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Kenta Kobayashi [1]

Abstract

In solving elliptic problems by the finite element method in a bounded domain with has a re-entrant corner, the rate of convergence could be improved by adding a singular function to the usual C^0 approximating basis. When the domain is enclosed by line segments which forms a corner of $\pi/2$ or $3\pi/2$, we have obtained an explicit an a priori H^1 error estimation of $O(h)$ for such a finite element solution of the Poisson equation. Particularly, we emphasize that all constants in our error estimates are numerically determined, which plays an essential role in the numerical verification of solutions for non-linear elliptic problems.
Key Words: Finite element method, A priori error estimation, Poisson equation

Mathematics Subject Classification (2000): 65N30, 65N15, 35J05

1 Introduction

In this paper, we consider the elliptic problem on polygonal domain Ω which is enclosed by line segments and right angles. Ω is assumed to be connected but it is not necessarily simply connected. First of all, we assume $\Omega = \Omega_0$, where Ω_0 is an inside of the L-shape domain shown in Fig.1. The general case is described in Section 3.

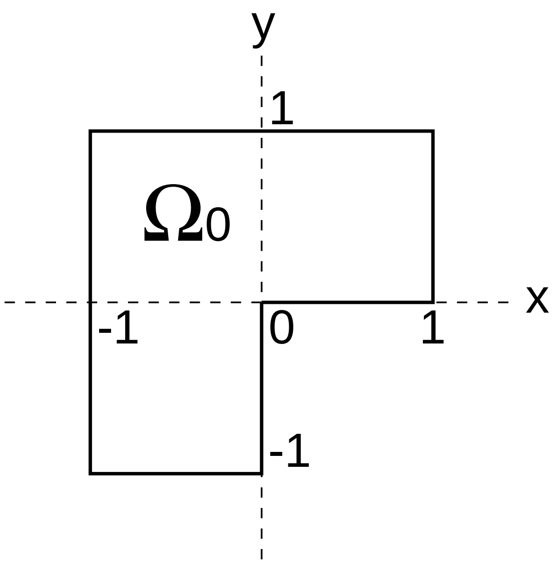

Figure 1: The shape of Ω_0

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For $f \in L^2(\Omega)$, we consider the weak solution of the following partial differential equation

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.1}$$

It is known that u has a singular function representation [9][10]

$$u(x,y) = w(x,y) + \lambda\sigma(x,y) \tag{1.2}$$

where $w(x,y) \in H^2(\Omega) \cap H_0^1(\Omega)$, λ is a constant, $\sigma(x,y) \in H_0^1(\Omega)$ and

$$\sigma(x,y) \sim r^{2/3} \sin\left(\frac{2}{3}\theta\right)$$

in a neighborhood of the origin. Here, (r,θ) is the polar coordinates of (x,y) where θ satisfies $0 \leq \theta < 2\pi$.

There is arbitrariness in the choice of σ. To simplify the calculation of H_0^1 inner product, we take

$$\sigma(x,y) = (1-x^2)(1-y^2)r^{2/3} \sin\left(\frac{2}{3}\theta\right),$$

in this paper.

Solving this problem by the finite element method, we use the square mesh with mesh size h. The mesh of $h = 1/8$ is shown in Fig.2.

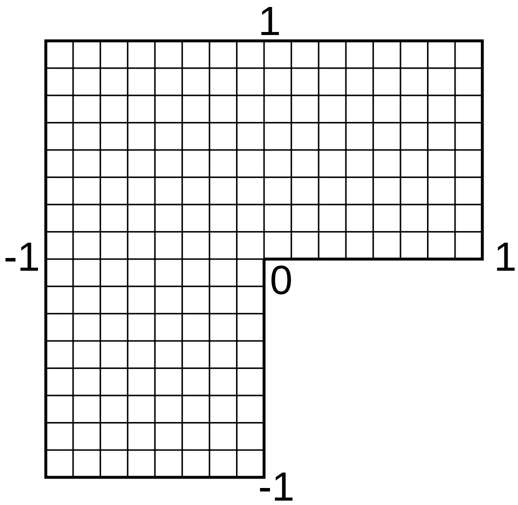

Figure 2: The square mesh when $h = 1/8$

We use the piecewise bilinear function,

$$\phi_{i,j}(x,y) \equiv \max\left(1 - \left|\frac{x}{h} - i\right|,\ 0\right) \cdot \max\left(1 - \left|\frac{y}{h} - j\right|,\ 0\right),$$

as the finite element basis.

We define Φ_h by the set of functions $\phi_{i,j}$ in $H_0^1(\Omega)$,

$$\Phi_h = \left\{\phi_{i,j} \,\middle|\, (ih, jh) \in \Omega\right\}.$$

Since u does not generally have H^2 regularity, we can not obtain $O(h)$ error estimates with H_0^1 norm by using this approximating basis Φ_h.

Therefore, we adopt $\Phi_h \cup \{\sigma\}$ as the finite element basis. In this case, it is known that the following error estimation holds [8][9][10],

$$\|u - u_h\|_{H_0^1(\Omega)} \leq Ch\|f\|_{L^2(\Omega)},$$

where u_h is the finite element solution. The following $O(h^2)$ estimation for the L^2-error is also obtained by the Aubin-Nitsche trick [6],

$$\|u - u_h\|_{L^2(\Omega)} \leq C^2h\|f\|_{L^2(\Omega)}.$$

The main purpose of this paper is to obtain this constant C.

The coefficient λ in (1.2) is often called the stress intensity factor in a context of mechanics. In our error estimation, the explicit evaluation of the coefficient λ is essential (Lemma 4.2 and Lemma 4.6 in section 4). For the coefficient λ, the flowing extraction formula holds [9][10]

$$\lambda = \frac{1}{\pi}\left\{\iint_\Omega f\eta s_- dxdy + \iint_\Omega u\Delta(\eta s_-)dxdy\right\},$$

where

$$s_- = r^{-2/3}\sin\left(\frac{2}{3}\theta\right),$$

and η is a cut-off function which equals one at the origin and zero on $\big\{(x,y)\big|\max(|x|,|y|) = 1\big\}$. Poincaré-Friedrichs inequality is needed to evaluate λ by this extraction formula. However, since Poincaré-Friedrichs inequality is reduced to a kind of a problem of eigenvalue bounds, it is not easy to obtain good estimation, except for the case that Ω is simple domain such as rectangle. In this paper, instead of using cut-off function, we use the maximum principle for the super harmonic functions to evaluate λ directly.

There are several approaches to deal with the lack of regularity at the re-entrant corner. The method in this paper is based on [8]. This method is a simple finite element method but is enough to obtain H_0^1 and L^2 error bounds. The dual singular function method (DSFM) [3][7] is presented to obtain better approximation for the coefficient λ. DSFM consists of a system of w and λ which is derived from extraction formula, and is often implemented as an iterative procedure. A multigrid version of this method appears in [4]. An efficient method using improved extraction formula was presented in [5] . Another useful method is based on the local mesh refinement [2]. The advantage of using mesh refinement is that calculation of the element matrix is easy, because the information about the singular function is not needed.

In many applications, it is useful to obtain explicit error estimation. For example, numerical verification method for nonlinear problems are based on explicit error bounds for linear equations [12][16].

We should mention that in [15] an explicit error estimation for non-convex domain is proposed. However, the error estimation in [15] needs specific information about the finite element solution of (1.1), and furthermore, the order of their error estimation is about $O(h^{0.6927})$ with H_0^1 norm. What we are going to present is much higher order error estimation which can be a priori calculated.

The present paper is organized as follows. In section 2, we present a priori error estimation when the case of Ω is simple L-shape domain. The general case is explained in section 3. Section 4 contains proof of lemmas which appear in section 2 and 3. We show numerical results in section 5 and conclude this paper by section 6.

Throughout this paper, we take the angle of polar coodinates in $[0, 2\pi)$. Let 1_A denotes the function which takes value 1 if condition A holds, and takes value 0 otherwise.

2 A priori error estimation

The main purpose of this section is to prove the following theorem. Lemmas appear in this section will be proved in section 4.

