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A Small Final Coalgebra Theorem

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Abstract

This paper presents an elementary and self-contained proof of an existence theorem of final coalgebras for endofunctors on the category of sets and functions.

1 Introduction

Graphs are fundamental algebraic structures in computer science. Recently labeled transition systems, namely, labeled directed graphs have been considered an appropriate model for concurrent computations. It is known that graph structures are often represented by coalgebra structures [5, 8]. Many kinds of coalgebras have been considered as objects with circularity in semantics, knowledge dynamics and situation theory.

In 1988 Aczel [2] pointed out that the axiom of anti-foundation (AFA) on axiomatic set theory claims that the universal class of all sets with the membership relation is the final graph structure on classes. Moreover Aczel and Mendler [3] proved a final coalgebra theorem for set-based endofunctors. As is well-known the collection of all philosophical concepts constitutes a proper class. Thus it is natural to consider the hyperset theory based on classes for situation semantics. On the other hand the investigation of algebraic structures within the well-founded set theory (ZFC) seems to be enough for usual applications to computer science. In fact Barr [4] showed the theorem of Aczel and Mendler [3] on the existence of final coalgebras for accessible endofunctors on the category **Set** of (well-founded) sets and functions.

CCS due to Milner [7] is a language for communicating concurrent processes, which has the equationally axiomatic system. Its semantics is given as labeled transition systems and observational equivalences. Labeled transition systems are expressed as coalgebra structures with respect to an endofunctor $\Phi(X) = \wp(A \times X)$ on **Set**, for reasons of nondeterministic behavior of concurrent processes. In general the category of coalgebras for an endofunctor on **Set** does not always have final coalgebras. It is well-known that for the powerset functor $\wp$, as a typical case, the final coalgebra does not exist because of Cantor's diagonal method. Rutten and Turi [9, 10, 11] showed the existence of final coalgebras following Barr [4]. Their study of final semantics of processes made use of preservation of kernel pairs for the functor $\wp_f(A \times -)$, which is known to formulate processes as coalgebra. Their construction of final coalgebras, however, requires the continuity of functors. Recently Adámek and Koubek [1] extensively studied final coalgebras related to labeled trees and completions, and proved a sufficient condition for the existence of the greatest fixed point of set functors.

In this paper we will give an elementary proof of the small final coalgebra theorem due to Barr [4]. The theorem may guarantee the existence of small final coalgebras by restricting the range of transitions to some cardinality. Some detailed analysis on trees (in other words, the

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subcoalgebras generated by single elements) and congruences [3] (or, bisimulation equivalences) on coalgebras are essential in this note. The discussion of the paper is elementary and self-contained.

From a categorical point of view the existence of a final object results from special adjoint functor theorem [6], for which the proof following Mac Lane [6] will be recalled in Appendix at the end of the note.

The paper is organized as follows. In Section 2 we review the definition of coalgebras for endofunctors on **Set**, and then state that the category of coalgebras and their homomorphisms is cocomplete and co–well-powered, and the class of all coalgebras defined on subsets of a given set forms a set. In Section 3 we recall some basic properties of subcoalgebras for endofunctors on **Set**. In particular, when the involved endofunctor preserves intersections of subsets, the notion of trees of coalgebras, which are the smallest subcoalgebras containing singleton sets, can be considered. In Section 4 we discuss congruences of coalgebras, which is a modification of bisimulation equivalence relations on labeled transition systems due to D. Park. Then a well-known fundamental fact [2, Theorem 2.4] and [3, Lemma 4.3] that every coalgebra has the maximum congruence will be proved. The terminology "congruence" was initially used for algebras, for examples, in [9] and [10]. However, we reuse this terminology for coalgebras in the sense of Aczel and Mendler[3]. In Section 5 we state the main result of the paper. First we introduce tree congruences on coalgebras using the notion of trees. Then we show that, whenever all trees of coalgebras are bounded to a set, there is a weak final coalgebra. Thus by the similar fashion to Aczel and Mendler [3] an existence theorem of final coalgebras is proved. In section 6 a few examples of coalgebras which satisfy the main theorem are listed.

2 Coalgebras

This section defines the notion of coalgebras for endofunctors on the category **Set** of sets and functions. Let $\Phi : \mathbf{Set} \to \mathbf{Set}$ denote an endofunctor throughout the paper. A Φ-*coalgebra* (A, a) is a pair of a set A and a function $a : A \to \Phi(A)$. A *homomorphism* $f : (A, a) \to (B, b)$ of a Φ-coalgebra (A, a) into another Φ-coalgebra (B, b) is a function $f : A \to B$ rendering the following square commutative:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ \Phi(A) & \xrightarrow[\Phi(f)]{} & \Phi(B). \end{array}$$

All Φ-coalgebras and all their homomorphisms form a category $\mathbf{Set}(\Phi)$ which is called the category of Φ-coalgebras.

