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Existence of Optimal Growth Paths in a Class of Dynamic Economic Problems

Keisuke Osumi

1 Introduction

Many current economic issues beg a rigorous and coherent explanation from the perspective of a dynamic context. For example, innovation and education are today central to theories of economic growth. Industrial innovation is regarded as one of the engines of economic growth, and is promoted by accumulations of human capital. These issues should be dealt with within a dynamic framework with a long-term horizon rather than a static system with a short time horizon. Since seminal papers by Ramsey(1928), there has been much written about the characterizations of optimal growth paths or equilibrium growth paths, there remain a lot of unanswered questions concerning the existence of optimal growth paths or equilibrium growth paths within a rigorous framework(see Brock and Haurie(1976), Chichilnisky(1981), and Montrucchio(1995)). Arguments regarding these issues motivate the systematic investigation of either infinite sequences of states within the discrete-time framework, or of continuous time-paths of states in a continuous-time context. The purpose of this paper is to develop a general mathematical framework in a unified and self-contained way that can be applied to a wide variety of dynamic economic models and to consider the existence problem of solutions in a deterministic economic growth context.

2 Existence Problem in a General Context

In the following development, we suppose that X is a function space and $J : X \to \boldsymbol{R}$. Also, in order to simplify the notation, we write $(\boldsymbol{x}(t))$ for $\boldsymbol{x}(t) : [0,\infty) \to \boldsymbol{R}^n$. In the first part of this chapter, we consider the existence of solutions for a class of the following problems.

$$
\begin{aligned}
&\text{maximize } J[(\boldsymbol{x}(t))] \\
&\text{subject to} \\
&\qquad (\boldsymbol{x}(t)) \in X,\ \boldsymbol{x}(0) = \bar{\boldsymbol{x}}_0 = \text{given.}
\end{aligned}
$$

As noticed for the above formulation, the existence of solutions depends on how X and $J : X \to \boldsymbol{R}$ are specified.

In this section, we consider the existence of solutions in a class of dynamic economic models. As is well known, if X is a compact set and the functional $J : X \to \boldsymbol{R}$ is continuous, there exists an optimal solution.

We introduce some important theorems:

Theorem 1 : *Suppose that X is a compact set in a metric space and $J(x) : X \to \boldsymbol{R}$ is a continuous mapping. Then*

(1) *$J(x)$ is bounded on X;*

(2)　*$J(x)$ attains the supremum and infimum on X.*

Proof: Since $J(X)$ is a compact set, it is bounded. Therefore, for $J(x)$, there exist the supremum and infimum on (X). Also since $J(X)$ is closed, it contains the supremum and the infimum.

Q. E. D.

We proceed to introduce the weaker concept of continuity.

Definition 1 : *We suppose that W is a normed space and $f : W \to \boldsymbol{R}$. If one of the following conditions is satisfied, $f : W \to \boldsymbol{R}$ is said to be upper semicontinuous.*

(1)　*For any $x_0 \in W$ and for any $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\|x - x_0\| < \delta,\ x \in W \ \Rightarrow\ f(x) - f(x_0) < \varepsilon.$$

(2)　*For any $x_0 \in W$ and for any sequence (x_n) of W converging to x_0,*

$$x_n \to x_0 \ \Rightarrow \lim_{n\to\infty} \sup f(x_n) \ \leqq \ f(x_0).$$

(3)　*For any $\alpha \in R$, $\{x \mid x \in W,\ f(x) \geqq \alpha\}$ is a closed set.*

We may state the following important theorem:

Theorem 2 : *Suppose that X is a compact set in a metric space. Then, if $J : X \to \boldsymbol{R}$ is upper semicontinuous, $J[x]$ attains the supremum.*

Proof: For any $a \in \boldsymbol{R}$, we define as follows:

$$E_a = \{x \mid x \in X,\ J[x] < a\}.$$

Since $J : X \to \boldsymbol{R}$ is upper semicontinuous, E_a is an open set in X. Also, since $X = \cup_{a \in R} E_a$ and X is a compact set, there exists a finite open covering of X $\{E_{a_1}, E_{a_2}, \cdots, E_{a_m}\}$. That is,

$$X = \cup_{i=1}^{m} E_{a_i}.$$

Next, define $\hat{a} = \max\{a_1, a_2, \cdots, a_m\}$. Then, for any $x \in X$,

$$J[x] < \hat{a}.$$

Therefore there exists $c = \sup\{J[x] \big| x \in X\}$. Suppose that there exists no $\hat{x} \in X$ such that $J[\hat{x}] = c \in R$. Then,

$$X = \cup_{m=1}^{\infty} \left\{ x \mid x \in X,\ J[x] < c - \frac{1}{m} \right\}.$$

Since X is a compact set, there exists a finite open covering such that

$$X = \cup_{i=1}^{l} \left\{ x \mid x \in X,\ J[x] < c - \frac{1}{n_i} \right\}.$$

Define as follows:

$$\hat{c} = \max\left\{c - \frac{1}{n_1}, c - \frac{1}{n_2}, \cdots, c - \frac{1}{n_l}\right\}.$$

Then, for any $x \in X$,

$$f(x) < \hat{c}.$$

Thus

$$c = \sup\{f(x) \big| x \in X\} \leqq \hat{c}.$$

Also, $\hat{c} = \max\left\{c - \frac{1}{n_1}, \cdots, c - \frac{1}{n_l}\right\} < c$. This is a contradiction.

Q. E. D.

Now, X is not always compact. We consider the following example such that X is not a compact set. We suppose that X is the set of all functions $(x(t))$ satisfying the following conditions.

(1) $(x(t))$ is continuous over $[0, 1]$.

(2) $x(0) = 0,\ x(1) = 1$.

(3) $\max\limits_{t} \big|f(t)\big| \leqq 1$.

Also, we consider the following functional:

$$J[(x(t))] = -\int_0^1 x(t)^2 dt.$$

Now, suppose that $x(t) = t^n$. Then,

$$J[(t^n)] = -\frac{1}{2n+1}.$$

Also

$$\sup_{(x(t)) \in X} J[(x(t))] = 0.$$

By the way, for any $(x(t)) \in X$,

$$J[(x(t))] < 0.$$

Therefore, the supremum is not attained.

In order to deal with the case where X is not compact, we need some concepts.We introduce a number of concepts and theorems. We suppose that M, S are normed spaces. Also, we denote the set of all linear continuous mappings from M to S by $B(M, S)$. We state the following theorem:

Lemma 1 : *Suppose that M is a normed space and S is Banach space. Then, $B(M, S)$ is a Banach space.*

Proof: We prove the completeness. Suppose that (T_n) is a Cauchy sequence of $B(M,S)$, that is,
$\forall\varepsilon>0,\ \exists n(\varepsilon)\in N,\ \forall n,m\geqq n(\varepsilon)$:

$$|T_n-T_m|<\varepsilon.$$

Then, for any x, $\lim\limits_{n\to\infty} T_n\,x=T\,x$. Furthermore

$$\begin{aligned}|T\,x-T_n\,x| &\leqq |T\,x-T_m\,x|+|T_m\,x-T_n\,x|\\ &\leqq |T\,x-T_m\,x|+|T_m-T_n|\,|x|.\end{aligned}$$

Here, letting $m\to\infty$, the right-hand side tends to 0. Therefore,

$$|T-T_n|\to 0.$$

Q. E. D.

Thus we may have the following result.