Theorem 2.1 *When $\Omega = \Omega_0$, as to the finite element solution u_h by using $\Phi_h \cup \{\sigma\}$ as the basis, the following error estimation holds,*

$$\|u - u_h\|_{H_0^1(\Omega)} \leq 1.156h\|f\|_{L^2(\Omega)},$$
$$\|u - u_h\|_{L^2(\Omega)} \leq 1.335h^2\|f\|_{L^2(\Omega)}.$$

Proof. We represent the exact solution as,

$$u = w + \lambda\sigma. \tag{2.1}$$

Where λ is a constant which depends on Ω and f, and w is a function which belongs to $H^2(\Omega) \cap H_0^1(\Omega)$.

Define w_h as the bilinear interpolation of w. Then, since $w \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$\|w - w_h\|_{H_0^1(\Omega)} \leq \frac{h}{\pi}|w|_{H^2(\Omega)}$$

holds [13].

Since Ω is a polygonal domain, the following equality holds [11].

$$|w|_{H^2(\Omega)} = \|\Delta w\|_{L^2(\Omega)}.$$

We immediately have,

$$\|w - w_h\|_{H_0^1(\Omega)} \leq \frac{h}{\pi}\|\Delta w\|_{L^2(\Omega)}.$$

Let

$$\widetilde{u}_h = w_h + \lambda\sigma$$

then, we have

$$\|u - \widetilde{u}_h\|_{H_0^1(\Omega)} \leq \frac{h}{\pi}\|\Delta w\|_{L^2(\Omega)}. \tag{2.2}$$

Form (2.1),

$$\|\Delta w\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} + |\lambda| \, \|\Delta\sigma\|_{L^2(\Omega)} \,. \tag{2.3}$$

From Lemma 4.1,

$$\left\|\Delta\left\{(1-x^2)(1-y^2)r^{2/3}\sin\left(\frac{2}{3}\theta\right)\right\}\right\|_{L^2(\Omega)} \leq \sqrt{\frac{4000}{81} - \frac{11713}{1782}\pi}. \tag{2.4}$$

For the coefficient λ, Lemma 4.2 implies

$$|\lambda| \leq \frac{1}{\pi}\left\|\left(r^{-2/3} - 2^{-2/3}r^{2/3}\right)\sin\left(\frac{2}{3}\theta\right)\right\|_{L^2(\Omega)} \|f(x,y)\|_{L^2(\Omega)} \tag{2.5}$$

and, from Lemma 4.3,

$$\left\|\left(r^{-2/3} - 2^{-2/3}r^{2/3}\right)\sin\left(\frac{2}{3}\theta\right)\right\|_{L^2(\Omega)} \leq \sqrt{\frac{3\cdot 2^{1/3}}{5}\pi}.$$

Consequently, from (2.2),(2.3),(2.4) and (2.5),

$$\begin{aligned}|u - \widetilde{u}_h|_{H_0^1(\Omega)} &\leq \frac{h}{\pi}\left(1 + \sqrt{\frac{4000}{81} - \frac{11713}{1782}\pi}\cdot\frac{1}{\pi}\sqrt{\frac{3\cdot 2^{1/3}}{5}\pi}\right)\|f\|_{L^2(\Omega)} \\ &= 1.1552884253\cdots h\|f\|_{L^2(\Omega)} \leq 1.156h\|f\|_{L^2(\Omega)}.\end{aligned}$$

Since the finite element solution u_h is the best approximation in H_0^1 space, we have

$$\|u - u_h\|_{H_0^1(\Omega)} \leq \|u - \widetilde{u}_h\|_{H_0^1(\Omega)} \leq 1.156h\|f\|_{L^2(\Omega)}.$$

$$\|u - u_h\|_{L^2(\Omega)} \leq 1.1553^2h^2\|f\|_{L^2(\Omega)} \leq 1.335h^2\|f\|_{L^2(\Omega)}$$

is also obtained by the Aubin-Nitsche trick. □

3 Generalization

We consider general cases in this section. We suppose that Ω is enclosed by line segments which forms a corner of $\pi/2$ or $3\pi/2$. Ω is assumed to be connected but it is not necessarily simply connected (Fig.3).

The following singular bases are used together with usual interpolating basis

$$T_k\sigma\left(\frac{x}{l_k}, \frac{y}{l_k}\right), \quad (k = 1, \cdots, n),$$

where n is a number of re-entrant corner,

$$\sigma(x,y) = \begin{cases} (1-x^2)(1-y^2)r^{2/3}\sin\left(\frac{2}{3}\theta\right), & \left((x,y)\in\Omega_0\right), \\ 0, & \text{(otherwise)}, \end{cases}$$

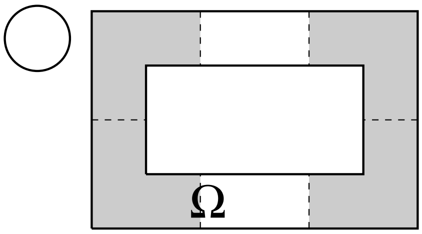

Figure 3: Admissible pattern

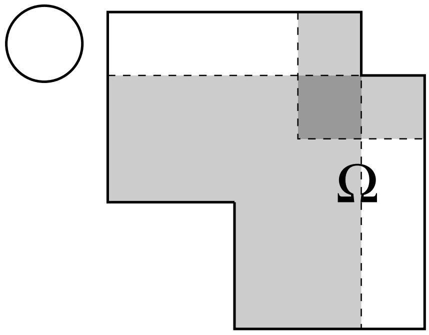

Figure 4: Admissible pattern

and T_k is a combination of parallel translation and rotation.

l_k denotes sizes of the singular bases. Different sizes of the singular bases are admissible. It is also admissible even if some part of the support of the singular bases are overlapped (Fig.4).

There are some restrictions on defining the singular bases. Let us now suppose that Γ is the support of a singular basis, and $\partial\Gamma$ consists of line segments $\gamma_1 \sim \gamma_6$ where γ_1 and γ_6 form the re-entrant corner (Fig.5). In this case, γ_1 and γ_6 must be contained in $\partial\Omega$ and $\gamma_2 \sim \gamma_5$ must coincide with the grid line of the mesh. Therefore, both Fig.6 and Fig.7 are not admissible.

In this situation, we have the following Theorem.

Theorem 3.1 *For the finite element solution u_h with the basis $\Phi_h \cup \left\{ T_k\sigma\big(x/l_k,\, y/l_k\big) \;\middle|\; k = 1, \cdots, n \right\}$, the following error estimation holds,*

$$\|u - u_h\|_{H_0^1(\Omega)} \le \left(0.319 + \sum_{k=1}^{n} \sqrt{0.971 + 1.469\frac{|\Omega|}{l_k^2}} \right) h\|f\|_{L^2(\Omega)},$$

$$\|u - u_h\|_{L^2(\Omega)} \le \left(0.319 + \sum_{k=1}^{n} \sqrt{0.971 + 1.469\frac{|\Omega|}{l_k^2}} \right)^2 h^2\|f\|_{L^2(\Omega)},$$

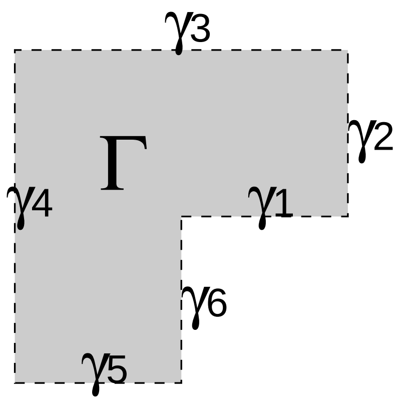

Figure 5: The support of a singular basis

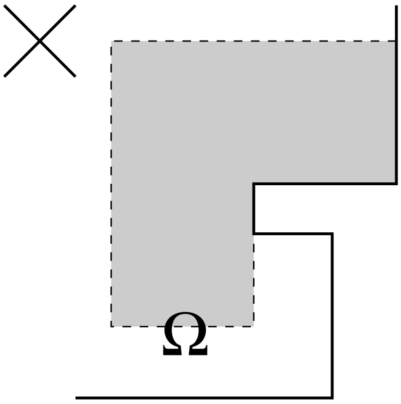

Figure 6: Nonadmissible pattern

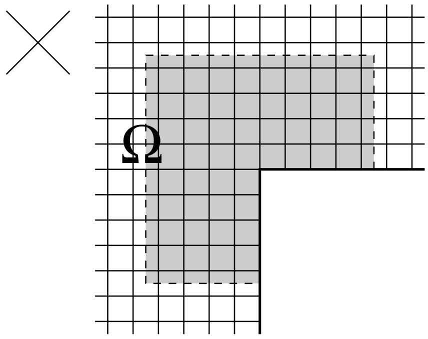

Figure 7: Nonadmissible pattern

where $|\Omega|$ *denotes the area of* Ω.

