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Abstract

We present a computer-assisted proof of global uniqueness of Stokes' wave of extreme form. Stokes' wave of extreme form is a water wave which forms a corner of 120 degrees at the crest, and is considered to be the limit of the positive solution of Nekrasov's equation which expresses periodic gravity waves of permanent form on the free surface. The numerical verification method plays an important role in the proof. As for the global uniqueness of Stokes' wave of extreme form, it is not only a longtime open problem, but also it is related to an important conjecture which is called Stokes' conjecture.

Key Words: Stokes' wave of extreme form, Nekrasov's equation, Numerical verification method, gravity wave, Stokes' conjecture

1 Introduction

We are concerned in this paper with Stokes' wave of extreme form (see, e.g., [8]), which is a positive solution of a nonlinear integral equation for the unknown $\theta : (0, \pi] \to I\!R$ and is written as follows:

$$(1.1)\qquad \begin{cases} \theta(s) = \displaystyle\int_0^\pi K(s,t) \frac{\sin\theta(t)}{\int_0^t \sin\theta(w)dw}\, dt, \\ 0 < \theta(s) < \dfrac{\pi}{2} \qquad s \in (0,\pi), \\ \theta(\pi) = 0, \end{cases}$$

where

$$K(s,t) = \frac{2}{3\pi}\sum_{k=0}^{\infty} \frac{\sin ks \sin kt}{k} = \frac{1}{3\pi}\log\left|\frac{\sin\frac{s+t}{2}}{\sin\frac{s-t}{2}}\right|.$$

Stokes' wave of extreme form is derived from the following equation by Nekrasov:

$$(1.2)\qquad \begin{cases} \theta(s) = \displaystyle\int_0^\pi K(s,t) \frac{\sin\theta(t)}{\mu^{-1} + \int_0^t \sin\theta(w)dw}\, dt, \\ 0 < \theta(s) < \dfrac{\pi}{2} \qquad s \in (0,\pi), \\ \theta(0) = \theta(\pi) = 0, \end{cases}$$

where μ is a positive parameter. The equation (1.1) is derived from this by letting $\mu \to \infty$.

Nekrasov's equation arises from the following physical situation. We consider a two-dimensional, irrotational motion of inviscid fluid having a free surface. In a coordinate

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system moving with the wave, the wave profile is assumed to be stationary. We further assume that the shape is periodic and that the flow is infinitely deep. We are then asked to determine the shape of the free surface. Details are omitted (see [8]) but we should note that the free surface is represented as $(x(s), y(s))$ $(0 < s < 2\pi)$, where x and y are determined by

$$(1.3) \qquad \frac{dx}{ds} = -\frac{L}{2\pi}e^{-H\theta(s)}\cos\theta(s), \qquad \frac{dy}{ds} = -\frac{L}{2\pi}e^{-H\theta(s)}\sin\theta(s).$$

Here, L is the wavelength and H is the Hilbert transform. The right hand sides are known if θ is. Thus, after integrating in s, we have a parametric representation of the free surface.

The wave is assumed (i) to be symmetric about its crest, (ii) to have a surface that is a single-valued 2π-periodic function of horizontal distance, and (iii) to have only one peak and one trough per period. As a consequence we have

$$(1.4) \qquad \theta(s) = -\theta(-s), \quad \theta(s+2\pi) = \theta(s), \quad \text{and} \quad 0 < \theta(s) < \frac{\pi}{2} \ \text{ on } \ (0, \pi).$$

We call such solutions positive solutions.

We finally note that

$$\mu = \frac{3gL}{2\pi c^2}e^{-3H\theta(0)}$$

where c denotes the propagation speed, and g the gravitational acceleration. We refer the reader to [8] for the derivation of (1.2) and (1.3).

The global nature of the positive solutions for (1.2) was shown first by Krasovskii[5]. He showed that, for any $\beta \in (0, \pi/6)$, the equation (1.2) has a solution (μ, θ) satisfying (1.4) and $\max\theta(s) = \beta$. Presumably Krasovskii imagined that his solutions formed the whole branch up to the extreme wave. However, the fact of the matter was not that simple: numerical experiments showed strong evidence, that there existed solutions in which the maximum slope was strictly bigger than $\pi/6$ (see [10]). The existence of a positive solution for every $\mu > 3$ (there is no positive solution when $\mu \leq 3$) was first proved by Keady & Norbury[4].

When $\mu \to \infty$, the speed at the crest tends to zero and the crest becomes sharper. Keady and Norbury guaranteed all waves having one crest and one trough except for the case of $\mu = \infty$. The waves of extreme form satisfies (1.1). The existence of a solution to (1.1) was proved by Toland[12]. He proved that as $\mu \to \infty$, the solution of (1.2) have a convergent subsequence and that the limit function satisfies (1.1).

About the extreme wave, Stokes[11] recognized the following two propositions which are nowadays called Stokes' conjectures:

(a) the crest forms a corner of angle $2\pi/3$, that is, $\lim_{s\downarrow 0}\theta(s) = \pi/6$.

(b) the wave profile between two consecutive crests is concave, that is, $\theta'(s) < 0$ for $s \in (0, \pi)$.

The conjecture (a) was proved by Amick *et al.*[1]. Toland & Plotnikov[13] proved that there exists a wave which satisfies $\theta'(s) < 0$. However, this does not settle the conjecture (b) completely. Because of the lack of the proof of the uniqueness, it is not excluded that there is a positive solution of (1.1) which does not satisfy $\theta'(s) < 0$.

Although the existence of solutions for Nekrasov's equation and Stokes' wave of extreme form are proved, despite the effort of many mathematicians for many decades, the global uniqueness seems to remain open. While writing this paper, the author heard that Fraenkel[2] proved the local uniqueness, i.e., the uniqueness in a certain neighborhood of the solution. On the other hand, we had succeeded to prove the global uniqueness of a positive solution for Nekrasov's equation when $\mu \leq 170$ (see [3]). With the technique in [3], we present in this paper a proof of the global uniqueness of Stokes' wave of extreme form. In the two proofs, we employ the numerical verification method (see, for instance, [9], [6]) to obtain rigorous mathematical results by numerical computations.

The present paper is organized as follows: In section 2, we explain the idea of the proof of uniqueness. Discretization is explained in section 3, and we show the numerical results in section 4.

Influence of discretization in section 3 and rounding error in section 4 are both rigorously evaluated. Consequently, though we use numerical computation, our result is rigorous.

In this paper, θ always denotes Stokes' wave of extreme form, namely, a solution of (1.1), s is assumed to run in $0 < s \leq \pi$. 1_A denotes the function which takes value 1 if condition A holds, and takes value 0 otherwise.

2 The idea of the proof of uniqueness

In this section, we first derive a formula about bounds of θ. Secondly, we present the specific lower bound of θ. Finally, we explain how to prove the uniqueness.

First of all, we prove the following theorem,

Theorem 2.1 *Assume that, for $\theta(s)$, an upper bound and a lower bound are given as*

$$0 \leq \underline{\theta}(s) \leq \theta(s) \leq \overline{\theta}(s) \leq \pi/2,$$

then, it holds that

$$J\big(\underline{\theta}, \overline{\theta}\big)(s) \leq \theta(s) \leq J\big(\overline{\theta}, \underline{\theta}\big)(s),$$

where

$$J\big(\phi, \varphi\big)(s) = \frac{1}{6\pi}\int_0^\pi \cot\frac{t}{2} \log\left(1 + \frac{\int_{|s-t|}^{\pi-|s+t-\pi|} \sin\phi(w)dw}{\int_0^{\min(s,|s-t|)} \sin\varphi(w)dw + \int_{\min(s,|s-t|)}^{|s-t|} \sin\phi(w)dw}\right) dt.$$

Proof. Note first that

$$\frac{\partial}{\partial t}K(s,t) = \frac{1}{6\pi}\left(\cot\frac{t+s}{2} - \cot\frac{t-s}{2}\right) = \frac{1}{3\pi} \cdot \frac{\sin s}{\cos t - \cos s}.$$

With this and (1.1) we compute as follows.

