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Abstract This paper is the three dimensional extension of the two dimensional work [4] and [7] on a computer assisted proof of the existence of nontrivial steady state solutions for Rayleigh–Bénard convection based on the fixed point theorem using a Newton like operator. The differences are emerging of complicated types of bifurcation, direct attack on the problem without stream functions, and increased complexity of numerical computation. The last one makes it hard to proceed the verification of solutions corresponding to the points on bifurcation diagram for three dimensional case. Actually, this work should be the first result for the three dimensional Navier-Stokes problems which seems to be very difficult to solve by theoretical approaches.

1 Introduction

The Rayleigh–Bénard convection describes the instability of fluid between two infinite solid plates with hot bottom and cool top. The motion of fluid is self-sustained as soon as gravitational energy release overcomes dissipation losses, which is called Rayleigh's mechanism by buoyancy (for detailed mechanism, see [1] and [3]). We will use the Oberbeck–Boussinesq equations as approximate equations for this convection problem after normalization of variables and parameters:

$$\frac{1}{\mathscr{P}}\left[\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right]+\nabla p = \Delta\mathbf{u}-(\mathscr{G}-\mathscr{R}T)\mathbf{e}_z, \tag{1a}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{1b}$$

$$\frac{\partial T}{\partial t}+(\mathbf{u}\cdot\nabla)T = \Delta T, \tag{1c}$$

where $\mathbf{u}=(u,v,w)$ is the velocity field, p the pressure, T the temperature, $\mathscr{R}$ Rayleigh number, $\mathscr{P}$ Prandtl number, and a parameter $\mathscr{G}$ containing gravity factors. We use stress free boundary conditions on the velocity field and Dirichlet boundary conditions on the temperature ($T_{z=0}=0$, $T_{z=\pi}=\pi$). Under a reference pressure p_a, the equilibrium state comes from the pure heat conduction:

$$\mathbf{u}=\mathbf{0},\quad T=\pi-z,\quad p=\mathscr{G}(\pi-z)-\frac{1}{2}\mathscr{R}(\pi-z)^2+p_a. \tag{2}$$

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Using the conduction solution (2) and eliminating time derivatives from (1), we obtain the steady state bifurcation equations for the perturbation $(\mathbf{u},\theta,p)$ to the equilibrium:

$$-\Delta\mathbf{u}+\frac{1}{\mathscr{P}}(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p-\mathscr{R}\theta\mathbf{e}_z=\mathbf{0}, \tag{3a}$$

$$\nabla\cdot\mathbf{u}=0, \tag{3b}$$

$$-\Delta\theta+(\mathbf{u}\cdot\nabla)\theta-w=0. \tag{3c}$$

Given positive wave numbers $a,b\le 1$, we assume that all fluid motions are essentially confined to

$$\Omega=\left\{(x,y,z)\in\mathbb{R}^3\,:\,0\le x\le\frac{2\pi}{a},0\le y\le\frac{2\pi}{b},0\le z\le\pi\right\},\quad |\Omega|=\frac{4\pi^3}{ab},$$

and impose parity conditions on new boundaries as in [5] together with periodic boundary conditions in horizontal directions [4]. From these boundary conditions, the velocity field, the perturbations of temperature and pressure can be represented by the Fourier series [5]:

$$\mathbf{u}=\sum_{\alpha\neq\mathbf{0}}[u_\alpha\phi_1^\alpha,v_\alpha\phi_2^\alpha,w_\alpha\phi_3^\alpha],\quad \theta=\sum_{\alpha_3\neq 0}\theta_\alpha\phi_3^\alpha,\quad p=\sum_{\alpha\neq\mathbf{0}}p_\alpha\phi_4^\alpha, \tag{4}$$

where $\alpha\equiv(\alpha_1,\alpha_2,\alpha_3)$ is the three dimensional multi-index of non–negative integers $\mathbb{Z}_0$, and u_α, v_α, w_α, θ_α, p_α are coefficients of $\mathbf{u}$, θ, p with respect to the base functions ϕ_i^α defined by,

$$\phi_1^\alpha(x,y,z)=K_\alpha\sin(a\alpha_1x)\cos(b\alpha_2y)\cos(\alpha_3z),\ \ \phi_2^\alpha(x,y,z)=K_\alpha\cos(a\alpha_1x)\sin(b\alpha_2y)\cos(\alpha_3z),$$
$$\phi_3^\alpha(x,y,z)=K_\alpha\cos(a\alpha_1x)\cos(b\alpha_2y)\sin(\alpha_3z),\ \ \phi_4^\alpha(x,y,z)=K_\alpha\cos(a\alpha_1x)\cos(b\alpha_2y)\cos(\alpha_3z),$$

where the normalization factor with respect to the usual $L^2(\Omega)$ inner product $\langle\cdot,\cdot\rangle$ is

$$K_\alpha=\sqrt{(2-\delta_{0\alpha_1})(2-\delta_{0\alpha_2})(2-\delta_{0\alpha_3})/|\Omega|},\quad \delta_{ij}=\text{Kronecker delta on } i,j.$$

The various kinds of norms for $\mathbf{u}$, θ and p in (4) can be written as:

$$\|\mathbf{u}\|_0^2=\sum_{\alpha\neq\mathbf{0}}\{u_\alpha^2+v_\alpha^2+w_\alpha^2\},\ \|\nabla\mathbf{u}\|_0^2=\sum_{\alpha\neq\mathbf{0}}\{u_\alpha^2+v_\alpha^2+w_\alpha^2\}A_\alpha^2,\ \left\|\nabla^2\mathbf{u}\right\|_0^2=\sum_{\alpha\neq\mathbf{0}}\{u_\alpha^2+v_\alpha^2+w_\alpha^2\}A_\alpha^4,$$
$$\|\theta\|_0^2=\sum_{\alpha_3\neq 0}\theta_\alpha^2,\qquad \|\nabla\theta\|_0^2=\sum_{\alpha_3\neq 0}\theta_\alpha^2A_\alpha^2,\qquad \left\|\nabla^2\theta\right\|_0^2=\sum_{\alpha_3\neq 0}\theta_\alpha^2A_\alpha^4,$$
$$\|p\|_0^2=\sum_{\alpha\neq\mathbf{0}}p_\alpha^2,\qquad \|\nabla p\|_0^2=\sum_{\alpha\neq\mathbf{0}}p_\alpha^2A_\alpha^2,\qquad \left\|\nabla^2 p\right\|_0^2=\sum_{\alpha\neq\mathbf{0}}p_\alpha^2A_\alpha^4,$$

where $A_\alpha\equiv\sqrt{(a\alpha_1)^2+(b\alpha_2)^2+\alpha_3^2}$ provided that the corresponding righthand sides converge.

We now define the divergence free and orthogonal system by:

$$\Phi^\alpha=\left[-\frac{a\alpha_1\alpha_3}{A_\alpha B_\alpha}\phi_1^\alpha,-\frac{b\alpha_2\alpha_3}{A_\alpha B_\alpha}\phi_2^\alpha,\frac{B_\alpha}{A_\alpha}\phi_3^\alpha\right],\ \alpha\in I_1,\quad \Psi^\alpha=\left[\frac{b\alpha_2}{B_\alpha}\phi_1^\alpha,-\frac{a\alpha_1}{B_\alpha}\phi_2^\alpha,0\right],\ \alpha\in I_2,$$

where $B_\alpha\equiv\sqrt{(a\alpha_1)^2+(b\alpha_2)^2}$ and indices subsets are $I_1\equiv\{[1,0,1]+\mathbb{Z}_0^3\}\cup\{[0,1,1]+\mathbb{Z}_0^3\}$, $I_2\equiv[1,1,0]+\mathbb{Z}_0^3$. Set $I_0=I_1\cup I_2$ and then define the function spaces V and W with associated usual H^1-norm as follows:

$$V=\Big\{\mathbf{u}=\sum_{\alpha\in I_0}\{\xi_\alpha\Phi^\alpha+\eta_\alpha\Psi^\alpha\}\,:\,\|\Delta\mathbf{u}\|_0<\infty\Big\}\subset H^2(\Omega)^3,$$
$$W=\Big\{\theta=\sum_{\alpha\in I_3}\theta_\alpha\phi_3^\alpha\,:\,\|\Delta\theta\|_0<\infty\Big\}\subset H^2(\Omega),\quad \text{where } I_3\equiv[0,0,1]+\mathbb{Z}_0^3.$$

Note that $\|\mathbf{u}\|_0^2=\sum_{\alpha\in I_0}\{\xi_\alpha^2+\eta_\alpha^2\}$, $\|\nabla\mathbf{u}\|_0^2=\sum_{\alpha\in I_0}\{\xi_\alpha^2+\eta_\alpha^2\}A_\alpha^2$, $\|\Delta\mathbf{u}\|_0^2=\sum_{\alpha\in I_0}\{\xi_\alpha^2+\eta_\alpha^2\}A_\alpha^4$ for all $\mathbf{u}\in V$, and $\|\theta\|_0^2=\sum_{\alpha\in I_3}\theta_\alpha^2$, $\|\nabla\theta\|_0^2=\sum_{\alpha\in I_3}\theta_\alpha^2A_\alpha^2$, $\|\Delta\theta\|_0^2=\sum_{\alpha\in I_3}\theta_\alpha^2A_\alpha^4$ for all $\theta\in W$.