Proof. Let

$$u(x,y) = w(x,y) + \sum_{k=1}^{n} \lambda_k T_k \sigma\left(\frac{x}{l_k}, \frac{y}{l_k}\right), \qquad w(x,y) \in H^2(\Omega) \cap H_0^1(\Omega).$$

be the exact solution.

We define $\widetilde{u}_h$ as follows,

$$\widetilde{u}_h(x,y) = w_h(x,y) + \sum_{k=1}^{n} \lambda_k T_k \sigma\left(\frac{x}{l_k}, \frac{y}{l_k}\right)$$

where w_h denotes the bilinear interpolation of w.

We also define

$$\overline{\sigma}(x,y) = \begin{cases} (1-r)^2 r^{2/3} \sin\left(\dfrac{2}{3}\theta\right), & (r < 1), \\ 0, & (r \geq 1). \end{cases}$$

Now, we exclude the grid of the square mesh from Ω and define it as Ω_*. Then,

$$
\begin{aligned}
\|u-\widetilde{u}_h\|_{H_0^1(\Omega)} &= \|w-w_h\|_{H_0^1(\Omega_*)} \le \frac{h}{\pi}|w|_{H^2(\Omega_*)} \\
&= \frac{h}{\pi}\left| u-\sum_{k=1}^{n}\lambda_k T_k \sigma\left(\frac{x}{l_k},\frac{y}{l_k}\right)\right|_{H^2(\Omega_*)} \\
&\le \frac{h}{\pi}\left| u-\sum_{k=1}^{n}\lambda_k T_k \overline{\sigma}\left(\frac{x}{l_k},\frac{y}{l_k}\right)\right|_{H^2(\Omega)} \\
&\qquad + \frac{h}{\pi}\left|\sum_{k=1}^{n}\lambda_k T_k\left(\sigma\left(\frac{x}{l_k},\frac{y}{l_k}\right)-\overline{\sigma}\left(\frac{x}{l},\frac{y}{l}\right)\right)\right|_{H^2(\Omega_*)} \\
&= \frac{h}{\pi}\left\|\Delta\left(u-\sum_{k=1}^{n}\lambda_k T_k \overline{\sigma}\left(\frac{x}{l_k},\frac{y}{l_k}\right)\right)\right\|_{L^2(\Omega)} \\
&\qquad + \frac{h}{\pi}\left\|\sum_{k=1}^{n}\lambda_k \Delta T_k\left(\sigma\left(\frac{x}{l_k},\frac{y}{l_k}\right)-\overline{\sigma}\left(\frac{x}{l_k},\frac{y}{l_k}\right)\right)\right\|_{L^2(\Omega_*)} \\
&\le \frac{h}{\pi}\|f(x,y)\|_{L^2(\Omega)} \\
&\quad + \frac{h}{\pi}\sum_{k=1}^{n}\frac{|\lambda_k|}{l_k}\left(\|\Delta\overline{\sigma}(x,y)\|_{L^2(\Omega_0)}+\left\|\Delta\Big(\sigma(x,y)-\overline{\sigma}(x,y)\Big)\right\|_{L^2(\Omega_0)}\right).
\end{aligned}
$$

From Lemma 4.4,

$$
\|\Delta\overline{\sigma}(x,y)\|_{L^2(\Omega_0)} = \frac{3\sqrt{\pi}}{2}.
$$

Lemma 4.5 implies that

$$
\left\|\Delta\Big(\sigma(x,y)-\overline{\sigma}(x,y)\Big)\right\|_{L^2(\Omega_0)} \le \sqrt{\frac{4000}{81}-\frac{232367}{46332}\pi}.
$$

Then, we have

$$
\|u-\widetilde{u}_h\|_{H_0^1(\Omega)} \le \frac{h}{\pi}\|f\|_{L^2(\Omega)} + \frac{h}{\pi}\left(\frac{3\sqrt{\pi}}{2}+\sqrt{\frac{4000}{81}-\frac{232367}{46332}\pi}\right)\sum_{k=1}^{n}\frac{|a_k|}{l_k}.
$$

From Lemma 4.6, we have the following evaluation

$$
\begin{aligned}
|\lambda_k| \le \Bigg\{ &\iint_{\Omega} 1_{0\le\theta<3\pi/2}\cdot\left|G_{l_k}\left(r,\cos\left(\frac{4}{3}\theta\right),\cos\left(\frac{4}{3}\theta\right)\right)\right|^2 dxdy \\
&+ \iint_{\Omega} 1_{3\pi/2\le\theta<2\pi}\cdot\left|G_{l_k}\left(r,\frac{1}{9}\Big(8+\cos(4\theta)\Big),1\right)\right|^2 dxdy\Bigg\}^{1/2}\|f(x,y)\|_{L^2(\Omega)},
\end{aligned}
$$

where

$$
G_l(r,X,Y) = \frac{l^{2/3}}{\sqrt{2}\pi}\sqrt{\sqrt{r^{-8/3}+l^{-8/3}-2r^{-4/3}l^{-4/3}X}+l^{-4/3}-r^{-4/3}Y}.
$$

It follows from Lemma 4.7 that

$$|\lambda_k| \le \frac{1}{\pi}\sqrt{\left(\frac{5}{2}-\frac{3\pi}{8}\right) l_k^2 + 2|\Omega|} \cdot \|f\|_{L^2(\Omega)}.$$

Then,

$$\begin{aligned}
&\|u-\widetilde{u}_h\|_{H_0^1(\Omega)} \\
&\le \frac{h}{\pi}\left(1+\left(\frac{3\sqrt{\pi}}{2}+\sqrt{\frac{4000}{81}-\frac{232367}{46332}\pi}\right)\sum_{k=1}^{n}\frac{1}{\pi}\sqrt{\left(\frac{5}{2}-\frac{3\pi}{8}\right)+\frac{2|\Omega|}{l_k^2}}\right) \\
&\qquad \times \|f\|_{L^2(\Omega)} \\
&= \left(0.31830988\cdots+\sum_{k=1}^{n}\sqrt{0.97070784\cdots+1.46865243\cdots\frac{|\Omega|}{l_k^2}}\right) h\|f\|_{L^2(\Omega)} \\
&\le \left(0.319+\sum_{k=1}^{n}\sqrt{0.971+1.469\frac{|\Omega|}{l_k^2}}\right) h\|f\|_{L^2(\Omega)}.
\end{aligned}$$

Since the finite element solution u_h is the best approximation in H_0^1 space,

$$\begin{aligned}
\|u-u_h\|_{H_0^1(\Omega)} &\le \|u-\widetilde{u}_h\|_{H_0^1(\Omega)} \\
&\le \left(0.319+\sum_{k=1}^{n}\sqrt{0.971+1.469\frac{|\Omega|}{l_k^2}}\right) h\|f\|_{L^2(\Omega)}.
\end{aligned}$$

We also obtain

$$\|u-u_h\|_{L^2(\Omega)} \le \left(0.319+\sum_{k=1}^{n}\sqrt{0.971+1.469\frac{|\Omega|}{l_k^2}}\right)^2 h^2\|f\|_{L^2(\Omega)}$$

by the Aubin-Nitsche trick. □

4 Lemmas

Lemma 4.1

$$\left\|\Delta\left\{(1-x^2)(1-y^2)r^{2/3}\sin\left(\frac{2}{3}\theta\right)\right\}\right\|_{L^2(\Omega_0)} \le \sqrt{\frac{4000}{81}-\frac{11713}{1782}\pi}.$$

Proof.