$$
\begin{aligned}
\theta(s) &= \int_0^\pi K(s,t)\frac{d}{dt}\log\left(\int_0^t \sin\theta(w)dw\right)dt \\
&= \frac{1}{3\pi}\int_0^\pi \frac{\sin s}{\cos s-\cos t}\cdot\log\left(\frac{\int_0^t \sin\theta(w)dw}{\int_0^s \sin\theta(w)dw}\right)dt \\
&\le \frac{1}{3\pi}\int_0^\pi \frac{\sin s}{\cos s-\cos t}\cdot\log\left(\frac{\int_0^t \sin\left(1_{w<s}\cdot\theta(w)+1_{w\ge s}\cdot\overline{\theta}(w)\right)dw}{\int_0^s \sin\theta(w)dw}\right)dt \\
&= \frac{1}{6\pi}\int_0^\pi \left(\cot\frac{t-s}{2}-\cot\frac{t+s}{2}\right)\log\left(\frac{\int_0^t \sin\left(1_{w<s}\cdot\theta(w)+1_{w\ge s}\cdot\overline{\theta}(w)\right)dw}{\int_0^s \sin\theta(w)dw}\right)dt \\
&= \frac{1}{6\pi}\int_{-\pi}^\pi \cot\frac{t-s}{2}\cdot\log\left(\frac{\int_0^{|t|} \sin\left(1_{w<s}\cdot\theta(w)+1_{w\ge s}\cdot\overline{\theta}(w)\right)dw}{\int_0^s \sin\theta(w)dw}\right)dt \\
&= \frac{1}{6\pi}\int_0^\pi \cot\frac{t}{2}\cdot\log\left(\frac{\int_0^{\pi-|s+t-\pi|} \sin\left(1_{w<s}\cdot\theta(w)+1_{w\ge s}\cdot\overline{\theta}(w)\right)dw}{\int_0^s \sin\theta(w)dw}\right)dt \\
&\qquad +\frac{1}{6\pi}\int_{-\pi}^0 \cot\frac{t}{2}\cdot\log\left(\frac{\int_0^{|s+t|} \sin\left(1_{w<s}\cdot\theta(w)+1_{w\ge s}\cdot\overline{\theta}(w)\right)dw}{\int_0^s \sin\theta(w)dw}\right)dt \\
&= \frac{1}{6\pi}\int_0^\pi \cot\frac{t}{2}\cdot\log\left(\frac{\int_0^{\pi-|s+t-\pi|} \sin\left(1_{w<s}\cdot\theta(w)+1_{w\ge s}\cdot\overline{\theta}(w)\right)dw}{\int_0^{|s-t|} \sin\left(1_{w<s}\cdot\theta(w)+1_{w\ge s}\cdot\overline{\theta}(w)\right)dw}\right)dt \\
&= \frac{1}{6\pi}\int_0^\pi \cot\frac{t}{2}\cdot\log\left(1+\frac{\int_{|s-t|}^{\pi-|s+t-\pi|} \sin\left(1_{w<s}\cdot\theta(w)+1_{w\ge s}\cdot\overline{\theta}(w)\right)dw}{\int_0^{\min(s,|s-t|)} \sin\theta(w)dw+\int_{\min(s,|s-t|)}^{|s-t|} \sin\overline{\theta}(w)dw}\right)dt \\
&\le \frac{1}{6\pi}\int_0^\pi \cot\frac{t}{2}\cdot\log\left(1+\frac{\int_{|s-t|}^{\pi-|s+t-\pi|} \sin\overline{\theta}(w)dw}{\int_0^{\min(s,|s-t|)} \sin\underline{\theta}(w)dw+\int_{\min(s,|s-t|)}^{|s-t|} \sin\overline{\theta}(w)dw}\right)dt.
\end{aligned}
$$

Hence, $J\big(\overline{\theta},\underline{\theta}\big)(s)$ is a renewed upper bound of $\theta(s)$. In the same way, $J\big(\underline{\theta},\overline{\theta}\big)(s)$ is shown to be a renewed lower bound of $\theta(s)$.

This theorem is used in section 4 to obtain sharp upper and lower bounds of $\theta(s)$ by iteration.

Next, we prove the following theorem about the specific lower bound of $\theta(s)$.

Theorem 2.2

$$\theta(s) \geq 0.00005 \cdot 1_{0<s\leq\pi/2}.$$

Proof. Let ε denote 0.00005. Assume that $0 < a < \pi/2$ satisfies the following condition;

$$\text{for all} \quad 0 < s \leq a, \quad \theta(s) \geq \varepsilon \quad \text{holds.}$$

Since Amick *et al.*[1] proved that $\lim_{s\downarrow 0}\theta(s) = \pi/6$, such an '$a$' certainly exists.

Then, from Theorem 2.1, for $a \leq s \leq \min(1.01a,\ \pi/2)$,

$$\begin{aligned}
\theta(s) &\geq J\big(\varepsilon\cdot 1_{s\leq a},\ \pi/2\big)(s)\\
&\geq \frac{1}{6\pi}\int_{s-a}^{s+a}\cot\frac{t}{2}\cdot\log\left(1+\frac{a-|s-t|}{|s-t|}\sin\varepsilon\right)dt\\
&= \frac{1}{3\pi}\int_0^a \frac{\sin s}{\cos t-\cos s}\cdot\log\left(1+\frac{a-t}{t}\sin\varepsilon\right)dt\\
&\geq \frac{1}{3\pi}\int_0^a \frac{1}{s-t}\cdot\log\left(1+\frac{a-t}{t}\sin\varepsilon\right)dt\\
&\geq \frac{1}{3\pi}\cdot\frac{1}{s}\int_0^a \log\left(1+\frac{a-t}{t}\sin\varepsilon\right)dt\\
&= \frac{1}{3\pi}\cdot\frac{a}{s}\cdot\frac{\sin\varepsilon}{1-\sin\varepsilon}\cdot\log\left(\frac{1}{\sin\varepsilon}\right)\\
&\geq \frac{1}{3\pi}\cdot\frac{\sin\varepsilon}{1.01}\cdot\log\left(\frac{1}{\sin\varepsilon}\right) = 0.0000520194\cdots \geq \varepsilon.
\end{aligned}$$

Therefore, if $\theta(s) \geq \varepsilon$ holds for $0 < s \leq a$, then its also holds for $0 < s \leq \min(1.01a,\ \pi/2)$. Inductively, $\theta(s) \geq \varepsilon$ holds for all $0 < s \leq \pi/2$.

Now, we will explain how to prove the uniqueness of (1.1). For this purpose, we use the following theorem.

Theorem 2.3 *The solution of* (1.1) *is globaly unique if the following condition holds,*

$$\sup_{0<s\leq\pi}\frac{G\big(\overline{\theta},\underline{\theta},g\big)(s)}{g(s)} < 1,$$

where $\underline{\theta}(s) \geq 0$ *and* $\overline{\theta}(s) \leq \pi/2$ *are lower and upper bound of the solution of* (1.1), $g(s)$ *is an arbitrary positive function, and*

$$G(\phi,\varphi,g)(s) = \frac{1}{6\pi}\int_0^\pi \cot\frac{t}{2}\cdot\left(\frac{\int_{|s-t|}^{\pi-|s+t-\pi|}\sin\phi(w)dw\cdot\int_0^{\min(s,|s-t|)}\cos\varphi(w)\cdot g(w)dw}{\int_0^{|s-t|}\sin\varphi(w)dw}\right.$$
$$\left.+\int_{|s-t|}^{\pi-|s+t-\pi|}\cos\varphi(w)\cdot g(w)dw\right)\cdot\frac{dt}{\int_0^{\pi-|s+t-\pi|}\sin\varphi(w)dw}.$$

Proof. Suppose that $\overline{\theta}(s) \geq 0$ and $\underline{\theta}(s) \leq \pi/2$ are given. We then define inductively the following functions:

$$\begin{cases} \quad \phi_0(s) = \overline{\theta}(s), \qquad \varphi_0(s) = \underline{\theta}(s), \\ \phi_{n+1}(s) = \min\Big(\phi_n(s),\ J(\phi_n,\varphi_n)(s)\Big), \qquad\qquad (n = 0, 1, \cdots), \\ \varphi_{n+1}(s) = \max\Big(\varphi_n(s),\ J(\varphi_n,\phi_n)(s)\Big), \qquad\qquad (n = 0, 1, \cdots). \end{cases}$$

Theorem 2.1 shows that for all $n \geq 0$, $0 \leq \varphi_n(s) \leq \theta(s) \leq \phi_n(s) \leq \pi/2$.

By these definitions,

$$\frac{\phi_{n+1}(s) - \varphi_{n+1}(s)}{g(s)} \leq \frac{J(\phi_n,\varphi_n)(s) - J(\varphi_n,\phi_n)(s)}{g(s)}$$

$$= \frac{1}{6\pi g(s)}\int_0^\pi \cot\frac{t}{2}\cdot\log\left(\frac{1 + \dfrac{\int_{|s-t|}^{\pi-|s+t-\pi|}\sin\phi_n(w)dw}{\int_0^{\min(s,|s-t|)}\sin\varphi_n(w)dw + \int_{\min(s,|s-t|)}^{|s-t|}\sin\phi_n(w)dw}}{1 + \dfrac{\int_{|s-t|}^{\pi-|s+t-\pi|}\sin\varphi_n(w)dw}{\int_0^{\min(s,|s-t|)}\sin\phi_n(w)dw + \int_{\min(s,|s-t|)}^{|s-t|}\sin\varphi_n(w)dw}}\right)dt.$$

$$= \frac{1}{6\pi g(s)}\int_0^\pi \cot\frac{t}{2}\cdot\log\left[1 + \left\{\frac{\int_{|s-t|}^{\pi-|s+t-\pi|}\sin\phi_n(w)dw}{\int_0^{\min(s,|s-t|)}\sin\varphi_n(w)dw + \int_{\min(s,|s-t|)}^{|s-t|}\sin\phi_n(w)dw}\right.\right.$$

$$\times\left(\int_0^{\min(s,|s-t|)}\big(\sin\phi_n(w) - \sin\varphi_n(w)\big)dw - \int_{\min(s,|s-t|)}^{|s-t|}\big(\sin\phi_n(w) - \sin\varphi_n(w)\big)dw\right)$$

$$\left. + \int_{|s-t|}^{\pi-|s+t-\pi|}\big(\sin\phi_n(w) - \sin\varphi_n(w)\big)dw\right\}$$

