \in \mathcal{P}$ and $e_j \in \mathcal{K}$. Then we define the Malliavin derivative of ϕ as follows:

$$\nabla\phi = \sum_{j=1}^{m}\frac{\partial\phi}{\partial x^j}(l_1,\ldots,l_m)l_j \in H, \quad \text{if } \phi \in \mathcal{P},$$
$$\nabla\phi = \sum_{j=1}^{m'}\nabla\phi_j \otimes e_j \in H\otimes\mathcal{K}, \quad \text{if } \phi \in \mathcal{P}(\mathcal{K}),$$

where, by the Riesz representation theorem, each l_j is considered an element of H, and for real separable Hilbert spaces E_1 and E_2, $E_1\otimes E_2$ represents the Hilbert space of all Hilbert-Schmidt operators $E_1 \to E_2$, and, for $e^1 \in E_1, e^2 \in E_2$, $e^1\otimes e^2$

represents the Hilbert-Schmidt operator: $E_1 \ni e \mapsto \langle e^1, e\rangle_{E_1} e^2 \in E_2$. $E_1 \otimes E_2$ has an inner product given by $\langle A, B\rangle_{E_1\otimes E_2} = \sum_{j=1}^{\infty} \langle Ae_j^1, Be_j^1\rangle_{E_2}$, where $(e_j^1)_{j=1,2,\dots}$ is a complete orthonormal system of E_1.

By closability of the operator ∇, we extend the domain $\mathcal{P}(\mathcal{K})$ to $\mathbb{D}^{k,p}(\mathcal{K})$ by completion under the norm:

$$\|\phi\|_{k,p} = \sum_{j=0}^{k} \left\{ E\left[\left\|\nabla^j \phi\right\|^p_{H^{\otimes j}\otimes\mathcal{K}} \right] \right\}^{1/p}.$$

$\mathbb{L}^a_{m,p}(\mathcal{K})$ is denoted by the set of $\mathcal{K}$-valued progressively measurable processes $u = (u(t))_{0\le t\le T}$ such that

- for each $t \in [0,T]$, $u(t) \in \mathbb{D}^{m,p}(\mathcal{K})$,
- for each $k = 1, 2, \dots, m$, $\nabla^k u(\cdot)$ admits a progressively measurable version,
- $\|u\|_{\mathbb{L}^a_{m,p}(\mathcal{K})} := E\left[\sum_{k=0}^{m} \left(\int_0^T \left\|\nabla^k u(t)\right\|^2_{H^{\otimes k}\otimes\mathcal{K}} dt \right)^{\frac{p}{2}} \right]^{\frac{1}{p}} < \infty.$

It is known that, if $\|u^i - u^j\|_{\mathbb{L}^a_{m,p}(\mathcal{K})} \to 0$ $(i, j \to \infty)$, then there exists $u \in \mathbb{L}^a_{m,p}(\mathcal{K})$ such that $\|u^i - u\|_{\mathbb{L}^a_{m,p}(\mathcal{K})} \to 0$ $(i \to \infty)$.

We define an attendant operator D on ∇ as follows. Let $(h^i)_{i=1}^{\infty}$ and $(k^j)_{j=1}^{\infty}$ be complete orthonormal systems of H and $\mathcal{K}$ respectively. Now we can get an isometric isomorphism between $H \otimes \mathcal{K}$ and $L^2([0,T], \mathcal{K}^n)$ by identifying $K = \sum_{i,j} a_{ij} h^i \otimes k^j \in H \otimes \mathcal{K}$ and $\dot{K}(\cdot) = \sum_{i,j} a_{ij} \dot{h}^i(\cdot) k^j \in L^2([0,T], \mathcal{K}^n)$. We denote by $\tilde{K}$ the isometric isomorphism $H \otimes \mathcal{K} \to L^2([0,T], \mathcal{K}^n)$ and $D := \tilde{K}\nabla$, that is, for $X \in \mathbb{D}^{1,2}(\mathcal{K})$, $DX \in L^2([0,T], \mathcal{K}^n)$. For $v \in H \otimes \mathcal{K}$, $\tilde{K}_u v$ represents the value of v at $u \in [0,T]$ and we denote $D_u X = \tilde{K}_u \nabla X$ for $X \in \mathbb{D}^{1,2}(\mathcal{K})$ and $u \in [0,T]$. Then we see that

$$(\nabla X)h = \int_0^T D_u X \cdot \dot{h}(u) du, \quad h \in H,$$
$$\|\nabla X\|^2_{H\otimes\mathcal{K}} = \int_0^T \|D_u X\|^2_{\mathcal{K}^n} du.$$

In the same manner, we can define k-th order operators $\tilde{K}^k \colon H^{\otimes k} \otimes \mathcal{K} \to L^2([0,T]^k, \mathcal{K}^{n^k})$ and $D^k = \tilde{K}^k \nabla^k$, $\tilde{K}^k_{u_1,\dots,u_k}$ and $D^k_{u_1,\dots,u_k}$ for $k = 2, 3, \dots$ and $u_1, \dots, u_k \in [0,T]$. Then it holds that for $h_1, \dots, h_k \in H$,

$$(\nabla^k X)(h_1 \otimes \cdots \otimes h_k) = \int_{[0,T]^k} du_1 \cdots du_k \sum_{j_1,\dots,j_k=1}^{n} (D^k_{u_1,\dots,u_k} X)^{j_1,\dots,j_k} \dot{h}_1^{j_1}(u_1) \cdots \dot{h}_k^{j_k}(u_k),$$

$$\left\|\nabla^k X\right\|^2_{H^{\otimes k}\otimes\mathcal{K}} = \int_{[0,T]^k} \left\|D^k_{u_1,\ldots,u_k} X\right\|^2_{\mathcal{K}^{n^k}} du_1\cdots du_k,$$

where $(D^k_{u_1,\ldots,u_k}X)^{j_1,\ldots,j_k}$ represents the $(j_1,\ldots,j_k)$-th component of $D^k_{u_1,\ldots,u_k}X$ and $\dot{h}^j_i$ does also the same.

Finally, we give a term on versions of stochastic processes. Let A be a subset of the Euclidean space and $g(t,\omega,x)$ be a function defined on $[0,T]\times\Omega\times A$. We also say that g admits a progressively measurable version if there exists a measurable function $\tilde{g}(t,\omega,x)$ defined on $[0,T]\times\Omega\times A$ such that

- for each $x\in A$, $(\tilde{g}(t,x))_{0\le t\le T}$ is a progressively measurable process,
- for each $(t,x)\in[0,T]\times A$, $g(t,x)=\tilde{g}(t,x)$, a.s.

Chapter 2

L^p Solutions to BSDEs with Linear Growth Generators

In this chapter, we will discuss the existence of L^p solutions of $\mathbb{R}$-valued BSDEs (1.2) with linear growth generators.

2.1 Assumptions

We use the following assumptions (H1)-(H3):

(H1) There exists a positive constant K and a non-negative predictable process $(g_t)_{0\le t\le T}$ such that

$$E\left[\left(\int_0^T g_s ds\right)^p\right] < \infty, \quad |f(t,\omega,y,z)| \le g_t(\omega) + K(|y|+|z|)$$
$$\text{for any } (t,\omega,y,z) \in [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d.$$

(H2) For each $(t,\omega) \in [0,T]\times\Omega$, $f(t,\omega,y,z)$ is continuous in (y,z).

(H3) $\xi \in L^p$.

In the case $p > 1$ and the generator is Lipschitz, the existence and uniqueness of L^p solution is known.

Theorem 2.1.1 (El Karoui et al. [7]). *Assume that f is uniformly Lipschitz in (y,z), i.e., there exists a positive constant C such that*

$$|f(t,\omega,y_1,z_1) - f(t,\omega,y_2,z_2)| \le C(|y_1-y_2| + |z_1-z_2|)$$
$$\textit{for any } \ (t,\omega)\in[0,T]\times\Omega,\ y_1,y_2\in\mathbb{R},\ z_1,z_2\in\mathbb{R}^d.$$

And assume (H3) holds and

$$E\left[\left(\int_0^T |f(s,0,0)|ds\right)^p\right] < \infty.$$

Then, BSDE (1.2) *has a unique* L^p *solution.*

It is also known that

Theorem 2.1.2 (El Karoui et al. [7])**.** *For* $i = 1, 2$*, let* f^i *be uniformly Lipschitz in* (y, z)*,* ξ^i *satisfy (H3) and*

$$E\left[\left(\int_0^T |f^i(s,0,0)|ds\right)^p\right] < \infty.$$

In addition, assume that each (Y^i, Z^i) *is the* L^p *solution to the BSDE with respect to* (f^i, ξ^i)*. If* $\xi^1 \geq \xi^2$ *a.s. and* $f^1(t, Y_t^2, Z_t^2) \geq f^2(t, Y_t^2, Z_t^2)$ $dt \times dP$*-a.e., then* $Y^1 \geq Y^2$ *a.s..*

Remark 2.1.3. In [7], the assertion of Theorems 2.1.1 and 2.1.2 are stated under the assumptions like

$$E\left[\left(\int_0^T |f(s,0,0)|^2 ds\right)^{\frac{p}{2}}\right] < \infty, \tag{2.1}$$

which is stronger than the ones in the theorems. Observing the proof in [7] carefully, we can weaken the assumption (2.1) to the one as we used.

2.2 A priori estimates

We prepare the following estimations which play a key role in the observation of this paper, by generalizing the ones in [6] used by Chen for specified solutions.

Proposition 2.2.1. (i) *Let* $p > 1$*. There exists a positive constant* C_p*, depending only on* p*, such that for any* L^p *solution* (Y, Z) *to the BSDE* (1.2) *it holds that*

$$\|Y\|_{\mathscr{S}^p}^p \leq C_p E\left[|\xi|^p + \int_0^T |Y_s|^{p-1}|f(s, Y_s, Z_s)|ds\right],$$

$$\|Z\|_{\mathscr{H}^p}^p \leq C_p \left\{E\left[|\xi|^p + \left(\int_0^T |Y_s||f(s, Y_s, Z_s)|ds\right)^{\frac{p}{2}}\right] + \|Y\|_{\mathscr{S}^p}^p\right\}.$$

Moreover, if f *satisfies (H1), then there exists a positive constant* C *depending only on* $p, K, T, E[|\xi|^p]$ *and* $E[(\int_0^T g_s ds)^p]$ *such that*

$$\|Z\|_{\mathscr{H}^p}^p \leq C(1 + \|Y\|_{\mathscr{S}^p}^{\frac{p}{2}} + \|Y\|_{\mathscr{S}^p}^p)$$

holds.

(ii) *Let $p > 1$. There exists a positive constant C_p depending only on p such that if (Y^i, Z^i) is an L^p solution to the BSDE with respect to (f^i, ξ^i), $i = 1, 2$, respectively, then*

$$\|\delta Y\|^p_{\mathscr{S}^p} \le C_p E\left[|\delta Y_T|^p + \int_0^T |\delta Y_s|^{p-1}|\delta f_s|ds\right],$$
$$\|\delta Z\|^p_{\mathscr{H}^p} \le C_p \left\{ E\left[|\delta Y_T|^p + \left(\int_0^T |\delta Y_s||\delta f_s|ds\right)^{\frac{p}{2}}\right] + \|\delta Y\|^p_{\mathscr{S}^p}\right\},$$

where $\delta Y := Y^1 - Y^2,\ \delta Z := Z^1 - Z^2,\ \delta f_s := f^1(s, Y^1_s, Z^1_s) - f^2(s, Y^2_s, Z^2_s)$.

Proof. The assertion (ii) follows from (i). Namely, put $\tilde f(t, y, z) = f^1(t, Y^2_t + y, Z^2_t + z) - f^2(t, Y^2_t, Z^2_t)$. Then, $\delta f_t = \tilde f(t, \delta Y_t, \delta Z_t)$ and the pair $(\delta Y, \delta Z) \in \mathscr{S}^p \times \mathscr{H}^p$ satisfies

$$\delta Y_t = \delta Y_T + \int_t^T \tilde f(s, \delta Y_s, \delta Z_s)ds - \int_t^T \delta Z_s \cdot dW_s, \quad 0 \le t \le T.$$

Thus, we only prove (i).

Let $p > 1$. We first estimate Y. As an elementary application of Itô's formula, we obtain

$$\begin{aligned} |Y_t|^p + \frac{p(p-1)}{2}\int_t^T |Y_s|^{p-2}\tilde{\mathbf{1}}(Y_s)|Z_s|^2 ds &\\ = |\xi|^p + p\int_t^T \operatorname{sgn}(Y_s)|Y_s|^{p-1} f(s, Y_s, Z_s)ds &\\ - p\int_t^T \operatorname{sgn}(Y_s)|Y_s|^{p-1} Z_s \cdot dW_s, &\qquad 0 \le t \le T, \end{aligned} \quad (2.2)$$

where

$$\tilde{\mathbf{1}}(y) := \begin{cases} \mathbf{1}_{\{y \ne 0\}}, & 1 < p < 2 \\ 1, & 2 \le p \end{cases}, \qquad \operatorname{sgn}(x) := \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}.$$

See also [4, Lemma 2.2]. Hence, we get

$$\begin{aligned} \sup_{0\le t\le T} |Y_t|^p \le |\xi|^p + p\int_0^T |Y_s|^{p-1}|f(s, Y_s, Z_s)|ds &\\ + 2p \sup_{0\le t\le T}\left|\int_0^t \operatorname{sgn}(Y_s)|Y_s|^{p-1} Z_s \cdot dW_s\right|. & \end{aligned} \quad (2.3)$$

By the Burkholder-Davis-Gundy inequality (the BDG inequality in short), there exists a positive constant C_1 such that

$$\begin{aligned}
&2pE\left[\sup_{0\le t\le T}\left|\int_0^t \operatorname{sgn}(Y_s)|Y_s|^{p-1}Z_s\cdot dW_s\right|\right]\\
&\le 2pC_1E\left[\left(\int_0^T |Y_s|^{2p-2}\tilde{\mathbf{1}}(Y_s)|Z_s|^2ds\right)^{\frac12}\right]\\
&\le 2pC_1E\left[\sup_{0\le t\le T}|Y_t|^{\frac p2}\left(\int_0^T |Y_s|^{p-2}\tilde{\mathbf{1}}(Y_s)|Z_s|^2ds\right)^{\frac12}\right]\\
&\le \frac12 E\left[\sup_{0\le t\le T}|Y_t|^p\right]+2p^2C_1^2E\left[\int_0^T |Y_s|^{p-2}\tilde{\mathbf{1}}(Y_s)|Z_s|^2ds\right],
\end{aligned}\tag{2.4}$$

where, to see the third inequality above, we have used the inequality

$$2ab\le \varepsilon a^2+\varepsilon^{-1}b^2,\qquad \varepsilon>0,\quad a,b\ge 0 \tag{$*$}$$

with $\varepsilon=1/2$.

By the Hölder inequality, we have

$$\begin{aligned}
&E\left[\left(\int_0^T |Y_s|^{2p-2}\tilde{\mathbf{1}}(Y_s)|Z_s|^2ds\right)^{\frac12}\right]\\
&\le E\left[\sup_{0\le t\le T}|Y_t|^{p-1}\left(\int_0^T |Z_s|^2ds\right)^{\frac12}\right]\\
&\le \left\{E\left[\sup_{0\le t\le T}|Y_t|^p\right]\right\}^{1-\frac1p}\left\{E\left[\left(\int_0^T |Z_s|^2ds\right)^{\frac p2}\right]\right\}^{\frac1p}<\infty.
\end{aligned}$$

Thus, $(\int_0^t \operatorname{sgn}(Y_s)|Y_s|^{p-1}Z_s\cdot dW_s)_{0\le t\le T}$ is a martingale. Then, taking the expectations of (2.2), we get

$$\begin{aligned}
&\frac{p(p-1)}{2}E\left[\int_0^T |Y_s|^{p-2}\tilde{\mathbf{1}}(Y_s)|Z_s|^2ds\right]\\
&\qquad\le E\left[|\xi|^p+p\int_0^T |Y_s|^{p-1}|f(s,Y_s,Z_s)|ds\right].
\end{aligned}\tag{2.5}$$

Then (2.3), (2.4) and (2.5) yield the estimation of Y.

Next, we estimate Z. By (2.2) with $p=2$, we deduce that

$$\int_0^T |Z_s|^2ds\le |\xi|^2+2\int_0^T |Y_s||f(s,Y_s,Z_s)|ds+2\sup_{0\le t\le T}\left|\int_0^t Y_sZ_s\cdot dW_s\right|.$$

Hence, it follows that

$$\left(\int_0^T |Z_s|^2 ds\right)^{\frac{p}{2}} \le C_2 \left\{ |\xi|^p + \left(\int_0^T |Y_s||f(s,Y_s,Z_s)|ds\right)^{\frac{p}{2}} + \sup_{0\le t\le T}\left|\int_0^t Y_s Z_s \cdot dW_s\right|^{\frac{p}{2}} \right\}, \quad (2.6)$$

where C_2 is a positive constant depending only on p. By the BDG inequality, there exists a positive constant C_3 depending only on p such that

$$\begin{aligned} C_2 E\left[\sup_{0\le t\le T}\left|\int_0^t Y_s Z_s \cdot dW_s\right|^{\frac{p}{2}}\right] &\le C_2 C_3 E\left[\left(\int_0^T |Y_s|^2|Z_s|^2 ds\right)^{\frac{p}{4}}\right] \\ &\le C_2 C_3 E\left[\sup_{0\le t\le T}|Y_t|^{\frac{p}{2}}\left(\int_0^T |Z_s|^2 ds\right)^{\frac{p}{4}}\right] \\ &\le 2C_2^2C_3^2 E\left[\sup_{0\le t\le T}|Y_t|^p\right] + \frac{1}{2}E\left[\left(\int_0^T |Z_s|^2 ds\right)^{\frac{p}{2}}\right], \end{aligned} \quad (2.7)$$

where, to see the third inequality above, we have used $(*)$ again with $\varepsilon = 1/2$. Then, we get the second estimation from (2.6) and (2.7).

We finally show the last assertion of (i). To do this, it is sufficient to estimate the second term of the estimation with respect to Z. By (H1) and the Hölder inequality, there exists positive constants $C_{p,K}, C_{p,K,T}$ and $C'_{p,K,T}$ which depend only on the subscripts such that

$$\begin{aligned} & E\left[\left(\int_0^T |Y_s||f(s,Y_s,Z_s)|ds\right)^{\frac{p}{2}}\right] \\ &\le C_{p,K}\left\{ E\left[\left(\int_0^T |Y_s|g_s ds\right)^{\frac{p}{2}}\right] \right. \\ &\qquad \left. + E\left[\left(\int_0^T |Y_s|^2 ds\right)^{\frac{p}{2}}\right] + E\left[\left(\int_0^T |Y_s||Z_s|ds\right)^{\frac{p}{2}}\right]\right\} \\ &\le C_{p,K,T}\left\{ \|Y\|_{\mathscr{S}^p}^{\frac{p}{2}} \left\{E\left[\left(\int_0^T g_s ds\right)^p\right]\right\}^{\frac{1}{2}} \right. \\ &\qquad \left. + \|Y\|_{\mathscr{S}^p}^p + E\left[\left(\int_0^T \left(\varepsilon^{-1}|Y_s|^2 + \varepsilon|Z_s|^2\right) ds\right)^{\frac{p}{2}}\right]\right\} \end{aligned}$$

$$\le C'_{p,K,T}\left(\|Y\|_{\mathscr{S}^p}^{\frac{p}{2}}\left\{E\left[\left(\int_0^T g_s ds\right)^p\right]\right\}^{\frac{1}{2}} + \varepsilon^{-\frac{p}{2}}\|Y\|_{\mathscr{S}^p}^p + \varepsilon^{\frac{p}{2}}\|Z\|_{\mathscr{H}^p}^p\right),$$

where, to see the second inequality above, we have used $(*)$ with $C_pC'_{p,K,T}\varepsilon^{\frac{p}{2}} = 1/2$. Then, we obtain the desired estimation. □

2.3 Existence of an L^p solution

2.3.1 Approximation of linear growth functions

According to [17], linear growth functions can be approximated by Lipschitz functions. Precisely speaking, when a generator f satisfies (H1) and (H2),

$$f_n(t,y,z) := \inf_{(u,v)\in\mathbb{R}^{d+1}}\{f(t,u,v) + n(|y-u| + |z-v|)\}, \qquad n \ge K \tag{2.8}$$

is a Lipschitz function and approximates the linear growth function f, where K is a constant appeared in (H1).

Lemma 2.3.1. *Assume (H1) and (H2) hold. Then,* (2.8) *is well-defined and the following properties i)-iv) hold:*

i) $|f_n(t,\omega,y,z)| \le g_t(\omega) + K(|y| + |z|)$ *for any* $(t,\omega,y,z) \in [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d$,

ii) $f_n \le f_{n+1} \le f, \quad n \ge K$,

iii) $|f_n(t,\omega,y_1,z_1) - f_n(t,\omega,y_2,z_2)| \le n(|y_1-y_2| + |z_1-z_2|)$ *for any* $(t,\omega) \in [0,T]\times\Omega$,

iv) *if* $(y_n,z_n) \to (y,z)$, *then* $f_n(t,\omega,y_n,z_n) \to f(t,\omega,y,z)$ *for any* $(t,\omega) \in [0,T]\times\Omega$.

2.3.2 Approximation of a solution

Let $p > 1$ and assumptions (H1)-(H3) hold. We consider the following one-dimensional BSDEs:

$$Y_t^n = \xi + \int_t^T f_n(s,Y_s^n,Z_s^n)ds - \int_t^T Z_s^n\cdot dW_s, \qquad n \ge K, \tag{2.9}$$

$$U_t = \xi + \int_t^T \{g_s + K(|U_s| + |V_s|)\}ds - \int_t^T V_s\cdot dW_s.$$

Theorem 2.1.1 assures the existence and uniqueness of L^p solution to these BSDEs. Thus, (Y^n,Z^n) and (U,V) are well-defined for $n \ge K$. Moreover, by Theorem 2.1.2 and Lemma 2.3.1-ii), we have

$$Y^n \le Y^{n+1} \le U, \qquad n \ge K. \tag{2.10}$$

Theorem 2.3.2. *(Y^n, Z^n) is a Cauchy sequence in $\mathscr{S}^p \times \mathscr{H}^p$.*

Proof. The assertion for $1 < p \le 2$ can be proved in the same manner as [6, Lemma 4]. Thus, we give the proof only for the case $p > 2$.

Since (Y^n) is non-decreasing, it admits the limit process Y. By (2.10), it follows that

$$Y^{\lceil K \rceil} \le Y^n, Y \le U, \qquad n \ge K,$$

where $\lceil \cdot \rceil$ represents the ceiling function. Thus, we have

$$|Y^n_\cdot| \le M, \quad |Y_\cdot| \le M, \qquad n \ge K, \tag{2.11}$$

where $\sup_{0 \le t \le T} |Y^{\lceil K \rceil}_t| \vee \sup_{0 \le t \le T} |U_t| =: M \in L^p$. Then, by the dominated convergence theorem, it follows that

$$E\left[\int_0^T |Y^n_s - Y_s|^{p-1} g_s ds\right] \to 0, \quad E\left[\int_0^T |Y^n_s - Y_s|^p ds\right] \to 0,$$

and thus, we get

$$E\left[\int_0^T |Y^n_s - Y^m_s|^{p-1} g_s ds\right] \to 0, \quad E\left[\int_0^T |Y^n_s - Y^m_s|^p ds\right] \to 0, \qquad \text{as } \ n, m \to \infty. \tag{2.12}$$

By Proposition 2.2.1-(ii), we have

$$\|Y^n - Y^m\|^p_{\mathscr{S}^p} \le C_p E\left[\int_0^T |Y^n_s - Y^m_s|^{p-1} |f_n(s, Y^n_s, Z^n_s) - f_m(s, Y^m_s, Z^m_s)| ds\right], \tag{2.13}$$

$$\|Z^n - Z^m\|^p_{\mathscr{H}^p} \le C_p \left\{ E\left[\left(\int_0^T |Y^n_s - Y^m_s| |f_n(s, Y^n_s, Z^n_s) - f_m(s, Y^m_s, Z^m_s)| ds\right)^{\frac{p}{2}}\right] + \|Y^n - Y^m\|^p_{\mathscr{S}^p} \right\}. \tag{2.14}$$

We first estimate the right hand side of (2.13). By Lemma 2.3.1-i), we get

$$E\left[\int_0^T |Y^n_s - Y^m_s|^{p-1} |f_n(s, Y^n_s, Z^n_s) - f_m(s, Y^m_s, Z^m_s)| ds\right] \le 2E\left[\int_0^T |Y^n_s - Y^m_s|^{p-1} g_s ds\right] + KE\left[\int_0^T |Y^n_s - Y^m_s|^{p-1} F_{n,m}(s) ds\right], \tag{2.15}$$

where $F_{n,m}(s) := |Y_s^n| + |Z_s^n| + |Y_s^m| + |Z_s^m|$. By (2.12), we know the first term of (2.15) converges to zero. Thus, we estimate the second term of this. By the Hölder inequality and $(*)$, we have

$$
\begin{aligned}
&KE\left[\int_0^T |Y_s^n - Y_s^m|^{p-1} F_{n,m}(s)ds\right] \\
&\leq KE\left[\left(\int_0^T |Y_s^n - Y_s^m|^{2p-2}ds\right)^{\frac{1}{2}} \left(\int_0^T \{F_{n,m}(s)\}^2 ds\right)^{\frac{1}{2}}\right] \\
&\leq KE\left[\sup_{0\leq t\leq T} |Y_t^n - Y_t^m|^{\frac{p}{2}} \left(\int_0^T |Y_s^n - Y_s^m|^{p-2}ds\right)^{\frac{1}{2}} \left(\int_0^T \{F_{n,m}(s)\}^2 ds\right)^{\frac{1}{2}}\right] \\
&\leq \varepsilon E\left[\sup_{0\leq t\leq T} |Y_t^n - Y_t^m|^p\right] + \varepsilon^{-1}K^2 E\left[\int_0^T |Y_s^n - Y_s^m|^{p-2}ds \int_0^T \{F_{n,m}(s)\}^2 ds\right] \\
&\leq \varepsilon \left\|Y^n - Y^m\right\|_{\mathscr{S}^p}^p \\
&\qquad + \varepsilon^{-1}K^2 \left\{E\left[\left(\int_0^T |Y_s^n - Y_s^m|^{p-2}ds\right)^{\frac{p}{p-2}}\right]\right\}^{1-\frac{2}{p}} \\
&\qquad\qquad \times \left\{E\left[\left(\int_0^T \{F_{n,m}(s)\}^2 ds\right)^{\frac{p}{2}}\right]\right\}^{\frac{2}{p}} \\
&\leq \varepsilon \left\|Y^n - Y^m\right\|_{\mathscr{S}^p}^p \\
&\qquad + \varepsilon^{-1}K^2 T^{\frac{2}{p}} \left\{E\left[\int_0^T |Y_s^n - Y_s^m|^p ds\right]\right\}^{1-\frac{2}{p}} \\
&\qquad\qquad \times \left\{E\left[\left(\int_0^T \{F_{n,m}(s)\}^2 ds\right)^{\frac{p}{2}}\right]\right\}^{\frac{2}{p}}. \qquad (2.16)
\end{aligned}
$$

By (2.11), we have

$$\sup_{n\geq K} \left\|Y^n\right\|_{\mathscr{S}^p} < \infty.$$

Thus, by Proposition 2.2.1-(i), we see that

$$\sup_{n,m\geq K} E\left[\left(\int_0^T \{F_{n,m}(s)\}^2 ds\right)^{\frac{p}{2}}\right] < \infty.$$

Letting ε such that $C_p\varepsilon = 1/2$, by (2.12), (2.13), (2.15) and (2.16), it follows that

$$\left\|Y^n - Y^m\right\|_{\mathscr{S}^p} \to 0, \quad \text{as} \;\; n, m \to \infty.$$

By Lemma 2.3.1-i) and the Schwartz inequality, we get the following estimation for the first term of the right hand side of (2.14):

$$
\begin{aligned}
&E\left[\left(\int_0^T |Y_s^n - Y_s^m||f_n(s,Y_s^n,Z_s^n) - f_m(s,Y_s^m,Z_s^m)|ds\right)^{\frac{p}{2}}\right] \\
&\le C\left\{E\left[\left(\int_0^T |Y_s^n - Y_s^m|g_s ds\right)^{\frac{p}{2}}\right] + E\left[\left(\int_0^T |Y_s^n - Y_s^m|F_{n,m}(s)ds\right)^{\frac{p}{2}}\right]\right\} \\
&\le C\left[\|Y^n - Y^m\|_{\mathscr{S}^p}^{\frac{p}{2}}\left\{E\left[\left(\int_0^T g_s ds\right)^p\right]\right\}^{\frac{1}{2}}\right. \\
&\qquad\left. + T^{\frac{p}{4}}\|Y^n - Y^m\|_{\mathscr{S}^p}^{\frac{p}{2}}\left\{E\left[\left(\int_0^T \{F_{n,m}(s)\}^2\right)^{\frac{p}{2}}\right]\right\}^{\frac{1}{2}}\right],
\end{aligned}
$$

where C is a positive constant depending only on p. Since $\|Y^n - Y^m\|_{\mathscr{S}^p} \to 0$, we obtain $\|Z^n - Z^m\|_{\mathscr{H}^p} \to 0$. □

By Theorem 2.3.2, we denote by (Y, Z) the limit of (Y^n, Z^n) in $\mathscr{S}^p \times \mathscr{H}^p$.