Proposition 2.1 *The category* $\mathbf{Set}(\Phi)$ *of* Φ*-coalgebras has all colimits.*

Proof. It suffices to prove the existence of coequalizers and coproducts because of [6, Theorem V **2.1**]. First let $f, g : (A, a) \to (B, b)$ be a pair of parallel homomorphisms of Φ-coalgebras. As the category **Set** has all small colimits there is a coequalizer $e : B \to Q$ of a pair of functions f and g in **Set**. Noticing that $\Phi(e)bf = \Phi(e)\Phi(f)a = \Phi(e)\Phi(g)a = \Phi(e)bg$ there is a unique function $q : Q \to \Phi(Q)$ such that $qe = \Phi(e)b$. It is an elementary exercise to show that $e : (B, b) \to (Q, q)$ is a coequalizer of f and g in $\mathbf{Set}(\Phi)$. Next suppose that $\{(A_\lambda, a_\lambda)\}_{\lambda \in \Lambda}$ is a family of Φ-coalgebras indexed by a set Λ. Let A be a coproduct (or disjoint union) of $\{A_\lambda\}_{\lambda\in\Lambda}$ and $i_\lambda : A_\lambda \to A$ the inclusion of coproducts for $\lambda \in \Lambda$. Define a function $a : A \to \Phi(A)$ to be a unique function such that a square

$$\begin{array}{ccc} A_\lambda & \xrightarrow{i_\lambda} & A \\ {\scriptstyle a_\lambda}\downarrow & & \downarrow{\scriptstyle a} \\ \Phi(A_\lambda) & \xrightarrow[\Phi(i_\lambda)]{} & \Phi(A) \end{array}$$

commutes for every $\lambda \in \Lambda$. It is also a routine work to show that a Φ-coalgebra (A, a) is a coproduct of $\{(A_\lambda, a_\lambda)\}_{\lambda\in\Lambda}$. □

The last result can be strengthened: the forgetful functor $\mathbf{Set}(\Phi) \to \mathbf{Set}$ creates colimits ([6, page 138] and [11, Theorem **10.1**]). This creation of colimits leads to a fundamental fact that every epimorphism of $\mathbf{Set}(\Phi)$ is a surjective function. (Of course the converse is trivial.)

Lemma 2.2 *If $f : X \to Y$ is an injection and X is a nonempty set, then $\Phi(f) : \Phi(X) \to \Phi(Y)$ is an injection.*

Proof. Choose $x_0 \in X$ and define a function $g : Y \to X$ by $g(y) = x$ if $y = f(x)$ for $x \in X$ and $g(y) = x_0$ if there is no $x \in X$ such that $y = f(x)$. Then it is clear that $gf = \mathrm{id}_X$ and $\Phi(g)\Phi(f) = \mathrm{id}_{\Phi(X)}$, which shows that $\Phi(f) : \Phi(X) \to \Phi(Y)$ is injective. □

Given a set M the class of all Φ-coalgebras (A, a) such that A is a nonempty subset of M is denoted by $\mathbf{Set}_M(\Phi)$. The following proposition points out that $\mathbf{Set}_M(\Phi)$ constitutes a set.

Proposition 2.3 *For every set M the class $\mathbf{Set}_M(\Phi)$ is a subset of $\wp(M) \times \wp(M \times \Phi(M))$, that is,*

$$\mathbf{Set}_M(\Phi) \subseteq \wp(M) \times \wp(M \times \Phi(M)).$$

Proof. Let (A, a) be a Φ-coalgebras in $\mathbf{Set}_M(\Phi)$ and $i : A \to M$ the inclusion. Then it is immediate that $A \in \wp(M)$ and $a \in \wp(M \times \Phi(M))$ since a function $a : A \to \Phi(A)$ can be identified with a subset $\{(x, \Phi(i)a(x)) | x \in A\}$ of $M \times \Phi(M)$ by the last lemma. □

Let $f : (A, a) \to (B, b)$ be a bijective homomorphism of Φ-coalgebras and $g : B \to A$ its inverse function. Then $\Phi(g)b = \Phi(g)bfg = \Phi(g)\Phi(f)ag = ag$ by $fg = \mathrm{id}_B$, $gf = \mathrm{id}_A$ and $bf = \Phi(f)a$. Hence all bijective homomorphisms of Φ-coalgebras are isomorphisms. Next assume that (A, a) is a Φ-coalgebra with $card(A) \leq card(M)$. Then there is an injective function $m : A \to M$. and so the restriction $r : A \to S$ of m is a bijection, where $S = m(A)$ (the image of m). It is easy to see that $r : (A, a) \to (S, \Phi(r)ar^{-1})$ is a homomorphism of Φ-coalgebras, so an isomorphism since r is bijective. Therefore every Φ-coalgebra (A, a) with $card(A) \leq card(M)$ is isomorphic to a Φ-coalgebra in $\mathbf{Set}_M(\Phi)$.

Proposition 2.4 *The category $\mathbf{Set}(\Phi)$ of Φ–coalgebras is co-well-powered.*

Proof. If (Q, q) is a quotient of a Φ–coalgebra (A, a), then $card(Q) \leq card(A)$ and so (Q, q) is isomorphic to a coalgebra in $\mathbf{Set}_A(\Phi)$. Hence $\mathbf{Set}_A(\Phi)$ contains all coalgebras which are isomorphic to a quotient of (A, a). □

3 Subcoalgebras

This section is devoted to state the notion and the basic properties of subcoalgebras. Trees, that is, the smallest subcoalgebras containing singleton sets, play an important role to prove the main theorem of the paper.