2 A priori error estimates

For a fixed number $N \geq 2$, define the finite dimensional subspaces V_N and W_N of V and W by:

$$V_N \equiv \{\, \mathbf{u} \in V \;:\; \xi_\alpha = \eta_\alpha = 0, \text{ if } |\alpha| \equiv \alpha_1 + \alpha_2 + \alpha_3 > N \}, \quad W_N \equiv \{\, \theta \in W \;:\; \theta_\alpha = 0, \text{ if } |\alpha| > N \}.$$

Set $X \equiv V \times W$ and $X_N \equiv V_N \times W_N$. Define the projections $P_N : V \to V_N$ and $Q_N : W \to W_N$ as in [7]:

$$\langle \nabla(\mathbf{u} - P_N \mathbf{u}), \nabla \mathbf{v} \rangle = 0, {}^\forall \mathbf{v} \in V_N, \quad \langle \nabla(\theta - Q_N \theta), \nabla \vartheta \rangle = 0, {}^\forall \vartheta \in W_N, \tag{5}$$

with respect to the inner product $\langle \cdot, \cdot \rangle$ on $L^2(\Omega)$ or properly extended spaces. Due to orthogonal relations of base functions in X, these projections P_N and Q_N are truncation operators:

$$P_N \mathbf{u} = \sum_{\alpha \in I_{0,N} \equiv I_0 \cap I_{,N}} \{\xi_\alpha \Phi^\alpha + \eta_\alpha \Psi^\alpha\}, \quad Q_N \theta = \sum_{\alpha \in I_{3,N} \equiv I_3 \cap I_{,N}} \theta_\alpha \phi_3^\alpha,$$

where $I_{,N} \equiv \left\{ \alpha \in \mathbb{Z}_0^3 \;:\; |\alpha| \leq N \right\}$. The sum $|\alpha|$ of multi-index $\alpha \in \mathbb{Z}_0^3$ can be considered as the result of the inner product between two vectors $[a^{-1}, b^{-1}, 1]$ and $[a\alpha_1, b\alpha_2, \alpha_3]$. This consideration with the Cauchy-Schwartz inequality gives us

$$|\alpha| \leq \|[a^{-1}, b^{-1}, 1]\| \cdot \|[a\alpha_1, b\alpha_2, \alpha_3]\| = C_0 A_\alpha, \quad C_0 \equiv \sqrt{a^{-2} + b^{-2} + 1}. \tag{6}$$

Note that C_0 depends only on the wave numbers a and b, so we can say that it depends only on Ω. From these characterization of projections and special estimation (6), we have

Theorem 1 *For any $(\mathbf{u}, \theta) \in X$ and $(P_N \mathbf{u}, Q_N \theta) \in X_N$ in (5), the following holds:*

$$\|\mathbf{u} - P_N \mathbf{u}\|_0 \leq \frac{C_0^2}{(N+1)^2} \|\Delta \mathbf{u}\|_0, \;\; \|\nabla(\mathbf{u} - P_N \mathbf{u})\|_0 \leq \frac{C_0}{N+1} \|\Delta \mathbf{u}\|_0, \tag{7a}$$

$$\|\theta - Q_N \theta\|_0 \leq \frac{C_0^2}{(N+1)^2} \|\Delta \theta\|_0, \;\; \|\nabla(\theta - Q_N \theta)\|_0 \leq \frac{C_0}{N+1} \|\Delta \theta\|_0. \tag{7b}$$

Proof Due to (6), we have $1 \leq \frac{C_0}{N+1} A_\alpha$ if $|\alpha| > N$. Hence we can establish the following estimates:

$$\|\mathbf{u} - P_N \mathbf{u}\|_0^2 = \sum_{\alpha \in I_0 - I_{0,N}} \{\xi_\alpha^2 + \eta_\alpha^2\} \leq \frac{C_0^4}{(N+1)^4} \sum_{\alpha \in I_0 - I_{0,N}} \{\xi_\alpha^2 + \eta_\alpha^2\} A_\alpha^4 \leq \frac{C_0^4}{(N+1)^4} \|\Delta \mathbf{u}\|_0^2,$$

$$\|\nabla(\mathbf{u} - P_N \mathbf{u})\|_0^2 = \sum_{\alpha \in I_0 - I_{0,N}} \{\xi_\alpha^2 + \eta_\alpha^2\} A_\alpha^2 \leq \frac{C_0^2}{(N+1)^2} \sum_{\alpha \in I_0 - I_{0,N}} \{\xi_\alpha^2 + \eta_\alpha^2\} A_\alpha^4 \leq \frac{C_0^2}{(N+1)^2} \|\Delta \mathbf{u}\|_0^2,$$

$$\|\theta - P_N \theta\|_0^2 = \sum_{\alpha \in I_3 - I_{3,N}} \theta_\alpha^2 \leq \frac{C_0^4}{(N+1)^4} \sum_{\alpha \in I_3 - I_{3,N}} \theta_\alpha^2 A_\alpha^4 \leq \frac{C_0^4}{(N+1)^4} \|\Delta \theta\|_0^2,$$

$$\|\nabla(\theta - P_N \theta)\|_0^2 = \sum_{\alpha \in I_3 - I_{3,N}} \theta_\alpha^2 A_\alpha^2 \leq \frac{C_0^2}{(N+1)^2} \sum_{\alpha \in I_3 - I_{3,N}} \theta_\alpha^2 A_\alpha^4 \leq \frac{C_0^2}{(N+1)^2} \|\Delta \theta\|_0^2.$$

These lead (7) after taking square root of them. □

As usual, the L^∞ norms $\|\mathbf{u}\|_\infty$ and $\|\theta\|_\infty$ of $\mathbf{u} \in V$ and $\theta \in W$ are defined by

$$\|\mathbf{u}\|_\infty \equiv \sup_{\mathbf{x} \in \Omega} |\mathbf{u}(\mathbf{x})|, \quad \|\theta\|_\infty \equiv \sup_{\mathbf{x} \in \Omega} |\theta(\mathbf{x})|, \quad |\mathbf{u}| = \sqrt{u^2 + v^2 + w^2}, \quad |\theta| = \sqrt{\theta^2},$$

$$\mathbf{u} = [u, v, w], \quad u = \sum_{\alpha \in I_0} u_\alpha \phi_1^\alpha, \quad v = \sum_{\alpha \in I_0} v_\alpha \phi_2^\alpha, \quad w = \sum_{\alpha \in I_0} w_\alpha \phi_3^\alpha, \quad \theta = \sum_{\alpha \in I_3} \theta_\alpha \phi_3^\alpha.$$

For a fixed $\alpha \in I_0$, the vector $\xi_\alpha \Phi^\alpha + \eta_\alpha \Psi^\alpha$ can be written of the form: $[u_\alpha \phi_1^\alpha, v_\alpha \phi_2^\alpha, w_\alpha \phi_3^\alpha]$ with

$$u_\alpha \equiv -\frac{a\alpha_1 \alpha_3}{A_\alpha B_\alpha} \xi_\alpha + \frac{b\alpha_2}{B_\alpha} \eta_\alpha, \quad v_\alpha \equiv -\frac{b\alpha_2 \alpha_3}{A_\alpha B_\alpha} \xi_\alpha - \frac{a\alpha_1}{B_\alpha} \eta_\alpha, \quad w_\alpha \equiv \frac{B_\alpha}{A_\alpha} \xi_\alpha, \quad u_\alpha^2 + v_\alpha^2 + w_\alpha^2 = \xi_\alpha^2 + \eta_\alpha^2.$$

Now, the square sum $f_\alpha \equiv (u_\alpha \phi_1^\alpha)^2 + (v_\alpha \phi_2^\alpha)^2 + (w_\alpha \phi_1^\alpha)^2$ can be bounded on Ω as follows, setting $f_1 \equiv \cos^2(a\alpha_1 x)$, $f_2 \equiv \cos^2(b\alpha_2 y)$, $f_3 \equiv \cos^2(\alpha_3 z)$,

$$\begin{aligned} f_\alpha &= K_\alpha^2 [u_\alpha^2 (1-f_1) f_2 f_3 + v_\alpha^2 f_1 (1-f_2) f_3 + w_\alpha^2 f_1 f_2 (1-f_3)] \\ &\le K_\alpha^2 [u_\alpha^2 + v_\alpha^2 + w_\alpha^2][(1-f_1) f_2 f_3 + f_1(1-f_2) f_3 + f_1 f_2 (1-f_3)] \le K_\alpha^2 [\xi_\alpha^2 + \eta_\alpha^2], \end{aligned}$$

since, for each i, f_i on Ω takes value in the interval $[0,1]$. Thus, we have $\|\xi_\alpha \Phi^\alpha + \eta_\alpha \Psi^\alpha\|_\infty \le K_\alpha \sqrt{\xi_\alpha^2 + \eta_\alpha^2}$ for any $\alpha \in I_0$.

