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On some relations and generators of multiple zeta values

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Contents

Part I
Overview

In this paper, we prove several relations for multiple zeta values and finite multiple zeta values, and consider generators of multiple zeta values. The multiple zeta values are multi-variate generalizations of the values of the Riemann zeta function at positive integers. These numbers have been appeared in various contexts in number theory, knot theory, arithmetic geometry, mathematical physics and so on. The multiple zeta values were first studied by L. Euler. Euler studied the multiple zeta values of depth 2. In 1990's, D. Zagier and M. Hoffman independently started their study of multiple zeta values of general depth. In 1994, Zagier made a conjecture about the dimensions of the vector spaces spanned by the multiple zeta values. This conjecture was partially solved by T. Terasoma, A. Goncharov and P. Deligne.

In recent years, two types of finite multiple zeta values, $\mathcal{A}$-finite multiple zeta values and symmetrized multiple zeta values have been studied. The $\mathcal{A}$-finite multiple zeta value is a collection of certain finite sums whose setting was given by Zagier, and the symmetrized multiple zeta value was introduced by Kaneko and Zagier to establish a crucial bridge between the multiple zeta value and $\mathcal{A}$-finite multiple zeta values.

In part II, we review the theory of multiple zeta values and finite multiple zeta values.

In part III, we prove a new linear relation by using the integral series identity. We also give the equivalent of the new relation and the Ohno-type relation. We also present an analogous result for finite multiple zeta values. The content of this part is based on [23].

In part IV, we consider a cyclic analogue of multiple zeta values (CMZVs), which has two kinds of expressions; series and integral expression. We prove an 'integral=series' type identity for CMZVs. By using this identity, we construct two classes of $\mathbb{Q}$-linear relations among CMZVs. One of them is a generalization of the cyclic sum formula for multiple zeta-star values. We also give an alternative proof of the derivation relation for multiple zeta values. The content of this part is based on [7].

In part V, we take another look at the so-called quasi-derivation relations in the theory of multiple zeta values, by giving a certain formula for the quasi-derivation operator. In doing so, we are not only able to prove the quasi-derivation relations in a simpler manner but also give an analog of the quasi-derivation relations for finite multiple zeta values. The content of this part is based on [18].

In part VI, we prove that any multiple t-values of maximal height (that is, all components of the index are greater than 1) can be written as a rational linear combination of multiple zeta values by using Glanois's theorem. The multiple t-value is an "odd variant" of multiple zeta value introduced by Hoffman. We also prove that each multiple zeta value is a $\mathbb{Q}$-linear combination of multiple t-values of all components of the index are 2 or 3.

Part II
Preliminaries

1 Multiple zeta values

1.1 Definition of MZVs

In this section, we review the theory of multiple zeta values.

Definition 1. For integers $k_1, \dots, k_d \in \mathbb{Z}_{\geq 1}$ with $k_d \geq 2$, the multiple zeta value (MZV for short) and the multiple zeta-star value (MZSV for short) are defined by

$$\zeta(k_1, \dots, k_d) = \sum_{1 \leq n_1 < \cdots < n_d} \frac{1}{n_1^{k_1} \cdots n_d^{k_d}}$$

and

$$\zeta^{\star}(k_1, \dots, k_d) = \sum_{1 \leq n_1 \leq \cdots \leq n_d} \frac{1}{n_1^{k_1} \cdots n_d^{k_d}}$$

respectively.

For an index $\mathbf{k} = (k_1, \dots, k_d) \in (\mathbb{Z}_{\geq 1})^d$, we call $|\mathbf{k}| := k_1 + \cdots + k_d$ the weight, d the depth. An index $\mathbf{k} = (k_1, \dots, k_d)$ is admissible if $k_d \geq 2$. Multiple zeta values can be written as a linear combination of multiple zeta-star values and vice versa. For example,

$$\begin{aligned} \zeta^{\star}(k_1, k_2) &= \zeta(k_1, k_2) + \zeta(k_1 + k_2), \\ \zeta^{\star}(k_1, k_2, k_3) &= \zeta(k_1, k_2, k_3) + \zeta(k_1 + k_2, k_3) + \zeta(k_1, k_2 + k_3) + \zeta(k_1 + k_2 + k_3), \\ &\cdots \end{aligned}$$

and

$$\begin{aligned} \zeta(k_1, k_2) &= \zeta^{\star}(k_1, k_2) - \zeta^{\star}(k_1 + k_2), \\ \zeta(k_1, k_2, k_3) &= \zeta^{\star}(k_1, k_2, k_3) - \zeta^{\star}(k_1 + k_2, k_3) - \zeta^{\star}(k_1, k_2 + k_3) + \zeta^{\star}(k_1 + k_2 + k_3), \\ &\cdots . \end{aligned}$$

1.2 Vector space spanned by MZVs

We introduce the vector space over $\mathbb{Q}$ spanned by MZVs.

Definition 2.

$$\mathcal{Z}_0 = \mathbb{Q}, \quad \mathcal{Z}_1 = \{0\},$$
$$\mathcal{Z}_k = \sum_{\substack{1 \le d < k \\ k_1+\cdots+k_d=k \\ k_1,\ldots,k_{d-1}\ge 1, k_d \ge 2}} \mathbb{Q}\cdot\zeta(k_1,\ldots,k_d) \quad (k \ge 2),$$
$$\mathcal{Z} = \sum_{k=0}^{\infty} \mathcal{Z}_k.$$

Since the MZVs with weight 2 is only $\zeta(2)$, and by the known relations $\zeta(1,2) = \zeta(3)$, $\zeta(1,1,2) = \zeta(4)$, $\zeta(1,3)+\zeta(2,2) = \zeta(4)$ and $4\zeta(1,3) = \zeta(4)$, we find $\mathcal{Z}_2 = \mathbb{Q}\cdot\zeta(2)$, $\mathcal{Z}_3 = \mathbb{Q}\cdot\zeta(3)$, and $\mathcal{Z}_4 = \mathbb{Q}\cdot\zeta(4)$. In [34], Zagier gave a conjecture on the dimension of $\mathcal{Z}_k$.

Conjecture 1 (Zagier [34])**.** *We have*

$$\dim_{\mathbb{Q}} \mathcal{Z}_k = d_k,$$

where d_k is the non-negative integer satisfying the following recurrence relation.

$$d_k = d_{k-2} + d_{k-3} \ (k \ge 3), \ d_0 = 1, \ d_1 = 0, \ d_2 = 1. \tag{1}$$

Goncharov [5], Terasoma [32] and Deligne–Goncharov [2] proved that the number d_k gives an upper bound of the dimension of $\mathcal{Z}_k$.

Theorem 1.1. *The inequality*

$$\dim_{\mathbb{Q}} \mathcal{Z}_k \le d_k$$

holds.

The following theorem is conjectured by Hoffman in [9], and proved by F. Brown in [1].

Theorem 1.2. *Every multiple zeta value is a $\mathbb{Q}$-linear combination of elements in*

$$\{\zeta(k_1,\ldots,k_d) \mid k_1,\ldots,k_d \in \{2,3\}\}.$$

We note that the number of indices $(k_1,\ldots,k_d)$ of weight k with all $k_i \in \{2,3\}$ is equal to d_k.

1.3 Two products of MZVs

The vector space $\mathcal{Z}$ is closed under two types of product. The one is called the harmonic (stuffle) product and the other the shuffle product. The harmonic product is obtained by expanding the product of the series expressions. For example,

$$\begin{aligned}\zeta(a)\zeta(b) &= \left(\sum_{0<m}\frac{1}{m^a}\right)\left(\sum_{0<n}\frac{1}{n^b}\right)\\ &= \sum_{0<m<n}\frac{1}{m^a n^b} + \sum_{0<n<m}\frac{1}{m^a n^b} + \sum_{0<m}\frac{1}{m^{a+b}}\\ &= \zeta(a,b) + \zeta(b,a) + \zeta(a+b).\end{aligned}$$

In general, the product of MZVs of weight k_1 and weight k_2 is a sum of MZVs of weight k_1+k_2. To describe the shuffle product, we need the iterated integral expression of MZVs. We consider the following iterated integral.

$$\begin{aligned}I(\epsilon_1,\cdots,\epsilon_k)&=\int_{0<t_1<\cdots<t_k<1}A_{\epsilon_1}(t_1)\cdots A_{\epsilon_k}(t_k)dt_1\cdots dt_k\\&=\int_0^1 A_{\epsilon_k}(t_k)dt_k\int_0^{t_k}A_{\epsilon_{k-1}}(t_{k-1})dt_{k-1}\cdots\int_0^{t_2}A_{\epsilon_1}(t_1)dt_1,\end{aligned}$$

where $\epsilon_j\in\{0,1\}$ with $\epsilon_1=1$ and $\epsilon_k=0$, and

$$A_0(t)=\frac{1}{t},\quad A_1(t)=\frac{1}{1-t}.$$

We note that the above iterated integral converges and $\zeta(k_1,\ldots,k_d)$ is represented as follows.

Theorem 1.3 (Iterated integral expression of MZV)**.**

$$\zeta(k_1,\ldots,k_d)=I(1,\underbrace{0,\ldots,0}_{k_1-1},1,\underbrace{0,\ldots,0}_{k_2-1},\cdots,1,\underbrace{0,\ldots,0}_{k_d-1}).$$

The shuffle product results from dividing the domain of integration of the product of two integrals. For example,

$$\begin{aligned}\zeta(2)^2=&I(1,0)^2\\=&\int_{0<t_1<t_2<1}\frac{dt_1}{1-t_1}\frac{dt_2}{t_2}\int_{0<s_1<s_2<1}\frac{ds_1}{1-s_1}\frac{ds_2}{s_2}\\=&\int_{0<t_1<t_2<s_1<s_2<1}\frac{dt_1}{1-t_1}\frac{dt_2}{t_2}\frac{ds_1}{1-s_1}\frac{ds_2}{s_2}+\int_{0<t_1<s_1<t_2<s_2<1}\frac{dt_1}{1-t_1}\frac{ds_1}{1-s_1}\frac{dt_2}{t_2}\frac{ds_2}{s_2}\\&+\int_{0<t_1<s_1<s_2<t_2<1}\frac{dt_1}{1-t_1}\frac{ds_1}{1-s_1}\frac{ds_2}{s_2}\frac{dt_2}{t_2}+\int_{0<s_1<t_1<t_2<s_2<1}\frac{ds_1}{1-s_1}\frac{dt_1}{1-t_1}\frac{dt_2}{t_2}\frac{ds_2}{s_2}\\&+\int_{0<s_1<t_1<s_2<t_2<1}\frac{ds_1}{1-s_1}\frac{dt_1}{1-t_1}\frac{ds_2}{s_2}\frac{dt_2}{t_2}+\int_{0<s_1<s_2<t_1<t_2<1}\frac{ds_1}{1-s_1}\frac{ds_2}{s_2}\frac{dt_1}{1-t_1}\frac{dt_2}{t_2}\\=&4I(1,1,0,0)+2I(1,0,1,0)\\=&4\zeta(1,3)+2\zeta(2,2).\end{aligned}$$

1.4 Algebraic setup of MZVs

We recall Hoffman's algebraic setup with a slightly different convention (see [9]). Let $\mathfrak{h}$ be the noncommutative polynomial algebra $\mathbb{Q}\langle x,y\rangle$. We call monomials in x and y the words (1 is the empty word). We also let $\mathfrak{h}^0:=\mathbb{Q}\oplus y\mathfrak{h}x$ and $\mathfrak{h}^1:=\mathbb{Q}\oplus y\mathfrak{h}$ subalgebras of $\mathfrak{h}$. Put $z_k=yx^{k-1}$ for $k\in\mathbb{Z}_{\geq1}$. The subalgebra $\mathfrak{h}^1$ can also be considered as a non-commutative polynomial algebra over $\mathbb{Q}$ freely generated by $\{z_k\}_{k\in\mathbb{Z}_{\geq 1}}$. We often identify an index $(k_1,\ldots,k_r)$ with the monomial $z_{k_1}\cdots z_{k_r}$. We define the $\mathbb{Q}$-linear map (called evaluation map) $Z:\mathfrak{h}^0\longrightarrow\mathbb{R}$ by

$$Z(1)=1,\quad Z(z_{k_1}\cdots z_{k_d})=\zeta(k_1,\ldots,k_d).$$

Let $*:\mathfrak{h}^1\times\mathfrak{h}^1\to\mathfrak{h}^1$ be the $\mathbb{Q}$-bilinear map defined by

$$1*w=w*1=w,$$
$$w_1z_{k_1}*w_2z_{k_2}=(w_1*w_2z_{k_2})z_{k_1}+(w_1z_{k_1}*w_2)z_{k_2}+(w_1*w_2)z_{k_1+k_2}$$

for $k_1,k_2\in\mathbb{Z}_{\geq 1}$, and the words w,w_1,w_2 in $\mathfrak{h}^1$. This product $*$ is called the harmonic product on $\mathfrak{h}^1$. It is known that the product $*$ is commutative and associative. Therefore $\mathfrak{h}^1$ is a $\mathbb{Q}$-commutative algebra with respect to $*$, and we denote this algebra by $\mathfrak{h}^1_*$. The subset $\mathfrak{h}^0$ is a subalgebra of $\mathfrak{h}^1$ with respect to $*$, which is denoted by $\mathfrak{h}^0_*$. For this product, we have

$$Z(w_1*w_2)=Z(w_1)Z(w_2) \tag{2}$$

for any $w_1,w_2\in\mathfrak{h}^0$.

Let $⧢:\mathfrak{h}\times\mathfrak{h}\to\mathfrak{h}$ be the $\mathbb{Q}$-bilinear map defined by

$$1⧢w=w⧢1=w,$$
$$u_1w_1⧢u_2w_2=u_1(w_1⧢u_2w_2)+u_2(u_1w_1⧢w_2)$$

for $u_1,u_2\in\{x,y\}$, and the words w,w_1,w_2 in $\mathfrak{h}$. The product $⧢$ is called the shuffle product on $\mathfrak{h}$. It is also known that the product $⧢$ is commutative and associative (see [29]), therefore $\mathfrak{h}$ is a $\mathbb{Q}$-commutative algebra with respect to $⧢$. We denote this algebra by $\mathfrak{h}_{⧢}$. The subsets $\mathfrak{h}^1$ and $\mathfrak{h}^0$ are subalgebras of $\mathfrak{h}$ with respect to $⧢$ and we denote them by $\mathfrak{h}^1_{⧢}$, $\mathfrak{h}^0_{⧢}$. For this product, we also have

$$Z(w_1⧢w_2)=Z(w_1)Z(w_2) \tag{3}$$

for any $w_1,w_2\in\mathfrak{h}^0$. From (2) and (3), we have the following equation which is called the finite double shuffle relations,

$$Z(w_1*w_2-w_1⧢w_2)=0$$

for $w_1,w_2\in\mathfrak{h}^0$.

1.5 Regularized double shuffle relations

From the isomorphisms $\mathfrak{h}^1_*\simeq\mathfrak{h}^0_*[y]$ and $\mathfrak{h}^1_{⧢}\simeq\mathfrak{h}^0_{⧢}[y]$ (see [9], [29]), the following proposition holds.

Proposition 1.4 (Ihara–Kaneko–Zagier [13]). *We have two algebra homomorphisms*

$$Z^*:\mathfrak{h}^1_*\longrightarrow\mathbb{R}[T]\quad\text{and}\quad Z^{⧢}:\mathfrak{h}^1_{⧢}\longrightarrow\mathbb{R}[T]$$

which are uniquely characterized by the properties that they both extend the evaluation map $Z:\mathfrak{h}^0\longrightarrow\mathbb{R}$ *and send* y *to* T.

Example 1.5. We have

$$\begin{aligned} y * yx &= yyx + yxy + yxx, \\ y \sqcup\!\sqcup\, yx &= 2yyx + yxy. \end{aligned} \tag{4}$$

Thus,

$$\begin{aligned} Z^*(yxy) =&\zeta(2)T - \zeta(1,2) - \zeta(3), \\ Z^{\sqcup\!\sqcup}(yxy) =&\zeta(2)T - 2\zeta(1,2). \end{aligned} \tag{5}$$

In [13], the relation between the two regularizations Z^* and $Z^{\sqcup\!\sqcup}$ is given, and as a result, the following "regularized double shuffle relations" of MZVs is proved.

Theorem 1.6 (Ihara–Kaneko–Zagier [13])**.** *For any* $w_1 \in \mathfrak{h}^1$ *and* $w_2 \in \mathfrak{h}^0$, *we have*

$$\begin{aligned} Z^*(w_1 \sqcup\!\sqcup\, w_2 - w_1 * w_2) = 0, \\ Z^{\sqcup\!\sqcup}(w_1 \sqcup\!\sqcup\, w_2 - w_1 * w_2) = 0. \end{aligned}$$

Example 1.7. From (4), we have

$$y \sqcup\!\sqcup\, yx - y * yx = yyx - yxx$$

and thus we obtain the classic relation

$$\zeta(1,2) = \zeta(3).$$

1.6 Derivation relations for MZVs

A derivation ∂ on $\mathfrak{h}$ is a $\mathbb{Q}$-linear endomorphism on $\mathfrak{h}$ satisfying Leibniz's rule $\partial(ww') = \partial(w)w' + w\partial(w')$. Such a derivation is uniquely determined by its images of generators x and y. Put $z = x + y$. For each $n \in \mathbb{Z}_{\geq 1}$, we define the derivation $\partial_n : \mathfrak{h} \to \mathfrak{h}$ by

$$\partial_n(x) = yz^{n-1}x, \quad \partial_n(y) = -yz^{n-1}x.$$

We note that $\partial_n(1) = 0$ and $\partial_n(z) = 0$. In [13], Ihara–Kaneko–Zagier proved the derivation relations for MZVs.

Theorem 1.8 (Ihara–Kaneko–Zagier [13])**.** *For* $n \in \mathbb{Z}_{\geq 1}$, *we have*

$$Z(\partial_n(\mathfrak{h}^0)) = 0.$$

Example 1.9.

$$\begin{aligned} \partial_2(yx) &= \partial_2(y)x + y\partial_2(x) = -yzxx + yyzx \\ &= -yxxx - yyxx + yyxx + yyyx = -yxxx + yyyx. \end{aligned}$$

Thus,

$$Z(\partial_2(yx)) = Z(-yxxx + yyyx) = -\zeta(4) + \zeta(1,1,2) = 0.$$

1.7 Ohno's relation for MZVs

To state the duality theorem and Ohno's relation, we define the dual index.

Definition 3. For an admissible index

$$\mathbf{k} = (\underbrace{1,\dots,1}_{a_1-1}, b_1+1, \dots, \underbrace{1,\dots,1}_{a_s-1}, b_s+1) \quad (a_q, b_q \geq 1),$$

we define the dual index of $\mathbf{k}$ by

$$\mathbf{k}^\dagger := (\underbrace{1,\dots,1}_{b_s-1}, a_s+1, \dots, \underbrace{1,\dots,1}_{b_1-1}, a_1+1).$$

The following theorem is a direct consequence of the iterated integral expression.

Theorem 1.10 (the duality theorem). *For an admissible index $\mathbf{k}$, we have*

$$\zeta(\mathbf{k}) = \zeta(\mathbf{k}^\dagger).$$

For two indices $\mathbf{k}$ and $\mathbf{l}$, we denote by $\mathbf{k}+\mathbf{l}$ the index obtained by componentwise addition, and always assume implicitly the depths of $\mathbf{k}$ and $\mathbf{l}$ are equal. We write $\mathbf{l} \geq 0$ if every component of $\mathbf{l}$ is a non-negative integer.

Theorem 1.11 (Ohno's relation, Ohno [26]). *For an admissible index $\mathbf{k}$ and $m \in \mathbb{Z}_{\geq 0}$, we have*

$$\sum_{\substack{|\mathbf{e}|=m \\ \mathbf{e}\geq 0}} \zeta(\mathbf{k}+\mathbf{e}) = \sum_{\substack{|\mathbf{e}'|=m \\ \mathbf{e}'\geq 0}} \zeta(\mathbf{k}^\dagger+\mathbf{e}').$$

If $m = 0$, Ohno's relation is reduced to the duality theorem. From Ohno's relation, we immediately obtain the following theorem, which is called the weak Ohno relation. This relation is known to be equivalent to the derivation relation [13].

Theorem 1.12 (weak Ohno relation). *For an admissible index $\mathbf{k}$ and $m \in \mathbb{Z}_{\geq 0}$, we have*

$$\sum_{\substack{|\mathbf{e}|=m \\ \mathbf{e}\geq 0}} \zeta(\mathbf{k}+\mathbf{e}) = \sum_{\substack{|\mathbf{e}'|=m \\ \mathbf{e}'\geq 0}} \zeta((\mathbf{k}^\dagger+\mathbf{e}')^\dagger).$$

Horikawa–Murahara–Oyama [12] showed the equivalence of the weak Ohno relation and Theorem 1.13, which we call the Ohno-type relation.

Definition 4. For $\mathbf{k} = (k_1, \dots, k_d) \in (\mathbb{Z}_{\geq 1})^d$, we define Hoffman's dual index of $\mathbf{k}$ by

$$\mathbf{k}^\vee = (\underbrace{1,\dots,1}_{k_1} + \underbrace{1,\dots,1}_{k_2} + 1, \dots, 1 + \underbrace{1,\dots,1}_{k_d}).$$

Theorem 1.13 (Ohno-type relation, Horikawa–Murahara–Oyama [12])**.** *For* $\mathbf{k} = (k_1, \ldots, k_d) \in (\mathbb{Z}_{\geq 1})^d$ *and* $m \in \mathbb{Z}_{\geq 0}$*, we have*

$$\sum_{\substack{|\mathbf{e}|=m \\ \mathbf{e}\geq 0}} \zeta^+(\mathbf{k}+\mathbf{e}) = \sum_{\substack{|\mathbf{e}'|=m \\ \mathbf{e}'\geq 0}} \zeta^+((\mathbf{k}^\vee + \mathbf{e}')^\vee).$$

Here and hereafter, we write $\zeta^+(k_1, \ldots, k_d) := \zeta(k_1, \ldots, k_{d-1}, k_d + 1)$.

Theorem 1.14 (Horikawa–Murahara–Oyama [12])**.** *The weak Ohno relation and the Ohno-type relation are equivalent.*

2 Integral-series identity of multiple zeta values

2.1 Circled harmonic product

For a non-empty index $\mathbf{k} = (k_1, \ldots, k_d)$, let $\mathbf{k}^\star$ be the formal sum of 2^{d-1} indices of the form $(k_1 \square \cdots \square k_d)$, where each $\square$ is replaced by "," or " + ". We put $\emptyset^\star = \emptyset$. We also define the $\mathbb{Q}$-bilinear "circled harmonic product" $\circledast : \mathfrak{h}^1 \times \mathfrak{h}^1 \to \mathfrak{h}^0$ by

$$w_1 z_k \circledast w_2 z_l := (w_1 * w_2) z_{k+l}$$

for $k, l \in \mathbb{Z}_{\geq 1}$ and $w_1, w_2 \in \mathfrak{h}^1$. From this definition, we have

$$\zeta(\mathbf{k} \circledast \mathbf{l}^\star) = \sum_{0<m_1<\cdots<m_d=n_s\geq\cdots\geq n_1>0} \frac{1}{m_1^{k_1}\cdots m_d^{k_d} n_1^{l_1} \cdots n_s^{l_s}}.$$

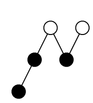
Example 2.1.

$$\begin{aligned} \zeta\left((1) \circledast (2,1)^\star\right) =& \zeta\left((1) \circledast (2,1)\right) + \zeta\left((1) \circledast (3)\right) \\ =& \zeta(2,2) + \zeta(4). \end{aligned} \tag{6}$$

2.2 2-poset and associated integrals

Definition 5. A 2-poset is a pair (X, δ_X), where $X = (X, \leq)$ is a finite partially ordered set (poset for short) and δ_X is a map from X to $\{0, 1\}$. The δ_X is called the label map of X. A 2-poset (X, δ_X) is called admissible if $\delta_X(x) = 0$ for all maximal elements $x \in X$ and $\delta_X(x) = 1$ for all minimal elements $x \in X$.

A 2-poset (X, δ_X) is depicted as a Hasse diagram in which an element x with $\delta(x) = 0$ (resp. $\delta(x) = 1$) is represented by $\circ$ (resp. $\bullet$). For example, the diagram



represents the 2-poset $X = \{x_1, x_2, x_3, x_4, x_5\}$ with order $x_1 < x_2 < x_3 > x_4 < x_5$ and label $(\delta_X(x_1), \dots, \delta_X(x_5)) = (1, 1, 0, 1, 0)$. This 2-poset is admissible.

For an admissible 2-poset X, we define the associated integral

$$I(X) := \int_{\Delta_X} \prod_{x \in X} \omega_{\delta_X(x)}(t_x),$$

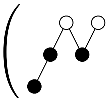
where

$$\Delta_X := \{(t_x)_x \in [0,1]^X \mid t_x < t_y \text{ if } x < y\}$$

and

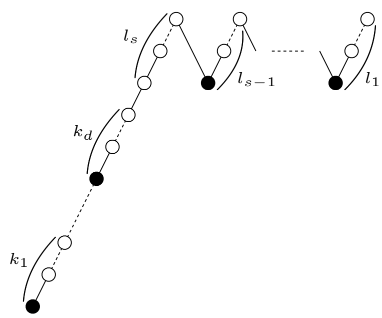
$$\omega_0(t) := \frac{dt}{t}, \quad \omega_1(t) := \frac{dt}{1-t}.$$

For example,

$I\Big($  $\Big) = \int_{t_1<t_2<t_3>t_4<t_5} \frac{dt_1}{1-t_1}\frac{dt_2}{1-t_2}\frac{dt_3}{t_3}\frac{dt_4}{1-t_4}\frac{dt_5}{t_5}.$

2.3 Integral-series identity

For non-empty indices $\mathbf{k} = (k_1, \dots, k_d)$ and $\mathbf{l} = (l_1, \dots, l_s)$, we define $\mu(\mathbf{k}, \mathbf{l})$ as a 2-poset corresponding to the following diagram.

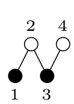
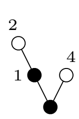
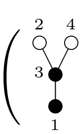


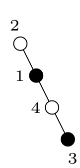
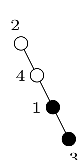
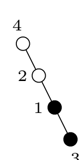
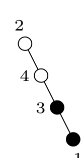
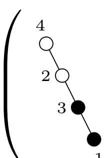
Kaneko–Yamamoto proved the integral-series identity for MZVs.

Theorem 2.2 (Kaneko–Yamamoto [19]). *For any non-empty indices* $\mathbf{k}$ *and* $\mathbf{l}$*, we have*

$$I(\mu(\mathbf{k}, \mathbf{l})) = \zeta(\mathbf{k} \circledast \mathbf{l}^\star).$$

Example 2.3. For $\mathbf{k} = (1), \mathbf{l} = (2, 1)$, we have

$I\Big($  $\Big) = I\Big($  $\Big) + I\Big($  $\Big)$

$= I\Big($  $\Big) + I\Big($  $\Big) + I\Big($  $\Big) + I\Big($  $\Big) + I\Big($  $\Big)$

$= \zeta(2, 2) + 4\zeta(1, 3).$

Therefore from (6), Theorem 2.2 gives a linear relation

$$\zeta(2,2)+4\zeta(1,3)=\zeta(2,2)+\zeta(4).$$

In [19], it is shown that Theorem 2.2 is equivalent to Theorem 1.6 under (2) and (3).

3 Finite multiple zeta values

3.1 $\mathcal{A}$-finite multiple zeta values

We consider the ring $\mathcal{A}$ defined by

$$\mathcal{A}:=\frac{\prod_p \mathbb{Z}/p\mathbb{Z}}{\bigoplus_p \mathbb{Z}/p\mathbb{Z}}=\{(a_p)_p \mid a_p\in\mathbb{Z}/p\mathbb{Z}\}/\sim .$$

Here, p runs over all prime numbers, and the relation $(a_p)_p\sim(b_p)_p$ means that the equality $a_p=b_p$ holds for all but finitely many primes p.

Definition 6. For $k_1,\dots,k_d\in\mathbb{Z}_{\geq 1}$, the $\mathcal{A}$-finite multiple zeta value ($\mathcal{A}$-FMZV) and the $\mathcal{A}$-finite multiple zeta-star value ($\mathcal{A}$-FMZSV) are defined by

$$\zeta_{\mathcal{A}}(k_1,\dots,k_d)=\left(\sum_{1\leq n_1<\cdots<n_d<p}\frac{1}{n_1^{k_1}\cdots n_d^{k_d}} \bmod p\right)_p\in\mathcal{A}$$

and

$$\zeta^\star_{\mathcal{A}}(k_1,\dots,k_d)=\left(\sum_{1\leq n_1\leq\cdots\leq n_d<p}\frac{1}{n_1^{k_1}\cdots n_d^{k_d}} \bmod p\right)_p\in\mathcal{A}$$

respectively.

We use the terms weight and depth similarly for FMZ(S)Vs. We give some examples of FMZ(S)Vs.

Example 3.1 (Hoffman [10], Zhao [36])**.** (1) For $k\in\mathbb{Z}_{\geq 1}$, we have

$$\zeta_{\mathcal{A}}(k)=\zeta^\star_{\mathcal{A}}(k)=0.$$

(2) For $k_1,k_2\in\mathbb{Z}_{\geq 1}$, we have

$$\zeta_{\mathcal{A}}(k_1,k_2)=\zeta^\star_{\mathcal{A}}(k_1,k_2)=(-1)^{k_1}\binom{k_1+k_2}{k_2}\left(\frac{B_{p-k_1-k_2}}{k_1+k_2} \bmod p\right)_p,$$

where B_n is the Bernoulli number defined by the following generating function:

$$\sum_{n\geq 0}B_n\frac{t^n}{n!}=\frac{te^t}{e^t-1}.$$

3.2 Symmetrized multiple zeta values

The symmetrized multiple zeta(-star) values (SMZ(S)Vs) was first introduced by Kaneko–Zagier [16, 20]. For any integers $k_1,\dots,k_d\in\mathbb{Z}_{\geq 1}$, we let

$$\zeta^*_{\mathcal{S}}(k_1,\dots,k_d)=\sum_{i=0}^{d}(-1)^{k_{i+1}+\cdots+k_d}\zeta^*(k_1,\dots,k_i)\zeta^*(k_d,\dots,k_{i+1})$$

and

$$\zeta^{ш}_{\mathcal{S}}(k_1,\dots,k_d)=\sum_{i=0}^{d}(-1)^{k_{i+1}+\cdots+k_d}\zeta^{ш}(k_1,\dots,k_i)\zeta^{ш}(k_d,\dots,k_{i+1}).$$

Here, $\zeta^*(\mathbf{k})=Z^*(w(\mathbf{k}))|_{T=0}$ and $\zeta^{ш}(\mathbf{k})=Z^{ш}(w(\mathbf{k}))|_{T=0}$ in the notation of §1.5, where $w(\mathbf{k})$ is a word in $\mathfrak{h}^1$ corresponding to $\mathbf{k}$.