Theorem 2.3.3. *(Y, Z) is an L^p solution to the BSDE* (1.2).

Proof. It is already seen that

$$\|Y^n - Y\|_{\mathscr{S}^p} \to 0, \quad \text{as } n \to \infty.$$

By the BDG inequality, we have

$$\sup_{0\le t\le T}\left|\int_0^t (Z_s^n - Z_s)\cdot dW_s\right| \to 0 \quad \text{in } L^p, \quad \text{as } n \to \infty.$$

Since $\|Y^n - Y\|_{\mathscr{S}^p} \to 0$, $\|Z^n - Z\|_{\mathscr{H}^p} \to 0$ as $n \to \infty$, we may assume

$$
\begin{aligned}
&Y_t^n \to Y_t, \quad 0 \le t \le T \quad a.s., \\
&Z^n \to Z, \quad dt \times dP\text{-}a.e.
\end{aligned}
$$

by choosing a subsequence if necessary. Thus, by Lemma 2.3.1-iv), we get

$$f_n(t, Y_t^n, Z_t^n) \to f(t, Y_t, Z_t), \quad dt \times dP\text{-}a.e.$$

Now, by Lemma 2.3.1-i), we have

$$|f_n(t, Y_t^n, Z_t^n)| \le g_t + K(|Y_t^n| + |Z_t^n|).$$

By the Hölder inequality, $Y^n \to Y$, $Z^n \to Z$ in L^1 with respect to $dt \times dP$, and then, we see that $(Y^n)_{n\ge K}$ and $(Z^n)_{n\ge K}$ are uniformly integrable with respect to

$\frac{dt}{T} \times dP$. Hence, $(f_n(\cdot, Y_\cdot^n, Z_\cdot^n))_{n \geq K}$ is uniformly integrable with respect to $\frac{dt}{T} \times dP$. Thus, we get

$$\int_0^T |f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)| ds \to 0 \quad \text{in } L^1.$$

Therefore, letting $n \to \infty$ in (2.9), we obtain (1.2). □

Chapter 3

Differentiability of solutions to BSDEs in the sense of the Mallivin calculus

3.1 BSDEs on Hilbert spaces

In this section, we present the results on the first differentiability of solutions in the sense of the Malliavin calculus.

In the rest of this paper, we assume $p \geq 2$. We introduce the assumption (A1):

1) $\xi \in \mathbb{D}^{1,p}(\mathbb{R}^d) \cap L^{2p}$, where $\xi \in L^q$ means $E[|\xi|^q] < \infty$,

2) $E[(\int_0^T |f(s,0,0)|ds)^{2p}] < \infty$,

3) for every $(t,\omega) \in [0,T] \times \Omega$, $f(t,\omega,\cdot,\cdot) \in C_b^1(\mathbb{R}^d \times \mathbb{R}^{n\times d}, \mathbb{R}^d)$ and

$$\sup_{\substack{t,\omega,y,z\\1\leq i\leq d\\1\leq j\leq n}} \{|\partial_{y^i} f(t,\omega,y,z)| + |\partial_{z^{ji}} f(t,\omega,y,z)|\} < \infty,$$

4) for each $(y,z) \in \mathbb{R}^d \times \mathbb{R}^{n\times d}$, $f(\cdot,y,z) \in \mathbb{L}^a_{1,p}(\mathbb{R}^d)$, and the version of the Malliavin derivative is denoted by $\nabla f(t,y,z)$,

5) $E[\int_0^T \|\nabla f(s,Y_s,Z_s)\|^p_{H\otimes\mathbb{R}^d}\, ds] < \infty$,

6) there exists a nonnegative progressively measurable process $(K_t)_{0\leq t\leq T}$ such that for any $(t,\omega) \in [0,T] \times \Omega$, $y_1, y_2 \in \mathbb{R}^d$, $z_1, z_2 \in \mathbb{R}^{n\times d}$,

$$\|\nabla f(t,\omega,y_1,z_1) - \nabla f(t,\omega,y_2,z_2)\|_{H\otimes\mathbb{R}^d} \leq K_t(\omega)(|y_1-y_2| + |z_1-z_2|),$$
$$E\left[\int_0^T K_s^{2p} ds\right] < \infty,$$

We say that the assumption (A1)$'$ is satisfied if

1)$'$ $\xi \in \mathbb{D}^{1,p}(\mathbb{R}^d)$,

2)$'$ $E[(\int_0^T |f(s,0,0)|ds)^p] < \infty$,

6)$'$ there exists a positive constant L such that for any $(t,\omega) \in [0,T]\times\Omega,\ y_1, y_2 \in \mathbb{R}^d,\ z_1, z_2 \in \mathbb{R}^{n\times d}$,

$$\|\nabla f(t,\omega,y_1,z_1) - \nabla f(t,\omega,y_2,z_2)\|_{H\otimes\mathbb{R}^d} \leq L(|y_1 - y_2| + |z_1 - z_2|),$$

are fulfilled instead of 1), 2) and 6) of (A1) respectively.

In repetition of the argument in [7], we see

Proposition 3.1.1. *Suppose (A1) holds. Let (Y,Z) be a unique L^{2p} solution of BSDE;*

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds - \int_t^T Z_s^* dW_s, \quad 0 \leq t \leq T. \tag{3.1}$$

Then, (Y,Z) belongs to $\mathbb{L}^a_{1,p}(\mathbb{R}^d) \times \mathbb{L}^a_{1,p}(\mathbb{R}^{n\times d})$ and $(\nabla Y, \nabla Z) \in \mathscr{S}^p(H\otimes\mathbb{R}^d) \times \mathscr{H}^p(H\otimes\mathbb{R}^{n\times d})$ solves the following $H\otimes\mathbb{R}^d$-valued BSDE:

$$\begin{aligned}\nabla Y_t = \nabla\xi - \int_{\cdot\wedge t}^{\cdot} Z_s^* ds& \\ + \int_t^T \{\nabla f(s,Y_s,Z_s) + \partial_y f(s,Y_s,Z_s)\nabla Y_s + \partial_z f(s,Y_s,Z_s)\nabla Z_s\}\, ds& \\ - \int_t^T (\nabla Z_s)^* dW_s, \quad 0 \leq t \leq T,&\end{aligned} \tag{3.2}$$

where $\partial_y f, \partial_z f$ are Fréchet derivatives with respect to $y \in \mathbb{R}^d, z \in \mathbb{R}^{n\times d}$ respectively, and $\int_{\cdot\wedge t}^{\cdot} Z_s^* ds$, $\partial_y f(s,Y_s,Z_s)\nabla Y_s$ and $\partial_z f(s,Y_s,Z_s)\nabla Z_s$ represent Hilbert-Schmidt operators;

$$\begin{aligned}&H \ni h \mapsto \int_t^T Z_s^* \dot{h}(s)ds \in \mathbb{R}^d,\\ &H \ni h \mapsto \partial_y f(s,Y_s,Z_s)\{(\nabla Y_s)h\} \in \mathbb{R}^d,\\ &H \ni h \mapsto \partial_y f(s,Y_s,Z_s)\{(\nabla Z_s)h\} \in \mathbb{R}^d.\end{aligned}$$

Moreover, $D_t Y_t = Z_t$ for almost all $t \in [0,T]$.

Remark 3.1.2. In El Karoui et al. [7, Proposition 5.3], the following assumption, stronger than the assumption (A1)-6), is used;

- for a.e. $\theta \in [0,T]$ there exists a nonnegative progressively measurable process $(K_\theta(t,\cdot))_{0\le t\le T}$ such that for any $(t,\omega)\in[0,T]\times\Omega,\ y_1,y_2\in\mathbb{R}^d,\ z_1,z_2\in\mathbb{R}^{n\times d}$,

$$|D_\theta f(t,\omega,y_1,z_1)-D_\theta f(t,\omega,y_2,z_2)|\le K_\theta(t,\omega)(|y_1-y_2|+|z_1-z_2|),$$
$$\int_0^T E\left[\left(\int_0^T |K_\theta(s)|^2 ds\right)^p\right]d\theta<\infty.$$

However, their argument works under the assumption (A1)-6).

As mentioned in the remark of [7, p.59], it holds

Corollary 3.1.3. *Suppose (A1)′ holds. Let (Y,Z) be a unique L^p solution of BSDE (3.1). Then, (Y,Z) belongs to $\mathbb{L}^a_{1,p}(\mathbb{R}^d)\times\mathbb{L}^a_{1,p}(\mathbb{R}^{n\times d})$ and $(\nabla Y,\nabla Z)\in\mathscr{S}^p(H\otimes\mathbb{R}^d)\times\mathscr{H}^p(H\otimes\mathbb{R}^{n\times d})$ solves (3.2).*

Moreover, $D_tY_t=Z_t$ for almost all $t\in[0,T]$.

Remark 3.1.4. In order to obtain the equality $D_tY_t=Z_t$, it is necessary to take a simultaneous null set with respect to time parameter t. To do this, in this paper, we prepare Lemma 3.2.2 in the next subsection, which makes us possible to obtain a version of the derivative process of a solution; $(D.Y,D.Z)$.

We now proceed to a general Hilbert space $\mathcal{K}$ to discuss higher order differentiability of solutions of real valued BSDEs. Then, we are going to consider BSDEs on Hilbert spaces and differentiability of solutions.

We can show a priori estimates and the existence and uniqueness of solution, in the same manner as [7];

Proposition 3.1.5. *There exists a positive constant C_p such that for $\xi\in L^p(\mathcal{K})$, i.e., $E[\|\xi\|^p_{\mathcal{K}}]<\infty$, and L^p solution (Y,Z) to (1.2),*

$$E\left[\sup_{0\le t\le T}\|Y_t\|^p_{\mathcal{K}}+\left(\int_0^T\|Z_s\|^2_{\mathcal{K}^n}\,ds\right)^{\frac{p}{2}}\right]\le C_pE\left[\|\xi\|^p_{\mathcal{K}}+\left(\int_0^T\|f(s,Y_s,Z_s)\|_{\mathcal{K}}\,ds\right)^p\right].$$

Theorem 3.1.6. *Suppose the following conditions hold;*

- $\xi\in L^p(\mathcal{K})$,
- $E\left[\left(\int_0^T\|f(s,0,0)\|_{\mathcal{K}}\,ds\right)^p\right]<\infty$,
- *There exists C such that for any $(t,\omega)\in[0,T]\times\Omega,\ y_1,y_2\in\mathcal{K},\ z_1,z_2\in\mathcal{K}^n$,*

$$\|f(t,\omega,y_1,z_1)-f(t,\omega,y_2,z_2)\|_{\mathcal{K}}\le C\left(\|y_1-y_2\|_{\mathcal{K}}+\|z_1-z_2\|_{\mathcal{K}^n}\right).$$

Then, there exists a unique L^p solution to the BSDE (1.2).

Remark 3.1.7. In the next subsection, we consider the following type of $\mathcal{K}$-valued BSDE;

$$Y_t = \xi + \zeta_t^T + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s \cdot dW_s, \quad 0 \le t \le T, \tag{3.3}$$

where $\zeta_t^T := -\zeta_T + \zeta_t$ with a given $\mathcal{K}$-valued continuous (resp. RCLL) process $(\zeta_t)_{0\le t\le T}$, which corresponds to $\int_0^{\cdot\wedge t} Z_s^* ds$ (resp. $\mathbf{1}_{[0,t]}(u)Z_u^*$). By letting $\tilde{Y}_t = Y_t - \zeta_t$ and $\tilde{f}(t, y, z) = f(t, y + \zeta_t, z)$, BSDE above is rewritten as

$$\tilde{Y}_t = \xi + \int_t^T \tilde{f}(s, \tilde{Y}_s, Z_s)ds - \int_t^T Z_s^* dW_s. \tag{3.4}$$

Thus, we can obtain a continuous (resp. RCLL) solution Y of (3.3) from a continuous solution $\tilde{Y}$ of (3.4).

3.2 Differentiability of solutions in the sense of the Malliavin calculus to linear BSDEs on Hilbert spaces

As in Proposition 3.1.1, the Malliavin derivative process $(\nabla Y, \nabla Z)$ of the solution (Y, Z) of (3.1) is also the solution of a linear BSDE (3.2) on Hilbert space $H \otimes \mathbb{R}^d$. Taking the Malliavin derivative of (3.2) formally, again a linear BSDE on a Hilbert space appears. We can see the same circumstances even if the solution is differentiated repeatedly. This formal argument indicates us that the higher order Malliavin derivatives of the solution are also solutions of associated linear BSDEs. In this section, thus, we focus on linear BSDEs on Hilbert spaces. We consider Malliavin differentiability of a solution to a linear BSDE and show the derivative process is also a solution of a linear BSDE on a Hilbert space.

First, we note that the existence of a version of the derivative of a solution. An equality containing the derivative of a solution, such as $(D_u Y_t, D_u Z_t)$, gives us a negligible set depending on t, not simultaneously, because the equality is given in the sense of $L^2(du)$. Therefore, it is critical to change t continuously under a fixed u. Thus, we will give the existence of a version of $(D_u Y_t, D_u Z_t)$ in order to take a simultaneous negligible set. The following lemma assures the existence of a version. We now mention that the assumptions of b and c in the lemma make sense only on canonical set-up; they correspond to ones on the derivatives of f with respect to y, z thus they are satisfied in later sections.

We prepare a term for simplification.

Definition 3.2.1. *Denote by $\mathcal{L}(\mathcal{H}, \mathcal{K})$ the space of all bounded linear operators from a Hilbert space $\mathcal{H}$ to a Hilbert one $\mathcal{K}$ with the operator norm, and denote*

$\mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H}, \mathcal{H})$.

Let $k \in \mathbb{N}$ *and* $(\alpha_t)_{0\le t\le T}$ *be an* $\mathcal{L}(\mathcal{K})$*-valued (or* $\mathcal{L}(\mathcal{K}^n, \mathcal{K})$*-valued) process. We say* $(\alpha_t)_{0\le t\le T}$ *is exchangeable if there exists an* $\mathcal{L}(\mathcal{K}^{n^k})$*-valued (resp.* $\mathcal{L}(\mathcal{K}^{n\times n^k}, \mathcal{K}^{n^k})$*-valued) process* $(\tilde{\alpha}_t)_{0\le t\le T}$ *such that for any* $\eta \in H^{\otimes k} \otimes \mathcal{K}$ *(resp.* $\eta \in H^{\otimes k} \otimes \mathcal{K}^n$*) and* $0 \le t \le T$*, it holds that* $\|\alpha_t\| = \|\tilde{\alpha}_t\|$ *and*

$$\tilde{K}^k_u(\alpha_t \eta) = \tilde{\alpha}_t \tilde{K}^k_u \eta, \quad a.e.\, u \in [0, T]^k,$$

where, for $u = (u_1, \dots, u_k) \in [0, T]^k$*,* $\tilde{K}^k_u$ *is the same as* $\tilde{K}^k_{u_1,\dots,u_k}$ *in Subsection 1.3.*

$(\tilde{\alpha}_t)_{0\le t\le T}$ *is said to be an exchangeable version of* $(\alpha_t)_{0\le\le T}$.

Lemma 3.2.2. *Let* $k \in \mathbb{N}$ *and let* $(Y, Z) \in \mathscr{S}^p(H^{\otimes k} \otimes \mathcal{K}) \times \mathscr{H}^p(H^{\otimes k} \otimes \mathcal{K}^n)$ *be a unique solution to the following* $H^{\otimes k} \otimes \mathcal{K}$*-valued linear BSDE;*

$$Y_t = \xi + \int_t^T \{a_s + b_s Y_s + c_s Z_s\} ds - \int_t^T Z_s \cdot dW_s,$$

where ξ *is an* $\mathcal{F}_T$*-measurable* $H^{\otimes k} \otimes \mathcal{K}$*-valued random variable,* $(a_t)_{0\le t\le T}$ *is an* $H^{\otimes k} \otimes \mathcal{K}$*-valued progressively measurable process,* $(b_t)_{0\le t\le T}$ *is an* $\mathcal{L}(\mathcal{K})$*-valued progressively measurable process and* $(c_t)_{0\le t\le T}$ *is an* $\mathcal{L}(\mathcal{K}^n, \mathcal{K})$*-valued progressively measurable process. Let* $q \in [2, p]$ *and suppose*

- $E[\int_{[0,T]^k} \|\tilde{K}^k_{u_1,\dots,u_k} \xi\|^q_{\mathcal{K}} du_1 \cdots du_k] < \infty$,
- $E[\int_{[0,T]^k} (\int_0^T \|\tilde{K}^k_{u_1,\dots,u_k} a_s\|_{\mathcal{K}} ds)^q du_1 \cdots du_k] < \infty$,
- $\sup_{t,\omega}(\|b_t(\omega)\|_{\mathcal{L}(\mathcal{K})} + \|c_t(\omega)\|_{\mathcal{L}(\mathcal{K}^n,\mathcal{K})}) < \infty$,
- $(b_t)_{0\le t\le T}$ *and* $(c_t)_{0\le t\le T}$ *admit exchangeable versions* $(\tilde{b}_t)_{0\le t\le T}$ *and* $(\tilde{c}_t)_{0\le t\le T}$*, respectively.*

Denote by $(\bar{\Omega}^{(k)}, \bar{\mathcal{F}}^{(k)}, \bar{P}^{(k)})$ *the completion of* $([0, T]^k \times \Omega, \mathcal{B}([0, T]^k) \otimes \mathcal{F}, \mu^{(k)} \otimes P)$*, where* $d\mu^{(k)} = \frac{du_1 \cdots du_k}{T^k}$*, and let* $\bar{\mathcal{F}_t}^{(k)} = (\mathcal{B}([0, T]^k) \otimes \mathcal{F}_t) \vee \mathcal{N}^{\bar{P}^{(k)}} = (\mathcal{B}([0, T]^k) \otimes \mathcal{F}^W_t) \vee \mathcal{N}^{\bar{P}^{(k)}}$ *for* $t \in [0, T]$*, where* $\mathcal{N}^{\bar{P}^{(k)}}$ *represents the collection of all* $\bar{P}^{(k)}$*-negligible sets. We may extend a random variable* X *on* $(\Omega, \mathcal{F}, P)$ *to one on* $(\bar{\Omega}^{(k)}, \bar{\mathcal{F}}^{(k)}, \bar{P}^{(k)})$ *by defining for* $(u, \omega) \in \bar{\Omega}^{(k)}$*,* $\bar{X}(u, \omega) = X(\omega)$*. Furthermore,* $\mathscr{S}^q(\mathcal{K}^{n^k}, \bar{P}^{(k)})$ *and* $\mathscr{H}^q(\mathcal{K}^{n\times n^k}, \bar{P}^{(k)})$ *represent the spaces* $\mathscr{S}^q(\mathcal{K}^{n^k})$ *and* $\mathscr{H}^q(\mathcal{K}^{n\times n^k})$ *defined in Subsection 1.3 under the probability space* $(\bar{\Omega}^{(k)}, \bar{\mathcal{F}}^{(k)}, \bar{P}^{(k)})$*, respectively. Then, there exists a pair* $(\bar{Y}, \bar{Z}) \in \mathscr{S}^q(\mathcal{K}^{n^k}, \bar{P}^{(k)}) \times \mathscr{H}^q(\mathcal{K}^{n\times n^k}, \bar{P}^{(k)})$ *such that*

$$\int_0^T \int_{[0,T]^k} \|\tilde{K}^k_u Y_t - \bar{Y}_t(u)\|^2_{\mathcal{K}^{n^k}} d\mu^{(k)}(u) dt$$

$$= \int_0^T \int_{[0,T]^k} \|\tilde{K}_u^k Z_t - \bar{Z}_t(u)\|^2_{\mathcal{K}^{n\times n^k}} d\mu^{(k)}(u)dt = 0 \quad (3.5)$$

and

$$\bar{Y}_t(u) = \tilde{K}_u^k \xi + \int_t^T \{\tilde{K}_u^k a_s + \tilde{b}_s \bar{Y}_s(u) + \tilde{c}_s \bar{Z}_s(u)\} ds - \int_t^T \{\bar{Z}_s(u)\}^* dW_s,$$
$$0 \le t \le T, \quad \bar{P}^{(k)}\text{-a.e.}$$

Proof. For simplicity of notation, we only prove when $k = 1$.

By Theorem 3.1.6, there exists a unique solution $(\bar{Y}, \bar{Z}) \in \mathscr{S}^q(\mathcal{K}^n, \bar{P}) \times \mathscr{H}^q(\mathcal{K}^{n\times n}, \bar{P})$ to the $\mathcal{K}^n$-valued BSDE on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}) := (\bar{\Omega}^{(1)}, \bar{\mathcal{F}}^{(1)}, \bar{P}^{(1)})$;

$$\bar{Y}_t = \tilde{K}\xi + \int_t^T \{\tilde{K}a_s + \tilde{b}_s \bar{Y}_s + \tilde{c}_s \bar{Z}_s\} ds - \int_t^T \bar{Z}_s^* dW_s, \quad 0 \le t \le T.$$

$H \otimes \mathcal{K}$-valued process $(\eta_t)_{0\le t\le T}$ and $H \otimes \mathcal{K}^n$-valued process $(\zeta_t)_{0\le t\le T}$ are defined by

$$\eta_t \colon H \ni h \mapsto \int_0^T \bar{Y}_t(u) \cdot \dot{h}(u) du \in \mathcal{K},$$
$$\zeta_t \colon H \ni h \mapsto \int_0^T \{\bar{Z}_t(u)\}^* \dot{h}(u) du \in \mathcal{K}^n.$$

Let $(h^j)_{j=1,2,\ldots}$ be a complete orthonormal system of H. Then, for all j, we get

$$\begin{aligned}
&\int_0^T \bar{Y}_t(u) \cdot \dot{h}^j(u) du \\
&= \int_0^T \tilde{K}_u \xi \cdot \dot{h}^j(u) du \\
&\quad + \int_t^T \left\{ \int_0^T \tilde{K}_u a_s \cdot \dot{h}^j(u) du + b_s \int_0^T \bar{Y}_s(u) \cdot \dot{h}^j(u) du + c_s \int_0^T \{\bar{Z}_s(u)\}^* \dot{h}^j(u) du \right\} ds \\
&\quad + \int_0^T \left(\int_t^T \{\bar{Z}_s(u)\}^* dW_s \right) \cdot \dot{h}^j(u) du.
\end{aligned}$$

By the representation of elements of $L^2([0,T], \mathcal{K}^n)$, we know that $\bar{Z}_s(u)$ is represented as $\sum_{i,j=1}^\infty a_{ij}(s) \dot{h}^i(u) (k^j)^*$, where $(k^j)_{j=1,2,\ldots}$ is a complete orthonormal system of $\mathcal{K}^n$. Then, for all j, we obtain

$$\int_0^T \left(\int_t^T \{\bar{Z}_s(u)\}^* dW_s \right) \cdot \dot{h}^j(u) du = \int_t^T \left(\int_0^T \{\bar{Z}_s(u)\}^* \dot{h}^j(u) du \right) \cdot dW_s.$$

Thus, for all j, we get

$$\eta_t h^j = \int_0^T \tilde{K}_u \xi \cdot \dot{h}^j(u) du + \int_t^T \left\{ \int_0^T \tilde{K}_u a_s \cdot \dot{h}^j(u) du + b_s(\eta_s h^j) + c_s(\zeta_s h^j) \right\} ds$$

$$- \int_t^T \zeta_s h^j \cdot dW_s.$$

Then, it follows

$$\eta_t = \xi + \int_t^T \{a_s + b_s \eta_s + c_s \zeta_s\} ds - \int_t^T \zeta_s \cdot dW_s.$$

By uniqueness of solution, we obtain $\eta = Y, \zeta = Z$ in $\mathscr{S}^q(H \otimes \mathcal{K}, \bar{P}) \times \mathscr{H}^q(H \otimes \mathcal{K}^n, \bar{P})$. Thus, (3.5) is satisfied. □

Remark 3.2.3. η in the proof is continuous because, by the identity

$$\frac{1}{T} E\left[\int_0^T \sup_{0 \le t \le T} \|\bar{Y}_t(u)\|_{\mathcal{K}}^2 du\right] = E_{\bar{P}}\left[\sup_{0 \le t \le T} \|\bar{Y}_t\|_{\mathcal{K}}^2\right] < \infty,$$

the dominated convergence theorem and the continuity of $\bar{Y}$, it follows

$$\|\eta_t - \eta_s\|_{H \otimes \mathcal{K}}^2 = \int_0^T \|\bar{Y}_t(u) - \bar{Y}_s(u)\|_{\mathcal{K}}^2 du \to 0 \ (t \to s).$$

Therefore, we can use the uniqueness of solution.