Lemma 2 *For any $(\mathbf{u}, \theta) \in X$, it holds that*

$$\begin{aligned} \|\mathbf{u}\|_\infty &\le \frac{\pi}{3}\sqrt{6 - \frac{2\pi^2}{5}}\, C_1 \|\Delta \mathbf{u}\|_0 < 1.50015 C_1 \|\Delta \mathbf{u}\|_0, \\ \|\theta\|_\infty &\le \frac{\pi}{3}\sqrt{6 - \frac{36\zeta(3)}{\pi^2} + \frac{\pi^2}{5}}\, C_1 \|\Delta \theta\|_0 < 1.98398 C_1 \|\Delta \theta\|_0, \end{aligned}$$

where $C_1 \equiv C_0^2 |\Omega|^{-\frac{1}{2}}$ depends only on Ω and $\zeta(s) \equiv \sum_{n=1}^\infty \frac{1}{n^s}$ is the Riemann zeta function for $s > 1$.

Proof From the above argument, $\|\mathbf{u}\|_\infty \le \sum_{\alpha \in I_0} \|\xi_\alpha \Phi^\alpha + \eta_\alpha \Psi^\alpha\|_\infty \le \sum_{\alpha \in I_0} K_\alpha \sqrt{\xi_\alpha^2 + \eta_\alpha^2}$. And the Cauchy-Schwartz inequality shows that

$$\|\mathbf{u}\|_\infty \le \sum_{\alpha \in I_0} K_\alpha \sqrt{\xi_\alpha^2 + \eta_\alpha^2} \le \sqrt{\sum_{\alpha \in I_0} K_\alpha^2 A_\alpha^{-4}} \sqrt{\sum_{\alpha \in I_0} \{\xi_\alpha^2 + \eta_\alpha^2\} A_\alpha^4} \le \sqrt{C} \|\Delta \mathbf{u}\|_0, \quad C \equiv \sum_{\alpha \in I_0} K_\alpha^2 A_\alpha^{-4}.$$

The indices subset I_0 can be decomposed into four mutually disjoint subsets:

$$\begin{aligned} I_0^{1,1,0} &\equiv [1,1,0] + \mathbb{Z}_0^2 \times \{0\}, \quad I_0^{1,0,1} \equiv [1,0,1] + \mathbb{Z}_0 \times \{0\} \times \mathbb{Z}_0, \\ I_0^{0,1,1} &\equiv [0,1,1] + \{0\} \times \mathbb{Z}_0^2, \quad I_0^{1,1,1} \equiv [1,1,1] + \mathbb{Z}_0^3. \end{aligned}$$

And the numbers $n(k)$ of non-negative integer solutions of $|\alpha| = k$ and the values K_α^2 on these subsets are:

$$n(k)\Big|_{I_0 - I_0^{1,1,1}} = k-1, \quad n(k)\Big|_{I_0^{1,1,1}} = \frac{(k-1)(k-2)}{2}, \quad K_\alpha^2\Big|_{I_0 - I_0^{1,1,1}} = \frac{4}{|\Omega|}, \quad K_\alpha^2\Big|_{I_0^{1,1,1}} = \frac{8}{|\Omega|}.$$

Due to (6) and the above relations, we can bound C as follows:

$$\begin{aligned} C &\le C_0^4 \sum_{\alpha \in I_0} K_\alpha^2 |\alpha|^{-4} = \frac{4C_0^4}{|\Omega|}\left[3\sum_{k=2}^\infty \frac{k-1}{k^4} + 2\sum_{k=3}^\infty \frac{(k-1)(k-2)}{2k^4}\right] = 4C_1^2 \sum_{k=1}^\infty \frac{(k-1)(k+1)}{k^4} \\ &= 4C_1^2 \sum_{k=1}^\infty \left[\frac{1}{k^2} - \frac{1}{k^4}\right] = 4C_1^2 \left[\frac{\pi^2}{6} - \frac{\pi^4}{90}\right] = \frac{\pi^2}{9}\left[6 - \frac{2\pi^2}{5}\right] C_1^2, \end{aligned}$$

which proves the first part of the lemma.
Next, taking account that $\|\theta\|_\infty \le \sum_{\alpha \in I_3} \left\|\theta_\alpha \phi_3^\alpha\right\|_\infty = \sum_{\alpha \in I_3} K_\alpha |\theta_\alpha|$, similar to the above, we have

$$\|\theta\|_\infty \le \sum_{\alpha \in I_3} K_\alpha |\theta_\alpha| \le \sqrt{\sum_{\alpha \in I_3} K_\alpha^2 A_\alpha^{-4}} \sqrt{\sum_{\alpha \in I_3} \theta_\alpha^2 A_\alpha^4} \le \sqrt{\tilde{C}} \|\Delta \theta\|_0, \quad \text{with} \quad \tilde{C} \equiv \sum_{\alpha \in I_3} K_\alpha^2 A_\alpha^{-4}.$$

The indices subset I_3 can be decomposed into four mutually disjoint subsets:

$$I_3^{0,0,1} \equiv [0,0,1] + \{0\}^2 \times \mathbb{Z}_0, I_3^{1,0,1} \equiv I_0^{1,0,1}, I_3^{0,1,1} \equiv I_0^{0,1,1}, I_3^{1,1,1} \equiv I_0^{1,1,1}.$$

And $n(k)$ and K_α^2 on these subsets are:

$$\begin{aligned} & n(k)\Big|_{I_3^{0,0,1}} = 1, \quad n(k)\Big|_{I_3^{1,0,1} \cup I_3^{0,1,1}} = k-1, \quad n(k)\Big|_{I_3^{1,1,1}} = \frac{(k-1)(k-2)}{2}, \\ & K_\alpha^2\Big|_{I_3^{0,0,1}} = \frac{2}{|\Omega|}, \quad K_\alpha^2\Big|_{I_3^{1,0,1} \cup I_3^{0,1,1}} = \frac{4}{|\Omega|}, \quad K_\alpha^2\Big|_{I_3^{1,1,1}} = \frac{8}{|\Omega|}. \end{aligned}$$

Thus, the bound $\tilde{C}$ can be obtained as follows:

$$\tilde{C} \le C_0^4 \sum_{\alpha\in I_3} K_\alpha^2 |\alpha|^{-4} = \frac{2C_0^4}{|\Omega|}\left[\sum_{k=1}^{\infty}\frac{1}{k^4} + 4\sum_{k=2}^{\infty}\frac{k-1}{k^4} + 4\sum_{k=3}^{\infty}\frac{(k-1)(k-2)}{2k^4}\right] = 2C_1^2 \sum_{k=1}^{\infty}\frac{2k^2-2k+1}{k^4}$$
$$= 2C_1^2 \sum_{k=1}^{\infty}\left[\frac{2}{k^2}-\frac{2}{k^3}+\frac{1}{k^4}\right] = 2C_1^2\left[\frac{\pi^2}{3} - 2\zeta(3) + \frac{\pi^4}{90}\right] = \frac{\pi^2}{9}\left[6 - \frac{36\zeta(3)}{\pi^2} + \frac{\pi^2}{5}\right] C_1^2,$$

which completes the proof. □

Since Ω is a rectangle with side lengths $\frac{2\pi}{a}$, $\frac{2\pi}{b}$, and π, we can also use the result in [6] as follows ($0 < a, b \le 1$):

$$\|\theta\|_\infty \le |\Omega|^{-\frac{1}{2}} \left[\gamma_0\|\theta\|_0 + \frac{\gamma_1 \pi c_1}{\sqrt{3}}\|\nabla\theta\|_0 + \frac{\gamma_2\pi^2 c_2}{3}\left\|\nabla^2\theta\right\|_0\right],$$
$$c_1 = \sqrt{4(a^{-2}+b^{-2})+1} \ge \sqrt{3}C_0,$$
$$c_2 = \sqrt{c_1^4 + \frac{4}{5}[16(a^{-4}+b^{-4})+1]} \ge \sqrt{\frac{19}{15}}c_1^2 \ge 3\sqrt{\frac{19}{15}}C_0^2,$$
$$\frac{\gamma_2\pi^2 c_2}{3} \ge 5.59879\,\gamma_2\frac{\pi}{3}\sqrt{6-\frac{36\zeta(3)}{\pi^2}+\frac{\pi^2}{5}}\,C_0^2 \ge 2.31862\,\frac{\pi}{3}\sqrt{6-\frac{36\zeta(3)}{\pi^2}+\frac{\pi^2}{5}}\,C_0^2.$$

Here, we have used the fact taht γ_2 is estimated as 0.41413 in [6]. Thus the coefficients in Lemma 2 are at least twice finer than those in [6].