Example 3.2. From (5), we have

$$\begin{aligned}\zeta^*(2,1)=&-\zeta(3)-\zeta(1,2),\\ \zeta^{ш}(2,1)=&-2\zeta(1,2).\end{aligned}$$

Example 3.3.

$$\begin{aligned}\zeta^*_{\mathcal{S}}(3,1,2)=&\zeta(2,1,3)-\zeta(3)\zeta^*(2,1)+\zeta^*(3,1)\zeta(2)+\zeta(3,1,2)\\ =&\zeta(2,1,3)-\zeta(3)\left(-\zeta(1,2)-\zeta(3)\right)+\left(-\zeta(1,3)-\zeta(4)\right)\zeta(2)+\zeta(3,1,2)\\ =&\zeta(2,1,3)+\zeta(3,1,2)+\zeta(1,3,2)+\zeta(1,2,3)+\zeta(4,2)+\zeta(1,5)+2\zeta(3,3)+\zeta(6)\\ &-\zeta(2,1,3)-\zeta(1,2,3)-\zeta(1,3,2)-\zeta(3,3)-\zeta(1,5)\\ &-\zeta(2,4)-\zeta(4,2)-\zeta(6)+\zeta(3,1,2)\\ =&\zeta(3,3)-\zeta(2,4)+2\zeta(3,1,2).\end{aligned}$$

In [16, 20], Kaneko–Zagier proved that the congruence

$$\zeta^*_{\mathcal{S}}(k_1,\dots,k_d)\equiv\zeta^{ш}_{\mathcal{S}}(k_1,\dots,k_d)\pmod{\zeta(2)}$$

holds in $\mathcal{Z}$. They then defined the symmetrized multiple zeta value (SMZV) $\zeta_{\mathcal{S}}(k_1,\dots,k_d)$ as an element in the quotient ring $\mathcal{Z}/\zeta(2)\mathcal{Z}$ by

$$\zeta_{\mathcal{S}}(k_1,\dots,k_d):=\zeta^*_{\mathcal{S}}(k_1,\dots,k_d)\mod\zeta(2).$$

For $k_1,\dots,k_d\in\mathbb{Z}_{\geq 1}$, we also define the SMZSVs in $\mathcal{Z}/\zeta(2)\mathcal{Z}$ by

$$\zeta^{\star}_{\mathcal{S}}(k_1,\dots,k_d):=\sum_{\substack{\square\text{ is either a comma ``,''}\\ \text{or a plus ``+''}}}\zeta^*_{\mathcal{S}}(k_1\square\cdots\square k_d)\mod\zeta(2).$$

3.3 Finite multiple zeta values

We denote by $\mathcal{Z}_{\mathcal{A}}$ the $\mathbb{Q}$-vector subspace of $\mathcal{A}$ spanned by 1 and all $\mathcal{A}$-FMZVs. We notice that $\mathcal{Z}_{\mathcal{A}}$ is a $\mathbb{Q}$-algebra with the harmonic product. Kaneko–Zagier conjectured the following.

Conjecture 2 (Kaneko–Zagier). *There exists an algebra isomorphism between $\mathcal{Z}_{\mathcal{A}}$ and $\mathcal{Z}/\zeta(2)\mathcal{Z}$ such that*

$$\begin{array}{ccc} \mathcal{Z}_{\mathcal{A}} & \to & \mathcal{Z}/\zeta(2)\mathcal{Z} \\ \Downarrow & & \Downarrow \\ \zeta_{\mathcal{A}}(k_1,\dots,k_d) & \mapsto & \zeta_{\mathcal{S}}(k_1,\dots,k_d). \end{array}$$

We define two $\mathbb{Q}$-linear maps $Z_{\mathcal{A}}\colon \mathfrak{h}^1 \to \mathcal{A}$ and $Z_{\mathcal{S}}\colon \mathfrak{h}^1 \to \mathcal{Z}/\zeta(2)\mathcal{Z}$ by $Z_{\mathcal{A}}(1) = 1$ and $Z_{\mathcal{A}}(yx^{k_1-1}\cdots yx^{k_d-1}) = \zeta_{\mathcal{A}}(k_1,\dots,k_d)$, and $Z_{\mathcal{S}}(1) = 1$ and $Z_{\mathcal{S}}(yx^{k_1-1}\cdots yx^{k_d-1}) = \zeta_{\mathcal{S}}(k_1,\dots,k_d)$, respectively. In view of Conjecture 2, we shall call $\mathcal{A}$-finite multiple zeta values and symmetrized multiple zeta values as finite multiple zeta values (FMZVs). In the following, the letter "$\mathcal{F}$" stands either for "$\mathcal{A}$" or "$\mathcal{S}$".

Now we mention the harmonic and shuffle product rules for FMZVs.

Theorem 3.4 (Hoffman [9], Kaneko–Zagier [20, 17]). *For any words $w_1 = z_{k_1}\cdots z_{k_d}, w_2 = z_{k'_1}\cdots z_{k'_s} \in \mathfrak{h}^1$, we have*

$$\begin{aligned} Z_{\mathcal{F}}(w_1 * w_2) &= Z_{\mathcal{F}}(w_1)Z_{\mathcal{F}}(w_2), \\ Z_{\mathcal{F}}(w_1 ⧢ w_2) &= (-1)^{|w_2|}Z_{\mathcal{F}}(z_{k_1}\cdots z_{k_d}z_{k'_s}\cdots z_{k'_1}), \end{aligned}$$

where $|w_2|$ is the total degree of w_2.

The duality theorems for $\mathcal{A}$-FMZVs and SMZVs are proved by Hoffman and Jarossay, respectively. We define the involutive automorphism ϕ on $\mathfrak{h}$ by

$$\phi(x) = z = x + y, \quad \phi(y) = -y.$$

Theorem 3.5 (Hoffman [10], Jarrossay [14]). *For any word $w \in \mathfrak{h}^1$, we have*

$$Z_{\mathcal{F}}(w) = Z_{\mathcal{F}}(\phi(w)).$$

The derivation relation for FMZVs was conjectured by Oyama and proved by Murahara [22].

Theorem 3.6 (Murahara [22]). *For $n \in \mathbb{Z}_{\geq 1}$, we have*

$$Z_{\mathcal{F}}(\partial_n(y\mathfrak{h}x)x^{-1}) = 0.$$

Part III
On a generalization of restricted sum formula

4 Main results

We prove a generalization (Theorem 4.4) of the so-called restricted sum formula by using the integral-series identity. We show that a seemingly weaker but simpler version, Theorem 4.1, is actually equivalent to Theorem 4.4. We prove this equivalence in the current section and give the proof of Theorem 4.4 in Section 5. In Section 6, we prove that Theorem 4.1 and the Ohno-type relation are equivalent.

We recall the notation $\zeta^+(k_1, \dots, k_d) := \zeta(k_1, \dots, k_{d-1}, k_d + 1)$.

Theorem 4.1. *For $(k_1, \dots, k_d) \in (\mathbb{Z}_{\geq 1})^d, t \in \mathbb{Z}_{\geq 0}$, we have*

$$\sum_{\substack{m_1+\cdots+m_d=d+t \\ m_i \geq 1\,(1\leq i\leq d)}} \sum_{\substack{|\mathbf{a}_{m_i}|=k_i+m_i-1 \\ (1\leq i\leq d)}} \zeta^+(\mathbf{a}_{m_1}, \dots, \mathbf{a}_{m_d})$$
$$= \sum_{l=0}^{t} \sum_{\substack{m_1+\cdots+m_{d-1}=t-l \\ m_i\geq 0\,(1\leq i\leq d-1)}} \sum_{\substack{|\mathbf{e}|=l \\ \mathbf{e}\geq 0}} \zeta^+\big((k_1, \{1\}^{m_1}, \dots, k_{d-1}, \{1\}^{m_{d-1}}, k_d) + \mathbf{e}\big).$$

Here and hereafter, each $\mathbf{a}_{m_i}$ denotes an m_i-tuple of positive integers. When $d = 1$, we understand the R.H.S. as $\zeta^+(k_1 + t)$.

Remark 4.2. *1) Professor Ohno pointed out that Theorem 4.1 (as well as Theorem 4.4) can directly be deduced from his weak Ohno relations (Theorem 1.12) by applying them to each l on the right. Since the Ohno-type relation (Theorem 1.13) is known to be equivalent to the weak Ohno relation (Theorem 1.14), our Theorem 6.1 shows that Theorem 4.4 and the weak Ohno relation are equivalent. Professor Ohno also commented that it would be of some interest that Theorem 4.4, apparently weaker than the weak Ohno relation, was actually equivalent.*

2) We formulated our theorems by employing the ζ^+-notation in order to make the similarity between Theorem 4.1 and Theorem 4.5 (FMZV case) visible.

Example 4.3. For $(k_1, k_2) = (2, 1)$, $t = 1$, we have

$$2\zeta(2, 1, 2) + \zeta(1, 2, 2) = \zeta(3, 2) + \zeta(2, 3) + \zeta(2, 1, 2).$$

Theorem 4.4. *For* $(k_1,\dots,k_d)\in(\mathbb{Z}_{\geq 1})^d$, $s,t\in\mathbb{Z}_{\geq 0}$, *we have*

$$\begin{aligned}&\sum_{\substack{m_1+\cdots+m_d=d+t\\ m_i\geq 1\,(1\leq i\leq d)}}\ \sum_{\substack{|\mathbf{a}_{m_i}|=k_i+m_i-1\\ (1\leq i\leq d)}}\zeta^+(\{1\}^s,\mathbf{a}_{m_1},\dots,\mathbf{a}_{m_d})\\ &=\sum_{l=0}^{t}\ \sum_{\substack{m_1+\cdots+m_{d-1}=t-l\\ m_i\geq 0\,(1\leq i\leq d-1)}}\ \sum_{\substack{|\mathbf{e}|=l\\ \mathbf{e}\geq 0}}\zeta^+\big((\{1\}^s,k_1,\{1\}^{m_1},\dots,k_{d-1},\{1\}^{m_{d-1}},k_d)+\mathbf{e}\big).\end{aligned}$$

When $d=1$, *we understand the R.H.S. as* $\sum_{\substack{|\mathbf{e}|=t\\ \mathbf{e}\geq 0}}\zeta^+\big((\{1\}^s,k_1)+\mathbf{e}\big)$.

This is a generalization of the restricted sum formula given in Eie, Liaw and Ong [3], which, as remarked in Remark 4.2, is a direct consequence of Ohno's relation. Here, we prove the equivalence of Theorem 4.1 and Theorem 4.4.

Proof of the equivalence of Theorem 4.1 and Theorem 4.4. It is clear that Theorem 4.4 implies Theorem 4.1 by setting $s=0$. So, we prove that Theorem 4.1 implies Theorem 4.4. Write $G(\mathbf{k},s,t)$ (resp. $H(\mathbf{k},s,t)$) for the left-hand side (resp. the right-hand side) of Theorem 4.4 and let $F(\mathbf{k},s,t):=G(\mathbf{k},s,t)-H(\mathbf{k},s,t)$. We prove $F(\mathbf{k},s,t)=0$ for $\mathbf{k}\in(\mathbb{Z}_{\geq 1})^d$, $s,t\in\mathbb{Z}_{\geq 0}$ by induction on s. If $s=0$, then $F(\mathbf{k},0,t)=0$ by Theorem 4.1. We assume $F(\mathbf{k},s,t)=0$ for some $s\in\mathbb{Z}_{\geq 0}$ and show $F(\mathbf{k},s+1,t)=0$.

$$\begin{aligned}G((1,\mathbf{k}),s,t)&=\sum_{\substack{m_0+\cdots+m_d=d+t+1\\ m_i\geq 1\,(0\leq i\leq d)}}\ \sum_{\substack{|\mathbf{a}_{m_i}|=k_i+m_i-1\\ (1\leq i\leq d)}}\zeta^+(\{1\}^{s+m_0},\mathbf{a}_{m_1},\dots,\mathbf{a}_{m_d})\\ &=\sum_{m_0=1}^{t+1}\ \sum_{\substack{m_1+\cdots+m_d=d+t-m_0+1\\ m_i\geq 1\,(1\leq i\leq d)}}\ \sum_{\substack{|\mathbf{a}_{m_i}|=k_i+m_i-1\\ (1\leq i\leq d)}}\zeta^+(\{1\}^{s+m_0},\mathbf{a}_{m_1},\dots,\mathbf{a}_{m_d})\\ &=\sum_{m_0=1}^{t+1}G(\mathbf{k},s+m_0,t-m_0+1)\\ &=\sum_{u=0}^{t}G(\mathbf{k},s+t-u+1,u),\end{aligned}$$

$$\begin{aligned}H((1,\mathbf{k}),s,t)&=\sum_{l=0}^{t}\ \sum_{\substack{m_0+\cdots+m_{d-1}=t-l\\ m_i\geq 0\,(0\leq i\leq d-1)}}\ \sum_{\substack{|\mathbf{e}|=l\\ \mathbf{e}\geq 0}}\zeta^+\big((\{1\}^{s+m_0+1},k_1,\{1\}^{m_1},\dots,\{1\}^{m_{d-1}},k_d)+\mathbf{e}\big)\\ &=\sum_{l=0}^{t}\sum_{m_0=0}^{t-l}\ \sum_{\substack{m_1+\cdots+m_{d-1}=t-l-m_0\\ m_i\geq 0\,(1\leq i\leq d-1)}}\ \sum_{\substack{|\mathbf{e}|=l\\ \mathbf{e}\geq 0}}\zeta^+\big((\{1\}^{s+m_0+1},k_1,\{1\}^{m_1},\dots,\{1\}^{m_{d-1}},k_d)+\mathbf{e}\big)\end{aligned}$$

$$= \sum_{m_0=0}^{t} \sum_{l=0}^{t-m_0} \sum_{\substack{m_1+\cdots+m_{d-1}=t-l-m_0 \\ m_i \geq 0\,(1\leq i\leq d-1)}} \sum_{\substack{|\mathbf{e}|=l \\ \mathbf{e}\geq 0}} \zeta^+\big((\{1\}^{s+m_0+1}, k_1, \{1\}^{m_1}, \dots, \{1\}^{m_{d-1}}, k_d) + \mathbf{e}\big)$$

$$= \sum_{m_0=0}^{t} H(\mathbf{k}, s+m_0+1, t-m_0)$$

$$= \sum_{u=0}^{t} H(\mathbf{k}, s+t-u+1, u).$$

Therefore, we have

$$F((1,\mathbf{k}), s, t) = \sum_{u=0}^{t} F(\mathbf{k}, s+t-u+1, u).$$

By replacing s to $s+1$ and t to $t-1$, we have

$$F((1,\mathbf{k}), s+1, t-1) = \sum_{u=0}^{t-1} F(\mathbf{k}, s+t-u+1, u).$$

Take subtraction of the previous two equations, we have

$$F(\mathbf{k}, s+1, t) = F((1,\mathbf{k}), s, t) - F((1,\mathbf{k}), s+1, t-1).$$

By applying this equation repeatedly and $F(\mathbf{k}, s, 0) = 0$ for arbitrary index $\mathbf{k}$ and $s \in \mathbb{Z}_{\geq 0}$, we obtain the following equation.

$$F(\mathbf{k}, s+1, t) = \sum_{t'=1}^{t} (-1)^{t'-1} F((\{1\}^{t'}, \mathbf{k}), s, t-t'+1).$$

Therefore, we obtain the desired result. □

We state an analogous result for finite multiple zeta values.

Theorem 4.5. *For $(k_1, \dots, k_d) \in (\mathbb{Z}_{\geq 1})^d, t \in \mathbb{Z}_{\geq 0}$, we have*

$$\sum_{\substack{m_1+\cdots+m_d=d+t \\ m_i\geq 1\,(1\leq i\leq d)}} \sum_{\substack{|\mathbf{a}_{m_i}|=k_i+m_i-1 \\ (1\leq i\leq d)}} \zeta_{\mathcal{F}}(\mathbf{a}_{m_1}, \dots, \mathbf{a}_{m_d})$$

$$= \sum_{l=0}^{t} \sum_{\substack{m_1+\cdots+m_{d-1}=t-l \\ m_i\geq 0\,(1\leq i\leq d-1)}} \sum_{\substack{|\mathbf{e}|=l \\ \mathbf{e}\geq 0}} \zeta_{\mathcal{F}}\big((k_1, \{1\}^{m_1}, \dots, k_{d-1}, \{1\}^{m_{d-1}}, k_d) + \mathbf{e}\big).$$

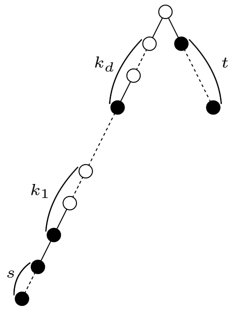
When $d = 1$, we understand the R.H.S. as $\zeta_{\mathcal{F}}(k_1 + t)$.

Remark 4.6. *We can also obtain the FMZVs version of the restricted sum fomula by replacing ζ^+ with $\zeta_{\mathcal{F}}$ in Theorem 4.4.*

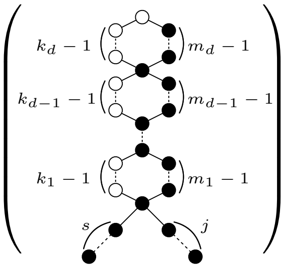
5 Proof of Theorem 4.4

For $\mathbf{k} = (\{1\}^s, k_1, \dots, k_d)$ and $\mathbf{l} = (\{1\}^{t+1})$, we have

$$I(\mu(\mathbf{k},\mathbf{l})) = I\left(\right.$$



$$\left.\right) = \sum_{\substack{m_1+\cdots+m_d+j=d+t\\(m_i\geq 1, j\geq 0)}} I\left(\right.$$



$$\left.\right)$$

$$= \sum_{j=0}^{t} \binom{s+j}{s} \sum_{\substack{m_1+\cdots+m_d=d+t-j\\ m_i\geq 1\,(1\leq i\leq d)}} \sum_{\substack{|\mathbf{a}_{m_i}|=k_i+m_i-1\\(1\leq i\leq d)}} \zeta^+(\{1\}^{s+j}, \mathbf{a}_{m_1}, \dots, \mathbf{a}_{m_d}).$$

In general, for $\mathbf{k}' = (k'_1, \dots, k'_s, k_1, \dots, k_d)$ and $\mathbf{l} = (\{1\}^{t+1})$, we have

$$\zeta(\mathbf{k}' \circledast \mathbf{l}^\star) = \sum_{l=0}^{t} \sum_{\substack{m'_1+\cdots+m'_s+m_1+\cdots+m_d=d+s+t-l\\ m'_i\geq 1\,(1\leq i\leq s),\ m_i\geq 1\,(1\leq i\leq d)}} \sum_{\substack{|\mathbf{e}|=l\\ \mathbf{e}\geq 0}} \zeta^+\big((\{1\}^{m'_1-1}, k'_1, \dots, \{1\}^{m'_s-1}, k'_s, \{1\}^{m_1-1}, k_1, \dots, \{1\}^{m_d-1}, k_d) + \mathbf{e}\big)$$

because the index $(\{1\}^{t+1})^\star$ is equal to the formal sum of all indices of weight $t+1$. Now, we put $k'_1 = \cdots = k'_s = 1$ here. Then, the index $(\{1\}^{m'_1-1}, k'_1, \dots, \{1\}^{m'_s-1}, k'_s, \{1\}^{m_1-1}, k_1, \dots, \{1\}^{m_d-1}, k_d)$ on the right becomes $(\{1\}^{u-1}, k_1, \{1\}^{m_1-1}, \dots, k_{d-1}, \{1\}^{m_{d-1}-1}, k_d)$ with $u = m'_1 + \cdots + m'_s$. For a fixed u, the number of $(s+1)$-tuple $(m'_1, \dots, m'_s)$ giving $u = m'_1 + \cdots + m'_s$ is $\binom{u-1}{s}$. Thus,

$$\zeta(\mathbf{k} \circledast \mathbf{l}^\star) = \sum_{l=0}^{t} \sum_{u=s+j}^{s+t-l+1} \binom{u-1}{s} \sum_{\substack{m_1+\cdots+m_{d-1}=d+t+s-l-u\\ m_i\geq 1}} \sum_{\substack{|\mathbf{e}|=l\\ \mathbf{e}\geq 0}} \zeta^+\big((\{1\}^{u-1}, k_1, \{1\}^{m_1-1}, \dots, k_d) + \mathbf{e}\big).$$

By writing $u = s + j + 1$,

$$\zeta(\mathbf{k} \circledast \mathbf{l}^\star) = \sum_{l=0}^{t} \sum_{j=0}^{t-l} \binom{s+j}{s} \sum_{\substack{m_1+\cdots+m_{d-1}=d+t-l-j-1\\ m_i\geq 1}} \sum_{\substack{|\mathbf{e}|=l\\ \mathbf{e}\geq 0}} \zeta^+\big((\{1\}^{s+j}, k_1, \{1\}^{m_1-1}, \dots, k_d) + \mathbf{e}\big)$$

$$= \sum_{j=0}^{t} \binom{s+j}{s} \sum_{l=0}^{t-j} \sum_{\substack{m_1+\cdots+m_{d-1}=d+t-l-j-1\\ m_i\geq 1}} \sum_{\substack{|\mathbf{e}|=l\\ \mathbf{e}\geq 0}} \zeta^+\big((\{1\}^{s+j}, k_1, \{1\}^{m_1-1}, \dots, k_d) + \mathbf{e}\big).$$

By the integral-series identity and by induction on t, Theorem 4.4 follows.

6 Alternative proof of Theorem 4.1/Proof of Theorem 4.5

6.1 Alternative proof of Theorem 4.1

In this subsection, we prove Theorem 4.1 by showing that it is equivalent to Theorem 1.13, i.e., we will show the following.

Theorem 6.1. *Theorem 4.1 and Theorem 1.13 are equivalent.*

Remark 6.2. *Tanaka [31] shows that the restricted sum formula in [3] is written as a linear combination of the derivation relations. Theorem 6.1 is a generalization of this result.*

The implications among "Generalized restricted sum formula", "Restricted sum formula", "Ohno-type relation" and "Weak Ohno relation" for MZVs can be summarized as follows.

Generalized restricted sum formula (Theorem 4.1, Theorem 4.4)	$\supset$	Restricted sum formula ([3])
$\Updownarrow$ (Theorem 6.1)		
Ohno-type relation ([12], Theorem 1.13)		
$\Updownarrow$ ([12])		
Weak Ohno relation (Theorem 1.12)		

Now we prove Theorem 6.1. The case $d = 1$ is obvious. For $d \geq 2$, the following Lemma 6.3 gives Theorem 6.1.

For $\mathbf{k} = (k_1, \ldots, k_d) \in (\mathbb{Z}_{\geq 1})^d$ with $d \geq 2$ and $u \in \mathbb{Z}_{\geq 0}$, let $\mathbf{k}_u$ be the formal sum

$$\sum_{\alpha_1+\cdots+\alpha_{d-1}=u,\, \alpha_i\geq 0} (k_1, \{1\}^{\alpha_1}, k_2, \{1\}^{\alpha_2}, \ldots, k_{d-1}, \{1\}^{\alpha_{d-1}}, k_d).$$

We also let

$$f_L(\mathbf{k}, t) := (\text{L.H.S. of Theorem 4.1 for } \mathbf{k}, t), \qquad f_R(\mathbf{k}, t) := (\text{R.H.S. of Theorem 4.1 for } \mathbf{k}, t),$$

$$g_L(\mathbf{k}, t) := \sum_{\substack{|\mathbf{e}|=t \\ \mathbf{e}\geq 0}} \zeta^+(\mathbf{k}+\mathbf{e}), \qquad g_R(\mathbf{k}, t) := \sum_{\substack{|\mathbf{e}'|=t \\ \mathbf{e}'\geq 0}} \zeta^+((\mathbf{k}^\vee+\mathbf{e}')^\vee),$$

$$f(\mathbf{k}, t) := f_L(\mathbf{k}, t) - f_R(\mathbf{k}, t), \qquad g(\mathbf{k}, t) := g_L(\mathbf{k}, t) - g_R(\mathbf{k}, t)$$

and we extend them linearly with respect to the indices. Under these settings, we have the following:

Lemma 6.3. *For* $\mathbf{k} \in (\mathbb{Z}_{\geq 1})^d$ *with* $d \geq 2$ *and* $t \in \mathbb{Z}_{\geq 0}$*, we have*

$$f(\mathbf{k}, t) = -\sum_{u=0}^{t} g(\mathbf{k}_u, t-u),$$
$$g(\mathbf{k}, t) = -\sum_{u=0}^{t} (-1)^u f(\mathbf{k}_u, t-u).$$

Proof. To prove the first equation, it is sufficient to show $f_R(\mathbf{k}, t) = \sum_{u=0}^{t} g_L(\mathbf{k}_u, t-u)$ and $f_L(\mathbf{k}, t) = \sum_{u=0}^{t} g_R(\mathbf{k}_u, t-u)$. The proof of the former is obvious as follows:

$$f_R(\mathbf{k}, t) = \sum_{l=0}^{t} \sum_{\substack{|\mathbf{e}|=l \\ \mathbf{e}\geq 0}} \zeta^+(\mathbf{k}_{t-l} + \mathbf{e}) = \sum_{l=0}^{t} g_L(\mathbf{k}_{t-l}, l).$$

To prove the latter, we denote the m-fold repetition of "1+"(resp."1,") by $\boxed{1+}^{m}$(resp. $\boxed{1,}^{m}$), and 1 by $\boxed{1}$. For example, $\zeta\left(\boxed{1,}^{2}\boxed{1+}^{3}\boxed{1,}^{0}\boxed{1}\right) = \zeta(1, 1, 1+1+1+1) = \zeta(1, 1, 4)$.

$$\begin{aligned}
g_R(\mathbf{k}_u, t-u) &= \sum_{\substack{|\mathbf{e}'|=t-u \\ \mathbf{e}'\geq 0}} \zeta^+\left((\mathbf{k}_u{}^{\vee} + \mathbf{e}')^{\vee}\right) \\
&= \sum_{\substack{\alpha_1+\cdots+\alpha_{d-1}=u,\, \alpha_i\geq 0 \\ |\mathbf{e}'|=t-u,\, \mathbf{e}'\geq 0}} \zeta^+\left(\left((k_1, \{1\}^{\alpha_1}, k_2, \{1\}^{\alpha_2}, \ldots, k_{d-1}, \{1\}^{\alpha_{d-1}}, k_d)^{\vee} + \mathbf{e}'\right)^{\vee}\right) \\
&= \sum_{\substack{\alpha_1+\cdots+\alpha_{d-1}=u,\, \alpha_i\geq 0 \\ |\mathbf{e}'|=t-u,\, \mathbf{e}'\geq 0}} \zeta^+\Bigg(\Bigg(\Bigg(\boxed{1+}^{k_1-1}\boxed{1,}^{\alpha_1+1}\boxed{1+}^{k_2-1}\boxed{1,}^{\alpha_2+1}\cdots \\
&\qquad\qquad \cdots\boxed{1+}^{k_{d-1}-1}\boxed{1,}^{\alpha_{d-1}+1}\boxed{1+}^{k_d-1}\boxed{1}\Bigg)^{\vee} + \mathbf{e}'\Bigg)^{\vee}\Bigg) \\
&= \sum_{\substack{\alpha_1+\cdots+\alpha_{d-1}=u,\, \alpha_i\geq 0 \\ |\mathbf{e}'|=t-u,\, \mathbf{e}'\geq 0}} \zeta^+\Bigg(\Bigg(\Bigg(\boxed{1,}^{k_1-1}\boxed{1+}^{\alpha_1+1}\boxed{1,}^{k_2-1}\boxed{1+}^{\alpha_2+1}\cdots \\
&\qquad\qquad \cdots\boxed{1,}^{k_{d-1}-1}\boxed{1+}^{\alpha_{d-1}+1}\boxed{1,}^{k_d-1}\boxed{1}\Bigg) + \mathbf{e}'\Bigg)^{\vee}\Bigg) \\
&= \sum_{\substack{\alpha_1+\cdots+\alpha_{d-1}=u \\ e_{1,1}+\cdots+e_{d,k_d-1}=t-u \\ \alpha_i\geq 0,\, e_{i,j}\geq 0}} \zeta^+\Bigg(\Bigg(\boxed{1+}^{e_{1,1}}\boxed{1,}\cdots\boxed{1+}^{e_{1,k_1-1}}\boxed{1,}\;\boxed{1+}^{e_{1,k_1}+1} \\
&\qquad\qquad \boxed{1+}^{\alpha_1}\boxed{1,}\boxed{1+}^{e_{2,1}}\boxed{1,}\cdots\boxed{1+}^{e_{2,k_2-2}}\boxed{1,}\boxed{1+}^{e_{2,k_2-1}+1} \\
&\qquad\qquad \cdots\cdots
\end{aligned}$$

$$\boxed{1+}^{\alpha_{d-1}} \boxed{1,}\ \boxed{1+}^{e_{d,1}} \boxed{1,} \cdots \boxed{1+}^{e_{d,k_d-2}} \boxed{1,}\ \ \boxed{1+}^{e_{d,k_d-1}} \boxed{1}\Bigg)^{\vee}\Bigg)$$

$$= \sum_{\substack{\alpha_1+\cdots+\alpha_{d-1}=u \\ e_{1,1}+\cdots+e_{d,k_d-1}=t-u \\ \alpha_i\geq 0,\, e_{i,j}\geq 0}} \zeta^{+}\Bigg(\boxed{1,}^{e_{1,1}} \boxed{1+} \cdots \boxed{1,}^{e_{1,k_1-1}} \boxed{1+}\ \ \boxed{1,}^{e_{1,k_1}+1}$$

$$\boxed{1,}^{\alpha_1} \boxed{1+}\ \boxed{1,}^{e_{2,1}} \boxed{1+} \cdots \boxed{1,}^{e_{2,k_2-2}} \boxed{1+}\ \boxed{1,}^{e_{2,k_2-1}+1}$$

$$\cdots\cdots$$

$$\boxed{1,}^{\alpha_{d-1}} \boxed{1+}\ \boxed{1,}^{e_{d,1}} \boxed{1+} \cdots \boxed{1,}^{e_{d,k_d-2}} \boxed{1+}\ \ \boxed{1,}^{e_{d,k_d-1}} \boxed{1}\Bigg).$$

Taking the sum over $u = 0, \dots, t$, we have

$$\sum_{u=0}^{t} g_R(\mathbf{k}_u, t-u) = \sum_{\substack{\alpha_1+\cdots+\alpha_{d-1}+e_{1,1}+\cdots+e_{d,k_d-1}=t \\ \alpha_i\geq 0, e_{i,j}\geq 0}} \zeta^{+}\Bigg(\underbrace{\boxed{1,}^{e_{1,1}} \boxed{1+} \cdots \boxed{1,}^{e_{1,k_1-1}} \boxed{1+}\ \ \boxed{1,}^{e_{1,k_1}+1}}_{\substack{weight=e_{1,1}+\cdots+e_{1,k_1}+k_1 \\ depth=e_{1,1}+\cdots+e_{1,k_1}+1}}$$

$$\underbrace{\boxed{1,}^{\alpha_1} \boxed{1+}\ \boxed{1,}^{e_{2,1}} \boxed{1+} \cdots \boxed{1,}^{e_{2,k_2-2}} \boxed{1+}\ \boxed{1,}^{e_{2,k_2-1}+1}}_{\substack{weight=\alpha_1+e_{2,1}+\cdots+e_{2,k_2-1}+k_2 \\ depth=\alpha_1+e_{2,1}+\cdots+e_{2,k_2-1}+1}}$$

$$\cdots\cdots$$

$$\underbrace{\boxed{1,}^{\alpha_{d-1}} \boxed{1+}\ \boxed{1,}^{e_{d,1}} \boxed{1+} \cdots \boxed{1,}^{e_{d,k_d-2}} \boxed{1+}\ \ \boxed{1,}^{e_{d,k_d-1}} \boxed{1}}_{\substack{weight=\alpha_{d-1}+e_{d,1}+\cdots+e_{d,k_d-1}+k_d \\ depth=\alpha_{d-1}+e_{d,1}+\cdots+e_{d,k_d-1}+1}}\Bigg)$$

$$= \sum_{\substack{m_1+\cdots+m_d=d+t \\ m_i\geq 1\ (1\leq i\leq d)}} \sum_{\substack{|\mathbf{a}_{m_i}|=k_i+m_i-1 \\ (1\leq i\leq d)}} \zeta^{+}(\mathbf{a}_{m_1}, \dots, \mathbf{a}_{m_d})$$

$$= f_L(\mathbf{k}, t).$$

We assume the first equation in the lemma and prove the second by induction on t. The case $t = 0$ is clear. Let $t > 0$ and assume $g(\mathbf{k}, t') = -\sum_{u=0}^{t'}(-1)^u f(\mathbf{k}_u, t'-u)$ for all integers t' with $0 \leq t' < t$. From the first equation, we have

$$\begin{aligned} g(\mathbf{k}, t) &= -f(\mathbf{k}, t) - \sum_{u=1}^{t} g(\mathbf{k}_u, t-u) \\ &= -f(\mathbf{k}, t) + \sum_{u=1}^{t}\sum_{u'=0}^{t-u}(-1)^{u'} f((\mathbf{k}_u)_{u'}, t-u-u'). \end{aligned}$$

Since $(\mathbf{k}_u)_{u'} = \binom{u+u'}{u}\mathbf{k}_{u+u'}$ and by writing $v = u + u'$,

$$\begin{aligned} g(\mathbf{k}, t) &= - f(\mathbf{k}, t) + \sum_{u=1}^{t}\sum_{v=u}^{t}(-1)^{v-u}\binom{v}{u} f(\mathbf{k}_v, t-v) \\ &= - f(\mathbf{k}, t) + \sum_{v=1}^{t}(-1)^{v}\sum_{u=1}^{v}(-1)^{u}\binom{v}{u} f(\mathbf{k}_v, t-v) \\ &= - f(\mathbf{k}, t) - \sum_{v=1}^{t}(-1)^{v} f(\mathbf{k}_v, t-v) \\ &= - \sum_{u=0}^{t}(-1)^{u} f(\mathbf{k}_u, t-u). \end{aligned}$$

□

Remark 6.4. *Let*

$$h(\mathbf{k}, t) := \sum_{\substack{|\mathbf{e}|=t \\ \mathbf{e}\geq 0}} \zeta(\mathbf{k}_+ + \mathbf{e}) - \sum_{\substack{|\mathbf{e}'|=t \\ \mathbf{e}'\geq 0}} \zeta(((\mathbf{k}_+)^{\dagger} + \mathbf{e}')^{\dagger}),$$

where $(k_1, \ldots, k_{d-1}, k_d)_+ := (k_1, \ldots, k_{d-1}, k_d + 1)$, *and we extend linearly with respect to the indices. Then we obtain*

$$f(\mathbf{k}, t) = -\sum_{u=0}^{t} h(\mathbf{k}_u, t-u)$$

in a same manner.