Let (A, a) be a Φ-coalgebra. A subset H of A is called a *subcoalgebra* of (A, a) if $a(H) \subseteq \Phi(i)(\Phi(H))$, where $i : H \to A$ is the inclusion of H into A. In other words, H is a subcoalgebra if and only if for each $x \in H$ there exists $z \in \Phi(H)$ such that $a(x) = \Phi(i)(z)$. It is also easy to verify that a subset H of A is a subcoalgebra of (A, a) if and only if there is a (unique) function $a_H : H \to \Phi(H)$ which makes the inclusion $i : H \to A$ a homomorphism $i : (H, a_H) \to (A, a)$ of Φ-coalgebras. (By the definition the empty set $\emptyset$ is always a subcoalgebra.)

Let H be a subcoalgebra of a Φ-coalgebra (A, a). Then a subset S of H is a subcoalgebra of H if and only if S is a subcoalgebra of (A, a).

Proposition 3.1 *Every homomorphic image of Φ-coalgebras is a subcoalgebra.*

Proof. Let $f : (A, a) \to (B, b)$ be a homomorphism of Φ-coalgebras. First note that a function $f : A \to B$ can be decomposed into the composite of a surjection $f' : A \to f(A)$ followed by an inclusion $j : f(A) \to B$. Hence for $x \in A$ we have $b(f(x)) = \Phi(f)(a(x)) = \Phi(j)(\Phi(f')(a(x)))$, which completes the proof. $\Box$

Let M be a set. A Φ-coalgebra (A, a) is *M-bounded* (with respect to subcoalgebras) if for each $x \in A$ there is a subcoalgebra H of (A, a) such that $x \in H$ and $card(H) \leq card(M)$. An endofunctor $\Phi : \mathbf{Set} \to \mathbf{Set}$ is called *M-bounded* (with respect to subcoalgebras) if all Φ-coalgebras are M-bounded.

Proposition 3.2 *The category $\mathbf{Set}(\Phi)$ of Φ-coalgebras has a generating set if and only if Φ is M-bounded for a set M.*

Proof. First assume that Φ is M-bounded for a set M. We will show that the set $\mathbf{Set}_M(\Phi)$ is a generating set of $\mathbf{Set}(\Phi)$. Let $f, g : (A, a) \to (B, b)$ be two different homomorphisms such that $f(x) \neq g(x)$ for a point $x \in A$. From M-boundedness we have a subcoalgebra (H, h) of (A, a) with $x \in H$ and $card(H) \leq card(M)$. By the discussion just before Proposition **2.4** (H, h) is isomorphic to a coalgebra (S, s) in $\mathbf{Set}_M(\Phi)$, that is, there is an isomorphism $t : (S, s) \to (H, h)$. Finally it is easy to see that $fit \neq git$, where $i : H \to A$ denotes the inclusion. This proves that $\mathbf{Set}_M(\Phi)$ is a generating set of $\mathbf{Set}(\Phi)$. Conversely assume that $\mathcal{G}$ is a generating set of $\mathbf{Set}(\Phi)$. Let (A, a) be a Φ-coalgebra. By the virtue of Lemma **4**(a) in Appendix there exists an epimorphism $e : G_X \to (A, a)$. Recall that an epimorphism of Φ-coalgebras is a surjection since the forgetful functor $\mathbf{Set}(\Phi) \to \mathbf{Set}$ creates colimits. Hence for each $x \in A$ there exists a Φ-coalgebra (S, s) in $\mathbf{Set}_M(\Phi)$ and a homomorphism $t : (S, s) \to (A, a)$ such that $x \in t(S)$. But $t(S)$ is a subcoalgebra of (A, a) and $card(t(S)) \leq card(S) \leq card(M)$, where

$$M = \bigcup_{(S,s) \in \mathbf{Set}_M(\Phi)} S,$$

which is a set because of the axiom of union. This shows that (A, a) is M-bounded. $\Box$

Making use of Theorem **5** in Appendix we obtain the following corollary due to Barr [4].

Corollary 3.3 *If a functor Φ is M–bounded then the category $\mathbf{Set}(\Phi)$ of Φ–coalgebras has a final coalgebra.*

In the rest of the paper we will show another proof of the above corollary using the notion of trees. A tree of a coalgebra is the minimum subcoalgebra containing a given point. In order to ensure the existence of trees, the set of all subcoalgebras of a coalgebra is expected to be closed under intersection. By this reason we briefly mention on endofunctors weakly preserving generalized pullbacks. The weak preservation of generalized pullbacks is stronger than that of kernel pairs in [10]. As will see, if an endofunctor $\Phi : \mathbf{Set} \to \mathbf{Set}$ weakly preserves generalized pullbacks, homomorphisms of Φ–coalgebras preserve trees.

The following defines the notions of generalized pullbacks and weak generalized pullbacks in categories.