Corollary 3 *Under the same assumptions of Theorem 1, the following holds for the same constant C_1 in Lemma 2:*

$$\|\mathbf{u} - P_N\mathbf{u}\|_\infty \le 2C_1\sqrt{\frac{1}{N} - \frac{1}{3(N+1)^3}}\,\|\Delta\mathbf{u}\|_0 < \frac{2C_1}{\sqrt{N}}\|\Delta\mathbf{u}\|_0, \tag{8a}$$
$$\|\theta - Q_N\theta\|_\infty \le 2C_1\sqrt{\frac{1}{N} - \frac{1}{2(N+1)^2} + \frac{1}{6N^3}}\,\|\Delta\theta\|_0 < \frac{2C_1}{\sqrt{N}}\|\Delta\theta\|_0. \tag{8b}$$

Proof We can proceed as that of Lemma 2 and need to change C for each case. For $\|\mathbf{u} - P_N\mathbf{u}\|_\infty$,

$$C \equiv \sum_{\alpha\in I_0 - I_{0,N}} K_\alpha^2 A_\alpha^{-4} \le C_0^4 \sum_{\alpha\in I_0-I_{0,N}} K_\alpha^2|\alpha|^{-4} = 4C_1^2\left[3\sum_{k=N+1}^{\infty}\frac{k-1}{k^4} + 2\sum_{k=N+1}^{\infty}\frac{(k-1)(k-2)}{2k^4}\right]$$
$$= 4C_1^2\sum_{k=N+1}^{\infty}\frac{(k-1)(k+1)}{k^4} = 4C_1^2\sum_{k=N+1}^{\infty}\left[\frac{1}{k^2}-\frac{1}{k^4}\right] < 4C_1^2\left[\int_N^\infty \frac{1}{x^2}\,dx - \int_{N+1}^\infty\frac{1}{x^4}\,dx\right]$$
$$= 4\left[\frac{1}{N} - \frac{1}{3(N+1)^3}\right]C_1^2 < \frac{4}{N}C_1^2.$$

Similarly, we have the following estimates for $\|\theta - Q_N\theta\|_\infty$:

$$\tilde{C} \equiv \sum_{\alpha\in I_3 - I_{3,N}} K_\alpha^2 A_\alpha^{-4} \le C_0^4 \sum_{\alpha\in I_3-I_{3,N}} K_\alpha^2|\alpha|^{-4} = 2C_1^2\left[\sum_{k=N+1}^{\infty}\frac{1}{k^4} + 4\sum_{k=N+1}^{\infty}\frac{k-1}{k^4} + 4\sum_{k=N+1}^{\infty}\frac{(k-1)(k-2)}{2k^4}\right]$$
$$= 2C_1^2\sum_{k=N+1}^{\infty}\frac{2k^2-2k+1}{k^4} = 2C_1^2\sum_{k=N+1}^{\infty}\left[\frac{2}{k^2}-\frac{2}{k^3}+\frac{1}{k^4}\right] < 2C_1^2\left[\int_N^\infty\frac{2}{x^2}\,dx - \int_{N+1}^\infty\frac{2}{x^3}\,dx + \int_N^\infty\frac{1}{x^4}\,dx\right]$$
$$= 2\left[\frac{2}{N} - \frac{1}{(N+1)^2} + \frac{1}{3N^3}\right]C_1^2 = 4\left[\frac{1}{N} - \frac{1}{2(N+1)^2} + \frac{1}{6N^3}\right]C_1^2 < \frac{4}{N}C_1^2.$$

The last inequality holds due to the assumption $N \ge 2$. □

3 A fixed point formulation

The steady state solution of (3) can be written of the form:

$$-\Delta \mathbf{u} + \nabla p = \mathbf{f}(\mathbf{u}, \theta), \tag{9a}$$
$$\nabla\cdot \mathbf{u} = 0, \tag{9b}$$
$$-\Delta \theta = g(\mathbf{u}, \theta), \tag{9c}$$

where the right hand sides of (9) are defined by

$$\mathbf{f}(\mathbf{u}, \theta) = -\frac{1}{\mathscr{P}}(\mathbf{u}\cdot\nabla)\mathbf{u} + \mathscr{R}\theta\mathbf{e}_z, \quad g(\mathbf{u}, \theta) = -(\mathbf{u}\cdot\nabla)\theta + w.$$

If $\mathbf{u} \in V$, then $P_N\mathbf{u}$ satisfies (9b) and $-\Delta P_N \mathbf{u}$ converges to $-\Delta \mathbf{u}$ in L^2 sense. Taking inner product for both sides of (9a) with $-\Delta P_N \mathbf{u}$, we have

$$\begin{aligned}\langle -\Delta \mathbf{u}, -\Delta P_N \mathbf{u}\rangle &= \langle -\Delta \mathbf{u}, -\Delta P_N \mathbf{u}\rangle + \langle p, \Delta\nabla\cdot P_N \mathbf{u}\rangle = \langle -\Delta \mathbf{u}, -\Delta P_N \mathbf{u}\rangle + \langle p, \nabla\cdot\Delta P_N \mathbf{u}\rangle \\ &= \langle -\Delta \mathbf{u}, -\Delta P_N \mathbf{u}\rangle + \langle \nabla p, -\Delta P_N \mathbf{u}\rangle = \langle \mathbf{f}, -\Delta P_N \mathbf{u}\rangle.\end{aligned}$$

This identity converges to $\langle -\Delta \mathbf{u}, -\Delta \mathbf{u}\rangle = \langle \mathbf{f}, -\Delta \mathbf{u}\rangle$ which means $\|\Delta \mathbf{u}\|_0 \le \|\mathbf{f}\|_0$. And taking inner product for both sides of (9c) with $-\Delta\theta$, we have $\langle -\Delta\theta, -\Delta\theta\rangle = \langle g, -\Delta\theta\rangle$ and $\|\Delta\theta\|_0 \le \|g\|_0$.

Now, setting $F(\mathbf{u}, \theta) \equiv (\mathbf{f}(\mathbf{u}, \theta), g(\mathbf{u}, \theta))$, the weak form of (9) is written as :

$$\langle \nabla(\mathbf{u}, \theta), \nabla(\mathbf{v}, \vartheta)\rangle = \langle F(\mathbf{u}, \theta), (\mathbf{v}, \vartheta)\rangle, \quad {}^{\forall}(\mathbf{v}, \vartheta) \in X.$$

We call the solution operator $\mathscr{S}$ for (9) as *Stokes operator*. Thus $(\mathbf{u}, \theta) = \mathscr{S}F(\mathbf{u}, \theta)$ means

$$\langle \nabla\mathscr{S}F(\mathbf{u}, \theta), \nabla(\mathbf{v}, \vartheta)\rangle = \langle F(\mathbf{u}, \theta), (\mathbf{v}, \vartheta)\rangle, \quad {}^{\forall}(\mathbf{v}, \vartheta) \in X. \tag{10}$$

Note that we always have $\mathscr{S}^{-1}(\mathbf{u}, \theta) = (-\Delta \mathbf{u} + \nabla p, -\Delta\theta)$ with an associated pressure $p = p(\mathbf{u}, \theta)$.

Usually, we use Newton's method (see [4]) to get an approximate solution $(\mathbf{u}_N, \theta_N) \in X_N$ of (9) and define the approximate pressure p_N by

$$\nabla p_N \equiv \mathbf{f}_N(\mathbf{u}_N, \theta_N) + \Delta \mathbf{u}_N,$$

where $\mathbf{f}_N$ is the truncation up to $I_{,N}$ of the expansion of $\mathbf{f}$. For the solution $(\mathbf{u}, \theta)$ of (9) with its associated pressure p, let $(\bar{\mathbf{u}}, \bar{\theta}) \equiv (\mathbf{u} - \mathbf{u}_N, \theta - \theta_N)$ and $\bar{p} \equiv p - p_N$. Then we have the following residual equations:

$$-\Delta \bar{\mathbf{u}} + \nabla \bar{p} = \mathbf{f}(\mathbf{u}_N + \bar{\mathbf{u}}, \theta_N + \bar{\theta}) + \Delta \mathbf{u}_N - \nabla p_N, \tag{11a}$$
$$\nabla\cdot \bar{\mathbf{u}} = 0, \tag{11b}$$
$$-\Delta \bar{\theta} = g(\mathbf{u}_N + \bar{\mathbf{u}}, \theta_N + \bar{\theta}) + \Delta\theta_N, \tag{11c}$$

Set $\bar{F}(\bar{\mathbf{u}}, \bar{\theta}) \equiv (\mathbf{f}(\mathbf{u}_N + \bar{\mathbf{u}}, \theta_N + \bar{\theta}) + \Delta \mathbf{u}_N - \nabla p_N, g(\mathbf{u}_N + \bar{\mathbf{u}}, \theta_N + \bar{\theta}) + \Delta\theta_N) \equiv (\bar{\mathbf{f}}(\bar{\mathbf{u}}, \bar{\theta}), \bar{g}(\bar{\mathbf{u}}, \bar{\theta}))$, then the Stokes operator $\mathscr{S}$ gives us a fixed point problem from (11):

$$(\bar{\mathbf{u}}, \bar{\theta}) = \mathscr{S}\bar{F}(\bar{\mathbf{u}}, \bar{\theta}) \equiv \mathscr{K}(\bar{\mathbf{u}}, \bar{\theta}). \tag{12}$$

Since $X \subset H^1(\Omega)^4$, $\mathscr{K}$ is a compact operator on X . Hence by Schauder's fixed point theorem, if we find a nonempty, closed, convex, and bounded set $U \subset X$ satisfying $\mathscr{K}U \subset U$, then there exists a solution of (12) in U which is called a candidate set.