6.2 Proof of Theorem 4.5

The following theorem is called Ohno-type relation for FMZVs. This is conjectured by Kaneko [16] and proved by Oyama [28].

Theorem 6.5 (Oyama [28])**.** *For* $\mathbf{k} \in (\mathbb{Z}_{\geq 1})^d$ *and* $l \in \mathbb{Z}_{\geq 0}$, *we have*

$$\sum_{\substack{|\mathbf{e}|=l \\ \mathbf{e}\geq 0}} \zeta_{\mathcal{F}}(\mathbf{k} + \mathbf{e}) = \sum_{\substack{|\mathbf{e}'|=l \\ \mathbf{e}'\geq 0}} \zeta_{\mathcal{F}}\big((\mathbf{k}^{\vee} + \mathbf{e}')^{\vee}\big).$$

Remark 6.6. *Horikawa–Murahara–Oyama [12] shows the equivalence of the derivation relation for FMZVs (Theorem 3.6) and the above theorem.*

The following theorem holds in exactly the same manner as in the previous subsection. Then, Theorem 4.5 holds.

Theorem 6.7. *Theorem 4.5 and Theorem 6.5 are equivalent.*

The relations among "Generalized restricted sum formula", "Ohno-type relation" and "Derivation relation" for FMZVs can be summarized as follows.

Generalized restricted sum formula (Theorem 4.5)

$\Updownarrow$ (Theorem 6.7)

Ohno-type relation ([28], Theorem 6.5)

$\Updownarrow$ ([12])

Derivation relation ([22])

Part IV

A cyclic analogue of multiple zeta values

7 Cyclic integral-series identity

In [25], Nakasuji–Phuksuwan–Yamasaki gave an integral expression of ribbon type Schur multiple zeta values, which is a generalization of the 'integral=series' identity established by Kaneko–Yamamoto [19]. The first main result of this part is a cyclic analogue of their results. Let $s \in \mathbb{Z}_{\geq 1}$, $r_1, \dots, r_s \in \mathbb{Z}_{\geq 1}$, and $k_{1,1}, \dots, k_{1,r_1}, \dots, k_{s,1}, \dots, k_{s,r_s} \in \mathbb{Z}_{\geq 1}$. A multi-index $[(k_{1,1}, \dots, k_{1,r_1}), \dots, (k_{s,1}, \dots, k_{s,r_s})]$ is called an admissible multi-index if

- for all $1 \leq i \leq s$, the index $(k_{i,1}, \dots, k_{i,r_i})$ is admissible or equal to (1),
- there exists $1 \leq i \leq s$ such that $(k_{i,1}, \dots, k_{i,r_i}) \neq (1)$.

For an admissible multi-index $\mathbf{k} = [(k_{1,1}, \dots, k_{1,r_1}), \dots, (k_{s,1}, \dots, k_{s,r_s})]$, we define the cyclic multiple zeta value (CMZV) by

$$\zeta_{\mathrm{cyc}}(\mathbf{k}) := \sum_{(n_{1,1},\dots,n_{s,r_s}) \in S} \prod_{i=1}^{s} \prod_{j=1}^{r_i} \frac{1}{n_{i,j}^{k_{i,j}}}, \tag{7}$$

where

$$S := \{(n_{1,1}, ..., n_{s,r_s}) \in (\mathbb{Z}_{\geq 1})^{r_1 + \dots + r_s} \mid n_{1,1} < \dots < n_{1,r_1} \geq n_{2,1} < \dots < n_{2,r_2} \geq \dots \geq n_{s,1} < \dots < n_{s,r_s} \geq n_{1,1}\}.$$

By definition, CMZVs satisfy the cyclic property

$$\zeta_{\mathrm{cyc}}(\mathbf{k}_1, \dots, \mathbf{k}_s) = \zeta_{\mathrm{cyc}}(\mathbf{k}_i, \dots, \mathbf{k}_s, \mathbf{k}_1, \dots, \mathbf{k}_{i-1}) \quad (1 \leq i \leq s).$$

For the convergence of CMZVs, see Remark 8.1.

Theorem 7.1 (Cyclic integral-series identity)**.** *Let* $\mathbf{k} = [(k_{1,1}, \dots, k_{1,r_1}), \dots, (k_{s,1}, \dots, k_{s,r_s})]$ *be an admissible multi-index. Put* $k_i := \sum_{j=1}^{r_i} k_{i,j}$. *Then we have*

$$\zeta_{\mathrm{cyc}}(\mathbf{k}) = \int_D \prod_{i=1}^{s} \prod_{j=1}^{k_i} a_{i,j} dt_{i,j}, \tag{8}$$

where

$$a_{i,j} := \begin{cases} \frac{1}{1-t_{i,j}} & j \in \{1, k_{i,1}+1, \dots, k_{i,1} + \dots + k_{i,r_i-1} + 1\}, \\ \frac{1}{t_{i,j}} & \textit{otherwise} \end{cases}$$

and

$$D := \{(t_{1,1}, ..., t_{s,k_s}) \in (0,1)^{k_1 + \dots + k_s} \mid t_{1,1} < \dots < t_{1,k_1} > t_{2,1} < \dots < t_{2,k_2} > \dots > t_{s,1} < \dots < t_{s,k_s} > t_{1,1}\}.$$

We call (7) (resp. (8)) as series (resp. integral) expression of $\zeta_{\mathrm{cyc}}(\mathbf{k})$.

The second and the third main theorems (Theorems 7.2 and 7.3) are two classes of $\mathbb{Q}$-linear relations among CMZVs. Theorem 7.2 is a generalization of the cyclic sum formula for MZSVs which was proved by Ohno–Wakabayashi [27]. We recall the notations $\mathfrak{h} := \mathbb{Q}\langle x, y\rangle$, $\mathfrak{h}^0 := \mathbb{Q}\oplus y\mathfrak{h}x$ and $\mathfrak{h}^1 := \mathbb{Q} \oplus y\mathfrak{h}$ in §1.4. In addition, we define subspaces $\mathfrak{h}_C$, $\mathfrak{h}_C^0$ and $\mathfrak{h}_C^1$ by $\mathfrak{h}_C := \mathfrak{h}x \oplus \mathfrak{h}y$, $\mathfrak{h}_C^0 := \mathfrak{h}^0 \cap \mathfrak{h}_C$ and $\mathfrak{h}_C^1 := \mathfrak{h}^1 \cap \mathfrak{h}_C$. We denote by $\mathfrak{h}^{\mathrm{cyc}}$ the subspace of $\oplus_{s=1}^{\infty}\mathfrak{h}^{\otimes s}$ spanned by

$$\bigcup_{s=1}^{\infty}\{u_1 \otimes \cdots \otimes u_s \in \mathfrak{h}^{\otimes s} \mid u_1, \ldots, u_s \in \mathfrak{h}_C^0 \cup \{y\} \text{ and there exists } j \text{ such that } u_j \neq y\}.$$

We define a $\mathbb{Q}$-linear map $Z_{\mathrm{cyc}} : \mathfrak{h}^{\mathrm{cyc}} \to \mathbb{R}$ by

$$Z_{\mathrm{cyc}}(z_{k_{1,1}} \cdots z_{k_{1,r_1}} \otimes \cdots\cdots \otimes z_{k_{s,1}} \cdots z_{k_{s,r_s}}) = \zeta_{\mathrm{cyc}}([(k_{1,1}, \ldots, k_{1,r_1}), \ldots\ldots, (k_{s,1}, \ldots, k_{s,r_s})]).$$

Theorem 7.2. *For $u_1 \otimes \cdots \otimes u_s \in \mathfrak{h}^{\mathrm{cyc}}$, we have*

$$\sum_{i=1}^{s} Z_{\mathrm{cyc}}(u_1\otimes\cdots\otimes u_{i-1}\otimes(y\underline{ш}u_i)\otimes u_{i+1}\otimes\cdots\otimes u_s) = \sum_{i=1}^{s} Z_{\mathrm{cyc}}(u_1\otimes\cdots\otimes u_i\otimes y\otimes u_{i+1}\otimes\cdots\otimes u_s),$$

where $y \,\underline{ш}\, u_i = y ш u_i - yu_i - u_i y$ (see Section 9.1 for the general definition of $\underline{ш}$).

Theorem 7.3. *For $u_1 \otimes \cdots \otimes u_s \in \mathfrak{h}^{\mathrm{cyc}}$ and $k \in \mathbb{Z}_{\geq 1}$, we have*

$$\sum_{i=1}^{s} Z_{\mathrm{cyc}}(u_1\otimes\cdots\otimes u_{i-1}\otimes(z_k \,\underline{*}\, u_i)\otimes u_{i+1}\otimes\cdots\otimes u_s) = \sum_{i=1}^{s} Z_{\mathrm{cyc}}(u_1\otimes\cdots\otimes u_i\otimes z_k\otimes u_{i+1}\otimes\cdots\otimes u_s),$$

where $z_k \,\underline{*}\, u_i = z_k * u_i - z_k u_i - u_i z_k$ (see Section 10.1 for the general definition of $\underline{*}$).

The proofs of Theorems 7.1, 7.2, and 7.3 are described in Sections 8, 9, and 10, respectively. In Section 11, we give an alternative proof of the cyclic sum formula for MZSVs (see [27]), the derivation relation for MZVs (see [13]) and the sum formula for MZVs as applications of Theorems 7.1 and 7.2.

8 Proof of cyclic integral-series identity

8.1 Nakasuji–Phuksuwan–Yamasaki's integral-series identity for ribbon type Schur MZVs

For the proof of the cyclic integral-series identity, let us introduce the notion of ribbon type Schur MZVs. Let

$$\mathbf{k} = [(k_{1,1}, \ldots, k_{1,r_1}), \ldots, (k_{s,1}, \ldots, k_{s,r_s})]$$

be an admissible multi-index with $(k_{1,1},\dots,k_{1,r_1})\neq(1)$. Put $k_i:=\sum_{j=1}^{r_i}k_{i,j}$. Then the ribbon type Schur MZV $\zeta_{\mathrm{ribbon}}(\mathbf{k})$ is defined by

$$\sum_{(n_{1,1},\dots,n_{s,r_s})\in S'}\prod_{i=1}^{s}\prod_{j=1}^{r_i}\frac{1}{n_{i,j}^{k_{i,j}}},$$

where

$$S':=\{(n_{1,1},...,n_{s,r_s})\in(\mathbb{Z}_{\geq1})^{r_1+\cdots+r_s}\mid n_{1,1}<\cdots<n_{1,r_1}\geq n_{2,1}<\cdots<n_{2,r_2}\geq\cdots\geq n_{s,1}<\cdots<n_{s,r_s}\}.$$

Remark 8.1. *The ribbon type Schur MZV $\zeta_{\mathrm{ribbon}}(\mathbf{k})$ converges for any admissible multi-index such that $(k_{1,1},\dots,k_{1,r_1})\neq(1)$ (see [25, Lemma 2.1]). We can show the convergence of $\zeta_{\mathrm{cyc}}(\mathbf{k})$ for any admissible multi-index in the following way. First, without loss of generality we can assume $\mathbf{k}_1\neq(1)$ by the cyclic property of CMZVs. Then, since the domain of the summation in $\zeta_{\mathrm{cyc}}(\mathbf{k})$ is contained in the one of $\zeta_{\mathrm{ribbon}}(\mathbf{k})$, $\zeta_{\mathrm{cyc}}(\mathbf{k})$ also converges.*

In [25, Section 6.1], Nakasuji–Phuksuwan–Yamasaki gave a following integral expression:

$$\zeta_{\mathrm{ribbon}}(\mathbf{k})=\int_{D'}\prod_{i=1}^{s}\prod_{j=1}^{k_i}a_{i,j}dt_{i,j},$$

where

$$a_{i,j}:=\begin{cases}\frac{1}{1-t_{i,j}} & j\in\{1,k_{i,1}+1,\dots,k_{i,1}+\cdots+k_{i,r_i-1}+1\},\\ \frac{1}{t_{i,j}} & \text{otherwise}\end{cases}$$

and

$$D':=\{(t_{1,1},...,t_{s,k_s})\in(0,1)^{k_1+\cdots+k_s}\mid t_{1,1}<\cdots<t_{1,k_1}>t_{2,1}<\cdots<t_{2,k_2}>\cdots>t_{s,1}<\cdots<t_{s,k_s}\}.$$

Note that $S=S'\cap\{n_{s,r_s}\geq n_{1,1}\}$ and $D=D'\cap\{t_{s,k_s}>t_{1,1}\}$.

8.2 Proof of cyclic integral-series identity

In this section, we prove Theorem 7.1. Let

$$\mathbf{k}=[(k_{1,1},\dots,k_{1,r_1}),\dots,(k_{s,1},\dots,k_{s,r_s})]$$

be an admissible multi-index. Put $\mathbf{k}_i:=(k_{i,1},\dots,k_{i,r_i})$ and $k_i:=\sum_{j=1}^{r_i}k_{i,j}$. We denote by $\zeta_{\mathrm{cycint}}(\mathbf{k})$ the integral expression appeared in Theorem 7.1. We prove $\zeta_{\mathrm{cyc}}(\mathbf{k})=\zeta_{\mathrm{cycint}}(\mathbf{k})$ by induction on s. By the cyclic property of CMZVs, without loss of generality, we can assume $\mathbf{k}_1\neq(1)$. The case $s=1$ is just a usual integral expression of a multiple zeta value. Note that we have

$$\begin{aligned}&\{(t_{1,1},\dots,t_{s,k_s})\in(0,1)^{k_1+\cdots+k_s}\mid t_{1,1}<\cdots<t_{1,k_1}>t_{2,1}<\cdots<t_{2,k_2}>\cdots>t_{s,1}<\cdots<t_{s,k_s}\}\\=&\{(t_{1,1},...,t_{s,k_s})\in(0,1)^{k_1+\cdots+k_s}\mid t_{1,1}<\cdots<t_{1,k_1}>t_{2,1}<\cdots<t_{2,k_2}>\cdots>t_{s,1}<\cdots<t_{s,k_s}\geq t_{1,1}\}\\&\sqcup\{(t_{1,1},...,t_{s,k_s})\in(0,1)^{k_1+\cdots+k_s}\mid t_{1,1}<\cdots<t_{1,k_1}>t_{2,1}<\cdots<t_{2,k_2}>\cdots>t_{s,1}<\cdots<t_{s,k_s}<t_{1,1}\}.\end{aligned}$$

Thus from (8.1),

$$\zeta_{\mathrm{ribbon}}([\mathbf{k}_1,\dots,\mathbf{k}_s]) = \zeta_{\mathrm{cycint}}([\mathbf{k}_1,\dots,\mathbf{k}_s]) + \zeta_{\mathrm{cycint}}([\mathbf{k}_s\mathbf{k}_1,\mathbf{k}_2,\dots,\mathbf{k}_{s-1}]). \tag{9}$$

Here we denote by $\mathbf{k}_s\mathbf{k}_1$ the concatenation of $\mathbf{k}_s$ and $\mathbf{k}_1$, i.e., $\mathbf{k}_s\mathbf{k}_1 := (k_{s,1},\dots,k_{s,r_s},k_{1,1},\dots,k_{1,r_1})$. From

$$\begin{aligned}
&\{(n_{1,1},\dots,n_{s,r_s}) \in (\mathbb{Z}_{\geq 1})^{r_1+\cdots+r_s} \mid n_{1,1}<\cdots<n_{1,r_1}\geq n_{2,1}<\cdots<n_{2,r_2}\geq\cdots\geq n_{s,1}<\cdots<n_{s,r_s}\}\\
=&\{(n_{1,1},\dots,n_{s,r_s}) \in (\mathbb{Z}_{\geq 1})^{r_1+\cdots+r_s} \mid n_{1,1}<\cdots<n_{1,r_1}\geq n_{2,1}<\cdots<n_{2,r_2}\geq\cdots\geq n_{s,1}<\cdots<n_{s,r_s}\geq n_{1,1}\}\\
&\sqcup\{(n_{1,1},\dots,n_{s,r_s}) \in (\mathbb{Z}_{\geq 1})^{r_1+\cdots+r_s} \mid n_{1,1}<\cdots<n_{1,r_1}\geq n_{2,1}<\cdots<n_{2,r_2}\geq\cdots\geq n_{s,1}<\cdots<n_{s,r_s}<n_{1,1}\},
\end{aligned}$$

we have

$$\zeta_{\mathrm{ribbon}}([\mathbf{k}_1,\dots,\mathbf{k}_s]) = \zeta_{\mathrm{cyc}}([\mathbf{k}_1,\dots,\mathbf{k}_s]) + \zeta_{\mathrm{cyc}}([\mathbf{k}_s\mathbf{k}_1,\mathbf{k}_2,\dots,\mathbf{k}_{s-1}]). \tag{10}$$

From the induction hypothesis, we have

$$\zeta_{\mathrm{cycint}}([\mathbf{k}_s\mathbf{k}_1,\mathbf{k}_2,\dots,\mathbf{k}_{s-1}]) = \zeta_{\mathrm{cyc}}([\mathbf{k}_s\mathbf{k}_1,\mathbf{k}_2,\dots,\mathbf{k}_{s-1}]). \tag{11}$$

From (9), (10), (11), we have

$$\zeta_{\mathrm{cycint}}([\mathbf{k}_1,\dots,\mathbf{k}_s]) = \zeta_{\mathrm{cyc}}([\mathbf{k}_1,\dots,\mathbf{k}_s]).$$

Thus Theorem 7.1 is proved.

9 Proof of Theorem 7.2

9.1 Inner shuffle product

We define the inner shuffle product $\underline{ш} : \mathfrak{h}\times\mathfrak{h}_C \to \mathfrak{h}_C$ by

$$w \,\underline{ш}\, x = w \,\underline{ш}\, y = 0,$$

$$w \,\underline{ш}\, uw'u' = u(w \sqcup w')u',$$

where $u,u' \in \{x,y\}$ and $w,w' \in \mathfrak{h}$. Note that we have $y \,\underline{ш}\, w = y \sqcup w - yw - wy$ for $w \in \mathfrak{h}_C$.

Definition 7. For $0<p<q<1$, let $f_{p,q} : \mathfrak{h}_C \to \mathbb{R}$ be a $\mathbb{Q}$-linear map defined by $f_{p,q}(x) = f_{p,q}(y) = 0$ and

$$f_{p,q}(u_1\cdots u_k) := \int_{p=t_1<t_2<\cdots<t_{k-1}<t_k=q} a_1\cdots a_k dt_2\cdots dt_{k-1}$$

for $k>1$, where $u_1,\dots,u_k \in \{x,y\}$ and

$$a_i = \begin{cases} \frac{1}{t_i} & u_i = x,\\ \frac{1}{1-t_i} & u_i = y.\end{cases}$$

Here, for $k=2$, we understand the right-hand side as a_1a_2.

Lemma 9.1. *For $0<p<q<1$, $k \in \mathbb{Z}_{\geq 0}$, $u_1, \ldots, u_k \in \{x, y\}$, and $w \in \mathfrak{h}_C$, we have*

$$f_{p,q}(u_1 \cdots u_k ш w) = f_{p,q}(w) \int_{p<t_1<\cdots<t_k<q} \prod_{i=1}^{k} a_i dt_i,$$

where

$$a_i = \begin{cases} \frac{1}{t_i} & u_i = x, \\ \frac{1}{1-t_i} & u_i = y. \end{cases}$$

Proof. Let $g_{p,q} : \mathfrak{h}_C \to \mathbb{R}$ be a $\mathbb{Q}$-linear map defined by

$$g_{p,q}(v_1 \cdots v_l) := \int_{p<t_1<\cdots<t_l<q} b_1 \cdots b_l dt_1 \cdots dt_l,$$

where $v_1, \ldots, v_l \in \{x, y\}$ and

$$b_i = \begin{cases} \frac{1}{t_i} & v_i = x, \\ \frac{1}{1-t_i} & v_i = y. \end{cases}$$

Then, for all $A, B \in \mathfrak{h}$, we have

$$g_{p,q}(A ш B) = g_{p,q}(A) g_{p,q}(B).$$

The case $w \in \{x, y\}$ is obvious since $u_1 \cdots u_k ш w = 0$ and $f_{p,q}(w) = 0$. Put $w = vWv'$ $(v, v' \in \{x, y\})$ and

$$\beta := \begin{cases} \frac{1}{p} & v = x \\ \frac{1}{1-p} & v = y \end{cases} \times \begin{cases} \frac{1}{q} & v' = x, \\ \frac{1}{1-q} & v' = y. \end{cases}$$

Then from the definition, we have

$$f_{p,q}(vAv') = g_{p,q}(A) \cdot \beta$$

for all $A \in \mathfrak{h}$. Thus we get

$$\begin{aligned} f_{p,q}(u_1 \cdots u_k ш w) &= f_{p,q}(v(u_1 \cdots u_k ш W)v') \\ &= g_{p,q}(u_1 \cdots u_k ш W) \cdot \beta \\ &= g_{p,q}(W) \cdot \beta \cdot g_{p,q}(u_1 \cdots u_k) \\ &= f_{p,q}(w) \cdot g_{p,q}(u_1 \cdots u_k). \end{aligned}$$

Since $g_{p,q}(u_1 \cdots u_k) = \int_{p<t_1<\cdots<t_k<q} \prod_{i=1}^{k} a_i dt_i$, the lemma has proved. ∎

Example 9.2. When $k = 2, u_1 = x, u_2 = x$ and $w = yx$, we have

$$\begin{aligned} f_{p,q}(x^2 ш yx) &= f_{p,q}(yxxx) \\ &= \frac{1}{1-p} \cdot \frac{1}{q} \cdot \int_{p<t_2<t_3<q} \frac{dt_2}{t_2} \cdot \frac{dt_3}{t_3} \\ &= f_{p,q}(yx) \cdot \int_{p<t_2<t_3<q} \frac{dt_2}{t_2} \cdot \frac{dt_3}{t_3}. \end{aligned}$$

9.2 Proof of Theorem 7.2

For $s \le s'$, we put

$$E(s,s',t) := \begin{cases} 1 & s \le t \le s', \\ 0 & \text{otherwise}. \end{cases}$$

Assume that $u_1, \ldots, u_s$ are monomials given by $u_i = u_{i,1} \cdots u_{i,k_i}$ with $u_{i,j} \in \{x, y\}$. By definition, we have

$$Z_{\mathrm{cyc}}(u_1 \otimes \cdots \otimes u_{i-1} \otimes (y \sqcup\!\sqcup u_i) \otimes u_{i+1} \otimes \cdots \otimes u_s) = \int_{D'} f_{p,q}(y \sqcup\!\sqcup u_i) \left(\prod_{\substack{1 \le c \le s \\ c \ne i}} \prod_{j=1}^{k_c} a_{c,j} dt_{c,j} \right) dpdq,$$

where the domain of integration $D' \subset \{(t_{1,1}, \ldots, t_{i-1,k_{i-1}}, p, q, t_{i+1,1}, \ldots, t_{s,k_s}) \in (0,1)^{k_1+\cdots+k_{i-1}+2+k_{i+1}+\cdots+k_s}\}$ is given by

$$D' := \{t_{1,1} < \cdots < t_{1,k_1} > \cdots < t_{i-1,k_{i-1}} > p < q > t_{i+1,1} < \cdots < t_{s,k_s} > t_{1,1}\},$$

and $a_{i,j}$ is given by

$$a_{i,j} = \begin{cases} \frac{1}{t_{i,j}} & u_{i,j} = x, \\ \frac{1}{1-t_{i,j}} & u_{i,j} = y. \end{cases}$$

By Lemma 9.1, we have

$$\begin{aligned} &\int_{D'} f_{p,q}(y \sqcup\!\sqcup u_i) \left(\prod_{\substack{1 \le c \le s \\ c \ne i}} \prod_{j=1}^{k_c} a_{c,j} dt_{c,j} \right) dpdq \\ &= \int_{D'} f_{p,q}(u_i) \left(\prod_{\substack{1 \le c \le s \\ c \ne i}} \prod_{j=1}^{k_c} a_{c,j} dt_{c,j} \right) dpdq \int_{0<t<1} E(p,q,t) \frac{dt}{1-t} \\ &= \int_{D} \left(\prod_{c=1}^{s} \prod_{j=1}^{k_c} a_{c,j} dt_{c,j} \right) \int_{0<t<1} E(t_{i,1}, t_{i,k_i}, t) \frac{dt}{1-t}. \end{aligned}$$

Thus we have

$$Z_{\mathrm{cyc}}(u_1 \otimes \cdots \otimes u_{i-1} \otimes (y \sqcup\!\sqcup u_i) \otimes u_{i+1} \otimes \cdots \otimes u_s) = \int_{D} \left(\prod_{c=1}^{s} \prod_{j=1}^{k_c} a_{c,j} dt_{c,j} \right) \int_{0<t<1} E(t_{i,1}, t_{i,k_i}, t) \frac{dt}{1-t}.$$

By definition, we have

$$Z_{\mathrm{cyc}}(u_1 \otimes \cdots \otimes u_i \otimes y \otimes u_{i+1} \otimes \cdots \otimes u_s) = \int_{D} \left(\prod_{c=1}^{s} \prod_{j=1}^{k_c} a_{c,j} dt_{c,j} \right) \int_{0<t<1} E(t_{(i+1 \bmod s),1}, t_{i,k_i}, t) \frac{dt}{1-t},$$

where $(i+1 \bmod s)$ means $i+1$ for $1 \le i < s$ and 1 for $i = s$. Thus Theorem 7.2 follows from

$$\sum_{i=1}^{s} E(t_{i,1}, t_{i,k_i}, t) = \sum_{i=1}^{s} E(t_{(i+1 \bmod s),1}, t_{i,k_i}, t) \quad (t \in (0,1)).$$

10 Proof of Theorem 7.3

10.1 Inner harmonic product

We rewrite the definition of the harmonic product $* : \mathfrak{h}^1 \times \mathfrak{h}^1 \to \mathfrak{h}^1$ described in §1.4, by

$$z_{k_1} \cdots z_{k_r} * z_{l_1} \cdots z_{l_s} := \sum_{d=\max(r,s)}^{r+s} \sum_{\substack{f:\{1,\dots,r\}\to\{1,\dots,d\} \\ g:\{1,\dots,s\}\to\{1,\dots,d\} \\ f,g:\text{strictly increasing} \\ \mathrm{Im} f \cup \mathrm{Im} g = \{1,\dots,d\}}} z_{m_1} \cdots z_{m_d},$$

where

$$m_i = \begin{cases} k_{f^{-1}(i)} & i \in \mathrm{Im} f \setminus \mathrm{Im} g, \\ l_{g^{-1}(i)} & i \in \mathrm{Im} g \setminus \mathrm{Im} f, \\ k_{f^{-1}(i)} + l_{g^{-1}(i)} & i \in \mathrm{Im} f \cap \mathrm{Im} g. \end{cases}$$

Similarly, we define an inner harmonic product $\underline{*} : \mathfrak{h}^1 \times \mathfrak{h}^1_C \to \mathfrak{h}^1_C$ by

$$z_{k_1} \cdots z_{k_r} \,\underline{*}\, z_{l_1} \cdots z_{l_s} := \sum_{d=\max(r,s)}^{r+s} \sum_{\substack{f:\{1,\dots,r\}\to\{1,\dots,d\} \\ g:\{1,\dots,s\}\to\{1,\dots,d\} \\ f,g:\text{strictly increasing} \\ \mathrm{Im} f \cup \mathrm{Im} g = \{1,\dots,d\} \\ g(1)\le f(i) \le g(s) \text{ for all } i}} z_{m_1} \cdots z_{m_d},$$

where the definition of m_i is same as the one in the previous definition. Note that we have $z_k \,\underline{*}\, w = z_k * w - z_k w - w z_k$ for $w \in \mathfrak{h}^1_C$ since

$$\begin{aligned} z_k * z_{l_1} \cdots z_{l_s} - z_k \,\underline{*}\, z_{l_1} \cdots z_{l_s} &= \sum_{d=\max(1,s)}^{1+s} \sum_{\substack{f:\{1\}\to\{1,\dots,d\} \\ g:\{1,\dots,s\}\to\{1,\dots,d\} \\ g:\text{strictly increasing} \\ \mathrm{Im} f \cup \mathrm{Im} g = \{1,\dots,d\} \\ f(1)<g(1) \text{ or } g(s)<f(1)}} z_{m_1} \cdots z_{m_d} \\ &= \sum_{\substack{f:\{1\}\to\{1,s+1\} \\ g:\{1,\dots,s\}\to\{1,\dots,s+1\} \\ g:\text{strictly increasing} \\ f(1)<g(1) \text{ or } g(s)<f(1)}} z_{m_1} \cdots z_{m_d} \\ &= z_k z_{l_1} \cdots z_{l_s} + z_{l_1} \cdots z_{l_s} z_k. \end{aligned}$$