$$\left.\times\frac{1}{\int_0^{\min(s,|s-t|)}\sin\phi_n(w)dw + \int_{\min(s,|s-t|)}^{\pi-|s+t-\pi|}\sin\varphi_n(w)dw}\right]dt$$

$$\leq \frac{1}{6\pi g(s)}\int_0^\pi \cot\frac{t}{2}\cdot\left(\frac{\int_{|s-t|}^{\pi-|s+t-\pi|}\sin\phi_n(w)dw\cdot\int_0^{\min(s,|s-t|)}\big(\sin\phi_n(w) - \sin\varphi_n(w)\big)dw}{\int_0^{\min(s,|s-t|)}\sin\varphi_n(w)dw + \int_{\min(s,|s-t|)}^{|s-t|}\sin\phi_n(w)dw}\right.$$

$$\left. + \int_{|s-t|}^{\pi-|s+t-\pi|}\big(\sin\phi_n(w) - \sin\varphi_n(w)\big)dw\right)$$

$$\times\frac{dt}{\int_0^{\min(s,|s-t|)}\sin\phi_n(w)dw + \int_{\min(s,|s-t|)}^{\pi-|s+t-\pi|}\sin\varphi_n(w)dw}$$

$$\leq \frac{1}{6\pi g(s)}\int_0^\pi \cot\frac{t}{2}\cdot\left(\frac{\int_{|s-t|}^{\pi-|s+t-\pi|}\sin\overline{\theta}(w)dw\cdot\int_0^{\min(s,|s-t|)}\cos\underline{\theta}(w)\cdot g(w)dw}{\int_0^{|s-t|}\sin\underline{\theta}(w)dw}\right.$$

$$\left.+\int_{|s-t|}^{\pi-|s+t-\pi|}\cos\underline{\theta}(w)\cdot g(w)dw\right)\cdot\frac{dt}{\int_0^{\pi-|s+t-\pi|}\sin\underline{\theta}(w)dw}\cdot\sup_{0\leq s\leq\pi}\frac{\phi_n(s)-\varphi_n(s)}{g(s)}$$

$$=\frac{G\big(\overline{\theta},\underline{\theta},g\big)(s)}{g(s)}\cdot\sup_{0<s\leq\pi}\frac{\phi_n(s)-\varphi_n(s)}{g(s)}.$$

This leads us to

$$\sup_{0<s\leq\pi}\frac{\phi_{n+1}(s)-\varphi_{n+1}(s)}{g(s)}\leq\sup_{0<s\leq\pi}\frac{G\big(\overline{\theta},\underline{\theta},g\big)(s)}{g(s)}\cdot\sup_{0<s\leq\pi}\frac{\phi_n(s)-\varphi_n(s)}{g(s)}.$$

If the assumption of this theorem holds, then

$$\sup_{0<s\leq\pi}\frac{\phi_n(s)-\varphi_n(s)}{g(s)}\to 0,\quad (n\to\infty),$$

and this means that the solution of (1.1) is unique.

The next problem is to choose a suitable function $g(s)$. The simple choice $g(s)\equiv$ *constant* was found not to work. Our choice, which will be exhibited in section 4, is more involved.

3 Discretization

The main purpose of this section is to explain how to discretize functional $J(\cdot,\cdot)$ and $G(\cdot,\cdot,\cdot)$ by step functions.

Let N be an arbitrary even number ≥ 4, and let Q_N denote the set of step functions which are constant in each intervals

$$\frac{k\pi}{N}<s\leq\frac{(k+1)\pi}{N}\quad (k=0,1,\cdots,N-1). \tag{3.1}$$

Assume that $\underline{\theta}(s)\geq 0$ and $\overline{\theta}(s)\leq\pi/2$ are lower and upper bounds of the solution $\theta(s)$, and that $g(s)$ is an arbitrary positive function. We also assume that $\underline{\theta}(s)$, $\overline{\theta}(s)$ and $g(s)$ belong to Q_N.

We will constructively define step functions $\overline{J}_*$, $\underline{J}_*$ and G_* in Q_N which satisfy

$$\overline{J}_*(\overline{\theta},\underline{\theta})(s)\geq\theta(s), \tag{3.2}$$

$$\underline{J}_*(\underline{\theta},\overline{\theta})(s)\leq\theta(s), \tag{3.3}$$

$$G_*(\overline{\theta},\underline{\theta},g)(s)\geq G(\overline{\theta},\underline{\theta},g)(s), \tag{3.4}$$

as discretizations of $\overline{J}(\cdot,\cdot)$, $\underline{J}(\cdot,\cdot)$ and $G(\cdot,\cdot,\cdot)$.

The values of $\overline{J}_*$, $\underline{J}_*$ and G_* in the subinterval (3.1) are denoted by $\overline{J}_k$, $\underline{J}_k$ and G_k. We may accordingly write as follows:

$$\overline{J}_*(\overline{\theta},\underline{\theta})(s) = \sum_{k=0}^{N-1} 1_{\frac{k\pi}{N}<s\le\frac{(k+1)\pi}{N}} \cdot \overline{J}_k(\overline{\theta},\underline{\theta}),$$
$$\underline{J}_*(\underline{\theta},\overline{\theta})(s) = \sum_{k=0}^{N-2} 1_{\frac{k\pi}{N}<s\le\frac{(k+1)\pi}{N}} \cdot \underline{J}_k(\underline{\theta},\overline{\theta}),$$
$$G_*(\overline{\theta},\underline{\theta},g)(s) = \sum_{k=0}^{N-1} 1_{\frac{k\pi}{N}<s\le\frac{(k+1)\pi}{N}} \cdot G_k(\overline{\theta},\underline{\theta},g).$$

The precise definitions of $\overline{J}_k(\overline{\theta},\underline{\theta})$ etc. are rather complicated and will be given later. To that end, we define the following symbols.

$$\begin{aligned}
\overline{\Theta}(u) &\equiv \sup_{u<x\le N} \sin\overline{\theta}\left(\frac{\pi}{N}x\right), \quad 0\le u<N,\\
\underline{\Theta}(u) &\equiv \inf_{0<x\le\min(u,N)} \sin\underline{\theta}\left(\frac{\pi}{N}x\right), \quad 0<u,\\
\gamma(u) &\equiv \sup_{u<x\le N} \cos\underline{\theta}\left(\frac{\pi}{N}x\right)\cdot g\left(\frac{\pi}{N}x\right), \quad 0\le u<N,\\
\overline{\Theta}(N) &\equiv \lim_{u\uparrow N}\overline{\Theta}(u), \quad \underline{\Theta}(0)\equiv\lim_{u\downarrow 0}\underline{\Theta}(u), \quad \gamma(N)\equiv\lim_{u\uparrow N}\gamma(u),\\
\overline{\Theta}_a &\equiv \overline{\Theta}(a), \quad \overline{\Psi}_a^b \equiv \int_a^b \overline{\Theta}(u)du,\\
\underline{\Theta}_a &\equiv \underline{\Theta}(a), \quad \underline{\Psi}_a^b \equiv \int_a^b \underline{\Theta}(u)du,\\
\gamma_a &\equiv \gamma(a), \quad \Gamma_a^b \equiv \int_a^b \gamma(u)du,\\
c_1 &\equiv \sup_{0<t\le N}\frac{\overline{\Psi}_0^t}{\underline{\Psi}_0^t}, \quad c_2\equiv\sup_{0<t\le N}\frac{\Gamma_0^t}{\underline{\Psi}_0^t}, \quad c_3\equiv\sup_{0<t\le N}\frac{t}{\underline{\Psi}_0^t},\\
f[w] &\equiv N-|w-N|.
\end{aligned}$$

For each s and integer k such that

$$\frac{k\pi}{N} < s \le \frac{(k+1)\pi}{N} \qquad (0\le k\le N-1),$$

the following inequality holds,

$$\underline{\Theta}\left(\tfrac{(k+1)w}{s}\right) \le \sin\theta(w) \le \overline{\Theta}\left(\tfrac{kw}{s}\right) \qquad (0<w\le\pi). \tag{3.5}$$

In fact, note first that $\sin\theta(w)\le\sin\overline{\theta}$. Since $\frac{kw}{s}<\frac{Nw}{\pi}$, we have $\overline{\Theta}\left(\frac{kw}{s}\right)\ge\sin\overline{\theta}(w)$, which implies the inequality on the right hand side. The one on the left hand side is proved similarly.