Remark 3.2.4. In the general theory of BSDE, a solution Z is constructed by the martingale representation theorem (see [7]). The theorem requires the filtration to be the Brownian. In the proof of the lemma, the filtration is not a one generated by the Brownian motion extended to $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ but a product σ-field containing the original Brownian filtration. Then, we can see that the martingale representation on the extended probability space version holds as usual.

In what follows, the notation of derivatives of solutions, such as $(D_u Y_t, D_u Z_t)$, $(D_u \nabla Y_t, D_u \nabla Z_t)$, $(D^2_{u,v} Y_t, D^2_{u,v} Z_t)$ and so on, are used in the sense of the lemma.

Hereafter, as in the proof of Lemma 3.2.2, $(\bar{\Omega}^{(1)}, \bar{\mathcal{F}}^{(1)}, \bar{P}^{(1)})$ is denoted by $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$.

We consider the following $\mathcal{K}$-valued linear BSDE;

$$Y_t = \xi + \zeta_t^T + \int_t^T \{A_s + B_s Y_s + \Gamma_s Z_s\} ds - \int_t^T Z_s \cdot dW_s, \quad 0 \le t \le T, \tag{3.6}$$

where $\zeta_t^T = -\zeta_T + \zeta_t$, $\zeta = (\zeta_t)_{0 \le t \le T}$ is a $\mathcal{K}$-valued continuous adapted process, $A = (A_t)_{0 \le t \le T}$ is a $\mathcal{K}$-valued progressively measurable process and $B = (B_t)_{0 \le t \le T}$ is an $\mathcal{L}(\mathcal{K})$-valued progressively measurable process and $\Gamma = (\Gamma_t)_{0 \le t \le T}$ is a $\mathcal{L}(\mathcal{K}^n, \mathcal{K})$-valued progressively measurable process.

We introduce the assumption (A2):

1) $\xi \in \mathbb{D}^{1,p}(\mathcal{K}) \cap L^{2p}(\mathcal{K})$,

2) $E[(\int_0^T \|A_s\|_{\mathcal{K}}\, ds)^{2p}] < \infty$,

3) $\sup_{t,\omega}(\|B_t(\omega)\|_{\mathcal{L}(\mathcal{K})} + \|\Gamma_t(\omega)\|_{\mathcal{L}(\mathcal{K}^n,\mathcal{K})}) < \infty$,

4) $\zeta \in \mathbb{L}^a_{1,p}(\mathcal{K}) \cap \mathscr{S}^{2p}(\mathcal{K})$,

5) for each $t \in [0,T]$, $A_t \in \mathbb{D}^{1,p}(\mathcal{K})$, and $(\nabla A_t)_{0\le t\le T}$ admits a progressively measurable version and satisfies $E[(\int_0^T \|\nabla A_s\|_{H\otimes\mathcal{K}}\, ds)^p] < \infty$,

6) for any $F \in \mathbb{D}^{1,p}(\mathcal{K})$ and $G \in \mathbb{D}^{1,p}(\mathcal{K}^n)$, BF and ΓG belong to $\mathbb{L}^a_{1,p}(\mathcal{K})$ and there exist an $\mathcal{L}(\mathcal{K}, H\otimes\mathcal{K})$-valued progressively measurable process ${}^\nabla B$, an $\mathcal{L}(\mathcal{K}^n, H\otimes\mathcal{K})$-valued progressively measurable process ${}^\nabla\Gamma$, an $\mathcal{L}(\mathcal{K})$-valued progressively measurable process $\tilde{B}$ and an $\mathcal{L}(\mathcal{K}^n,\mathcal{K})$-valued progressively measurable process $\tilde{\Gamma}$ such that $\nabla(B_tF) = {}^\nabla B_tF + \tilde{B}_t\nabla F$ and $\nabla(\Gamma_tG) = {}^\nabla\Gamma_tG + \tilde{\Gamma}_t\nabla G$,

7) $\sup_{t,\omega}\left(\left\|\tilde{B}_t(\omega)\right\|_{\mathcal{L}(\mathcal{K})} + \left\|\tilde{\Gamma}_t(\omega)\right\|_{\mathcal{L}(\mathcal{K}^n,\mathcal{K})}\right) < \infty$,
$$E\left[\left(\int_0^T \left\{\|{}^\nabla B_s\|^2_{\mathcal{L}(\mathcal{K},H\otimes\mathcal{K})} + \|{}^\nabla\Gamma_s\|^2_{\mathcal{L}(\mathcal{K}^n,H\otimes\mathcal{K})}\right\} ds\right)^p\right] < \infty.$$

We say that the assumption (A2)$'$ is satisfied if, in addition to (A2),

8) $E[\int_0^T(\int_0^T \|D_uA_s + \tilde{K}_u({}^\nabla B_sY_s) + \tilde{K}_u({}^\nabla\Gamma_sZ_s)\|_{\mathcal{K}^n}ds)^2du] < \infty$,

9) if $t < u$, $\tilde{K}_u({}^\nabla B_tY_t) = \tilde{K}_u({}^\nabla\Gamma_tZ_t) = 0$,

10) $D_\cdot\zeta \in \mathscr{S}^2_{rc}(\mathcal{K}^n, \bar{P})$

are fulfilled.

We say the assumption (A2)$''$ is satisfied if

1)$'$ $\xi \in \mathbb{D}^{1,p}(\mathcal{K})$,

2)$'$ $E[(\int_0^T \|A_s\|_{\mathcal{K}}\, ds)^p] < \infty$,

4)$'$ $\zeta \in \mathbb{L}^a_{1,p}(\mathcal{K}) \cap \mathscr{S}^p(\mathcal{K})$,

7)$'$ $\sup_{t,\omega}\left(\left\|\tilde{B}_t\right\|_{\mathcal{L}(\mathcal{K})} + \left\|\tilde{\Gamma}_t\right\|_{\mathcal{L}(\mathcal{K}^n,\mathcal{K})} + \|{}^\nabla B_s\|_{\mathcal{L}(\mathcal{K},H\otimes\mathcal{K})} + \|{}^\nabla\Gamma_s\|_{\mathcal{L}(\mathcal{K}^n,H\otimes\mathcal{K})}\right) < \infty$,

are fulfilled instead of 1), 2), 4) and 7) of (A2)$'$ respectively.

Theorem 3.2.5. *Suppose (A2) holds. Let (Y,Z) be a unique L^{2p} solution to the BSDE (3.6). Then, (Y,Z) belongs to $\mathbb{L}^a_{1,p}(\mathcal{K}) \times \mathbb{L}^a_{1,p}(\mathcal{K}^n)$ and $(\nabla Y, \nabla Z) \in \mathscr{S}^p(H\otimes\mathcal{K}) \times \mathscr{H}^p(H\otimes\mathcal{K}^n)$ solves the following $H\otimes\mathcal{K}$-valued linear BSDE;*

$$\nabla Y_t = \nabla\xi + \nabla\zeta_t^T - \int_{\cdot\wedge t}^{\cdot} Z_s ds + \int_t^T \left\{ \nabla A_s + {}^{\nabla}B_s Y_s + {}^{\nabla}\Gamma_s Z_s + \tilde{B}_s \nabla Y_s + \tilde{\Gamma}_s \nabla Z_s \right\} ds - \int_t^T \nabla Z_s \cdot dW_s, \quad 0 \le t \le T, \quad (3.7)$$

where $\int_{\cdot\wedge t}^{\cdot} Z_s ds$ *represents a Hilbert-Schmidt operator* $H \ni h \mapsto \int_t^T Z_s \cdot \dot{h}(s) ds \in \mathcal{K}$.

Moreover, under (A2)$'$, and if there exist an $\mathcal{L}(\mathcal{K}^n)$*-valued progressively measurable process* $(\bar{B}_t)_{0\le t\le T}$ *and an* $\mathcal{L}(\mathcal{K}^{n\times n}, \mathcal{K}^n)$*-valued progressively measurable process* $(\bar{\Gamma}_t)_{0\le t\le T}$ *such that for any* $\kappa \in \mathcal{K}^n$, $\boldsymbol{\kappa} \in \mathcal{K}^{n\times n}$, $h \in H$ *and* $0 \le t, u \le T$,

$$\tilde{B}_t(\kappa \cdot \dot{h}(u)) = (\bar{B}_t\kappa) \cdot \dot{h}(u), \qquad \tilde{\Gamma}_t(\boldsymbol{\kappa}^* \dot{h}(u)) = (\bar{\Gamma}_t \boldsymbol{\kappa})^* \dot{h}(u),$$
$$\|\tilde{B}_t\|_{\mathcal{L}(\mathcal{K})} = \|\bar{B}_t\|_{\mathcal{L}(\mathcal{K}^n)}, \qquad \|\tilde{\Gamma}_t\|_{\mathcal{L}(\mathcal{K}^n, \mathcal{K})} = \|\bar{\Gamma}_t\|_{\mathcal{L}(\mathcal{K}^{n\times n}, \mathcal{K}^n)},$$

then $(D.Y, D.Z) \in \mathscr{S}^2_{rc}(\mathcal{K}^n, \bar{P}) \times \mathscr{H}^2(\mathcal{K}^{n\times n}, \bar{P})$ *and* $D_t Y_t = D_t \zeta_t + Z_t$ *for almost all* $t \in [0, T]$.

Example 3.2.6. Let $(W_t)_{0\le t\le T}$ be a one-dimensional Brownian motion and r, θ be real constants. Suppose $\xi \in D^{1,p}(\mathcal{K}) \cap L^{2p}(\mathcal{K})$, $\phi \in \mathbb{L}^a_{1,p}(\mathcal{K}^n) \cap \mathscr{H}^{2p}(\mathcal{K}^n)$. Then, the following $\mathcal{K}$-valued BSDE;

$$Y_t = \xi - \int_{\cdot\wedge t}^{\cdot} \phi_s ds - \int_t^T (rY_s + \theta Z_s) ds - \int_t^T Z_s dW_s$$

satisfies (A2)$'$. This type of BSDE is given by taking the Malliavin derivative of the portfolio under the Black-Scholes model, where r represents an interest rate of nonrisky asset and θ does a risk premium (see [7]).

Before the proof, we mention the following lemma on differentiability of stochastic integration in the sense of the Malliavin calculus. This lemma is an extension of the result of El Karoui et al. [7, Lemma 5.1] to Hilbert space valued processes.

Lemma 3.2.7. (1) *Assume* $\zeta = (\zeta_t)_{0\le t\le T} \in \mathscr{H}^p(\mathcal{K}^n)$ *and* $\xi := \int_0^T \zeta_s \cdot dW_s \in \mathbb{D}^{1,p}(\mathcal{K})$. *Then,* ζ *admits a version* $\tilde{\zeta} = (\tilde{\zeta}_t)_{0\le t\le T} \in \mathbb{L}^a_{1,p}(\mathcal{K}^n)$ *such that* $\xi = \int_0^T \tilde{\zeta}_s \cdot dW_s$.

(2) *If* $\zeta = (\zeta_t)_{0\le t\le T} \in \mathbb{L}^a_{1,p}(\mathcal{K}^n)$, *then* $\xi(t) := \int_0^t \zeta_s \cdot dW_s \in \mathbb{D}^{1,p}(\mathcal{K})$ *for every* $t \in [0, T]$ *and*

$$\nabla\xi(t) = \int_0^t \nabla\zeta_s \cdot dW_s + \int_0^{\cdot\wedge t} \zeta_s ds,$$

where $\int_0^{\cdot\wedge t} \zeta_s ds \in H \otimes \mathcal{K}$ *represents a Hilbert-Schmidt operator* $H \ni h \mapsto \int_0^t \zeta_s \cdot \dot{h}(s) ds$.

Proof. For (2), see Shigekawa [25, Proposition 6.1]. Hence we prove (1).

First, we show the following; for any $\xi \in \mathbb{D}^{1,p}(\mathcal{K})$ which is represented as a stochastic integral of a $\zeta \in \mathbb{L}^a_{1,p}(\mathcal{K}^n)$, i.e. $\xi = \int_0^T \zeta_s \cdot dW_s$,

$$c_p \left\|\zeta\right\|_{\mathbb{L}^a_{1,p}(\mathcal{K}^n)} \leq \left\|\xi\right\|_{\mathbb{D}^{1,p}(\mathcal{K})} \leq C_p \left\|\zeta\right\|_{\mathbb{L}^a_{1,p}(\mathcal{K}^n)}, \tag{3.8}$$

where c_p and C_p are positive constants depending only on p.

In calculation below, all notations c, C represent just positive constants depending only on p and they may change from place to place. By the martingale moment inequality, we get

$$cE\left[\left(\int_0^T \|\zeta_s\|^2_{\mathcal{K}^n} ds\right)^{\frac{p}{2}}\right] \leq E\left[\|\xi\|^p_{\mathcal{K}}\right] \leq CE\left[\left(\int_0^T \|\zeta_s\|^2_{\mathcal{K}^n} ds\right)^{\frac{p}{2}}\right].$$

By (2), we see that

$$\nabla\xi = \int_0^T \nabla\zeta_s \cdot dW_s + \int_0^{\cdot} \zeta_s ds.$$

Thus, we obtain

$$\begin{aligned} E\left[\|\nabla\xi\|^p_{H\otimes\mathcal{K}}\right] &\leq CE\left[\left\|\int_0^{\cdot} \zeta_s ds\right\|^p_{H\otimes\mathcal{K}} + \left\|\int_0^T \nabla\zeta_s \cdot dW_s\right\|^p_{H\otimes\mathcal{K}}\right] \\ &\leq C\left(E\left[\left(\int_0^T \|\zeta_s\|^2_{\mathcal{K}^n} ds\right)^{\frac{p}{2}}\right] + CE\left[\left(\int_0^T \|\nabla\zeta_s\|^2_{H\otimes\mathcal{K}^n} ds\right)^{\frac{p}{2}}\right]\right), \end{aligned}$$

and it follows that

$$\|\xi\|_{\mathbb{D}^{1,p}(\mathcal{K})} \leq C_p \|\zeta\|_{\mathbb{L}^a_{1,p}(\mathcal{K}^n)}.$$

By using

$$\int_0^T \nabla\zeta_s \cdot dW_s = \nabla\xi - \int_0^{\cdot} \zeta_s ds$$

and the martingale moment inequality, we obtain

$$\begin{aligned} E\left[\left(\int_0^T \|\nabla\zeta_s\|^2_{H\otimes\mathcal{K}^n} ds\right)^{\frac{p}{2}}\right] &\leq CE\left[\left\|\int_0^T \nabla\zeta_s \cdot dW_s ds\right\|^p_{H\otimes\mathcal{K}}\right] \\ &\leq C\left\{E\left[\|\nabla\xi\|^p_{H\otimes\mathcal{K}}\right] + E\left[\left(\int_0^T \|\zeta_s\|^2_{\mathcal{K}^n} ds\right)^{\frac{p}{2}}\right]\right\} \\ &\leq C\left\{E\left[\|\nabla\xi\|^p_{H\otimes\mathcal{K}}\right] + CE\left[\left\|\int_0^T \zeta_s \cdot dW_s\right\|^p_{\mathcal{K}}\right]\right\} \end{aligned}$$

$$\leq C\left(E\left[\|\xi\|_{\mathcal{K}}^p\right]+E\left[\|\nabla\xi\|_{H\otimes\mathcal{K}}^p\right]\right).$$

Thus, we get

$$c_p\|\zeta\|_{\mathbb{L}^a_{1,p}(\mathcal{K}^n)}\leq\|\xi\|_{\mathbb{D}^{1,p}(\mathcal{K})}.$$

Thus inequalities (3.8) have been shown.

We now proceed to the proof of the assertion (2). Put $A=\{\xi\in\mathcal{P}(\mathcal{K});E[\xi]=0\}$ and $B=\{\xi=\int_0^T\zeta_s\cdot dW_s;\zeta\in\mathbb{L}^a_{1,p}(\mathcal{K}^n)\}$. Then, we know $A,B\subset\mathbb{D}^{1,p}(\mathcal{K})\cap\{\xi;E[\xi]=0\}$. For any $\xi\in A$, we see that

$$\xi=\int_0^T E\left[D_s\xi|\,\mathcal{F}_s\right]\cdot dW_s,$$

where we have used the Hilbert space version of the Clark-Ocone formula, which is easily obtained from well-known real-valued one. Thus, we get $A\subset B$. Since A is dense in $\mathbb{D}^{1,p}(\mathcal{K})\cap\{\xi;E[\xi]=0\}$ with respect to $\mathbb{D}^{1,p}(\mathcal{K})$-topology, so is B. Then, for any $\xi=\int_0^T\zeta_s\cdot dW_s\in\mathbb{D}^{1,p}(\mathcal{K})$, there exists a sequence $\xi^m=\int_0^T\zeta^m_s\cdot dW_s\in B$ such that $\|\xi^m-\xi\|_{\mathbb{D}^{1,p}(\mathcal{K})}\to0$ $(m\to\infty)$. By (3.8), $\left\|\zeta^l-\zeta^m\right\|_{\mathbb{L}^a_{1,p}(\mathcal{K}^n)}\to0$ $(l,m\to\infty)$. Thus, we can take $\tilde\zeta$ as the limit of $\zeta^m\in\mathbb{L}^a_{1,p}(\mathcal{K}^n)$. □

We give the proof of Theorem 3.2.5, modifying the argument of El Karoui et al. [7, Proposition 5.3].

Proof of Theorem 3.2.5. Let $(Y^k,Z^k)\in\mathscr{S}^{2p}(\mathcal{K})\times\mathscr{H}^{2p}(\mathcal{K}^n)$ be the Picard iterative sequence; $(Y^0,Z^0)=(0,0)$ and, for $k\geq1$,

$$Y^{k+1}_t=\xi+\zeta^T_t+\int_t^T\left\{A_s+B_sY^k_s+\Gamma_sZ^k_s\right\}ds-\int_t^T Z^{k+1}_s\cdot dW_s,\quad 0\leq t\leq T. \tag{3.9}$$

In exactly the same manner as real valued case ([7, Corollary 2.1]), $\left\|Y^k-Y\right\|_{\mathscr{S}^{2p}(\mathcal{K})}$ and $\left\|Z^k-Z\right\|_{\mathscr{H}^{2p}(\mathcal{K}^n)}$ tend to zero as $k\to\infty$.

Obviously, $(Y^0,Z^0)\in\mathbb{L}^a_{1,p}(\mathcal{K})\times\mathbb{L}^a_{1,p}(\mathcal{K}^n)$. Now we show $(Y^k,Z^k)\in\mathbb{L}^a_{1,p}(\mathcal{K})\times\mathbb{L}^a_{1,p}(\mathcal{K}^n)$ for all $k\geq0$ by induction. Assume $(Y^k,Z^k)\in\mathbb{L}^a_{1,p}(\mathcal{K})\times\mathbb{L}^a_{1,p}(\mathcal{K}^n)$. By (3.9) and that $\xi+\zeta^T_t+\int_t^T\{A_s+B_sY^k_s+\Gamma_sZ^k_s\}ds\in\mathbb{D}^{1,p}(\mathcal{K})$, we see that $Y^{k+1}_t=E[\xi+\zeta^T_t+\int_t^T\{A_s+B_sY^k_s+\Gamma_sZ^k_s\}ds|\mathcal{F}_t]\in\mathbb{D}^{1,p}(\mathcal{K})$. Then we get $Y^{k+1}\in\mathbb{L}^a_{1,p}(\mathcal{K})$. Again by (3.9), we see that $\int_0^T Z^{k+1}_s\cdot dW_s=\xi+\zeta^T_0+\int_0^T\{A_s+B_sY^k_s+\Gamma_sZ^k_s\}ds-Y^{k+1}_0\in\mathbb{D}^{1,p}(\mathcal{K})$. Then, by Lemma 3.2.7, it follows $Z^{k+1}\in\mathbb{L}^a_{1,p}(\mathcal{K}^n)$.

Taking the Malliavin derivative of (3.9), by Lemma 3.2.7, we get

$$\nabla Y^{k+1}_t=\nabla\xi+\nabla\zeta^T_t-\int_{\cdot\wedge t}^{\cdot}Z^{k+1}_s ds$$

$$+\int_t^T \left\{\nabla A_s + {}^{\nabla}B_s Y_s^k + {}^{\nabla}\Gamma_s Z_s^k + \tilde{B}_s \nabla Y_s^k + \tilde{\Gamma}_s \nabla Z_s^k\right\} ds - \int_t^T \nabla Z_s^{k+1} \cdot dW_s.$$

We now show that (Y^k, Z^k) converges in $\mathbb{L}_{1,p}^a(\mathcal{K}) \times \mathbb{L}_{1,p}^a(\mathcal{K}^n)$. Let $(Y^\nabla, Z^\nabla) \in \mathscr{S}^p(H \otimes \mathcal{K}) \times \mathscr{H}^p(H \otimes \mathcal{K}^n)$ be the unique solution of the following $H \otimes \mathcal{K}$-valued linear BSDE:

$$Y_t^\nabla = \nabla\xi + \nabla\zeta_t^T - \int_{\cdot\wedge t}^{\cdot} Z_s ds$$
$$+ \int_t^T \left\{\nabla A_s + {}^{\nabla}B_s Y_s + {}^{\nabla}\Gamma_s Z_s + \tilde{B}_s Y_s^\nabla + \tilde{\Gamma}_s Z_s^\nabla\right\} ds - \int_t^T Z_s^\nabla \cdot dW_s.$$

For $k \geq 0$, define

$$E_{k+1} = E\left[\sup_{0\leq t\leq T} \left\|\nabla Y_t^{k+1} - Y_t^\nabla\right\|_{H\otimes\mathcal{K}}^p + \left(\int_0^T \left\|\nabla Z_s^{k+1} - Z_s^\nabla\right\|_{H\otimes\mathcal{K}}^2 ds\right)^{\frac{p}{2}}\right].$$

By Proposition 3.1.5, we obtain

$$E_{k+1}$$
$$\leq C_p E\left[\left(\int_0^T \left\|{}^{\nabla}B_s(Y_s^k - Y_s) + {}^{\nabla}\Gamma_s(Z_s^k - Z_s) + \tilde{B}_s(\nabla Y_s^k - Y_s^\nabla) + \tilde{\Gamma}_s(\nabla Z_s^k - Z_s^\nabla)\right\|_{H\otimes\mathcal{K}} ds\right)^p\right.$$
$$\left. + \left(\int_0^T \left\|Z_s^{k+1} - Z_s\right\|_{\mathcal{K}^n}^2 ds\right)^{\frac{p}{2}}\right],$$

where C_p represents a positive constant depending only on p. Since

$$\left\|{}^{\nabla}B_s(Y_s^k - Y_s) + {}^{\nabla}\Gamma_s(Z_s^k - Z_s) + \tilde{B}_s(\nabla Y_s^k - Y_s^\nabla) + \tilde{\Gamma}_s(\nabla Z_s^k - Z_s^\nabla)\right\|_{H\otimes\mathcal{K}}$$
$$\leq \left\|{}^{\nabla}B_s\right\|_{\mathcal{L}(\mathcal{K},H\otimes\mathcal{K})} \left\|Y_s^k - Y_s\right\|_{\mathcal{K}} + \left\|{}^{\nabla}\Gamma_s\right\|_{\mathcal{L}(\mathcal{K}^n,H\otimes\mathcal{K})} \left\|Z_s^k - Z_s\right\|_{\mathcal{K}^n}$$
$$+ \left\|\tilde{B}_s\right\|_{\mathcal{L}(\mathcal{K})} \left\|\nabla Y_s^k - Y_s^\nabla\right\|_{H\otimes\mathcal{K}} + \left\|\tilde{\Gamma}_s\right\|_{\mathcal{L}(\mathcal{K}^n,\mathcal{K})} \left\|\nabla Z_s^k - Z_s^\nabla\right\|_{H\otimes\mathcal{K}^n},$$

we obtain

$$E_{k+1} \leq C_p(\alpha_k + \beta_k + \gamma_k), \tag{3.10}$$

where

$$\alpha_k := E\left[\left(\int_0^T \left\|{}^{\nabla}B_s\right\|_{\mathcal{L}(\mathcal{K},H\otimes\mathcal{K})} \left\|Y_s^k - Y_s\right\|_{\mathcal{K}} ds\right)^p + \left(\int_0^T \left\|{}^{\nabla}\Gamma_s\right\|_{\mathcal{L}(\mathcal{K}^n,H\otimes\mathcal{K})} \left\|Z_s^k - Z_s\right\|_{\mathcal{K}^n} ds\right)^p\right],$$
$$\beta_k := E\left[\left(\int_0^T \left\|\tilde{B}_s\right\|_{\mathcal{L}(\mathcal{K})} \left\|\nabla Y_s^k - Y_s^\nabla\right\|_{\mathcal{K}} ds\right)^p\right.$$

$$+\left(\int_0^T \left\|\tilde{\Gamma}_s\right\|_{\mathcal{L}(\mathcal{K}^n,\mathcal{K})} \left\|\nabla Z_s^k - Z_s^{\nabla}\right\|_{H\otimes\mathcal{K}^n} ds\right)^p\Bigg],$$

$$\gamma_k := E\left[\left(\int_0^T \left\|Z_s^{k+1} - Z_s\right\|_{\mathcal{K}^n}^2 ds\right)^{\frac{p}{2}}\right].$$

We note that $\gamma_k = \left\|Z^{k+1} - Z\right\|_{\mathscr{H}^p(\mathcal{K}^n)}^p$ tends to 0 as $k \to \infty$.