Furthermore, we have $u_1 \,\underline{*}\, (u_2 \,\underline{*}\, u_3) = (u_1 * u_2) \,\underline{*}\, u_3$ for $u_1 \in \mathfrak{h}^1$ and $u_2, u_3 \in \mathfrak{h}^1_C$ since both $z_{k_1} \cdots z_{k_r} \,\underline{*}\, (z_{l_1} \cdots z_{l_s} \,\underline{*}\, z_{m_1} \cdots z_{m_t})$ and $(z_{k_1} \cdots z_{k_r} * z_{l_1} \cdots z_{l_s}) \,\underline{*}\, z_{m_1} \cdots z_{m_t}$ are equal to

$$\sum_{d=\max(r,s,t)}^{r+s+t} \sum_{\substack{f:\{1,\dots,r\}\to\{1,\dots,d\} \\ g:\{1,\dots,s\}\to\{1,\dots,d\} \\ h:\{1,\dots,t\}\to\{1,\dots,d\} \\ f,g,h:\text{strictly increasing} \\ \mathrm{Im}f\cup\mathrm{Im}g\cup\mathrm{Im}h=\{1,\dots,d\} \\ h(1)\le f(i)\le h(t)\ \text{for all } i \\ h(1)\le g(i)\le h(t)\ \text{for all } i}} z_{n_1} \cdots z_{n_d},$$

where

$$n_i = \begin{cases} k_{f^{-1}(i)} & i \in \mathrm{Im}f \\ 0 & i \notin \mathrm{Im}f \end{cases} + \begin{cases} l_{g^{-1}(i)} & i \in \mathrm{Im}g \\ 0 & i \notin \mathrm{Im}g \end{cases} + \begin{cases} m_{h^{-1}(i)} & i \in \mathrm{Im}h, \\ 0 & i \notin \mathrm{Im}h. \end{cases}$$

Definition 8. For positive integers $p \le q$, define a $\mathbb{Q}$-linear map $\lambda_{p,q} : \mathfrak{h}^1_C \to \mathbb{R}$ by

$$\lambda_{p,q}(z_{k_1} \cdots z_{k_r}) := \sum_{p=n_1<\cdots<n_r=q} n_1^{-k_1} \cdots n_r^{-k_r},$$

where $k_1, \dots, k_r \in \mathbb{Z}_{\ge 1}$. Here, for $q - p < r - 1$, we understand the right-hand side as 0, and for $r = 1$, we understand the right-hand side as

$$\sum_{p=n=q} n^{-k_1} = \begin{cases} p^{-k_1} & p = q, \\ 0 & p < q. \end{cases}$$

Lemma 10.1. *For $w \in \mathfrak{h}^1_C$ and positive integers $p \le q, k_1, \dots, k_r$, we have*

$$\lambda_{p,q}(z_{k_1} \cdots z_{k_r} \,\underline{*}\, w) = \lambda_{p,q}(w) \sum_{p\le n_1<\cdots<n_r\le q} n_1^{-k_1} \cdots n_r^{-k_r}.$$

Proof. Put $w := z_{l_1} \cdots z_{l_s}$. Then we have

$$\lambda_{p,q}(w) \sum_{p\le n_1<\cdots<n_r\le q} n_1^{-k_1} \cdots n_r^{-k_r} = \sum_{(n_1,\dots,n_r,n'_1,\dots,n'_s)\in X} n_1^{-k_1} \cdots n_r^{-k_r} {n'_1}^{-l_1} \cdots {n'_s}^{-l_s}$$

where

$$X := \{(n_1, \dots, n_r, n'_1, \dots, n'_s) \in \mathbb{Z}^{r+s} \mid p \le n_1 < \cdots < n_r \le q, p = n'_1 < \cdots < n'_s = q\}.$$

Then we decompose X by the pattern of order of $(n_1, \dots, n_r, n'_1, \dots, n'_s)$, i.e.,

$$X = \bigsqcup_{d=\max(r,s)}^{r+s} \bigsqcup_{\substack{f:\{1,\dots,r\}\to\{1,\dots,d\} \\ g:\{1,\dots,s\}\to\{1,\dots,d\} \\ f,g:\text{strictly increasing} \\ \mathrm{Im}f\cup\mathrm{Im}g=\{1,\dots,d\} \\ g(1)\le f(i)\le g(s)\ \text{for all } i}} X_{d,f,g},$$

where $X_{d,f,g}$ is the set of $(n_1,\dots,n_r,n'_1,\dots,n'_s)$ such that $n_i=o_{f(i)}$ for $1\le i\le r$ and $n'_j=o_{g(j)}$ for $1\le j\le s$ where o_k is the k-th smallest element of $\{n_1,\dots,n_r\}\cup\{n'_1,\dots,n'_s\}$. Then we have

$$\sum_{(n_1,\dots,n_r,n'_1,\dots,n'_s)\in X_{d,f,g}} n_1^{-k_1}\cdots n_r^{-k_r}{n'_1}^{-l_1}\cdots {n'_s}^{-l_s}=\sum_{p=o_1<\cdots<o_d=q} o_1^{-m_1}\cdots o_d^{-m_d}=\lambda_{p,q}(z_{m_1}\cdots z_{m_d}),$$

where

$$m_i=\begin{cases} k_{f^{-1}(i)} & i\in \mathrm{Im}f\setminus \mathrm{Im}g,\\ l_{g^{-1}(i)} & i\in \mathrm{Im}g\setminus \mathrm{Im}f,\\ k_{f^{-1}(i)}+l_{g^{-1}(i)} & i\in \mathrm{Im}f\cap \mathrm{Im}g.\end{cases}$$

Thus

$$\sum_{(n_1,\dots,n_r,n'_1,\dots,n'_s)\in X} n_1^{-k_1}\cdots n_r^{-k_r}{n'_1}^{-l_1}\cdots {n'_s}^{-l_s}=\lambda_{p,q}(z_{k_1}\cdots z_{k_r}\,\underline{*}\,w).$$

Hence the lemma is proved. ☐

Example 10.2. When $r=1$ and $w=z_{l_1}z_{l_2}$, we have

$$\begin{aligned}\lambda_{p,q}(z_k\,\underline{*}\,z_{l_1}z_{l_2})&=\lambda_{p,q}(z_{l_1+k}z_{l_2})+\lambda_{p,q}(z_{l_1}z_kz_{l_2})+\lambda_{p,q}(z_{l_1}z_{l_2+k})\\ &=\sum_{p=n_1<n_2=q}\frac{1}{n_1^{l_1+k}n_2^{l_2}}+\sum_{p=n_1<n_2<n_3=q}\frac{1}{n_1^{l_1}n_2^{k}n_3^{l_2}}+\sum_{p=n_1<n_2=q}\frac{1}{n_1^{l_1}n_2^{l_2+k}}\\ &=\sum_{p=n_1<n_2=q}\frac{1}{n_1^{l_1}n_2^{l_2}}\sum_{p\le n\le q}\frac{1}{n^k}\\ &=\lambda_{p,q}(z_{l_1}z_{l_2})\sum_{p\le n\le q}\frac{1}{n^k}.\end{aligned}$$

10.2 Proof of Theorem 7.3

For $p\le q$, we put

$$E(p,q,n):=\begin{cases}1 & p\le n\le q,\\ 0 & \text{otherwise.}\end{cases}$$

By the definition of Z_{cyc},

$$\begin{aligned}&Z_{\mathrm{cyc}}(u_1\otimes\cdots\otimes u_{i-1}\otimes(z_k\,\underline{*}\,u_i)\otimes u_{i+1}\otimes\cdots\otimes u_s)\\ &=\sum_{(n_{1,1},\dots,n_{i-1,r_{i-1}},p,q,n_{i+1,1},\dots,n_{s,r_s})\in S'}\left(\prod_{\substack{1\le c\le s\\ c\ne i}}\prod_{j=1}^{r_c}\frac{1}{n_{c,j}^{k_{c,j}}}\right)\lambda_{p,q}(z_k\,\underline{*}\,u_i),\end{aligned}$$

where the domain of the summation $S'\subset(\mathbb{Z}_{\ge1})^{r_1+\cdots+r_{i-1}+2+r_{i+1}+\cdots+r_s}$ of $n_{1,1},\dots,n_{i-1,r_{i-1}}$, p, q, $n_{i+1,1},\dots,n_{s,r_s}$ is defined by

$$\{n_{1,1}<\cdots<n_{1,r_1}\ge n_{2,1}<\cdots<n_{i-1,r_{i-1}}\ge p\le q\ge n_{i+1,1}<\cdots<n_{s,r_s}\ge n_{1,1}\}.$$

By Lemma 10.1,

$$\sum_{(n_{1,1},\dots,n_{i-1,r_{i-1}},p,q,n_{i+1,1},\dots,n_{s,r_s})\in S'}\left(\prod_{\substack{1\le c\le s\\ c\neq i}}\prod_{j=1}^{r_c}\frac{1}{n_{c,j}^{k_{c,j}}}\right)\lambda_{p,q}(z_k\,\underline{*}\,u_i)$$
$$=\sum_{(n_{1,1},\dots,n_{i-1,r_{i-1}},p,q,n_{i+1,1},\dots,n_{s,r_s})\in S'}\left(\prod_{\substack{1\le c\le s\\ c\neq i}}\prod_{j=1}^{r_c}\frac{1}{n_{c,j}^{k_{c,j}}}\right)\lambda_{p,q}(z_k)\sum_{n\in\mathbb{Z}_{\ge1}}\frac{E(p,q,n)}{n^k}$$
$$=\sum_{(n_{1,1},\dots,n_{s,r_s})\in S}\left(\prod_{i=1}^{s}\prod_{j=1}^{r_i}\frac{1}{n_{i,j}^{k_{i,j}}}\right)\sum_{n\in\mathbb{Z}_{\ge1}}\frac{E(n_{i,1},n_{i,r_i},n)}{n^k}.$$

Thus, we have

$$Z_{\mathrm{cyc}}(u_1\otimes\cdots\otimes u_{i-1}\otimes(z_k\underline{*}u_i)\otimes u_{i+1}\otimes\cdots\otimes u_s)=\sum_{(n_{1,1},\dots,n_{s,r_s})\in S}\left(\prod_{i=1}^{s}\prod_{j=1}^{r_i}\frac{1}{n_{i,j}^{k_{i,j}}}\right)\sum_{n\in\mathbb{Z}_{\ge1}}\frac{E(n_{i,1},n_{i,r_i},n)}{n^k}.$$

By definition, we have

$$Z_{\mathrm{cyc}}(u_1\otimes\cdots\otimes u_i\otimes z_k\otimes u_{i+1}\otimes\cdots\otimes u_s)=\sum_{(n_{1,1},\dots,n_{s,r_s})\in S}\left(\prod_{i=1}^{s}\prod_{j=1}^{r_i}\frac{1}{n_{i,j}^{k_{i,j}}}\right)\sum_{n\in\mathbb{Z}_{\ge1}}\frac{E(n_{(i+1\bmod s),1},n_{i,r_i},n)}{n^k},$$

where $(i+1 \bmod s)$ means $i+1$ for $1\le i<s$ and 1 for $i=s$. Thus Theorem 7.3 follows from

$$\sum_{i=1}^{s}E(n_{i,1},n_{i,r_i},n)=\sum_{i=1}^{s}E(n_{(i+1\bmod s),1},n_{i,r_i},n)\quad(n\in\mathbb{Z}_{\ge1}).$$

11 Applications of Theorems 7.1 and 7.2

11.1 Proof of cyclic sum formula for MZSVs

In this section, we give an alternative proof of the following theorem due to Ohno–Wakabayashi [27] as an application of Theorem 7.2.

Theorem 11.1 ([27, Theorem 1], Cyclic sum formula for MZSVs)**.** *For $k_1,\dots,k_s\in\mathbb{Z}_{\ge1}$ such that $k_1+\cdots+k_s>s$, we have*

$$\sum_{i=1}^{s}\sum_{j=1}^{k_i-1}\zeta^\star(k_i-j,k_{i+1},\dots,k_s,k_1,\dots,k_{i-1},j+1)=k\zeta(k+1).$$

where $k=k_1+\cdots+k_s$.

Lemma 11.2. *For $k_1, \dots, k_s, l \in \mathbb{Z}_{\geq 1}$ such that $k_s > 1$, we have*

$$\zeta_{\mathrm{cyc}}([(k_1), \dots, (k_s)]) = \zeta(k_1 + \cdots + k_s),$$
$$\zeta_{\mathrm{cyc}}([(l, k_s), (k_{s-1}), \dots, (k_1)]) = \zeta^{\star}(l, k_1, \dots, k_s) - \zeta(l + k_1 + \cdots + k_s).$$

Proof. This is an immediate consequence of the series expression of ζ_{cyc}. □

Proof of Theorem 11.1. Fix $k_1, \dots, k_s \in \mathbb{Z}_{\geq 1}$ such that $k_1 + \cdots + k_s > s$. Put $k := k_1 + \cdots + k_s$. By Theorem 7.2, we have

$$\sum_{i=1}^{s} Z_{\mathrm{cyc}}(z_{k_s} \otimes \cdots \otimes z_{k_{i+1}} \otimes (y \sqcup\!\sqcup z_{k_i}) \otimes z_{k_{i-1}} \otimes \cdots \otimes z_{k_1}) = \sum_{i=1}^{s} Z_{\mathrm{cyc}}(z_{k_s} \otimes \cdots \otimes z_{k_{i+1}} \otimes y \otimes z_{k_i} \otimes \cdots \otimes z_{k_1}). \tag{12}$$

By the previous lemma, we have

$$Z_{\mathrm{cyc}}(z_{k_s} \otimes \cdots \otimes z_{k_{i+1}} \otimes y \otimes z_{k_i} \otimes \cdots \otimes z_{k_1}) = \zeta(k+1) \tag{13}$$

for $1 \leq i \leq s$. Since

$$y \sqcup\!\sqcup z_l = \sum_{j=1}^{l-1} z_{l-j} z_{j+1}$$

for $l \in \mathbb{Z}_{\geq 1}$, we have

$$Z_{\mathrm{cyc}}(z_{k_s} \otimes \cdots \otimes z_{k_{i+1}} \otimes (y \sqcup\!\sqcup z_{k_i}) \otimes z_{k_{i-1}} \otimes \cdots \otimes z_{k_1}) = \sum_{j=1}^{k_i - 1} \zeta^{\star}(k_i - j, k_{i+1}, \dots, k_s, k_1, \dots, k_{i-1}, j+1) - (k_i - 1)\zeta(k+1) \tag{14}$$

for $1 \leq i \leq s$. From (12), (13) and (14), we have

$$\sum_{i=1}^{s} \sum_{j=1}^{k_i - 1} \zeta^{\star}(k_i - j, k_{i+1}, \dots, k_s, k_1, \dots, k_{i-1}, j+1) = k\zeta(k+1).$$

Thus the claim is proved. □

11.2 Algebraic preliminary

For $m \geq 1$, we define derivation maps ∂_m and δ_m on $\mathfrak{h}$ by

$$\delta_m(x) = 0,\ \delta_m(y) = yx^{m-1}(x+y),$$

$$\partial_m(x) = y(x+y)^{m-1}x,\ \partial_m(y) = -y(x+y)^{m-1}x.$$

By definition, we have

$$\delta_m(z_k) = \delta_m(yx^{k-1}) = yx^{m-1}(x+y)x^{k-1} = z_m z_k + z_{m+k}.$$

Thus,

$$\delta_m(z_{k_1}\cdots z_{k_d})=\sum_{i=1}^{d} z_{k_1}\cdots z_{k_{i-1}}(z_m z_{k_i}+z_{m+k_i})z_{k_{i+1}}\cdots z_{k_d}.$$

Therefore,

$$\delta_m(w)=z_m * w - w z_m = z_m \underline{*}\, w + z_m w \tag{15}$$

for $w\in\mathfrak{h}^1_C$. We denote by $[A,B]$ the commutator $AB-BA$.

Lemma 11.3. *For $m\geq 1$, we have*

$$\sum_{j=1}^{m-1}[\delta_j,\partial_{m-j}]=(m-1)(\partial_m+\delta_m).$$

Proof. Put $z:=x+y$. We define a derivation $s:\mathfrak{h}\to\mathfrak{h}$ by

$$s(x)=x^2,\ \ s(z)=z^2.$$

Then we can easily check that

$$[s,\delta_m]=m\delta_{m+1},$$
$$[s,\partial_m]=m\partial_{m+1}.$$

We prove the lemma by induction on m. We can check the case $m\leq 2$ by direct calculation. Take $m\geq 3$ and assume that

$$\sum_{\substack{p+q=m-1\\ 1\leq p,q\leq m-2}}[\delta_p,\partial_q]=(m-2)(\partial_{m-1}+\delta_{m-1}).$$

From the Jacobi identity, we have

$$\begin{aligned}0=&\sum_{\substack{p+q=m-1\\ 1\leq p,q\leq m-2}}([s,[\delta_p,\partial_q]]+[\partial_q,[s,\delta_p]]+[\delta_p,[\partial_q,s]])\\ =&(m-2)(m-1)(\partial_m+\delta_m)\\ &-\sum_{\substack{p+q=m-1\\ 1\leq p,q\leq m-2}}p[\delta_{p+1},\partial_q]-\sum_{\substack{p+q=m-1\\ 1\leq p,q\leq m-2}}q[\delta_p,\partial_{q+1}]\\ =&(m-2)(m-1)(\partial_m+\delta_m)-(m-2)\sum_{\substack{p+q=m\\ 1\leq p,q\leq m-1}}[\delta_p,\partial_q].\end{aligned}$$

Since $m>2$, we obtain

$$\sum_{\substack{p+q=m\\ 1\leq p,q\leq m-1}}[\delta_p,\partial_q]=(m-1)(\partial_m+\delta_m).$$

Thus the claim is proved. ∎

11.3 Proof of the derivation relation for MZVs

In this subsection, we give an alternative proof of the derivation relation

$$Z(\partial_m(w)) = 0 \qquad (m \in \mathbb{Z}_{\geq 1}, w \in \mathfrak{h}_C^0)$$

due to Ihara–Kaneko–Zagier [13]. We put $\{1\}^m := y^m$ and

$$\{1\}_\star^m := \begin{cases} 1 & m = 0, \\ y(x+y)^{m-1} & m > 0. \end{cases}$$

Lemma 11.4. *For $m \geq 1$, we have*

$$m\{1\}_\star^m = \sum_{i=1}^{m} z_i * \{1\}_\star^{m-i}, \tag{16}$$

$$m\{1\}^m = \sum_{i=1}^{m} (-1)^{i-1} z_i * \{1\}^{m-i}, \tag{17}$$

$$\sum_{i=0}^{m} (-1)^i \{1\}_\star^{m-i} * \{1\}^i = \begin{cases} 1 & m = 0, \\ 0 & m > 0. \end{cases} \tag{18}$$

Proof. We first prove equation (16). By definition,

$$\{1\}_\star^n = \sum_{d=1}^{n} \sum_{\substack{k_1+\cdots+k_d=n \\ k_i \geq 1}} z_{k_1} \cdots z_{k_d}$$

for $n \geq 1$. Thus, we have

$$\begin{aligned}
&\sum_{i=1}^{m} z_i * \{1\}_\star^{m-i} \\
&= \sum_{i=1}^{m-1} \sum_{d=1}^{m-i} \sum_{\substack{k_1+\cdots+k_d=m-i \\ k_i \geq 1}} z_i * z_{k_1} \ldots z_{k_d} + z_m \\
&= \sum_{i=1}^{m-1} \sum_{d=1}^{m-i} \left(\sum_{j=1}^{d} \sum_{\substack{k_1+\cdots+k_d=m-i \\ k_l \geq 1}} z_{k_1} \ldots z_{k_j+i} \ldots z_{k_d} + \sum_{j=0}^{d} \sum_{\substack{k_1+\cdots+k_d=m-i \\ k_l \geq 1}} z_{k_1} \ldots z_{k_j} z_i z_{k_{j+1}} \ldots z_{k_d} \right) + z_m \\
&= \sum_{d=1}^{m-1} \sum_{i=1}^{m-d} \sum_{j=1}^{d} \sum_{\substack{k_1+\cdots+k_d=m-i \\ k_l \geq 1}} z_{k_1} \ldots z_{k_j+i} \ldots z_{k_d} \\
&\quad + \sum_{d=1}^{m-1} \sum_{i=1}^{m-d} \sum_{j=0}^{d} \sum_{\substack{k_1+\cdots+k_d=m-i \\ k_l \geq 1}} z_{k_1} \ldots z_{k_j} z_i z_{k_{j+1}} \ldots z_{k_d} + z_m
\end{aligned} \tag{19}$$

Here we have

$$\begin{aligned}
\text{the first term of (19)} &= \sum_{d=1}^{m-1}\sum_{j=1}^{d}\sum_{i=1}^{m-d}\sum_{\substack{k_1+\cdots+k_d=m-i\\ k_l\geq 1}} z_{k_1}\dots z_{k_j+i}\dots z_{k_d}\\
&= \sum_{d=1}^{m-1}\sum_{j=1}^{d}\sum_{\substack{k_1+\cdots+k_d=m\\ k_l\geq 1}} (k_j-1) z_{k_1}\dots z_{k_d}\\
&= \sum_{d=1}^{m-1}\sum_{\substack{k_1+\cdots+k_d=m\\ k_l\geq 1}} (m-d) z_{k_1}\dots z_{k_d},\\
\text{the second term of (19)} &= \sum_{d=1}^{m-1}\sum_{j=0}^{d}\sum_{i=1}^{m-d}\sum_{\substack{k_1+\cdots+k_d=m-i\\ k_l\geq 1}} z_{k_1}\dots z_{k_j}z_i z_{k_{j+1}}\dots z_{k_d}\\
&= \sum_{d=1}^{m-1}\sum_{\substack{k_1+\cdots+k_{d+1}=m\\ k_l\geq 1}} (d+1) z_{k_1}\dots z_{k_{d+1}}\\
&= \sum_{d=2}^{m}\sum_{\substack{k_1+\cdots+k_d=m\\ k_l\geq 1}} d z_{k_1}\dots z_{k_d}\\
&= \sum_{d=1}^{m}\sum_{\substack{k_1+\cdots+k_d=m\\ k_l\geq 1}} d z_{k_1}\dots z_{k_d} - z_m.
\end{aligned}$$

Then we get (16).

We next prove equation (17). We note that

$$\begin{aligned}
z_1 * z_1^{m-1} &= m z_1^m + \sum_{i=1}^{m-1} z_1^{i-1} z_2 z_1^{m-1-i},\\
z_2 * z_1^{m-2} &= \sum_{i=1}^{m-1} z_1^{i-1} z_2 z_1^{m-1-i} + \sum_{i=1}^{m-2} z_1^{i-1} z_3 z_1^{m-2-i},\\
&\dots\\
z_{m-1} * z_1 &= \sum_{i=1}^{2} z_1^{i-1} z_{m-1} z_1^{2-i} + z_m,\\
z_m * 1 &= z_m
\end{aligned}$$

hold. By taking the alternating sum, we have (17).

Equation (18) follows from [21, Proposition 7.1]. □

By Lemma 10.1 and the series expression of ζ_{cyc}, we have

$$Z_{\mathrm{cyc}}(w \otimes \overbrace{y \otimes \cdots \otimes y}^{m \text{ times}}) = Z(\{1\}_\star^m \underline{*}\, w)$$

for $w \in \mathfrak{h}_C^0$. By Theorem 7.2, we have

$$Z_{\mathrm{cyc}}(y \sqcup\!\sqcup\, w \otimes \overbrace{y \otimes \cdots \otimes y}^{m \text{ times}}) - (m+1) Z_{\mathrm{cyc}}(w \otimes \overbrace{y \otimes \cdots \otimes y}^{m+1 \text{ times}}) = 0.$$

Thus, we get the following corollary of Theorem 7.2.

Corollary 11.5. *For $w \in \mathfrak{h}_C^0$ and $m \geq 0$, we have*

$$Z(F(w,m)) = 0,$$

where $F(w,m) := \{1\}_\star^m \underline{*}\, (y \sqcup\!\sqcup\, w) - (m+1)\{1\}_\star^{m+1} \underline{*}\, w$.

In fact, Corollary 11.5 is essentially the derivation relation. More precisely, the following theorem holds.

Theorem 11.6. *For $w \in \mathfrak{h}_C^0$ and $m \geq 1$, we have*

$$\sum_{i=1}^{m} (-1)^{i-1} F(y^{i-1} \underline{*}\, w, m-i) = \partial_m(w).$$

Remark 11.7. *By Theorem 11.6 and* (18)*, we can obtain*

$$F(w, m-1) = \sum_{i=1}^{m} \partial_i(\{1\}_\star^{m-i} \underline{*}\, w).$$

Thus Theorem 11.6 implies that the family of relations obtaind by Corollary 11.5 essentially coincides with the one obtained by the derivation relations.

We prove this theorem in the rest of this section. We prepare some lemmas.

Lemma 11.8. *For $m \geq 1$, we have*

$$\sum_{j=1}^{m-1} \partial_{m-j}(z_j) = -(m-1) z_m.$$

Proof. By Lemma 11.3, we have

$$\sum_{j=1}^{m-1} [\delta_j, \partial_{m-j}](x+y) = (m-1)(\partial_m + \delta_m)(x+y).$$

Since

$$\sum_{j=1}^{m-1}[\delta_j, \partial_{m-j}](x+y) = -\sum_{j=1}^{m-1}\partial_{m-j}(yx^{j-1}(x+y))$$
$$= -\sum_{j=1}^{m-1}\partial_{m-j}(z_j)(x+y)$$

and

$$(\partial_m + \delta_m)(x+y) = z_m(x+y),$$

we have

$$\sum_{j=1}^{m-1}\partial_{m-j}(z_j) = -(m-1)z_m. \qquad \square$$

Lemma 11.9. *For $m \geq 1$, we have*

$$\sum_{i=0}^{m-1}(-1)^i(m-i)\{1\}_\star^{m-i} * \{1\}^i = z_m.$$

Proof. This follows from the following calculation:

$$\begin{aligned}
\sum_{i=0}^{m-1}(-1)^i(m-i)\{1\}_\star^{m-i} * \{1\}^i =& \sum_{i=0}^{m}(-1)^i(m-i)\{1\}_\star^{m-i} * \{1\}^i \\
=& \sum_{i=0}^{m}(-1)^i\left(\sum_{j=1}^{m-i} z_j * \{1\}_\star^{m-i-j}\right) * \{1\}^i \qquad \text{(by (16))} \\
=& \sum_{j=1}^{m} z_j * \sum_{i=0}^{m-j}(-1)^i\{1\}_\star^{m-j-i} * \{1\}^i \\
=& \sum_{j=1}^{m} z_j * \begin{cases} 1 & j = m, \\ 0 & j \neq m \end{cases} \qquad \text{(by (18))} \\
=& z_m.
\end{aligned}$$

$\square$

Proof of Theorem 11.6. Put

$$G_m(w) := \sum_{i=1}^{m}(-1)^{i-1}F(y^{i-1} \underline{*}\, w, m-i).$$

By definition, we have

$$G_m(w) = G'_m(w) + G''_m(w),$$

where

$$G'_m(w) = \sum_{i=1}^{m} (-1)^{i-1} \{1\}_{\star}^{m-i} \underline{*} \left(y \sqcup\!\sqcup (\{1\}^{i-1} \underline{*}\, w) \right)$$

and

$$\begin{aligned}
G''_m(w) &= -\sum_{i=1}^{m} (-1)^{i-1} (m-i+1) \{1\}_{\star}^{m-i+1} \underline{*} \left(\{1\}^{i-1} \underline{*}\, w \right) \\
&= -\left(\sum_{i=1}^{m} (-1)^{i-1} (m-i+1) \{1\}_{\star}^{m-i+1} * \{1\}^{i-1} \right) \underline{*}\, w \\
&= -\, z_m \underline{*}\, w.
\end{aligned}$$

Here the last equality follows from Lemma 11.9. Thus we have

$$G_m(w) = \sum_{i=1}^{m} (-1)^{i-1} \{1\}_{\star}^{m-i} \underline{*} \left(y \sqcup\!\sqcup (\{1\}^{i-1} \underline{*}\, w) \right) - z_m \underline{*}\, w. \tag{20}$$

Now we prove $G_m(w) = \partial_m(w)$ by induction on m.

We first prove the case $m = 1$. By the definition of G_1 and (15), we have

$$\begin{aligned}
G_1(w) &= y \sqcup\!\sqcup w - z_1 \underline{*}\, w \\
&= y \sqcup\!\sqcup w + yw - \delta_1(w).
\end{aligned}$$

Here, by the definition of the shuffle product,

$$y \sqcup\!\sqcup w + yw = \theta(w),$$

where θ is the derivation map defined by $\theta(x) = yx$ and $\theta(y) = y^2$. Since $(\theta - \delta_1)(x) = \partial_1(x)$ and $(\theta - \delta_1)(y) = \partial_1(y)$, we have $\theta - \delta_1 = \partial_1$. Thus $G_1(w) = (\theta - \delta_1)(w) = \partial_1(w)$. Hence the case $m = 1$ is proved.

For $m \geq 2$, we assume that $G_{m-j}(w) = \partial_{m-j}(w)$ for all $1 \leq j \leq m-1$. By Lemma 11.3, we have

$$\begin{aligned}
(m-1)(\partial_m + \delta_m)(w) &= \sum_{j=1}^{m-1} (\delta_j \partial_{m-j} - \partial_{m-j} \delta_j)(w) \\
&= \sum_{j=1}^{m-1} (\delta_j G_{m-j} - G_{m-j} \delta_j)(w) \\
&= \sum_{j=1}^{m-1} \left(z_j \underline{*}\, G_{m-j}(w) + z_j G_{m-j}(w) - G_{m-j}(z_j \underline{*}\, w) - G_{m-j}(z_j w) \right).
\end{aligned} \tag{21}$$

Here the last equality follows from (15). Now we have

$$\begin{aligned}
&\sum_{j=1}^{m-1}\left(z_j \,\underline{*}\, G_{m-j}(w) - G_{m-j}(z_j \,\underline{*}\, w)\right)\\
=&\sum_{j=1}^{m-1}\sum_{i=1}^{m-j}(-1)^{i-1}(z_j * \{1\}_\star^{m-i-j}) \,\underline{*}\, (y \sqcup\!\sqcup (\{1\}^{i-1} \,\underline{*}\, w))\\
&-\sum_{j=1}^{m-1}\sum_{i=1}^{m-j}(-1)^{i-1}\{1\}_\star^{m-i-j} \,\underline{*}\, (y \sqcup\!\sqcup ((z_j * \{1\}^{i-1}) \,\underline{*}\, w))\\
&-\sum_{j=1}^{m-1} z_j \,\underline{*}\, (z_{m-j} \,\underline{*}\, w) + \sum_{j=1}^{m-1} z_{m-j} \,\underline{*}\, (z_j \,\underline{*}\, w) && \text{(by (20))}\\
=&\sum_{i=1}^{m-1}\sum_{j=1}^{m-i}(-1)^{i-1}(z_j * \{1\}_\star^{m-i-j}) \,\underline{*}\, (y \sqcup\!\sqcup (\{1\}^{i-1} \,\underline{*}\, w))\\
&+\sum_{k=2}^{m}\sum_{j=1}^{k-1}(-1)^{k-1}\{1\}_\star^{m-k} \,\underline{*}\, (y \sqcup\!\sqcup (((-1)^{j-1} z_j * \{1\}^{k-j-1}) \,\underline{*}\, w)) && (k := i+j)\\
=&\sum_{i=1}^{m-1}(-1)^{i-1}(m-i)\{1\}_\star^{m-i} \,\underline{*}\, (y \sqcup\!\sqcup (\{1\}^{i-1} \,\underline{*}\, w))\\
&+\sum_{k=2}^{m}(-1)^{k-1}(k-1)\{1\}_\star^{m-k} \,\underline{*}\, (y \sqcup\!\sqcup (\{1\}^{k-1} \,\underline{*}\, w)) && \text{(by (16), (17))}\\
=&(m-1)\sum_{i=1}^{m}(-1)^{i-1}\{1\}_\star^{m-i} \,\underline{*}\, (y \sqcup\!\sqcup (\{1\}^{i-1} \,\underline{*}\, w))\\
=&(m-1)G_m(w) + (m-1)z_m \,\underline{*}\, w. && (22)
\end{aligned}$$

We also have

$$\begin{aligned}
\sum_{j=1}^{m-1}\left(z_j G_{m-j}(w) - G_{m-j}(z_j w)\right) =& \sum_{j=1}^{m-1}\left(z_j \partial_{m-j}(w) - \partial_{m-j}(z_j w)\right)\\
=& -\sum_{j=1}^{m-1}\partial_{m-j}(z_j)w\\
=&(m-1)z_m w. && (23)
\end{aligned}$$

Here the last equality follows from Lemma 11.8. From (15), (21), (22), and (23), we have

$$\begin{aligned}
(m-1)(\partial_m + \delta_m)(w) =&(m-1)G_m(w) + (m-1)z_m \,\underline{*}\, w + (m-1)z_m w\\
=&(m-1)(G_m(w) + \delta_m(w)).
\end{aligned}$$

Thus the claim $\partial_m(w) = G_m(w)$ is proved. □

11.4 Proof of the sum formula

Let $k > d > 0$. Put $\mathbf{k} := [(k-d+1), \overbrace{(1), \dots, (1)}^{d-1}]$. From the series expression (7), we have

$$\zeta_{\mathrm{cyc}}(\mathbf{k}) = \zeta(k).$$

On the other hand, from the integral expression (8), we have

$$\begin{aligned}\zeta_{\mathrm{cyc}}(\mathbf{k}) =& \int_{t_{1,1} < \cdots < t_{1,k-d+1} > t_{2,1} > \cdots > t_{d,1} > t_{1,1}} \frac{dt_{1,1}}{1-t_{1,1}} \frac{dt_{1,2}}{t_{1,2}} \cdots \frac{dt_{1,k-d+1}}{t_{1,k-d+1}} \frac{dt_{2,1}}{1-t_{2,1}} \cdots \frac{dt_{d,1}}{1-t_{d,1}} \\ =& Z(y^{d-1} \sqcup\!\sqcup\, z_{k-d+1}) \\ =& \sum_{\substack{k_1+\cdots+k_d=k \\ k_1,\dots,k_{d-1} \geq 1, k_d \geq 2}} \zeta(k_1, \dots, k_d).\end{aligned}$$

Then we get the sum formula:

$$\sum_{\substack{k_1+\cdots+k_d=k \\ k_1,\dots,k_{d-1} \geq 1, k_d \geq 2}} \zeta(k_1, \dots, k_d) = \zeta(k).$$

Part V

Quasi-derivation relations for multiple zeta values revisited

12 Quasi-derivation relations for MZVs

The *quasi-derivation relations* in the theory of multiple zeta values is a generalization, proposed by Kaneko [15] and established by Tanaka [30], of a set of linear relations known as *derivation relations* (see §1.6).