Define $\mathbf{P}_N : X \to X_N$ by $\mathbf{P}_N = (P_N, Q_N)$, then (5) can be simplified as: for $(\mathbf{u}, \theta) \in X$

$$\langle \nabla((\mathbf{u}, \theta) - \mathbf{P}_N(\mathbf{u}, \theta)), \nabla(\mathbf{v}, \vartheta)\rangle = 0, \quad {}^{\forall}(\mathbf{v}, \vartheta) \in X_N. \tag{13}$$

Then (12) can be decomposed into two parts:

$$\mathbf{P}_N(\bar{\mathbf{u}}, \bar{\theta}) = \mathbf{P}_N\mathscr{K}(\bar{\mathbf{u}}, \bar{\theta}), \tag{14a}$$
$$(I - \mathbf{P}_N)(\bar{\mathbf{u}}, \bar{\theta}) = (I - \mathbf{P}_N)\mathscr{K}(\bar{\mathbf{u}}, \bar{\theta}). \tag{14b}$$

The Fréchet derivative $F'(\mathbf{u},\theta)$ of F at $(\mathbf{u},\theta)$ has the form: for any $(\bar{\mathbf{u}},\bar{\theta})\in X$,

$$
\begin{aligned}
F'(\mathbf{u},\theta)(\bar{\mathbf{u}},\bar{\theta}) &\equiv \left(\mathbf{f}'(\mathbf{u},\theta)(\bar{\mathbf{u}},\bar{\theta}), g'(\mathbf{u},\theta)(\bar{\mathbf{u}},\bar{\theta})\right),\\
\mathbf{f}'(\mathbf{u},\theta)(\bar{\mathbf{u}},\bar{\theta}) &\equiv -\frac{1}{\mathscr{P}}\left[(\mathbf{u}\cdot\nabla)\bar{\mathbf{u}}+(\bar{\mathbf{u}}\cdot\nabla)\mathbf{u}\right]+\mathscr{R}\bar{\theta}\mathbf{e}_z,\\
g'(\mathbf{u},\theta)(\bar{\mathbf{u}},\bar{\theta}) &\equiv -\left[(\mathbf{u}\cdot\nabla)\bar{\theta}+(\bar{\mathbf{u}}\cdot\nabla)\theta\right]+\bar{w}.
\end{aligned}
$$

Now, define $\mathscr{L}_N : X_N \to X_N$ by

$$
\mathscr{L}_N \equiv \mathbf{P}_N\left[I-\mathscr{S}F'(\mathbf{u}_N,\theta_N)\right]\Big|_{X_N},
$$

and assume $\mathscr{L}_N$ is regular or one-to-one and onto. And we can express $\mathscr{L}_N$ as:

$$
\mathscr{L}_N = \mathbf{P}_N\mathscr{S}[\mathscr{S}^{-1}-F'(\mathbf{u}_N,\theta_N)]\Big|_{X_n} = \mathbf{P}_N\mathscr{S}\mathscr{L}_0, \quad \mathscr{L}_0 \equiv [\mathscr{S}^{-1}-F'(\mathbf{u}_N,\theta_N)]\Big|_{X_n}.
$$

Define the Newton–like iteration operator $\mathscr{N} : X \to X_N$ for (12) and the new map $\mathscr{T}$ as follows:

$$
\mathscr{N} \equiv \mathbf{P}_N - \mathscr{L}_N^{-1}\mathbf{P}_N(I-\mathscr{K}), \quad \mathscr{T} \equiv \mathscr{N} + (I-\mathbf{P}_N)\mathscr{K}.
$$

The second part of $\mathscr{T}$ is expected to be small or contractive if the truncation number N is sufficiently large. The operator $\mathscr{N}$ is also compact since it maps X into the finite dimensional space X_N, and so is $\mathscr{T}$.

Lemma 4 *The problem (14) is equivalent to the following fixed point problem:*

$$
(\bar{\mathbf{u}},\bar{\theta}) = \mathscr{T}(\bar{\mathbf{u}},\bar{\theta}). \tag{15}
$$

Proof Assume $(\bar{\mathbf{u}},\bar{\theta})\in X$ satisfies (14), then $\mathscr{N}(\bar{\mathbf{u}},\bar{\theta}) = \mathbf{P}_N(\bar{\mathbf{u}},\bar{\theta})$ which means

$$
\mathscr{T}(\bar{\mathbf{u}},\bar{\theta}) = \mathbf{P}_N(\bar{\mathbf{u}},\bar{\theta}) + (I-\mathbf{P}_N)\mathscr{K}(\bar{\mathbf{u}},\bar{\theta}) = \mathbf{P}_N(\bar{\mathbf{u}},\bar{\theta}) + (I-\mathbf{P}_N)(\bar{\mathbf{u}},\bar{\theta}) = (\bar{\mathbf{u}},\bar{\theta}).
$$

Thus $(\bar{\mathbf{u}},\bar{\theta})$ satisfies (15). On the other hand, if $(\bar{\mathbf{u}},\bar{\theta})$ satisfies (15), then

$$
\begin{aligned}
\mathbf{P}_N(\bar{\mathbf{u}},\bar{\theta}) &= \mathbf{P}_N\mathscr{T}(\bar{\mathbf{u}},\bar{\theta}) = \mathbf{P}_N\mathscr{N}(\bar{\mathbf{u}},\bar{\theta}) = \mathbf{P}_N(\bar{\mathbf{u}},\bar{\theta}) - \mathscr{L}_N^{-1}\mathbf{P}_N(I-\mathscr{K})(\bar{\mathbf{u}},\bar{\theta})\\
&\Rightarrow \mathscr{L}_N^{-1}\mathbf{P}_N(I-\mathscr{K})(\bar{\mathbf{u}},\bar{\theta}) = 0 \Rightarrow \mathbf{P}_N(I-\mathscr{K})(\bar{\mathbf{u}},\bar{\theta}) = 0 \Rightarrow \mathbf{P}_N(\bar{\mathbf{u}},\bar{\theta}) = \mathbf{P}_N\mathscr{K}(\bar{\mathbf{u}},\bar{\theta}),\\
(I-\mathbf{P}_N)(\bar{\mathbf{u}},\bar{\theta}) &= (I-\mathbf{P}_N)\mathscr{T}(\bar{\mathbf{u}},\bar{\theta}) = (I-\mathbf{P}_N)\mathscr{K}(\bar{\mathbf{u}},\bar{\theta}).
\end{aligned}
$$

Here the second implication is due to the assumption on the regularity of $\mathscr{L}_N$. □

From Lemma 4, we have an alternative verification condition of the form: $\mathscr{T}U \subset U$ for a candidate set U which is nonempty, closed, convex, and bounded in X.

Now, for given real numbers $\overline{\xi}_\alpha, \overline{\eta}_\alpha, \overline{\theta}_\alpha \geq 0$, set real intervals as $[\overline{\xi}_\alpha] \equiv [-\overline{\xi}_\alpha, \overline{\xi}_\alpha]$, $[\overline{\eta}_\alpha] \equiv [-\overline{\eta}_\alpha, \overline{\eta}_\alpha]$, $[\overline{\theta}_\alpha] \equiv [-\overline{\theta}_\alpha, \overline{\theta}_\alpha]$, and define

$$
U_N \equiv \left\{ \left(\sum_{\alpha\in I_{0,N}} \{\xi_\alpha \Phi^\alpha + \eta_\alpha \Psi^\alpha\}, \sum_{\alpha\in I_{3,N}} \theta_\alpha \phi_3^\alpha \right) \in X_N \ : \ \xi_\alpha \in [\overline{\xi}_\alpha], \eta_\alpha \in [\overline{\eta}_\alpha], \theta_\alpha \in [\overline{\theta}_\alpha] \right\}. \tag{16}
$$

And for given $m_1, m_2 \geq 0$, we define

$$
U_* \equiv \left\{ (\mathbf{u},\theta) \in X_N^\perp \ : \ \begin{array}{l} \|\mathbf{u}\|_0 \leq \frac{C_0^2}{(N+1)^2} m_1,\ \|\nabla\mathbf{u}\|_0 \leq \frac{C_0}{N+1} m_1,\ \|\mathbf{u}\|_\infty \leq \frac{2C_1}{\sqrt{N}} m_1, \\ \|\theta\|_0 \leq \frac{C_0^2}{(N+1)^2} m_2,\ \|\nabla\theta\|_0 \leq \frac{C_0}{N+1} m_2 \end{array} \right\}. \tag{17}
$$

Here $X_N^\perp$ is the orthogonal complement of X_N in X with respect to the projection $\mathbf{P}_N$ defined by (13). Now, set $U \equiv U_N \oplus U_*$, then we obtain:

Theorem 5 *Let U_N, U_* and U be sets defined as above. If*

$$
\mathscr{N}U \subset U_N, \tag{18a}
$$

$$
(I-\mathbf{P}_N)\mathscr{K}U \subset U_*. \tag{18b}
$$

then there exists a fixed point of $\mathscr{T}$ in U.