Corollary 1 : *Suppose that M is a normed space. Then, $B(M,\boldsymbol{R})$ is a Banach space.*

Definition 2 : *We suppose that M is a normed space. $B(M,\boldsymbol{R})=M^*$ is called the conjugate space of M.*

Definition 3 *A Banach space M is said to be reflexive if $M=(M^*)^*$.*

We may state the folowing fact.

Theorem 3 : *Given a Banach space M, the following conditions are equivalent.*

(1) *A Banach space M is reflexive;*

(2) *Any bounded closed convex set of M is compact in the sense of weak topology.*

Definition 4 : *We suppose that x_0 is an element of a Banach space M and $g\in M^*$. Also, we define a neighborhood of x_0 as follows:*

$$B(x_0;\varepsilon;g)=\{x\mid x\in M,\ |g(x-x_0)|<\varepsilon\}.$$

The topology introduced by this specification in E is called a weak topology.

Theorem 4 : *Suppose that M is a Banach space and the above weak topology is introduced in M. Then, a sequence (x_n) of M converges to x_0 in the sense of weak topology if and only if for any $g\in M^*$,*

$$n\to\infty\ \Rightarrow\ g(x_n)\to g(x_0).$$

Now, suppose that for a sequence (x_n) of a Banach space M,

$$n\to\infty\ \Rightarrow\ x_n\to x_0.$$

Then for any $g \in M^*$,

$$\begin{aligned} & |g(x_n) - g(x_0)| \\ = \; & |g(x_n - x_0)| \\ \geqq \; & \|g\| \, \|x_n - x_0\|. \end{aligned}$$

Therefore, if a sequence (x_n) in a Banach space converges to x_0 in an ordinary sense, by the above theorem, (x_n) converges to x_0 in the sense of weak topology. The converse does not always hold. We may state the folowing facts.

Lemma 2 : *Given a Banach space M, the following conditions are equivalent.*

(1) *A Banach space M is reflexive;*

(2) *Any bounded closed convex set of M is compact in the sense of weak topology.*

A set M is said to be weakly compact if it is compact in the sense of weak topology.

Lemma 3 : *Suppose that X is a convex set in a Banach space. Then, X is a closed set in the sense of the norm if and only if it is a closed set in the sense of weak topology.*

Proof: Trivilally, if X is a closed set in the sense of weak topology, it is a closed set in the sense of the norm. Suppose that X is closed in the sense of the norm. Also, suppose that (x_n) is a sequence of X which converges to x in the sense of weak topology. Suppose that $x \notin X$. By the separation theorem, there exists a linear continuous mapping $g \in X^*$ such that

$$g(x) < \inf\{g(y) \,|\, y \in X\}.$$

Define as follows:

$$m = \inf\{f(y) \,|\, y \in X\}.$$

Then $g(x) < m \leqq g(x_n)$. Here

$$g(x_n) \to g(x).$$

Therefore, $g(x) < m \leqq g(x)$. This is a contradiction. Thus, $x \in X$. Therefore, X is a closed set in the sense of weak topology.

Q. E. D.

Lemma 4 : *Suppose X is closed and convex in a Banach space and $J : X \to \boldsymbol{R}$ is a concave function. Then, $J : X \to \boldsymbol{R}$ is upper semicontinuous in the sense of the ordinary norm if and only if it is upper semicontinuous in the sense of weak topology.*

Proof: For any $c \in \boldsymbol{R}$, the following set is convex.

$$E_c = \{x \mid x \in X, \; f(x) \geqq c\}.$$

By Lemma 3, E_c is closed in the sense of weak topology if and only if E_c is closed in the norm.

Q. E. D.

We may state the following important result.

Theorem 5 : *Suppose that X is a compact convex set in the sense of weak topology in Banach space. Also, suppose that $J : X \to [-\infty, \infty)$ satisfies the following conditions.*

(1) $\exists x_0 \in X : \; J[x_0] > -\infty$.

(2) $J : X \to [-\infty, \infty)$ *is concave.*

(3) $J : X \to [-\infty, \infty)$ *is upper semicontinuous.*

Then there exists $\hat{x} \in X$ such that $f(\hat{x}) = \max_{x \in X} f(x)$.

Proof: By condition (1), there exists $a \in \boldsymbol{R}$ such that

$$A = \{x \mid x \in X, \; J(x) \geqq a\} \neq \phi.$$

Since $J : X \to \boldsymbol{R} \cup \{-\infty\}$ is upper semicontinuous, A is closed and convex in X. Thus, A is a compact convex set in the sense of weak topology. Also, since $J : X \to \boldsymbol{R} \cup \{-\infty\}$ is an upper semicontinuous function, by Lemma 4, it is an upper semicontinuous function in the sense of weak topology. Therefore, given a weak topology, we can apply theorem 2 to this proof.

Q. E. D.

2.1 Existence Problem in a Class of Continuous-Time Economic Models

In this section, we make use of the argument of Montrucchio(1995) to show the existence of optimal economic growth path. In order to proceed, we need some concepts and results.

Definition 5 : $g : \Omega \to \boldsymbol{R} \cup \{-\infty\}$ *is said to be $\alpha -$ concave if*

(1) $g(x) + \frac{1}{2}\alpha|x|^2$ *is concave on Ω, that is, for any $x, y \in \Omega$ and for any $\theta \in [0,1]$,*

$$\begin{aligned} & g(\theta x + (1-\theta)y) + \frac{1}{2}\alpha\|\theta x + (1-\theta)y\|^2 \\ \geqq \; & \theta g(x) + (1-\theta)g(y) + \frac{1}{2}\alpha\theta\|x\|^2 + \frac{1}{2}\alpha(1-\theta)\|y\|^2 \end{aligned}$$

where $\|\cdot\|$ is a norm in Ω.