$$\begin{aligned}
&\Delta\left\{(1-x^2)(1-y^2)r^{2/3}\sin\left(\frac{2}{3}\theta\right)\right\} \\
&\qquad = \frac{2}{3}r^{2/3}\left\{(4r^2-10)\sin\left(\frac{2}{3}\theta\right)+r^2\sin\left(\frac{10}{3}\theta\right)\right\} \\
&\qquad = \frac{2}{3}r^{2/3}\sin\left(\frac{2}{3}\theta\right)\left\{9r^2-10-20r^2\sin^2\left(\frac{2}{3}\theta\right)+16r^2\sin^4\left(\frac{2}{3}\theta\right)\right\}.
\end{aligned}$$

When $1 \le r \le \sqrt{2}$,

$$
\begin{aligned}
&\Delta\left\{(1-x^2)(1-y^2)r^{2/3}\sin\left(\frac{2}{3}\theta\right)\right\} \\
&= \frac{2}{3}r^{2/3}\sin\left(\frac{2}{3}\theta\right)\left\{9r^2-10-4r^2\sin^2\left(\frac{2}{3}\theta\right)\right. \\
&\qquad\qquad \left. -16r^2\sin^2\left(\frac{2}{3}\theta\right)\left(1-\sin^2\left(\frac{2}{3}\theta\right)\right)\right\} \\
&\le \frac{2}{3}r\sin\left(\frac{2}{3}\theta\right)\left\{8-4r^2\sin^2\left(\frac{2}{3}\theta\right)\right\},
\end{aligned}
$$

and

$$
\begin{aligned}
&\Delta\left\{(1-x^2)(1-y^2)r^{2/3}\sin\left(\frac{2}{3}\theta\right)\right\} \\
&= -\frac{2}{3}r^{2/3}\sin\left(\frac{2}{3}\theta\right)\left\{10-4r^2\sin^2\left(\frac{2}{3}\theta\right)-r^2\left(3-4\sin^2\left(\frac{2}{3}\theta\right)\right)^2\right\} \\
&\ge -\frac{2}{3}r\sin\left(\frac{2}{3}\theta\right)\left\{10-4r^2\sin^2\left(\frac{2}{3}\theta\right)\right\},
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\left|\Delta\left\{(1-x^2)(1-y^2)r^{2/3}\sin\left(\frac{2}{3}\theta\right)\right\}\right| \\
&\le \frac{2}{3}r\sin\left(\frac{2}{3}\theta\right)\left\{10-4r^2\sin^2\left(\frac{2}{3}\theta\right)\right\} \\
&= \frac{20\sqrt{10}}{9\sqrt{3}}-\frac{8}{3}\left(r\sin\left(\frac{2}{3}\theta\right)-\sqrt{\frac{5}{6}}\right)^2\left(r\sin\left(\frac{2}{3}\theta\right)+\sqrt{\frac{10}{3}}\right) \le \frac{20\sqrt{10}}{9\sqrt{3}}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\left\|\Delta\left\{(1-x^2)(1-y^2)r^{2/3}\sin\left(\frac{2}{3}\theta\right)\right\}\right\|^2_{L^2(\Omega_0)} \\
&\le \frac{4}{9}\int_0^1\int_0^{3\pi/2} r^{4/3}\left\{(4r^2-10)\sin\left(\frac{2}{3}\theta\right)+r^2\sin\left(\frac{10}{3}\theta\right)\right\}^2 r\,d\theta\,dr \\
&\qquad\qquad + \left(|\Omega_0|-\frac{3}{4}\pi\right)\left(\frac{20\sqrt{10}}{9\sqrt{3}}\right)^2 \\
&= \frac{127}{22}\pi+\left(3-\frac{3}{4}\pi\right)\frac{4000}{243} = \frac{4000}{81}-\frac{11713}{1782}\pi.
\end{aligned}
$$

□

Lemma 4.2 *When* $\Omega = \Omega_0$,

$$
|\lambda| \le \frac{1}{\pi}\left\|\left(r^{-2/3}-2^{-2/3}r^{2/3}\right)\sin\left(\frac{2}{3}\theta\right)\right\|_{L^2(\Omega)}\|f(x,y)\|_{L^2(\Omega)}.
$$

Proof. For any $0 < \varepsilon < 1$, let g_ε be a weak solution of the following equation,

$$\begin{cases} -\Delta g_\varepsilon = -\Delta \widetilde{g_\varepsilon} & \text{in} \quad \Omega, \\ g_\varepsilon = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{4.1}$$

where,

$$\widetilde{g_\varepsilon}(x,y) = \begin{cases} \dfrac{1}{\pi}\Big((2\varepsilon^{-4/3} - 2^{-2/3})r^{2/3} - \varepsilon^{-8/3}r^2\Big)\sin\left(\dfrac{2}{3}\theta\right), & (r < \varepsilon), \\ \dfrac{1}{\pi}\Big(r^{-2/3} - 2^{-2/3}r^{2/3}\Big)\sin\left(\dfrac{2}{3}\theta\right), & (\varepsilon \le r). \end{cases}$$

From the fact that $\widetilde{g_\varepsilon} \in C^1(\Omega)$ and $-\Delta\widetilde{g_\varepsilon} \ge 0$, both g_ε and $\widetilde{g_\varepsilon} - g_\varepsilon$ are superharmonic function. Since g_ε and $\widetilde{g_\varepsilon} - g_\varepsilon$ takes non-negative value on $\partial\Omega$,

$$0 \le g_\varepsilon(x,y) \le \widetilde{g_\varepsilon}(x,y).$$

Consequently,

$$|g_\varepsilon(x,y)| \le |\widetilde{g_\varepsilon}(x,y)|.$$

From (1.1) and (4.1), taking g_ε and u as test functions,

$$\begin{aligned}
\iint_\Omega f g_\varepsilon \, dxdy &= \iint_\Omega \nabla u \cdot \nabla g_\varepsilon \, dxdy = -\iint_\Omega u\Delta\widetilde{g_\varepsilon}\, dxdy \\
&= \frac{32\varepsilon^{-8/3}}{9\pi}\iint_\Omega 1_{r<\varepsilon} \cdot u(x,y)\sin\left(\frac{2}{3}\theta\right) dxdy \\
&= \frac{32\varepsilon^{-8/3}\lambda}{9\pi}\int_0^\varepsilon\int_0^{3\pi/2}(1-x^2)(1-y^2)r^{2/3}\sin^2\left(\frac{2}{3}\theta\right) r\, d\theta dr \\
&\qquad\qquad + \frac{32\varepsilon^{-8/3}}{9\pi}\int_0^\varepsilon\int_0^{3\pi/2} w(x,y)\sin\left(\frac{2}{3}\theta\right) r\, d\theta dr \\
&= \left(1 - \frac{4}{7}\varepsilon^2 + \frac{\varepsilon^4}{20}\right)\lambda \\
&\qquad + \frac{16\varepsilon^{-8/3}}{9\pi}\int_0^\varepsilon\int_0^{3\pi/2}(\varepsilon^2 - r^2)\frac{\partial}{\partial r}w(x,y)\,\sin\left(\frac{2}{3}\theta\right) d\theta dr.
\end{aligned}$$

Then

$$
\begin{aligned}
&\left(1 - \frac{4}{7}\varepsilon^2 + \frac{\varepsilon^4}{20}\right)|\lambda| \\
&\qquad \le \|f\|_{L^2(\Omega)}\|g_\varepsilon\|_{L^2(\Omega)} \\
&\qquad\qquad + \frac{16\varepsilon^{-8/3}}{9\pi}\left(\int_0^\varepsilon\int_0^{3\pi/2}\left|(\varepsilon^2 - r^2)\sin\left(\frac{2}{3}\theta\right)\right|^{7/6} r^{-1/6}\,d\theta dr\right)^{6/7} \\
&\qquad\qquad\qquad \times \left(\int_0^\varepsilon\int_0^{3\pi/2}\left|\frac{\partial}{\partial r}w(x,y)\right|^7 r\,d\theta dr\right)^{1/7} \\
&\qquad \le \|f\|_{L^2(\Omega)}\|\widetilde{g}_\varepsilon\|_{L^2(\Omega)} \\
&\qquad\qquad + \frac{16\varepsilon^{1/21}}{9\pi}\left(\int_0^1\int_0^{3\pi/2}\left|(1 - r^2)\sin\left(\frac{2}{3}\theta\right)\right|^{7/6} r^{-1/6}\,d\theta dr\right)^{6/7} \\
&\qquad\qquad\qquad \times \left(\iint_\Omega\left(\left|\frac{\partial}{\partial x}w(x,y)\right| + \left|\frac{\partial}{\partial y}w(x,y)\right|\right)^7 dxdy\right)^{1/7}.
\end{aligned}
$$