In order to introduce the quasi-derivation relations, we first define a $\mathbb{Q}$-linear map $\theta := \theta^{(c)} \colon \mathfrak{h} \to \mathfrak{h}$ with a parameter $c \in \mathbb{Q}$ (we often drop c from the notation) by setting

$$\theta(u) = uz = u(x+y) \quad \text{for} \quad u = x,\, y$$

and requiring

$$\theta(ww') = \theta(w)w' + w\theta(w') + cH(w)\partial_1(w')$$

for $w, w' \in \mathfrak{h}$, where H is the $\mathbb{Q}$-linear map from $\mathfrak{h}$ to itself defined by $H(w) = \deg(w) \cdot w$ for any monomial $w \in \mathfrak{h}$ ($\deg(w)$ is the degree of w). This is well defined because H is a derivation on $\mathfrak{h}$. Now we define the quasi-derivation map $\partial_n^{(c)}$. Write $\mathrm{ad}(\theta)$ the adjoint operator by θ, i.e., $\mathrm{ad}(\theta)(\partial) := [\theta, \partial] = \theta\partial - \partial\theta$.

Definition 9. For each positive integer n and any rational number c, we define a $\mathbb{Q}$-linear map $\partial_n^{(c)} \colon \mathfrak{h} \to \mathfrak{h}$ by

$$\partial_n^{(c)} := \frac{1}{(n-1)!}\mathrm{ad}(\theta)^{n-1}(\partial_1).$$

Then the quasi-derivation relations of Tanaka [30] is stated as

$$Z(\partial_n^{(c)}(w)) = 0$$

for all $n \geq 1$, $c \in \mathbb{Q}$, and $w \in y\mathfrak{h}x$. Our aim in this part is to take another look at this relation, or rather at the operator $\partial_n^{(c)}$.

Remark 12.1. *1) We have changed the definition of $\theta = \theta^{(c)}$ by shifting the original ([15, 30]) by the derivation $w \to [z, w]/2 = (zw - wz)/2$. However, we can check that this does not change $\partial_n^{(c)}(w)$. Note also that the convention of the order of the product in $\mathfrak{h}$ there is opposite from ours.*

2) As noted in [13] , the special case $c = 0$ gives the original derivation ∂_n: $\partial_n = \partial_n^{(0)}$.

3) From $\theta(z^r) = rz^{r+1}$ $(r \geq 1)$ and $\partial_n(z) = 0$, we see that $\partial_n^{(c)}(wz) = \partial_n^{(c)}(w)z$ and $\partial_n^{(c)}(zw) = z\partial_n^{(c)}(w)$. We need to use this at several points later.

13 Main Theorem

We present a formula for $\partial_n^{(c)}(w)$ when w is in $\mathfrak{h}x$. To describe the formula, we define a product $\diamond$ on $\mathfrak{h}$ introduced in Hirose–Murahara–Onozuka [8] by

$$w_1 \diamond w_2 := \phi\big(\phi(w_1) * \phi(w_2)\big) \quad (w_1, w_2 \in \mathfrak{h}), \tag{24}$$

where ϕ is an involutive automorphism of $\mathfrak{h}$ determined by

$$\phi(x) = z = x + y \ \text{ and } \ \phi(y) = -y.$$

This is an associative and commutative binary operation with $1 \diamond w = w \diamond 1 = w$ for any $w \in \mathfrak{h}$. In [8], the definition of $\diamond$ is given in an inductive manner like the definition of $*$ in [6]. Later we only use the shuffle-type equality

$$xw_1 \diamond yw_2 = x(w_1 \diamond yw_2) + y(xw_1 \diamond w_2), \tag{25}$$

which holds for any $w_1, w_2 \in \mathfrak{h}$.

We define a specific element $q_n = q_n^{(c)}$ in $\mathfrak{h}$ for each $n \geq 1$ as follows.

Definition 10. Let $\tilde{\theta} = \tilde{\theta}^{(c)}$ be the map from $\mathfrak{h}$ to itself given by

$$\tilde{\theta}(w) := \theta(w) + cH(w)y \ \ (w \in \mathfrak{h}).$$

For each positive integer n, we define

$$q_n := \frac{1}{(n-1)!}\tilde{\theta}^{n-1}(y).$$

We thus have $q_1 = y$ and $q_n = \tilde{\theta}(q_{n-1})/(n-1)$ for $n \geq 2$.

Note that $q_n = q_n^{(c)}$ is in $y\mathfrak{h}$, as can be seen inductively by the definition. We shall give an explicit formula for q_n in the next section. Here is our main theorem.

Theorem 13.1. *For all $n \geq 1$ and $c \in \mathbb{Q}$, we have*

$$\partial_n^{(c)}(wx) = (w \diamond q_n)x \quad (w \in \mathfrak{h}).$$

Assuming the theorem, it is straightforward to deduce the quasi-derivation relations from Kawashima's relations (strictly speaking, its "linear part"). Recall the linear part of Kawashima's relations [21] asserts that

$$Z(\phi(w_1 * w_2)x) = 0$$

for any $w_1, w_2 \in y\mathfrak{h}$. Using this and the definition (24) of $\diamond$, we see that

$$Z\big(\partial_n^{(c)}(ywx)\big) = Z\big((yw \diamond q_n)x\big) = Z\big(\phi\big(\phi(yw) * \phi(q_n)\big)x\big) = 0$$

because both $\phi(yw)$ and $\phi(q_n)$ are in $y\mathfrak{h}$. This is the quasi-derivation relations.

Another immediate corollary to the theorem is the commutativity of the operators $\partial_n^{(c)}$, that is, $\partial_{n_1}^{(c_1)}$ and $\partial_{n_2}^{(c_2)}$ commute with each other for any $n_1, n_2 \geq 1$ and $c_1, c_2 \in \mathbb{Q}$. This was proved in [30] but the argument was quite involved. Here we may show

$$[\partial_{n_1}^{(c_1)}, \partial_{n_2}^{(c_2)}](w) = 0$$

first for $w \in \mathfrak{h}x$ as

$$\begin{aligned}
[\partial_{n_1}^{(c_1)}, \partial_{n_2}^{(c_2)}](wx) &= (\partial_{n_1}^{(c_1)}\partial_{n_2}^{(c_2)} - \partial_{n_2}^{(c_2)}\partial_{n_1}^{(c_1)})(wx) \\
&= ((w \diamond q_{n_2}) \diamond q_{n_1})x - ((w \diamond q_{n_1}) \diamond q_{n_2})x \\
&= 0
\end{aligned}$$

because the product $\diamond$ is associative and commutative, and then for the general case by induction on the degree of w by noting $\partial_n^{(c)}(wz) = \partial_n^{(c)}(w)z$ as remarked before.

Proof of Theorem 13.1. We need some lemmas. Recall $z = x + y$.

Lemma 13.2. *For $w_1, w_2 \in \mathfrak{h}$, we have*

$$zw_1 \diamond w_2 = w_1 \diamond zw_2 = z(w_1 \diamond w_2).$$

Proof. This follows from $\phi(z) = x$, $\phi(x) = z$ and $xw_1 * w_2 = w_1 * xw_2 = x(w_1 * w_2)$. See also [8]. ☐

Lemma 13.3. *For $w \in \mathfrak{h}$, we have $\partial_1(w) = w \diamond y - wy$.*

Proof. We proceed by induction on $\deg(w)$. The case $\deg(w) = 0$ is obvious because $\partial_1(1) = 0$. Suppose $\deg(w) \geq 1$. By linearity, it is enough to prove the equation when w is of the form zw' and xw'. If $w = zw'$, we have, by using the induction hypothesis and Lemma 13.2,

$$\partial_1(w) = \partial_1(zw') = z\partial_1(w') = z(w' \diamond y - w'y) = zw' \diamond y - zw'y = w \diamond y - wy.$$

When $w = xw'$, we similarly compute (using equation (25))

$$\begin{aligned}
\partial_1(w) = \partial_1(xw') &= yxw' + x\partial_1(w') = yxw' + x(w' \diamond y - w'y) \\
&= y(xw' \diamond 1) + x(w' \diamond y) - xw'y = xw' \diamond y - xw'y \\
&= w \diamond y - wy.
\end{aligned}$$ ☐

Lemma 13.4. *For $u \in \mathbb{Q}x + \mathbb{Q}y$, we have*

$$\tilde{\theta}(uw) = u\big(\tilde{\theta}(w) + zw + c(w \diamond y)\big).$$

Proof. We only need to show the equation for $u = x$ and y. By the definition of $\tilde{\theta}$, we have

$$\begin{aligned}\tilde{\theta}(uw) &= \theta(uw) + cH(uw)y \\ &= uzw + u\theta(w) + cu\partial_1(w) + cuwy + cuH(w)y \\ &= u\big(\tilde{\theta}(w) + zw + c(\partial_1(w) + wy)\big).\end{aligned}$$

From Lemma 13.3, we complete the proof. □

We need one more preparatory result, which may be of interest in its own right.

Proposition 13.5. *The $\mathbb{Q}$-linear map $\tilde{\theta}$ is a derivation on $\mathfrak{h}$ with respect to the product $\diamond$, i.e., the equation*

$$\tilde{\theta}(w_1 \diamond w_2) = \tilde{\theta}(w_1) \diamond w_2 + w_1 \diamond \tilde{\theta}(w_2) \tag{26}$$

holds for any $w_1, w_2 \in \mathfrak{h}$.

Proof. We prove this by induction on $\deg(w_1) + \deg(w_2)$. The case $\deg(w_1) + \deg(w_2) = 0$ holds trivially:

$$\tilde{\theta}(1 \diamond 1) = \tilde{\theta}(1) = 0 = \tilde{\theta}(1) \diamond 1 + 1 \diamond \tilde{\theta}(1).$$

When $\deg(w_1) + \deg(w_2) \geq 1$, we first prove when w_1 is of the form $w_1 = zw_1'$. By the definition of $\tilde{\theta}$ and Lemmas 13.2 and 13.4, we have

$$\tilde{\theta}(zw_1' \diamond w_2) = \tilde{\theta}(z(w_1' \diamond w_2)) = z\big(\tilde{\theta}(w_1' \diamond w_2) + z(w_1' \diamond w_2) + c(w_1' \diamond w_2 \diamond y)\big).$$

On the other hand, we have

$$\begin{aligned}&\tilde{\theta}(zw_1') \diamond w_2 + zw_1' \diamond \tilde{\theta}(w_2) \\ &= z\big(\tilde{\theta}(w_1') + zw_1' + c(w_1' \diamond y)\big) \diamond w_2 + z\big(w_1' \diamond \tilde{\theta}(w_2)\big) \\ &= z\big(\tilde{\theta}(w_1') \diamond w_2 + w_1' \diamond \tilde{\theta}(w_2) + z(w_1' \diamond w_2) + c(w_1' \diamond w_2 \diamond y)\big).\end{aligned}$$

Hence by the induction hypothesis we obtain

$$\tilde{\theta}(zw_1' \diamond w_2) = \tilde{\theta}(zw_1') \diamond w_2 + zw_1' \diamond \tilde{\theta}(w_2).$$

Since the binary operator $\diamond$ is commutative and bilinear, it suffices then to prove equation (26) only in the case where $w_1 = xw_1'$ and $w_2 = yw_2'$. By using equation (25) and Lemma 13.4, we have

$$\begin{aligned}&\tilde{\theta}(xw_1' \diamond yw_2') \\ &= \tilde{\theta}\left(x(w_1' \diamond yw_2') + y(xw_1' \diamond w_2')\right) \\ &= x\big(\tilde{\theta}(w_1' \diamond yw_2') + z(w_1' \diamond yw_2') + c(w_1' \diamond yw_2' \diamond y)\big) \\ &\quad + y\big(\tilde{\theta}(xw_1' \diamond w_2') + z(xw_1' \diamond w_2') + c(xw_1' \diamond w_2' \diamond y)\big)\end{aligned}$$

and

$$\begin{aligned}
&\tilde{\theta}(xw_1') \diamond yw_2' + xw_1' \diamond \tilde{\theta}(yw_2') \\
&= x\big(\big(\tilde{\theta}(w_1') + zw_1' + c(w_1' \diamond y)\big) \diamond yw_2'\big) + y\big(\tilde{\theta}(xw_1') \diamond w_2'\big) \\
&\quad + x\big(w_1' \diamond \tilde{\theta}(yw_2')\big) + y\big(xw_1' \diamond \big(\tilde{\theta}(w_2') + zw_2' + c(w_2' \diamond y)\big)\big) \\
&= x\big(\tilde{\theta}(w_1') \diamond yw_2' + w_1' \diamond \tilde{\theta}(yw_2') + z(w_1' \diamond yw_2') + c(w_1' \diamond yw_2' \diamond y)\big) \\
&\quad + y\big(\tilde{\theta}(xw_1') \diamond w_2' + xw_1' \diamond \tilde{\theta}(w_2') + z(xw_1' \diamond w_2') + c(xw_1' \diamond w_2' \diamond y)\big).
\end{aligned}$$

From these, we see by the induction hypothesis that

$$\tilde{\theta}(xw_1' \diamond yw_2') = \tilde{\theta}(xw_1') \diamond yw_2' + xw_1' \diamond \tilde{\theta}(yw_2')$$

holds. □

Now we prove Theorem 13.1 by induction on n. When $n = 1$, we have

$$\partial_1^{(c)}(wx) = \partial_1(wx) = \partial_1(w)x + wyx = (\partial_1(w) + wy)x = (w \diamond y)x = (w \diamond q_1)x$$

by Lemma 13.3. When $n \geq 2$, we have

$$\begin{aligned}
\partial_n^{(c)}(wx) &= \frac{1}{n-1} ad(\theta)(\partial_{n-1}^{(c)})(wx) \\
&= \frac{1}{n-1}\left(\theta\partial_{n-1}^{(c)}(wx) - \partial_{n-1}^{(c)}\theta(wx)\right).
\end{aligned}$$

By the induction hypothesis, we have

$$\begin{aligned}
\theta\partial_{n-1}^{(c)}(wx) &= \theta((w \diamond q_{n-1})x) \\
&= \theta(w \diamond q_{n-1})x + (w \diamond q_{n-1})xz + cH(w \diamond q_{n-1})yx \\
&= \tilde{\theta}(w \diamond q_{n-1})x + (w \diamond q_{n-1})xz
\end{aligned}$$

and

$$\begin{aligned}
\partial_{n-1}^{(c)}\theta(wx) &= \partial_{n-1}^{(c)}\left(\theta(w)x + wxz + cH(w)yx\right) \\
&= (\theta(w) \diamond q_{n-1})x + (w \diamond q_{n-1})xz + c(H(w)y \diamond q_{n-1})x \\
&= (\tilde{\theta}(w) \diamond q_{n-1})x + (w \diamond q_{n-1})xz.
\end{aligned}$$

We therefore obtain by Proposition 13.5,

$$\begin{aligned}
\partial_n^{(c)}(wx) &= \frac{1}{n-1}\big(\tilde{\theta}(w \diamond q_{n-1}) - (\tilde{\theta}(w) \diamond q_{n-1})\big)x = \frac{1}{n-1}\big(w \diamond \tilde{\theta}(q_{n-1})\big)x \\
&= (w \diamond q_n)x,
\end{aligned}$$

which completes the proof. □

14 Explicit formula for q_n

We now describe the element $q_n = q_n^{(c)}$ in an explicit manner. For any index $\mathbf{l} = (l_1, \dots, l_s) \in \mathbb{N}^s$, we define $a(\mathbf{l}) = a(l_1, \dots, l_s) \in \mathbb{Q}$ (or $\in \mathbb{Z}[c]$ if we view c as a variable) inductively by $a(1) := 1$ and

$$a(\mathbf{l}) := \sum_{i=1}^{s} \big(l_i - 1 - (l_1 + \cdots + l_{i-1})c\big)\, a(\mathbf{l}^{(i)}),$$

where

$$\mathbf{l}^{(i)} = \begin{cases} (l_1, \dots, l_{i-1}, l_{i+1}, \dots, l_s) & \text{if } l_i = 1, \\ (l_1, \dots, l_{i-1}, l_i - 1, l_{i+1}, \dots, l_s) & \text{if } l_i > 1. \end{cases}$$

Proposition 14.1. *For $n \geq 1$, we have*

$$q_n = -\frac{1}{(n-1)!} \sum_{|\mathbf{l}|=n} a(\mathbf{l}) w(\mathbf{l}), \tag{27}$$

where the sum runs over all indices $\mathbf{l} = (l_1, \dots, l_s) \in (\mathbb{Z}_{\geq 1})^s$ of any length s and of weight $|\mathbf{l}| := l_1 + \cdots + l_s = n$, and $w(\mathbf{l}) = \phi(yx^{l_1 - 1} \cdots yx^{l_s - 1}) = (-1)^s yz^{l_1 - 1} \cdots yz^{l_s - 1}$.

Proof. Let q'_n denote the right-hand side of (27). We prove (27) by induction on n. When $n = 1$, we easily see $q'_1 = y$.

Suppose $n \geq 2$. We want to show that $q'_n = \tilde{\theta}(q'_{n-1})/(n-1)$. Since $\theta(z^m) = mz^{m+1}$ and $\partial_1(z) = 0$, we have

$$\theta(yz^{k-1}) = yz^k + (k-1)yz^k = kyz^k,$$

and so

$$\begin{aligned}
&\theta(yz^{k_1-1} \cdots yz^{k_r-1}) \\
&= \sum_{j=1}^{r} yz^{k_1-1} \cdots yz^{k_{j-1}-1} \cdot k_j yz^{k_j} \cdot yz^{k_{j+1}-1} \cdots yz^{k_r-1} \\
&\quad + c \sum_{1 \leq i < j \leq r} yz^{k_1-1} \cdots H(yz^{k_i-1}) \cdots \partial_1(yz^{k_j-1}) \cdots yz^{k_r-1} \\
&= \sum_{j=1}^{r} k_j\, yz^{k_1-1} \cdots yz^{k_{j-1}-1} yz^{k_j} yz^{k_{j+1}-1} \cdots yz^{k_r-1} \\
&\quad - c \sum_{1 \leq i < j \leq r} yz^{k_1-1} \cdots (k_i yz^{k_i-1}) \cdots y(z-y)z^{k_j-1} yz^{k_{j+1}-1} \cdots yz^{k_r-1} \\
&= \sum_{j=1}^{r} k_j\, yz^{k_1-1} \cdots yz^{k_{j-1}-1} yz^{k_j} yz^{k_{j+1}-1} \cdots yz^{k_r-1} \\
&\quad - c \sum_{j=2}^{r} (k_1 + \cdots + k_{j-1}) yz^{k_1-1} \cdots yz^{k_{j-1}-1} y(z-y) z^{k_j-1} yz^{k_{j+1}-1} \cdots yz^{k_r-1}.
\end{aligned}$$

Since $cH(yz^{k_1-1}\cdots yz^{k_r-1})y = c(k_1+\cdots+k_r)yz^{k_1-1}\cdots yz^{k_r-1}y$, we finally obtain for $\mathbf{k} = (k_1,\ldots,k_r)$,

$$\begin{aligned}
&\tilde{\theta}(w(\mathbf{k}))\\
&= (-1)^r\tilde{\theta}(yz^{k_1-1}\cdots yz^{k_r-1})\\
&= (-1)^r\sum_{j=1}^{r}\bigl(k_j - c(k_1+\cdots+k_{j-1})\bigr)yz^{k_1-1}\cdots yz^{k_{j-1}-1}yz^{k_j}yz^{k_{j+1}-1}\cdots yz^{k_r-1}\\
&\quad - (-1)^{r+1}c\sum_{j=1}^{r}(k_1+\cdots+k_j)yz^{k_1-1}\cdots yz^{k_j-1}\cdot y\cdot yz^{k_{j+1}-1}\cdots yz^{k_r-1}.
\end{aligned}$$

If we write

$$\tilde{\theta}(q'_{n-1}) = -\frac{1}{(n-2)!}\sum_{|\mathbf{l}|=n} a'(\mathbf{l})w(\mathbf{l}),$$

we see from this that the coefficient $a'(\mathbf{l})$ of $w(\mathbf{l}) = (-1)^s yz^{l_1-1}\cdots yz^{l_s-1}$ is given exactly by $a(\mathbf{l})$ as defined recursively. □

15 Quasi-derivation relations for finite multiple zeta values

In this section, we briefly discuss how the quasi-derivation relations look like for "finite" multiple zeta values. We recall $Z_{\mathcal{F}}$ is the $\mathbb{Q}$-linear map from $y\mathfrak{h}$ to either algebra assigning the monomial $yx^{k_1-1}\cdots yx^{k_r-1}$ to $\zeta_{\mathcal{A}}(k_1,\ldots,k_r)$ or $\zeta_{\mathcal{S}}(k_1,\ldots,k_r)$.

As a consequence of our Theorem 13.1, we have the following.

Theorem 15.1 (Quasi-derivation relations for finite multiple zeta values)**.** *For all $n \geq 1$ and $c \in \mathbb{Q}$, we have*

$$Z_{\mathcal{F}}(\partial_n^{(c)}(w)x^{-1}) = Z_{\mathcal{F}}(wx^{-1})Z_{\mathcal{F}}(q_n^{(c)}) \quad (w \in y\mathfrak{h}x).$$

Proof. This is almost immediate from Theorem 13.1 if one notes $Z_{\mathcal{F}} \circ \phi = Z_{\mathcal{F}}$ and $Z_{\mathcal{F}}$ is a $*$-homomorphism (see Theorem 3.4 and 3.5). □

Remark 15.2. *When $c = 0$, we can easily compute that $q_n^{(0)} = yz^{n-1}$. Since $Z_{\mathcal{F}}(yz^{n-1}) = Z_{\mathcal{F}}\bigl(\phi(yz^{n-1})\bigr) = -Z_{\mathcal{F}}(yx^{n-1}) = -\zeta_{\mathcal{F}}(n) = 0$ for $\mathcal{F} = \mathcal{A}$ or $\mathcal{S}$, we see that Theorem 15.1 generalizes the derivation relations (Theorem 3.6).*

Part VI

On Hoffman's t-values of maximal height

16 Multiple t-values and main result

In [11], Hoffman introduced and studied an "odd variant" of multiple zeta value

$$t(k_1,\ldots,k_d) := \sum_{\substack{0<n_1<\cdots<n_d \\ n_i:odd}} \frac{1}{n_1^{k_1}\cdots n_d^{k_d}},$$

which we call multiple t-value (MtV). Any multiple t-value can be written as a $\mathbb{Q}$-linear combination of "Euler sums", which is defined by

$$\zeta\left(\begin{array}{c} k_1,\cdots,k_d \\ \epsilon_1,\cdots,\epsilon_d \end{array}\right) := \sum_{0<n_1<\cdots<n_d} \frac{\epsilon_1^{n_1}\cdots\epsilon_d^{n_d}}{n_1^{k_1}\cdots n_d^{k_d}},$$

for $k_1,\ldots,k_d \in \mathbb{Z}_{\geq 1}$, $\epsilon_1,\ldots,\epsilon_d \in \{\pm 1\}$ with $(k_d,\epsilon_d) \neq (1,1)$.

If we write

$$t(k_1,\ldots,k_d) = 2^{-d} \sum_{0<n_1<\cdots<n_d} \frac{(1-(-1)^{n_1})\cdots(1-(-1)^{n_d})}{n_1^{k_1}\cdots n_d^{k_d}},$$

we see by expanding this that

$$t(k_1,\ldots,k_d) = 2^{-d} \sum_{\epsilon_m\in\{\pm 1\}} \epsilon_1\cdots\epsilon_d \zeta\left(\begin{array}{c} k_1,\ldots,k_d \\ \epsilon_1,\ldots,\epsilon_d \end{array}\right). \tag{28}$$

Multiple zeta values are special cases of Euler sums. Our first main result shows that in some cases multiple t-values are linear combination of multiple zeta values.

Theorem 16.1. *When all k_i are greater than 1, $t(k_1,\ldots,k_d)$ is a $\mathbb{Q}$-linear combination of multiple zeta values.*

In particular, if all k_i are either 2 or 3, $t(k_1,\ldots,k_d)$ belongs to the space spanned by the MZVs. Hoffman conjectured and Brown proved that $\zeta(k_1,\ldots,k_d)$'s with all k_i equal 2 or 3 span the space of MZVs. Our second main result gives an alternative, analogous generators of MZVs by means of MtVs.

Theorem 16.2. *Every multiple zeta value is a $\mathbb{Q}$-linear combination of elements in*

$$\{t(k_1,\ldots,k_d) \mid k_1,\ldots,k_d \in \{2,3\}\}.$$

To prove this theorem, we need the following theorem, just as Brown's proof of [1, Theorem 1.1] needed Zagier's [35, Theorem 1].

Theorem 16.3. *Let $\tilde{t}(k_1,\dots,k_d) := 2^{k_1+\cdots+k_d}t(k_1,\dots,k_d)$ and set*

$$K(a,b) := \tilde{t}(\underbrace{2,\dots,2}_{a},3,\underbrace{2,\dots,2}_{b}) \quad \textit{and} \quad K(n) := \tilde{t}(\underbrace{2,\dots,2}_{n}),$$

for any $a,b,n \in \mathbb{Z}_{\geq 0}$. Then, for all integers $a,b \geq 0$, we have

$$K(a,b) = 2\sum_{r=1}^{a+b+1}(-1)^{r-1}\left[\binom{2r}{2a+1} + \left(1-2^{-2r}\right)\binom{2r}{2b+1}\right]K(a+b-r+1)\zeta(2r+1). \quad (29)$$

We prove Theorems 16.1 and 16.2 by proving their motivic counterparts. In Section 17, we introduce motivic Euler sums. In Section 18, we prove Theorem 16.1 by using Glanois's theorem. In Section 19, we prove Theorem 16.3. The proof of this theorem is based on the same argument as in the proof of [35, Theorem 1]. In Section 20, we prove Theorem 16.2 in a similar way as in Brown's proof of [1, Theorem 1.1].

17 Motivic Setup

For $a \in \mathbb{Z}_{\geq 0}, k_i \in \mathbb{Z}_{\geq 1}$ and $\epsilon_i \in \{\pm 1\}$ $(1 \leq i \leq d)$, the (general) motivic Euler sum is defined in terms of motivic iterated integral as follows:

$$\zeta_a^{\mathfrak{m}}\begin{pmatrix} k_1,\dots,k_d \\ \epsilon_1,\dots,\epsilon_d \end{pmatrix} = (-1)^d I^{\mathfrak{m}}\left(0;0^a,\eta_1,0^{k_1-1},\eta_2,0^{k_2-1},\dots,\eta_d,0^{k_d-1};1\right),$$

where $\eta_i := \epsilon_i\cdots\epsilon_d$. When $a=0$, we denote $\zeta_0^{\mathfrak{m}}$ by $\zeta^{\mathfrak{m}}$. Also when $\epsilon_1=\cdots=\epsilon_d=1$, we denote $\zeta_a^{\mathfrak{m}}\begin{pmatrix} k_1,\dots,k_d \\ 1,\dots,1 \end{pmatrix}$ by $\zeta_a^{\mathfrak{m}}(k_1,\dots,k_d)$, which we call the motivic MZV. For the definition of motivic iterated integrals and their properties, we refer the reader to [4]. Let $\mathcal{H}^2$ (resp. $\mathcal{H}^1$) be the $\mathbb{Q}$-vector space spanned by the motivic Euler sums (resp. motivic MZVs). Let $\mathcal{H}_N^2$ denote the $\mathbb{Q}$-vector space spanned by all motivic Euler sums of weight N. The space $\mathcal{H}^2$ naturally has the structure of a graded $\mathbb{Q}$-algebra

$$\mathcal{H}^2 = \bigoplus_{N\geq 0}\mathcal{H}_N^2$$

with the shuffle product. There is a surjective homomorphism called the period map

$$\mathrm{per}: \mathcal{H}^2 \longrightarrow \mathbb{R},$$

which sends $\zeta^{\mathfrak{m}}\begin{pmatrix} k_1,\dots,k_d \\ \epsilon_1,\dots,\epsilon_d \end{pmatrix}$ to $\zeta\begin{pmatrix} k_1,\dots,k_d \\ \epsilon_1,\dots,\epsilon_d \end{pmatrix}$.

Let $\mathcal{A}$ be the quotient algebra $\mathcal{H}^2/\zeta^{\mathfrak{m}}(2)\mathcal{H}^2$, and $\mathcal{A}_N$ denotes its weight N part. We use $I^{\mathfrak{a}}$ (resp. $\zeta^{\mathfrak{a}}$) to denote the image of $I^{\mathfrak{m}}$ (resp. $\zeta^{\mathfrak{m}}$) under the canonical surjection $\mathcal{H}^2 \to \mathcal{A}$. Let

$\mathcal{L} := \mathcal{A}_{>0}/\mathcal{A}_{>0} \cdot \mathcal{A}_{>0}$ be the 'linearized quotient' of $\mathcal{A}_{>0}$. We denote by $\mathcal{L}_N$ the weight N part of $\mathcal{L}$ and by $I^{\mathfrak{l}}$ (resp. $\zeta^{\mathfrak{l}}$) the image of $I^{\mathfrak{a}}$ (resp. $\zeta^{\mathfrak{a}}$) under the canonical surjection $\mathcal{A}_{>0} \to \mathcal{L}$.