We may state the following:

Lemma 5 : $g : \Omega \to \boldsymbol{R} \cup \{-\infty\}$ *is α concave on Ω if and only if for $x, y \in \Omega$ and for any $\theta \in (0,1)$,*

$$g(\theta x + (1-\theta)y) \geqq \theta g(x) + (1-\theta)g(y) + \frac{1}{2}\alpha\theta(1-\theta)\|x-y\|^2.$$

Proof: In the following development, we denote the inner product between a and b by (a, b).

$$\begin{aligned} & \theta\|x\|^2 + (1-\theta)\|y\|^2 - \|\theta x + (1-\theta)y\|^2 \\ = \; & \theta(x,x) + (1-\theta)(y,y) - (\theta x + (1-\theta)y, \; \theta x + (1-\theta)y) \end{aligned}$$

$$
\begin{aligned}
&= \theta(x,x) + (1-\theta)(y,y) - \theta^2(x,x) - (1-\theta)^2(y,y) - 2\theta(1-\theta)(x,y) \\
&= \theta(1-\theta)(x,x) + (1-\theta)\theta(y,y) - 2\theta(1-\theta)(x,y) \\
&= \theta(1-\theta)\{(x,x) + (y,y) - 2(x,y)\} \\
&= \theta(1-\theta)(x-y, x-y) \\
&= \theta(1-\theta)\|x-y\|^2.
\end{aligned}
$$

Q. E. D.

Lemma 6 : *Suppose that* $g : \Omega \to \boldsymbol{R} \cup \{-\infty\}$ *satisfies the following conditions:*

(1) $g : \Omega \to \boldsymbol{R} \cup \{-\infty\}$ *is* α *concave, where* $\alpha > 0$;

(2) $g : \Omega \to \boldsymbol{R} \cup \{-\infty\}$ *is upper semicontinuous.*

Suppose that there exists $x_0 \in \Omega$ *such that* $g(x_0) > 0$*. Then the following set is bounded in a sense of norm in Hilbert space.*

$$T(g(x_0)) = \{x \mid x \in \Omega,\ g(x) \geqq g(x_0)\}.$$

Furthermore, $\sup_{x \in \Omega} g(x) < \infty$.

Proof: Since there exists $x_0 \in \Omega$ such that $g(x_0) > 0$ and $g : \Omega \to \boldsymbol{R} \cup \{-\infty\}$ is an upper semicontinuous,

$\forall \varepsilon > 0, \exists \delta > 0$:

$$\|x - x_0\| \leqq \delta,\ x \in \Omega \ \Rightarrow\ g(x) - g(x_0) < \varepsilon.$$

Let $\varepsilon_0 > 0$ be given. Then there exists $\delta_0 > 0$ such that

$$\|x - x_0\| \leqq \delta_0,\ x \in \Omega \ \Rightarrow\ g(x) < g(x_0) + \varepsilon_0.$$

Here we set $g(x_0) + \varepsilon_0 = M > 0$.

Now suppose that $T(g(x_0))$ is not bounded. Then there exists a sequence (x_n) such that

$$x_n \in T(g(x_0)),\ x_n \neq x_0,\ \|x_n\| \to \infty\ (n \to \infty),\ \|x_n - x_0\| > \delta_0.$$

Define

$$\alpha_n = \frac{\delta_0}{\|x_n - x_0\|},\ z_n = \alpha_n x_n + (1 - \alpha_n) x_0.$$

Then

$$z_n - x_0 = \alpha_n (x_n - x_0) = \frac{\delta_0}{\|x_n - x_0\|}(x_n - x_0).$$

Thus

$$\|z_n - x_0\| = \delta_0.$$

Since $g : \Omega \to \boldsymbol{R} \cup \{-\infty\}$ is α concave, by Lemma 5,

$$
\begin{aligned}
& g(z_n) \\
&= g(\alpha_n x_n + (1-\alpha_n)x_0) \\
&\geqq \alpha_n g(x_n) + (1-\alpha_n) g(x_0) + \frac{1}{2}\alpha\alpha_n(1-\alpha_n)\|x_n - x_0\|^2 \\
&\geqq g(x_0) + \frac{1}{2}\alpha\alpha_n(1-\alpha_n)\|x_n - x_0\|^2.
\end{aligned}
$$

Since $\|z_n - x_0\| = \delta_0$, $g(z_n) < M$. Therefore

$$\begin{aligned} M &> g(x_0) + \frac{1}{2}\alpha\alpha_n(1-\alpha_n)\|x_n - x_0\|^2 \\ &= g(x_0) + \frac{1}{2}\alpha\frac{\delta_0}{\|x_n - x_0\|}\left\{1 - \frac{\delta_0}{\|x_n - x_0\|}\right\}\|x_n - x_0\|^2 \\ &= g(x_0) + \frac{1}{2}\alpha\delta_0\left\{\|x_n - x_0\| - \delta_0\right\}. \end{aligned}$$

Also, $\|x_n - x_0\| \to \infty$ $(n \to \infty)$. This is a contradiction. Therefore $T(g(x_0))$ is bounded, that is, $\exists\nu > 0$, $\forall x \in T(g(x_0))$:

$$\|x\| < \nu.$$

Next, we prove that $\sup_{x \in Omega} g(x) < +\infty$. The following fact is trivial.

$$\sup_{x\in\Omega} g(x) = \sup_{x \in T(g(x_0))} g(x).$$

If $\|x - x_0\| \leqq \delta_0$, $x \in \Omega$, $g(x) < M$. Thus consider $x \in T(g(x_0))$ such that

$$\|x - x_0\| > \delta_0.$$

Define η by

$$\eta = \frac{\delta_0}{\|x - x_0\|}.$$

Since $g : \Omega \to \boldsymbol{R} \cup \{-\infty\}$ is α concave,

$$g(\eta x + (1-\eta)x_0) \geqq \eta g(x) + (1-\eta)g(x_0) + \frac{1}{2}\alpha\eta(1-\eta)\|x - x_0\|^2.$$

Hence,

$$\begin{aligned} g(x) &\leqq \eta^{-1}g(\eta x + (1-\eta)x_0) - \eta^{-1}(1-\eta)g(x_0) - \frac{1}{2}\alpha(1-\eta)\|x - x_0\|^2 \\ &\leqq \eta^{-1}g(\eta x + (1-\eta)x_0). \end{aligned}$$

Also

$$\begin{aligned} &\eta x + (1-\eta)x_0 - x_0 \\ = \;&\eta(x - x_0) = \frac{\delta_0}{\|x - x_0\|}(x - x_0). \end{aligned}$$

Therefore

$$\|\eta x + (1-\eta)x_0 - x_0\| = \delta_0.$$

Hence

$$g(x) \leqq \frac{\|x - x_0\|}{\delta_0}M \leqq \frac{\|x_0\| + \nu}{\delta_0}M.$$

Thus since $g(x)$ is bounded from above, there exists $\sup_{x\in\Omega} g(x)$.