Definition 3.4 *Let Λ be a set. A generalized pullback of a Λ–indexed set of arrows $f_\lambda : A_\lambda \to B$ with a common codomain B is an object P together with a Λ–indexed set of arrows $p_\lambda : P \to A_\lambda$ satisfying the following:*

(a) *$f_\lambda p_\lambda = f'_\lambda p'_\lambda$ for any $\lambda, \lambda' \in \Lambda$,*

(b) *For any set of arrows $g_\lambda : X \to A_\lambda$ such that $f_\lambda g_\lambda = f'_\lambda g'_\lambda$ for any $\lambda \in \Lambda, \lambda'$, there exists a unique function $g : X \to P$ such that $p_\lambda g = g_\lambda$ for all $\lambda \in \Lambda$.*

The object P together with Λ–indexed set of arrows $p_\lambda : P \to A_\lambda$ is called a generalized weak pullback if in the preceding formulation the requirement of uniqueness is omitted.

A functor weakly preserves generalized pullbacks if and only if it maps every generalized pullback (of any Λ–indexed set of arrows with a common domain) to a weak generalized pullback.

Lemma 3.5 *Let (A, a) be a Φ–coalgebra. If Φ : **Set** $\to$ **Set** weakly preserves generalized pullbacks, then for every family $\{H_\lambda\}_\lambda$ of subcoalgebras of (A, a) its intersection $H = \cap_\lambda H_\lambda$ is a subcoalgebra of (A, a).*

Proof. Let a set H together with a Λ–indexed set of functions $j_\lambda : H \to H_\lambda$ be a generalized pullback of a Λ–indexed set of injections $i_\lambda : H_\lambda \to A$. It is trivial that $H = \cap_\lambda H_\lambda$. Since $\Phi(H)$ is a weak pullback of $\{\Phi(i_\lambda) \mid \lambda \in \Lambda\}$ by the assumption, there exists $a_H : H \to \Phi(H)$ such that $\Phi(j_\lambda) a_H = a_{H_\lambda} j_\lambda$ for every λ. So that

$$\Phi(j) a_H = \Phi(i_\lambda)\Phi(j_\lambda) a_H = \Phi(i_\lambda) a_{H_\lambda} j_\lambda = a i_\lambda j_\lambda = ai.$$

Hence (H, a_H) is a subcoalgebra of (A, a). □

Let Φ : **Set** $\to$ **Set** be an endofunctor weakly preserving generalized pullbacks. For a Φ-coalgebra (A, a) consider the set of all subcoalgebras of (A, a) containing an element $x \in A$. Then by the last lemma their intersection is the smallest subcoalgebra containing x, which is called the *tree* of (A, a) generated by x and denoted by $[x]_A$.

Proposition 3.6 *Let $f : (A, a) \to (B, b)$ be a Φ–homomorphism and K a subcoalgebra of (B, b). If Φ weakly preserves generalized pullbacks, then the inverse image $f^{-1}(K)$ of K is a subcoalgebra of (A, a).*

Proof. We can draw the following diagram:

$$
\begin{tikzcd}
 & A \arrow[rr, "f"] \arrow[dd, "a"] & & B \arrow[dd, "b"] \\
f^{-1}K \arrow[ur, hook, "j"] \arrow[rr, "f'"] \arrow[dd, dotted, "h"'] & & K \arrow[ur, hook, "i"'] \arrow[dd] & \\
 & \Phi(A) \arrow[rr, "\Phi(f)"] & & \Phi(B), \\
\Phi f^{-1}K \arrow[ur, "\Phi(j)"] \arrow[rr, "\Phi(f')"'] & & \Phi K \arrow[ur, "\Phi(i)"'] &
\end{tikzcd}
$$

where i and j are injections and f' is the restriction of f to $f^{-1}(K)$. The upper square is a pullback and the right square and the back square commute. By assumption the lower square is a weak pullback, so that there exists a function $h : f^{-1}(K) \to \Phi f^{-1}(K)$ such that $ai = \Phi(i)h$ and $kf' = \Phi(f')h$. Hence $(f^{-1}(K), h)$ is a subcoalgebra of (A, a).

Corollary 3.7 *Assume that Φ weakly preserves generalized pullbacks.*

(a) *If $f : (A, a) \to (B, b)$ is a homomorphism of Φ–coalgebras, then $f[x]_A = [f(x)]_B$ for all $x \in A$.*

(b) *If (H, a_H) is a subcoalgebra of (A, a), then $[x]_A = [x]_H$ for all $x \in H$.*

p

Proof. (a) Clearly $[f(x)]_B \subseteq f([x]_A)$. By Proposition **3.5** we have $[x]_A \subseteq f^{-1}([f(x)]_B)$ and so $f([x]_A) \subseteq [f(x)]_B$. (b) Trivial since the inclusion $i : H \to A$ is a homomorphism of Φ–coalgebras. □

Example 3.8 *The powerset functor $\wp$:* **Set** $\to$ **Set** *weakly preserves generalized pullbacks.*