From $w \in H^2$ and the Sobolev embedding theorem [1],

$$
\frac{\partial w}{\partial x} \in L^7(\Omega), \quad \frac{\partial w}{\partial y} \in L^7(\Omega).
$$

Then, we have the conclusion when $\varepsilon \to 0$. □

Lemma 4.3

$$
\left\|\left(r^{-2/3} - 2^{-2/3}r^{2/3}\right)\sin\left(\frac{2}{3}\theta\right)\right\|_{L^2(\Omega_0)} \le \sqrt{\frac{3\cdot 2^{1/3}}{5}\pi}.
$$

Proof.

$$
\begin{aligned}
&\left\|\left(r^{-2/3} - 2^{-2/3}r^{2/3}\right)\sin\left(\frac{2}{3}\theta\right)\right\|^2_{L^2(\Omega_0)} \\
&\qquad \le \int_0^{\sqrt{2}}\int_0^{3\pi/2}\left(r^{-2/3} - 2^{-2/3}r^{2/3}\right)^2\sin^2\left(\frac{2}{3}\theta\right) r\,d\theta dr = \frac{3\cdot 2^{1/3}}{5}\pi.
\end{aligned}
$$

□

Lemma 4.4

$$
\left\|\Delta\left\{1_{r<1}\cdot(1-r)^2 r^{2/3}\sin\left(\frac{2}{3}\theta\right)\right\}\right\|_{L^2(\Omega_0)} = \frac{3\sqrt{\pi}}{2}.
$$

Proof. We have

$$\left\|\Delta\left\{1_{r<1}\cdot(1-r)^2r^{2/3}\sin\left(\frac{2}{3}\theta\right)\right\}\right\|^2_{L^2(\Omega_0)}$$
$$=\left\|1_{r<1}\cdot\frac{2}{3}\Big(10r-7\Big)r^{-1/3}\sin\left(\frac{2}{3}\theta\right)\right\|^2_{L^2(\Omega_0)}$$
$$=\frac{4}{9}\int_0^1\int_0^{3\pi/2}\Big(10r-7\Big)^2r^{-2/3}\sin^2\left(\frac{2}{3}\theta\right)r\,d\theta dr=\frac{9}{4}\pi,$$

by calculating directly. □

Lemma 4.5

$$\left\|\Delta\left\{\Big((1-x^2)(1-y^2)-1_{r<1}\cdot(1-r)^2\Big)r^{2/3}\sin\left(\frac{2}{3}\theta\right)\right\}\right\|_{L^2(\Omega_0)}$$
$$\leq\sqrt{\frac{4000}{81}-\frac{232367}{46332}\pi}.$$

Proof.

$$\Delta\left\{\Big((1-x^2)(1-y^2)-(1-r)^2\Big)r^{2/3}\sin\left(\frac{2}{3}\theta\right)\right\}$$
$$=\frac{2}{3}r^{2/3}\left\{\left(4r^2-20+\frac{7}{r}\right)\sin\left(\frac{2}{3}\theta\right)+r^2\sin\left(\frac{10}{3}\theta\right)\right\}$$

When $r\geq 1$, same as proof of Lemma 4.1,

$$\left|\Delta\left\{(1-x^2)(1-y^2)r^{2/3}\sin\left(\frac{2}{3}\theta\right)\right\}\right|\leq\frac{20\sqrt{10}}{9\sqrt{3}},$$

Then

$$\left\|\Delta\left\{\Big((1-x^2)(1-y^2)-1_{r<1}\cdot(1-r)^2\Big)r^{2/3}\sin\left(\frac{2}{3}\theta\right)\right\}\right\|^2_{L^2(\Omega_0)}$$
$$\leq\frac{4}{9}\int_0^1\int_0^{3\pi/2}r^{4/3}\left\{\left(4r^2-20+\frac{7}{r}\right)\sin\left(\frac{2}{3}\theta\right)+r^2\sin\left(\frac{10}{3}\theta\right)\right\}^2r\,d\theta\,dr$$
$$+\left(|\Omega_0|-\frac{3}{4}\pi\right)\left(\frac{20\sqrt{10}}{9\sqrt{3}}\right)^2$$
$$=\frac{4193}{572}\pi+\left(3-\frac{3}{4}\pi\right)\frac{4000}{243}=\frac{4000}{81}-\frac{232367}{46332}\pi.$$

□

Lemma 4.6

$$|\lambda_k| \le \left\{ \iint_\Omega 1_{0\le\theta<3\pi/2} \cdot \left| G_{l_k}\left(r, \cos\left(\frac{4}{3}\theta\right), \cos\left(\frac{4}{3}\theta\right)\right)\right|^2 dxdy \right.$$

$$\left. + \iint_\Omega 1_{3\pi/2\le\theta<2\pi} \cdot \left| G_{l_k}\left(r, \frac{1}{9}\Big(8+\cos(4\theta)\Big), 1\right)\right|^2 dxdy \right\}^{1/2} \|f(x,y)\|_{L^2(\Omega)}.$$

Proof. Using the parallel translation and the rotation, we move re-entrant corner to the origin and re-entrant angle to $[0, 3\pi/2]$.

For $\varepsilon < l_k$, let g_ε be a weak solution of following equation,

$$\begin{cases} -\Delta g_\varepsilon = -1_{0<\theta<3\pi/2} \cdot 1_{r<\varepsilon} \cdot \Delta \widetilde{g_\varepsilon} & \text{in} \quad \Omega, \\ \quad g_\varepsilon = 0, & \text{on} \quad \partial\Omega, \end{cases}$$

where,

$$\widetilde{g_\varepsilon}(x,y) = \begin{cases} \Big(2\varepsilon^{-4/3} r^{4/3} - \varepsilon^{-8/3} r^{8/3}\Big) G_{l_k}\left(r, \cos\left(\frac{4}{3}\theta\right), \cos\left(\frac{4}{3}\theta\right)\right), & \\ & (r<\varepsilon,\ 0\le\theta\le 3\pi/2), \\ G_{l_k}\left(r, \cos\left(\frac{4}{3}\theta\right), \cos\left(\frac{4}{3}\theta\right)\right), & (r\ge\varepsilon,\ 0\le\theta\le 3\pi/2), \\ G_{l_k}\left(r, \frac{1}{9}\Big(8+\cos(4\theta)\Big), 1\right), & (3\pi/2<\theta<2\pi). \end{cases}$$

From the fact that $\widetilde{g_\varepsilon} \in C^1(\Omega)$ and Lemma 4.8, 4.9 and 4.10, both g_ε and $\widetilde{g_\varepsilon} - g_\varepsilon$ are superharmonic function. Since g_ε and $\widetilde{g_\varepsilon} - g_\varepsilon$ takes non-negative value on $\partial\Omega$,

$$0 \le g_\varepsilon(x,y) \le \widetilde{g_\varepsilon}(x,y).$$

Consequently,

$$|g_\varepsilon(x,y)| \le |\widetilde{g_\varepsilon}(x,y)|.$$

Then, same as in Lemma 4.2,

$$\iint_\Omega f g_\varepsilon \, dxdy = \iint_\Omega \nabla u \cdot \nabla g_\varepsilon \, dxdy = -\iint_\Omega 1_{0<\theta<3\pi/2} \cdot 1_{r<\varepsilon} \cdot u \Delta \widetilde{g_\varepsilon} \, dxdy$$

$$= -\iint_\Omega 1_{0<\theta<3\pi/2} \cdot 1_{r<\varepsilon} \cdot u(x,y)$$

$$\times \Delta \left\{ \Big(2\varepsilon^{-4/3} r^{4/3} - \varepsilon^{-8/3} r^{8/3}\Big) G_{l_k}\left(r, \cos\left(\frac{4}{3}\theta\right), \cos\left(\frac{4}{3}\theta\right)\right)\right\} dxdy.$$