Using (3.5) and Theorem 2.1, we have

$$
\begin{aligned}
\theta(s) &\le \frac{1}{6\pi}\int_0^{\pi}\cot\frac{t}{2}\log\left(1+\frac{\int_{|s-t|}^{\pi-|s+t-\pi|}\overline{\Theta}\left(\frac{kw}{s}\right)dw}{\int_0^{\min(s,|s-t|)}\underline{\Theta}\left(\frac{(k+1)w}{s}\right)dw+\int_{\min(s,|s-t|)}^{|s-t|}\overline{\Theta}\left(\frac{kw}{s}\right)dw}\right)dt\\
&= \frac{1}{6\pi}\int_0^{\frac{\pi}{s}}\frac{s}{\tan\frac{st}{2}}\log\left(1+\frac{\int_{|t-1|}^{\frac{\pi}{s}-\left|1+t-\frac{\pi}{s}\right|}\overline{\Theta}(kw)dw}{\int_0^{\min(1,|t-1|)}\underline{\Theta}((k+1)w)dw+\int_{\min(1,|t-1|)}^{|t-1|}\overline{\Theta}(kw)dw}\right)dt\\
&\le \frac{1}{6\pi}\int_0^{\frac{N}{k}}\lim_{\sigma\downarrow\frac{k\pi}{N}}\frac{\sigma}{\tan\frac{\sigma t}{2}}\log\left(1+\frac{\int_{|t-1|}^{\frac{\pi}{\sigma}-\left|1+t-\frac{\pi}{\sigma}\right|}\overline{\Theta}(kw)dw}{\int_0^{\min(1,|t-1|)}\underline{\Theta}((k+1)w)dw+\int_{\min(1,|t-1|)}^{|t-1|}\overline{\Theta}(kw)dw}\right)dt\\
&\le \sum_{l=0}^{N-1}\overline{A}_{k,l}(\overline{\theta},\underline{\theta})+\sum_{l=k}^{N-1}\min\left(\overline{B}_{k,l}(\overline{\theta},\underline{\theta}),\ \overline{C}_{k,l}(\overline{\theta},\underline{\theta})\right)\equiv\overline{J}_k(\overline{\theta},\underline{\theta}),
\end{aligned}
$$

where $\overline{A}_{k,l}$, $\overline{B}_{k,l}$ and $\overline{C}_{k,l}$ are defined as follows:

First, $\overline{A}_{k,l}(\overline{\theta},\underline{\theta})$ is defined for $l=0$ as

$$
\frac{1}{6\pi}\int_0^{\frac{1}{N}}\frac{4\,\overline{\Theta}_{k\left(1-\frac{1}{N}\right)}}{\frac{1}{k+1}\,\underline{\Psi}_0^{(k+1)\left(1-\frac{1}{N}\right)}+\left(\frac{1}{N}-t\right)\underline{\Theta}_{k+1}}dt;
$$

for $k=0$, $1\le l$,

$$
\frac{1}{6\pi}\int_{\frac{l}{N}}^{\frac{l+1}{N}}\frac{2N}{l}\log\left(1+\frac{\frac{2l}{N}\overline{\Theta}_0}{\underline{\Psi}_0^{\left(1-\frac{l+1}{N}\right)}+\left(\frac{l+1}{N}-t\right)\underline{\Theta}_{1-\frac{l}{N}}}\right)dt;
$$

and for $1\le k$, $1\le l$,

$$
\frac{1}{6\pi}\int_{\frac{l}{N}}^{\frac{l+1}{N}}\frac{\frac{k\pi}{N}}{\tan\frac{kl\pi}{2N^2}}\log\left(1+\frac{\min\left(\frac{1}{k}\,\overline{\Psi}_{k\left(1-\frac{l+1}{N}\right)}^{f\left[k\left(1+\frac{l+1}{N}\right)\right]},\ \frac{2l}{N}\overline{\Theta}_{k\left(1-\frac{l+1}{N}\right)}\right)}{\frac{1}{k+1}\,\underline{\Psi}_0^{(k+1)\left(1-\frac{l+1}{N}\right)}+\left(\frac{l+1}{N}-t\right)\underline{\Theta}_{(k+1)\left(1-\frac{l}{N}\right)}}\right)dt.
$$

$\overline{B}_{k,l}(\overline{\theta},\underline{\theta})$ is defined as ∞ if $l=0$;

for $k=0,\ 1\le l\le \frac{N}{2}-1$,

$$\frac{1}{6\pi}\int_{\frac{N}{l+1}}^{\frac{N}{l}} \frac{2(l+1)}{N}\log\left(1+\frac{2\overline{\Theta}_0}{\underline{\Psi}_0^1+(t-2)\,\overline{\Theta}_0}\right)dt;$$

for $k=0,\ \frac{N}{2}\le l$,

$$\frac{1}{6\pi}\int_{\frac{N}{l+1}}^{\frac{N}{l}} \frac{2(l+1)}{N}\log\left(1+\frac{2\overline{\Theta}_0}{\underline{\Psi}_0^{\frac{N}{l+1}-1}+\left(t-\frac{N}{l+1}\right)\,\underline{\Theta}_{\frac{N}{l}-1}}\right)dt;$$

for $k\ge 1,\ 1\le l\le \frac{N}{2}-1$,

$$\frac{1}{6\pi}\int_{\frac{N}{l+1}}^{\frac{N}{l}} \frac{\frac{k\pi}{N}}{\tan\frac{k\pi}{2(l+1)}}\log\left(1+\frac{\frac{1}{k}\,\overline{\Psi}_{k\left(\frac{N}{l+1}-1\right)}^{f\left[k\left(\frac{N}{l+1}+1\right)\right]}}{\frac{1}{k+1}\,\underline{\Psi}_0^{k+1}+\frac{1}{k}\,\overline{\Psi}_k^{k\left(\frac{N}{l+1}-1\right)}+\left(t-\frac{N}{l+1}\right)\,\overline{\Theta}_{k\left(\frac{N}{l}-1\right)}}\right)dt;$$

and for $k\ge 1,\ \frac{N}{2}\le l$,

$$\frac{1}{6\pi}\int_{\frac{N}{l+1}}^{\frac{N}{l}} \frac{\frac{k\pi}{N}}{\tan\frac{k\pi}{2(l+1)}}\log\left(1+\frac{\frac{1}{k}\,\overline{\Psi}_{k\left(\frac{N}{l+1}-1\right)}^{f\left[k\left(\frac{N}{l+1}+1\right)\right]}}{\frac{1}{k+1}\,\underline{\Psi}_0^{(k+1)\left(\frac{N}{l+1}-1\right)}+\left(t-\frac{N}{l+1}\right)\,\underline{\Theta}_{(k+1)\left(\frac{N}{l}-1\right)}}\right)dt.$$

$\overline{C}_{k,l}(\overline{\theta},\underline{\theta})$ is defined as ∞ if $l=N-1$;

for $k=0,\ l\le \frac{N}{2}-1$,

$$\frac{1}{6\pi}\int_{\frac{N}{l+1}}^{\frac{N}{l}} \frac{2}{t}\cdot\frac{2\overline{\Theta}_0}{\underline{\Psi}_0^1+(t-2)\,\overline{\Theta}_0}dt;$$

for $k=0,\ \frac{N}{2}\le l\le N-2$,

$$\frac{1}{6\pi}\int_{\frac{N}{l+1}}^{\frac{N}{l}} \frac{2}{t}\cdot\frac{2\overline{\Theta}_0}{\underline{\Psi}_0^{\frac{N}{l+1}-1}+\left(t-\frac{N}{l+1}\right)\,\underline{\Theta}_{\frac{N}{l}-1}}dt;$$

for $k\ge 1,\ l\le \frac{N}{2}-1$,

$$\frac{1}{6\pi}\int_{\frac{N}{l+1}}^{\frac{N}{l}} \frac{\cos\frac{k\pi}{l+1}+1}{\frac{N}{k\pi}\sin\frac{k\pi}{l+1}+t-\frac{N}{l+1}}\cdot\frac{\frac{1}{k}\,\overline{\Psi}_{k\left(\frac{N}{l+1}-1\right)}^{f\left[k\left(\frac{N}{l+1}+1\right)\right]}}{\frac{1}{k+1}\,\underline{\Psi}_0^{k+1}+\frac{1}{k}\,\overline{\Psi}_k^{k\left(\frac{N}{l+1}-1\right)}+\left(t-\frac{N}{l+1}\right)\,\overline{\Theta}_{k\left(\frac{N}{l}-1\right)}}dt;$$

and for $\quad k \geq 1, \;\; \frac{N}{2} \leq l \leq N-2,$

$$\frac{1}{6\pi}\int_{\frac{N}{l+1}}^{\frac{N}{l}} \frac{\cos\frac{k\pi}{l+1}+1}{\frac{N}{k\pi}\sin\frac{k\pi}{l+1}+t-\frac{N}{l+1}} \cdot \frac{\frac{1}{k}\,\overline{\Psi}_{k\left(\frac{N}{l+1}-1\right)}^{f\left[k\left(\frac{N}{l+1}+1\right)\right]}}{\frac{1}{k+1}\,\underline{\Psi}_{0}^{(k+1)\left(\frac{N}{l+1}-1\right)}+\left(t-\frac{N}{l+1}\right)\,\underline{\Theta}_{(k+1)\left(\frac{N}{l}-1\right)}}dt.$$

We have thus defined $\overline{J}_k(\overline{\theta},\underline{\theta})$.