Q. E. D.

Lemma 7 : *Suppose that Ω and $g : \Omega \to \boldsymbol{R} \cup \{-\infty\}$ satisfy the following conditions:*

(1) *Ω is closed and convex in a Hilbert space.*

(2) *$g : \Omega \to \boldsymbol{R} \cup \{-\infty\}$ is α concave ($\alpha > 0$).*

(3) *$g : \Omega \to \boldsymbol{R} \cup \{-\infty\}$ is upper semicontinuous.*

(4) *there exists x_0 such that $g(x_0) > 0$.*

Then, there exists a unique $x^* \in \Omega$ such that

$$\forall x \in \Omega : \ g(x^*) \geqq g(x).$$

Proof: For x_0 such that $g(x_0) > 0$, $T(g(x_0)) = \{x \mid x \in \Omega, \ g(x) \geqq g(x_0)\}$ is bounded. Since $g : \Omega \to \boldsymbol{R} \cup \{-\infty\}$ is upper semicontinuous, $T(g(x_0))$ is a closed set. Also, $g : \Omega \to \boldsymbol{R} \cup \{-\infty\}$ is concave. Therefore, $T(g(x_0))$ is convex. Since reflexivity is guaranteed in a Hilbert space, $T(g(x_0))$ is weakly compact. Therefore, by Theorem 5, there exists $x^* \in \Omega$ such that

$$\forall x \in \Omega : \ g(x^*) \geqq g(x).$$

Uniqueness is guaranteed by the strict concavity of $g : \Omega \to \boldsymbol{R} \cup \{-\infty\}$.

Q. E. D.

Lemma 8 : *We suppose that $g : \Omega \to \boldsymbol{R} \cup \{-\infty\}$ is αconcave. We consider $x^* \in \Omega$ such that for any $x \in \Omega$,*

$$g(x^*) \geqq g(x).$$

Then

$$\forall x \in \Omega : \ g(x) \leqq g(x^*) - \frac{1}{2}\alpha\|x - x^*\|^2.$$

Proof: Since $g : \Omega \to \boldsymbol{R} \cup \{-\infty\}$ is α concave, for any $x \in \Omega$ and for any $\theta \in [0, 1]$,

$$g(\theta x^* + (1-\theta)x) \geqq \theta g(x^*) + (1-\theta)g(x) + \frac{1}{2}\alpha\theta(1-\theta)\|x^* - x\|^2.$$

Since $g(x^*) \geqq g(x)$ for any $x \in \Omega$,

$$(1-\theta)g(x^*) \geqq (1-\theta)g(x) + \frac{1}{2}\alpha\theta(1-\theta)\|x^* - x\|^2.$$

Also for $\theta \in [0, 1)$,

$$g(x^*) \geqq g(x) + \frac{1}{2}\alpha\theta\|x^* - x\|^2.$$

Therefore,

$$g(x^*) \geqq g(x) + \frac{1}{2}\alpha\|x^* - x\|^2.$$

Q. E. D.

We proceed to deal with the following problem:

$$\text{maximize } T((\boldsymbol{k}(t))) = \int_0^\infty v(\boldsymbol{k}(t), \dot{\boldsymbol{k}}(t))\,\mu(dt)$$

subject

(1) $(\boldsymbol{k}(t), \dot{\boldsymbol{k}}(t)) \in X$ for almost every t, $\boldsymbol{k}(0) = \boldsymbol{k}_0$;

(2) $\boldsymbol{k}(t) : \boldsymbol{R}_+ \to \boldsymbol{R}^n$, $\dot{\boldsymbol{k}}(t) : \boldsymbol{R}_+ \to \boldsymbol{R}^n$ are measurable functions.

We introduce the following assumptions.

(A.1) X is closed and convex.

(A.2) $v : X \to \boldsymbol{R} \cup \{-\infty\}$ is upper semicontinuous.

(A.3) $\exists \alpha > 0, \beta > 0 : v(\boldsymbol{k}, \boldsymbol{z}) + \frac{1}{2}\alpha|\boldsymbol{k}|^2 + \frac{1}{2}\beta|\boldsymbol{z}|^2$ is a concave function.

Now, we may state the following important theorem:

Theorem 6 : *For any $\boldsymbol{k}_0 \in K$, there exists a unique optimal economic growth path $(\boldsymbol{k}^*(t))$ from $\boldsymbol{k}_0$. Also*

$$\int_0^\infty |\boldsymbol{k}^*(t)|^2 \mu(dt) < +\infty, \quad \int_0^\infty |\dot{\boldsymbol{k}}^*(t)|^2 \mu(dt) < +\infty.$$

Proof: Since $v(\boldsymbol{k}, \boldsymbol{z})$ is an upper semicontinuous function on X and a concave function, there exists $(\boldsymbol{k}^*, \boldsymbol{z}^*)$ such that

$$\forall(\boldsymbol{k}, \boldsymbol{z}) \in X : v(\boldsymbol{k}^*, \boldsymbol{z}^*) \geqq v(\boldsymbol{k}, \boldsymbol{z}).$$

Also, since $v(\boldsymbol{k}, \boldsymbol{z})$ is α concave with respect to k and β concave with respect to z, for any $(\boldsymbol{k}, \boldsymbol{z}) \in X$,

$$v(\boldsymbol{k}, \boldsymbol{z}) \leqq v(\boldsymbol{k}^*, \boldsymbol{z}^*) - \frac{1}{2}\alpha\|\boldsymbol{k} - \boldsymbol{k}^*\|^2 - \frac{1}{2}\beta\|\boldsymbol{z} - \boldsymbol{z}^*\|^2.$$

Therefore, for any feasible economic growth path $(\boldsymbol{k}(t))$,

$$\begin{aligned} & \int_0^\infty e^{-\rho t} v(\boldsymbol{k}(t), \dot{\boldsymbol{k}}(t))\mu(dt) \\ \leqq & \int_0^\infty e^{-\rho t} v(\boldsymbol{k}^*, \boldsymbol{z}^*)\mu(dt) - \frac{1}{2}\alpha \int_0^\infty e^{-\rho t}\|\boldsymbol{k}(t) - \boldsymbol{k}^*\|^2 \mu(dt) \\ & \qquad - \frac{1}{2}\beta \int_0^\infty e^{-\rho t}\|\dot{\boldsymbol{k}}(t) - \boldsymbol{z}^*\|^2 \mu(dt). \end{aligned}$$

Now, suppose that $\int_0^\infty e^{-\rho t}\|\boldsymbol{k}(t)\|^2 \mu(dt) = +\infty$ or $\int_0^\infty e^{-\rho t}\|\dot{\boldsymbol{k}}(t)\|\mu(dt) = +\infty$. Then, $\int_0^\infty e^{-\rho t} v(\boldsymbol{k}(t), \dot{\boldsymbol{k}}(t))\mu(dt) = -\infty$. Thus, $(\boldsymbol{k}(t))$ is not an optimal economic growth path. Therefore, we can make a restriction to the set of feasible growth paths. We denote L^2_μ for the Hilbert space of all functions $(\boldsymbol{k}(t))$ such that

(1) $\boldsymbol{k}(t): \boldsymbol{R}_+ \to \boldsymbol{R}^n$ is a measurable function ;