Proof Clearly, $(\mathbf{0},0)\in U$ which means U is non-empty. Due to the definition, U is closed, convex and bounded in X. Under the condition (18), we have

$$
\mathscr{T}U \subset \mathscr{N}U + (I-\mathbf{P}_N)\mathscr{K}U \subset U_N + U_* = U.
$$

Since $\mathscr{T}$ is compact, there exists a fixed point of $\mathscr{T}$ in U by Schauder's fixed point theorem. □

4 Computable verification conditions

To construct the candidate set U in X satisfying (18), we use an algorithm based on iterative scheme as in [7].

First, set the initial values $\overline{\xi}_\alpha^{(0)} = \overline{\eta}_\alpha^{(0)} = \overline{\theta}_\alpha^{(0)} = 0$, and $m_1^{(0)} = m_2^{(0)} = 0$, which means $U^{(0)}$ contains only one element $(\mathbf{0}, 0)$. For $k \geq 0$, with a fixed inflation factor $0 < \delta \ll 1$, set

$$\overline{\xi}_\alpha^{(k+\frac{1}{2})} = \overline{\xi}_\alpha^{(k)}(1+\delta), \quad \overline{\eta}_\alpha^{(k+\frac{1}{2})} = \overline{\eta}_\alpha^{(k)}(1+\delta), \quad \overline{\theta}_\alpha^{(k+\frac{1}{2})} = \overline{\theta}_\alpha^{(k)}(1+\delta), \quad m_i^{(k+\frac{1}{2})} = m_i^{(k)}(1+\delta), i = 1, 2,$$

which define δ–inflations $U_N^{(k+\frac{1}{2})}$ and $U_*^{(k+\frac{1}{2})}$ of $U_N^{(k)}$ and $U_*^{(k)}$ respectively. Set the δ–inflation $U^{(k+\frac{1}{2})}$ of $U^{(k)}$ as the direct sum of $U_N^{(k+\frac{1}{2})}$ and $U_*^{(k+\frac{1}{2})}$, i.e., $U^{(k+\frac{1}{2})} \equiv U_N^{(k+\frac{1}{2})} \oplus U_*^{(k+\frac{1}{2})}$. Now, $U^{(k+1)}$ can be constructed as the direct sum of $U_N^{(k+1)}$ and $U_*^{(k+1)}$ as follows:

$$U_N^{(k+1)} \equiv \mathscr{N} U^{(k+\frac{1}{2})}, \quad m_1^{(k+1)} \equiv \left\|\bar{\mathbf{f}}(U^{(k+\frac{1}{2})})\right\|_0, \quad m_2^{(k+1)} \equiv \left\|\bar{g}(U^{(k+\frac{1}{2})})\right\|_0, \tag{19}$$

where $\|f(U)\|_0 \equiv \sup\{\, \|f(\mathbf{u},\theta)\|_0 : (\mathbf{u},\theta) \in U\,\}$ for any function f. Note that $U^{(k+1)}$ cannot be calculated exactly, but its over–estimated enclosure can be obtained and will be set as $U_N^{(k+1)}$ in the actual calculation on a computer. Thus the verification condition in a computer is:

Theorem 6 *For some k, if the following conditions*

$$\overline{\xi}_\alpha^{(k+1)} < \overline{\xi}_\alpha^{(k+\frac{1}{2})}, \quad \overline{\eta}_\alpha^{(k+1)} < \overline{\eta}_\alpha^{(k+\frac{1}{2})}, \quad \overline{\theta}_\alpha^{(k+1)} < \overline{\theta}_\alpha^{(k+\frac{1}{2})}, \quad m_i^{(k+1)} < m_i^{(k+\frac{1}{2})}, i = 1, 2, \tag{20}$$

hold, then the set $U^{(k+\frac{1}{2})}$ contains an element $(\bar{\mathbf{u}}, \bar{\theta})$ satisfying $(\bar{\mathbf{u}}, \bar{\theta}) = \mathscr{T}(\bar{\mathbf{u}}, \bar{\theta})$.

Proof Due to Theorem 5, it is sufficient to check (18) holds for $U^{(k+\frac{1}{2})}$. By the condition (20) and the definition (19), we have $\mathscr{N} U^{(k+\frac{1}{2})} = U_N^{(k+1)} \subset U_N^{(k+\frac{1}{2})}$. And for any $(\mathbf{u}, \theta) \in (I - \mathbf{P}_N)\mathscr{K} U^{(k+\frac{1}{2})}$, there exists $(\bar{\mathbf{u}}, \bar{\theta}) \in U^{(k+\frac{1}{2})}$ such that $(\mathbf{u}, \theta) = (I - \mathbf{P}_N)\mathscr{S}\bar{F}(\bar{\mathbf{u}}, \bar{\theta})$. Using Theorem 1, Corollary 3 and (19), we obtain

$$\begin{aligned}
\|\mathbf{u}\|_0 &= \left\|(I-P_N)\Pi_1 \mathscr{S}\bar{F}(\bar{\mathbf{u}},\bar{\theta})\right\|_0 \leq \frac{C_0^2}{(N+1)^2}\left\|\bar{\mathbf{f}}(\bar{\mathbf{u}},\bar{\theta})\right\|_0 \leq \frac{C_0^2}{(N+1)^2} m_1^{(k+1)} < \frac{C_0^2}{(N+1)^2} m_1^{(k+\frac{1}{2})}, \\
\|\nabla\mathbf{u}\|_0 &= \left\|\nabla(I-P_N)\Pi_1 \mathscr{S}\bar{F}(\bar{\mathbf{u}},\bar{\theta})\right\|_0 \leq \frac{C_0}{N+1}\left\|\bar{\mathbf{f}}(\bar{\mathbf{u}},\bar{\theta})\right\|_0 \leq \frac{C_0}{N+1} m_1^{(k+1)} < \frac{C_0}{N+1} m_1^{(k+\frac{1}{2})}, \\
\|\mathbf{u}\|_\infty &= \left\|(I-P_N)\Pi_1 \mathscr{S}\bar{F}(\bar{\mathbf{u}},\bar{\theta})\right\|_\infty \leq \frac{2C_1}{\sqrt{N}}\left\|\bar{\mathbf{f}}(\bar{\mathbf{u}},\bar{\theta})\right\|_0 \leq \frac{2C_1}{\sqrt{N}} m_1^{(k+1)} < \frac{2C_1}{\sqrt{N}} m_1^{(k+\frac{1}{2})}, \\
\|\theta\|_0 &= \left\|(I-Q_N)\Pi_2 \mathscr{S}\bar{F}(\bar{\mathbf{u}},\bar{\theta})\right\|_0 \leq \frac{C_0^2}{(N+1)^2}\left\|\bar{g}(\bar{\mathbf{u}},\bar{\theta})\right\|_0 \leq \frac{C_0^2}{(N+1)^2} m_2^{(k+1)} < \frac{C_0^2}{(N+1)^2} m_2^{(k+\frac{1}{2})}, \\
\|\nabla\theta\|_0 &= \left\|\nabla(I-Q_N)\Pi_2 \mathscr{S}\bar{F}(\bar{\mathbf{u}},\bar{\theta})\right\|_0 \leq \frac{C_0}{N+1}\left\|\bar{g}(\bar{\mathbf{u}},\bar{\theta})\right\|_0 \leq \frac{C_0}{N+1} m_2^{(k+1)} < \frac{C_0}{N+1} m_2^{(k+\frac{1}{2})},
\end{aligned}$$

where $\Pi_1 : X \to V$ and $\Pi_2 : X \to W$ are the natural projections from X to V and W, respectively. These estimates mean that $(\mathbf{u}, \theta) \in U_*^{(k+\frac{1}{2})}$ and $(I - \mathbf{P}_N)\mathscr{K} U^{(k+\frac{1}{2})} \subset U_*^{(k+\frac{1}{2})}$ holds. □