By the Schwarz inequality, we get

$$\begin{aligned}
\alpha_k \leq &\left\{E\left[\left(\int_0^T \left\|{}^{\nabla}B_s\right\|_{\mathcal{L}(\mathcal{K},H\otimes\mathcal{K})}^2 ds\right)^p\right]\right\}^{\frac{1}{2}} \left\{E\left[\left(\int_0^T \left\|Y_s^k - Y_s\right\|_{\mathcal{K}}^2 ds\right)^p\right]\right\}^{\frac{1}{2}} \\
&+ \left\{E\left[\left(\int_0^T \left\|{}^{\nabla}\Gamma_s\right\|_{\mathcal{L}(\mathcal{K}^n,H\otimes\mathcal{K})}^2 ds\right)^p\right]\right\}^{\frac{1}{2}} \left\{E\left[\left(\int_0^T \left\|Z_s^k - Z_s\right\|_{\mathcal{K}^n}^2 ds\right)^p\right]\right\}^{\frac{1}{2}} \\
\leq &\left\{E\left[\left(\int_0^T \left\|{}^{\nabla}B_s\right\|_{\mathcal{L}(\mathcal{K},H\otimes\mathcal{K})}^2 ds\right)^p\right]\right\}^{\frac{1}{2}} \times T^{\frac{p}{2}} \left\|Y^k - Y\right\|_{\mathscr{S}^{2p}(\mathcal{K})}^p \\
&+ \left\{E\left[\left(\int_0^T \left\|{}^{\nabla}\Gamma_s\right\|_{\mathcal{L}(\mathcal{K}^n,H\otimes\mathcal{K})}^2 ds\right)^p\right]\right\}^{\frac{1}{2}} \times \left\|Z^k - Z\right\|_{\mathscr{H}^{2p}(\mathcal{K}^n)}^p \\
\to &\ 0 \ (k \to \infty).
\end{aligned}$$

Then by Schwarz inequality, we obtain also

$$\begin{aligned}
\beta_k &\leq \left(\left\|\tilde{B}\right\|_\infty^p + \left\|\tilde{\Gamma}\right\|_\infty^p\right) E\left[\left(\int_0^T \left\|\nabla Y_s^k - Y_s^{\nabla}\right\|_{H\otimes\mathcal{K}} ds\right)^p + \left(\int_0^T \left\|\nabla Z_s^k - Z_s^{\nabla}\right\|_{H\otimes\mathcal{K}} ds\right)^p\right] \\
&\leq \left(\left\|\tilde{B}\right\|_\infty^p + \left\|\tilde{\Gamma}\right\|_\infty^p\right) (T^p + T^{\frac{p}{2}}) E_k,
\end{aligned}$$

where $\|\cdot\|_\infty$ represents the supremum with respect to $(t,\omega) \in [0,T] \times \Omega$. We can divide $[0,T]$ into subdivisions and consider the convergence of the approximation on each of them. Thus, we can assume T is sufficiently small such that

$$r := C_p\left(\left\|\tilde{B}\right\|_\infty^p + \left\|\tilde{\Gamma}\right\|_\infty^p\right) (T^p + T^{\frac{p}{2}}) < 1.$$

Fix an $\varepsilon > 0$ arbitrarily. Then, by $\alpha_k, \gamma_k \to 0$ $(k \to \infty)$, there exists $N \in \mathbb{N}$ such that, for any $k > N$, $C_p(\alpha_k + \gamma_k) < \varepsilon$. Thus, by (3.10), for any $k > N$, we obtain

$$\begin{aligned}
E_{k+1} &\leq rE_k + \varepsilon \\
&\leq r(rE_{k-1} + \varepsilon) + \varepsilon \\
&\ \vdots \\
&\leq r^{k+1-N}E_N + \varepsilon(1 + r + \cdots + r^{k-N}) \\
&\leq r^{k+1-N}E_N + \frac{\varepsilon}{1-r}.
\end{aligned}$$

Then, it follows $E_k \to 0$ $(k \to \infty)$. Therefore, we see that (Y^k, Z^k) converges to (Y, Z) in $\mathbb{L}^a_{1,p}(\mathcal{K}) \times \mathbb{L}^a_{1,p}(\mathcal{K}^n)$ and $(\nabla Y, \nabla Z)$ is equal to (Y^∇, Z^∇).

Since $Z \in \mathscr{H}^{2p}(\mathcal{K})$ and by (A2)′-8), we see

$$E\left[\int_0^T \left(\int_0^T \left\|\bar{A}^1_s(u)\right\|_{\mathcal{K}} ds\right)^2 du\right] \le C\left(E\left[\int_0^T \left(\int_0^T \left\|\bar{A}^2_s(u)\right\|_{\mathcal{K}} ds\right)^2 du + T^2 \int_0^T \|Z_u\|^2_{\mathcal{K}} du\right]\right) < \infty,$$

where C represents a positive constant, $\bar{A}^1_t(u) = \bar{A}^2_t(u) + \bar{B}_t(D_u\zeta_t + \mathbf{1}_{[0,t]}(u)Z_u)$ and $\bar{A}^2_t(u) = D_uA_t + \tilde{K}_u(^\nabla B_tY_t) + \tilde{K}_u(^\nabla\Gamma_tZ_t)$. Thus, putting $\bar{Y}_t(u) = D_uY_t - D_u\zeta_t - \mathbf{1}_{[0,t]}(u)Z_u$, by (3.7) and Lemma 3.2.2, $(\bar{Y}(\cdot), D.Z)$ belongs to $\mathscr{S}^2(\mathcal{K}) \times \mathscr{H}^2(\mathcal{K})$ and satisfies

$$\bar{Y}_t(u) = D_u\xi - D_u\zeta_T - Z_u + \int_t^T \{\bar{A}^1_s(u) + \bar{B}_s\bar{Y}_s(u) + \bar{\Gamma}_sD_uZ_s\}ds - \int_t^T (D_uZ_s)^*dW_s, \quad 0 \le t \le T, \quad du \otimes dP\text{-a.e.}$$

Since

$$E\left[\int_0^T \sup_{0\le t\le T} \mathbf{1}_{[0,t]}(u) \|Z_u\|^2_{\mathcal{K}} du\right] \le \|Z\|^2_{\mathscr{H}^2(\mathcal{K})} < \infty$$

and by (A2)′-10), we obtain $(D.Y, D.Z) \in \mathscr{S}^2_{rc}(\mathcal{K}, \bar{P}) \times \mathscr{H}^2(\mathcal{K}, \bar{P})$. Then, for almost all $u \in [0, T]$, (D_uY, D_uZ) satisfies

$$D_uY_t - D_u\zeta_t = D_u\xi - D_u\zeta_T - \mathbf{1}_{(t,T]}(u)Z_u + \int_t^T \{\bar{A}^2_s(u) + \bar{B}_sD_uY_s + \bar{\Gamma}_sD_uZ_s\}ds - \int_t^T (D_uZ_s)^*dW_s, \quad 0 \le t \le T.$$

Fix $u \in (0, T]$ satisfying the above identity and let $t < u \le s,\ t, s \in [0, T]$. Then, we obtain

$$D_uY_t - D_u\zeta_t = D_uY_s - D_u\zeta_s - Z_u + \int_t^s \{\bar{A}^2_\sigma(u) + \bar{B}_\sigma D_uY_\sigma + \bar{\Gamma}_\sigma D_uZ_\sigma\}d\sigma - \int_t^s (D_uZ_\sigma)^*dW_\sigma.$$

By $D_uY_t = D_u\zeta_t = 0$ and, for $\sigma < u$, $\bar{A}^2_\sigma(u) = 0$ and $D_uZ_\sigma = 0$, we see that

$$D_uY_s = D_u\zeta_s + Z_u - \int_u^s \{\bar{A}^2_\sigma(u) + \bar{B}_\sigma D_uY_\sigma + \bar{\Gamma}_\sigma D_uZ_\sigma\}d\sigma - \int_u^s (D_uZ_\sigma)^*dW_\sigma.$$

Then, taking $s = u$ yields that $D_uY_u = D_u\zeta_u + Z_u$. □

By the proof of Theorem 3.2.5, we see the following instantly.

Corollary 3.2.8. *Under the assumption (A2)$''$, the unique L^p solution (Y, Z) to the BSDE* (1.2) *belongs to $\mathbb{L}^a_{1,p}(\mathcal{K}) \times \mathbb{L}^a_{1,p}(\mathcal{K}^n)$ and $(\nabla Y, \nabla Z) \in \mathscr{S}^p(H \otimes \mathcal{K}) \times \mathscr{H}^p(H \otimes \mathcal{K}^n)$ solves* (3.7).

Moverover, under the same assumptions on $\tilde{B}, \tilde{\Gamma}$ as in Theorem 3.2.5, $(D.Y, D.Z) \in \mathscr{S}^2_{rc}(\mathcal{K}^n, \bar{P}) \times \mathscr{H}^2(\mathcal{K}^{n\times n}, \bar{P})$ and $D_tY_t = D_t\zeta_t + Z_t$ for almost all $t \in [0, T]$.

3.3 Second Differentiability of Solutions to BSDEs

In this subsection, we consider the second Malliavin differentiability of solutions to real valued BSDEs (3.1) by using the result in the previous subsection. The same kind of result is shown in Lin [18, Theorem 2.2].

We introduce the assumption (A3):

1) (A1) with $2p$, instead of p, holds,

2) $\xi \in \mathbb{D}^{2,p}(\mathbb{R}^d)$,

3) for each $(y, z) \in \mathbb{R}^d \times \mathbb{R}^{n\times d}$, $f(\cdot, y, z) \in \mathbb{L}^a_{2,p}(\mathbb{R}^d)$ and the the version of the second Malliavin derivative is denoted by $\nabla^2 f(t, y, z)$,

4) for each $(y, z) \in \mathbb{R}^d \times \mathbb{R}^{n\times d}$, $\partial_{y^i} f(\cdot, y, z), \partial_{z^{ji}} f(\cdot, y, z) \in \mathbb{L}^a_{1,p}(\mathbb{R}^d)$ and the versions of the Malliavin derivatives are denoted by $\nabla\partial_{y^i} f(t, y, z)$, $\nabla\partial_{z^{ji}} f(t, y, z)$ respectively,

5) for each $(t, \omega) \in [0, T] \times \Omega$, $\nabla^2 f(t, \omega, \cdot, \cdot)$ is continuous, and there exists a nonnegative random variable $M \in L^p$ such that for any $(t, \omega) \in [0, T] \times \Omega$, $y \in \mathbb{R}^d$, $z \in \mathbb{R}^{n\times d}$

$$\left\|\nabla^2 f(t, \omega, y, z)\right\|_{H^{\otimes 2}\otimes\mathbb{R}^d} \leq M,$$

6) for each $(t, \omega) \in [0, T] \times \Omega$, $f(t, \omega, \cdot, \cdot) \in C^2_b(\mathbb{R}^d \times \mathbb{R}^{n\times d}, \mathbb{R}^d)$ and

$$\sup_{\substack{t,\omega,y,z \\ 1\leq i,i_1,i_2\leq d \\ 1\leq j_1,j_2\leq n}} \left\{ \left|\frac{\partial^2 f}{\partial y^i \partial y^j}(t, \omega, y, z)\right| + \left|\frac{\partial^2 f}{\partial z^{j_1 i_1} \partial z^{j_2 i_2}}(t, \omega, y, z)\right| + \left|\frac{\partial^2 f}{\partial y^i \partial z^{j_1 i_1}}(t, \omega, y, z)\right| \right\} < \infty,$$

7) for each $(t, \omega) \in [0, T] \times \Omega$, $\nabla f(t, \omega, \cdot, \cdot) \in C^1_b(\mathbb{R}^d \times \mathbb{R}^{n\times d}, H \otimes \mathbb{R}^d)$.

We say the assumption (A3)$'$ is satisfied if

1)$'$ (A1)$'$ with $2p$, instead of p, holds.

By Proposition 3.1.1 and Theorem 3.2.5, we obtain the following theorem.

Theorem 3.3.1. *Suppose (A3) holds. Let* (Y, Z) *be a unique* L^{4p} *solution to the BSDE (3.1). Then,* (Y, Z) *belongs to* $\mathbb{L}^a_{2,p}(\mathbb{R}^d) \times \mathbb{L}^a_{2,p}(\mathbb{R}^{n\times d})$*, and* $(\nabla Y, \nabla Z) \in \mathscr{S}^{2p}(H \otimes \mathbb{R}^d) \times \mathscr{H}^{2p}(H \otimes \mathbb{R}^d)$ *solves (3.2) and* $(\nabla^2 Y, \nabla^2 Z) \in \mathscr{S}^p(H^{\otimes 2} \otimes \mathbb{R}^d) \times \mathscr{H}^p(H^{\otimes 2} \otimes \mathbb{R}^{n\times d})$ *solves the following* $H^{\otimes 2} \otimes \mathbb{R}^d$*-valued linear BSDE:*

$$
\begin{aligned}
&\nabla^2 Y^\alpha_t \\
&= \nabla^2 \xi^\alpha - \nabla \left(\int_{\cdot\wedge t}^{\cdot} Z^\alpha_s ds \right) - \int_{\cdot\wedge t}^{\cdot} \nabla Z^\alpha_s ds \\
&\quad + \int_t^T \Bigg\{ \nabla^2 f^\alpha(s, Y_s, Z_s) \\
&\qquad + \sum_{i=1}^n \nabla Y^i_s \otimes \nabla \partial_{y^i} f^\alpha(s, Y_s, Z_s) + \sum_{i=1}^n \sum_{j=1}^d \nabla Z^{ji}_s \otimes \nabla \partial_{z^{ji}} f^\alpha(s, Y_s, Z_s) \\
&\qquad + \sum_{i=1}^n \nabla \partial_{y^i} f^\alpha(s, Y_s, Z_s) \otimes \nabla Y^i_s + \sum_{i=1}^n \sum_{j=1}^d \nabla \partial_{z^{ji}} f^\alpha(s, Y_s, Z_s) \otimes \nabla Z^{ji}_s \\
&\qquad + \sum_{i,j=1}^d \partial_{y^i} \partial_{y^j} f^\alpha(s, Y_s, Z_s) \nabla Y^i_s \otimes \nabla Y^j_s + \sum_{i,j=1}^d \sum_{k=1}^n \partial_{y^i} \partial_{z^{kj}} f^\alpha(s, Y_s, Z_s) \nabla Y^i_s \otimes \nabla Z^{kj}_s \\
&\qquad + \sum_{i,j=1}^d \sum_{k=1}^n \partial_{z^{kj}} \partial_{y^i} f^\alpha(s, Y_s, Z_s) \nabla Z^{jk}_s \otimes \nabla Y^i_s + \sum_{i,j=1}^d \sum_{k,l=1}^n \partial_{z^{ki}} \partial_{z^{lj}} f^\alpha(s, Y_s, Z_s) \nabla Z^{ki}_s \otimes \nabla Z^{lj}_s \\
&\qquad\qquad + \sum_{i=1}^n \partial_{y^i} f^\alpha(s, Y_s, Z_s) \nabla^2 Y^i_s + \sum_{j=1}^n \sum_{j=1}^d \partial_{z^{ji}} f^\alpha(s, Y_s, Z_s) \nabla^2 Z^{ji} \Bigg\} ds \\
&\quad - \int_t^T \nabla^2 Z^\alpha_s \cdot dW_s, \quad 0 \le t \le T, \quad 1 \le \alpha \le d, \qquad (3.11)
\end{aligned}
$$

where, superscripts of ξ, f, Y, Z *represent corresponding components of them respectively and* $Z^\alpha = (Z^{1\alpha}, \ldots, Z^{n\alpha})^*$.

Moreover, $(D.\nabla Y, D.\nabla Z) \in \mathscr{S}^2_{rc}((H \otimes \mathbb{R}^d)^n, \bar{P}) \times \mathscr{H}^2((H \otimes \mathbb{R}^{n\times d})^n, \bar{P})$ *and* $D_t \nabla Y_t = \nabla Z_t$ *for almost all* $t \in [0, T]$.

Proof. For simplicity of notation, we give the proof in the case when $d = n = 1$.

By Proposition 3.2, $(\nabla Y, \nabla Z) \in \mathscr{S}^{2p}(H) \times \mathscr{H}^{2p}(H)$ solves the following H-valued BSDE:

$$
\begin{aligned}
\nabla Y_t &= \nabla \xi - \int_{\cdot\wedge t}^{\cdot} Z_s ds \\
&\qquad + \int_t^T \{ \nabla f(s, Y_s, Z_s) + \partial_y f(s, Y_s, Z_s) \nabla Y_s + \partial_z f(s, Y_s, Z_s) \nabla Z_s \} ds
\end{aligned}
$$

$$- \int_t^T \nabla Z_s dW_s, \quad 0 \le t \le T.$$

Then, the correspondence to (3.6) is as follows:

$$\zeta_t = -\int_0^{\cdot \wedge t} Z_s ds, \quad A_t = \nabla f(t, Y_t, Z_t), \quad B_t = \partial_y f(t, Y_t, Z_t), \quad \Gamma_t = \partial_z f(t, Y_t, Z_t).$$

It follows that

$$\nabla \zeta_t = -\nabla \left(\int_0^{\cdot \wedge t} Z_s ds \right),$$
$$\nabla A_t = \nabla^2 f(t, Y_t, Z_t) + \nabla Y_t \otimes \nabla \partial_y f(t, Y_t, Z_t) + \nabla Z_t \otimes \nabla \partial_z f(t, Y_t, Z_t),$$
$${}^{\nabla} B_t h = \nabla \partial_y f(t, Y_t, Z_t) \otimes h + \partial_y^2 f(t, Y_t, Z_t) \nabla Y_t \otimes h + \partial_z \partial_y f(t, Y_t, Z_t) \nabla Z_t \otimes h, \quad \forall h \in H,$$
$$\tilde{B}_t = \partial_y f(t, Y_t, Z_t),$$
$${}^{\nabla} \Gamma_t g = \nabla \partial_z f(t, Y_t, Z_t) \otimes g + \partial_y \partial_z f(t, Y_t, Z_t) \nabla Y_t \otimes g + \partial_z^2 f(t, Y_t, Z_t) \nabla Z_t \otimes g, \quad \forall g \in H,$$
$$\tilde{\Gamma}_t = \partial_z f(t, Y_t, Z_t).$$

We shall show $(\nabla Y, \nabla Z) \in \mathbb{L}^a_{1,p}(H) \times \mathbb{L}^a_{1,p}(H)$ by applying Theorem 3.2.5. By the assumption and (3.2), we see that (A2)-1)-4),6) are satisfied. Thus, it suffices to show (A2)′-5),7).

By (A1)-3),6) with $2p$ and (A3)-5), it follows

$$E\left[\left(\int_0^T \|\nabla A_s\|_{H^{\otimes 2}} ds \right)^p \right] \le C \Bigg(T E[M^p] + \left\{ E\left[\int_0^T K_s^{2p} ds \right] \right\}^{\frac{1}{2}} \left(T^p \|\nabla Y\|^p_{\mathscr{S}^{2p}(H)} + \|\nabla Z\|^p_{\mathscr{H}^{2p}(H)} \right) \Bigg) < \infty,$$

where C_1 is a positive constant. Thus, (A2)′-5) is satisfied.

(A2)′-7) follows from the following inequality obtained form (A1)-3),6) with $2p$;

$$E\left[\left(\int_0^T \left\{ \left\| {}^{\nabla} B_s \right\|_{\mathcal{L}(H, H^{\otimes 2})} + \left\| {}^{\nabla} \Gamma_s \right\|_{\mathcal{L}(H, H^{\otimes 2})} \right\}^2 ds \right)^p \right]$$
$$\le C_2 \Bigg(E\left[\int_0^T K_s^{2p} ds \right] + \left(\left\| \partial_y^2 f \right\|_\infty^{2p} + \| \partial_y \partial_z f \|_\infty^{2p} \right) \left(T^{2p} \|\nabla Y\|^{2p}_{\mathscr{S}^{2p}(H)} + \|\nabla Z\|^{2p}_{\mathscr{H}^{2p}(H)} \right) \Bigg)$$
$$< \infty,$$

where C_2 is a positive constant and $\|\cdot\|_\infty$ represents the supremum. Thus, (A2)′-7) is satisfied. Therefore, $(\nabla^2 Y, \nabla^2 Z) \in \mathscr{S}^p(H^{\otimes 2}) \times \mathscr{H}^p(H^{\otimes 2})$ solves (3.11).

Next, we check (A2)′-8) in order to get $D_t \nabla Y_t = \nabla Z_t$. We see

$$D_u A_t = D_u \nabla f(t, Y_t, Z_t) + D_u Y_t \nabla \partial_y f(t, Y_t, Z_t) + D_u Z_t \nabla \partial_z f(t, Y_t, Z_t).$$

Then, by the Schwarz inequality, (A1)-6) and (A3)-5),

$$\begin{aligned}
E\left[\int_0^T \left(\int_0^T \|D_u A_s\|_H \, ds\right)^2 du\right] &\leq C_3 \left\{ E\left[\int_0^T \left(\int_0^T \|D_u \nabla f(s, Y_s, Z_s)\|_{\mathcal{K}} \, ds\right)^2 du\right] \right. \\
&\quad + E\left[\int_0^T \left(\int_0^T |D_u Y_s| \, \|\nabla \partial_y f(s, Y_s, Z_s)\|_H \, ds\right)^2 du\right] \\
&\quad \left. + E\left[\int_0^T \left(\int_0^T |D_u Z_s| \, \|\nabla \partial_z f(s, Y_s, Z_s)\|_H \, ds\right)^2 du\right] \right\} \\
&\leq C_3 \left\{ T^2 E[M^2] + E\left[\int_0^T \|\nabla Y_s\|_H^2 \, ds \int_0^T K_s^2 ds\right] \right. \\
&\quad \left. + E\left[\int_0^T \|\nabla Z_s\|_H^2 \, ds \int_0^T K_s^2 ds\right] \right\} \\
&\leq C_3 \left\{ T^2 E[M^2] + T^{\frac{3}{2}} \|\nabla Y\|_{\mathscr{S}^4(H)}^2 \left(E\left[\int_0^T K_s^4 ds\right]\right)^{\frac{1}{2}} \right. \\
&\quad \left. + T^{\frac{1}{2}} \|\nabla Z\|_{\mathscr{H}^4(H)}^2 \left(E\left[\int_0^T K_s^4 ds\right]\right)^{\frac{1}{2}} \right\} \\
&< \infty,
\end{aligned}$$

where C_3 represents a positive constant.

Since

$$\begin{aligned}
&\tilde{K}_u({}^{\nabla} B_t \nabla Y_t) \\
&\quad = D_u \partial_y f(t, Y_t, Z_t) \nabla Y_t + \partial_y^2 f(t, Y_t, Z_t) D_u Y_t \nabla Y_t + \partial_y \partial_z f(t, Y_t, Z_t) D_u Z_t \nabla Y_t, \\
&\tilde{K}_u({}^{\nabla} \Gamma_t \nabla Z_t) \\
&\quad = D_u \partial_z f(t, Y_t, Z_t) \nabla Z_t + \partial_y \partial_z f(t, Y_t, Z_t) D_u Y_t \nabla Z_t + \partial_z^2 f(t, Y_t, Z_t) D_u Z_t \nabla Z_t,
\end{aligned}$$

we see that

$$\begin{aligned}
&E\left[\int_0^T \left(\int_0^T \left\{ \left\|\tilde{K}_u({}^{\nabla} B_t \nabla Y_t)\right\|_H + \left\|\tilde{K}_u({}^{\nabla} \Gamma_t \nabla Z_t)\right\|_H \right\} ds\right)^2 du\right] \\
&\leq C_4 \left\{ E\left[\int_0^T \left(\int_0^T |D_u \partial_y f(s, Y_s, Z_s)| \, \|\nabla Y_s\|_H \, ds\right)^2 du\right] \right.
\end{aligned}$$

$$
\begin{aligned}
&+ E\left[\int_0^T \left(\int_0^T |D_u Y_s| \, \|\nabla Y_s\|_H \, ds\right)^2 du\right] + E\left[\int_0^T \left(\int_0^T |D_u Z_s| \, \|\nabla Y_s\|_H \, ds\right)^2 du\right] \\
&+ E\left[\int_0^T \left(\int_0^T |D_u \partial_z f(s, Y_s, Z_s)| \, \|\nabla Z_s\|_H \, ds\right)^2 du\right] \\
&+E\left[\int_0^T \left(\int_0^T |D_u Y_s| \, \|\nabla Z_s\|_H \, ds\right)^2 du\right] + E\left[\int_0^T \left(\int_0^T |D_u Z_s| \, \|\nabla Z_s\|_H \, ds\right)^2 du\right]\Bigg\} \\
\leq C_4 \Bigg\{ &\left(T^{\frac{3}{2}} \|\nabla Y\|^2_{\mathscr{S}^4(H)} + T^{\frac{1}{2}} \|\nabla Z\|^2_{\mathscr{H}^4(H)}\right) \left(E\left[\int_0^T K_s^4 ds\right]\right)^{\frac{1}{2}} \\
&+T^2 \|\nabla Y\|^4_{\mathscr{S}^4(H)} + \|\nabla Z\|^4_{\mathscr{H}^4(H)} + 2T \|\nabla Y\|^2_{\mathscr{S}^4(H)} \|\nabla Z\|^2_{\mathscr{H}^4(H)} \Bigg\} \\
< \infty,&
\end{aligned}
$$

where C_4 is a positive constant.

Thus (A2)$'$-8) is satisfied. Then, we get $(D.\nabla Y, D.\nabla Z) \in \mathscr{S}^2_{rc}(H) \times \mathscr{H}^2(H)$ and $D_t \nabla Y_t = -D_t \zeta_t + \nabla Z_t$ by Theorem 3.2.5. Since $D_u Z_t = 0$ for $t < u$, we see

$$
\|D_u \zeta_t\|^2_H = \begin{cases} 0, & t \leq u, \\ \displaystyle\int_u^t |D_u Z_s|^2 ds, & u < t. \end{cases}
$$

Then we obtain $D_t \nabla Y_t = \nabla Z_t$. □

By the proof of Theorem 3.3.1, we see also the following.