(2) $\displaystyle\int_0^\infty \|\boldsymbol{k}(t)\|^2 \mu(dt) < \infty.$

Now we extend $v(\boldsymbol{k}, \boldsymbol{z}) : X \to \boldsymbol{R}$ as follows:

$$v^*(\boldsymbol{k}, \boldsymbol{z}) = \begin{cases} v(\boldsymbol{k}, \boldsymbol{z}) & ((\boldsymbol{k}, \boldsymbol{z}) \in X) \\ -\infty & ((\boldsymbol{k}, \boldsymbol{z}) \notin X). \end{cases}$$

$$\text{maximize } T((\boldsymbol{k}(t)), (\boldsymbol{z}(t))) = \int_0^\infty e^{-\rho t} v^*(\boldsymbol{k}(t), \boldsymbol{z}(t))\, \mu(dt)$$

subject

$$(\boldsymbol{k}(t)), (\boldsymbol{z}(t))) \in H \subset L^2_\mu \times L^2_\mu,$$

where we denote H for the convex set of all $(\boldsymbol{k}(t)), (\boldsymbol{z}(t)))$ in $L^2_\mu \times L^2_\mu$ such that

$$\boldsymbol{k}(t) = \boldsymbol{k}_0 + \int_0^t \boldsymbol{z}(\tau) d\tau \quad (\text{for almost every } t \in \boldsymbol{R}_+).$$

Here $\displaystyle\int_0^t \boldsymbol{z}(\tau)d\tau = \left(\int_0^t z_1(\tau)d\tau, \cdots, \int_0^t z_n(\tau)d\tau\right).$

Now since $T((\boldsymbol{k}(t)), (\boldsymbol{z}(t)))$ is bounded from above by $v(\boldsymbol{k}^*, \boldsymbol{z}^*)\rho^{-1}$, $T((\boldsymbol{k}(t)), (\boldsymbol{z}(t)))$ is well-defined. Also, since $v(\boldsymbol{k}, \boldsymbol{z}) \leqq v(\boldsymbol{k}^*, \boldsymbol{z}^*)$, by Fatou's lemma, if

$$(\boldsymbol{k}^{(n)}(t)), (\boldsymbol{z}^{(n)}(t)) \to (\hat{\boldsymbol{k}}(t)), (\hat{\boldsymbol{z}}(t)),$$

$$\begin{aligned} \int_0^T e^{-\rho t} v^*(\hat{\boldsymbol{k}}(t), \hat{\boldsymbol{z}}(t))\, \mu(dt) &\geqq \int_0^\infty \lim_n \sup e^{-\rho t} v^*((\boldsymbol{k}^{(n)}(t)), (\boldsymbol{z}^{(n)}(t)))\, \mu(dt) \\ &\geqq \lim_n \sup \int_0^\infty e^{-\rho t} v^*(\boldsymbol{k}^{(n)}(t), \boldsymbol{z}^{(n)}(t))\, \mu(dt) \\ &= \lim_n \sup T((\boldsymbol{k}^{(n)}(t)), (\boldsymbol{z}^{(n)}(t))). \end{aligned}$$

Therefore, $T((\boldsymbol{k}(t)), (\boldsymbol{z}(t)))$ is upper semicontinuous over $L^2_\mu \times L^2_\mu$. Furthermore, the following function is concave over $L^2_\mu \times L^2_\mu$.

$$T((\boldsymbol{k}(t)), (\boldsymbol{z}(t))) + \frac{1}{2}\alpha\|(\boldsymbol{k}(t))\|_2^2 + \frac{1}{2}\beta\|(\boldsymbol{z}(t))\|_2^2.$$

Here, $\|(\boldsymbol{v}(t))\|_2^2 = \int_0^\infty \|\boldsymbol{v}(t)\|^2\, \mu(dt).$

Now, we show that H is closed. We suppose that $((\boldsymbol{k}^{(n)}(t)), (\boldsymbol{z}^{(n)}(t))) \in H$ and

$$(\boldsymbol{k}^{(n)}(t)) \to (\hat{\boldsymbol{k}}(t)), \ (\boldsymbol{z}^{(n)}(t)) \to (\hat{\boldsymbol{z}}(t)).$$

For any $t \in \boldsymbol{R}_+$,

$$\int_0^\infty |\hat{\boldsymbol{z}}(\tau) - \boldsymbol{z}^{(n)}(\tau)|^2 \, \mu(d\tau)$$
$$\geqq \int_0^t |\hat{\boldsymbol{z}}(\tau) - \boldsymbol{z}^{(n)}(\tau)|^2 \, \mu(d\tau).$$

Thus, as $(\boldsymbol{z}^{(n)}(t)) \to (\hat{\boldsymbol{z}}(t))$, the following expression tends to 0.

$$\int_0^t |\hat{\boldsymbol{z}}(\tau) - \boldsymbol{z}^{(n)}(\tau)|^2 d\tau.$$

Therefore, $(\boldsymbol{z}^{(n)}(t))$ converges to $(\hat{\boldsymbol{z}}(t))$ in Hilbert space $L^2(0,t)$. Thus,

$$\int_0^t \boldsymbol{z}^{(n)}(\tau)\, d\tau \;\to\; \int_0^t \hat{\boldsymbol{z}}(\tau)\, d\tau.$$

Hence

$$\boldsymbol{k}^{(n)}(t) = \boldsymbol{k}_0 + \int_0^t \boldsymbol{z}^{(n)}(\tau)\, d\tau \;\to\; \boldsymbol{k}_0 + \int_0^t \hat{\boldsymbol{z}}(\tau)\, d\tau.$$

Also, since $(\boldsymbol{k}^{(n)}(t)) \to (\hat{\boldsymbol{k}}(t))$ in L^2_μ, there exists a subsequence such that $(\boldsymbol{k}^{n_i}(t)) \to (\hat{\boldsymbol{k}}(t))$ (μ, a, e, t) . Therefore

$$\hat{\boldsymbol{k}}(t) = \boldsymbol{k}_0 + \int_0^t \hat{\boldsymbol{z}}(\tau)\, d\tau \quad \text{(for almost every } t \in \boldsymbol{R}_+).$$

Q. E. D.

3 Existence Problem in a Discrete-Time Economic Model

3.1 Preliminary Results

We define some concepts.

Definition 6 : *A correspondence $T : X \to Y$ is said to be lower hemi-continuous at $x \in X$ if*

(1) $T(x)$ *is nonempty;*

(2) $\forall y \in T(x),\ \forall (x_n)$ *with* $x_n \to x,\ \exists N \geqq 1,\ \exists (y_n)$ *with* $y_n \to y$:

$$(y_n) \in T(x_n)\ (n \geqq N).$$

Definition 7 : *Suppose that for any $x \in S$, $T(x)$ is compact. A correspondence $T : X \to Y$ is said to be upper hemi-continuous at $x \in X$ if*

(1) $T(x)$ *is nonempty;*

(2) $\forall (x_n)$ *with* $x_n \to x$, $\forall (y_n)$ *satisfying* $(y_n) \in T(x_n)$ $\exists (y_{n_i})$:

$$\lim_{i \to \infty} y_{n_i} \in T(x)$$

where (y_{n_i}) *is a subsequence of* (y_n).