Similarly we have

$$\begin{aligned}
\theta(s) &\geq \frac{1}{6\pi}\int_0^{\pi}\cot\frac{t}{2}\log\left(1+\frac{\int_{|s-t|}^{\pi-|s+t-\pi|}\underline{\Theta}\left(\frac{(k+1)w}{s}\right)dw}{\int_0^{\min(s,|s-t|)}\overline{\Theta}\left(\frac{kw}{s}\right)dw+\int_{\min(s,|s-t|)}^{|s-t|}\underline{\Theta}\left(\frac{(k+1)w}{s}\right)dw}\right)dt\\
&= \frac{1}{6\pi}\int_0^{\frac{\pi}{s}}\frac{s}{\tan\frac{st}{2}}\log\left(1+\frac{\int_{|t-1|}^{\frac{\pi}{s}-\left|1+t-\frac{\pi}{s}\right|}\underline{\Theta}\big((k+1)w\big)dw}{\int_0^{\min(1,|t-1|)}\overline{\Theta}\big(kw\big)dw+\int_{\min(1,|t-1|)}^{|t-1|}\underline{\Theta}\big((k+1)w\big)dw}\right)dt\\
&\geq \frac{1}{6\pi}\int_0^{\frac{N}{k+1}}\frac{\frac{(k+1)\pi}{N}}{\tan\frac{(k+1)t\pi}{2N}}\log\left(1+\frac{\int_{|t-1|}^{\frac{N}{k+1}-\left|1+t-\frac{N}{k+1}\right|}\underline{\Theta}\big((k+1)w\big)dw}{\int_0^{\min(1,|t-1|)}\overline{\Theta}\big(kw\big)dw+\int_{\min(1,|t-1|)}^{|t-1|}\underline{\Theta}\big((k+1)w\big)dw}\right)dt\\
&\geq \sum_{l=0}^{N-1}\underline{A}_{k,l}(\underline{\theta},\overline{\theta})+\sum_{l=k+1}^{N-1}\max\left(\underline{B}_{k,l}(\underline{\theta},\overline{\theta}),\;\underline{C}_{k,l}(\underline{\theta},\overline{\theta})\right)\equiv \underline{J}_k(\underline{\theta},\overline{\theta}),
\end{aligned}$$

where $\underline{A}_{k,l}$, $\underline{B}_{k,l}$ and $\underline{C}_{k,l}$ are defined as follows:

for $\quad k=0, \;\; \underline{A}_{k,l}(\underline{\theta},\overline{\theta}) \quad$ is defined as

$$\frac{1}{6\pi}\int_{\frac{l}{N}}^{\frac{l+1}{N}}\frac{\frac{\pi}{N}}{\tan\frac{(l+1)\pi}{2N^2}}\log\left(1+\frac{\max\left(\underline{\Psi}_{1-\frac{l}{N}}^{1+\frac{l}{N}},\quad \frac{2(l+1)}{N}\,\underline{\Theta}_{1+\frac{l+1}{N}}\right)}{(1-t)\,\overline{\Theta}_0}\right)dt;$$

and for $\quad k \geq 1,$

$$\frac{1}{6\pi}\int_{\frac{l}{N}}^{\frac{l+1}{N}}\frac{\frac{(k+1)\pi}{N}}{\tan\frac{(k+1)(l+1)\pi}{2N^2}}\log\left(1+\frac{\max\left(\frac{1}{k+1}\,\underline{\Psi}_{(k+1)\left(1-\frac{l}{N}\right)}^{f\left[(k+1)\left(1+\frac{l}{N}\right)\right]},\quad A\right)}{\frac{1}{k}\,\overline{\Psi}_0^{k\left(1-\frac{l+1}{N}\right)}+\left(\frac{l+1}{N}-t\right)\,\overline{\Theta}_{k\left(1-\frac{l+1}{N}\right)}}\right)dt,$$

where we have put $\quad A = 2\min\left(\frac{l+1}{N},\frac{N}{k+1}-1\right)\cdot\underline{\Theta}_{\min\left((k+1)\left(1+\frac{l+1}{N}\right),\,N,\,2N-(k+1)\left(1+\frac{l}{N}\right)\right)}.$

$\underline{B}_{k,l}(\underline{\theta},\overline{\theta})$ is defined as 0 for $l = k+1$;

for $k = 0$, $k+2 \le l \le \frac{N}{2}-1$,

$$\frac{1}{6\pi}\int_{\frac{N}{l+1}}^{\frac{N}{l}} \frac{\frac{\pi}{N}}{\tan\frac{\pi}{2l}} \log\left(1 + \frac{\underline{\Psi}_{\frac{N}{l}-1}^{\frac{N}{l}+1}}{\overline{\Theta}_0 + \underline{\Psi}_1^{\frac{N}{l+1}-1} + \left(t - \frac{N}{l+1}\right)\underline{\Theta}_{\frac{N}{l+1}-1}}\right) dt;$$

for $k = 0$, $\frac{N}{2} \le l$,

$$\frac{1}{6\pi}\int_{\frac{N}{l+1}}^{\frac{N}{l}} \frac{\frac{\pi}{N}}{\tan\frac{\pi}{2l}} \log\left(1 + \frac{\underline{\Psi}_{\frac{N}{l}-1}^{\frac{N}{l}+1}}{(t-1)\,\overline{\Theta}_0}\right) dt;$$

for $k \ge 1$, $k+2 \le l \le \frac{N}{2}-1$,

$$\frac{1}{6\pi}\int_{\frac{N}{l+1}}^{\frac{N}{l}} \frac{\frac{(k+1)\pi}{N}}{\tan\frac{(k+1)\pi}{2l}} \log\left(1 + \frac{\frac{1}{k+1}\underline{\Psi}_{(k+1)\left(\frac{N}{l}-1\right)}^{f\left[(k+1)\left(\frac{N}{l}+1\right)\right]}}{\frac{1}{k}\overline{\Psi}_0^{k} + \frac{1}{k+1}\underline{\Psi}_{k+1}^{(k+1)\left(\frac{N}{l+1}-1\right)} + \left(t - \frac{N}{l+1}\right)\underline{\Theta}_{(k+1)\left(\frac{N}{l+1}-1\right)}}\right) dt;$$

and for $k \ge 1$, $\frac{N}{2} \le l$,

$$\frac{1}{6\pi}\int_{\frac{N}{l+1}}^{\frac{N}{l}} \frac{\frac{(k+1)\pi}{N}}{\tan\frac{(k+1)\pi}{2l}} \log\left(1 + \frac{\frac{1}{k+1}\underline{\Psi}_{(k+1)\left(\frac{N}{l}-1\right)}^{f\left[(k+1)\left(\frac{N}{l}+1\right)\right]}}{\frac{1}{k}\overline{\Psi}_0^{k\left(\frac{N}{l+1}-1\right)} + \left(t - \frac{N}{l+1}\right)\overline{\Theta}_{k\left(\frac{N}{l+1}-1\right)}}\right) dt.$$

$\underline{C}_{k,l}(\underline{\theta},\overline{\theta})$ is defined as 0 for $l = k+1$;

for $k = 0$, $k+2 \le l \le \frac{N}{2}-1$,

$$\frac{1}{6\pi}\int_{\frac{N}{l+1}}^{\frac{N}{l}} \frac{\frac{\pi}{l(l+1)}}{\left(t-\frac{N}{l+1}\right)\tan\frac{\pi}{2l} + \left(\frac{N}{l}-t\right)\tan\frac{\pi}{2(l+1)}} \times \frac{\overline{\Theta}_0 + \underline{\Psi}_1^{\frac{N}{l+1}-1}}{\overline{\Theta}_0 + \underline{\Psi}_1^{\frac{N}{l+1}-1} + \left(t - \frac{N}{l+1}\right)\underline{\Theta}_{\frac{N}{l+1}-1}} \log\left(1 + \frac{\underline{\Psi}_{\frac{N}{l}-1}^{\frac{N}{l}+1}}{\overline{\Theta}_0 + \underline{\Psi}_1^{\frac{N}{l+1}-1}}\right) dt;$$

for $k = 0$, $\frac{N}{2} \le l$,

$$\frac{1}{6\pi}\int_{\frac{N}{l+1}}^{\frac{N}{l}} \frac{\frac{\pi}{l(l+1)}}{\left(t-\frac{N}{l+1}\right)\tan\frac{\pi}{2l} + \left(\frac{N}{l}-t\right)\tan\frac{\pi}{2(l+1)}} \cdot \frac{\left(\frac{N}{l+1}-1\right)\overline{\Theta}_0}{(t-1)\,\overline{\Theta}_0} \log\left(1 + \frac{\underline{\Psi}_{\frac{N}{l}-1}^{\frac{N}{l}+1}}{\left(\frac{N}{l+1}-1\right)\overline{\Theta}_0}\right) dt;$$

for $\quad k \geq 1, \quad k+2 \leq l \leq \frac{N}{2}-1,$

$$\frac{1}{6\pi}\int_{\frac{N}{l+1}}^{\frac{N}{l}} \frac{\frac{(k+1)\pi}{l(l+1)}}{\left(t-\frac{N}{l+1}\right)\tan\frac{(k+1)\pi}{2l}+\left(\frac{N}{l}-t\right)\tan\frac{(k+1)\pi}{2(l+1)}}$$
$$\times \frac{\frac{1}{k}\overline{\Psi}_0^{k}+\frac{1}{k+1}\underline{\Psi}_{k+1}^{(k+1)\left(\frac{N}{l+1}-1\right)}}{\frac{1}{k}\overline{\Psi}_0^{k}+\frac{1}{k+1}\underline{\Psi}_{k+1}^{(k+1)\left(\frac{N}{l+1}-1\right)}+\left(t-\frac{N}{l+1}\right)\underline{\Theta}_{(k+1)\left(\frac{N}{l+1}-1\right)}}$$
$$\times \log\left(1+\frac{\frac{1}{k+1}\underline{\Psi}_{(k+1)\left(\frac{N}{l}-1\right)}^{f\left[(k+1)\left(\frac{N}{l}+1\right)\right]}}{\frac{1}{k}\overline{\Psi}_0^{k}+\frac{1}{k+1}\underline{\Psi}_{k+1}^{(k+1)\left(\frac{N}{l+1}-1\right)}}\right)dt;$$

and for $\quad k \geq 1, \quad \frac{N}{2} \leq l,$

$$\frac{1}{6\pi}\int_{\frac{N}{l+1}}^{\frac{N}{l}} \frac{\frac{(k+1)\pi}{l(l+1)}}{\left(t-\frac{N}{l+1}\right)\tan\frac{(k+1)\pi}{2l}+\left(\frac{N}{l}-t\right)\tan\frac{(k+1)\pi}{2(l+1)}}$$
$$\times \frac{\frac{1}{k}\overline{\Psi}_0^{k\left(\frac{N}{l+1}-1\right)}}{\frac{1}{k}\overline{\Psi}_0^{k\left(\frac{N}{l+1}-1\right)}+\left(t-\frac{N}{l+1}\right)\overline{\Theta}_{k\left(\frac{N}{l+1}-1\right)}}\log\left(1+\frac{\frac{1}{k+1}\underline{\Psi}_{(k+1)\left(\frac{N}{l}-1\right)}^{f\left[(k+1)\left(\frac{N}{l}+1\right)\right]}}{\frac{1}{k}\overline{\Psi}_0^{k\left(\frac{N}{l+1}-1\right)}}\right)dt.$$