Corollary 3.3.2. *Suppose (A3)$'$ holds. Let (Y, Z) be a unique L^{2p} solution to the BSDE* (3.1)*. Then, (Y, Z) belongs to $\mathbb{L}^a_{2,p}(\mathbb{R}^d) \times \mathbb{L}^a_{2,p}(\mathbb{R}^{n\times d})$, and $(\nabla Y, \nabla Z) \in \mathscr{S}^{2p}(H \otimes \mathbb{R}^d) \times \mathscr{H}^{2p}(H \otimes \mathbb{R}^d)$ solves* (3.2) *and $(\nabla^2 Y, \nabla^2 Z) \in \mathscr{S}^p(H^{\otimes 2} \otimes \mathbb{R}^d) \times \mathscr{H}^p(H^{\otimes 2} \otimes \mathbb{R}^{n\times d})$ solves* (3.11)*.*

Moreover, $D_t \nabla Y_t = \nabla Z_t$ for almost all $t \in [0, T]$.

3.4 Higher Order Differentiability of Solutions of BSDEs

In this section, we consider the higher order differentiability of solutions of BSDEs. When considering higher than the second order differentiability, there is an obstacle from the chain rule.

Let us take the one more differentiation of (3.11) formally. Then by the chain rule, we get a term $\int_t^T \partial_z^3 f(s, Y_s, Z_s) \nabla Z_s \otimes \nabla Z_s \otimes \nabla Z_s ds$, whose integrability corresponds to (A2)-5). It is impossible to show the integrability of it because we can't control it with the norm $\|\cdot\|_{\mathscr{H}^p}$. The same circumstances occur when considering higher order differentiability of solution. When a term contains more than two derivatives of Z, it is impossible to estimate it appropriately. Then, we need some restriction which enable the estimation of such a term.

In what follows, we deal with the cases where additional assumptions are made on either f or Z.

3.4.1 Under Additional Conditions on f

In this subsection, we consider the higher order Malliavin differentiability of solutions with assumptions of generator f.

We set $\mathscr{S}^\infty(\mathcal{K}) = \bigcap_{p\geq 2} \mathscr{S}^p(\mathcal{K})$, $\mathscr{H}^\infty(\mathcal{K}) = \bigcap_{p\geq 2} \mathscr{H}^p(\mathcal{K})$, $\mathbb{D}^\infty(\mathcal{K}) = \bigcap_{k\geq 1, p\geq 2} \mathbb{D}^{k,p}(\mathcal{K})$, $\mathbb{L}^a_\infty(\mathcal{K}) = \bigcap_{k\geq 1, p\geq 2} \mathbb{L}^a_{k,p}(\mathcal{K})$.

We introduce some more notations. Let $\mathbb{Z}_+$ be the set of nonnegative integers. For

$$\beta = (\beta_1, \dots, \beta_d, \beta_{11}, \dots, \beta_{1d}, \dots, \beta_{n1}, \dots, \beta_{nd}) \in \mathbb{Z}_+^{d+nd},$$

we denote $|\beta| = \sum_{i=1}^d (\beta_i + \sum_{j=1}^n \beta_{ji})$. Then, we write the derivative of a function $g(y,z)$ defined on $\mathbb{R}^d \times \mathbb{R}^{n\times d}$ as

$$\partial^\beta g = \left(\frac{\partial}{\partial y^1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial y^d}\right)^{\beta_d} \left(\frac{\partial}{\partial z^{11}}\right)^{\beta_{11}} \cdots \left(\frac{\partial}{\partial z^{nd}}\right)^{\beta_{nd}} g, \quad \beta \in \mathbb{Z}_+^{d+nd}.$$

We introduce the assumption (A4). Especially, (A4)-3) plays a key role to overcome the above-mentioned obstacle.
(A4):

1) $\xi \in \mathbb{D}^\infty(\mathbb{R}^d)$,

2) for any $p \geq 2$, $E[(\int_0^T |f(s,0,0)|ds)^p] < \infty$,

3) for each $(t,\omega) \in [0,T]\times\Omega$, $f(t,\omega,\cdot,\cdot) \in C_b^\infty(\mathbb{R}^d \times \mathbb{R}^{n\times d}, \mathbb{R}^d)$ and
$$\frac{\partial^3 f}{\partial z^{j_1 i_1}\partial z^{j_2 i_2}\partial z^{j_3 i_3}}(t,\omega,y,z) = 0, \quad \begin{array}{l} 1 \leq \forall i_1, i_2, i_3 \leq d,\ 1 \leq \forall j_1, j_2, j_3 \leq n, \\ \forall (t,\omega,y,z) \in [0,T]\times\Omega\times\mathbb{R}^d\times\mathbb{R}^{n\times d}, \end{array}$$
$$\sup_{t,\omega,y,z} |\partial^\beta f(t,\omega,y,z)| < \infty, \quad \forall \beta \in \mathbb{Z}_+^{d+nd} \text{ with } |\beta| \geq 1,$$

4) for each $\beta \in \mathbb{Z}_+^{d+nd}$ and $(y,z) \in \mathbb{R}^d \times \mathbb{R}^{n\times d}$, $\partial^\beta f(\cdot,y,z) \in \mathbb{L}^a_\infty(\mathbb{R}^d)$ and the version of the Malliavin derivative is denoted by $\nabla\partial^\beta f(t,y,z)$,

5) for any $k \geq 1$ and $p \geq 2$,
$$E\left[\int_0^T \left\|\nabla^k f(s,Y_s,Z_s)\right\|^p_{H^{\otimes k}\otimes\mathbb{R}^d} ds\right] < \infty,$$

6) for each $k \geq 1$, $\beta \in \mathbb{Z}_+^{d+nd}$ and $(t,\omega) \in [0,T] \times \Omega$, $\nabla^k \partial^\beta f(t,\omega,\cdot,\cdot) \in C_b^\infty(\mathbb{R}^d \times \mathbb{R}^{n\times d}, H^{\otimes k} \otimes \mathbb{R}^d)$, and

$$\sup_{\substack{t,\omega,y,z\\1\le i\le d\\1\le j\le n}} \left(\left\|\partial_{y^i} \nabla^k \partial^\beta f(t,\omega,y,z)\right\|_{H^{\otimes k}\otimes\mathbb{R}^d} + \left\|\partial_{z^{ji}} \nabla^k \partial^\beta f(t,\omega,y,z)\right\|_{H^{\otimes k}\otimes\mathbb{R}^d} \right) < \infty.$$

Theorem 3.4.1. *Suppose (A4) holds. Let (Y,Z) be a unique solution to the BSDE (3.1). Then, (Y,Z) belongs to $\mathbb{L}^a_\infty(\mathbb{R}^d) \times \mathbb{L}^a_\infty(\mathbb{R}^{n\times d})$.*

Moreover, for each $k \geq 0$, $(D.\nabla^k Y, D.\nabla^k Z) \in \mathscr{S}^2_{rc}(H^{\otimes k}\otimes\mathbb{R}^d, \bar{P}) \times \mathscr{H}^2(H^{\otimes k}\otimes \mathbb{R}^{n\times d}, \bar{P})$ and $D_t\nabla^k Y_t = \nabla^k Z_t$ for almost all $t \in [0,T]$.

Example 3.4.2. We now give some examples under the Black-Scholes model.

Let $b \in \mathbb{R}$, $\sigma > 0$, $(W_t)_{0\le t\le T}$ be a one-dimensional Brownian motion and let $(S_t)_{0\le t\le T}$, which is a risky asset, obey the SDE;

$$dS_t = bS_t dt + \sigma S_t dW_t.$$

The interest rate of the nonrisky asset is denoted by $r > 0$. Denote by Δ_t the number of the risky asset held at t. Let Φ be a function defined on $\mathbb{R}$ satisfying $\sup_{x\in\mathbb{R}} |\Phi(x)|/(1+|x|)^k < \infty$ for some $k \in \mathbb{N}$. In the theory of mathematical finance, it is known that the self-financing portfolio Y replicating an European contingent claim $\Phi(S_T)$ satisfies the following BSDE;

$$\begin{cases} dY_t = (rY_t + \theta\Delta_t\sigma S_t)dt + \Delta_t\sigma S_t dW_t, \\ Y_T = \Phi(S_T), \end{cases}$$

where $\theta = (b-r)/\sigma$. By regarding $\Delta_t\sigma S_t$ as Z_t, we obtain

$$Y_t = \Phi(S_T) - \int_t^T (rY_t + \theta Z_t)dt - \int_t^T Z_s dW_s. \tag{3.12}$$

It is also known that

$$Y_t = c(t,S_t), \quad \Delta_t = \frac{\partial c}{\partial x}(t,S_t), \tag{3.13}$$

where

$$c(t,x) = e^{-r(T-t)} \int_{\mathbb{R}} \Phi\left(xe^{\sigma y + \left(r - \frac{\sigma^2}{2}\right)(T-t)} \right) g(T-t,y)dy, \quad 0 \le t < T, \tag{3.14}$$

$$g(t,y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{|y|^2}{2}},$$

which satisfies the Black-Scholes partial differential equation.

(1) Let $\Phi \in C^\infty(\mathbb{R})$ and assume that for any $m \in \mathbb{Z}_+$, there exists a $k \in \mathbb{N}$ such that $\sup_{x\in\mathbb{R}} |\frac{\partial^m \Phi}{\partial x^m}(x)|/(1+|x|)^k < \infty$. Then by (3.2) and (3.13), we see that $(Y,Z) \in \mathbb{L}^a_\infty(\mathbb{R}) \times \mathbb{L}^a_\infty(\mathbb{R})$ and $D_t \nabla^k Y_t = \nabla^k Z_t$. On the other hand, since (3.12) satisfies (A4), Theorem 3.4.1 yields the same result.

(2) Let $\Phi(S_T) = (S_T - K)^+ := \max\{S_T - K, 0\}$, which is called an European call option. We know $\Phi(S_T)$ is differentiable and $\nabla\Phi(S_T) = \mathbf{1}_{(0,\infty)}(S_T - K)\nabla S_T$. Hence, $\Phi(S_T)$ would not be smooth. Thus, (Y_T, Z_T) fails to belong to $\mathbb{D}^\infty \times \mathbb{D}^\infty$.

Proof of Theorem 3.4.1. For simplicity of letter, we give the proof in the case when $d = n = 1$. In the proof, notation C represents just a positive constant which may change form place to place.

Let $(Y,Z) \in \mathscr{S}^\infty(\mathbb{R}) \times \mathscr{H}^\infty(\mathbb{R})$. By Corollary 3.1.3, for any $p \geq 2$,

- $(Y,Z) \in \mathbb{L}^a_{1,p}(\mathbb{R}) \times \mathbb{L}^a_{1,p}(\mathbb{R})$ and $(\nabla Y, \nabla Z) \in \mathscr{S}^p(H) \times \mathscr{H}^p(H)$,
- $(D.Y, D.Z) \in \mathscr{S}^2_{rc}(\mathbb{R}, \bar{P}) \times \mathscr{H}^2(\mathbb{R}, \bar{P})$.

We show the following Claim 1 and 2 for $k \geq 2$ by induction:

Claim 1 Let $p \geq 2$. Then, $(\nabla^{k-1}Y, \nabla^{k-1}Z) \in \mathbb{L}^a_{1,p}(H^{\otimes(k-1)}) \times \mathbb{L}^a_{1,p}(H^{\otimes(k-1)})$ and $(\nabla^k Y, \nabla^k Z) \in \mathscr{S}^p(H^{\otimes k}) \times \mathscr{H}^p(H^{\otimes k})$ is a unique solution to the BSDE;

$$\nabla^k Y_t = \nabla^k \xi - \sum_{i=0}^{k-1} \nabla^i \left(\int_{\cdot\wedge t}^{\cdot} \nabla^{k-1-i} Z_s ds \right) + \int_t^T \left\{ A^k_s + B^k_s \nabla^k Y_s + \Gamma^k_s \nabla^k Z_s \right\} ds - \int_t^T \nabla^k Z_s dW_s, \quad 0 \leq t \leq T,$$

where $B^k_t = \partial_y f(t, Y_t, Z_t)$, $\Gamma^k_t = \partial_z f(t, Y_t, Z_t)$ and A^k is defined inductively as

$$A^1_t = \nabla f(t, Y_t, Z_t),$$
$$A^k_t = \nabla A^{k-1}_t + {}^\nabla B^{k-1}_t \nabla^{k-1} Y_t + {}^\nabla \Gamma^{k-1}_t \nabla^{k-1} Z_t, \quad k \geq 2.$$

Moreover, it holds that

$$\begin{aligned} A^k_t = \nabla^k f(t, Y_t, Z_t) &+ \sum_{1,k} (\nabla^{\alpha_1} Y_t \otimes \cdots \otimes \nabla^{\alpha_{m-1-r}} Y_t \\ &\otimes \nabla^{\beta_1} Z_t \otimes \cdots \otimes \nabla^{\beta_r} Z_t \otimes \nabla^\gamma \partial_y^{m-1-r} \partial_z^r f(t, Y_t, Z_t))^\Sigma \\ &+ \sum_{2,k} \partial_y^{m-r} \partial_z^r f(t, Y_t, Z_t) \\ &\times (\nabla^{\alpha_1} Y_t \otimes \cdots \otimes \nabla^{\alpha_{m-r}} Y_t \otimes \nabla^{\beta_1} Z_t \otimes \cdots \otimes \nabla^{\beta_r} Z_t)^\Sigma, \end{aligned} \quad (3.15)$$

where the notations of summation represent

$$\sum_{1,k} := \sum_{m=2}^{k} \sum_{r=0}^{(m-1)\wedge 2} \sum_{\substack{\alpha\in\mathbb{N}_{\mathrm{ND}}^{m-1-r}\\ \beta\in\mathbb{N}_{\mathrm{ND}}^{r}\\ \gamma\in\mathbb{N}\\ |\alpha|+|\beta|+\gamma=k}}, \quad \sum_{2,k} := \sum_{m=2}^{k}\sum_{r=0}^{m\wedge 2} \sum_{\substack{\alpha\in\mathbb{N}_{\mathrm{ND}}^{m-r}\\ \beta\in\mathbb{N}_{\mathrm{ND}}^{r}\\ |\alpha|+|\beta|=k}},$$

$\mathbb{N}_{\mathrm{ND}}^k$ denotes the set of $\alpha \in \mathbb{N}^k$ such that $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_k$, and the sum of all components of the tensor of $\nabla^{\alpha_1} F_1, \dots, \nabla^{\alpha_k} F_k$ is denoted by $(\nabla^{\alpha_1} F_1 \otimes \cdots \otimes \nabla^{\alpha_k} F_k)^{\Sigma}$; for example, for $h_1, h_2, h_3 \in H$,

$$\begin{aligned}&(\nabla^2 F_1 \otimes \nabla F_2)^{\Sigma}(h_1 \otimes h_2 \otimes h_3)\\ &= \nabla^2 F_1(h_1\otimes h_2)\nabla F_2(h_3) + \nabla^2 F_1(h_1\otimes h_3)\nabla F_2(h_2) + \nabla^2 F_1(h_2\otimes h_3)\nabla F_2(h_1).\end{aligned}$$

We note that the number of terms in each term of the sum is determined only by α, β, γ.

Claim 2 Let $p \ge 2$. Then,

$$E\left[\int_0^T \left(\int_0^T \|\tilde{K}_u A_s^k\|_{H^{\otimes(k-1)}} ds\right)^2 du\right] < \infty,$$

and then $(D_\cdot \nabla^{k-1} Y, D_\cdot \nabla^{k-1} Z) \in \mathscr{S}^2(H^{\otimes(k-1)}, \bar{P}) \times \mathscr{H}^2(H^{\otimes(k-1)}, \bar{P})$ and $D_t \nabla^{k-1} Y_t = \nabla^{k-1} Z_t$ for almost all $t \in [0, T]$.

We show the case $k = 2$. Let $p \ge 2$. By Corollary 3.3.2, $(\nabla^2 Y, \nabla^2 Z) \in \mathscr{S}^p(H^{\otimes 2}) \times \mathscr{H}^p(H^{\otimes 2})$ solves (3.11). Then, Claim 1 holds. As in the proof of Theorem 3.3.1, Claim 2 holds.

Then, we assume that $k > 2$ and the cases when $1, 2, \dots, k-1$ hold.

We shall prove Claim 1. Let $p \ge 2$. By the inductive assupmtion, $(\nabla^{k-1} Y, \nabla^{k-1} Z) \in \mathscr{S}^{2p}(H^{\otimes(k-1)}) \times \mathscr{H}^{2p}(H^{\otimes(k-1)})$ is a unique solution to the BSDE;

$$\begin{aligned}\nabla^{k-1} Y_t &= \nabla^{k-1}\xi - \sum_{i=0}^{k-2} \nabla^i \left(\int_{\cdot\wedge t}^{\cdot} \nabla^{k-2-i} Z_s ds\right)\\ &\quad + \int_t^T \left\{A_s^{k-1} + B_s^{k-1}\nabla^{k-1} Y_s + \Gamma_s^{k-1} \nabla^{k-1} Z_s\right\} ds - \int_t^T \nabla^{k-1} Z_s dW_s, \quad 0 \le t \le T,\end{aligned} \tag{3.16}$$

where $B_t^{k-1} = \partial_y f(t, Y_t, Z_t)$, $\Gamma_t^{k-1} = \partial_z f(t, Y_t, Z_t)$,

$$\begin{aligned}A_t^{k-1} =& \nabla^k f(t, Y_t, Z_t)\\ &+ \sum_{1,k-1} (\nabla^{\alpha_1} Y_t \otimes \cdots \otimes \nabla^{\alpha_{m-1-r}} Y_t\end{aligned}$$

$$\otimes \nabla^{\beta_1} Z_t \otimes \cdots \otimes \nabla^{\beta_r} Z_t \otimes \nabla^{\gamma} \partial_y^{m-1-r} \partial_z^r f(t, Y_t, Z_t))^{\Sigma}$$
$$+ \sum_{2,k-1} \partial_y^{m-r} \partial_z^r f(t, Y_t, Z_t)$$
$$\times (\nabla^{\alpha_1} Y_t \otimes \cdots \otimes \nabla^{\alpha_{m-r}} Y_t \otimes \nabla^{\beta_1} Z_t \otimes \cdots \otimes \nabla^{\beta_r} Z_t)^{\Sigma}.$$

We show $(\nabla^{k-1}Y, \nabla^{k-1}Z) \in \mathbb{L}^a_{1,p}(H^{\otimes(k-1)}) \times \mathbb{L}^a_{1,p}(H^{\otimes(k-1)})$ by applying Theorem 3.2.5. By (3.16) and (A4)-6), we see (A2)-3) is satisfied . The correspondence to (3.6) is as follows;

$$\xi = \nabla^{k-1}\xi,$$
$$\zeta_t = -\sum_{i=0}^{k-2} \nabla^i \left(\int_0^{\cdot\wedge t} \nabla^{k-2-i} Z_s ds \right).$$

We note that $\nabla^i \left(\int_0^{\cdot\wedge t} \nabla^{k-2-i} Z_s ds \right)$ represents a Hilbert-Schmidt operator such that

$$H^{\otimes(k-1)} \ni h_1 \otimes \cdots \otimes h_{k-1}$$
$$\mapsto \int_0^t (\nabla^{k-2} Z_s)(h_1 \otimes \cdots \otimes h_i \otimes h_{i+2} \otimes \cdots \otimes h_{k-1}) \dot{h}_{i+1}(s) ds \in \mathbb{R}.$$

Thus (A2)-1),4) are satisfied. We see that for any $F, G \in H^{\otimes(k-1)}$,

$${}^{\nabla}B_t^{k-1} F = \nabla \partial_y f(t, Y_t, Z_t) \otimes F + \partial_y^2 f(t, Y_t, Z_t) \nabla Y_t \otimes F + \partial_z \partial_y f(t, Y_t, Z_t) \nabla Z_t \otimes F,$$
$$\tilde{B}_t^{k-1} = \partial_y f(t, Y_t, Z_t),$$
$${}^{\nabla}\Gamma_t^{k-1} G = \nabla \partial_z f(t, Y_t, Z_t) \otimes G + \partial_y \partial_z f(t, Y_t, Z_t) \nabla Y_t \otimes G + \partial_z^2 f(t, Y_t, Z_t) \nabla Z_t \otimes G,$$
$$\tilde{\Gamma}_t^{k-1} = \partial_z f(t, Y_t, Z_t).$$

Hence, (A2)-6) is satisfied.

From careful calculation, we obtain that $\nabla A_t^{k-1} + {}^{\nabla}B_t^{k-1} \nabla^{k-1} Y_t + {}^{\nabla}\Gamma_t^{k-1} \nabla^{k-1} Z_t$ is equal to the right-hand side of (3.15). Then let $p \geq 2$. By (A4)-6), we obtain

$$E\left[\left(\int_0^T \left\| \nabla A_s^{k-1} \right\|_{H^{\otimes k}} ds \right)^p \right]$$
$$\leq C \left\{ E\left[\int_0^T \left\| \nabla^k f(s, Y_s, Z_s) \right\|^p_{H^{\otimes k}} ds \right] \right.$$
$$\left. + \sum_{\substack{m,r \in \mathbb{Z}_+ \\ r \leq 2 \\ 1 \leq m+r \leq k}} \sum_{\substack{\alpha \in (\mathbb{Z}_+ \cap [1,k-1])^{m+r} \\ m+r \leq |\alpha| \leq k}} E\left[\left(\int_0^T \prod_{j=1}^m \left\| \nabla^{\alpha_j} Y_s \right\|_{H^{\otimes \alpha_j}} \prod_{j'=1}^r \left\| \nabla^{\alpha_{j'+m}} Z_s \right\|_{H^{\otimes \alpha_{j'+m}}} ds \right)^p \right] \right\},$$

where the products above are defined to take 1 when $m = 0$ or $r = 0$. By the Hölder inequality, for each term of the summation above, we see

$$\begin{aligned}
&E\left[\left(\int_0^T \prod_{j=1}^m \|\nabla^{\alpha_j} Y_t\|_{H^{\otimes\alpha_j}} \prod_{j'=1}^r \|\nabla^{\alpha_{j'+m}} Z_s\|_{H^{\otimes\alpha_{j'+m}}}\, ds\right)^p\right] \\
&\leq \left\{E\left[\prod_{j=1}^m \sup_{0\leq t\leq T} \|\nabla^{\alpha_j} Y_t\|^{2p}_{H^{\otimes\alpha_j}}\right]\right\}^{\frac12} \left\{E\left[\left(\int_0^T \prod_{j'=1}^r \|\nabla^{\alpha_{j'+m}} Z_s\|_{H^{\otimes\alpha_{j'+m}}}\, ds\right)^{2p}\right]\right\}^{\frac12} \\
&\leq (1+T^{\frac p2}) \prod_{j=1}^m \|\nabla^{\alpha_j} Y\|^p_{\mathscr{S}^{2p_j}(H^{\otimes\alpha_j})} \prod_{j'=1}^r \|\nabla^{\alpha_{j'+m}} Z\|^p_{\mathscr{H}^{2q_{j'}}(H^{\otimes\alpha_{j'+m}})} \qquad (3.17)\\
&< \infty,
\end{aligned}$$

where $1/p = \sum_{j=1}^m 1/p_j = \sum_{j'=1}^r 1/q_{j'}$. Then, by (A4)-5),

$$E\left[\left(\int_0^T \left\|\nabla A_s^{k-1}\right\|_{H^{\otimes k}}\, ds\right)^p\right] < \infty.$$

(A2)′-5) is satisfied.

We check (A2)′-7). We now recall that for any $F, G \in H^{\otimes(k-1)}$,

$$\begin{aligned}
&{}^\nabla B_t^{k-1} F = \nabla\partial_y f(t,Y_t,Z_t)\otimes F + \partial_y^2 f(t,Y_t,Z_t)\nabla Y_t \otimes F + \partial_z\partial_y f(t,Y_t,Z_t)\nabla Z_t\otimes F,\\
&\tilde B_t^{k-1} = \partial_y f(t,Y_t,Z_t),\\
&{}^\nabla\Gamma_t^{k-1} G = \nabla\partial_z f(t,Y_t,Z_t)\otimes G + \partial_y\partial_z f(t,Y_t,Z_t)\nabla Y_t\otimes G + \partial_z^2 f(t,Y_t,Z_t)\nabla Z_t\otimes G,\\
&\tilde\Gamma_t^{k-1} = \partial_z f(t,Y_t,Z_t).
\end{aligned}$$

Then, we get $\sup_{t,\omega}(\|\tilde B_t(\omega)\|_{\mathcal{L}(\mathcal{K})} + \|\tilde\Gamma_t(\omega)\|_{\mathcal{L}(\mathcal{K})}) < \infty$. In addition, the following inequalities;

$$\begin{aligned}
&\left\|{}^\nabla B_t^{k-1}\right\|_{\mathcal{L}(H^{\otimes(k-1)},H^{\otimes k})} \leq \|\nabla\partial_y f(t,Y_t,Z_t)\|_H + \left\|\partial_y^2 f\right\|_\infty \|\nabla Y_t\|_H + \|\partial_y\partial_z f\|_\infty \|\nabla Z_t\|_H,\\
&\left\|{}^\nabla\Gamma_t^{k-1}\right\|_{\mathcal{L}(H^{\otimes(k-1)},H^{\otimes k})} \leq \|\nabla\partial_z f(t,Y_t,Z_t)\|_H + \|\partial_y\partial_z f\|_\infty \|\nabla Z_t\|_H + \left\|\partial_z^2 f\right\|_\infty \|\nabla Z_t\|_H
\end{aligned}$$

and (A4)-3),6) yield that

$$\begin{aligned}
&E\left[\left(\int_0^T \left\{\left\|{}^\nabla B_s^{k-1}\right\|^2_{\mathcal{L}(\mathcal{K},H\otimes\mathcal{K})} + \left\|{}^\nabla\Gamma_s^{k-1}\right\|^2_{\mathcal{L}(\mathcal{K},H\otimes\mathcal{K})}\right\} ds\right)^p\right]\\
&\leq C\left(1 + \|\nabla Y\|^{2p}_{\mathscr{S}^{2p}(H)} + \|\nabla Z\|^{2p}_{\mathscr{H}^{2p}(H)}\right) < \infty. \qquad (3.18)
\end{aligned}$$

Thus (A2)′-7) is satisfied.

Now, we see that (A2)-2),3) are satisfied because the properties corresponding to them are shown in previous k on (A2)-7) and Claim 2.