Definition 8 : *(1) A correspondence* $T : X \to Y$ *is said to be continuous at* $x \in X$ *if it is lower hemi-continuous and upper hemi-continuous.*

(2) A correspondence $T : X \to Y$ *is said to be continuous if it is lower hemi-continuous and upper hemi-continuous at every* $x \in X$.

In order to obtain the main theorem, we show some theorems.

Theorem 7 : *Let* $B(S)$ *be the set of all bounded continuous functions* $f : S \to \boldsymbol{R}$, *with the sup norm. Then,* $B(S)$ *is a complete normed space.*

Proof: We prove only the completeness of $B(S)$. Suppose that (f_n) is a Cauchy sequence of $B(S)$. Then

$$\forall \epsilon > 0,\ \exists n_\epsilon \in N,\ \forall m \geqq n_\epsilon,\ \forall n \geqq n_\epsilon :$$

$$\| f_n - f_m \| < \epsilon.$$

(1) Let $x \in S$ be given. Then, for the sequence $(f_n(x))$,

$$|f_n(x) - f_m(x)| \leqq \sup_{x' \in S} |f_n(x') - f_m(x')| = \| f_n - f_m \| .$$

Thus, the sequence $(f_n(x))$ is a Cauchy sequence in $\boldsymbol{R}$. Also, since the set of real numbers is complete, the sequence $(f_n(x))$ converges to a limit point $f(x)$. Thus, we may have the function $f : S \to R$.

(2) Next we will prove that

$$\lim_{n \to \infty} \| f_n - f \| = 0.$$

Let $\epsilon > 0$ be given. We may choose n_ϵ such that

$$\forall m \geqq n_\epsilon,\ \forall n \geqq n_\epsilon :$$

$$\| f_n - f_m \| < \frac{\epsilon}{2}.$$

$\forall m \geqq n_\epsilon,\ \forall n \geqq n_\epsilon$:

$$\begin{aligned} |f_n(x) - f(x)| &\leqq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| \\ &\leqq \| f_n - f_m \| + |f_m(x) - f(x)| \\ &\leqq \frac{\epsilon}{2} + \| + |f_m(x) - f(x)|. \end{aligned}$$

Also, we can choose $m \geqq n_\epsilon$ such that

$$|f_m(x) - f(x)| \leqq \frac{\epsilon}{2}.$$

Thus, $\forall \epsilon > 0, \exists n_\epsilon \in N,\ \forall n \geqq n_\epsilon$:

$$\| f_n - f \| < \epsilon.$$

(3) We will show that $f : S \to \boldsymbol{R}$ is bounded and continuous. Let $\epsilon > 0$ be given. Then, for $\epsilon > 0$, there exists n_ϵ such that for any $n \geqq n_\epsilon$,

$$\| f - f_n \| < \frac{\epsilon}{3}, f_n \in B(S).$$

Also choose δ such that for $y \in B(x;\delta)$,

$$|f_n(x) - f_n(y)| \leqq \frac{\epsilon}{3}.$$

Then, for any $y \in B(x;\delta)$,

$$\begin{aligned} |f(x) - f(y)| &\leqq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &\leqq 2 \| f - f_n \| + |f_n(x) - f_n(y)| \leqq \epsilon. \end{aligned}$$

Q. E. D.

Theorem 8 *:Suppose that S is contained in $\boldsymbol{R}^l$, $\hat{B}(S)$ is a set of all bounded functions $f : S \to \boldsymbol{R}$ with the sup norm. Also, let $A : \hat{B}(S) \to \hat{B}(S)$ be given by*

(1) *Suppose that $g, h \in \hat{B}(S)$ and $g(x) \leqq h(x)$ for any $x \in S$. Then for any $x \in S$: $(Ag)(x) \leqq (Ah)(x)$.*

(2) *There exists $\alpha \in (0,1)$ such that for any $g \in \hat{B}(S)$, for any $c \geqq 0$ and for any $x \in S$,*

$$(A(g+c))(x) \leqq (Ag)(x) + \alpha c.$$

Then, $A : \hat{B}(S) \to \hat{B}(S)$ satisfies the following condition.
$\exists \alpha \in (0,1),\ \forall (g,h) \in \hat{B}(S) \times \hat{B}(S)$:
$d(A(g), A(h)) \leqq \alpha d(g,h)$.

Proof: For any $g, h \in \hat{B}(S)$ and for any $x \in S$,

$$g(x) - h(x) \leqq \| g - h \|,$$

that is,

$$g(x) \leqq h(x) + \| g - h \| .$$

Thus, for any $x \in S$,

$$(Ag)(x) \leqq (A(h + \| g - h \|))(x) \leqq (Ah)(x) + \alpha \| g - h \| .$$

Similarly,

$$(Ah)(x) \leqq (Ag)(x) + \alpha \| g - h \| .$$

Thus, for any $x \in S$,

$$-\alpha \| g - h \| \leqq Ah(x) - Ag(x) \leqq \alpha \| g - h \| .$$

Therefore, for any $x \in S$,

$$|Ah(x) - Ag(x)| \leqq \alpha \parallel g - h \parallel .$$

Thus

$$\parallel Ag - Ah \parallel \leqq \alpha \parallel g - h \parallel .$$

Hence, $A : \hat{B}(S) \to \hat{B}(S)$ satisfies the following condition
$\exists \alpha \in (0,1),\ \forall (g,h) \in \hat{B}(S) \times \hat{B}(S)$:
$d(A(g), A(h)) \leqq \alpha d(g,h)$.

Q. E. D.

3.2 Existence of Optimal Growth Paths

In this subsection, by following the argument of Stokey and Lucas (2001), we deal with the existence problem in the discrete model. Let $\boldsymbol{k}$ be the economic state variable. Then we define some notation. S =the set of possible values for the economic state variable $\boldsymbol{k}$. $T(\boldsymbol{k})$ =the set of all feasible values of S from $\boldsymbol{k}$ through feasible transformation.

Definition 9 : *Given $\bar{\boldsymbol{k}}_0 \in S$, $(\boldsymbol{k}_t)$ is said to be a feasible growth path from $\bar{\boldsymbol{k}}_0$ if*

(1) $\boldsymbol{k}_0 = \bar{\boldsymbol{k}}_0$,

(2) $\boldsymbol{k}_{t+1} \in T(\boldsymbol{k}_t)$ *(for $t = 0, 1, \cdots$).*

Here, we denote $F(\boldsymbol{k}_0)$ for the set of all feasible growth paths from k_0.