Proof. First recall that the pullback construction in the category of sets. A generalized pullback of a Λ–indexed set of functions $f_\lambda : A_\lambda \to B$ is a set P together with a Λ–indexed set of functions $p_\lambda : P \to A_\lambda$ such that

$$P = \{c \in \prod_{\lambda \in \Lambda} A_\lambda \mid f_\lambda \pi_\lambda(c) = f_{\lambda'} \pi_{\lambda'}(c) \ \text{ for all } \lambda, \lambda' \in \Lambda\},$$

where $\pi_\lambda : \prod_\lambda A_\lambda \to A_\lambda$ is the λ–th projection and $p_\lambda : P \to A_\lambda$ is the restriction of the λ–th projection π_λ to P (that is, the composite of the inclusion $P \to \prod_\lambda A_\lambda$ followed by the λ–th projection π_λ). We will prove that a set of functions $\wp(p_\lambda) : \wp(P) \to \wp(A_\lambda)$ is a weak pullback of a set of functions $\wp(f_\lambda) : \wp(A_\lambda) \to \wp(B)$. Assume that a set of functions $g_\lambda : X \to \wp(A_\lambda)$ such that $\wp(f_\lambda)g_\lambda = \wp(f_{\lambda'})g_{\lambda'}$ for any λ, λ', or equivalently, $f_\lambda(g_\lambda(x)) = f_{\lambda'}(g_{\lambda'}(x))$ for $x \in X$. Then define a function $g(x) = A \cap \prod_\lambda g_\lambda(x)$ for $x \in X$. It suffices to show that $g_{\lambda_0} = \wp(p_{\lambda_0})g$ for each λ_0, that is, $g_{\lambda_0}(x) = \pi_{\lambda_0}(g(x))$ for $x \in X$. It is easy to see that

$$\pi_{\lambda_0}(g(x)) \subseteq \pi_{\lambda_0}(\prod_\lambda g_\lambda(x)) = g_{\lambda_0}(x).$$

We prove the converse $g_{\lambda_0}(x) \subseteq \pi_{\lambda_0}(g(x))$. Take any $a_{\lambda_0} \in g_{\lambda_0}(x)$. By hypothesis $f_\lambda(g_\lambda(x)) = f_{\lambda_0}(g_{\lambda_0}(x))$ we can choose $a_\lambda \in A_\lambda$ for $\lambda (\neq \lambda_0)$ such that $f_\lambda(a_\lambda) = f_{\lambda_0}(a_{\lambda_0})$, which gives a point $a = (a_\lambda)_\lambda \in g(x)$ such that $a_{\lambda_0} = \pi_{\lambda_0}(a) \in \pi_{\lambda_0}(g(x))$. The proof completes. □

4 Congruences

This section discusses the notion of congruences on coalgebras initiated by Aczel and Mendler [3]. The notion of congruences in [3] is a modification of bisimulation equivalence relations on labeled transition systems. The aim of this section is to show a usual fact ,[2, Theorem 2.4] and [3, Lemma 4.3], that every coalgebra has the maximum congruence.

A (binary) *relation* on a set A is a subset of $A \times A$. Hence boolean operations such as union and intersections can be applied to relations. An *equivalence* relation θ on a set A is a relation on A such that $(x,x) \in \theta$ (reflexive), if $(x,y) \in \theta$ then $(y,x) \in \theta$ (symmetric), and if $(x,y) \in \theta \wedge (y,z) \in \theta$ then $(x,z) \in \theta$ (transitive) for all $x,y,z \in A$. For any relation α the smallest equivalence relation containing α (that is, the reflexive, symmetric and transitive closure of α) will be denoted by α^*. Given an equivalence relation θ on A there is a surjection of A onto a (quotient) set Q such that $(x,y) \in \theta$ if and only if $e(x) = e(y)$. We call such a surjection $e : A \to Q$ a *quotient* function with respect to θ. Since a quotient function is unique up to isomorphisms, an equivalence relation $\Phi(\theta)$ on $\Phi(A)$ is uniquely defined as follows:

$$(u,v) \in \Phi(\theta) \text{ if and only if } \Phi(e)(u) = \Phi(e)(v).$$

Proposition 4.1 *Let θ and θ' be equivalence relations on A. If $\theta \subseteq \theta'$, then $\Phi(\theta) \subseteq \Phi(\theta')$.*

Proof. Let $e : A \to Q$ and $e' : A \to Q'$ be quotient functions with respect to θ and θ', respectively. Since $\theta \subseteq \theta'$, there is a function $k : Q \to Q'$ such that $ke = e'$. Hence, if $(u,v) \in \Phi(\theta)$, then $\Phi(e)(u) = \Phi(e)(v)$ by the definition and so

$$\Phi(e')(u) = \Phi(k)\Phi(e)(u) = \Phi(k)\Phi(e)(v) = \Phi(e')(v),$$

which shows $(u,v) \in \Phi(\theta')$. □

The congruence relations for universal algebras have been invented to constitute quotient algebras and they are required equivalence relations preserving involved operations of algebras.

Definition 4.2 *Let (A,a) be a Φ-coalgebra. An equivalence relation θ on A is a congruence on (A,a) if $(x,y) \in \theta$ implies $(a(x), a(y)) \in \Phi(\theta)$ for all pairs (x,y).* □

Proposition 4.3 *If $f : (A,a) \to (B,b)$ is a homomorphism of Φ-coalgebras, then an equivalence relation $\kappa(f) = \{(x,y) \in A \times A | f(x) = f(y)\}$ on A is a congruence on (A,a).*