From the above results and Theorem 3.2.5, we obtain $(\nabla^{k-1}Y, \nabla^{k-1}Z) \in \mathbb{L}^a_{1,p}(H^{\otimes(k-1)}) \times \mathbb{L}^a_{1,p}(H^{\otimes(k-1)})$ and $(\nabla^k Y, \nabla^k Z) \in \mathscr{S}^p(H^{\otimes k}) \times \mathscr{H}^p(H^{\otimes k})$ is a solution to the BSDE;

$$\begin{aligned}
\nabla^k Y_t = \nabla^k \xi - \sum_{i=0}^{k-1} \nabla^i \left(\int_{\cdot\wedge t}^{\cdot} \nabla^{k-1-i} Z_s ds \right) \\
+ \int_t^T \left\{ \nabla A_s^{k-1} + {}^{\nabla}B_s^{k-1}\nabla^{k-1}Y_s + {}^{\nabla}\Gamma_s^{k-1}\nabla^{k-1}Z_s + \tilde{B}_s^{k-1}\nabla^k Y_s + \tilde{\Gamma}_s^{k-1}\nabla^k Z_s \right\} ds \\
- \int_t^T \nabla^k Z_s dW_s, \quad 0 \le t \le T.
\end{aligned}$$

Define $A_t^k = \nabla A_t^{k-1} + {}^{\nabla}B_t^{k-1}\nabla^{k-1}Y_t + {}^{\nabla}\Gamma_t^{k-1}\nabla^{k-1}Z_t$. As mentioned above, A_t^k is written as the form (3.15). Claim 1 is proved.

We shall prove Claim 2. By (3.15) and (A4)-5),6), we get

$$\begin{aligned}
& E\left[\int_0^T \left(\int_0^T \left\| \tilde{K}_u A_s^k \right\|_{H^{\otimes(k-1)}} ds \right)^2 du \right] \\
& \le C \Bigg\{ E\left[\int_0^T \left\| \nabla^k f(s, Y_s, Z_s) \right\|^2_{H^{\otimes k}} ds \right] \\
& \quad + \sum_1 E\left[\int_0^T (I_1^{m,r,\alpha}(u))^2 du \right] + \sum_2 E\left[\int_0^T (I_2^{m,\alpha}(u))^2 du \right] \\
& \quad + \sum_3 E\left[\int_0^T (I_3^{m,r,\alpha}(u))^2 du \right] + \sum_4 E\left[\int_0^T (I_4^{m,\alpha}(u))^2 du \right] \\
& \quad + \sum_5 E\left[\int_0^T (I_5^{m,r,\alpha}(u))^2 du \right] \Bigg\},
\end{aligned} \tag{3.19}$$

where

$$\begin{aligned}
I_1^{m,r,\alpha}(u) = \int_0^T \left\| D_u \nabla^{\alpha_1 - 1} \partial_y^m \partial_z^r f(s, Y_s, Z_s) \right\|_{H^{\otimes(\alpha_1 - 1)}} \\
\times \prod_{j=1}^m \left\| \nabla^{\alpha_{j+1}} Y_s \right\|_{H^{\otimes \alpha_j}} \prod_{j'=1}^r \left\| \nabla^{\alpha_{j'+m+1}} Z_s \right\|_{H^{\otimes \alpha_{j'+m}}} ds,
\end{aligned}$$

$I_2^{m,\alpha}(u) = I_1^{m,2,\alpha}(u)$,

$$\sum_1 = \sum_{\substack{m,r\in\mathbb{Z}_+ \\ r\le 1 \\ 1\le m+r\le k-1}} \sum_{\substack{\alpha\in(\mathbb{Z}_+\cap[1,k-1])^{m+r+1} \\ |\alpha|=k}}, \quad \sum_2 = \sum_{\substack{m\in\mathbb{Z}_+ \\ 0\le m\le k-3}} \sum_{\substack{\alpha\in(\mathbb{Z}_+\cap[1,k-2])^{m+3} \\ |\alpha|=k}},$$

$$I_3^{m,r,\alpha}(u) = \int_0^T \left\| D_u \nabla^{\alpha_1 - 1} Y_s \right\|_{H^{\otimes(\alpha_1-1)}} \prod_{j=2}^{m} \|\nabla^{\alpha_j} Y_s\|_{H^{\otimes \alpha_j}} \prod_{j'=1}^{r} \|\nabla^{\alpha_{j'+m}} Z_s\|_{H^{\otimes \alpha_{j'+m}}} \, ds,$$

$$I_4^{m,\alpha}(u) = \int_0^T \left\| D_u \nabla^{\alpha_1 - 1} Y_s \right\|_{H^{\otimes(\alpha_1-1)}} \prod_{j=2}^{m} \|\nabla^{\alpha_j} Y_s\|_{H^{\otimes \alpha_j}} \prod_{j'=1}^{2} \|\nabla^{\alpha_{j'+m}} Z_s\|_{H^{\otimes \alpha_{j'+m}}} \, ds,$$

$$\sum_3 = \sum_{\substack{m,r \in \mathbb{Z}_+ \\ m \geq 1 \\ r \leq 1 \\ 1 \leq m+r \leq k}} \sum_{\substack{\alpha \in (\mathbb{Z}_+ \cap [1,k-1])^{m+r} \\ m+r \leq |\alpha| \leq k}}, \quad \sum_4 = \sum_{\substack{m,r \in \mathbb{Z}_+ \\ m \geq 1 \\ 1 \leq m \leq k-2}} \sum_{\substack{\alpha \in (\mathbb{Z}_+ \cap [1,k-2])^{m+2} \\ m+2 \leq |\alpha| \leq k}},$$

$$I_5^{m,r,\alpha}(u) = \int_0^T \left\| D_u \nabla^{\alpha_{1+m} - 1} Z_s \right\|_{H^{\otimes(\alpha_{1+m}-1)}} \prod_{j=1}^{m} \|\nabla^{\alpha_j} Y_s\|_{H^{\otimes \alpha_j}} \prod_{j'=2}^{r} \|\nabla^{\alpha_{j'+m}} Z_s\|_{H^{\otimes \alpha_{j'+m}}} \, ds,$$

$$\sum_5 = \sum_{\substack{m,r \in \mathbb{Z}_+ \\ 1 \leq r \leq 2 \\ 1 \leq m+r \leq k}} \sum_{\substack{\alpha \in (\mathbb{Z}_+ \cap [1,k-1])^{m+r} \\ m+r \leq |\alpha| \leq k}}.$$

In above, $\prod_{j=a}^{b} x_j$ is defined to be 1 if $a > b$. For each term of the summation $\sum_1$ in (3.19), we get by the Schwarz inequality,

$$\begin{aligned}
E\left[\int_0^T (I_1^{m,r,\alpha}(u))^2 du\right] &\leq E\left[\int_0^T du \int_0^T \left\| D_u \nabla^{\alpha_1 - 1} \partial_y^m \partial_z^r f(s, Y_s, Z_s) \right\|^2_{H^{\otimes(\alpha_1-1)}} ds \right. \\
&\qquad \left. \times \int_0^T \prod_{j=1}^{m} \|\nabla^{\alpha_{j+1}} Y_s\|^2_{H^{\otimes \alpha_{j+1}}} \prod_{j'=1}^{r} \|\nabla^{\alpha_{j'+m+1}} Z_s\|^2_{H^{\otimes \alpha_{j'+m+1}}} \, ds \right] \\
&\leq E\left[\int_0^T \left\| \nabla^{\alpha_1} \partial_y^m \partial_z^r f(s, Y_s, Z_s) \right\|^2_{H^{\otimes \alpha_1}} ds \right. \\
&\qquad \left. \times \int_0^T \prod_{j=1}^{m} \|\nabla^{\alpha_{j+1}} Y_s\|^2_{H^{\otimes \alpha_{j+1}}} \prod_{j'=1}^{r} \|\nabla^{\alpha_{j'+m+1}} Z_s\|^2_{H^{\otimes \alpha_{j'+m+1}}} \, ds \right] \\
&\leq CE\left[\prod_{j=1}^{m} \sup_{0 \leq t \leq T} \|\nabla^{\alpha_{j+1}} Y_t\|^2_{H^{\otimes \alpha_{j+1}}} \int_0^T \prod_{j'=1}^{r} \|\nabla^{\alpha_{j'+m+1}} Z_s\|^2_{H^{\otimes \alpha_{j'+m+1}}} \, ds\right] \\
&\leq C(1+T) \left\{ E\left[\prod_{j=1}^{m} \sup_{0 \leq t \leq T} \|\nabla^{\alpha_{j+1}} Y_t\|^4_{H^{\otimes \alpha_{j+1}}}\right]\right\}^{\frac{1}{2}} \\
&\qquad \times \prod_{j'=1}^{r} \left\{ E\left[\left(\int_0^T \|\nabla^{\alpha_{j'+m+1}} Z_s\|^2_{H^{\otimes \alpha_{j'+m+1}}} \, ds\right)^2\right]\right\}^{\frac{1}{2}},
\end{aligned}$$

where, to see in the fourth inequality, we use $r \leq 1$. And for each term of the summation $\sum_3$ in (3.19), we get

$$E\left[\int_0^T (I_3^{m,r,\alpha}(u))^2 du\right] \le E\left[\int_0^T du \int_0^T \left\|D_u \nabla^{\alpha_1 - 1} Y_s\right\|^2_{H^{\otimes(\alpha_1-1)}} ds \right.$$
$$\left. \times \int_0^T \prod_{j=2}^m \|\nabla^{\alpha_j} Y_s\|^2_{H^{\otimes \alpha_j}} \prod_{j'=1}^r \|\nabla^{\alpha_{j'+m}} Z_s\|^2_{H^{\otimes \alpha_{j'+m}}} ds\right]$$
$$\le E\left[\int_0^T \|\nabla^{\alpha_1} Y_t\|^2_{H^{\otimes \alpha_1}} ds \right.$$
$$\left. \times \int_0^T \prod_{j=2}^m \|\nabla^{\alpha_j} Y_s\|^2_{H^{\otimes \alpha_j}} \prod_{j'=1}^r \|\nabla^{\alpha_{j'+m}} Z_s\|^2_{H^{\otimes \alpha_{j'+m}}} ds\right]$$
$$\le (1+T)\left\{E\left[\prod_{j=1}^m \sup_{0\le t\le T} \|\nabla^{\alpha_j} Y_t\|^4_{H^{\otimes \alpha_j}}\right]\right\}^{\frac{1}{2}}$$
$$\times \prod_{j'=1}^r \left\{E\left[\left(\int_0^T \|\nabla^{\alpha_{j'+m}} Z_s\|^2_{H^{\otimes \alpha_{j'+m}}} ds\right)^2\right]\right\}^{\frac{1}{2}},$$

where, to see in the third inequality, we also use $r \le 1$. Hence, in the same manner as (3.17), we obtain

$$\sum_1 E\left[\int_0^T (I_1^{m,r,\alpha}(u))^2 du\right] + \sum_3 E\left[\int_0^T (I_3^{m,r,\alpha}(u))^2 du\right] < \infty.$$

For each term of the summation $\sum_2$ in (3.19), we see by Schwarz inequality,

$$E\left[\int_0^T (I_2^{m,\alpha}(u))^2 du\right] \le E\left[\int_0^T du \left(\int_0^T \left\|D_u \nabla^{\alpha_1-1} \partial_y^m \partial_z^2 f(s, Y_s, Z_s)\right\|_{H^{\otimes(\alpha_1-1)}} \|\nabla^{\alpha_{2+m}} Z_s\|_{H^{\otimes \alpha_{2+m}}} \right.\right.$$
$$\left.\left. \times \prod_{j=1}^m \|\nabla^{\alpha_{j+1}} Y_s\|_{H^{\otimes \alpha_j}} \|\nabla^{\alpha_{3+m}} Z_s\|_{H^{\otimes \alpha_{3+m}}} ds\right)^2\right]$$
$$\le E\left[\int_0^T du \int_0^T \left\|D_u \nabla^{\alpha_1-1} \partial_y^m \partial_z^r f(s, Y_s, Z_s)\right\|^2_{H^{\otimes(\alpha_1-1)}} \|\nabla^{\alpha_{2+m}} Z_s\|^2_{H^{\otimes \alpha_{2+m}}} ds \right.$$
$$\left. \times \int_0^T \prod_{j=1}^m \|\nabla^{\alpha_{j+1}} Y_s\|^2_{H^{\otimes \alpha_{j+1}}} \|\nabla^{\alpha_{3+m}} Z_s\|^2_{H^{\otimes \alpha_{3+m}}} ds\right]$$
$$\le E\left[\int_0^T \left\|\nabla^{\alpha_1} \partial_y^m \partial_z^2 f(s, Y_s, Z_s)\right\|^2_{H^{\otimes \alpha_1}} \|\nabla^{\alpha_{2+m}} Z_s\|^2_{H^{\otimes \alpha_{2+m}}} ds\right.$$

$$
\begin{aligned}
&\qquad \times \int_0^T \prod_{j=1}^{m} \|\nabla^{\alpha_{j+1}} Y_s\|^2_{H^{\otimes \alpha_{j+1}}} \|\nabla^{\alpha_{3+m}} Z_s\|^2_{H^{\otimes \alpha_{3+m}}}\, ds\Bigg] \\
&\le CE\left[\prod_{j=1}^{m} \sup_{0\le t\le T} \|\nabla^{\alpha_{j+1}} Y_t\|^2_{H^{\otimes \alpha_{j+1}}} \prod_{j'=1}^{2} \int_0^T \|\nabla^{\alpha_{j'+m+1}} Z_s\|^2_{H^{\otimes \alpha_{j'+m+1}}}\, ds\right] \\
&\le C\left\{E\left[\prod_{j=1}^{m} \sup_{0\le t\le T} \|\nabla^{\alpha_{j+1}} Y_t\|^4_{H^{\otimes \alpha_{j+1}}}\right]\right\}^{\frac{1}{2}} \\
&\qquad \times \left\{E\left[\prod_{j'=1}^{2} \left(\int_0^T \|\nabla^{\alpha_{j'+m+1}} Z_s\|^2_{H^{\otimes \alpha_{j'+m+1}}}\, ds\right)^2\right]\right\}^{\frac{1}{2}} \\
&< \infty.
\end{aligned}
$$

And for each term of the summation $\sum_4$ in (3.19), we get

$$
\begin{aligned}
E\left[\int_0^T (I_4^{m,\alpha}(u))^2 du\right] &\le E\Bigg[\int_0^T du \Bigg(\int_0^T \left\|D_u \nabla^{\alpha_1 - 1} Y_s\right\|_{H^{\otimes(\alpha_1-1)}} \|\nabla^{\alpha_{1+m}} Z_s\|_{H^{\otimes \alpha_{1+m}}} \\
&\qquad \times \prod_{j=2}^{m} \|\nabla^{\alpha_j} Y_s\|_{H^{\otimes \alpha_j}} \|\nabla^{\alpha_{2+m}} Z_s\|_{H^{\otimes \alpha_{2+m}}}\, ds\Bigg)^2\Bigg] \\
&\le E\Bigg[\int_0^T du \int_0^T \left\|D_u \nabla^{\alpha_1 - 1} Y_s\right\|^2_{H^{\otimes(\alpha_1-1)}} \|\nabla^{\alpha_{1+m}} Z_s\|^2_{H^{\otimes \alpha_{1+m}}}\, ds \\
&\qquad \times \int_0^T \prod_{j=2}^{m} \|\nabla^{\alpha_j} Y_s\|^2_{H^{\otimes \alpha_j}} \|\nabla^{\alpha_{2+m}} Z_s\|^2_{H^{\otimes \alpha_{2+m}}}\, ds\Bigg] \\
&\le E\Bigg[\int_0^T \|\nabla^{\alpha_1} Y_s\|^2_{H^{\otimes \alpha_1}} \|\nabla^{\alpha_{1+m}} Z_s\|^2_{H^{\otimes \alpha_{1+m}}}\, ds \\
&\qquad \times \int_0^T \prod_{j=2}^{m} \|\nabla^{\alpha_j} Y_s\|^2_{H^{\otimes \alpha_j}} \|\nabla^{\alpha_{2+m}} Z_s\|^2_{H^{\otimes \alpha_{2+m}}}\, ds\Bigg] \\
&\le E\left[\prod_{j=1}^{m} \sup_{0\le t\le T} \|\nabla^{\alpha_j} Y_t\|^2_{H^{\otimes \alpha_j}} \prod_{j'=1}^{2} \int_0^T \|\nabla^{\alpha_{j'+m}} Z_s\|^2_{H^{\otimes \alpha_{j'+m}}}\, ds\right] \\
&\le \left\{E\left[\prod_{j=1}^{m} \sup_{0\le t\le T} \|\nabla^{\alpha_j} Y_t\|^4_{H^{\otimes \alpha_j}}\right]\right\}^{\frac{1}{2}} \\
&\qquad \times \left\{E\left[\prod_{j'=1}^{2} \left(\int_0^T \|\nabla^{\alpha_{j'+m}} Z_s\|^2_{H^{\otimes \alpha_{j'+m}}}\, ds\right)^2\right]\right\}^{\frac{1}{2}}
\end{aligned}
$$

$$< \infty.$$

Thus, we get

$$\sum_2 E\left[\int_0^T (I_2^{m,\alpha}(u))^2 du\right] + \sum_4 E\left[\int_0^T (I_4^{m,\alpha}(u))^2 du\right] < \infty.$$

For each term of the summation $\sum_5$ in (3.19), we get

$$\begin{aligned}
E\left[\int_0^T (I_5^{m,r,\alpha}(u))^2 du\right] &\le E\left[\int_0^T du \int_0^T \left\|D_u \nabla^{\alpha_{1+m}-1} Z_s\right\|^2_{H^{\otimes(\alpha_{1+m}-1)}} ds \right.\\
&\qquad \left. \times \int_0^T \prod_{j=1}^m \|\nabla^{\alpha_j} Y_s\|^2_{H^{\otimes \alpha_j}} \prod_{j'=2}^r \|\nabla^{\alpha_{j'+m}} Z_s\|^2_{H^{\otimes \alpha_{j'+m}}} ds\right]\\
&\le E\left[\int_0^T \|\nabla^{\alpha_{1+m}} Z_s\|^2_{H^{\otimes \alpha_{1+m}}} ds \right.\\
&\qquad \left. \times \int_0^T \prod_{j=1}^m \|\nabla^{\alpha_j} Y_s\|^2_{H^{\otimes \alpha_j}} \prod_{j'=2}^r \|\nabla^{\alpha_{j'+m}} Z_s\|^2_{H^{\otimes \alpha_{j'+m}}} ds\right]\\
&\le E\left[\prod_{j=1}^m \sup_{0\le t\le T} \|\nabla^{\alpha_j} Y_t\|^2_{H^{\otimes \alpha_j}} \prod_{j'=1}^r \int_0^T \|\nabla^{\alpha_{j'+m}} Z_s\|^2_{H^{\otimes \alpha_{j'+m}}} ds\right]\\
&\le \left\{E\left[\prod_{j=1}^m \sup_{0\le t\le T} \|\nabla^{\alpha_j} Y_t\|^4_{H^{\otimes \alpha_j}}\right]\right\}^{\frac{1}{2}}\\
&\qquad \times \left\{E\left[\prod_{j'=1}^r \left(\int_0^T \|\nabla^{\alpha_{j'+m}} Z_s\|^2_{H^{\otimes \alpha_{j'+m}}} ds\right)^2\right]\right\}^{\frac{1}{2}}\\
&< \infty.
\end{aligned}$$

Then, we see

$$\sum_5 E\left[\int_0^T (I_5^{m,r,\alpha}(u))^2 du\right] < \infty.$$

From the above results, we obtain

$$E\left[\int_0^T \left(\int_0^T \left\|\tilde{K}_u A_s^k\right\|_{H^{\otimes(k-1)}} ds\right)^2 du\right] < \infty,$$

which implies (A2)′-8). Thus, by Theorem 3.2.5, we get $(D.\nabla^{k-1}Y, D.\nabla^{k-1}Z) \in \mathscr{S}^2_{rc}(H^{\otimes(k-1)}, \bar{P}) \times \mathscr{H}^2(H^{\otimes(k-1)}, \bar{P})$ and $D_t\nabla^{k-1}Y_t = \nabla^{k-1}Z_t$ for almost all $t \in [0,T]$. Claim 2 is proved. □

Remark 3.4.3. The key point of the proof of Theorem 3.4.1 is seen in the estimation of (3.17) and (3.19). The degree of the integrand in $\mathscr{H}^p$-norm is two and we have no tools to estimate integrals in which the degree of the integrands are greater than two; for example, $E[(\int_0^T \left\|\nabla^k Z_s\right\|^3 ds)^p]$, $E[(\int_0^T \left\|\nabla^k Z_s\right\|^4 ds)^p]$ and so on. Thus, if (3.19) contains derivatives of Z more than two, we fail to estimate $E[(\int_0^T \|\nabla A_s\| ds)^p]$. (A4)-3) assures that there appears at most two derivatives of Z in (3.19); $r \leq 2$.

3.4.2 Under Boundedness Assumption of Z

In this subsection, we consider higher order Malliavin differentiability of solutions assuming boundedness of the first derivative of Z.

We introduce the assumption (A5) by the following:

1) $\xi \in \mathbb{D}^\infty(\mathbb{R}^d)$ and for each $k \geq 1$, $\sup_{(u,\omega)\in\bar{\Omega}^{(k)}} |D_u^k \xi(\omega)| < \infty$,

2) for any $p \geq 2$, $E[(\int_0^T |f(s,0,0)|ds)^p] < \infty$,

3) for each $(t,\omega) \in [0,T] \times \Omega$, $f(t,\omega,\cdot,\cdot) \in C_b^\infty(\mathbb{R}^d \times \mathbb{R}^{n\times d}, \mathbb{R}^d)$ and for any $|\beta| \geq 1$,
$$\operatorname*{ess.sup}_{t,\omega,y,z} |\partial^\beta f(t,\omega,y,z)| < \infty,$$

4) for each $\beta \in \mathbb{Z}_+^{d+nd}$ and $(y,z) \in \mathbb{R}^d \times \mathbb{R}^{n\times d}$, $\partial^\beta f(\cdot,y,z) \in \mathbb{L}_\infty^a(\mathbb{R}^d)$, the version of the Malliavin derivative is denoted by $\nabla^k \partial^\beta f(t,y,z)$ for $k \geq 1$,

5) for each $k \geq 1$, $\beta \in \mathbb{Z}_+^{d+nd}$ and $(t,\omega) \in [0,T] \times \Omega$, $\nabla^k \partial^\beta f(t,\omega,\cdot,\cdot) \in C_b^\infty(\mathbb{R}^d \times \mathbb{R}^{n\times d}, H^{\otimes k} \otimes \mathbb{R}^d)$ and
$$\sup_{\substack{t,\omega,y,z\\1\leq i\leq d\\1\leq j\leq n}} \left(\left\|\partial_{y^i} \nabla^k \partial^\beta f(t,\omega,y,z)\right\|_{H^{\otimes k}\otimes\mathbb{R}^d} + \left\|\partial_{z^{ji}} \nabla^k \partial^\beta f(t,\omega,y,z)\right\|_{H^{\otimes k}\otimes\mathbb{R}^d}\right) < \infty,$$

6) for each $k \geq 1$ and $\beta \in \mathbb{Z}_+^{d+nd}$,
$$\sup_{\substack{t,y,z\\(u,\omega)\in\bar{\Omega}^{(k)}}} |D_u^k \partial^\beta f(t,\omega,y,z)| < \infty.$$

Theorem 3.4.4. *Suppose (A5) holds. Let $(Y,Z) \in \mathscr{S}^\infty(\mathbb{R}^d) \times \mathscr{H}^\infty(\mathbb{R}^{n\times d})$ be a unique solution to the BSDE* (3.1)*. If $\sup_{u,t,\omega} |D_u Z_t(\omega)| < \infty$, then (Y,Z) belongs to $\mathbb{L}_\infty^a(\mathbb{R}^d) \times \mathbb{L}_\infty^a(\mathbb{R}^{n\times d})$.*

Moreover, for each $k \geq 0$, $(D.\nabla^k Y, D.\nabla^k Z) \in \mathscr{S}_{rc}^\infty((H^{\otimes k}\otimes\mathbb{R}^d)^n, \bar{P}) \times \mathscr{H}^\infty((H^{\otimes k}\otimes \mathbb{R}^{n\times d})^n, \bar{P})$ and $D_t \nabla^k Y_t = \nabla^k Z_t$ for almost all $t \in [0,T]$.

We introduce some examples on Theorem 3.4.4.

Example 3.4.5. Let $(W_t)_{0\le t\le T}$ be a one-dimensional Brownian motion and a function $g\colon [0,T]\times\mathbb{R}\ni (t,x)\mapsto g(t,x)\in\mathbb{R}$ belong to $C^{1,2}([0,T]\times\mathbb{R})$. Set $Y_t = g(t,W_t)$ and $Z_t = \frac{\partial g}{\partial x}(t,W_t)$. Then (Y,Z) is a solution to the BSDE;

$$Y_t = g(T,W_T) - \int_t^T \left(\frac{\partial g}{\partial t} + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}\right)(s,W_s)ds - \int_t^T Z_s dW_s,$$

and it holds

$$D_u Z_t = \frac{\partial^2 g}{\partial x^2}(t,W_t)\mathbf{1}_{[0,t]}(u).$$

We now can construct some examples satisfying the assumptions (A4) or (A5) by above formulae.
(1) Let $g(t,x) = t\sin x$. Then, (Y,Z), defined as above, is a unique solution to the BSDE;

$$Y_t = T\sin W_T + \int_t^T \left(-\sin W_t + \frac{Y_t}{2}\right)ds - \int_t^T Z_s dW_s,$$

which satisfies (A5) and $\sup_{u,t,\omega}|D_u Z_t(\omega)| < \infty$ as well as (A4).
(2) Let $g(t,x) = x\arctan(2x) - \frac{1}{4}\log(1+4x^2)$. Then, (Y,Z), defined as above, is a unique solution to the BSDE;

$$Y_t = W_T\arctan(2W_T) - \frac{1}{4}\log(1+4W_T^2) - \int_t^T \cos^2 Z_s ds - \int_t^T Z_s dW_s.$$

(A4)-3) is not satisfied but (A5) and $\sup_{u,t,\omega}|D_u Z_t(\omega)| < \infty$ are satisfied since

$$D_u Z_t = \frac{4}{1+4W_t^2}\mathbf{1}_{[0,t]}(u).$$

First, we introduce the following lemma given by extracting the argument of the result of Zhen et al. [26, Proposition 2].