Next, note that

$$G(\overline{\theta},\underline{\theta},g)(s)=\frac{1}{6\pi}\int_0^{\pi}\cot\frac{t}{2}\times\left(\frac{\int_{|s-t|}^{\pi-|s+t-\pi|}\sin\overline{\theta}(w)dw\cdot\int_0^{\min(s,|s-t|)}\cos\underline{\theta}(w)\cdot g(w)dw}{\int_0^{|s-t|}\sin\underline{\theta}(w)dw}\right.$$
$$\left.+\int_{|s-t|}^{\pi-|s+t-\pi|}\cos\underline{\theta}(w)\cdot g(w)dw\right)\times\frac{dt}{\int_0^{\pi-|s+t-\pi|}\sin\underline{\theta}(w)dw}$$
$$=\frac{1}{6\pi}\int_0^{\frac{\pi}{s}}\frac{s}{\tan\frac{st}{2}}\times\left(\frac{\int_{|1-t|}^{\frac{\pi}{s}-\left|1+t-\frac{\pi}{s}\right|}\sin\overline{\theta}(sw)dw\cdot\int_0^{\min(1,|1-t|)}\cos\underline{\theta}(sw)\cdot g(sw)dw}{\int_0^{|1-t|}\sin\underline{\theta}(sw)dw}\right.$$
$$\left.+\int_{|1-t|}^{\frac{\pi}{s}-\left|1+t-\frac{\pi}{s}\right|}\cos\underline{\theta}(sw)\cdot g(sw)dw\right)\times\frac{dt}{\int_0^{\frac{\pi}{s}-\left|1+t-\frac{\pi}{s}\right|}\sin\underline{\theta}(sw)dw}$$

$$
\leq \frac{1}{6\pi}\int_0^1 \lim_{\sigma\downarrow\frac{k\pi}{N}} \frac{\sigma}{\tan\frac{\sigma t}{2}} \times \left\{ \min\left(\frac{\int_{1-t}^{\frac{\pi}{\sigma}-\left|1+t-\frac{\pi}{\sigma}\right|} \overline{\Theta}(kw)dw}{\int_0^{\frac{N}{k+1}-\left|1+t-\frac{N}{k+1}\right|} \underline{\Theta}((k+1)w)dw}, c_1 \right) \right.
$$

$$
\times \min\left(\frac{\int_0^{1-t}\gamma(kw)dw}{\int_0^{1-t}\underline{\Theta}((k+1)w)dw},\ c_2 \right) + \min\left(\frac{\int_{1-t}^{\frac{\pi}{\sigma}-\left|1+t-\frac{\pi}{\sigma}\right|}\gamma(kw)dw}{\int_0^{\frac{N}{k+1}-\left|1+t-\frac{N}{k+1}\right|}\underline{\Theta}((k+1)w)dw},\ c_2 \right) \Bigg\} dt
$$

$$
+ \frac{1}{6\pi}\int_1^{\frac{N}{k}} \lim_{\sigma\downarrow\frac{k\pi}{N}} \frac{\sigma}{\tan\frac{\sigma t}{2}} \times \left\{ \min\left(\frac{\int_{t-1}^{\frac{\pi}{\sigma}-\left|1+t-\frac{\pi}{\sigma}\right|}\overline{\Theta}(kw)dw}{\int_0^{\max\left(\frac{N}{k+1}-\left|1+t-\frac{N}{k+1}\right|,0\right)}\underline{\Theta}((k+1)w)dw},\ c_1,\ \frac{2\overline{\Theta}(0)}{1+t}c_3 \right) \right.
$$

$$
\times \min\left(\frac{\int_0^{\min(1,t-1)}\gamma(kw)dw}{\int_0^{t-1}\underline{\Theta}((k+1)w)dw}, c_2 \right)
$$

$$
\left. + \min\left(\frac{\int_{t-1}^{\frac{\pi}{\sigma}-\left|1+t-\frac{\pi}{\sigma}\right|}\gamma(kw)dw}{\int_0^{\max\left(\frac{N}{k+1}-\left|1+t-\frac{N}{k+1}\right|,0\right)}\underline{\Theta}((k+1)w)dw},\ c_2,\ \frac{2\gamma(0)}{1+t}c_3 \right) \right\} dt
$$

$$
\leq \sum_{l=0}^{N-1} D_{k,l}(\overline{\theta},\underline{\theta},g) + \sum_{l=k}^{N-1} \min\left(E_{k,l}(\overline{\theta},\underline{\theta}),\ F_{k,l}(\overline{\theta},\underline{\theta})\right) \equiv G_k(\overline{\theta},\underline{\theta},g),
$$

where $D_{k,l}$, $E_{k,l}$ and $F_{k,l}$ are defined as follows:

for $k = l = 0$, $D_{k,l}(\overline{\theta},\underline{\theta},g)$ is defined as

$$
\frac{1}{6\pi}\int_0^{\frac{1}{N}} \frac{4}{\underline{\Psi}_0^1} \left\{ \overline{\Theta}_0 \cdot \min\left(\frac{\gamma_0}{\underline{\Psi}_0^{1-\frac{1}{N}}},\ c_2 \right) + \gamma_0 \right\} dt;
$$

for $k = 0$, $l \geq 1$,

$$
\frac{1}{6\pi}\int_{\frac{l}{N}}^{\frac{l+1}{N}} \frac{2N}{l} \left\{ \min\left(\frac{\frac{2l}{N}\overline{\Theta}_0}{\underline{\Psi}_0^{1+\frac{l}{N}}},\ c_1 \right) \cdot \min\left(\frac{\left(1-\frac{l}{N}\right)\gamma_0}{\underline{\Psi}_0^{1-\frac{l+1}{N}}},\ c_2 \right) + \min\left(\frac{\frac{2l}{N}\gamma_0}{\underline{\Psi}_0^{1+\frac{l}{N}}},\ c_2 \right) \right\} dt;
$$

for $k \geq 1$, $l = 0$,

$$
\frac{1}{6\pi}\int_0^{\frac{1}{N}} \frac{4}{\frac{1}{k+1}\underline{\Psi}_0^{\min\left(k+1,\,2N-(k+1)\left(1+\frac{1}{N}\right)\right)}}
$$

$$
\times \left\{ \overline{\Theta}_{k\left(1-\frac{1}{N}\right)} \cdot \min\left(\frac{\frac{1}{k}\Gamma_0^k}{\frac{1}{k+1}\underline{\Psi}_0^{(k+1)\left(1-\frac{1}{N}\right)}},\ c_2 \right) + \gamma_{k\left(1-\frac{1}{N}\right)} \right\} dt;
$$

and for $k \geq 1, \;\; l \geq 1,$

$$\frac{1}{6\pi}\int_{\frac{l}{N}}^{\frac{l+1}{N}} \frac{\frac{k\pi}{N}}{\tan\frac{kl\pi}{2N^2}} \left\{ \min\left(\frac{\min\left(\frac{2l}{N}\,\overline{\Theta}_{k\left(1-\frac{l+1}{N}\right)},\ \frac{1}{k}\,\overline{\Psi}_{k\left(1-\frac{l+1}{N}\right)}^{f\left[k\left(1+\frac{l+1}{N}\right)\right]}\right)}{\frac{1}{k+1}\,\underline{\Psi}_0^{\min\left((k+1)\left(1+\frac{l}{N}\right),\,2N-(k+1)\left(1+\frac{l+1}{N}\right)\right)}},\ c_1\right) \right.$$
$$\times \min\left(\frac{\frac{1}{k}\,\Gamma_0^{k\left(1-\frac{l}{N}\right)}}{\frac{1}{k+1}\,\underline{\Psi}_0^{(k+1)\left(1-\frac{l+1}{N}\right)}},\ c_2\right)$$
$$\left. + \min\left(\frac{\min\left(\frac{2l}{N}\,\gamma_{k\left(1-\frac{l+1}{N}\right)},\ \frac{1}{k}\,\Gamma_{k\left(1-\frac{l+1}{N}\right)}^{f\left[k\left(1+\frac{l+1}{N}\right)\right]}\right)}{\frac{1}{k+1}\,\underline{\Psi}_0^{\min\left((k+1)\left(1+\frac{l}{N}\right),\,2N-(k+1)\left(1+\frac{l+1}{N}\right)\right)}},\ c_2\right)\right\} dt.$$