Definition 10 : *For any t, we denote $V(\boldsymbol{x}, \boldsymbol{y}, t)$ for the utility obtained by the transformation from $\boldsymbol{x}$ to $\boldsymbol{y}$.*

We introduce the following assumption.

Assumption 1 :

(1) *S is a subset of $\boldsymbol{R}^n$.*

(2) *For any $\boldsymbol{k} \in S$, $T(\boldsymbol{k})$ is nonempty and compact. Also, the correspondence $T(\boldsymbol{k}) : S \to S$ is continuous.*

(3) *For any t, $V(\boldsymbol{x}, \boldsymbol{y}, t) = \delta^t v(\boldsymbol{x}, \boldsymbol{y})$, where $0 < \delta < 1$. Also, $v(\boldsymbol{x}, \boldsymbol{y}) : S \times S \to \mathbb{R}$ is bounded and continuous.*

In this section, we deal with the following problem.

$$\sup \quad \lim_{n \to \infty} \sum_{t=0}^{n} \delta^t v(\boldsymbol{k}_t, \boldsymbol{k}_{t+1})$$

$$\text{subject to} \quad (\boldsymbol{k}_t) \in F(\boldsymbol{k}_0).$$

In this case, however, we may show that for any $\boldsymbol{k}_0 \in S$ and for any $(\boldsymbol{k}_t) \in F(\boldsymbol{k}_0)$,

$$\lim_{n \to \infty} \sum_{t=0}^{n} \delta^t v(\boldsymbol{k}_t, \boldsymbol{k}_{t+1})$$

is finite.

We introduce some concepts.

Definition 11 :

(1) *Given* $\boldsymbol{k}_0 \in S$, *for* $(\boldsymbol{k}_t) \in F(\boldsymbol{k}_0)$,

$$w((\boldsymbol{k}_t)) = \lim_{n\to\infty} \sum_{t=0}^{n} \delta^t v(\boldsymbol{k}_t, \boldsymbol{k}_{t+1})$$

where $w((\boldsymbol{k}_t))$ *is finite. Given* k_0, *we denote* $w : F(\boldsymbol{k}_0) \to \boldsymbol{R}$.

(2) *Given* $\boldsymbol{k}_0$, *we define the function* $W(\boldsymbol{k}_0) : S \to \bar{\boldsymbol{R}}$ *by*

$$W(\boldsymbol{k}_0) = \sup w((\boldsymbol{k}_t)) \text{ over } (\boldsymbol{k}_t) \in F(\boldsymbol{k}_0).$$

In this case, if $U(\boldsymbol{k}) : S \to \bar{\boldsymbol{R}}$ is a solution of the functional equation, then it is the supremum function, too.

Definition 12 : *For* $U(\boldsymbol{k}) \in B(S)$,

(1) $h(\boldsymbol{k}) = \sup_{\boldsymbol{y}\in T(\boldsymbol{k})}\{v(\boldsymbol{k},\boldsymbol{y}) + \delta U(\boldsymbol{y})\}$.

(2) $G(\boldsymbol{k}) = \{\boldsymbol{y} | \boldsymbol{y} \in T(\boldsymbol{k}), h(\boldsymbol{k}) = v(\boldsymbol{k},\boldsymbol{y}) + \delta U(\boldsymbol{y})\}$.

(3) $U(\boldsymbol{k}) : S \to \bar{\boldsymbol{R}}$ *is said to be a solution of the functional equation if it satisfies*

$$U(\boldsymbol{k}) = \sup_{\boldsymbol{y}\in T(\boldsymbol{k})} \{v(\boldsymbol{k},\boldsymbol{y}) + \delta U(\boldsymbol{y})\}.$$

Lemma 9 : $G(\boldsymbol{k})$ *is upper hemi-continuous and* $h(\boldsymbol{k})$ *is continuous.*

Proof: For any $\boldsymbol{k} \in S$, $G(\boldsymbol{k})$ is nonempty and $G(\boldsymbol{k}) \subset T(\boldsymbol{k})$. Also, since $T(\boldsymbol{k})$ is compact, $G(\boldsymbol{k})$ is bounded. Suppose that $(\boldsymbol{y}_n)$ is a converging sequence in $G(\boldsymbol{k})$. Let the limit be $\boldsymbol{y}$. Since $T(\boldsymbol{k})$ is compact, $\boldsymbol{y} \in T(\boldsymbol{k})$. Also, for any $n \in N$,

$$v(\boldsymbol{k},\boldsymbol{y}_n) + \delta U(\boldsymbol{y}_n) = h(\boldsymbol{k}).$$

Let $\boldsymbol{k}$ be given. Since $v(\boldsymbol{k},\boldsymbol{y}) + \delta U(\boldsymbol{y}) : S \to \boldsymbol{R}$ is continuous, $v(\boldsymbol{k},\boldsymbol{y}) + \delta U(\boldsymbol{y}) = h(\boldsymbol{k})$. Thus, $\boldsymbol{y} \in G(\boldsymbol{k})$. Therefore, $G(\boldsymbol{k})$ is closed. Next, we wish to show that $G(\boldsymbol{k})$ is upper hemi-continuous. For any $\boldsymbol{k} \in S$, let $(\boldsymbol{k}_n)$ be any sequence of S converging to $\boldsymbol{k}$. Choose any sequence $(\boldsymbol{y}_n)$ such that

$$\boldsymbol{y}_n \in G(\boldsymbol{k}_n).$$

Since $T(\boldsymbol{k})$ is a continuous correspondence, $T(\boldsymbol{k})$ is upper hemi-continuous, there exists a subsequence $(\boldsymbol{y}_{n_i})$ of $(\boldsymbol{y}_n)$ converging to $y \in T(\boldsymbol{k})$. Also, let $\boldsymbol{z} \in T(\boldsymbol{k})$. Since $T(\boldsymbol{k})$ is lower hemi-continuous, there exists $(\boldsymbol{z}_{n_i})$ converging to $\boldsymbol{z}$ such that $\boldsymbol{z}_{n_i} \in T(\boldsymbol{k}_{n_i})$. Thus,

$$v(\boldsymbol{k}_{n_i},\boldsymbol{y}_{n_i}) + \delta U(\boldsymbol{y}_{n_i}) \geqq v(\boldsymbol{k}_{n_i},\boldsymbol{z}_{n_i}) + \delta U(\boldsymbol{z}_{n_i}).$$

Since $v(\boldsymbol{k},\boldsymbol{y}) + \delta U(\boldsymbol{y}) : S \times S \to \boldsymbol{R}$ is continuous,

$$v(\boldsymbol{k},\boldsymbol{y}) + \delta U(\boldsymbol{y}) \geqq v(\boldsymbol{k},\boldsymbol{z}) + \delta U(\boldsymbol{z}).$$

Thus, for any $\boldsymbol{z} \in T(\boldsymbol{k})$,

$$v(\boldsymbol{k}, \boldsymbol{y}) + \delta U(\boldsymbol{y}) \geqq v(\boldsymbol{k}, \boldsymbol{z}) + \delta U(\boldsymbol{z}).$$

Therefore, $\boldsymbol{y} \in G(\boldsymbol{k})$. Hence, $G(\boldsymbol{k})$ is upper hemi-continuous.