Proof. Let $f = me$ be an image factorization of f into the composite of a surjection $e : A \to Q$ followed by an injection $m : Q \to B$. Then e is a quotient function with respect to $\kappa(f)$. Note that $\Phi(m)$ is injective by Proposition **3.2**. Hence, if $f(x) = f(y)$, then

$$\Phi(m)\Phi(e)a(x) = \Phi(f)a(x) = bf(x) = bf(y) = \Phi(f)a(y) = \Phi(m)\Phi(e)a(y),$$

and so $\Phi(e)a(x) = \Phi(e)a(y)$ using the injectivity of $\Phi(m)$. □

Proposition 4.4 *Given a congruence θ on (A,a) and a quotient function $e : A \to Q$ with respect to θ there is a unique function $q : Q \to \Phi(Q)$ such that $e : (A,a) \to (Q,q)$ is a homomorphism of Φ-coalgebras.*

Proof. A function $q : Q \to \Phi(Q)$ can be defined as follows:

$$\text{For } w \in Q : q(w) = \Phi(e)a(x) \text{ if } w = e(x).$$

This definition is well-defined, since if $e(x) = e(y)$ then $(x, y) \in \theta$ and so $(a(x), a(y)) \in \Phi(\theta)$, since θ is a congruence. It is trivial that $qe = \Phi(e)a$. The uniqueness of q follows from the surjectivity of e. This completes the proof. □

The Φ-coalgebra (Q, q) constructed in the above proposition is called a *quotient* Φ-coalgebra of (A, a) with respect to a congruence θ and denoted by $(A/\theta, a/\theta)$.

Lemma 4.5 *If θ_0 and θ_1 are congruences on (A, a), then $(\theta_0 \cup \theta_1)^*$ is a congruence on (A, a).*

Proof. We have to see that $(x, y) \in (\theta_0 \cup \theta_1)^*$ implies $(a(x), a(y)) \in \Phi((\theta_0 \cup \theta_1)^*)$. So it suffices to show that $(x, y) \in \theta_0$ implies $(a(x), a(y)) \in \Phi((\theta_0 \cup \theta_1)^*)$. But, if $(x, y) \in \theta_0$, then $(a(x), a(y)) \in \Phi(\theta_0)$ and consequently $(a(x), a(y)) \in \Phi((\theta_0 \cup \theta_1)^*)$, because $\Phi(\theta_0) \subseteq \Phi((\theta_0 \cup \theta_1)^*)$ by Theorem **5.1**. □

Theorem 4.6 *Every Φ-coalgebra (A, a) has the maximum congruence Ξ_A.*

Proof. Define a relation Ξ_A on A to be a union (supremum) of all congruences on (A, a), that is,

$$\Xi_A = \bigcup_{\theta \in S} \theta,$$

where S is the set of all congruences on (A, a). First we show that Ξ_A is an equivalence relation on A. As the identity relation id_A on A is a congruence, it is clear that $\mathrm{id}_A \subseteq \Xi_A$ (reflexive). Assume that $(x, y) \in \Xi_A$. Then there is a congruence θ such that $(x, y) \in \theta$ and so $(y, x) \in \theta$ since θ is a equivalence relation. Hence $(y, x) \in \Xi$ (symmetric). Next assume that $(x, y) \in \Xi_A$ and $(y, z) \in \Xi_A$. Then $(x, y) \in \theta_0$ and $(y, z) \in \theta_1$ for some $\theta_0, \theta_1 \in S$. Hence

$$(x, y) \in \theta_0 \subseteq (\theta_0 \cup \theta_1)^* \text{ and } (y, z) \in \theta_1 \subseteq (\theta_0 \cup \theta_1)^*$$

and so $(x, z) \in (\theta_0 \cup \theta_1)^*$ by the trasitivity of $(\theta_0 \cup \theta_1)^*$. As $(\theta_0 \cup \theta_1)^*$ is a congruence by the last lemma we conclude $(x, z) \in \Xi_A$ (transitive). Finally it suffices to prove that Ξ_A is a congruence. But, if $(x, y) \in \Xi_A$, then $(x, y) \in \theta$ for some congruence θ on A and so $(a(x), a(y)) \in \Phi(\theta) \subseteq \Phi(\Xi_A)$ by Theorem **5.1**. This shows that Ξ_A is a congruence. □

Theorem 4.7 *For every Φ-coalgebra (A, a) there is at most one homomorphism from any Φ-coalgebra into $(A/\Xi_A, a/\Xi_A)$.*

Proof. Let $e : A \to A/\Xi_A$ be a quotient function with respect to Ξ_A. Assume that $f, g : (B, b) \to (A/\Xi_A, a/\Xi_A)$ are two homomorphisms. Construct a coequalizer $e_1 : (A/\Xi_A, a/\Xi_A) \to (R, r)$ of f and g (which does exist by Proposition **2.1**). Then for any $u \in B$ there is $x, y \in A$ such that $f(u) = e(x)$ and $g(u) = e(y)$. Moreover $e_1 e(x) = e_1 f(u) = e_1 g(u) = e_1 e(y)$, which means that $(x, y) \in \kappa(e_1 e)$. As $\kappa(e_1 e) \subseteq \Xi_A$ by Proposition **4.3** it follows that $(x, y) \in \Xi_A$ and $e(x) = e(y)$. Hence $f(u) = e(x) = e(y) = g(u)$, which proves that $f = g$. □

The following corollary is an immediate consequence from the last theorem.