$E_{k,l}(\overline{\theta}, \underline{\theta}, g)$ is defined as ∞ for $l = 0$;

for $k = 0, \;\; l \geq 1,$

$$\frac{1}{6\pi}\int_{\frac{N}{l+1}}^{\frac{N}{l}} \frac{2(l+1)}{N} \left\{ \min\left(\frac{2\,\overline{\Theta}_0}{\underline{\Psi}_0^{1+\frac{N}{l+1}}},\ c_1\right) \cdot \min\left(\frac{\min\left(1, \frac{N}{l}-1\right)\,\gamma_0}{\underline{\Psi}_0^{\frac{N}{l+1}-1}},\ c_2\right) \right.$$
$$\left. + \min\left(\frac{2\,\gamma_0}{\underline{\Psi}_0^{1+\frac{N}{l+1}}},\ c_2\right)\right\} dt;$$

and for $k \geq 1, \;\; l \geq 1,$

$$\frac{1}{6\pi}\int_{\frac{N}{l+1}}^{\frac{N}{l}} \frac{\frac{k\pi}{N}}{\tan\frac{k\pi}{2(l+1)}} \left\{ \min\left(\frac{\frac{1}{k}\,\overline{\Psi}_{k\left(\frac{N}{l+1}-1\right)}^{f\left[k\left(1+\frac{N}{l+1}\right)\right]}}{\frac{1}{k+1}\,\underline{\Psi}_0^{\min\left((k+1)\left(1+\frac{N}{l+1}\right),\,\max\left(2N-(k+1)\left(1+\frac{N}{l}\right),0\right)\right)}},\ c_1\right) \right.$$
$$\times \min\left(\frac{\frac{1}{k}\,\Gamma_0^{k\min\left(1,\frac{N}{l}-1\right)}}{\frac{1}{k+1}\,\underline{\Psi}_0^{(k+1)\left(\frac{N}{l+1}-1\right)}},\ c_2\right)$$
$$\left. + \min\left(\frac{\frac{1}{k}\,\Gamma_{k\left(\frac{N}{l+1}-1\right)}^{f\left[k\left(1+\frac{N}{l+1}\right)\right]}}{\frac{1}{k+1}\,\underline{\Psi}_0^{\min\left((k+1)\left(1+\frac{N}{l+1}\right),\,\max\left(2N-(k+1)\left(1+\frac{N}{l}\right),0\right)\right)}},\ c_2\right)\right\} dt.$$

$F_{k,l}(\overline{\theta}, \underline{\theta}, g)$ is defined for $k = 0$ as,

$$\frac{1}{6\pi}\int_{\frac{N}{l+1}}^{\frac{N}{l}} \frac{2}{t} \left\{ \frac{2\,\overline{\Theta}_0}{1+t} c_3 \cdot \min\left(\frac{\min\left(1, \frac{N}{l}-1\right)\,\gamma_0}{\frac{1}{k+1}\,\underline{\Psi}_0^{(k+1)\left(\frac{N}{l+1}-1\right)}},\ c_2\right) + \frac{2\,\gamma_0}{1+t} c_3 \right\} dt;$$

and for $k > 0$,

$$\frac{1}{6\pi}\int_{\frac{N}{l+1}}^{\frac{N}{l}} \frac{\cos\frac{k\pi}{l+1}+1}{\frac{N}{k\pi}\sin\frac{k\pi}{l+1}+t-\frac{N}{l+1}} \left\{ \frac{2\overline{\Theta}_0}{1+t}c_3 \cdot \min\left(\frac{\frac{1}{k}\Gamma_0^{k\min\left(1,\frac{N}{l}-1\right)}}{\frac{1}{k+1}\underline{\Psi}_0^{(k+1)\left(\frac{N}{l+1}-1\right)}},\ c_2 \right) + \frac{2\gamma_0}{1+t}c_3 \right\} dt.$$

The inequalities (3.2)-(3.4) are verified elementarily, though the proof is lengthy. In the proof, we use the following facts.

1. $\dfrac{x}{\tan xy}\log(1+cy), \quad (x>0,\ y>0,\ xy<\pi/2,\ c:\text{a positive constant})$
   is monotone decreasing in x and y.

2. $x - |c-x|$ is a non-decreasing function of x.

3. For an arbitrary non-increasing and non-negative function $P(w)$ and a positive constant $k \le n$,
   $\displaystyle\int_{k(1-x)}^{f[k(1+x)]} P(w)dw$ is non-decreasing in $0 \le x \le 1$.

4. For an arbitrary non-increasing and non-negative function $P(w)$ and a positive constant $k \le n$,
   $\displaystyle\int_{k(x-1)}^{f[k(1+x)]} P(w)dw$ is non-increasing in $1 \le x$.

5. $\dfrac{\log(1+\overline{x})}{\overline{x}} \cdot x \le \log(1+x) \le x$, for $0 < x \le \overline{x}$.

6. For an arbitrary non-increasing and non-negative function $P(w)$, a positive constant $c \ge 1$ and $x \in [1,c]$,
   $\dfrac{\displaystyle\int_{x-1}^{c-|1+x-c|} P(w)dw}{c-|1+x-c|} \le \dfrac{c-|1+x-c|-(x-1)}{c-|1+x-c|}\cdot P(0) \le \dfrac{(1+x)-(x-1)}{1+x}\cdot P(0) = \dfrac{2P(0)}{1+x}.$

4 Numerical result

The IEEE754 standard defines several modes for controlling the behavior of floating point operations. We can choose a rounding mode: round-up, round-down, or round-to-nearest (see [6], [9]). In our computation, we use both round-up and round-down appropriately to obtain rigorous mathematical results. We used a Pentium4 PC with Red Hat Linux 8.0 and `gcc`.

We control rounding mode to make $\overline{J}_*$ and G_* as large as possible, and $\underline{J}_*$ as small as possible.

Set $N = 3000$, and we calculate following sequence of functions,

$$
(4.1) \qquad \begin{cases} \overline{\theta}_0(s) = \dfrac{\pi}{2}, \\ \underline{\theta}_0(s) = 0.00005 \cdot 1_{0<s\leq \pi/2}, \\ \overline{\theta}_{n+1}(s) = \min\Big(\overline{\theta}_n(s),\ \overline{J}_*(\overline{\theta}_n, \underline{\theta}_n)(s)\Big), \\ \underline{\theta}_{n+1}(s) = \max\Big(\underline{\theta}_n(s),\ \underline{J}_*(\underline{\theta}_n, \overline{\theta}_n)(s)\Big). \end{cases}
$$

From (3.2),(3.3) and Theorem 2.2, any solution of (1.1) must lie between $\overline{\theta}_n(s)$ and $\underline{\theta}_n$.
$\overline{\theta}_n(s) - \underline{\theta}_n(s)$ decreases as n grows (Fig.1). The shape of $\overline{\theta}_0(s), \overline{\theta}_5(s), \overline{\theta}_{10}(s), \cdots, \overline{\theta}_{45}(s)$ and
$\underline{\theta}_0(s), \underline{\theta}_5(s), \underline{\theta}_{10}(s), \cdots, \underline{\theta}_{45}(s)$ are shown in Fig.2, Fig.3 and Fig.4.

To prove uniqueness, we calculate the following sequence of functions,

$$
(4.2) \qquad \begin{cases} g_0(s) = 1, \\ g_{n+1}(s) = \dfrac{\overline{G}\big(\overline{\theta}_{45}, \underline{\theta}_{45}, g_n\big)(s)}{\sup_{0<s\leq\pi} \overline{G}\big(\overline{\theta}_{45}, \underline{\theta}_{45}, g_n\big)(s)}, \end{cases}
$$

and the following inequality was verified:

$$
\sup_{0<s\leq\pi} \frac{\overline{G}\big(\overline{\theta}_{45}, \underline{\theta}_{45}, g_4\big)(s)}{g_4(s)} = 0.99865799 \cdots < 1.
$$

Then, Theorem 2.3 and the results of previous section implies that the solution of (1.1), namely Stokes' wave of extreme form, is globally unique.

The shape of $g_4(s)$ is shown in Fig.5. The whole calculation took approximately 815 minutes.

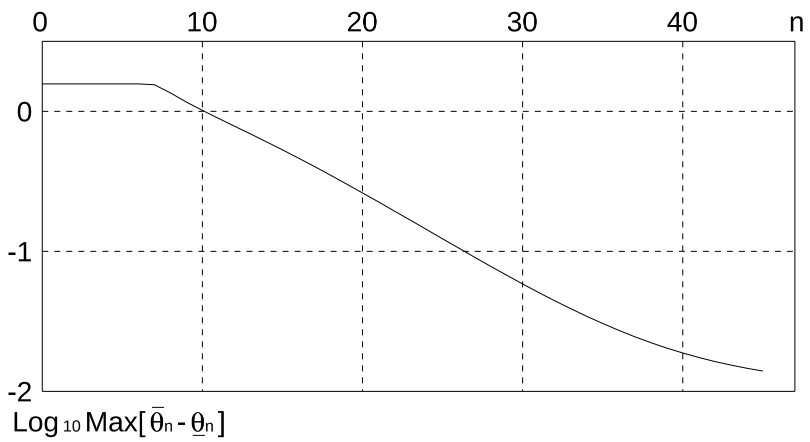

Figure 1: n vs. $\log_{10} \max_s \big(\overline{\theta}_n(s) - \underline{\theta}_n(s)\big)$.