From Lemma 4.11,

$$
\begin{aligned}
&\iint_\Omega f g_\varepsilon \, dxdy \\
&= \frac{32\varepsilon^{-8/3} l_k^{2/3}}{9\pi}(1+O(\varepsilon^{4/3})) \iint_\Omega 1_{r<\varepsilon} \cdot u(x,y) \sin\left(\frac{2}{3}\theta\right) dxdy \\
&= \frac{32\varepsilon^{-8/3} l_k^{2/3} \lambda_k}{9\pi}(1+O(\varepsilon^{4/3})) \\
&\qquad \times \int_0^\varepsilon \int_0^{3\pi/2} \left(1-\frac{x^2}{l_k^2}\right)\left(1-\frac{y^2}{l_k^2}\right)\left(\frac{r}{l_k}\right)^{2/3} \sin^2\left(\frac{2}{3}\theta\right) r \, d\theta dr \\
&\qquad\qquad + \frac{32\varepsilon^{-8/3} l_k^{2/3}}{9\pi}(1+O(\varepsilon^{4/3})) \int_0^\varepsilon \int_0^{3\pi/2} w(x,y) \sin\left(\frac{2}{3}\theta\right) r \, d\theta dr \\
&= (1+O(\varepsilon^{4/3}))\lambda_k \\
&\qquad + \frac{32\varepsilon^{-8/3} l_k^{2/3}}{9\pi}(1+O(\varepsilon^{4/3})) \int_0^\varepsilon \int_0^{3\pi/2} w(x,y) \sin\left(\frac{2}{3}\theta\right) r \, d\theta dr.
\end{aligned}
$$

Then, same as in Lemma 4.2,

$$
\begin{aligned}
(1+O(\varepsilon^{4/3}))|\lambda_k| &\le \|f\|_{L^2(\Omega)} \|g_\varepsilon\|_{L^2(\Omega)} + O(\varepsilon^{1/21}) \\
&\le \|f\|_{L^2(\Omega)} \|\widetilde{g_\varepsilon}\|_{L^2(\Omega)} + O(\varepsilon^{1/21}).
\end{aligned}
$$

We have the conclusion when $\varepsilon \to 0$. □

Lemma 4.7

$$
\begin{aligned}
&\left\{ \iint_\Omega 1_{0\le\theta<3\pi/2} \cdot \left| G_l\left(r, \cos\left(\frac{4}{3}\theta\right), \cos\left(\frac{4}{3}\theta\right)\right)\right|^2 dxdy \right. \\
&\qquad \left. + \iint_\Omega 1_{3\pi/2\le\theta<2\pi} \cdot \left| G_l\left(r, \frac{1}{9}\Big(8+\cos(4\theta)\Big), 1\right)\right|^2 dxdy \right\}^{1/2} \\
&\qquad\qquad \le \frac{1}{\pi}\sqrt{\left(\frac{5}{2}-\frac{3\pi}{8}\right) l^2 + 2|\Omega|}
\end{aligned}
$$

Proof.

$$\begin{aligned}
&\left|G_l\left(r,\cos\left(\frac{4}{3}\theta\right),\cos\left(\frac{4}{3}\theta\right)\right)\right|^2 \\
&\quad = \frac{l^{4/3}}{2\pi^2}\left\{\sqrt{r^{-8/3}+l^{-8/3}-2r^{-4/3}l^{-4/3}\cos\left(\frac{4}{3}\theta\right)}\right. \\
&\qquad\qquad \left. + l^{-4/3}-r^{-4/3}\cos\left(\frac{4}{3}\theta\right)\right\} \\
&\quad \le \frac{l^{4/3}}{2\pi^2}\left\{\left|l^{-4/3}-r^{-4/3}\right|+\sqrt{2r^{-4/3}l^{-4/3}\left(1-\cos\left(\frac{4}{3}\theta\right)\right)}\right. \\
&\qquad\qquad \left. + l^{-4/3}-r^{-4/3}\cos\left(\frac{4}{3}\theta\right)\right\} \\
&\quad = \frac{l^{4/3}}{\pi^2}\left\{\max\left(l^{-4/3}-r^{-4/3},\ 0\right)\right. \\
&\qquad\qquad \left. +r^{-2/3}l^{-2/3}\sin\left(\frac{2}{3}\theta\right)+r^{-4/3}\sin^2\left(\frac{2}{3}\theta\right)\right\},
\end{aligned}$$

$$\begin{aligned}
&\left|G_l\left(r,\frac{1}{9}\Big(8+\cos(4\theta)\Big),1\right)\right|^2 \\
&= \frac{l^{4/3}}{2\pi^2}\left\{\sqrt{r^{-8/3}+l^{-8/3}-\frac{2}{9}r^{-4/3}l^{-4/3}\Big(8+\cos(4\theta)\Big)}+l^{-4/3}-r^{-4/3}\right\} \\
&\le \frac{l^{4/3}}{2\pi^2}\left\{\left|l^{-4/3}-r^{-4/3}\right|+\sqrt{\frac{2}{9}r^{-4/3}l^{-4/3}\Big(1-\cos(4\theta)\Big)}+l^{-4/3}-r^{-4/3}\right\} \\
&= \frac{l^{4/3}}{\pi^2}\left\{\max\left(l^{-4/3}-r^{-4/3},\ 0\right)+\frac{1}{3}r^{-2/3}l^{-2/3}|\sin(2\theta)|\right\}.
\end{aligned}$$

It follows from these inequalities that,

$$\begin{aligned}
&\iint_\Omega 1_{0\le\theta<3\pi/2}\cdot\left|G_l\left(r,\cos\left(\frac{4}{3}\theta\right),\cos\left(\frac{4}{3}\theta\right)\right)\right|^2\,dxdy \\
&\qquad + \iint_\Omega 1_{3\pi/2\le\theta<2\pi}\cdot\left|G_l\left(r,\frac{1}{9}\Big(8+\cos(4\theta)\Big),1\right)\right|^2\,dxdy \\
&\le \frac{l^{4/3}}{\pi^2}\int_0^l\int_0^{3\pi/2}\left\{r^{-2/3}l^{-2/3}\sin\left(\frac{2}{3}\theta\right)+r^{-4/3}\sin^2\left(\frac{2}{3}\theta\right)\right\}r\,d\theta dr \\
&\qquad + \frac{l^{4/3}}{\pi^2}\int_0^l\int_{3\pi/2}^{2\pi}\frac{1}{3}r^{-2/3}l^{-2/3}|\sin(2\theta)|\,r\,d\theta dr \\
&\qquad + \left(|\Omega|-\frac{3\pi l^2}{4}\right)\frac{l^{4/3}}{\pi^2}\cdot 2l^{-4/3}
\end{aligned}$$

$$= \frac{l^2}{\pi^2}\left(\frac{5}{2} - \frac{3\pi}{8}\right) + \frac{2}{\pi^2}|\Omega|.$$

□

Lemma 4.8
$$-\Delta G_l\left(r, \cos\left(\frac{4}{3}\theta\right), \cos\left(\frac{4}{3}\theta\right)\right) = 0 \qquad in \quad \Omega.$$

Proof.
$$G_l\left(r, \cos\left(\frac{4}{3}\theta\right), \cos\left(\frac{4}{3}\theta\right)\right)$$
is a constant times imaginary part of
$$\left(z^{-4/3} - l^{-4/3}\right)^{1/2}, \quad z = re^{i\theta}.$$
Therefore, this function is harmonic in Ω. □

Lemma 4.9 *For $r < \varepsilon < l$ and $0 < \theta < 3\pi/2$,*
$$-\Delta\left(\left(2\varepsilon^{-4/3}r^{4/3} - \varepsilon^{-8/3}r^{8/3}\right)G_l\left(r, \cos\left(\frac{4}{3}\theta\right), \cos\left(\frac{4}{3}\theta\right)\right)\right) \geq 0 \quad in \quad \Omega.$$