Let $\boldsymbol{k}$ be given. Also, take any sequence $(\boldsymbol{k}_n)$ of S converging to $\boldsymbol{k}$ and choose $\boldsymbol{y}_n \in T(\boldsymbol{k}_n)$. In this case,

$$\sup_{n \geqq m} h(\boldsymbol{k}_n) \geqq h(\boldsymbol{k}_m) \geqq \inf_{n \geqq m} h(\boldsymbol{k}_n).$$

Define

$$h_0 = \lim_{m \to \infty} \sup_{n \geqq m} h(\boldsymbol{k}_n), \; h_1 = \lim_{m \to \infty} \inf_{n \geqq m} h(\boldsymbol{k}_n).$$

Thus, there exist subsequences $\boldsymbol{y}'_{n_i} \in T(\boldsymbol{k}'_{n_i})$, $\boldsymbol{y}''_{n_i} \in T(\boldsymbol{k}''_{n_i})$ of $(\boldsymbol{y}_n)$ such that

$$h_0 = \lim\{v(\boldsymbol{k}'_{n_i}, \boldsymbol{y}'_{n_i}) + \delta U(\boldsymbol{y}'_{n_i})\}, \; h_1 = \lim\{v(\boldsymbol{k}''_{n_i}, \boldsymbol{y}''_{n_i}) + \delta U(\boldsymbol{y}''_{n_i})\}.$$

Furthermore, since $G(\boldsymbol{k})$ is upper hemi-continuous, we may choose a subsequence $(\hat{\boldsymbol{k}}_n), (\hat{\boldsymbol{y}}_n)$ of $(\boldsymbol{k}'_n), (\boldsymbol{y}'_n)$ and $(\boldsymbol{k}''_n), (\boldsymbol{y}''_n)$ such that

$$\lim \hat{\boldsymbol{y}}_n = \boldsymbol{y}, \boldsymbol{y} \in T(\boldsymbol{k}).$$

Thus,

$$h_0 = h(\boldsymbol{k}, \boldsymbol{y}) = h_1.$$

Q. E. D.

Definition 13 : *For the supremum function $W(\boldsymbol{k}) \in B(S)$,*

(1) $\hat{h}(\boldsymbol{k}) = \sup_{\boldsymbol{y} \in T(\boldsymbol{k})}\{v(\boldsymbol{k}, \boldsymbol{y}) + \delta W(\boldsymbol{y})\}$.

(2) $\hat{G}(\boldsymbol{k}) = \{\boldsymbol{y} | \boldsymbol{y} \in T(\boldsymbol{k}), \hat{h}(\boldsymbol{k}) = v(\boldsymbol{k}, \boldsymbol{y}) + \delta W(\boldsymbol{y})\}$.

Theorem 9 : *There exists the supremum function $W(\boldsymbol{k}) : S \to \bar{\boldsymbol{R}}$ and the optimal path of the above problem.*

Proof: It is sufficient for us to show that there exists $W(\boldsymbol{k}) : S \to \bar{\boldsymbol{R}}$ such that

$$W(\boldsymbol{k}) = \sup_{\boldsymbol{y} \in T(\boldsymbol{k})} \{v(\boldsymbol{k}, \boldsymbol{y}) + \delta W(\boldsymbol{y})\}.$$

We proceed to show the existence of $W(\boldsymbol{k}) : S \to \bar{\boldsymbol{R}}$ of the above functional equation in the space $B(S)$ of bounded continuous functions with sup norm. For any bounded continuous function $g : S \to \boldsymbol{R}$, we define the correspondence $T : B(S) \to B(S)$ by

$$(Tg)(\boldsymbol{k}) = \max_{\boldsymbol{y} \in T(\boldsymbol{k})} \{v(\boldsymbol{k}, \boldsymbol{y}) + \delta g(\boldsymbol{y})\}.$$

(1) For any $g \in B(S)$, there exists $B > 0$ such that for any $(\boldsymbol{k}, \boldsymbol{y}) \in S \times S$,

$$-B < v(\boldsymbol{k}, \boldsymbol{y}) + \delta g(\boldsymbol{y}) < B.$$

Thus, for any $\boldsymbol{k} \in S$,

$$-B < (Tg)(\boldsymbol{k}) = \max_{\boldsymbol{y} \in T(\boldsymbol{k})} \{v(\boldsymbol{k}, \boldsymbol{y}) + \delta g(\boldsymbol{y})\} < B.$$

Therefore, $(Tg)(\boldsymbol{k})$ is bounded.

(2) Also, by the above lemma, $(Tg)(\boldsymbol{k}) : S \to R$ is continuous. Thus, $T : B(S) \to B(S)$.

(3) Suppose that $g, f \in B(S)$ and $g(\boldsymbol{k}) \leqq f(\boldsymbol{x})$ for any $\boldsymbol{k} \in S$. Then, given $\boldsymbol{k}' \in S$, for any $\boldsymbol{y} \in T(\boldsymbol{k}')$,

$$v(\boldsymbol{k}', \boldsymbol{y}) + \delta g(\boldsymbol{y}) \leqq v(\boldsymbol{k}', \boldsymbol{y}) + \delta f(\boldsymbol{y}).$$

Thus, for any $\boldsymbol{k}' \in S$,

$$Tg(\boldsymbol{k}') = \max_{\boldsymbol{y} \in T(\boldsymbol{k}')} \{v(\boldsymbol{k}', \boldsymbol{y}) + \delta g(\boldsymbol{y})\} \leqq \max_{\boldsymbol{y} \in T(\boldsymbol{k}')} \{v(\boldsymbol{k}', \boldsymbol{y}) + \delta f(\boldsymbol{y})\} = Tf(\boldsymbol{k}').$$

Also, for any $\boldsymbol{k}' \in S$ and for any $a \geqq 0$,

$$(T(g+a))(\boldsymbol{k}') = \max_{\boldsymbol{y} \in T(\boldsymbol{k}')} \{v(\boldsymbol{k}', \boldsymbol{y}) + \delta(g(\boldsymbol{y}) + a)\} = \max_{\boldsymbol{y} \in T(\boldsymbol{k}')} \{v(\boldsymbol{k}', \boldsymbol{y}) + \delta g(\boldsymbol{y})\} + \delta a = Tg(\boldsymbol{k}') + \delta a.$$

Thus, T satisfies the conditions of Theorem 8. Also, as shown in the above lemma, $B(S)$ is a Banach space. Therefore, there exists a fixed point $W \in B(S)$.

Q. E. D.

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