Motivic Euler sums and motivic iterated integrals satisfy the following properties:

(I1) $I^{\mathfrak{m}}(a_0; a_1) = 1$.

(I2) $I^{\mathfrak{m}}(a_0; a_1, \ldots, a_n; a_{n+1}) = 0$ if $a_0 = a_{n+1}$, $n \geq 1$.

(I3) $\zeta_a^{\mathfrak{m}}\left(\begin{array}{c} k_1, \ldots, k_d \\ \epsilon_1, \ldots, \epsilon_d \end{array}\right) = (-1)^a \sum_{\substack{i_1+\cdots+i_d=a \\ i_j \geq 0\ (1\leq j\leq d)}} \binom{k_1+i_1-1}{i_1} \cdots \binom{k_d+i_d-1}{i_d} \zeta^{\mathfrak{m}}\left(\begin{array}{ccc} k_1+i_1 & , \ldots, & k_d+i_d \\ \epsilon_1 & , \ldots, & \epsilon_d \end{array}\right)$.

(I4) Path composition:

$$I^{\mathfrak{m}}(a_0; a_1, \ldots, a_n; a_{n+1}) = \sum_{i=1}^{n} I^{\mathfrak{m}}(a_0; a_1, \ldots, a_i; x)\, I^{\mathfrak{m}}(x; a_{i+1}, \ldots, a_n; a_{n+1}). \quad (x \in \{0, \pm 1\})$$

(I5) Path reversal: $I^{\mathfrak{m}}(a_0; a_1, \ldots, a_n; a_{n+1}) = (-1)^n I^{\mathfrak{m}}(a_{n+1}; a_n, \ldots, a_1; a_0)$.

(I6) Homothety: $I^{\mathfrak{m}}(0; -a_1, \ldots, -a_n; -a_{n+1}) = I^{\mathfrak{m}}(0; a_1, \ldots, a_n; a_{n+1})$.

Define the motivic multiple $\tilde{t}$-values for $\mathbf{k} = (k_1, \ldots, k_d)$ as

$$\tilde{t}^{\mathfrak{m}}(k_1, \ldots, k_d) := 2^{|\mathbf{k}|-d} \sum_{\epsilon_m \in \{\pm 1\}} \epsilon_1 \cdots \epsilon_d \zeta^{\mathfrak{m}}\left(\begin{array}{c} k_1, \ldots, k_d \\ \epsilon_1, \ldots, \epsilon_d \end{array}\right).$$

We recall $|\mathbf{k}| := k_1 + \cdots + k_d$ is the *weight* of the index $\mathbf{k}$. In view of (28), the image of $\tilde{t}^{\mathfrak{m}}(k_1, \ldots, k_d)$ under the period map is $\tilde{t}(k_1, \ldots, k_d)$.

From [4, Subsection 2.3.2], the coaction $\Delta : \mathcal{H}^2 \longrightarrow \mathcal{A} \otimes_{\mathbb{Q}} \mathcal{H}^2$ is given by

$$\begin{aligned} &\Delta I^{\mathfrak{m}}(a_0; a_1, \ldots, a_n; a_{n+1}) \\ &= \sum_{\substack{0 \leq k \leq n \\ i_0=0<i_1<\cdots<i_k<i_{k+1}=n+1}} \left(\prod_{p=0}^{k} I^{\mathfrak{a}}\left(a_{i_p}; a_{i_p+1}, \ldots, a_{i_{p+1}-1}; a_{i_{p+1}}\right)\right) \otimes I^{\mathfrak{m}}(a_0; a_{i_1}, \ldots, a_{i_k}; a_{n+1}). \end{aligned} \tag{30}$$

For any odd $r \geq 1$, we define the derivation operator:

$$D_r : \mathcal{H}^2 \longrightarrow \mathcal{L}_r \otimes_{\mathbb{Q}} \mathcal{H}^2$$

as the composite of $\Delta - (1 \otimes id)$ with $\pi_r \otimes id$, where π_r is the projection $\mathcal{A} \to \mathcal{L} \to \mathcal{L}_r$.

By (30), the action of D_r on $I^{\mathfrak{m}}(a_0; a_1, \ldots, a_n; a_{n+1})$ is given by

$$\begin{aligned} &D_r\left(I^{\mathfrak{m}}(a_0; a_1, \ldots, a_n; a_{n+1})\right) \\ &= \sum_{p=0}^{n-r} I^{\mathfrak{l}}(a_p; a_{p+1}, \ldots, a_{p+r}; a_{p+r+1}) \otimes I^{\mathfrak{m}}(a_0; a_1, \ldots, a_p, a_{p+r+1}, \ldots, a_n; a_{n+1}). \end{aligned}$$

Brown used the following theorem to lift a particular relation of MZVs proved by Zagier to the corresponding motivic relation.

Theorem 17.1 (Brown [1, Theorem3.3])**.** *Let $N \geq 2$. The kernel of $D_{<N} = \oplus_{1<2r+1<N} D_{2r+1}$ is one-dimensional in weight N spanned by $\zeta^{\mathfrak{m}}(N)$:*

$$\ker D_{<N} \cap \mathcal{H}_N = \mathbb{Q}\zeta^{\mathfrak{m}}(N).$$

We employ this theorem as well as the following theorem to prove our main result.

Theorem 17.2 (Glanois [4, Corollary 2.4])**.** *Let ξ be a motivic Euler sum. Then ξ is a motivic multiple zeta value if and only if:*

$$D_1(\xi) = 0 \quad \text{and} \quad D_r(\xi) \in \mathcal{L}_r \otimes \mathcal{H}^1 \text{ for } r : \text{odd}.$$

18 Proof of Theorem 16.1

In this section, we prove the following theorem. By applying the period map, we obtain Theorem 16.1.

Theorem 18.1. *When all k_i are greater than 1,*

$$\tilde{t}^{\mathfrak{m}}(k_1, \ldots, k_d) \in \mathcal{H}^1.$$

First, we prove Lemma 18.2 and Proposition 18.3, which are used for the calculation in the proof of Proposition 18.4.

Lemma 18.2. *Let $k_0, \ldots, k_i \in \mathbb{Z}_{\geq 1}$. For $\alpha, \beta \in \{\pm 1\}$,*

$$\sum_{\eta_1,\ldots,\eta_i \in \{\pm 1\}} I^{\mathfrak{l}}(\alpha; 0^{k_0-1}, \eta_1, 0^{k_1-1}, \ldots, \eta_i, 0^{k_i-1}; \beta) = 0.$$

Proof. If $\alpha = \beta$, by property (I2), we have

$$\sum_{\eta_m \in \{\pm 1\}} I^{\mathfrak{l}}(\alpha; 0^{k_0-1}, \eta_1, 0^{k_1-1}, \ldots, \eta_i, 0^{k_i-1}; \alpha) = 0.$$

If $\alpha \neq \beta$, we compute

$$\begin{aligned}
&\sum_{\eta_m \in \{\pm 1\}} I^{\mathfrak{l}}(\alpha; 0^{k_0-1}, \eta_1, 0^{k_1-1}, \ldots, \eta_i, 0^{k_i-1}; -\alpha) \\
=&\sum_{\eta_m \in \{\pm 1\}} I^{\mathfrak{l}}(\alpha; 0^{k_0-1}, \eta_1, 0^{k_1-1}, ..., \eta_i, 0^{k_i-1}; 0) + \sum_{\eta_m \in \{\pm 1\}} I^{\mathfrak{l}}(0; 0^{k_0-1}, \eta_1, 0^{k_1-1}, ..., \eta_i, 0^{k_i-1}; -\alpha) \quad \text{(by (I4))} \\
=&\sum_{\eta_m \in \{\pm 1\}} I^{\mathfrak{l}}(\alpha; 0^{k_0-1}, \eta_1, 0^{k_1-1}, ..., \eta_i, 0^{k_i-1}; 0) + \sum_{\eta_m \in \{\pm 1\}} I^{\mathfrak{l}}(0; 0^{k_0-1}, -\eta_1, 0^{k_1-1}, ..., -\eta_i, 0^{k_i-1}; \alpha) \quad \text{(by (I6))} \\
=&\sum_{\eta_m \in \{\pm 1\}} I^{\mathfrak{l}}(\alpha; 0^{k_0-1}, \eta_1, 0^{k_1-1}, ..., \eta_i, 0^{k_i-1}; 0) + \sum_{\eta_m \in \{\pm 1\}} I^{\mathfrak{l}}(0; 0^{k_0-1}, \eta_1, 0^{k_1-1}, ..., \eta_i, 0^{k_i-1}; \alpha) \\
=&\sum_{\eta_m \in \{\pm 1\}} I^{\mathfrak{l}}(\alpha; 0^{k_0-1}, \eta_1, 0^{k_1-1}, ..., \eta_i, 0^{k_i-1}; \alpha) \quad \text{(by (I4))} \\
=&0.
\end{aligned}$$

□

Proposition 18.3. *Let $a \in \mathbb{Z}_{\geq 0}$, $\mathbf{k} = (k_1, \dots, k_d) \in (\mathbb{Z}_{\geq 2})^d$ and $N := a + k_1 + \cdots + k_d$. We have*

$$\zeta_a^{\mathfrak{m}}(k_1, \dots, k_d) = (-1)^d 2^{N-d} \sum_{\eta_m \in \{\pm 1\}} I^{\mathfrak{m}}(0; 0^a, \eta_1, 0^{k_1-1}, \eta_2, 0^{k_2-1}, \dots, \eta_d, 0^{k_d-1}; \alpha). \tag{31}$$

Proof. In the following, we write $\mathbf{k}_{i,j} = (k_i, \dots, k_j)$ the subindex of $\mathbf{k} = (k_1, \dots, k_d)$ and $\delta_\bullet = 1$ if $\bullet$ is true and $\delta_\bullet = 0$ if not. We prove (31) by induction on N. Let $1 < r < N$ with r: odd. By applying D_r, we have

$$\begin{aligned}
&(-1)^d 2^{d-N} D_r(\text{R.H.S. of (31)}) \\
&= \sum_{\eta_m \in \{\pm 1\}} D_r \left(I^{\mathfrak{m}}(0; 0^a, \eta_1, 0^{k_1-1}, \eta_2, 0^{k_2-1}, \dots, \eta_d, 0^{k_d-1}; \alpha) \right) \\
&= \sum_{1 \leq j \leq d} \delta_{|\mathbf{k}_{1,j}| \leq r \leq a + |\mathbf{k}_{1,j}|} \sum_{\eta_m \in \{\pm 1\}} \Bigg(I^{\mathfrak{l}}(0; 0^{r - |\mathbf{k}_{1,j}|}, \eta_1, 0^{k_1-1}, \dots, \eta_j, 0^{k_j-1}; \eta_{j+1}) \qquad (32) \\
&\qquad \otimes I^{\mathfrak{m}}(0; 0^{a + |\mathbf{k}_{1,j}| - r}, \eta_{j+1}, 0^{k_{j+1}-1}, \dots, \eta_d, 0^{k_d-1}; \alpha) \Bigg) \\
&\quad + \sum_{1 \leq i < j \leq d} \delta_{|\mathbf{k}_{i+1,j}| \leq r < |\mathbf{k}_{i,j}| - 1} \sum_{\eta_m \in \{\pm 1\}} \Bigg(I^{\mathfrak{l}}(0; 0^{r - |\mathbf{k}_{i+1,j}|}, \eta_{i+1}, 0^{k_{i+1}-1}, \dots, \eta_j, 0^{k_j-1}; \eta_{j+1}) \qquad (33) \\
&\qquad \otimes I^{\mathfrak{m}}(0; 0^a, \eta_1, 0^{k_1-1}, \dots, \eta_i, 0^{|\mathbf{k}_{i,j}| - r - 1}, \eta_{j+1}, 0^{k_{j+1}-1}, \dots, \eta_d, 0^{k_d-1}; \alpha) \Bigg) \\
&\quad + \sum_{1 \leq i < j \leq d} \delta_{|\mathbf{k}_{i,j-1}| \leq r < |\mathbf{k}_{i,j}| - 1} \sum_{\eta_m \in \{\pm 1\}} \Bigg(I^{\mathfrak{l}}(\eta_i; 0^{k_i-1}, \eta_{i+1}, 0^{k_{i+1}-1}, \dots, \eta_j, 0^{r - |\mathbf{k}_{i,j-1}|}; 0) \qquad (34) \\
&\qquad \otimes I^{\mathfrak{m}}(0; 0^a, \eta_1, 0^{k_1-1}, \dots, \eta_i, 0^{|\mathbf{k}_{i,j}| - r - 1}, \eta_{j+1}, 0^{k_{j+1}-1}, \dots, \eta_d, 0^{k_d-1}; \alpha) \Bigg) \\
&\quad + \sum_{1 \leq i < j \leq d} \delta_{r = |\mathbf{k}_{i,j}| - 1} \sum_{\eta_m \in \{\pm 1\}} \Bigg(I^{\mathfrak{l}}(\eta_i; 0^{k_i-1}, \eta_{i+1}, 0^{k_{i+1}-1}, \dots, \eta_j, 0^{k_j-1}; \eta_{j+1}) \qquad (35) \\
&\qquad \otimes I^{\mathfrak{m}}(0; \eta_1, 0^{k_1-1}, \dots, \eta_i, \eta_{j+1}, 0^{k_{j+1}-1}, \dots, \eta_d, 0^{k_d-1}; \alpha) \Bigg).
\end{aligned}$$

The sum (32) is calculated as follows.

$$\begin{aligned}
(32) &= \sum_{1 \leq j \leq d} \delta_{|\mathbf{k}_{1,j}| \leq r \leq a + |\mathbf{k}_{1,j}|} \sum_{\eta_{j+1}, \dots, \eta_d \in \{\pm 1\}} \left(\sum_{\eta_1, \dots, \eta_j \in \{\pm 1\}} I^{\mathfrak{l}}(0; 0^{r - |\mathbf{k}_{1,j}|}, \eta_1, 0^{k_1-1}, \dots, \eta_j, 0^{k_j-1}; \eta_{j+1}) \right) \\
&\qquad \otimes I^{\mathfrak{m}}(0; 0^{a + |\mathbf{k}_{1,j}| - r}, \eta_{j+1}, 0^{k_{j+1}-1}, \dots, \eta_d, 0^{k_d-1}; \alpha) \\
&= (-1)^d 2^{d-N} \sum_{1 \leq j \leq d} \delta_{|\mathbf{k}_{1,j}| \leq r \leq a + |\mathbf{k}_{1,j}|} \zeta^{\mathfrak{l}}_{r - |\mathbf{k}_{1,j}|}(k_1, \dots, k_j) \otimes \zeta^{\mathfrak{m}}_{a + |\mathbf{k}_{1,j}| - r}(k_{j+1}, \dots, k_d).
\end{aligned}$$

The second equality follows from the induction hypothesis. The sums (33), (34) and (35) are calculated in a similar manner.

$$(33)=\sum_{1\le i<j\le d}\delta_{|\mathbf{k}_{i+1,j}|\le r<|\mathbf{k}_{i,j}|-1}\sum_{\substack{\eta_1,\dots,\eta_i\\ \eta_{j+1},\dots,\eta_d\in\{\pm1\}}}\left(\sum_{\eta_{i+1},\dots,\eta_j\in\{\pm1\}} I^{\mathfrak{l}}(0;0^{r-|\mathbf{k}_{i+1,j}|},\eta_{i+1},0^{k_{i+1}-1},\dots,\eta_j,0^{k_j-1};\eta_{j+1})\right)$$
$$\otimes I^{\mathfrak{m}}(0;0^a,\eta_1,0^{k_1-1},\dots,\eta_i,0^{|\mathbf{k}_{i,j}|-r-1},\eta_{j+1},0^{k_{j+1}-1},\dots,\eta_d,0^{k_d-1};\alpha)$$
$$=(-1)^d2^{d-N}\sum_{1\le i<j\le d}\delta_{|\mathbf{k}_{i+1,j}|\le r<|\mathbf{k}_{i,j}|-1}\zeta^{\mathfrak{l}}_{r-|\mathbf{k}_{i+1,j}|}(k_{i+1},...,k_j)\otimes\zeta^{\mathfrak{m}}_a(k_1,...,k_{i-1},|\mathbf{k}_{i,j}|-r,k_{j+1},...,k_d).$$

$$(34)=\sum_{1\le i<j\le d}\delta_{|\mathbf{k}_{i,j-1}|\le r<|\mathbf{k}_{i,j}|-1}\sum_{\substack{\eta_1,\dots,\eta_i\\ \eta_{j+1},\dots,\eta_d\in\{\pm1\}}}\left(\sum_{\eta_{i+1},\dots,\eta_j\in\{\pm1\}} I^{\mathfrak{l}}(\eta_i;0^{k_i-1},\eta_{i+1},0^{k_{i+1}-1},\dots,\eta_j,0^{r-|\mathbf{k}_{i,j-1}|};0)\right)$$
$$\otimes I^{\mathfrak{m}}(0;0^a,\eta_1,0^{k_1-1},\dots,\eta_i,0^{|\mathbf{k}_{i,j}|-r-1},\eta_{j+1},0^{k_{j+1}-1},\dots,\eta_d,0^{k_d-1};\alpha)$$
$$=-(-1)^d2^{d-N}\sum_{1\le i<j\le d}\delta_{|\mathbf{k}_{i,j-1}|\le r<|\mathbf{k}_{i,j}|-1}\zeta^{\mathfrak{l}}_{r-\sum_{l=i}^{j-1}k_l}(k_{j-1},...,k_i)\otimes\zeta^{\mathfrak{m}}_a(k_1,...,k_{i-1},|\mathbf{k}_{i,j}|-r,k_{j+1},...,k_d).$$

$$(35)=\sum_{1\le i<j\le d}\delta_{|\mathbf{k}_{i,j}|-1=r}\sum_{\substack{\eta_1,\dots,\eta_i\\ \eta_{j+1},\dots,\eta_d\in\{\pm1\}}}\left(\sum_{\eta_{i+1},\dots,\eta_j\in\{\pm1\}} I^{\mathfrak{l}}(\eta_i;0^{k_i-1},\eta_{i+1},0^{k_{i+1}-1},\dots,\eta_j,0^{k_j-1};\eta_{j+1})\right)$$
$$\otimes I^{\mathfrak{m}}(0;\eta_1,0^{k_1-1},\dots,\eta_i,\eta_{j+1},0^{k_{j+1}-1},\dots,\eta_d,0^{k_d-1};\alpha)$$
$$=0.\qquad\text{(by Lemma 18.2)}$$

Therefore, we obtain $D_r(L.H.S.\text{ of }(31))=D_r(R.H.S.\text{ of }(31))$. By Theorem 17.1, there exists $c\in\mathbb{Q}$ such that $(L.H.S.\text{ of }(31))-(R.H.S.\text{ of }(31))=c\zeta^{\mathfrak{m}}(N)$. Taking the period map and by distribution relations for Euler sums, we obtain $c=0$. Thus the proposition is proved. □

Proposition 18.4. *For* $\mathbf{k}=(k_1,\dots,k_d)\in(\mathbb{Z}_{\ge2})^d$, *we have*

$$\begin{aligned}&D_r\left(\tilde{t}^{\mathfrak{m}}(k_1,\dots,k_d)\right)\\ &=\sum_{1\le j\le d}\delta_{|\mathbf{k}_{1,j}|=r}\tilde{t}^{\mathfrak{l}}(k_1,\dots,k_j)\otimes\tilde{t}^{\mathfrak{m}}(k_{j+1},\dots,k_d)\\ &+\sum_{1\le i<j\le d}\delta_{|\mathbf{k}_{i+1,j}|\le r<|\mathbf{k}_{i,j}|-1}\zeta^{\mathfrak{l}}_{r-|\mathbf{k}_{i+1,j}|}(k_{i+1},\dots,k_j)\otimes\tilde{t}^{\mathfrak{m}}(k_1,\dots,k_{i-1},|\mathbf{k}_{i,j}|-r,k_{j+1},\dots,k_d)\\ &-\sum_{1\le i<j\le d}\delta_{|\mathbf{k}_{i,j-1}|\le r<|\mathbf{k}_{i,j}|-1}\zeta^{\mathfrak{l}}_{r-|\mathbf{k}_{i,j-1}|}(k_{j-1},\dots,k_i)\otimes\tilde{t}^{\mathfrak{m}}(k_1,\dots,k_{i-1},|\mathbf{k}_{i,j}|-r,k_{j+1},\dots,k_d).\end{aligned}$$

Proof. By the definition of motivic Euler sums and motivic multiple $\tilde{t}$-values, we have

$$(-1)^d2^{d-|\mathbf{k}|}\tilde{t}^{\mathfrak{m}}(k_1,\dots,k_d)=\sum_{\eta_m\in\{\pm1\}}\eta_1 I^{\mathfrak{m}}(0;\eta_1,0^{k_1-1},\eta_2,0^{k_2-1},\dots,\eta_d,0^{k_d-1};1).$$

By applying D_r, we have

$$
\begin{aligned}
&(-1)^d 2^{d-|\mathbf{k}|} D_r\left(\tilde{t}^{\mathfrak{m}}(k_1,\dots,k_d)\right)\\
=&\sum_{\eta_m\in\{\pm1\}} \eta_1 D_r\left(I^{\mathfrak{m}}(0;\eta_1,0^{k_1-1},\eta_2,0^{k_2-1},\dots,\eta_d,0^{k_d-1};1)\right)\\
=&\sum_{1\le j\le d}\delta_{|\mathbf{k}_{1,j}|=r}\sum_{\eta_m\in\{\pm1\}}\eta_1\Bigg(I^{\mathfrak{l}}(0;\eta_1,0^{k_1-1},\dots,\eta_j,0^{k_j-1};\eta_{j+1}) \qquad (36)\\
&\qquad\otimes I^{\mathfrak{m}}(0;\eta_{j+1},0^{k_{j+1}-1},\dots,\eta_d,0^{k_d-1};1)\Bigg)\\
&+\sum_{1\le i<j\le d}\delta_{|\mathbf{k}_{i+1,j}|\le r<|\mathbf{k}_{i,j}|-1}\sum_{\eta_m\in\{\pm1\}}\eta_1\Bigg(I^{\mathfrak{l}}(0;0^{r-|\mathbf{k}_{i+1,j}|},\eta_{i+1},0^{k_{i+1}-1},\dots,\eta_j,0^{k_j-1};\eta_{j+1}) \qquad (37)\\
&\qquad\otimes I^{\mathfrak{m}}(0;\eta_1,0^{k_1-1},\dots,\eta_i,0^{|\mathbf{k}_{i,j}|-r-1},\eta_{j+1},0^{k_{j+1}-1},\dots,\eta_d,0^{k_d-1};1)\Bigg)\\
&+\sum_{1\le i<j\le d}\delta_{|\mathbf{k}_{i,j-1}|\le r<|\mathbf{k}_{i,j}|-1}\sum_{\eta_m\in\{\pm1\}}\eta_1\Bigg(I^{\mathfrak{l}}(\eta_i;0^{k_i-1},\eta_{i+1},0^{k_{i+1}-1},\dots,\eta_j,0^{r-|\mathbf{k}_{i,j-1}|};0) \qquad (38)\\
&\qquad\otimes I^{\mathfrak{m}}(0;\eta_1,0^{k_1-1},\dots,\eta_i,0^{|\mathbf{k}_{i,j}|-r-1},\eta_{j+1},0^{k_{j+1}-1},\dots,\eta_d,0^{k_d-1};1)\Bigg)\\
&+\sum_{1\le i<j\le d}\delta_{|\mathbf{k}_{i,j}|-1=r}\sum_{\eta_m\in\{\pm1\}}\eta_1\Bigg(I^{\mathfrak{l}}(\eta_i;0^{k_i-1},\eta_{i+1},0^{k_{i+1}-1},\dots,\eta_j,0^{k_j-1};\eta_{j+1}) \qquad (39)\\
&\qquad\otimes I^{\mathfrak{m}}(0;\eta_1,0^{k_1-1},\dots,\eta_i,\eta_{j+1},0^{k_{j+1}-1},\dots,\eta_d,0^{k_d-1};1)\Bigg)
\end{aligned}
$$

First, we calculate the sum (36).

$$
\begin{aligned}
(36)=&\sum_{1\le j\le d}\delta_{|\mathbf{k}_{1,j}|=r}\sum_{\eta_{j+1},\dots,\eta_d\in\{\pm1\}}\left(\sum_{\eta_1,\dots,\eta_j\in\{\pm1\}}\eta_1\eta_{j+1}I^{\mathfrak{l}}(0;\eta_1\eta_{j+1},0^{k_1-1},\dots,\eta_j\eta_{j+1},0^{k_j-1};1)\right)\\
&\qquad\otimes\eta_{j+1}I^{\mathfrak{m}}(0;\eta_{j+1},0^{k_{j+1}-1},\dots,\eta_d,0^{k_d-1};1)\\
=&\sum_{1\le j\le d}\delta_{|\mathbf{k}_{1,j}|=r}\left(\sum_{\eta_1,\dots,\eta_j\in\{\pm1\}}\eta_1 I^{\mathfrak{l}}(0;\eta_1,0^{k_1-1},\dots,\eta_j,0^{k_j-1};1)\right)\\
&\qquad\otimes\left(\sum_{\eta_{j+1},\dots,\eta_d\in\{\pm1\}}\eta_{j+1}I^{\mathfrak{m}}(0;\eta_{j+1},0^{k_{j+1}-1},\dots,\eta_d,0^{k_d-1};1)\right)\\
=&(-1)^d 2^{d-|\mathbf{k}|}\sum_{1\le j\le d}\delta_{|\mathbf{k}_{1,j}|=r}\tilde{t}^{\mathfrak{m}}(k_1,\dots,k_j)\otimes\tilde{t}^{\mathfrak{m}}\left(k_{j+1},\dots,k_d\right).
\end{aligned}
$$

The last equality follows from the induction hypothesis. Next, we calculate the sum (37).

$$
\begin{aligned}
(37) =& \sum_{1\le i<j\le d} \delta_{|\mathbf{k}_{i+1,j}|\le r<|\mathbf{k}_{i,j}|-1} \sum_{\substack{\eta_1,\dots,\eta_i\\ \eta_{j+1},\dots,\eta_d\in\{\pm1\}}} \left(\sum_{\eta_{i+1},\dots,\eta_j\in\{\pm1\}} I^{\mathfrak{l}}(0;0^{r-|\mathbf{k}_{i+1,j}|},\eta_{i+1},0^{k_{i+1}-1},\dots,\eta_j,0^{k_j-1};\eta_{j+1}) \right)\\
&\otimes \eta_1 I^{\mathfrak{m}}(0;\eta_1,0^{k_1-1},\dots,\eta_i,0^{|\mathbf{k}_{i,j}|-r-1},\eta_{j+1},0^{k_{j+1}-1},\dots,\eta_d,0^{k_d-1};1)\\
=&(-1)^d 2^{d-|\mathbf{k}|} \sum_{1\le i<j\le d} \delta_{|\mathbf{k}_{i+1,j}|\le r<|\mathbf{k}_{i,j}|-1} \zeta^{\mathfrak{l}}_{r-|\mathbf{k}_{i+1,j}|}(k_{i+1},\dots,k_j)\otimes \tilde{t}^{\mathfrak{m}}\left(k_1,\dots,k_{i-1},|\mathbf{k}_{i,j}|-r,k_{j+1},\dots,k_d\right).
\end{aligned}
$$

The last equality follows from Proposition 18.3 and the induction hypothesis. The sum (38) is calculated in a similar way.

$$
\begin{aligned}
(38) =& \sum_{1\le i<j\le d} \delta_{|\mathbf{k}_{i,j-1}|\le r<|\mathbf{k}_{i,j}|-1} \sum_{\substack{\eta_1,\dots,\eta_i\\ \eta_{j+1},\dots,\eta_d\in\{\pm1\}}} \left(\sum_{\eta_{i+1},\dots,\eta_j\in\{\pm1\}} I^{\mathfrak{l}}(\eta_i;0^{k_i-1},\eta_{i+1},0^{k_{i+1}-1},\dots,\eta_j,0^{r-|\mathbf{k}_{i,j-1}|};0) \right)\\
&\otimes \eta_1 I^{\mathfrak{m}}(0;\eta_1,0^{k_1-1},\dots,\eta_i,0^{|\mathbf{k}_{i,j}|-r-1},\eta_{j+1},0^{k_{j+1}-1},\dots,\eta_d,0^{k_d-1};1)\\
=&-(-1)^d 2^{d-|\mathbf{k}|} \sum_{1\le i<j\le d} \delta_{|\mathbf{k}_{i,j-1}|\le r<|\mathbf{k}_{i,j}|-1} \zeta^{\mathfrak{l}}_{r-|\mathbf{k}_{i,j-1}|}(k_{j-1},\dots,k_i)\otimes \tilde{t}^{\mathfrak{m}}\left(k_1,\dots,k_{i-1},|\mathbf{k}_{i,j}|-r,k_{j+1},\dots,k_d\right).
\end{aligned}
$$

Finally, we calculate the sum (39).

$$
\begin{aligned}
(39) =& \sum_{1\le i<j\le d} \delta_{|\mathbf{k}_{i,j}|-1=r} \sum_{\substack{\eta_1,\dots,\eta_i\\ \eta_{j+1},\dots,\eta_d\in\{\pm1\}}} \left(\sum_{\eta_{i+1},\dots,\eta_j\in\{\pm1\}} I^{\mathfrak{l}}(\eta_i;0^{k_i-1},\eta_{i+1},0^{k_{i+1}-1},\dots,\eta_j,0^{k_j-1};\eta_{j+1}) \right)\\
&\otimes \eta_1 I^{\mathfrak{m}}(0;\eta_1,0^{k_1-1},\dots,\eta_i,\eta_{j+1},0^{k_{j+1}-1},\dots,\eta_d,0^{k_d-1};1)\\
=&0. \qquad \text{(by Lemma 18.2)}
\end{aligned}
$$

By the above calculation, the proposition is proved. □

Proof of Theorem 18.1. Let $\mathbf{k}=(k_1,\dots,k_d)\in(\mathbb{Z}_{\ge2})^d$. If $r=1$, there are no terms on the right-hand side of Proposition 18.4. Hence $D_1\left(\tilde{t}^{\mathfrak{m}}(\mathbf{k})\right)=0$. For $r\ge3$: odd, we have $D_r\left(\tilde{t}^{\mathfrak{m}}(\mathbf{k})\right)\in\mathcal{L}_r\otimes\mathcal{H}^1$ by induction on weight. Thus by Theorem 17.2, we obtain Theorem 18.1. □

19 Evaluation of $\tilde{t}(2,\ldots,2,3,2,\ldots,2)$

Let $G(x,y)$ and $\hat{G}(x,y)$ be the generating functions

$$G(x,y) := \sum_{a,b\geq 0} (-1)^{a+b} K(a,b) x^{2a+1} y^{2b+1},$$
$$\hat{G}(x,y) := \sum_{a,b\geq 0} (-1)^{a+b} \hat{K}(a,b) x^{2a+1} y^{2b+1},$$

where $\hat{K}(a,b)$ denotes the right-hand side of (29).

Proposition 19.1. *We have*

$$G(x,y) = \frac{xy}{\frac{1}{4}-x^2} \cos \pi y \sum_{m=1}^{\infty} \frac{1}{m-\frac{1}{2}} \cdot \frac{\left(-\frac{1}{2}+x\right)_m \left(-\frac{1}{2}-x\right)_m}{\left(\frac{1}{2}+y\right)_m \left(\frac{1}{2}-y\right)_m},$$

where $(a)_m := a(a+1)\cdots(a+m-1)$ is the ascending Pochhammer symbol.