Lemma 3.4.6. *Let (Y,Z) be an L^2 solution to the BSDE (3.1) and suppose*

$$y\cdot f(t,y,z) \le y\cdot a_t + b_t|y|^2 + c_t|y||z|, \quad (t,y,z)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^{n\times d},$$

where $(a_t)_{0\le\le T}$ is a $\mathbb{R}^d$-valued progressively measurable process satisfying $E[\int_0^T|a_s|^2ds] < \infty$, and $(b_t)_{0\le t\le T}$ and $(c_t)_{0\le t\le T}$ are real progressively measurable processes satisfying $\sup_{t,\omega}(|b_t(\omega)|+|c_t(\omega)|) < \infty$. Then, there exists a $\gamma > 0$ such that

$$|Y_t|^2 e^{\gamma t} + \frac{1}{2}E\left[\int_t^T e^{\gamma s}|Z_s|^2 ds\middle|\mathcal{F}_t\right] \le E\left[|\xi|^2 e^{\gamma T} + \int_t^T e^{\gamma s}|a_s|^2 ds\middle|\mathcal{F}_t\right], \quad 0\le t\le T.$$

Proof. Applying the Itô formula to the function $[0,T]\times\mathbb{R}^d \ni (t,y) \mapsto |y|^2 e^{\gamma t} \in \mathbb{R}$ with $y = Y_t$ and (3.1) yields that for any $\gamma > 0$,

$$|Y_t|^2 e^{\gamma t} + \int_t^T e^{\gamma s}\left(\gamma |Y_s|^2 + |Z_s|^2\right) ds = |\xi|^2 e^{\gamma T} + \int_t^T 2e^{\gamma s} Y_s \cdot f(s, Y_s, Z_s) ds - 2\int_t^T e^{\gamma s} Y_s \cdot Z_s dW_s.$$

By the identities $2ab \le a^2 + b^2$ and $2ab \le 2a^2 + \frac{1}{2}b^2$, we get

$$\begin{aligned}2Y_s \cdot f(s, Y_s, Z_s) &\le 2Y_s \cdot a_s + 2b_s|Y_s|^2 + 2c_s|Y_s||Z_s| \\ &\le 2|Y_s||a_s| + 2|b_s||Y_s|^2 + 2|c_s||Y_s||Z_s| \\ &\le |Y_s|^2 + |a_s|^2 + 2\|b\|_\infty |Y_s|^2 + 2\|c\|_\infty^2 |Y_s|^2 + \frac{1}{2}|Z_s|^2,\end{aligned}$$

where $\|\cdot\|_\infty$ represents the supremum with respect to $(s,\omega) \in [0,T]\times\Omega$. Then, we obtain

$$|Y_t|^2 e^{\gamma t} + \int_t^T e^{\gamma s}\left\{(\gamma - 2\|b\|_\infty - 2\|c\|_\infty^2 - 1)|Y_s|^2 + \frac{1}{2}|Z_s|^2\right\} ds \le |\xi|^2 e^{\gamma T} + \int_t^T e^{\gamma s}|a_s|^2 ds - 2\int_t^T e^{\gamma s} Y_s \cdot Z_s dW_s.$$

By choosing $\gamma \ge 2\|b\|_\infty + 2\|c\|_\infty^2 + 1$ and taking the conditional expectation, we get

$$|Y_t|^2 e^{\gamma t} + \frac{1}{2}E\left[\int_t^T e^{\gamma s}|Z_s|^2 ds \middle| \mathcal{F}_t\right] \le E\left[|\xi|^2 e^{\gamma T} + \int_t^T e^{\gamma s}|a_s|^2 ds \middle| \mathcal{F}_t\right].$$

□

Proof of Theorem 3.4.4. For simplicity of notation, we give the proof in the case $d = n = 1$. In the proof, notation C represents a positive constant which may change from place to place.

Let $(Y,Z) \in \mathscr{S}^\infty(\mathbb{R}) \times \mathscr{H}^\infty(\mathbb{R})$ be a unique solution to the BSDE (3.1).

Step 1: We show that for any $p \ge 2$,

i) $(Y,Z) \in \mathbb{L}^a_{1,p}(\mathbb{R}) \times \mathbb{L}^a_{1,p}(\mathbb{R})$ and $(\nabla Y, \nabla Z) \in \mathscr{S}^p(H) \times \mathscr{H}^p(H)$,

ii) $(D.Y, D.Z) \in \mathscr{S}^2_{rc}(\mathbb{R}, \bar{P}) \times \mathscr{H}^2(\mathbb{R}, \bar{P})$ and for almost all $0 \le u, t \le T$,

$$|D_u Y_t| + E\left[\int_t^T |D_u Z_s|^2 ds \middle| \mathcal{F}_t\right] + \|\nabla Y_t\|_H + |Z_t| \le C,$$

iii) $(D.Y, D.Z) \in \mathscr{S}^p_{rc}(\mathbb{R}, \bar{P}) \times \mathscr{H}^p(\mathbb{R}, \bar{P})$.

Let $p \geq 2$. (A5) implies (A3)$'$. Hence, by Corollary 3.1.3, i) holds.

We show ii). $(\nabla Y, \nabla Z) \in \mathscr{S}^p(H) \times \mathscr{H}^p(H)$ solves (3.2). $(D.Y, D.Z)$ belongs to $\mathscr{S}^2_{rc}(\mathbb{R}, \bar{P}) \times \mathscr{H}^2(\mathbb{R}, \bar{P})$ and solves

$$\begin{aligned} D_u Y_t = D_u \xi - \mathbf{1}_{(t,T]}(u) Z_u \\ + \int_t^T \{D_u f(s, Y_s, Z_s) + \partial_y f(s, Y_s, Z_s) D_u Y_s + \partial_z f(s, Y_s, Z_s) D_u Z_s\} ds \\ - \int_t^T D_u Z_s dW_s, \quad 0 \leq t \leq T, \quad \text{a.e.} u \in [0, T]. \end{aligned} \tag{3.20}$$

Take $u \in [0, T]$ satisfying (3.20). Then, $(D_u Y, D_u Z) \in \mathscr{S}^2_{rc}(\mathbb{R}) \times \mathscr{H}^2(\mathbb{R})$ solves

$$\begin{aligned} D_u Y_t = D_u \xi \\ + \int_t^T \{D_u f(s, Y_s, Z_s) + \partial_y f(s, Y_s, Z_s) D_u Y_s + \partial_z f(s, Y_s, Z_s) D_u Z_s\} ds \\ - \int_t^T D_u Z_s dW_s, \quad u \leq t \leq T. \end{aligned}$$

Lemma 3.4.6 yields that there exists a $\gamma > 0$ such that

$$\begin{aligned} |D_u Y_t|^2 e^{\gamma t} + \frac{1}{2} E\left[\int_t^T e^{\gamma s} |D_u Z_s|^2 ds \middle| \mathcal{F}_t \right] \\ \leq E\left[|D_u \xi|^2 e^{\gamma T} + \int_t^T e^{\gamma s} |D_u f(s, Y_s, Z_s)| ds \middle| \mathcal{F}_t \right], \quad u \leq t \leq T. \end{aligned}$$

By (A5)-1),6), we obtain

$$|D_u Y_t|^2 + E\left[\int_t^T |D_u Z_s|^2 ds \middle| \mathcal{F}_t \right] \leq C, \quad u \leq t \leq T.$$

If $0 \leq t < u$, then $D_u Y_t = D_u Z_t = 0$ and

$$\begin{aligned} E\left[\int_t^T |D_u Z_s|^2 ds \middle| \mathcal{F}_t \right] &= E\left[\int_u^T |D_u Z_s|^2 ds \middle| \mathcal{F}_t \right] \\ &= E\left[E\left[\int_u^T |D_u Z_s|^2 ds \middle| \mathcal{F}_u \right] \middle| \mathcal{F}_t \right]. \end{aligned}$$

Thus, we get

$$|D_u Y_t|^2 + E\left[\int_t^T |D_u Z_s|^2 ds \middle| \mathcal{F}_t \right] \leq C, \quad 0 \leq t \leq T. \tag{3.21}$$

By taking integrals with respect to u, we obtain

$$\|\nabla Y_t\|_H^2 \leq C, \quad 0 \leq t \leq T.$$

Since $D_t Y_t = Z_t$ for almost all $t \in [0,T]$ and by (3.21), $|Z_t| \leq C$ for almost all $t \in [0,T]$. Thus, we get for almost all $u, t \in [0,T]$,

$$|D_u Y_t| + E\left[\int_t^T |D_u Z_s|^2 ds \middle| \mathcal{F}_t\right] + \|\nabla Y_t\|_H + |Z_t| \leq C.$$

We will show iii). By ii), $(D.Y, D.Z) \in \mathscr{S}^2_{rc}(\mathbb{R}, \bar{P}) \times \mathscr{H}^2(\mathbb{R}, \bar{P})$ is a unique solution to the BSDE (3.20). Namely, putting $\bar{Y}^1_t(u) = D_u Y_t - \mathbf{1}_{[0,t]}(u) Z_u$, $(\bar{Y}^1(\cdot), D.Z) \in \mathscr{S}^2(\mathbb{R}, \bar{P}) \times \mathscr{H}^2(\mathbb{R}, \bar{P})$ is a unique solution to the BSDE;

$$\begin{aligned}\bar{Y}^1_t(u) = D_u\xi - Z_u + \int_t^T &\{D_u f(s, Y_s, Z_s) + \partial_y f(s, Y_s, Z_s)\mathbf{1}_{[0,s]}(u) Z_u \\ &+ \partial_y f(s, Y_s, Z_s)\bar{Y}^1_s(u) + \partial_z f(s, Y_s, Z_s) D_u Z_s\} ds \\ &- \int_t^T D_u Z_s dW_s, \quad 0 \leq t \leq T, \quad \text{a.e.} u \in [0,T].\end{aligned}$$

Let $p \geq 2$. By ii) and (A5)-3),6), we get

$$E\left[\int_0^T \left(\int_0^T |D_u f(s, Y_s, Z_s) + \partial_y f(s, Y_s, Z_s)\mathbf{1}_{[0,s]}(u) Z_u| ds\right)^p du\right] < \infty,$$
$$E\left[\int_0^T \sup_{0\leq t\leq T} |\mathbf{1}_{[0,t]}(u) Z_u|^p du\right] < \infty.$$

Thus, we obtain $(D.Y, D.Z) \in \mathscr{S}^p_{rc}(\mathbb{R}, \bar{P}) \times \mathscr{H}^p(\mathbb{R}, \bar{P})$.

Step 2: We show the following Claims 1-4 for $k \geq 2$ by induction:

Claim 1 Let $p \geq 2$. Then, $(\nabla^{k-1} Y, \nabla^{k-1} Z) \in \mathbb{L}^a_{1,p}(H^{\otimes(k-1)}) \times \mathbb{L}^a_{1,p}(H^{\otimes(k-1)})$ and $(\nabla^k Y, \nabla^k Z) \in \mathscr{S}^p(H^{\otimes k}) \times \mathscr{H}^p(H^{\otimes k})$ is a unique solution to the BSDE;

$$\begin{aligned}\nabla^k Y_t = \nabla^k \xi - \sum_{i=0}^{k-1} \nabla^i \left(\int_{\cdot\wedge t}^{\cdot} \nabla^{k-1-i} Z_s ds\right) \\ + \int_t^T \left\{A^k_s + B^k_s \nabla^k Y_s + \Gamma^k_s \nabla^k Z_s\right\} ds - \int_t^T \nabla^k Z_s dW_s, \quad 0 \leq t \leq T,\end{aligned}$$

where $B^k_t = \partial_y f(t, Y_t, Z_t)$, $\Gamma^k_t = \partial_z f(t, Y_t, Z_t)$ and A^k is defined inductively as

$$\begin{aligned}A^1_t &= \nabla f(t, Y_t, Z_t), \\ A^k_t &= \nabla A^{k-1}_t + {}^{\nabla}B^{k-1}_t \nabla^{k-1} Y_t + {}^{\nabla}\Gamma^{k-1}_t \nabla^{k-1} Z_t, \quad k \geq 2.\end{aligned}$$

Moreover, it holds that

$$\begin{aligned} A_t^k &= \nabla^k f(t, Y_t, Z_t) \\ &\quad + \sum_{1,k} (\nabla^{\alpha_1} Y_t \otimes \cdots \otimes \nabla^{\alpha_{m-1-r}} Y_t \\ &\qquad \otimes \nabla^{\beta_1} Z_t \otimes \cdots \otimes \nabla^{\beta_r} Z_t \otimes \nabla^{\gamma} \partial_y^{m-1-r} \partial_z^r f(t, Y_t, Z_t))^{\Sigma} \\ &\quad + \sum_{2,k} \partial_y^{m-r} \partial_z^r f(t, Y_t, Z_t) \\ &\qquad \times (\nabla^{\alpha_1} Y_t \otimes \cdots \otimes \nabla^{\alpha_{m-r}} Y_t \otimes \nabla^{\beta_1} Z_t \otimes \cdots \otimes \nabla^{\beta_r} Z_t)^{\Sigma}, \end{aligned} \tag{3.22}$$

where the notations of summation represent

$$\sum_{1,k} := \sum_{m=2}^{k} \sum_{r=0}^{m-1} \sum_{\substack{\alpha \in \mathbb{N}_{\mathrm{ND}}^{m-1-r} \\ \beta \in \mathbb{N}_{\mathrm{ND}}^{r} \\ \gamma \in \mathbb{N} \\ |\alpha|+|\beta|+\gamma = k}}, \quad \sum_{2,k} := \sum_{m=2}^{k} \sum_{r=0}^{m} \sum_{\substack{\alpha \in \mathbb{N}_{\mathrm{ND}}^{m-r} \\ \beta \in \mathbb{N}_{\mathrm{ND}}^{r} \\ |\alpha|+|\beta| = k}},$$

$\mathbb{N}_{\mathrm{ND}}^k$ and the superscript Σ represent the same as in the proof of Theorem 3.4.1.

Claim 2 The following holds;

$$E\left[\int_0^T \left(\int_0^T \|\tilde{K}_u A_s^k\|_{H^{\otimes(k-1)}} ds\right)^2 du\right] < \infty.$$

In addition, $(D.\nabla^{k-1}Y, D.\nabla^{k-1}Z) \in \mathscr{S}_{rc}^2(H^{\otimes(k-1)}, \bar{P}) \times \mathscr{H}^2(H^{\otimes(k-1)}, \bar{P})$ and $D_t \nabla^{k-1} Y_t = \nabla^{k-1} Z_t$ for almost all $t \in [0,T]$.

Claim 3 For almost all $0 \le t \le T$, $u \in [0,T]^k$ and $v \in [0,T]^{k-1}$,

$$|D_u^k Y_t| + E\left[\int_t^T |D_u^k Z_s|^2 ds \middle| \mathcal{F}_t\right] + |D_v^{k-1} Z_t| + \left\|\nabla^k Y_t\right\|_{H^{\otimes k}} + \left\|\nabla^{k-1} Z_t\right\|_{H^{\otimes(k-1)}} \le C.$$

Claim 4 $(D.\nabla^{k-1}Y, D.\nabla^{k-1}Z) \in \mathscr{S}_{rc}^\infty(H^{\otimes(k-1)}, \bar{P}) \times \mathscr{H}^\infty(H^{\otimes(k-1)}, \bar{P})$.

We show the case when $k = 2$. Let $p \ge 2$. By Corollary 3.3.2, $(\nabla^2 Y, \nabla^2 Z) \in \mathscr{S}^p(H^{\otimes 2}) \times \mathscr{H}^p(H^{\otimes 2})$ solves (3.11). Then, Claim 1 holds. As in the proof of Theorem 3.4.1, Claim 2 holds. We will show Claim 3 and 4.

We now prove Claim 3. For a.e.$(u,v) \in [0,T]^2$, $(D_{u,v}^2 Y, D_{u,v}^2 Z) \in \mathscr{S}^2(\mathbb{R}) \times \mathscr{H}^2(\mathbb{R})$ solves

$$D_{u,v}^2 Y_t = D_{u,v}^2 \xi + \int_t^T \{\tilde{K}_{u,v}^2 A_s^2 + \partial_y f(s, Y_s, Z_s) D_{u,v}^2 Y_s + \partial_z f(s, Y_s, Z_s) D_{u,v}^2 Z_s\} ds$$

$$-\int_t^T D^2_{u,v} Z_s dW_s, \quad u \vee v \le t \le T,$$

where

$$\begin{aligned}\tilde{K}^2_{u,v} A^2_s = & D^2_{u,v} f(s, Y_s, Z_s) + D_u Y_s D_v \partial_y f(s, Y_s, Z_s) + D_u Z_s D_v \partial_z f(s, Y_s, Z_s) \\ & + D_u \partial_y f(s, Y_s, Z_s) D_v Y_s + D_u \partial_z f(s, Y_s, Z_s) D_v Z_s \\ & + \partial^2_y f(s, Y_s, Z_s) D_u Y_s D_v Y_s + \partial_y \partial_z f(s, Y_s, Z_s) D_u Y_s D_v Z_s \\ & + \partial_z \partial_y f(s, Y_s, Z_s) D_u Z_s D_v Y_s + \partial^2_z f(s, Y_s, Z_s) D_u Z_s D_v Z_s.\end{aligned}$$

Fix a.e.$(u, v) \in [0, T]^2$ as above. By Lemma 3.4.6, there exists a $\gamma_2 > 0$ such that

$$\begin{aligned}|D^2_{u,v} Y_t|^2 e^{\gamma_2 t} + \frac{1}{2} E\left[\int_t^T e^{\gamma_2 s} |D^2_{u,v} Z_s|^2 ds \middle| \mathcal{F}_t\right] \\ \le E\left[|D^2_{u,v}\xi|^2 e^{\gamma_2 T} + \int_t^T e^{\gamma_2 s} |\tilde{K}^2_{u,v} A^2_s|^2 ds \middle| \mathcal{F}_t\right], \quad u \vee v \le t \le T. \quad (3.23)\end{aligned}$$

Since $\sup_{u,t,\omega} |D_u Z_t(\omega)| < \infty$, by (A5)-3),5),6) and Step 1-ii), we see for almost all $s \in [0, T]$,

$$|\tilde{K}^2_{u,v} A^2_s|^2 \le C.$$

Hence, we get

$$E\left[\int_t^T |\tilde{K}^2_{u,v} A^2_s|^2 ds \middle| \mathcal{F}_t\right] \le C.$$

By Step 1-ii), (A5)-1) and (3.23), we see

$$|D^2_{u,v} Y_t| + E\left[\int_t^T |D^2_{u,v} Z_s|^2 ds \middle| \mathcal{F}_t\right] \le C, \quad u \vee v \le t \le T.$$

Thus, in the same manner as Step 1-ii), we obtain for almost all $u, v, t \in [0, T]$,

$$|D^2_{u,v} Y_t| + E\left[\int_t^T |D^2_{u,v} Z_s|^2 ds \middle| \mathcal{F}_t\right] \le C.$$

And then, for almost all $v, t \in [0, T]$, we get

$$|D_v Z_t| = |D_v D_t Y_t| \le C.$$

Integration of $|D^2_{u,v} Y_t|^2$ and $|D_v Z_t|^2$ with respect to u and v yield also that

$$\left\|\nabla^2 Y_t\right\|_{H^{\otimes 2}} + \|\nabla Z_t\|_H \le C.$$

Claim 3 is proved.

We now prove Claim 4. $(D.\nabla Y, D.\nabla Z) \in \mathscr{S}^2_{rc}(H,\bar{P}) \times \mathscr{H}^2(H,\bar{P})$ is a unique solution to the BSDE;

$$\begin{aligned}
D_u\nabla Y_t = D_u\nabla\xi - D_u\left(\int_{\cdot\wedge t}^{\cdot} Z_s ds\right) - \mathbf{1}_{(t,T]}(u)\nabla Z_u \\
+ \int_t^T \{\tilde{K}_u A_s^2 + \partial_y f(s,Y_s,Z_s)D_u\nabla Y_s + \partial_z f(s,Y_s,Z_s)D_u\nabla Z_s\}ds \\
- \int_t^T D_u\nabla Z_s dW_s, \quad 0 \le t \le T, \quad du\otimes dP\text{-a.e.}
\end{aligned}$$

Namely, putting $\bar{Y}_t^2(u) = D_u\nabla Y_t - D_u(\int_0^{\cdot\wedge t} Z_s ds) - \mathbf{1}_{[0,t]}(u)\nabla Z_u$, $(\bar{Y}^2(\cdot), D_u\nabla Z) \in \mathscr{S}^2(H,\bar{P}) \times \mathscr{H}^2(H,\bar{P})$ is a unique solution to the BSDE;

$$\begin{aligned}
\bar{Y}_t^2(u) = D_u\nabla\xi - D_u\left(\int_0^{\cdot} Z_s ds\right) - \nabla Z_u \\
+ \int_t^T \Bigg\{\tilde{K}_u A_s^2 + \partial_y f(s,Y_S,Z_s)\left(D_u\left(\int_0^{\cdot\wedge s} Z_r dr\right) + \mathbf{1}_{[0,s]}(u)\nabla Z_u\right) \\
+ \partial_y f(s,Y_s,Z_s)\bar{Y}_s^2(u) + \partial_z f(s,Y_s,Z_s)D_u\nabla Z_s\Bigg\} ds \\
- \int_t^T D_u\nabla Z_s dW_s, \quad 0 \le t \le T, \quad du\otimes dP\text{-a.e.}
\end{aligned}$$

Let $p \ge 2$. Since $\|D_u(\int_0^{\cdot\wedge s} Z_r dr)\|_H^2 = \int_0^s |D_u Z_r|^2 dr$ and by Claim 3 and (A5)-3),5),6), we see

$$\begin{aligned}
&E\left[\int_0^T \left(\int_0^T \left\|\tilde{K}_u A_s^2 + \partial_y f(s,Y_S,Z_s)\left(D_u\left(\int_0^{\cdot\wedge s} Z_r dr\right) + \mathbf{1}_{[0,s]}(u)\nabla Z_u\right)\right\|_H ds\right)^p du\right] \\
&\le C\left(1 + E\left[\int_0^T \left\{\sup_{0\le t\le T}|D_u Y_t|^p + \left(\int_0^T |D_u Z_s|^2 ds\right)^{\frac{p}{2}}\right\} du\right]\right) < \infty, \\
&E\left[\int_0^T \sup_{0\le t\le T}\left\|D_u\left(\int_0^{\cdot\wedge t} Z_s ds\right) + \mathbf{1}_{[0,t]}(u)\nabla Z_u\right\|_H^p du\right] < \infty.
\end{aligned}$$

Thus, we obtain $(D.\nabla Y, D.\nabla Z) \in \mathscr{S}^p_{rc}(H,\bar{P}) \times \mathscr{H}^p(H,\bar{P})$. The proofs of Claims 1-4 for $k = 2$ completes.

Next, assume $k > 2$ and Claims 1-4 for $2, 3, \ldots, k-1$ hold.

We will show Claim 1. Let $p \ge 2$. By the inductive assumption, $(\nabla^{k-1}Y, \nabla^{k-1}Z) \in \mathscr{S}^{2p}(H^{\otimes(k-1)}) \times \mathscr{H}^{2p}(H^{\otimes(k-1)})$ is a unique solution to the BSDE;

$$\nabla^{k-1}Y_t = \nabla^{k-1}\xi - \sum_{i=0}^{k-2}\nabla^i\left(\int_{\cdot\wedge t}^{\cdot} \nabla^{k-2-i}Z_s ds\right)$$

$$+\int_t^T \left\{A_s^{k-1} + B_s^{k-1}\nabla^{k-1}Y_s + \Gamma_s^{k-1}\nabla^{k-1}Z_s\right\} ds - \int_t^T \nabla^{k-1}Z_s dW_s, \quad 0 \le t \le T, \tag{3.24}$$

where $B_t^{k-1} = \partial_y f(t, Y_t, Z_t)$, $\Gamma_t^{k-1} = \partial_z f(t, Y_t, Z_t)$,

$$\begin{aligned} A_t^{k-1} =& \nabla^k f(t, Y_t, Z_t) \\ &+ \sum_{1,k-1} (\nabla^{\alpha_1} Y_t \otimes \cdots \otimes \nabla^{\alpha_{m-1-r}} Y_t \\ &\qquad \otimes \nabla^{\beta_1} Z_t \otimes \cdots \otimes \nabla^{\beta_r} Z_t \otimes \nabla^{\gamma} \partial_y^{m-1-r} \partial_z^r f(t, Y_t, Z_t))^{\Sigma} \\ &+ \sum_{2,k-1} \partial_y^{m-r} \partial_z^r f(t, Y_t, Z_t) \\ &\qquad \times (\nabla^{\alpha_1} Y_t \otimes \cdots \otimes \nabla^{\alpha_{m-r}} Y_t \otimes \nabla^{\beta_1} Z_t \otimes \cdots \otimes \nabla^{\beta_r} Z_t)^{\Sigma}. \end{aligned}$$

We show $(\nabla^{k-1}Y, \nabla^{k-1}Z) \in \mathbb{L}^a_{1,p}(H^{\otimes(k-1)}) \times \mathbb{L}^a_{1,p}(H^{\otimes(k-1)})$ by applying Theorem 3.2.5. By (3.24) and (A5)-5), we see (A2)-3) is satisfied . The correspondence to (3.6) is as follows;

$$\begin{aligned} \xi &= \nabla^{k-1}\xi, \\ \zeta_t &= \sum_{i=0}^{k-2} \nabla^i \left(\int_0^{\cdot \wedge t} \nabla^{k-2-i} Z_s ds \right), \end{aligned}$$

which satisfy (A2)-1),4). We see that for any $F, G \in H^{\otimes(k-1)}$,

$$\begin{aligned} &{}^{\nabla}B_t^{k-1}F = \nabla \partial_y f(t, Y_t, Z_t) \otimes F + \partial_y^2 f(t, Y_t, Z_t) \nabla Y_t \otimes F + \partial_z \partial_y f(t, Y_t, Z_t) \nabla Z_t \otimes F, \\ &\tilde{B}_t^{k-1} = \partial_y f(t, Y_t, Z_t), \\ &{}^{\nabla}\Gamma_t^{k-1}G = \nabla \partial_z f(t, Y_t, Z_t) \otimes G + \partial_y \partial_z f(t, Y_t, Z_t) \nabla Y_t \otimes G + \partial_z^2 f(t, Y_t, Z_t) \nabla Z_t \otimes G, \\ &\tilde{\Gamma}_t^{k-1} = \partial_z f(t, Y_t, Z_t). \end{aligned}$$

Hence, (A2)-6) is satisfied.