Proof.
$$-\Delta\left\{\left(2\varepsilon^{-4/3}r^{4/3} - \varepsilon^{-8/3}r^{8/3}\right)G_l\left(r, \cos\left(\frac{4}{3}\theta\right), \cos\left(\frac{4}{3}\theta\right)\right)\right\}$$
$$= \frac{4\sqrt{2}}{9a^2b^2c\,l^2\pi} \cdot \frac{(a+c-1)^2(a+c+1)^2(a-b+c)}{\left(a+c-\cos\left(\frac{4}{3}\theta\right)\right)^{3/2}}$$

where
$$a = r^{4/3}l^{-4/3}, \qquad b = \varepsilon^{4/3}l^{-4/3}, \qquad c = \sqrt{a^2 + 1 - 2a\cos\left(\frac{4}{3}\theta\right)}.$$

Since
$$a - b + c = a - b + \sqrt{a^2 + 1 - 2a\cos\left(\frac{4}{3}\theta\right)} \geq a - b + |a-1| \geq 1 - b \geq 0,$$
this lemma is proved. □

Lemma 4.10 *For $3\pi/2 < \theta < 2\pi$,*
$$-\Delta G_l\left(r, \frac{1}{9}\Big(8 + \cos(4\theta)\Big), 1\right) \geq 0 \qquad in \quad \Omega.$$

Proof.

$$-\Delta G_l\left(r, \frac{1}{9}\Big(8+\cos(4\theta)\Big), 1\right)$$
$$= \frac{2\sqrt{2}\sqrt{a+d-1}}{9a^2d^3l^2\pi}\Big(2d^3+2(a-1)^2d-4(a-1)^3+d^2\Big)$$

where

$$a = r^{4/3}l^{-4/3}, \qquad d = \sqrt{a^2+1-\frac{2}{9}a\Big(8+\cos(4t)\Big)}.$$

Since

$$d \geq \sqrt{a^2+1-2a} = |a-1|$$

holds,

$$2d^3+2(a-1)^2d-4(a-1)^3+d^2 \geq 4|a-1|^3-4(a-1)^3+2(a-1)^2d+d^2 \geq 0.$$

Then, this lemma is proved. □

Lemma 4.11 *When $r < \varepsilon$,*

$$-\Delta\left\{\Big(2\varepsilon^{-4/3}r^{4/3}-\varepsilon^{-8/3}r^{8/3}\Big)G_l\left(r, \cos\left(\frac{4}{3}\theta\right), \cos\left(\frac{4}{3}\theta\right)\right)\right\}$$
$$= \frac{32\varepsilon^{-8/3}l^{2/3}}{9\pi}\sin\left(\frac{2}{3}\theta\right)\cdot(1+O(\varepsilon^{4/3})).$$

Proof. Define

$$a = r^{4/3}l^{-4/3}, \qquad b = \varepsilon^{4/3}l^{-4/3}, \qquad c = \sqrt{a^2+1-2a\cos\left(\frac{4}{3}\theta\right)},$$

then we have,

$$-\Delta\left\{\Big(2\varepsilon^{-4/3}r^{4/3}-\varepsilon^{-8/3}r^{8/3}\Big)G_l\left(r, \cos\left(\frac{4}{3}\theta\right), \cos\left(\frac{4}{3}\theta\right)\right)\right\}$$
$$= \frac{4\sqrt{2}}{9a^2b^2c\,l^2\pi}\cdot\frac{(a+c-1)^2(a+c+1)^2(a-b+c)}{\left(a+c-\cos\left(\frac{4}{3}\theta\right)\right)^{3/2}}.$$

We can easily confirm the following expressions

$$c = 1+O(\varepsilon^{4/3}),$$
$$a+c-1 = \frac{2a}{c-a+1}\left(1-\cos\left(\frac{4}{3}\theta\right)\right) = 2a\sin^2\left(\frac{2}{3}\theta\right)\cdot(1+O(\varepsilon^{4/3})),$$
$$a+c+1 = 2+O(\varepsilon^{4/3}),$$
$$a-b+c = 1+O(\varepsilon^{4/3}),$$
$$a+c-\cos\left(\frac{4}{3}\theta\right) = \frac{c+a+1}{c-a+1}\left(1-\cos\left(\frac{4}{3}\theta\right)\right)$$
$$= 2\sin^2\left(\frac{2}{3}\theta\right)\cdot(1+O(\varepsilon^{4/3})).$$

Then,

$$-\Delta\left\{\left(2\varepsilon^{-4/3}r^{4/3}-\varepsilon^{-8/3}r^{8/3}\right)G_l\left(r,\cos\left(\frac{4}{3}\theta\right),\cos\left(\frac{4}{3}\theta\right)\right)\right\}$$
$$=\frac{32\varepsilon^{-8/3}l^{2/3}}{9\pi}\sin\left(\frac{2}{3}\theta\right)\cdot(1+O(\varepsilon^{4/3})).$$

□

5 Numerical result

In this section, numerical results are shown to confirm the validity of the error estimation. All calculations were carried out on a Pentium IV PC at 2.2GHz with Borland C++ compiler. There are some difficulties in calculating H_0^1 inner product between singular basis and bilinear basis because gradient of singular basis diverges at the origin. To deal with this difficulty, the following Double Exponential transformation (DE transformation)[14] is used to calculate the integral at each of the square elements,

$$\int_{y_l}^{y_l+h}\int_{x_k}^{x_k+h}p(x,y)\,dxdy$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}p\Big(x_k+\varphi(x),y_l+\varphi(y)\Big)\,\varphi'(x)\varphi'(y)\,dxdy, \tag{5.1}$$

where $p(x,y)$ is an integrand and

$$\varphi(t)=\frac{h}{2}\left(\tanh\left(\frac{\pi}{2}\sinh t\right)+1\right).$$

(5.1) is approximated by the trapezoidal rule as follows,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}p\Big(x_k+\varphi(x),y_l+\varphi(y)\Big)\,\varphi'(x)\varphi'(y)\,dxdy$$
$$\approx\frac{L^2}{N^2}\sum_{j=-N}^{N}\sum_{i=-N}^{N}\left(x_k+\varphi\left(\frac{kL}{N}\right),y_l+\varphi\left(\frac{kL}{N}\right)\right)\varphi'\left(\frac{kL}{N}\right)\varphi'\left(\frac{kL}{N}\right).$$

We took $L=4$ and $N=100$ which is sufficient to obtain double floating point precision.

We consider following Poisson equation on two different shapes of Ω,

$$\begin{cases}-\Delta u=1 & \text{in} \quad \Omega,\\ \quad u=0 & \text{on} \quad \partial\Omega.\end{cases} \tag{5.2}$$

The first result is that Ω is L-shape domain which is shown in Fig.8. Fig.9 shows an arrangement of the singular bases and Fig.10 shows the shape of the numerical solution. The numerical results are presented in Table 5.1, where h is the mesh size and U_h denotes the numerical solution.