To determine the finite dimensional set $U_N^{(k+1)}$ in (19), we need to compute $\mathscr{N}$ on U. At first, from definitions of $\mathscr{L}_N$ and $\mathscr{K}$, we can rewrite $\mathscr{N}$ as follows:

$$\mathscr{N} = \mathscr{L}_N^{-1}\left[\mathscr{L}_N \mathbf{P}_N - \mathbf{P}_N + \mathbf{P}_N \mathscr{K}\right] = \mathscr{L}_N^{-1}\mathbf{P}_N \mathscr{S} F_0, \quad F_0 \equiv \bar{F} - F'(\mathbf{u}_N, \theta_N)\mathbf{P}_N.$$

For any fixed $(\bar{\mathbf{u}}, \bar{\theta}) \in U$, set $(\mathbf{u}_h, \theta_h) \equiv \mathscr{N}(\bar{\mathbf{u}}, \bar{\theta})$ and operate $\mathscr{L}_N$ on both sides, then

$$\mathbf{P}_N \mathscr{S}\mathscr{L}_0(\mathbf{u}_h, \theta_h) = \mathscr{L}_N(\mathbf{u}_h, \theta_h) = \mathbf{P}_N \mathscr{S} F_0(\bar{\mathbf{u}}, \bar{\theta}).$$

Using the projection property (13) of $\mathbf{P}_N$, we can derive

$$\langle \nabla \mathscr{S}\mathscr{L}_0(\mathbf{u}_h, \theta_h), \nabla(\mathbf{v}_N, \vartheta_N)\rangle = \left\langle \nabla \mathscr{S} F_0(\bar{\mathbf{u}}, \bar{\theta}), \nabla(\mathbf{v}_N, \vartheta_N)\right\rangle, \quad {}^\forall(\mathbf{v}_N, \vartheta_N) \in X_N$$

Due to (10), this can be written as

$$\langle \mathscr{L}_0(\mathbf{u}_h,\theta_h),(\mathbf{v}_N,\vartheta_N)\rangle = \langle F_0(\bar{\mathbf{u}},\bar{\theta}),(\mathbf{v}_N,\vartheta_N)\rangle, \quad {}^\forall(\mathbf{v}_N,\vartheta_N)\in X_N. \tag{21}$$

The left hand side of (21) can be written as:

$$\begin{aligned}\langle \mathscr{L}_0(\mathbf{u}_h,\theta_h),(\mathbf{v}_N,\vartheta_N)\rangle &= \langle \mathscr{S}^{-1}(\mathbf{u}_h,\theta_h) - F'(\mathbf{u}_N,\theta_N)(\mathbf{u}_h,\theta_h),(\mathbf{v}_N,\vartheta_N)\rangle \\ &= \langle \nabla(\mathbf{u}_h,\theta_h),\nabla(\mathbf{v}_N,\vartheta_N)\rangle - \langle F'(\mathbf{u}_N,\theta_N)(\mathbf{u}_h,\theta_h),(\mathbf{v}_N,\vartheta_N)\rangle.\end{aligned}$$

This gives us the interval version of the Jacobian matrix in Newton's method with respect to the base functions Φ^α, Ψ^α, and ϕ_3^α. The right hand side of (21) forms an interval vector whose elements can be enclosed with upper and lower bounds. Thus the operator $\mathscr{L}_N$ is regular when the solution $(\mathbf{u}_h,\theta_h)\in X_N$ for (21) exists under the guaranteed computation with interval arithmetic.

In order to compute $m_i^{(k+1)}$ in (19), we need to estimate $\|\bar{\mathbf{f}}(\mathbf{u},\theta)\|_0$ and $\|\bar{g}(\mathbf{u},\theta)\|_0$ for any $(\mathbf{u},\theta)\in U$. Pick up an element $(\mathbf{u},\theta)\equiv(\mathbf{u}_h+\mathbf{u}_*,\theta_h+\theta_*)\in U$ with $(\mathbf{u}_h,\theta_h)\in U_N$, $(\mathbf{u}_*,\theta_*)\in U_*$, then we have

$$\begin{aligned}\bar{\mathbf{f}}(\mathbf{u},\theta) &= \mathbf{f}(\mathbf{u}_N+\mathbf{u}_h+\mathbf{u}_*,\theta_N+\theta_h+\theta_*) + \Delta\mathbf{u}_N - \nabla p_N \\ &= \mathbf{f}(\mathbf{u}_N+\mathbf{u}_h,\theta_N+\theta_h) + \Delta\mathbf{u}_N - \nabla p_N \\ &- \frac{1}{\mathscr{P}}[((\mathbf{u}_N+\mathbf{u}_h)\cdot\nabla)\mathbf{u}_* + (\mathbf{u}_*\cdot\nabla)(\mathbf{u}_N+\mathbf{u}_h) + (\mathbf{u}_*\cdot\nabla)\mathbf{u}_*] + \mathscr{R}\theta_*\mathbf{e}_z, \\ \bar{g}(\mathbf{u},\theta) &= g(\mathbf{u}_N+\mathbf{u}_h+\mathbf{u}_*,\theta_N+\theta_h+\theta_*) + \Delta\theta_N \\ &= g(\mathbf{u}_N+\mathbf{u}_h,\theta_N+\theta_h) + \Delta\theta_N \\ &- [((\mathbf{u}_N+\mathbf{u}_h)\cdot\nabla)\theta_* + (\mathbf{u}_*\cdot\nabla)(\theta_N+\theta_h) + (\mathbf{u}_*\cdot\nabla)\theta_*] + w_*.\end{aligned}$$

These forms enable us to estimate the desired norms as follows:

$$\begin{aligned}\|\bar{\mathbf{f}}(\mathbf{u},\theta)\|_0 &\le \|\mathbf{f}(\mathbf{u}_N+\mathbf{u}_h,\theta_N+\theta_h)+\Delta\mathbf{u}_N-\nabla p_N\|_0 \\ &+ \frac{1}{\mathscr{P}}[\|\mathbf{u}_N+\mathbf{u}_h\|_\infty\|\nabla\mathbf{u}_*\|_0 + \|\mathbf{u}_*\|_0\|\nabla(\mathbf{u}_N+\mathbf{u}_h)\|_\infty + \|\mathbf{u}_*\|_\infty\|\nabla\mathbf{u}_*\|_0] + \mathscr{R}\|\theta_*\|_0 \\ &\le \|\mathbf{f}(\mathbf{u}_N+\mathbf{u}_h,\theta_N+\theta_h)+\Delta\mathbf{u}_N-\nabla p_N\|_0 \\ &+ \frac{1}{\mathscr{P}}\left[\frac{C_0}{N+1}\|\mathbf{u}_N+\mathbf{u}_h\|_\infty + \frac{C_0^2}{(N+1)^2}\|\nabla(\mathbf{u}_N+\mathbf{u}_h)\|_\infty\right]m_1 + \frac{2C_0C_1}{(N+1)\sqrt{N}}m_1^2 + \mathscr{R}\frac{C_0^2}{(N+1)^2}m_2, \\ \|\bar{g}(\mathbf{u},\theta)\|_0 &\le \|g(\mathbf{u}_N+\mathbf{u}_h,\theta_N+\theta_h)+\Delta\theta_N\|_0 \\ &+ [\|\mathbf{u}_N+\mathbf{u}_h\|_\infty\|\nabla\theta_*\|_0 + \|\mathbf{u}_*\|_0\|\nabla(\theta_N+\theta_h)\|_\infty + \|\mathbf{u}_*\|_\infty\|\nabla\theta_*\|_0] + \|w_*\|_0 \\ &\le \|g(\mathbf{u}_N+\mathbf{u}_h,\theta_N+\theta_h)+\Delta\theta_N\|_0 \\ &+ \frac{C_0}{N+1}\|\mathbf{u}_N+\mathbf{u}_h\|_\infty m_2 + \frac{C_0^2}{(N+1)^2}\|\nabla(\theta_N+\theta_h)\|_\infty m_1 + \frac{2C_0C_1}{(N+1)\sqrt{N}}m_1m_2 + \frac{C_0^2}{(N+1)^2}m_1.\end{aligned}$$

Note that upper bounds of L^2 and L^∞ norms for $(\mathbf{u}_N+\mathbf{u}_h,\theta_N+\theta_h)\in U_N\subset X_N$ can be computed by interval arithmetic, and these calculation may have additional inflations due to crude estimates. Thus we estimate $\|(I-P_N)\Pi_1\mathscr{S}\bar{F}(\bar{\mathbf{u}},\bar{\theta})\|_0$, $\|\nabla(I-P_N)\Pi_1\mathscr{S}\bar{F}(\bar{\mathbf{u}},\bar{\theta})\|_0$, $\|(I-P_N)\Pi_1\mathscr{S}\bar{F}(\bar{\mathbf{u}},\bar{\theta})\|_\infty$ $\|(I-Q_N)\Pi_2\mathscr{S}\bar{F}(\bar{\mathbf{u}},\bar{\theta})\|_0$, and $\|\nabla(I-Q_N)\Pi_2\mathscr{S}\bar{F}(\bar{\mathbf{u}},\bar{\theta})\|_0$ after decomposition of $\Pi_1\mathscr{S}\bar{F}(\bar{\mathbf{u}},\bar{\theta})$ and $\Pi_2\mathscr{S}\bar{F}(\bar{\mathbf{u}},\bar{\theta})$ into finite and infinite parts, which gives us more accurate values of them and efficient estimates in real computations.