Proof. By a straightforward calculation,

$$\begin{aligned}
G(x,y) &= xy \sum_{m=1}^{\infty} \prod_{0<k<m} \left(1 - \frac{x^2}{(k-\frac{1}{2})^2}\right) \cdot \frac{1}{(m-\frac{1}{2})^3} \cdot \prod_{l>m} \left(1 - \frac{y^2}{(l-\frac{1}{2})^2}\right) \\
&= xy \cos \pi y \sum_{m=1}^{\infty} \frac{\left(1-\frac{x^2}{(\frac{1}{2})^2}\right)\left(1-\frac{x^2}{(\frac{3}{2})^2}\right)\cdots\left(1-\frac{x^2}{(\frac{2m-3}{2})^2}\right)}{\left(1-\frac{y^2}{(\frac{1}{2})^2}\right)\left(1-\frac{y^2}{(\frac{3}{2})^2}\right)\cdots\left(1-\frac{y^2}{(\frac{2m-1}{2})^2}\right)} \cdot \frac{1}{(m-\frac{1}{2})^3} \\
&= xy \cos \pi y \sum_{m=1}^{\infty} \frac{1}{m-\frac{1}{2}} \cdot \frac{\left((\frac{1}{2})^2-x^2\right)\left((\frac{3}{2})^2-x^2\right)\cdots\left((\frac{2m-3}{2})^2-x^2\right)}{\left((\frac{1}{2})^2-y^2\right)\left((\frac{3}{2})^2-y^2\right)\cdots\left((\frac{2m-1}{2})^2-y^2\right)} \\
&= \frac{xy}{(-\frac{1}{2})^2-x^2} \cos \pi y \sum_{m=1}^{\infty} \frac{1}{m-\frac{1}{2}} \cdot \frac{\left(-\frac{1}{2}+x\right)_m \left(-\frac{1}{2}-x\right)_m}{\left(\frac{1}{2}+y\right)_m \left(\frac{1}{2}-y\right)_m}.
\end{aligned}$$

□

Proposition 19.2. *We have*

$$\hat{G}(x,y) = \cos \pi y \left[A(x+y) - A(x-y)\right] + \cos \pi x \left[B(x+y) - B(x-y)\right],$$

where $A(z)$ and $B(z)$ denote the power series

$$A(z) := \sum_{r=1}^{\infty} \zeta(2r+1) z^{2r}, \quad B(z) := \sum_{r=1}^{\infty} (1-2^{-2r}) \zeta(2r+1) z^{2r}. \tag{40}$$

Proof. This also follows from a straightforward calculation,

$$
\begin{aligned}
\hat{G}(x,y) &= 2\sum_{a,b\geq 0}\sum_{r=1}^{a+b+1}(-1)^{a+b-r+1}\left[\tilde{A}^r_{a,b}+B^r_{a,b}\right]K(a+b-r+1)\zeta(2r+1)x^{2a+1}y^{2b+1}\\
&= 2\sum_{r\geq 1,s\geq 0}(-1)^s\left[\sum_{a=0}^{r-1}\binom{2r}{2a+1}K(s)\zeta(2r+1)x^{2a+1}y^{2r+2s-2a-1}\right.\\
&\qquad\qquad\left.+\sum_{b=0}^{r-1}(1-2^{-2r})\binom{2r}{2b+1}K(s)\zeta(2r+1)x^{2r+2s-2b-1}y^{2b+1}\right]\\
&= 2\sum_{s\geq 0}(-1)^sK(s)y^{2s}\sum_{r\geq 1}\zeta(2r+1)\sum_{a=0}^{r-1}\binom{2r}{2a+1}x^{2a+1}y^{2r-2a-1}\\
&\quad+2\sum_{s\geq 0}(-1)^sK(s)x^{2s}\sum_{r\geq 1}(1-2^{-2r})\zeta(2r+1)\sum_{b=0}^{r-1}\binom{2r}{2b+1}x^{2r-2b-1}y^{2b+1}\\
&= 2\cos\pi y\sum_{r\geq 1}\zeta(2r+1)\cdot\frac{1}{2}\left[(x+y)^{2r}-(x-y)^{2r}\right]\\
&\quad+2\cos\pi x\sum_{r\geq 1}(1-2^{-2r})\zeta(2r+1)\cdot\frac{1}{2}\left[(x+y)^{2r}-(x-y)^{2r}\right]\\
&= \cos\pi y\left[\sum_{r\geq 1}\zeta(2r+1)(x+y)^{2r}-\sum_{r\geq 1}\zeta(2r+1)(x-y)^{2r}\right]\\
&\quad+\cos\pi x\left[\sum_{r\geq 1}(1-2^{-2r})\zeta(2r+1)(x+y)^{2r}-\sum_{r\geq 1}(1-2^{-2r})\zeta(2r+1)(x-y)^{2r}\right]\\
&= \cos\pi y\left[A(x+y)-A(x-y)\right]+\cos\pi x\left[B(x+y)-B(x-y)\right].
\end{aligned}
$$

□

For later use, we mention that the two series in (40) can be written as

$$
A(z)=\sum_{n=1}^{\infty}\frac{z^2}{n(n^2-z^2)},\qquad B(z)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}z^2}{n(n^2-z^2)},
$$

and also by using digamma function $\psi(x)=\Gamma'(x)/\Gamma(x)$ as

$$
\begin{aligned}
A(z) &= \psi(1)-\frac{1}{2}\left(\psi(1+z)+\psi(1-z)\right),\\
B(z) &= A(z)-A(z/2).
\end{aligned}
$$

Proposition 19.3. *Both $G(x,y)$ and $\hat{G}(x,y)$ are entire functions on $\mathbb{C}\times\mathbb{C}$ and are bounded by constant multiples of $e^{\pi X}\log X$ as $X\to\infty$, where $X=max\{|x|,|y|\}$, and also by multiples (depending on x) of $e^{\pi|\Im(y)|}$ as $|y|\to\infty$ with $x\in\mathbb{C}$ fixed.*

Proof. By the definition of $K(a,b)$, it holds the estimate

$$0 < K(a,b) < \frac{1}{a+\frac{1}{2}} K(n) = \frac{1}{a+\frac{1}{2}} \frac{\pi^{2n}}{(2n)!} \quad (a+b+1=n). \tag{41}$$

The last equality follows from

$$t(\underbrace{2,\ldots,2}_{n}) = \frac{\pi^{2n}}{2^{2n}(2n)!}, \tag{42}$$

which was proved by Hoffman [11]. By (41) and the definition of $G(x,y)$, we have

$$\max_{|x|,|y|\le M} |G(x,y)| < \sum_{n=1}^{\infty} \sum_{a=0}^{n-1} \frac{1}{a+\frac{1}{2}} \frac{\pi^{2n} M^{2n}}{(2n)!} = O\left(e^{\pi M} \log M\right).$$

Therefore, we obtain

$$\max_{|x|,|y|\le M} |G(x,y)| = O\left(e^{\pi M} \log M\right). \tag{43}$$

We next show the estimate

$$|G(x,y)| = O_x\left(e^{\pi |\Im(y)|}\right). \tag{44}$$

Here, the notation $O_x(\bullet)$ means $O(\bullet)$ for fixed x. For $|\Im(y)| \ge 2$, by $\left|\left(\frac{1}{2}+y\right)_m \left(\frac{1}{2}-y\right)_m\right| > \left\{\left(\frac{1}{2}\right)_m\right\}^2$ for large m, and $\left(-\frac{1}{2}+x\right)_m \left(-\frac{1}{2}-x\right)_m = O_x\left(\left\{\left(\frac{1}{2}\right)_{m-1}\right\}^2\right)$,

$$\begin{aligned}
|G(x,y)| &< \left|\frac{x}{\frac{1}{4}-x^2}\right| |\cos \pi y| \sum_{m=1}^{\infty} \frac{1}{m-\frac{1}{2}} \cdot \frac{|y| \left|\left(-\frac{1}{2}+x\right)_m \left(-\frac{1}{2}-x\right)_m\right|}{\left|\left(\frac{1}{2}+y\right)_m \left(\frac{1}{2}-y\right)_m\right|} \\
&< \left|\frac{x}{\frac{1}{4}-x^2}\right| |\cos \pi y| \sum_{m=1}^{\infty} \frac{1}{m-\frac{1}{2}} \cdot \frac{\left(m-\frac{1}{2}\right)\left\{\left(\frac{1}{2}\right)_{m-1}\right\}^2}{\left\{\left(\frac{1}{2}\right)_m\right\}^2} \\
&= \left|\frac{x}{\frac{1}{4}-x^2}\right| |\cos \pi y| \, \tilde{t}(2) \\
&= O_x\left(e^{\pi |\Im(y)|}\right).
\end{aligned}$$

Then, we obtain

$$|G(x,y)| = O_x\left(e^{\pi |\Im(y)|}\right) \quad \text{for } |\Im(y)| \ge 2. \tag{45}$$

By (43), (45) and the Phragmén-Lindelöf theorem, we obtain the estimate (44).

For $\hat{G}(x,y)$, we have

$$\max_{|x|,|y|\le M} \left|\hat{G}(x,y)\right| = O\left(e^{\pi M} \log M\right)$$

and

$$\left|\hat{G}(x,y)\right| = O_x\left(e^{\pi|\Im(y)|}\right),$$

from the estimate $\psi(x) = O(\log x) + O(1/\mathrm{dist}(x,\mathbb{Z}))$, where $\mathrm{dist}(x,\mathbb{Z})$ denotes the distance between x and it's nearlest integer. □

Proposition 19.4. *For $x \in \mathbb{C}$, we have*

$$G(x,x) = \hat{G}(x,x).$$

Proof. By a straightforward calculation,

$$\begin{aligned}
G(x,x) &= \frac{x^2}{\frac{1}{4}-x^2}\cos\pi x\sum_{m=1}^{\infty}\frac{1}{m-\frac{1}{2}}\cdot\frac{\left(-\frac{1}{2}+x\right)_m\left(-\frac{1}{2}-x\right)_m}{\left(\frac{1}{2}+x\right)_m\left(\frac{1}{2}-x\right)_m}\\
&= \frac{x^2}{\frac{1}{4}-x^2}\cos\pi x\sum_{m=1}^{\infty}\frac{1}{m-\frac{1}{2}}\cdot\frac{\left(-\frac{1}{2}+x\right)\left(-\frac{1}{2}-x\right)}{\left(x+m-\frac{1}{2}\right)\left(-x+m-\frac{1}{2}\right)}\\
&= \cos\pi x\sum_{m=1}^{\infty}\frac{1}{m-\frac{1}{2}}\cdot\frac{x^2}{\left(x+m-\frac{1}{2}\right)\left(-x+m-\frac{1}{2}\right)}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\hat{G}(x,x) &= \cos\pi x\,[A(2x)-A(0)] + \cos\pi x\,[B(2x)-B(0)]\\
&= \cos\pi x\,[A(2x)+B(2x)] \qquad \text{(by } A(0)=B(0)=0\text{)}\\
&= \cos\pi x\left[\sum_{n=1}^{\infty}\frac{(2x)^2}{n\left(n^2-(2x)^2\right)} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2x)^2}{n\left(n^2-(2x)^2\right)}\right]\\
&= \cos\pi x\left[2\sum_{\substack{n=1\\ n:odd}}^{\infty}\frac{(2x)^2}{n\left(n^2-(2x)^2\right)}\right]\\
&= \cos\pi x\left[2\sum_{m=1}^{\infty}\frac{(2x)^2}{(2m-1)\left((2m-1)^2-(2x)^2\right)}\right].
\end{aligned}$$

Therefore, we have the proposition. □

Proposition 19.5. *For $k \in \mathbb{N}$ and $x \in \mathbb{C}$, we have*

$$G(x,k) = \hat{G}(x,k).$$

Proof. We calculate $\hat{G}(x,k)$ first. By Proposition 19.2,

$$\hat{G}(x,k) = (-1)^k\left[A(x+k)-A(x-k)\right] + \cos\pi x\left[B(x+k)-B(x-k)\right].$$

By $A(z+1)-A(z)=-\frac{1}{2}\left(\frac{1}{z+1}+\frac{1}{z}\right)$,

$$\begin{aligned}
&A(x+k)-A(x-k)\\
=&(A(x+k)-A(x+k-1))+(A(x+k-1)-A(x+k-2))+\cdots+(A(x-k+1)-A(x-k))\\
=&-\frac{1}{2}\left(\frac{1}{x+k}+\frac{1}{x+k-1}\right)-\frac{1}{2}\left(\frac{1}{x+k-1}+\frac{1}{x+k-2}\right)-\cdots-\frac{1}{2}\left(\frac{1}{x-k+1}+\frac{1}{x-k}\right)\\
=&-\left(\frac{1}{2}\cdot\frac{1}{x+k}+\frac{1}{x+k-1}+\frac{1}{x+k-2}+\cdots+\frac{1}{x-k+1}+\frac{1}{2}\cdot\frac{1}{x-k}\right)\\
=&-\sum_{|j|\le k}^{*}\frac{1}{x-j}.
\end{aligned}$$

Here, the notation $\displaystyle\sum_{|j|\le k}^{*}$ means $\displaystyle\sum_{|j|<k}+\frac{1}{2}\sum_{|j|=k}$.

Since

$$\begin{aligned}
B(x+1)-B(x-1)&=A(x+1)-A(x-1)-\left[A\left(\frac{x+1}{2}\right)-A\left(\frac{x-1}{2}\right)\right]\\
&=[A(x+1)-A(x)]+[A(x)-A(x-1)]-\left[A\left(\frac{x+1}{2}\right)-A\left(\frac{x-1}{2}\right)\right]\\
&=-\frac{1}{2}\left(\frac{1}{x+1}+\frac{1}{x}\right)-\frac{1}{2}\left(\frac{1}{x}+\frac{1}{x-1}\right)+\frac{1}{2}\left(\frac{2}{x+1}+\frac{2}{x-1}\right)\\
&=\frac{1}{2}\left(\frac{1}{x+1}-\frac{2}{x}+\frac{1}{x-1}\right),
\end{aligned}$$

we have

$$\begin{aligned}
&B(x+k)-B(x-k)\\
=&(B(x+k)-B(x+k-2))+(B(x+k-2)-B(x+k-4))+\cdots+(B(x-k+2)-B(x-k))\\
=&\frac{1}{2}\cdot\frac{1}{x+k}-\frac{1}{x+k-1}+\frac{1}{x+k-2}-\frac{1}{x+k-3}+\cdots-\frac{1}{x-k+1}+\frac{1}{2}\cdot\frac{1}{x-k}\\
=&\sum_{|j|\le k}^{*}\frac{(-1)^{k-j}}{x-j}.
\end{aligned}$$

Therefore,

$$(-1)^k\hat{G}(x,k)=-\sum_{|j|\le k}^{*}\frac{1}{x-j}+\cos\pi x\sum_{|j|\le k}^{*}\frac{(-1)^j}{x-j}.$$

Here,

$$\sum_{|j|\le k}^{*}\frac{1}{x-j}=\frac{1}{x}+\sum_{1\le j\le k}^{*}\left(\frac{1}{x-j}+\frac{1}{x+j}\right)=\frac{1}{x}-\sum_{l=1}^{\infty}\left(\sum_{\substack{|j|\le k\\ j\ne 0}}^{*}\frac{1}{j^{2l}}\right)x^{2l-1},$$

$$\sum_{|j|\le k}{}^{*}\,\frac{(-1)^j}{x-j}=\frac{1}{x}+\sum_{1\le j\le k}{}^{*}\,(-1)^j\left(\frac{1}{x-j}+\frac{1}{x+j}\right)=\frac{1}{x}-\sum_{l=1}^{\infty}\left(\sum_{\substack{|j|\le k\\ j\neq 0}}{}^{*}\,\frac{(-1)^j}{j^{2l}}\right)x^{2l-1}.$$

Therefore, we obtain

$$\begin{aligned}
&(-1)^k\hat{G}(x,k)\\
&=-\sum_{|j|\le k}{}^{*}\,\frac{1}{x-j}+\cos\pi x\sum_{|j|\le k}{}^{*}\,\frac{(-1)^j}{x-j}\\
&=-\left(\frac{1}{x}-\sum_{l=1}^{\infty}\left(\sum_{\substack{|j|\le k\\ j\neq 0}}{}^{*}\,\frac{1}{j^{2l}}\right)x^{2l-1}\right)+\left(\sum_{l=0}^{\infty}(-1)^l\frac{\pi^{2l}}{(2l)!}x^{2l}\right)\left(\frac{1}{x}-\sum_{l=1}^{\infty}\left(\sum_{\substack{|j|\le k\\ j\neq 0}}{}^{*}\,\frac{(-1)^j}{j^{2l}}\right)x^{2l-1}\right)\\
&=\sum_{l=1}^{\infty}\left(s_k^*(l)-\sum_{m=0}^{l-1}(-1)^m a_k^*(l-m)\frac{\pi^{2m}}{(2m)!}+(-1)^l\frac{\pi^{2l}}{(2l)!}\right)x^{2l-1}\\
&=\sum_{l=1}^{\infty}\left(s_k^*(l)-\sum_{m=0}^{l}(-1)^m a_k^*(l-m)\frac{\pi^{2m}}{(2m)!}\right)x^{2l-1},
\end{aligned}$$

where

$$s_k^*(l):=\sum_{\substack{|j|\le k\\ j\neq 0}}{}^{*}\,\frac{1}{j^{2l}},\quad a_k^*(l):=\sum_{\substack{|j|\le k\\ j\neq 0}}{}^{*}\,\frac{(-1)^j}{j^{2l}}.$$

Let $\hat{\gamma}_k(l)$ be the coefficient of x^{2l-1} in $(-1)^k\hat{G}(x,k)$, i.e.,

$$\hat{\gamma}_k(l):=s_k^*(l)-\sum_{m=0}^{l}(-1)^m a_k^*(l-m)\frac{\pi^{2m}}{(2m)!}.$$

Put

$$\hat{\beta}_k(l):=\frac{2}{k^{2l}}-2(-1)^k\sum_{m=0}^{l}\frac{(-1)^m}{k^{2(l-m)}}\cdot\frac{\pi^{2m}}{(2m)!},$$

then it is clear that

$$\hat{\gamma}_k(l)=\sum_{1\le j\le k}{}^{*}\,\hat{\beta}_j(l)$$

and

$$\hat{\beta}_k(l)-\frac{1}{k^2}\hat{\beta}_k(l-1)=2(-1)^{k+l-1}\frac{\pi^{2l}}{(2l)!},\quad \hat{\beta}_k(0)=2\left(1-(-1)^k\right).$$

We calculate $G(x,k)$ next.

$$\begin{aligned}
G(x,k) =& \frac{xk}{\frac{1}{4}-x^2}(-1)^k \sum_{m=1}^{\infty} \frac{1}{m-\frac{1}{2}} \cdot \frac{\left(-\frac{1}{2}+x\right)_m \left(-\frac{1}{2}-x\right)_m}{\left(\frac{1}{2}+k\right)_m \left(\frac{1}{2}-k\right)_m} \\
=& x \sum_{m=1}^{\infty} \frac{k}{m-\frac{1}{2}} \cdot \frac{\left(\left(\frac{1}{2}\right)^2-x^2\right)\left(\left(\frac{3}{2}\right)^2-x^2\right)\cdots\left(\left(m-\frac{3}{2}\right)^2-x^2\right)}{\left(\frac{1}{2}\right)_{m+k}\left(\frac{1}{2}\right)_{m-k}} \\
=& x \sum_{m=1}^{\infty} \frac{k}{m-\frac{1}{2}} \cdot \frac{\left(m-k+\frac{1}{2}\right)\cdots\left(m-\frac{3}{2}\right)}{\left(m-\frac{1}{2}\right)\cdots\left(m+k-\frac{1}{2}\right)} \left(1-\frac{x^2}{\left(\frac{1}{2}\right)^2}\right)\left(1-\frac{x^2}{\left(\frac{3}{2}\right)^2}\right)\cdots\left(1-\frac{x^2}{\left(m-\frac{3}{2}\right)^2}\right).
\end{aligned}$$

Let $\gamma_k(l)$ be the coefficient of x^{2l-1} in $(-1)^k G(x,k)$, we have

$$\begin{aligned}
\gamma_k(l) =& (-1)^k \sum_{m=1}^{\infty} \frac{k}{m-\frac{1}{2}} \cdot \frac{\left(m-k+\frac{1}{2}\right)\cdots\left(m-\frac{3}{2}\right)}{\left(m-\frac{1}{2}\right)\left(m+\frac{1}{2}\right)\cdots\left(m+k-\frac{1}{2}\right)} \\
& \times \sum_{0<i_1<\cdots<i_{l-1}<m} (-1)^{l-1} \frac{1}{(i_1-\frac{1}{2})^2\cdots(i_{l-1}-\frac{1}{2})^2} \\
=& (-1)^{k+l-1} k \sum_{0<i_1<\cdots<i_{l-1}} \frac{1}{(i_1-\frac{1}{2})^2\cdots(i_{l-1}-\frac{1}{2})^2} \\
& \times \sum_{i_{l-1}<m} \frac{\left(m-k+\frac{1}{2}\right)\cdots\left(m-\frac{3}{2}\right)}{\left(m-\frac{1}{2}\right)^2\left(m+\frac{1}{2}\right)\cdots\left(m+k-\frac{1}{2}\right)}.
\end{aligned}$$

By using the partial fraction expansion

$$\frac{(m-\frac{3}{2})(m-\frac{5}{2})\cdots(m-k+\frac{1}{2})}{(m+\frac{1}{2})(m+\frac{3}{2})\cdots(m+k-\frac{1}{2})} = \sum_{i=1}^{k} (-1)^{k-i} \binom{i+k-1}{i-1}\binom{k}{i} \cdot \frac{1}{m+i-\frac{1}{2}},$$

$$\begin{aligned}
\gamma_k(l) =& (-1)^{k+l-1} k \sum_{0<i_1<\cdots<i_{l-1}} \frac{1}{(i_1-\frac{1}{2})^2\cdots(i_{l-1}-\frac{1}{2})^2} \\
& \times \sum_{i_{l-1}<m} \sum_{i=1}^{k} (-1)^{k-i} \binom{i+k-1}{i-1}\binom{k}{i} \cdot \frac{1}{(m-\frac{1}{2})^2(m+i-\frac{1}{2})} \\
=& (-1)^l k \sum_{i=1}^{k} (-1)^{i-1} \binom{i+k-1}{i-1}\binom{k}{i} \alpha_i(l)
\end{aligned}$$

where

$$\alpha_u(l) := \sum_{0<i_1<\cdots<i_{l-1}<i_l} \frac{1}{(i_1-\frac{1}{2})^2\cdots(i_{l-1}-\frac{1}{2})^2(i_l-\frac{1}{2})^2\cdot(i_l-\frac{1}{2}+u)}.$$

When $l=0$, we understand $\alpha_u(0)=\frac{1}{u-\frac{1}{2}}$.

The sum $\alpha_u(l)$ satisfies the following recurrence relation.

Lemma 19.6. *For $u, l \in \mathbb{Z}_{\geq 1}$, it holds that*

$$\alpha_u(l) = \frac{1}{u} \cdot \frac{\pi^{2l}}{(2l)!} - \frac{1}{u^2} \sum_{u'=1}^{u} \alpha_{u'}(l-1).$$

Proof.

$$\begin{aligned}
\sum_{i>a} \frac{1}{(i-\frac{1}{2})^2(i-\frac{1}{2}+u)} &= \frac{1}{u} \sum_{i>a} \frac{1}{(i-\frac{1}{2})^2} - \frac{1}{u^2} \sum_{i>a} \left(\frac{1}{i-\frac{1}{2}} - \frac{1}{i-\frac{1}{2}+u} \right) \\
&= \frac{1}{u} \sum_{i>a} \frac{1}{(i-\frac{1}{2})^2} - \frac{1}{u^2} \left(\frac{1}{a+1-\frac{1}{2}} + \cdots + \frac{1}{a+u-\frac{1}{2}} \right) \\
&= \frac{1}{u} \sum_{i>a} \frac{1}{(i-\frac{1}{2})^2} - \frac{1}{u^2} \sum_{u'=1}^{u} \frac{1}{a-\frac{1}{2}+u'}.
\end{aligned}$$

By setting $a = 0$ and (42), we obtain the lemma for $l = 1$. For $l \geq 2$, the sum $\alpha_u(l)$ is calculated as follows.

$$\begin{aligned}
\alpha_u(l) &= \sum_{0<i_1<\cdots<i_{l-1}} \frac{1}{(i_1-\frac{1}{2})^2 \cdots (i_{l-1}-\frac{1}{2})^2} \sum_{i_{l-1}<i} \frac{1}{(i-\frac{1}{2})^2(i-\frac{1}{2}+u)} \\
&= \sum_{0<i_1<\cdots<i_{l-1}} \frac{1}{(i_1-\frac{1}{2})^2 \cdots (i_{l-1}-\frac{1}{2})^2} \left(\frac{1}{u} \sum_{i>i_{l-1}} \frac{1}{(i-\frac{1}{2})^2} - \frac{1}{u^2} \sum_{u'=1}^{u} \frac{1}{i_{l-1}-\frac{1}{2}+u'} \right) \\
&= \frac{1}{u} \sum_{0<i_1<\cdots<i_l} \frac{1}{(i_1-\frac{1}{2})^2 \cdots (i_l-\frac{1}{2})^2} - \frac{1}{u^2} \sum_{u'=1}^{u} \sum_{0<i_1<\cdots<i_{l-1}} \frac{1}{(i_1-\frac{1}{2})^2 \cdots (i_{l-1}-\frac{1}{2})^2 (i_{l-1}-\frac{1}{2}+u')} \\
&= \frac{1}{u} \frac{\pi^{2l}}{(2l)!} - \frac{1}{u^2} \sum_{u'=1}^{u} \alpha_{u'}(l-1). \qquad \text{(by (42))}
\end{aligned}$$

Thus the lemma is proved. □

Put

$$\beta_k(l) := 2(-1)^l k \sum_{i=1}^{k} (-1)^{i-1} \binom{k+i-1}{i-1} \binom{k-1}{i-1} \alpha_i(l),$$

then we get $\sum_{j=1}^{k}{}^{*}\ \beta_j(l) = \gamma_k(l)$. In fact,

$$\begin{aligned}
\sum_{j=1}^{k}{}^{*}\ \beta_j(l) &= 2\sum_{j=1}^{k}{}^{*}\ (-1)^l j \sum_{i=1}^{j}(-1)^{i-1}\binom{j+i-1}{i-1}\binom{j-1}{i-1}\alpha_i(l) \\
&= 2\sum_{j=1}^{k}{}^{*}\ \sum_{i=1}^{j}(-1)^l(-1)^{i-1}j\binom{j+i-1}{i-1}\binom{j-1}{i-1}\alpha_i(l) \\
&= 2\sum_{j=1}^{k}\sum_{i=1}^{j}(-1)^l(-1)^{i-1}j\binom{j+i-1}{i-1}\binom{j-1}{i-1}\alpha_i(l) \\
&\qquad -\sum_{i=1}^{k}(-1)^l(-1)^{i-1}k\binom{k+i-1}{i-1}\binom{k-1}{i-1}\alpha_i(l) \\
&= 2\sum_{i=1}^{k}\sum_{j=i}^{k}(-1)^l(-1)^{i-1}j\binom{j+i-1}{i-1}\binom{j-1}{i-1}\alpha_i(l) \\
&\qquad -\sum_{i=1}^{k}(-1)^l(-1)^{i-1}k\binom{k+i-1}{i-1}\binom{k-1}{i-1}\alpha_i(l) \\
&= \sum_{i=1}^{k}(-1)^{l+i-1}\left(2\sum_{j=i}^{k}j\binom{j+i-1}{i-1}\binom{j-1}{i-1} - k\binom{k+i-1}{i-1}\binom{k-1}{i-1}\right)\alpha_i(l) \\
&= \sum_{i=1}^{k}(-1)^{l+i-1}i\left(2\sum_{j=i}^{k}\binom{j+i-1}{i-1}\binom{j}{i} - \binom{k+i-1}{i-1}\binom{k}{i}\right)\alpha_i(l).
\end{aligned}$$

By using the equation

$$\sum_{j=i}^{k}\binom{j+i-1}{i-1}\binom{j}{i} = \frac{1}{2}\binom{k+i}{i}\binom{k}{i},$$

we obtain

$$\begin{aligned}
\sum_{j=1}^{k}{}^{*}\ \beta_j(l) &= \sum_{i=1}^{k}(-1)^{l+i-1}i\left(\binom{k+i}{i}\binom{k}{i} - \binom{k+i-1}{i-1}\binom{k}{i}\right)\alpha_i(l) \\
&= \sum_{i=1}^{k}(-1)^{i}\left((k+i)\binom{k+i-1}{i-1}\binom{k}{i} - i\binom{k+i-1}{i-1}\binom{k}{i}\right)\alpha_i(l) \\
&= \sum_{i=1}^{k}(-1)^{i}k\binom{k+i-1}{i-1}\binom{k}{i}\alpha_i(l) \\
&= \gamma_k(l).
\end{aligned}$$

Lemma 19.7. *The sum $\beta_k(l)$ satisfies the following recurrence relation:*

$$\beta_k(l) - \frac{1}{k^2}\beta_k(l-1) = 2(-1)^{k+l-1}\frac{\pi^{2l}}{(2l)!}.$$

Proof.

$$\begin{aligned}
\beta_k(l) =& 2(-1)^l k \sum_{i=1}^{k}(-1)^{i-1}\binom{k+i-1}{i-1}\binom{k-1}{i-1}\alpha_i(l) \\
=& 2(-1)^l k \sum_{i=1}^{k}(-1)^{i-1}\binom{k+i-1}{i-1}\binom{k-1}{i-1}\left(\frac{1}{i}\cdot\frac{\pi^{2l}}{(2l)!} - \frac{1}{i^2}\sum_{j=1}^{i}\alpha_j(l-1)\right) \\
=& 2(-1)^l \sum_{i=1}^{k}(-1)^{i-1}\binom{k+i-1}{i-1}\binom{k}{i}\left(\frac{\pi^{2l}}{(2l)!} - \frac{1}{i}\sum_{j=1}^{i}\alpha_j(l-1)\right) \\
=& 2(-1)^l \sum_{i=1}^{k}(-1)^{i-1}\binom{k+i-1}{i-1}\binom{k}{i}\cdot\frac{\pi^{2l}}{(2l)!} \\
&- 2(-1)^l \sum_{j=1}^{k}\left(\sum_{i=j}^{k}(-1)^{i-1}\binom{k+i-1}{i-1}\binom{k}{i}\frac{1}{i}\right)\alpha_j(l-1).
\end{aligned}$$

Here, by the equations

$$\sum_{i=1}^{k}(-1)^{i-1}\binom{k+i-1}{i-1}\binom{k}{i} = (-1)^{k-1}$$

and

$$\sum_{i=j}^{k}(-1)^{i-1}\frac{1}{i}\binom{k+i-1}{i-1}\binom{k}{i} = (-1)^{j-1}\frac{1}{k}\binom{k+j-1}{j-1}\binom{k-1}{j-1},$$

we have

$$\begin{aligned}
\beta_k(l) &= 2(-1)^l(-1)^{k-1}\cdot\frac{\pi^{2l}}{(2l)!} - 2(-1)^l\sum_{j=1}^{k}(-1)^{j-1}\frac{1}{k}\binom{k+j-1}{j-1}\binom{k-1}{j-1}\alpha_j(l-1) \\
&= 2(-1)^l(-1)^{k-1}\cdot\frac{\pi^{2l}}{(2l)!} + \frac{1}{k^2}\beta_k(l-1).
\end{aligned}$$

□

Finally, by the equation

$$\sum_{i=1}^{k}(-1)^{i-1}\binom{k+i-1}{i-1}\binom{k-1}{i-1}\cdot\frac{1}{i-\frac{1}{2}} = \frac{1-(-1)^k}{k},$$

we have

$$\begin{aligned}\beta_k(0) &= 2k\sum_{i=1}^{k}(-1)^{i-1}\binom{k+i-1}{i-1}\binom{k-1}{i-1}\alpha_i(0)\\ &= 2k\sum_{i=1}^{k}(-1)^{i-1}\binom{k+i-1}{i-1}\binom{k-1}{i-1}\frac{1}{i-\frac{1}{2}}\\ &= 2k\cdot\frac{1-(-1)^k}{k}\\ &= 2\left(1-(-1)^k\right).\end{aligned}$$

Since the initial values are equal and the recurrence formulas are the same, we conclude $\beta_k(l) = \hat{\beta}_k(l)$. Hence $\gamma_k(l) = \hat{\gamma}_k(l)$ and $G(x,k) = \hat{G}(x,k)$. □

Proof of Theorem 16.3. For fixed x, let $f(y) = G(x,y) - \hat{G}(x,y)$. By Propositions 19.3 and 19.5, $f(y)$ is an entire function which vanishes at all integers and satisfies $f(y) = O\left(e^{\pi|\Im(y)|}\right)$. By [35, Lemma 2], $f(y)$ is a constant multiple of $\sin(\pi y)$. By Proposition 19.4, $f(y)$ vanishes identically. Then we have $G(x,y) = \hat{G}(x,y)$ and the theorem is proved. □

20 Proof of Theorem 16.2

In this section, we prove the motivic version of Theorem 16.2. By applying the period map, we obtain Theorem 16.2.