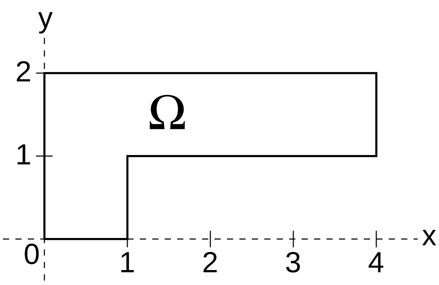

Figure 8: L-Shape domain

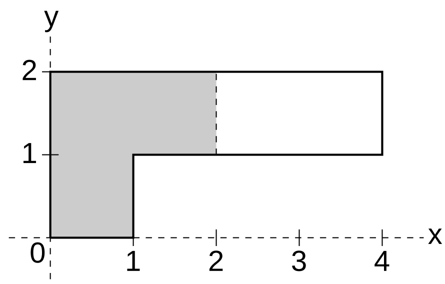

Figure 9: An arrangement of the singular bases

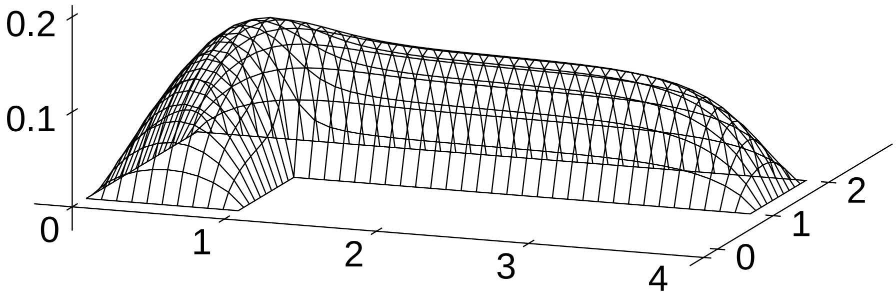

Figure 10: The numerical solution when $h = 1/10$

Table 5.1: Numerical results for the L-shape domain

h	$\|U_h - U_{h/2}\|_{H_0^1(\Omega)}$	$\|U_h - U_{h/2}\|_{L^2(\Omega)}$	Degree of freedom	Condition number
1/10	3.9052×10^{-2}	1.0684×10^{-3}	443	3.8210×10^{3}
1/20	1.9576×10^{-2}	2.6905×10^{-4}	1882	4.9653×10^{4}
1/30	1.3066×10^{-2}	1.2008×10^{-4}	4322	2.2077×10^{5}
1/40	9.8064×10^{-3}	6.7750×10^{-5}	7762	6.3104×10^{5}
1/50	7.8494×10^{-3}	4.3464×10^{-5}	12202	1.4173×10^{6}

Table 5.2: A priori error estimation for the L-shape domain

h	$\|u - u_h\|_{H_0^1(\Omega)}$	$\|u - u_h\|_{L^2(\Omega)}$
1/10	$7.161561 \cdots \times 10^{-1}$	$2.293667 \cdots \times 10^{-1}$
1/20	$3.580780 \cdots \times 10^{-1}$	$5.734168 \cdots \times 10^{-2}$
1/30	$2.387187 \cdots \times 10^{-1}$	$2.548519 \cdots \times 10^{-2}$
1/40	$1.790390 \cdots \times 10^{-1}$	$1.433542 \cdots \times 10^{-2}$
1/50	$1.432312 \cdots \times 10^{-1}$	$9.174669 \cdots \times 10^{-3}$

Table 5.3: Numerical results for the H-shape domain

h	$\|U_h - U_{h/2}\|_{H_0^1(\Omega)}$	$\|U_h - U_{h/2}\|_{L^2(\Omega)}$	Degree of freedom	Condition number
1/10	6.5080×10^{-2}	1.9355×10^{-3}	715	3.4622×10^{2}
1/20	3.4414×10^{-2}	5.4674×10^{-4}	3025	3.9510×10^{3}
1/30	2.3745×10^{-2}	2.6310×10^{-4}	6935	1.7397×10^{4}
1/40	1.8253×10^{-2}	1.5691×10^{-4}	12445	5.0101×10^{4}
1/50	1.4882×10^{-2}	1.0512×10^{-4}	19555	1.1370×10^{5}

Since the exact solution of (5.2) is not known, we calculated H_0^1 and L^2 norm of $U_h - U_{h/2}$ to obtain the rough estimation of the error. The true errors are expected to be several times lager than these value. To calculate error norm, we employed DE transformation again. The condition numbers of the element matrix were also calculated. The definition of the condition number is the ratio of the largest to the smallest eigenvalue.

From Theorem 3.1, H_0^1 and L^2 error estimations for this equation are obtained as follows,

$$\|u - u_h\|_{H_0^1(\Omega)} \leq \left(0.319 + \sqrt{0.971 + 1.469 \cdot \frac{5}{1^2}}\right) \sqrt{5}h,$$

$$\|u - u_h\|_{L^2(\Omega)} \leq \left(0.319 + \sqrt{0.971 + 1.469 \cdot \frac{5}{1^2}}\right)^2 \sqrt{5}h^2.$$

The right-hand side of these inequalities are presented in Table 5.2. Since a priori error estimate can be applied to arbitrary f and Ω, in general, the actual error is often smaller than a priori error estimation.

The next result is a case that Ω is an H-shape domain which is shown in Fig.11. Fig.12 shows an arrangement of the singular bases and Fig.13 shows the shape of the numerical solution. The numerical results are presented in Table 5.3.

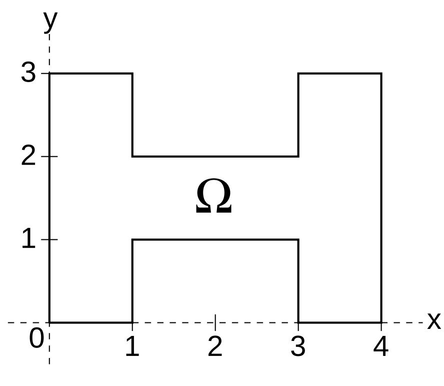

Figure 11: H-Shape domain

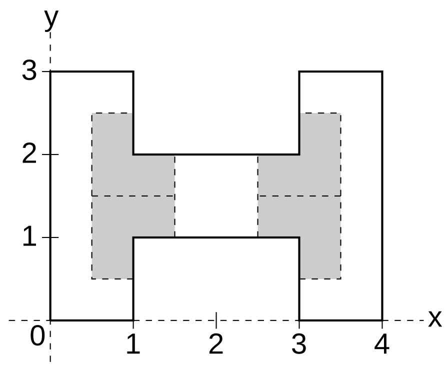

Figure 12: An arrangement of the singular bases

Table 5.4: A priori error estimation for the H-shape domain

h	$\|u-u_h\|_{H_0^1(\Omega)}$	$\|u-u_h\|_{L^2(\Omega)}$
1/10	$7.926879\cdots$	$22.21567\cdots$
1/20	$3.963439\cdots$	$5.553918\cdots$
1/30	$2.642293\cdots$	$2.468408\cdots$
1/40	$1.981719\cdots$	$1.388479\cdots$
1/50	$1.585375\cdots$	$0.888626\cdots$

From Theorem 3.1, H_0^1 and L^2 error estimation for this equation are obtained as follows,

$$\|u-u_h\|_{H_0^1(\Omega)} \leq \left(0.319 + 4\sqrt{0.971 + 1.469 \cdot \frac{8}{0.5^2}}\right)\sqrt{8}h,$$

$$\|u-u_h\|_{L^2(\Omega)} \leq \left(0.319 + 4\sqrt{0.971 + 1.469 \cdot \frac{8}{0.5^2}}\right)^2 \sqrt{8}h^2.$$

The right-hand side of these inequalities are presented in Table 5.4.

As we can see in Table 5.3 and Table 5.4, the value of a priori error estimation becomes lager when the number of the re-entrant corner increases and when the support of the singular function become smaller (n and l_k in Theorem 3.1).

6 Concluding remark

We presented a constructive a priori error estimation for finite element solution in a polygonal domain by using singular functions. The results are only when the domain is bounded by line segments which forms a right angle. However, it seems to be possible to extend this method to the general polygonal domain with triangular mesh. A priori error estimation for another important method such as DSFM or method of mesh refinement is further work.

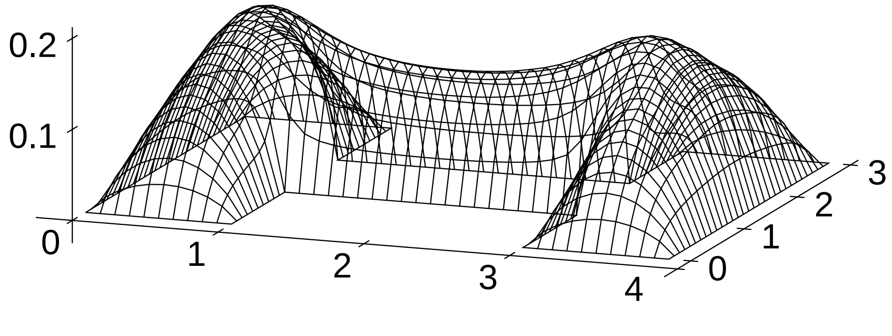

Figure 13: The numerical solution when $h = 1/10$

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