Corollary 4.8 *If the category* **Set**(Φ) *of Φ-coalgebras has a weak final coalgebra, then it has a final coalgebra.* □

5 Tree Congruences

This section proves the main theorem of the paper. To treat freely with trees of coalgebras we assume that an endofunctor $\Phi : \mathbf{Set} \to \mathbf{Set}$ preserves intersections throughout this section. First we introduce tree congruences on coalgebras using the notion of trees. Then we show that, whenever all trees of coalgebras are bounded to a set, there is a weak final coalgebra. Thus by the similar fashion to Aczel and Mendler [3] an existence theorem of final coalgebras is proved.

Let (A, a) be a Φ-coalgebra. Define a relation ξ_A on A as follows: $(x, y) \in \xi_A$ for $x, y \in A$ if and only if there is an isomorphism $f : [x]_A \to [y]_A$ of Φ-coalgebras such that $f(x) = y$. Obviously ξ_A is an equivalence relation on A, which we will call the *tree congruence* on (A, a) by virtue of the following

Theorem 5.1 *For each Φ-coalgebra (A, a) the equivalence relation ξ_A on A is a congruence on (A, a).*

Proof. Let $e : A \to Q$ be a quotient function with respect to ξ_A. It suffices to show that $(x, y) \in \xi_A$ implies $\Phi(e)a(x) = \Phi(e)a(y)$. Assume that $(x, y) \in \xi_A$. Let $i : [x]_A \to A$ and $j : [y]_A \to A$ be inclusions, respectively. There is an isomorphism $k : [x]_A \to [y]_A$ with $k(x) = y$.

$$\begin{array}{ccccc}
A & \xleftarrow{i} & [x]_A & \xrightarrow{jk} & A \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle h_x} & & \downarrow{\scriptstyle a} \\
\Phi(A) & \xleftarrow[\Phi(i)]{} & \Phi([x]_A) & \xrightarrow[\Phi(jk)]{} & \Phi(A).
\end{array}$$

First note that $ei = ejk$. For each $z \in [x]_A (= H)$ we have

$$\begin{array}{rcll}
[i(z)]_A & = & [z]_H & (3.4(\mathrm{a})) \\
 & \cong & jk[z]_H & (3.2(\mathrm{b})) \\
 & = & [jk(z)]_A & (3.4(\mathrm{b})),
\end{array}$$

which indicates that $(i(z), jk(z)) \in \xi_A$ and so $ei(z) = ejk(z)$. Therefore it follows that

$$\begin{array}{rcll}
\Phi(e)a(x) & = & \Phi(e)ai(x) & \\
 & = & \Phi(e)\Phi(i)h_x(x) & (i \text{ is a homomorphism.}) \\
 & = & \Phi(e)\Phi(jk)(x) & (ei = ejk) \\
 & = & \Phi(e)ajk(x) & (jk \text{ is a homomorphism.}) \\
 & = & \Phi(e)a(y) & (y = jk(x)).
\end{array}$$

The proof is completed. □

Note that the tree congruence ξ_A is not necessarily identical with the maximum congruence Ξ_A. For example, consider a homomorphism $f : (A, a) \to (B, b)$ of $\wp$-coalgebras, where $A = \{x, y\}$, $a(x) = A$, $a(y) = \{y\}$, $B = \{z\}$, $b(z) = B$, and $f(x) = f(y) = z$. Then $(x, y) \in \kappa(f)$, but $[x]_A = A$ and $[y]_A = \{y\}$ are not mutually isomorphic.

Theorem 5.2 *If every tree of a Φ-coalgebra (A, a) is isomorphic to a subcoalgebra of a Φ-coalgebra (T, t), then there is at least one homomorphism $f : (A, a) \to (T/\xi_T, t/\xi_T)$.*

Proof. Let $e : (T, t) \to (T/\xi_T, t/\xi_T)$ be a quotient homomorphism by ξ_T. For every $x \in A$ there is an injective homomorphism $k : [x]_A \to (T, t)$ by the assumption. Define a function $f : A \to T/\xi_T$ by $f(x) = ek(x)$.

$$\begin{array}{ccccccc}
A & \xleftarrow{i} & [x]_A & \xrightarrow{k} & T & \xrightarrow{e} & T/\xi_T \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle h_x} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t/\xi_T} \\
\Phi(A) & \xleftarrow[\Phi(i)]{} & \Phi([x]_A) & \xrightarrow[\Phi(k)]{} & \Phi(T) & \xrightarrow[\Phi(e)]{} & \Phi(T/\xi_T).
\end{array}$$

Note that this definition of $f(x)$ is independent on the choice of an injective homomorphism k. (Let $k' : [x]_A \to T$ be another injective homomorphism. Then by Proposition **3.2**(b) and Definition **3.4**(b) it is trivial that $[k(x)]_R \cong [x]_A \cong [k'(x)]_R$. Hence $ek(x) = ek'(x)$.) Next we show that $fi = ek$. For each $z \in [x]_A$ the composite mk of the inclusion $m : [z]_A \to [x]_A$ followed by k is an injective homomorphism into T and so $f(z) = ekm(z)$. Hence $fi(z) = f(z) = ekm(z) = ek(z)$, which shows that $fi = ek$. Finally we show that $f : A \to T/\xi_T$ is a homomorphism, that is, $a\Phi(f) = f(t/\xi_T)$. But we have

$$\begin{aligned}
\Phi(f)a(x) &= \Phi(f)ai(x) & \\
&= \Phi(f)\Phi(i)h_x(x) & (i \text{ is a homomorphism.}) \\
&= \Phi(ek)h_x(x) & (fi = ek) \\
&= (t/\xi_T)ek(x) & (ek \text{ is a homomorphism.}) \\
&= (t/\xi_T)f(x) & (f(x) = ek(x)). \qquad \Box
\end{aligned}$$