By messy but not difficult calculation, we obtain that $\nabla A_t^{k-1} + {}^{\nabla}B_t^{k-1}\nabla^{k-1}Y_t + {}^{\nabla}\Gamma_t^{k-1}\nabla^{k-1}Z_t$ is equal to the right-hand side of (3.22). Then by (A5)-3),5),6), we get

$$\begin{aligned} &E\left[\left(\int_0^T \left\|\nabla A_s^{k-1}\right\|_{H^{\otimes k}} ds\right)^p\right] \\ &\le C \left\{ 1 + \sum_{\substack{m,r\in\mathbb{Z}_+ \\ 1\le m+r\le 2}} \sum_{\substack{\alpha\in(\mathbb{Z}_+\cap[1,k-1])^{m+r} \\ m+r\le|\alpha|\le k}} E\left[\left(\int_0^T \prod_{j=1}^m \|\nabla^{\alpha_j} Y_s\|_{H^{\otimes\alpha_j}} \prod_{j'=1}^r \|\nabla^{\alpha_{j'+m}} Z_s\|_{H^{\otimes\alpha_{j'+m}}} ds\right)^p\right] \right. \end{aligned}$$

$$+\sum_{\substack{m,r\in\mathbb{Z}_+\\3\le m+r\le k}}\sum_{\substack{\alpha\in(\mathbb{Z}_+\cap[1,k-2])^{m+r}\\m+r\le|\alpha|\le k}} E\left[\left(\int_0^T\prod_{j=1}^m\|\nabla^{\alpha_j}Y_s\|_{H^{\otimes\alpha_j}}\prod_{j'=1}^r\|\nabla^{\alpha_{j'+m}}Z_s\|_{H^{\otimes\alpha_{j'+m}}}ds\right)^p\right]\Bigg\}, \tag{3.25}$$

where the products in (3.25) are defined to take 1 when $m=0$ or $r=0$. By the Hölder inequality, for each term of the first summation in (3.25), we see

$$\begin{aligned}
&E\left[\left(\int_0^T\prod_{j=1}^m\|\nabla^{\alpha_j}Y_t\|_{H^{\otimes\alpha_j}}\prod_{j'=1}^r\|\nabla^{\alpha_{j'+m}}Z_s\|_{H^{\otimes\alpha_{j'+m}}}ds\right)^p\right]\\
&\le\left\{E\left[\sup_{0\le t\le T}\prod_{j=1}^m\|\nabla^{\alpha_j}Y_t\|^{2p}_{H^{\otimes\alpha_j}}\right]\right\}^{\frac12}\left\{E\left[\left(\int_0^T\prod_{j'=1}^r\|\nabla^{\alpha_{j'+m}}Z_s\|_{H^{\otimes\alpha_{j'+m}}}ds\right)^{2p}\right]\right\}^{\frac12}\\
&\le(1+T^{\frac p2})\prod_{j=1}^m\|\nabla^{\alpha_j}Y\|^p_{\mathscr{S}^{2p_j}(H^{\otimes\alpha_j})}\prod_{j'=1}^r\|\nabla^{\alpha_{j'+m}}Z\|^p_{\mathscr{H}^{2q_{j'}}(H^{\otimes\alpha_{j'+m}})}\\
&<\infty,
\end{aligned}$$

where $1/p=\sum_{j=1}^m 1/p_j=\sum_{j'=1}^r 1/q_{j'}$. By the inductive assumption Claim 3, all $\|\nabla^{\alpha_j}Y_s\|_{H^{\otimes\alpha_j}}$ and $\|\nabla^{\alpha_{j'+m}}Z_s\|_{H^{\otimes\alpha_{j'+m}}}$ in the second summation in (3.25) are bounded; because for each $1\le i\le m+r$, $\alpha_i\le k-2$. Therefore, the second summation in (3.25) is bounded. Thus, we get

$$E\left[\left(\int_0^T\|\nabla A_s^{k-1}\|_{H^{\otimes k}}ds\right)^p\right]<\infty.$$

(A2)-5) is satisfied.

Now, we see that (A2)-2),3) are satisfied because the properties corresponding to them are shown in previous k on (A2)-7) and Claim 2.

From the above results, applying Theorem 3.2.5 yields that $(\nabla^{k-1}Y,\nabla^{k-1}Z)\in\mathbb{L}^a_{1,p}(H^{\otimes(k-1)})\times\mathbb{L}^a_{1,p}(H^{\otimes(k-1)})$ and that $(\nabla^kY,\nabla^kZ)\in\mathscr{S}^p(H^{\otimes k})\times\mathscr{H}^p(H^{\otimes k})$ is a unique solution to the BSDE;

$$\begin{aligned}
\nabla^kY_t&=\nabla^k\xi-\sum_{i=0}^{k-1}\nabla^i\left(\int_{\cdot\wedge t}^{\cdot}\nabla^{k-1-i}Z_sds\right)\\
&+\int_t^T\left\{\nabla A_s^{k-1}+{}^{\nabla}B^{k-1}\nabla^{k-1}Y_s+{}^{\nabla}\Gamma^{k-1}\nabla^{k-1}Z_s+\tilde B_s^{k-1}\nabla^kY_s+\tilde\Gamma_s^{k-1}\nabla^kZ_s\right\}ds\\
&\qquad-\int_t^T\nabla^kZ_sdW_s,\quad 0\le t\le T.
\end{aligned}$$

Define $A_t^k = \nabla A_t^{k-1} + {}^\nabla B_t^{k-1}\nabla^{k-1}Y_t + {}^\nabla\Gamma_t^{k-1}\nabla^{k-1}Z_t$. As mentioned above, A_t^k is written as (3.22). Claim 1 is proved.

We show Claim 2. By (3.22) and (A5)-3),5),6), we get

$$\begin{aligned}
&E\left[\int_0^T\left(\int_0^T \|\tilde{K}_u A_s^k\|_{H^{\otimes(k-1)}}ds\right)^2 du\right]\\
&\le C\left\{1+\sum_{1.1}E\left[\int_0^T (I_1^{m,r,\alpha}(u))^2du\right]+\sum_{1.2}E\left[\int_0^T (I_1^{m,r,\alpha}(u))^2du\right]\right.\\
&\qquad\left.+\sum_{2.1}E\left[\int_0^T (I_2^{m,r,\alpha}(u))^2du\right]+\sum_{2.2}E\left[\int_0^T (I_2^{m,r,\alpha}(u))^2du\right]\right\}, \qquad (3.26)
\end{aligned}$$

where

$$I_1^{m,r,\alpha}(u)=\int_0^T \left\|D_u\nabla^{\alpha_1-1}Y_s\right\|_{H^{\otimes(\alpha_1-1)}}\prod_{j=2}^m\|\nabla^{\alpha_j}Y_s\|_{H^{\otimes\alpha_j}}\prod_{j'=1}^r\|\nabla^{\alpha_{j'+m}}Z_s\|_{H^{\otimes\alpha_{j'+m}}}ds,$$

$$\sum_{1.1}=\sum_{\substack{m,r\in\mathbb{Z}_+\\ m\ge1\\ 1\le m+r\le 2}}\ \sum_{\substack{\alpha\in(\mathbb{Z}_+\cap[1,k-1])^{m+r}\\ m+r\le|\alpha|\le k}}, \qquad \sum_{1.2}=\sum_{\substack{m,r\in\mathbb{Z}_+\\ m\ge1\\ 3\le m+r\le k}}\ \sum_{\substack{\alpha\in(\mathbb{Z}_+\cap[1,k-2])^{m+r}\\ m+r\le|\alpha|\le k}},$$

$$I_2^{m,r,\alpha}(u)=\int_0^T \left\|D_u\nabla^{\alpha_{1+m}-1}Z_s\right\|_{H^{\otimes(\alpha_{1+m}-1)}}\prod_{j=1}^m\|\nabla^{\alpha_j}Y_s\|_{H^{\otimes\alpha_j}}\prod_{j'=2}^r\|\nabla^{\alpha_{j'+m}}Z_s\|_{H^{\otimes\alpha_{j'+m}}}ds,$$

$$\sum_{2.1}=\sum_{\substack{m,r\in\mathbb{Z}_+\\ r\ge1\\ 1\le m+r\le 2}}\ \sum_{\substack{\alpha\in(\mathbb{Z}_+\cap[1,k-1])^{m+r}\\ m+r\le|\alpha|\le k}}, \qquad \sum_{2.2}=\sum_{\substack{m,r\in\mathbb{Z}_+\\ r\ge1\\ 3\le m+r\le k}}\ \sum_{\substack{\alpha\in(\mathbb{Z}_+\cap[1,k-2])^{m+r}\\ m+r\le|\alpha|\le k}},$$

defining a product $\prod_{j=a}^b x_j = 1$ if $b<a$.

If $m,r\in\mathbb{Z}_+$, $m\ge1$, $1\le m+r\le 2$ and $\alpha\in(Z_+\cap[1,k-1])^{m+r}$, $m+r\le|\alpha|\le k$, then by the Schwarz inequality and the inductive assumption (1),

$$\begin{aligned}
E\left[\int_0^T (I_1^{m,r,\alpha}(u))^2du\right] &\le E\left[\int_0^T du\int_0^T \left\|D_u\nabla^{\alpha_1-1}Y_s\right\|^2_{H^{\otimes(\alpha_1-1)}}ds\right.\\
&\qquad\left.\times\int_0^T\prod_{j=2}^m\|\nabla^{\alpha_j}Y_s\|^2_{H^{\otimes\alpha_j}}\prod_{j'=1}^r\|\nabla^{\alpha_{j'+m}}Z_s\|^2_{H^{\otimes\alpha_{j'+m}}}ds\right]\\
&\le (T+T^2)E\left[\prod_{j=1}^m\sup_{0\le t\le T}\|\nabla^{\alpha_j}Y_s\|^2_{H^{\otimes\alpha_j}}\int_0^T\prod_{j'=1}^r\|\nabla^{\alpha_{j'+m}}Z_s\|^2_{H^{\otimes\alpha_{j'+m}}}ds\right]\\
&\le (T+T^2)\left\{E\left[\prod_{j=1}^m\sup_{0\le t\le T}\|\nabla^{\alpha_j}Y_s\|^4_{H^{\otimes\alpha_j}}\right]\right\}^{\frac12}
\end{aligned}$$

$$\times \left\{ E\left[\left(\int_0^T \prod_{j'=1}^{r} \|\nabla^{\alpha_{j'+m}} Z_s\|^2_{H^{\otimes \alpha_{j'+m}}} \, ds \right)^2 \right] \right\}^{\frac{1}{2}}$$
$$\le (T+T^2) \prod_{j=1}^{m} \|\nabla^{\alpha_j} Y\|^2_{\mathscr{S}^{2p_j}(H^{\otimes \alpha_j})} \prod_{j'=1}^{r} \|\nabla^{\alpha_{j'+m}} Z\|^2_{\mathscr{H}^4(H^{\otimes \alpha_{j'+m}})}$$
$$< \infty,$$

where, to see in the fourth inequality above, $r \le 1$ is used, and $1/2 = \sum_{j=1}^m 1/p_j$. Hence, we get

$$\sum_{1.1} E\left[\int_0^T (I_1^{m,r,\alpha}(u))^2 du \right] < \infty.$$

If $m, r \in \mathbb{Z}_+$, $r \ge 1$, $1 \le m+r \le 2$ and $\alpha \in (\mathbb{Z}_+ \cap [1, k-1])^{m+r}$, $m+r \le |\alpha| \le k$, then by the inductive assumption Claim 1,

$$E\left[\int_0^T (I_2^{m,r,\alpha}(u))^2 du \right] \le E\left[\int_0^T du \int_0^T \left\| D_u \nabla^{\alpha_{1+m}-1} Z_s \right\|^2_{H^{\otimes(\alpha_{1+m}-1)}} ds \right.$$
$$\left. \times \int_0^T \prod_{j=1}^{m} \|\nabla^{\alpha_j} Y_s\|^2_{H^{\otimes \alpha_j}} \prod_{j'=2}^{r} \|\nabla^{\alpha_{j'+m}} Z_s\|^2_{H^{\otimes \alpha_{j'+m}}} \, ds \right]$$
$$\le E\left[\int_0^T \|\nabla^{\alpha_{1+m}} Z_s\|^2_{H^{\otimes \alpha_{1+m}}} \, ds \right.$$
$$\left. \times \int_0^T \prod_{j=1}^{m} \|\nabla^{\alpha_j} Y_s\|^2_{H^{\otimes \alpha_j}} \prod_{j'=2}^{r} \|\nabla^{\alpha_{j'+m}} Z_s\|^2_{H^{\otimes \alpha_{j'+m}}} \, ds \right]$$
$$\le (1+T) \left\{ E\left[\left(\int_0^T \|\nabla^{\alpha_{1+m}} Z_s\|^2_{H^{\otimes \alpha_{1+m}}} \, ds \right)^2 \right] \right\}^{\frac{1}{2}}$$
$$\times \left\{ E\left[\prod_{j=1}^{m} \sup_{0 \le t \le T} \|\nabla^{\alpha_j} Y_s\|^4_{H^{\otimes \alpha_j}} \left(\int_0^T \prod_{j'=2}^{r} \|\nabla^{\alpha_{j'+m}} Z_s\|^2_{H^{\otimes \alpha_{j'+m}}} \, ds \right)^2 \right] \right\}^{\frac{1}{2}}$$
$$\le (1+T) \prod_{j=1}^{m} \|\nabla^{\alpha_j} Y\|^2_{\mathscr{S}^4(H^{\otimes \alpha_j})} \prod_{j'=1}^{r} \|\nabla^{\alpha_{j'+m}} Z\|_{\mathscr{H}^4(H^{\otimes \alpha_{j'+m}})}$$
$$< \infty.$$

Thus, we get

$$\sum_{2.1} E\left[\int_0^T (I_2^{m,r,\alpha}(u))^2 du \right] < \infty.$$

If $\alpha_i \leq k-2$ for $1 \leq i \leq m+r$, then by the inductive assumption Claim 3, $\|\nabla^{\alpha_j} Y_s\|_{H^{\otimes \alpha_j}}$ and $\|\nabla^{\alpha_{j'+m}} Z_s\|_{H^{\otimes \alpha_{j'+m}}}$ are bounded. Then, we see

$$\sum_{1.2} E\left[\int_0^T (I_1^{m,r,\alpha}(u))^2 du\right] + \sum_{2.2} E\left[\int_0^T (I_2^{m,r,\alpha}(u))^2 du\right] < \infty.$$

Hence, we obtain

$$E\left[\int_0^T \left(\int_0^T \|\tilde{K}_u A_s^k\| ds\right)^2 du\right] < \infty,$$

which implies (A2)′-8). Thus, by Theorem 3.3.1, we get $(D.\nabla^{k-1}Y, D.\nabla^{k-1}Z) \in \mathscr{S}_{rc}^2(H^{\otimes(k-1)}, \bar{P}) \times \mathscr{H}^2(H^{\otimes(k-1)}, \bar{P})$ and $D_t\nabla^{k-1}Y_t = \sum_{i=1}^{k-1} D_t\nabla^{i-1}(\int_0^{\cdot\wedge t} \nabla^{k-1-i} Z_s ds) + \nabla^{k-1}Z_t$. Since for $1 \leq i \leq k-1$,

$$\left\| D_u \nabla^{i-1}\left(\int_0^{\cdot\wedge t} \nabla^{k-1-i} Z_s ds\right)\right\|_{H^{\otimes(k-1)}}^2 = \begin{cases} 0, & t \leq u, \\ \displaystyle\int_u^t \left\| D_u \nabla^{k-2} Z_s\right\|_{H^{\otimes(k-2)}}^2 ds, & u < t, \end{cases}$$

we obtain $D_t\nabla^{k-1}Y_t = \nabla^{k-1}Z_t$ for almost all $t \in [0,T]$. Claim 2 is proved.

We prove Claim 3. For a.e.$u = (u_1, \ldots, u_k) \in [0,T]^k$, $(D_u^k Y, D_u^k Z) \in \mathscr{S}_{rc}^2(\mathbb{R}) \times \mathscr{H}^2(\mathbb{R})$ solves

$$D_u^k Y_t = D_u^k \xi + \int_t^T \{\tilde{K}_u^k A_s^k + \partial_y f(s, Y_s, Z_s) D_u^k Y_s + \partial_z f(s, Y_s, Z_s) D_u^k Z_s\} ds \\ - \int_t^T D_u^k Z_s dW_s, \quad \bar{u} \leq t \leq T,$$

where $\bar{u} = \max\{u_1, \ldots, u_k\}$. By Lemma 3.4.6, there exists a $\gamma_k > 0$ such that

$$|D_u^k Y_t|^2 e^{\gamma_k t} + \frac{1}{2} E\left[\int_t^T e^{\gamma_k s} |D_u^k Z_s|^2 ds \,\middle|\, \mathcal{F}_t\right] \\ \leq E\left[|D_u^k \xi|^2 e^{\gamma_k T} + \int_t^T e^{\gamma_k s} |\tilde{K}_u^k A_s^k|^2 ds \,\middle|\, \mathcal{F}_t\right], \quad \bar{u} \leq t \leq T. \quad (3.27)$$

By (3.22), we see

$$|\tilde{K}_u^k A_s^k|^2 \leq C\left\{1 + \sum_{1,k} \sum_{\sigma \in S_k} J_u^{1,m,r,\alpha,\beta,\sigma}(s) + \sum_{2,k} \sum_{\sigma \in S_k} J_u^{2,m,r,\alpha,\beta,\sigma}(s)\right\}, \quad (3.28)$$

where S_k represents the symmetric group of degree k and

$$J_u^{1,m,r,\alpha,\beta,\sigma}(s) = \prod_{j=1}^{m-1-r} |D_{u_j^{\sigma,\alpha}}^{\alpha_j} Y_s|^2 \prod_{j'=1}^{r} |D_{u_{j'}^{\sigma,\beta}}^{\beta_{j'}} Z_s|^2 |D_{u^{\sigma,\gamma}}^{\gamma} \partial_y^{m-1-r} \partial_z^r f(t, Y_t, Z_t)|^2,$$

$$J_u^{2,m,r,\alpha,\beta,\sigma}(s) = \prod_{j=1}^{m-r} |D^{\alpha_j}_{u_j^{\sigma,\alpha}} Y_s|^2 \prod_{j'=1}^{r} |D^{\beta_{j'}}_{u_{j'}^{\sigma,\beta}} Z_s|^2,$$

$$\alpha_0 = 1,\ \beta_0 = 1,$$

$$u_j^{\sigma,\alpha} = (u_{\sigma(\sum_{i=1}^j \alpha_{i-1})}, u_{\sigma(\sum_{i=1}^j \alpha_{i-1}+1)}, \ldots, u_{\sigma(\sum_{i=1}^j \alpha_{i-1}+\alpha_j-1)}),$$

$$u_{j'}^{\sigma,\beta} = (u_{\sigma(|\alpha|+\sum_{i=1}^{j'} \beta_{i-1})}, u_{\sigma(|\alpha|+\sum_{i=1}^{j'} \beta_{i-1}+1)}, \ldots, u_{\sigma(|\alpha|+\sum_{i=1}^{j'} \beta_{i-1}+\beta_{j'}-1)}),$$

$$u^{\sigma,\gamma} = (u_{\sigma(|\alpha|+|\beta|+1)}, u_{\sigma(|\alpha|+|\beta|+2)}, \ldots, u_{\sigma(|\alpha|+|\beta|+\gamma)}).$$

We divide the first summation of (3.28) into three ones;

$$\sum_{1,k}\sum_{\sigma\in S_k} J_u^1(s) = \sum_{\substack{1,k\\ r=0}}\sum_{\sigma\in S_k} J_u^{1,m,r,\alpha,\beta,\sigma}(s) + \sum_{\substack{1,k\\ r=1}}\sum_{\sigma\in S_k} J_u^{1,m,r,\alpha,\beta,\sigma}(s) + \sum_{\substack{1,k\\ r\geq 2}}\sum_{\sigma\in S_k} J_u^{1,m,r,\alpha,\beta,\sigma}(s).$$

If $r = 0$, derivatives of Z do not appear in $J_u^{1,m,r,\alpha,\beta,\sigma}(s)$ and each $\alpha_j \leq k-1$. By the inductive assumption and (A5)-6), we get

$$\sum_{\substack{1,k\\ r=0}}\sum_{\sigma\in S_k} E\left[\int_t^T J_u^{1,m,r,\alpha,\beta,\sigma}(s)ds\middle|\mathcal{F}_t\right] \leq C.$$

If $r = 1$, $\beta \in \mathbb{N}$ and $\beta_1 \leq k-1$. By the inductive assumption and (A5)-6), we get

$$\begin{aligned}\sum_{\substack{1,k\\ r=1}}\sum_{\sigma\in S_k} E\left[\int_t^T J_u^{1,m,r,\alpha,\beta,\sigma}(s)ds\middle|\mathcal{F}_t\right] &\leq C\sum_{\substack{1,k\\ r=0}}\sum_{\sigma\in S_k} E\left[\int_t^T |D^{\beta_1}_{u_1^{\sigma,\beta}} Z_s|^2 ds\middle|\mathcal{F}_t\right]\\ &\leq C.\end{aligned}$$

If $r \geq 2$, each $\beta_{j'} \leq k-2$. By the inductive assumption and (A5)-6), we get

$$\sum_{\substack{1,k\\ r\geq 2}}\sum_{\sigma\in S_k} E\left[\int_t^T J_u^{1,m,r,\alpha,\beta,\sigma}(s)ds\middle|\mathcal{F}_t\right] \leq C.$$

In the same manner as above, we can obtain

$$\sum_{2,k}\sum_{\sigma\in S_k} E\left[\int_t^T J_u^{2,m,r,\alpha,\beta,\sigma}(s)ds\middle|\mathcal{F}_t\right] \leq C.$$

Thus by (3.28), we get

$$E\left[\int_t^T |\tilde{K}_u^k A_s^k|^2 ds\middle|\mathcal{F}_t\right] \leq C.$$

Hence by (3.27) and (A5)-1), we obtain

$$|D_u^k Y_t|^2 + E\left[\int_t^T |D_u^k Z_s|^2 ds \middle| \mathcal{F}_t\right] \le C, \quad \bar{u} \le t \le T.$$

In the same manner as Step 1-ii), we see

$$|D_u^k Y_t|^2 + E\left[\int_t^T |D_u^k Z_s|^2 ds \middle| \mathcal{F}_t\right] \le C, \quad 0 \le t \le T.$$

And then, for almost all $(v,t) \in [0,T]^{k-1} \times [0,T]$, we get

$$|D_v^{k-1} Z_t| = |D_v^{k-1} D_t Y_t| \le C.$$

Integration of $|D_u^k Y_t|^2$ and $|D_v^{k-1} Z_t|^2$ with respect to u and v yield also that

$$\left\|\nabla^k Y_t\right\|_{H^{\otimes k}}^2 + \left\|\nabla^{k-1} Z_t\right\|_{H^{\otimes(k-1)}}^2 \le C.$$

Claim 3 is now proved.

We show Claim 4. $(D.\nabla^{k-1}Y, D.\nabla^{k-1}Z) \in \mathscr{S}_{rc}^2(H^{\otimes(k-1)}, \bar{P}) \times \mathscr{H}^2(H^{\otimes(k-1)}, \bar{P})$ is a unique solution to the BSDE;

$$\begin{aligned} D_u\nabla^{k-1}Y_t &= D_u\nabla^{k-1}\xi - \zeta_T(u) + \zeta_t(u) \\ &\quad + \int_t^T \{\tilde{K}_u A_s^k + \partial_y f(s,Y_s,Z_s) D_u\nabla^{k-1}Y_s + \partial_z f(s,Y_s,Z_s) D_u\nabla^{k-1}Z_s\} ds \\ &\qquad - \int_t^T D_u\nabla^{k-1}Z_s dW_s, \quad 0 \le t \le T, \quad du \otimes dP\text{-a.e.}, \end{aligned}$$

where $\zeta_t(u) = \sum_{i=1}^{k-1} D_u\nabla^{i-1}(\int_0^{\cdot\wedge t} \nabla^{k-1-i} Z_s ds) + \mathbf{1}_{[0,t]}(u)\nabla^{k-1}Z_u$. Namely, putting $\bar{Y}_t^k(u) = D_u\nabla^{k-1}Y_t - \zeta_t(u)$, $(\bar{Y}^2(\cdot), D_u\nabla Z) \in \mathscr{S}^2(H^{\otimes(k-1)}, \bar{P}) \times \mathscr{H}^2(H^{\otimes(k-1)}, \bar{P})$ is a unique solution to the BSDE;

$$\begin{aligned} \bar{Y}_t^k(u) &= D_u\nabla^{k-1}\xi - \zeta_T(u) \\ &\quad + \int_t^T \{\tilde{K}_u A_s^k + \partial_y f(s,Y_S,Z_s)\zeta_s(u) \\ &\qquad + \partial_y f(s,Y_s,Z_s)\bar{Y}_s^k(u) + \partial_z f(s,Y_s,Z_s) D_u\nabla^{k-1}Z_s\} ds \\ &\qquad\quad - \int_t^T D_u\nabla^{k-1}Z_s dW_s, \quad 0 \le t \le T, \quad du \otimes dP\text{-a.e.} \end{aligned}$$

Let $p \ge 2$. By Claim 2, we get $D_u\nabla^{k-2}Z_s = D_u D_s \nabla^{k-2} Y_s$ and by Claim 3,

$$E\left[\int_0^T \sup_{0 \le t \le T} \|\zeta_t(u)\|_{H^{\otimes(k-1)}}^p du\right]$$

$$\leq CE\left[\int_0^T\left\{\left(\int_0^T\|D_u\nabla^{k-2}Z_s\|^2_{H^{\otimes(k-2)}}ds\right)^{\frac{p}{2}}+\|\nabla^{k-1}Z_u\|^p_{H^{\otimes(k-1)}}\right\}du\right]$$
$$<\infty.$$

Thus in the same manner as (3.26) and (A5)-3)-6), we get

$$E\left[\int_0^T\left(\int_0^T\|\tilde{K}_uA^k_s+\partial_yf(s,Y_S,Z_s)\zeta_s(u)\|_{H^{\otimes(k-1)}}ds\right)^p du\right]$$
$$\leq C\left(1+\sum_{\alpha=0}^{k-2}E\left[\int_0^T\left\{\sup_{0\leq t\leq T}\|D_u\nabla^\alpha Y_t\|^{2p}_{H^{\otimes\alpha}}+\left(\int_0^T\|D_u\nabla^\alpha Z_s\|^2_{H^{\otimes\alpha}}\,ds\right)^p\right\}du\right]\right)<\infty.$$

Hence, we obtain $(D.\nabla^{k-1}Y, D.\nabla^{k-1}Z)\in\mathscr{S}^p_{rc}(H^{\otimes(k-1)},\bar{P})\times\mathscr{H}^p(H^{\otimes(k-1)},\bar{P})$. □

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