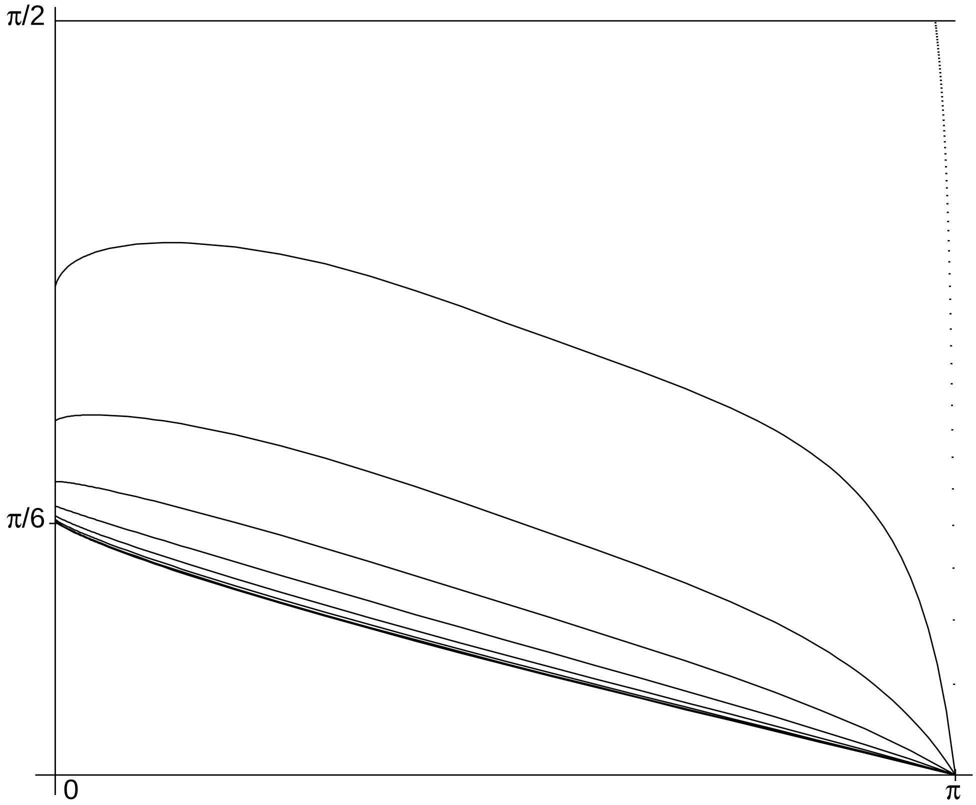

Figure 2: The shape of $\overline{\theta}_0(s)$, $\overline{\theta}_5(s)$,$\cdots$, $\overline{\theta}_{40}(s)$ and $\overline{\theta}_{45}(s)$.

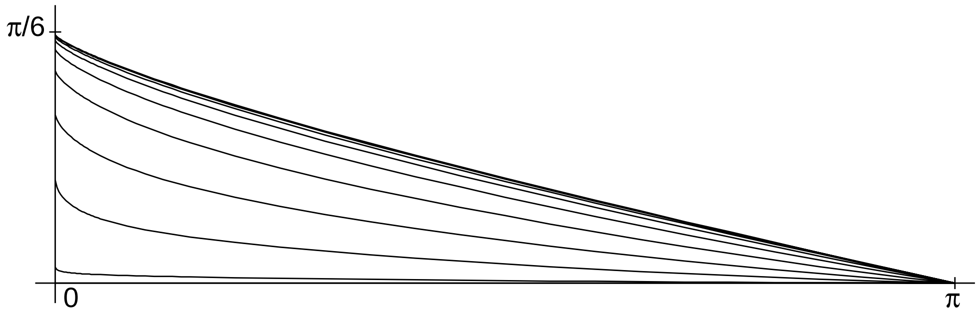

Figure 3: The shape of $\underline{\theta}_0(s)$, $\underline{\theta}_5(s)$,$\cdots$, $\underline{\theta}_{40}(s)$ and $\underline{\theta}_{45}(s)$.

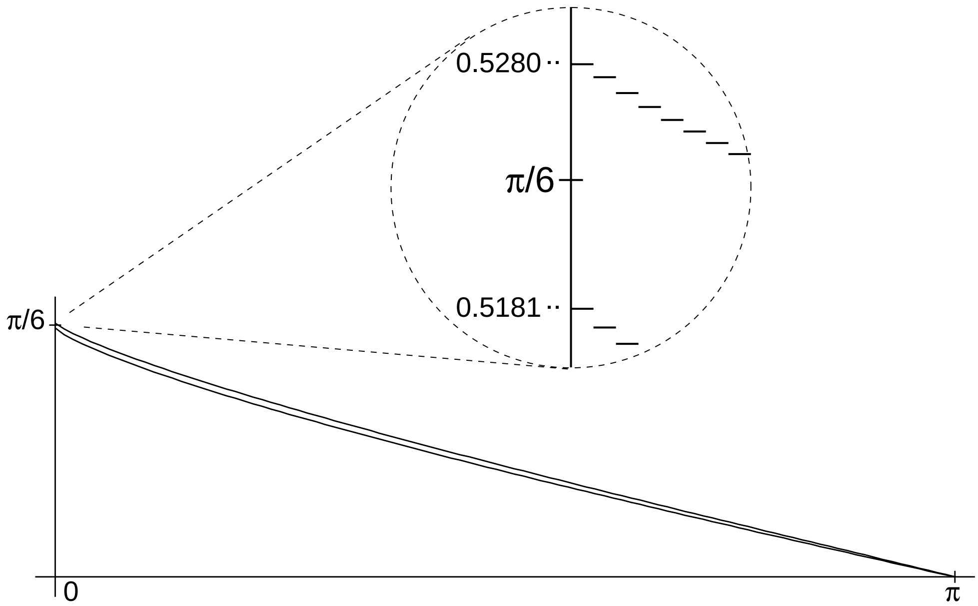

Figure 4: The shape of $\overline{\theta}_{45}(s)$ and $\underline{\theta}_{45}(s)$.

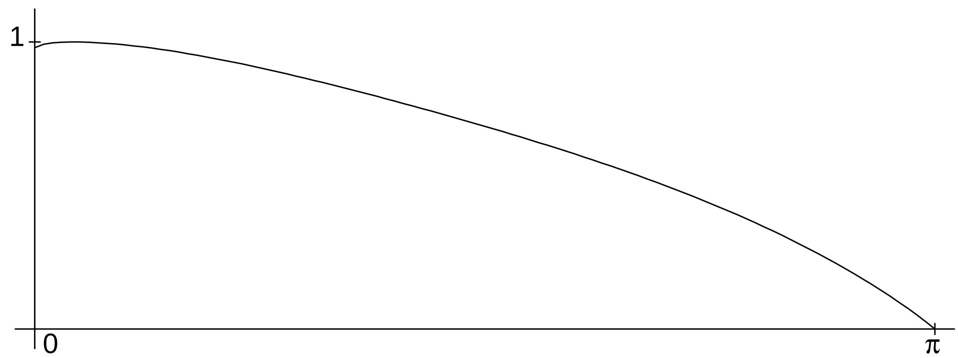

Figure 5: The shape of $g_4(s)$.

In order for our computer program to be openly tested, the author is ready to send his program on request. Also, it is posted in his internet homepage.

5 Conclusion

We have proved the global uniqueness of Stokes' wave of extreme form by the numerical verification method. Thus, combined with the results of [12] and [13], the second Stokes conjecture (the conjecture (b) of section 1) is solved affirmatively. Besides, our method seems to have applications to other nonlinear problems.

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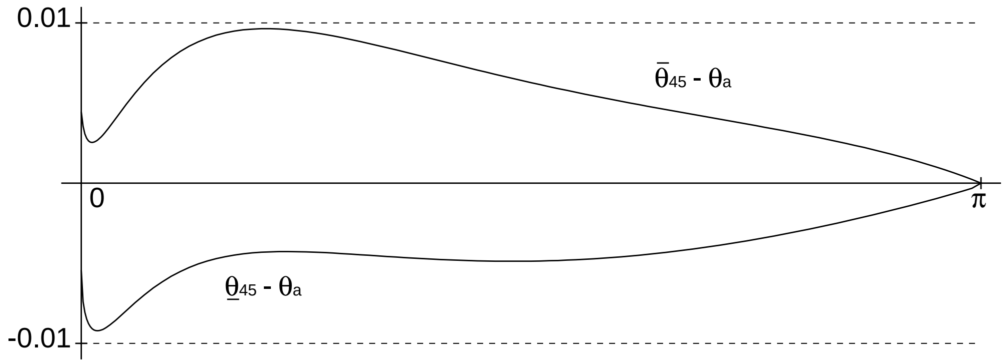

Figure 6: $\overline{\theta}_{45}(s) - \theta_a(s)$ and $\underline{\theta}_{45}(s) - \theta_a(s)$

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