Theorem 20.1. *The set of elements*

$$\left\{\tilde{t}^{\mathfrak{m}}(k_1,\ldots,k_d) \mid k_1,\ldots,k_d \in \{2,3\}\right\}$$

are a basis of the $\mathbb{Q}$-vector space of motivic multiple zeta values.

Let us define

$$\begin{aligned} H^{\mathfrak{m}}(a,b) &:= \zeta^{\mathfrak{m}}(\underbrace{2,\ldots,2}_{a},3,\underbrace{2,\ldots,2}_{b}), \quad H^{\mathfrak{m}}(n) := \zeta^{\mathfrak{m}}(\underbrace{2,\ldots,2}_{n}),\\ K^{\mathfrak{m}}(a,b) &:= \tilde{t}^{\mathfrak{m}}(\underbrace{2,\ldots,2}_{a},3,\underbrace{2,\ldots,2}_{b}), \quad K^{\mathfrak{m}}(n) := \tilde{t}^{\mathfrak{m}}(\underbrace{2,\ldots,2}_{n}),\end{aligned}$$

for any $a,b,n \in \mathbb{N}$.

Brown shows that [35, Theorem 1] lifts to motivic MZVs [1, Theorem 4.3].

Theorem 20.2 (Brown [1, Theorem 4.3])**.** *For all integers $a,b \geq 0$, we have*

$$H^{\mathfrak{m}}(a,b) = 2\sum_{r=1}^{a+b+1}(-1)^{r-1}\left(-A^r_{a,b} + B^r_{a,b}\right)H^{\mathfrak{m}}(a+b+1-r)\zeta^{\mathfrak{m}}(2r+1), \tag{46}$$

where

$$A^r_{a,b} := \binom{2r}{2a+2} \text{ and } B^r_{a,b} := \left(1 - 2^{-2r}\right)\binom{2r}{2b+1}$$

for any $a, b, r \in \mathbb{Z}_{\geq 0}$.

The proof of the following theorem is similar to the above theorem.

Theorem 20.3. *For all integers* $a, b \geq 0$, *we have*

$$K^{\mathfrak{m}}(a,b) = 2\sum_{r=1}^{a+b+1}(-1)^{r-1}\left(\tilde{A}^r_{a,b} + B^r_{a,b}\right)K^{\mathfrak{m}}(a+b-r+1)\zeta^{\mathfrak{m}}(2r+1), \tag{47}$$

where

$$\tilde{A}^r_{a,b} := \binom{2r}{2a+1} \text{ and } B^r_{a,b} := \left(1 - 2^{-2r}\right)\binom{2r}{2b+1}$$

for any $a, b, r \in \mathbb{Z}_{\geq 0}$.

Definition 11. We define $c_{2^a32^b}$ (resp. $\tilde{c}_{2^a32^b}$) $\in \mathbb{Q}$ the coefficient of $\zeta^{\mathfrak{m}}(2a+2b+3)$ in (46) (resp. (47)). We also define $c_{12^n} \in \mathbb{Q}$ the coefficient of $\zeta^{\mathfrak{m}}(2a+2b+3)$ in $\zeta^{\mathfrak{m}}_1(2^n) = -2\sum_{i=0}^{n-1} H^{\mathfrak{m}}(i, n-1-i)$.

From this definition, we have

$$c_{2^a32^b} = 2(-1)^{a+b}\left(-A^{a+b+1}_{a,b} + B^{a+b+1}_{a,b}\right),\ \tilde{c}_{2^a32^b} = 2(-1)^{a+b}\left(\tilde{A}^{a+b+1}_{a,b} + B^{a+b+1}_{a,b}\right). \tag{48}$$

From [1, Lemma 3.8 and Corollary 4.4], we have the following Lemma.

Lemma 20.4. *From definition of* $c_{2^a32^b}$ *and* $\tilde{c}_{2^a32^b}$, *these are in* $\mathbb{Z}[\frac{1}{2}]$. *Furthermore,* $c_{2^a32^b}$, $\tilde{c}_{2^a32^b}$ *and* c_{12^n} *satisfy*

(1) $c_{2^a32^b} - c_{2^b32^a} \in 2\mathbb{Z}$

(2) $\tilde{c}_{2^a32^b} - c_{2^b32^a} \in 2\mathbb{Z}$

(3) $c_{12^n} = 2(-1)^n$

(4) $v_2(c_{32^{a+b}}) \leq v_2(c_{2^a32^b}) \leq 0$ *where* v_p *is p-adic valuation.*

Lemma 20.5 (Brown [1, Lemma 4.2])**.** *For any* $a, b \geq 0$, *and* $1 \leq r \leq a+b+1$, *we have*

$$\sum_{\substack{\alpha \leq a \\ \beta \leq b}} A^r_{\alpha,\beta} - \sum_{\substack{\alpha \leq a \\ \beta < b}} A^r_{\beta,\alpha} + \delta_{b \geq r} - \delta_{a \geq r} = 0$$

and

$$\sum_{\substack{\alpha \leq a \\ \beta \leq b}} B^r_{\alpha,\beta} - \sum_{\substack{\alpha \leq a \\ \beta < b}} B^r_{\beta,\alpha} = B^r_{a,b},$$

where all sums are over sets of indices $\alpha, \beta \geq 0$ *satisfying* $\alpha + \beta + 1 = r$.

Lemma 20.6. *Let $a, b \geq 0$ and $1 \leq r \leq a+b$. Then*

$$D_{2r+1}(\tilde{t}^{\mathfrak{m}}(2^a 3 2^b)) = \pi(\tilde{\xi}^r_{a,b}) \otimes \tilde{t}^{\mathfrak{m}}(2^{a+b+1-r})$$

where $\tilde{\xi}^r_{a,b}$ is given by (sum over all indices $\alpha, \beta \geq 0$ satisfying $\alpha+\beta+1 = r$)

$$\tilde{\xi}^r_{a,b} = \delta_{a+1\leq r}\tilde{t}^{\mathfrak{m}}(2^a 3 2^{r-a-1}) + \sum_{\substack{\alpha<a\\ \beta\leq b}} \zeta^{\mathfrak{m}}(2^\alpha 3 2^\beta) - \sum_{\substack{\alpha\leq a\\ \beta<b}} \zeta^{\mathfrak{m}}(2^\beta 3 2^\alpha) + (\delta_{b\geq r} - \delta_{a\geq r})\,\zeta^{\mathfrak{m}}_1(2^r).$$

Proof. By Proposition 18.4. □

Proof of Theorem 20.3. The proof is by induction on the weight. Suppose that (47) holds for all $a+b<N$. Let $a, b \geq 0$ such that $a+b=N$. By Lemma 20.6,

$$D_{2r+1}(\tilde{t}^{\mathfrak{m}}(2^a 3 2^b)) = \pi(\tilde{\xi}^r_{a,b}) \otimes \tilde{t}^{\mathfrak{m}}(2^{a+b+1-r})$$

for $1 \leq r \leq N$.

$$\begin{aligned}
\pi(\tilde{\xi}^r_{a,b}) =& \delta_{a+1\leq r}\pi(\tilde{t}^{\mathfrak{m}}(2^a 3 2^{r-a-1})) + \sum_{\substack{\alpha<a\\ \beta\leq b}} \pi(\zeta^{\mathfrak{m}}(2^\alpha 3 2^\beta)) - \sum_{\substack{\alpha\leq a\\ \beta<b}} \pi(\zeta^{\mathfrak{m}}(2^\beta 3 2^\alpha)) + (\delta_{b\geq r} - \delta_{a\geq r})\,\pi(\zeta^{\mathfrak{m}}_1(2^r))\\
=& \left(\delta_{a+1\leq r}\tilde{c}_{2^a 3 2^{r-a-1}} + \sum_{\substack{\alpha<a\\ \beta\leq b}} c_{2^\alpha 3 2^\beta} - \sum_{\substack{\alpha\leq a\\ \beta<b}} c_{2^\beta 3 2^\alpha} + (\delta_{b\geq r} - \delta_{a\geq r}) c_{12^r}\right)\pi(\zeta^{\mathfrak{m}}(2r+1)) \qquad \text{(By (48))}\\
=& \Bigg(2(-1)^{r-1}\left(\tilde{A}^r_{a,r-a-1} + B^r_{a,r-a-1}\right) + \sum_{\substack{\alpha<a\\ \beta\leq b}} 2(-1)^r(A^r_{\alpha,\beta} - B^r_{\alpha,\beta}) - \sum_{\substack{\alpha\leq a\\ \beta<b}} 2(-1)^r(A^r_{\beta,\alpha} - B^r_{\beta,\alpha})\\
&+ 2(-1)^r\delta_{b\geq r} - 2(-1)^r\delta_{a\geq r}\Bigg)\pi(\zeta^{\mathfrak{m}}(2r+1))\\
&\qquad\qquad \text{(by the induction hypothesis and Lemma 20.4(3))}\\
=& 2(-1)^{r-1}\Bigg(\left(\tilde{A}^r_{a,r-a-1} + B^r_{a,r-a-1}\right)\\
&- \Bigg(\sum_{\substack{\alpha<a\\ \beta\leq b}}(A^r_{\alpha,\beta} - B^r_{\alpha,\beta}) - \sum_{\substack{\alpha\leq a\\ \beta<b}}(A^r_{\beta,\alpha} - B^r_{\beta,\alpha}) + \delta_{b\geq r} - \delta_{a\geq r}\Bigg)\Bigg)\pi(\zeta^{\mathfrak{m}}(2r+1))\\
=& 2(-1)^{r-1}\Bigg(\tilde{A}^r_{a,r-a-1} - \Bigg(\sum_{\substack{\alpha<a\\ \beta\leq b}} A^r_{\alpha,\beta} - \sum_{\substack{\alpha\leq a\\ \beta<b}} A^r_{\beta,\alpha} + \delta_{b\geq r} - \delta_{a\geq r}\Bigg)
\end{aligned}$$

$$+\left(\sum_{\substack{\alpha<a\\ \beta\le b}} B^r_{\alpha,\beta} + B^r_{a,r-a-1} - \sum_{\substack{\alpha\le a\\ \beta<b}} B^r_{\beta,\alpha}\right)\Bigg)\pi(\zeta^{\mathfrak{m}}(2r+1))$$

$$=2(-1)^{r-1}\left(\tilde{A}^r_{a,r-a-1} + B^r_{a,b}\right)\pi(\zeta^{\mathfrak{m}}(2r+1))$$

$$=2(-1)^{r-1}\left(\tilde{A}^r_{a,b} + B^r_{a,b}\right)\pi(\zeta^{\mathfrak{m}}(2r+1)).$$

Therefore, we obtain

$$D_{2r+1}(\tilde{t}^{\mathfrak{m}}(2^a32^b)) = 2(-1)^{r-1}\left(\tilde{A}^r_{a,b} + B^r_{a,b}\right)\pi(\zeta^{\mathfrak{m}}(2r+1)) \otimes \tilde{t}^{\mathfrak{m}}(2^{a+b+1-r}).$$

It follows that

$$\Theta = K^{\mathfrak{m}}(a,b) - 2\sum_{r=1}^{a+b+1}(-1)^{r-1}\left[\tilde{A}^r_{a,b} + B^r_{a,b}\right]K^{\mathfrak{m}}(a+b-r+1)\zeta^{\mathfrak{m}}(2r+1)$$

satisfies $D_{2r+1}(\Theta) = 0$ for all $r \le a+b$. By Theorem 17.1, there is an $\alpha \in \mathbb{Q}$ such that $\Theta = \alpha\zeta^{\mathfrak{m}}(2a+2b+3)$. Taking the period map and by Theorem 16.3, we see that α is equal to 0. Thus the theorem is proved. □

Definition 12. For $w \in \{2,3\}^\times$, we define the level of w to be $\deg_3 w$. We define $\mathbb{Q}$-subspaces of $\mathcal{H}^2$ as follows:

$$\mathcal{H}^{\{2,3\}} := \langle t^{\mathfrak{m}}(w) \mid w \in \{2,3\}^\times \rangle_{\mathbb{Q}}$$
$$F_l\mathcal{H}^{\{2,3\}} := \langle t^{\mathfrak{m}}(w) \mid w \in \{2,3\}^\times \ s.t.\ \deg_3 w \le l \rangle_{\mathbb{Q}}$$

and define the quotient subspace

$$\mathrm{gr}_l\mathcal{H}^{\{2,3\}} := F_l\mathcal{H}^{\{2,3\}}/F_{l-1}\mathcal{H}^{\{2,3\}}.$$

From [1, Lemma 5.5], we obtain the map

$$\mathrm{gr}_l D_{2r+1} : \mathrm{gr}_l\mathcal{H}^{\{2,3\}} \longrightarrow \mathcal{L}_{2r+1} \otimes_{\mathbb{Q}} \mathrm{gr}_{l-1}\mathcal{H}^{\{2,3\}}.$$

By Theorem 18.1, the inclusion $\mathcal{H}^{\{2,3\}} \subseteq \mathcal{H}^1$ holds.

Definition 13. For all $N, l \ge 1$, let $\partial_{N,l}$ be the linear map

$$\partial_{N,l} : \mathrm{gr}_l\mathcal{H}^{\{2,3\}}_N \longrightarrow \bigoplus_{1<2r+1\le N} \mathrm{gr}_{l-1}\mathcal{H}^{\{2,3\}}_{N-2r-1}$$

defined by first applying $\bigoplus_{1<2r+1\le N}\mathrm{gr}_l D_{2r+1}|_{\mathrm{gr}_l\mathcal{H}^{\{2,3\}}_N}$ and then sending all $\zeta^{\mathfrak{l}}(2r+1)$ to 1 by the projection $\tilde{\pi}_{2r+1} : \mathbb{Q}\zeta^{\mathfrak{l}}(2r+1) \to \mathbb{Q}$.

Definition 14. For $l \geq 1$, let $B_{N,l}$ and $B'_{N,l}$ denote respectively the sets $\{w \in \{2,3\}^\times \mid |w| = N, \deg_3 w = l\}$ and $\{w \in \{2,3\}^\times \mid |w| \leq N-3, \deg_3 w = l-1\}$, in reverse lexicographic order for the ordering $3 < 2$. The $B'_{N,l}$ includes the empty word if $l = 1$.

Example 20.7. For $N = 10$ and $l = 2$, the sets $B_{10,2}$ and $B'_{10,2}$ are as follows.

$$\begin{aligned} B_{10,2} &= \{\ 2233\ ,\ 2323\ ,\ 2332\ ,\ 3223\ ,\ 3232\ ,\ 3322\ \}, \\ B'_{10,2} &= \{\ 223\ ,\ 232\ ,\ 23\ ,\ 322\ ,\ 32\ ,\ 3\ \}. \end{aligned}$$

Definition 15. For $l \geq 1$, let $M_{N,l}$ be the matrix $(f^w_{w'})_{w \in B_{N,l}, w' \in B'_{N,l}}$, where $f^w_{w'}$ are the coefficient of $\tilde{t}^{\mathfrak{m}}(w')$ in $\partial_{N,l}\tilde{t}^{\mathfrak{m}}(w)$, and w corresponds to the rows and w' the columns.

Example 20.8. For $N = 10$ and $l = 2$, the matrix $M_{10,2}$ is as follows. The words in the first column and row are the elements of $B_{10,2}$ and $B'_{10,2}$ in order, respectively.

	223	232	23	322	32	3
2233	$c_3 - c_{12}$	0	$c_{23} - c_{32} - c_{122}$	0	0	$\tilde{c}_{223} - c_{322}$
2323	0	$c_3 - c_{12}$	$\tilde{c}_{23}$	0	0	$\tilde{c}_{232} - c_{232}$
2332	0	0	c_{32}	0	$\tilde{c}_{23} - c_{32}$	0
3223	$\tilde{c}_3 + c_{12} - c_3$	0	$\tilde{c}_{32} + c_{122} - c_{23}$	$c_3 - c_{12}$	$c_{23} - c_{122}$	$\tilde{c}_{322}$
3232	0	$\tilde{c}_3 + c_{12} - c_3$	0	0	$\tilde{c}_{32}$	c_{232}
3322	0	0	0	$\tilde{c}_3 + c_{12} - c_3$	$c_{32} + c_{122} - c_{23}$	c_{322}

Theorem 20.9. *Let $w \in B_{N,l}$. Then*

$$\partial_{N,l}\tilde{t}^{\mathfrak{m}}(w) \equiv \sum_{\substack{w=uv \\ \deg_3 v = 1}} c_v \tilde{t}^{\mathfrak{m}}(u) \mod 2\mathbb{Z}.$$

Proof. Let $k_1, \ldots, k_d \in \{2,3\}$ with $l = \#\{i \mid k_i = 3\}$ and $w = (k_1, \ldots, k_d)$. We consider the action of $\mathrm{gr}_l D_r$ on $\tilde{t}^{\mathfrak{m}}(k_1, \ldots, k_d)$. From Proposition 18.4, we have

$$\begin{aligned} &\mathrm{gr}_l D_r \left(\tilde{t}^{\mathfrak{m}}(k_1, \ldots, k_d)\right) \\ &= \sum_{1 \leq j \leq d} \delta_{|\mathbf{k}_{1,j}| = r} \tilde{\pi}_r \left(\tilde{t}^{\mathfrak{l}}(k_1, \ldots, k_j)\right) \tilde{t}^{\mathfrak{m}}(k_{j+1}, \ldots, k_d) \\ &+ \sum_{1 \leq i < j \leq d} \delta_{|\mathbf{k}_{i+1,j}| \leq r < |\mathbf{k}_{i,j}| - 1} \tilde{\pi}_r \left(\zeta^{\mathfrak{l}}_{r - |\mathbf{k}_{i+1,j}|}(k_{i+1}, \ldots, k_j)\right) \tilde{t}^{\mathfrak{m}}(k_1, \ldots, k_{i-1}, |\mathbf{k}_{i,j}| - r, k_{j+1}, \ldots, k_d) \quad (49) \\ &- \sum_{1 \leq i < j \leq d} \delta_{|\mathbf{k}_{i,j-1}| \leq r < |\mathbf{k}_{i,j}| - 1} \tilde{\pi}_r \left(\zeta^{\mathfrak{l}}_{r - |\mathbf{k}_{i,j-1}|}(k_{j-1}, \ldots, k_i)\right) \tilde{t}^{\mathfrak{m}}(k_1, \ldots, k_{i-1}, |\mathbf{k}_{i,j}| - r, k_{j+1}, \ldots, k_d). \quad (50) \end{aligned}$$

If $r = k_1 + \cdots k_d$, we have $\mathrm{gr}_l D_r \left(\tilde{t}^{\mathfrak{m}}(k_1, \ldots, k_d)\right) = \tilde{\pi}_r \left(\tilde{t}^{\mathfrak{l}}(k_1, \ldots, k_d)\right) \tilde{t}^{\mathfrak{m}}(\emptyset) = \tilde{c}_w \tilde{t}^{\mathfrak{m}}(\emptyset) \equiv c_w \tilde{t}^{\mathfrak{m}}(\emptyset)$. Hence, we assume that $r < k_1 + \cdots + k_d$.

First, we consider the sum (49). There are the following three types of nonzero terms of (49):

$(k_i,...,k_j)$	r	term
$(3,\underbrace{2,...,2}_{n})$	$r=2n+1$	$\zeta_1^{\mathfrak{l}}(2,...,2)\otimes\tilde{t}^{\mathfrak{m}}(k_1,...,k_{i-1},2,k_{j+1},...,k_d)$
$(2,\underbrace{2,...,2}_{\alpha},3,\underbrace{2,...,2}_{\beta})$	$r=2\alpha+2\beta+3$	$\zeta^{\mathfrak{l}}(\underbrace{2,...,2}_{\alpha},3,\underbrace{2,...,2}_{\beta})\otimes\tilde{t}^{\mathfrak{m}}(k_1,...,k_{i-1},2,k_{j+1},...,k_d)$
$(3,\underbrace{2,...,2}_{\alpha},3,\underbrace{2,...,2}_{\beta})$	$r=2\alpha+2\beta+3$	$\zeta^{\mathfrak{l}}(\underbrace{2,...,2}_{\alpha},3,\underbrace{2,...,2}_{\beta})\otimes\tilde{t}^{\mathfrak{m}}(k_1,...,k_{i-1},3,k_{j+1},...,k_d)$

All other types of terms are zero in $\mathcal{L}_{2r+1}\otimes_{\mathbb{Q}}\mathrm{gr}_{l-1}\mathcal{H}^{\{2,3\}}$.

Therefore, the sum (49) becomes as follows.

$$\begin{aligned}
(49) &= \sum_{1\le i<j\le d}\delta_{|\mathbf{k}_{i+1,j}|+1=r}\tilde{\pi}_r\left(\zeta_1^{\mathfrak{l}}(2,\ldots,2)\right)\tilde{t}^{\mathfrak{m}}(k_1,\ldots,k_{i-1},2,k_{j+1},\ldots,k_d)\\
&\quad+\sum_{1\le i<j\le d}\delta_{|\mathbf{k}_{i+1,j}|=r}\tilde{\pi}_r\left(\zeta^{\mathfrak{l}}(k_{i+1},\ldots,k_j)\right)\tilde{t}^{\mathfrak{m}}(k_1,\ldots,k_{i-1},k_i,k_{j+1},\ldots,k_d)\\
&= \sum_{1\le i<j\le d}\delta_{|\mathbf{k}_{i+1,j}|+1=r}c_{12^{j-i}}\tilde{t}^{\mathfrak{m}}(k_1,\ldots,k_{i-1},2,k_{j+1},\ldots,k_d)\\
&\quad+\sum_{1\le i<j\le d}\delta_{|\mathbf{k}_{i+1,j}|=r}c_{(k_{i+1},\ldots,k_j)}\tilde{t}^{\mathfrak{m}}(k_1,\ldots,k_{i-1},k_i,k_{j+1},\ldots,k_d)\\
&= \sum_{1\le i<j\le d}\delta_{|\mathbf{k}_{i+1,j}|+1=r}c_{12^{j-i}}\tilde{t}^{\mathfrak{m}}(k_1,\ldots,k_{i-1},2,k_{j+1},\ldots,k_d)\\
&\quad+\sum_{1\le i<j<d}\delta_{|\mathbf{k}_{i+1,j}|=r}c_{(k_{i+1},\ldots,k_j)}\tilde{t}^{\mathfrak{m}}(k_1,\ldots,k_{i-1},k_i,k_{j+1},\ldots,k_d)\\
&\quad+\sum_{1\le i<d}\delta_{|\mathbf{k}_{i+1,d}|=r}c_{(k_{i+1},\ldots,k_d)}\tilde{t}^{\mathfrak{m}}(k_1,\ldots,k_i).
\end{aligned}$$

The sum (50) is calculated similarly.

$$\begin{aligned}
(50) &= -\sum_{1\le i<j\le d}\delta_{|\mathbf{k}_{i,j-1}|+1=r}\tilde{\pi}_r\left(\zeta_1^{\mathfrak{l}}(2,\ldots,2)\right)\tilde{t}^{\mathfrak{m}}(k_1,\ldots,k_{i-1},2,k_{j+1},\ldots,k_d)\\
&\quad-\sum_{1\le i<j\le d}\delta_{|\mathbf{k}_{i,j-1}|=r}\tilde{\pi}_r\left(\zeta^{\mathfrak{l}}(k_{j-1},\ldots,k_i)\right)\tilde{t}^{\mathfrak{m}}(k_1,\ldots,k_{i-1},k_j,k_{j+1},\ldots,k_d)\\
&= -\sum_{1\le i<j\le d}\delta_{|\mathbf{k}_{i,j-1}|+1=r}c_{12^{j-i}}\tilde{t}^{\mathfrak{m}}(k_1,\ldots,k_{i-1},2,k_{j+1},\ldots,k_d)\\
&\quad-\sum_{1\le i<j\le d}\delta_{|\mathbf{k}_{i,j-1}|=r}c_{(k_{j-1},\ldots,k_i)}\tilde{t}^{\mathfrak{m}}(k_1,\ldots,k_{i-1},k_j,k_{j+1},\ldots,k_d)\\
&= -\sum_{1\le i<j\le d}\delta_{|\mathbf{k}_{i,j-1}|+1=r}c_{12^{j-i}}\tilde{t}^{\mathfrak{m}}(k_1,\ldots,k_{i-1},2,k_{j+1},\ldots,k_d)\\
&\quad-\sum_{1<j\le d}\delta_{|\mathbf{k}_{1,j-1}|=r}c_{(k_{j-1},\ldots,k_1)}\tilde{t}^{\mathfrak{m}}(k_j,k_{j+1},\ldots,k_d)
\end{aligned}$$

$$
\begin{aligned}
&\quad - \sum_{1<i<j\le d} \delta_{|\mathbf{k}_{i,j-1}|=r} c_{(k_{j-1},\dots,k_i)} \tilde{t}^{\mathfrak{m}}(k_1,\dots,k_{i-1},k_j,k_{j+1},\dots,k_d) \\
= &\quad - \sum_{1\le i<j\le d} \delta_{|\mathbf{k}_{i,j-1}|+1=r} c_{12^{j-i}} \tilde{t}^{\mathfrak{m}}(k_1,\dots,k_{i-1},2,k_{j+1},\dots,k_d) \\
&\quad - \sum_{1\le j<d} \delta_{|\mathbf{k}_{1,j}|=r} c_{(k_j,\dots,k_1)} \tilde{t}^{\mathfrak{m}}(k_{j+1},\dots,k_d) \\
&\quad - \sum_{1\le i<j<d} \delta_{|\mathbf{k}_{i+1,j}|=r} c_{(k_j,\dots,k_{i+1})} \tilde{t}^{\mathfrak{m}}(k_1,\dots,k_i,k_{j+1},\dots,k_d).
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
&\mathrm{gr}_l D_r\left(\tilde{t}^{\mathfrak{m}}(k_1,\dots,k_d)\right) \\
&\quad = \sum_{1\le j<d} \delta_{|\mathbf{k}_{1,j}|=r} \left(\tilde{c}_{(k_1,\dots,k_j)} - c_{(k_j,\dots,k_1)}\right) \tilde{t}^{\mathfrak{m}}(k_{j+1},\dots,k_d) \\
&\quad + \sum_{1\le i<j\le d} \delta_{|\mathbf{k}_{i+1,j}|+1=r} c_{12^{j-i}} \tilde{t}^{\mathfrak{m}}(k_1,\dots,k_{i-1},2,k_{j+1},\dots,k_d) \\
&\quad + \sum_{1\le i<j<d} \delta_{|\mathbf{k}_{i+1,j}|=r} \left(c_{(k_{i+1},\dots,k_j)} - c_{(k_j,\dots,k_{i+1})}\right) \tilde{t}^{\mathfrak{m}}(k_1,\dots,k_i,k_{j+1},\dots,k_d) \\
&\quad + \sum_{1\le i<d} \delta_{|\mathbf{k}_{i+1,d}|=r} c_{(k_{i+1},\dots,k_d)} \tilde{t}^{\mathfrak{m}}(k_1,\dots,k_{i-1},k_i) \\
&\quad - \sum_{1\le i<j\le d} \delta_{|\mathbf{k}_{i,j-1}|+1=r} c_{12^{j-i}} \tilde{t}^{\mathfrak{m}}(k_1,\dots,k_{i-1},2,k_{j+1},\dots,k_d).
\end{aligned}
$$

By Lemma 20.4, all but the fourth term in the above expression are in $2\mathbb{Z}$. Thus the theorem is proved. □

Corollary 20.10. *The matrices $M_{N,l}$ mod $2\mathbb{Z}$ are upper-trianglar. Every diagonal entry is equal to $c_{32^{r-1}}$ mod $2\mathbb{Z}$, and every entry above the diagonal in the same column is equal to $c_{2^a32^b}$, where $a+b+1=r$.*

Proof. We prove this corollary by the same way to the proof of [1, Corollary 6.2]. Let $l \ge 1$ and consider the map

$$
\begin{aligned}
B'_{N,l} &\longrightarrow B_{N,l} \\
u &\longmapsto u32^{r-1}
\end{aligned}
$$

where $r \ge 1$ is the unique integer such that $|u32^{r-1}| = N$. This map is a bijection and preserves the ordering of both $B'_{N,l}$ and $B_{N,l}$. Then the diagonal entries of $M_{N,l}$ are of the form $f_u^{u32^{r-1}}$. Let $u \in B'_{N,l}$. By Theorem 20.9, $f_u^{u32^{r-1}}$are equal to $c_{32^{r-1}}$ mod $2\mathbb{Z}$. If $w < u32^{r-1}$, w is of the form $w = u2^a32^b (a+b=r-1)$, then $f_u^w \equiv c_{2^a32^b}$. If $w > u32^{r-1}$, then we have $f_u^w \equiv 0$. □

Theorem 20.11. *For $N, l \geq 1$, the matrices $M_{N,l}$ are invertible.*

Proof. By Lemma 20.4(4) and [1, Lemma 7.1]. □

Theorem 20.12. *The set of elements*

$$\left\{\tilde{t}^{\mathfrak{m}}(k_1, \ldots, k_d) \mid k_i \in \{2, 3\}\right\}$$

are linearly independent.

Proof. We proceed by induction on the level. The elements of level zero are of the form $t^{\mathfrak{m}}(2^n)$ for $n \geq 0$, which are linearly independent. Now suppose that elements in

$$\left\{\tilde{t}^{\mathfrak{m}}(w) \mid w \in \{2, 3\}^{\times}, \deg_3 w \leq l - 1\right\}$$

are independent. By Theorem 20.11, the map $\partial_{N,l}$ is injective. Therefore,

$$\left\{\tilde{t}^{\mathfrak{m}}(w) \mid w \in B_{N,l}\right\}$$

are independent for $N \in \mathbb{Z}_{\geq 0}$. Since the weight is grading in $\mathcal{H}^2$,

$$\left\{\tilde{t}^{\mathfrak{m}}(w) \mid w \in \{2, 3\}^{\times}, \deg_3 w = l\right\}$$

are independent. Thus the theorem is proved. □

This theorem implies that

$$\dim \mathcal{H}_N^{\{2,3\}} = d_N.$$

Here, recall that d_N was defined in (1). Goncharov [5], Terasoma [32] and Deligne–Goncharov [2] show that the inequality $\dim_{\mathbb{Q}} \mathcal{H}_N^1 \leq d_N$ (see Theorem 1.1). By this result and the inclusion $\mathcal{H}_N^{\{2,3\}} \subseteq \mathcal{H}_N^1$, we have $\mathcal{H}_N^{\{2,3\}} = \mathcal{H}_N^1$. Therefore, Theorem 20.1 holds.

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