5 Numerical results

For the interval arithmetic, we use the PROFIL package [2] on Linux Intel Pentium 4 (3.8 GHz) machine.

We set $a^2=\frac{1}{8}$, $b^2=\frac{3}{8}$ and $\mathscr{P}=10$ in the numerical experiments with 1% inflation factor. Then the critical Rayleigh number $\mathscr{R}_c=6.75$ can be attained at some special mode α (see [5] for detail). We show several approximation results in figures 1-3. In these figures, the isothermal lines are drawn after adding the conduction solution (2) on the left, and contour lines of speed with streamlines are shown on the right. Note that streamlines for each type never change their shape during the change of relative Rayleigh numbers in short range. We present figures at the same relative Rayleigh number $\mathscr{R}/\mathscr{R}_c=1.1$ for easy comparison.

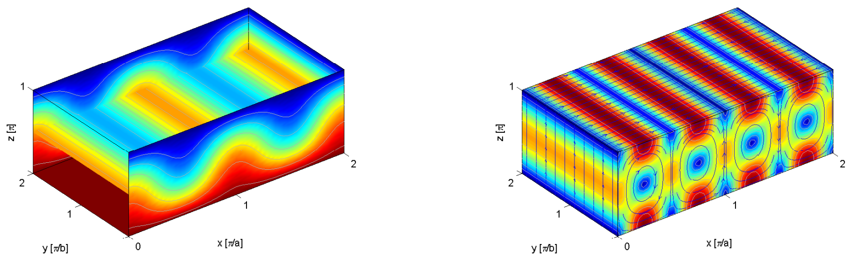

Fig. 1 Isothermal lines, and contour lines of speed with streamlines for roll type at $\mathscr{R}/\mathscr{R}_c = 1.1$.

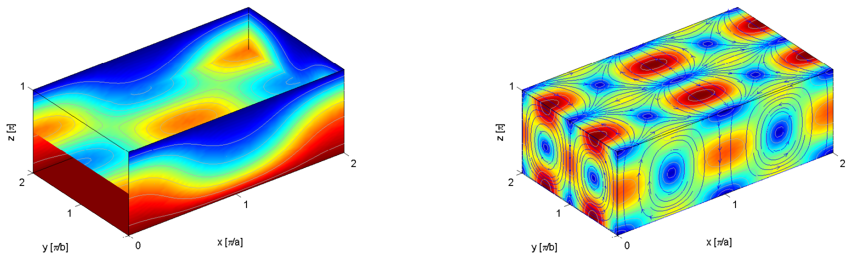

Fig. 2 Isothermal lines, and contour lines of speed with streamlines for rectangular type at $\mathscr{R}/\mathscr{R}_c = 1.1$.

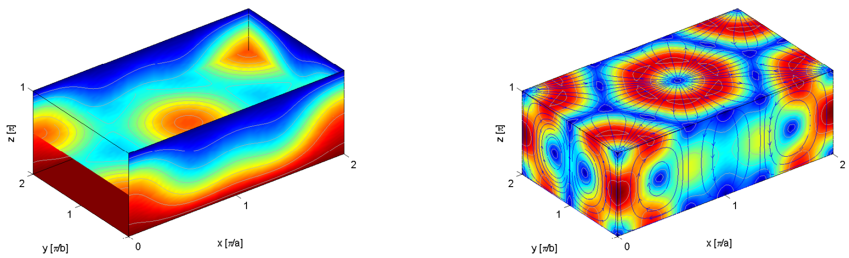

Fig. 3 Isothermal lines, and contour lines of speed with streamlines for hexagonal type at $\mathscr{R}/\mathscr{R}_c = 1.1$.

In Table 1, Table 2 and Table 3, we illustrate the verification results for each type of solutions with several relative Rayleigh numbers. In these tables, we show the relative Rayleigh number $\mathscr{R}/\mathscr{R}_c$, the truncation number N, the converged step k, L^∞ norms of approximate solutions $(\mathbf{u}_N, \theta_N)$, L^∞ norms of finite parts $(\mathbf{u}_h, \theta_h)$, and the bounds m_1, m_2 of infinite parts. The converged step means the inflated candidate set at step $k - \frac{1}{2}$ includes the new one at step k, namely, the verification was completed at the concerning iteration steps. In the roll type case (Table 1), the problem size becomes much smaller due to the elimination of one space variable which comes from the fact that the solutions are independent of that variable. For other types (Table 2, Table 3), we can find out the basic symmetry of solutions which make it possible to reduce the size of unknown coefficients.

From these tables, we can make a bifurcation diagram Fig. 4 with respect to the relative Rayleigh number $\mathscr{R}/\mathscr{R}_c$ and sum $\|\nabla \mathbf{u}_N\|_\infty + \|\nabla \theta_N\|_\infty$ of approximate solutions' L^∞ norms.

6 Conclusion

We could verify several kinds of bifurcating solutions. This should be the first result on the fact that there actually exist exact solutions around approximate solutions drawn in the figures corresponding to the interesting

$\mathscr{R}/\mathscr{R}_c$	N	k	$\|\nabla \mathbf{u}_N\|_\infty$	$\|\nabla \theta_N\|_\infty$	$\|\nabla \mathbf{u}_h\|_\infty$	$\|\nabla \theta_h\|_\infty$	m_1	m_2
1.01	16	12	0.74	0.31	5×10^{-11}	2×10^{-10}	1.44×10^{-10}	1.92×10^{-10}
1.05	18	29	1.66	0.72	2×10^{-8}	2×10^{-8}	4.06×10^{-9}	3.57×10^{-9}
1.5	44	19	5.58	2.47	3×10^{-11}	3×10^{-10}	1.72×10^{-11}	3.33×10^{-11}

Table 1 Verification results for roll type solutions.

$\mathscr{R}/\mathscr{R}_c$	N	k	$\|\nabla \mathbf{u}_N\|_\infty$	$\|\nabla \theta_N\|_\infty$	$\|\nabla \mathbf{u}_h\|_\infty$	$\|\nabla \theta_h\|_\infty$	m_1	m_2
1.01	16	7	0.93	0.38	2×10^{-11}	3×10^{-11}	2.37×10^{-12}	1.01×10^{-11}
1.05	16	10	2.20	0.96	2×10^{-8}	2×10^{-7}	2.91×10^{-9}	1.09×10^{-8}
1.1	24	11	3.26	1.47	4×10^{-10}	3×10^{-9}	8.33×10^{-13}	5.84×10^{-12}
1.2	28	33	4.99	2.27	2×10^{-10}	2×10^{-9}	4.60×10^{-12}	3.96×10^{-11}

Table 2 Verification results for rectangular type solutions.

$\mathscr{R}/\mathscr{R}_c$	N	k	$\|\nabla \mathbf{u}_N\|_\infty$	$\|\nabla \theta_N\|_\infty$	$\|\nabla \mathbf{u}_h\|_\infty$	$\|\nabla \theta_h\|_\infty$	m_1	m_2
1.01	16	7	1.12	0.47	2×10^{-10}	6×10^{-10}	4.53×10^{-11}	1.93×10^{-10}
1.05	16	15	2.70	1.21	2×10^{-6}	2×10^{-6}	3.22×10^{-8}	1.08×10^{-7}
1.1	24	19	4.09	1.88	4×10^{-9}	3×10^{-8}	2.50×10^{-11}	1.68×10^{-10}

Table 3 Verification results for hexagonal type solutions.

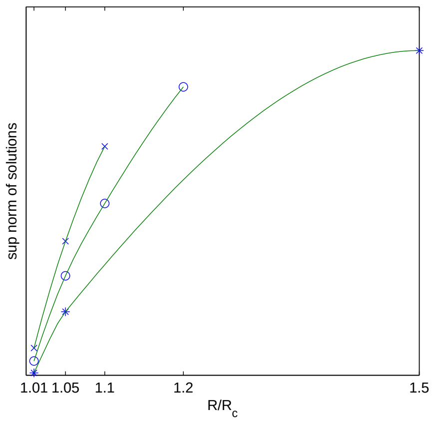

Fig. 4 Bifurcation diagram. $\times$ for hexagonal, $\circ$ for rectangular, and $*$ for roll type cases.

natural phenomena. Due to the limit of our computational power up to now, we proved only small part (Fig. 4) of bifurcation diagram except for the roll type solutions. This could be enhanced using parallel computation after update of PROFIL package into parallel version (not available now) or increasing physical memories for computation on some large scale computers.

More interesting problem would be the verification of the bifurcation point such that suggested in [5], which is more complicated problem than the usual. We believe that these interesting and important problems could be resolved in the near future.

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