For a set M the coproduct of all coalgebras in $\mathbf{Set}_M(\Phi)$ will be denoted by (T_M, t_M), that is,

$$(T_M, t_M) = \coprod_{(A,a)\in\mathbf{Set}_M(\Phi)} (A, a)$$

and $i_A : (A, a) \to (T_M, t_M)$ denotes the inclusion of the coproduct for a Φ-coalgebra $(A, a) \in \mathbf{Set}_M(\Phi)$. A Φ-coalgebra (A, a) is called *M-bounded* if there is an injection of A into M. It is obvious that for an M-bounded Φ-coalgebra (A, a) there is an injective homomorphism $k : (A, a) \to (T_M, t_M)$, that is, $card(A) \leq card(M)$. Hence we have the following

Corollary 5.3 *If all trees of Φ-coalgebras are M-bounded for a set M, then for each Φ-coalgebra (A, a) there is at least one homomorphism $f : (A, a) \to (T_M/\xi_{T_M}, t_M/\xi_{T_M})$, that is, the quotient coalgebra $(T_M/\xi_{T_M}, t_M/\xi_{T_M})$ of (T_M, t_M) is a weak final coalgebra in* $\mathbf{Set}(\Phi)$.
$\Box$

In a category of coalgebras a final coalgebra is a coalgebra such that there is a unique homomorphism from each coalgebra into it. Combining with Corollary **4.8** and the last corollary we have the following

Theorem 5.4 *If there is a set M such that all trees of Φ-coalgebras are M-bounded, then the category* $\mathbf{Set}(\Phi)$ *of Φ-coalgebras has a final coalgebra.* $\Box$

6 Examples

This section illustrates a few examples of coalgebras which satisfy the main theorem **5.4** and so have a final coalgebra.

Let M be a set. The M-bounded power set functor $\wp_M : \mathbf{Set} \to \mathbf{Set}$ is a functor such that

$$\wp_M(A) = \{S | S \subseteq A \wedge card(S) \leq card(M)\}$$

for all sets A, where $card(M)$ denotes the cardinality of M. For a set M n-th product M^n is defined by $M^0 = 1$ (a singleton set) and $M^{n+1} = M^n \times M$ for $n \geq 0$. The set M^* of all finite strings of elements in M is formally defined by $M^* = \cup_{n\geq 0} M^n$.

Theorem 6.1 *All trees of $\wp_M$-coalgebras are M^*-bounded.*

Proof. Let (A, a) be a $\wp_M$-coalgebra and $x \in A$. Define a subset $[x]_n$ of A by $[x]_0 = \{x\}$ and $[x]_{n+1} = \cup_{y\in[x]_n} a(y)$ for $n \geq 0$. Set $[x]_\infty = \cup_{n\geq 0}[x]_n$. From $card([x]_{n+1}) \leq card([x]_n \times M)$ it follows that

$$card([x]_\infty) \leq card(\cup_{n\geq 0} M^n) = card(M^*).$$

Finally it suffices to see that $[x]_A = [x]_\infty$. By induction we have $[x]_n \subseteq [x]_A$ for all $n \geq 0$ and so $[x]_\infty \subseteq [x]_A$. Because $[x]_0 \subseteq [x]_A$ and if $[x]_n \subseteq [x]_A$ then $[x]_{n+1} = \cup_{y\in[x]_n} a(y) \subseteq [x]_A$. Finally note that $[x]_\infty$ is a subcoalgebra of (A, a) since $a(y) \subseteq [x]_{n+1} \subseteq [x]_\infty$ (i.e. $a(y) \in \wp_M([x]_\infty)$) for $y \in [x]_n$. Hence $[x]_A \subseteq [x]_\infty$. □

Combining with Theorem **5.4** and the last theorem we have the following

Corollary 6.2 *The category* $\mathbf{Set}(\wp_M)$ *has a final coalgebra.* □

Note that $\wp_1(X) = 1 + X$ for a singleton set $1(= \{\emptyset\})$.

Let Ψ and Φ be endofunctors on **Set**. A natural transformation $\nu : \Psi \to \Phi$ is *strict* if for every injection $f : X \to Y$ a natural square

$$\begin{array}{ccc}
\Psi(X) & \xrightarrow{\Psi(f)} & \Psi(Y) \\
{\scriptstyle \nu_X}\downarrow & & \downarrow{\scriptstyle \nu_Y} \\
\Phi(X) & \xrightarrow[\Phi(f)]{} & \Phi(Y)
\end{array}$$

is a pullback.

Proposition 6.3 *Let* $\nu : \Psi \to \Phi$ *be a natural transformation between endofunctors* Ψ *and* Φ *on* **Set**. *If* Φ *preserves intersections and* $\nu : \Psi \to \Phi$ *is strict, then* Ψ *also preserves intersections.*

Proof. It follows from easy diagram chasing. □

Lemma 6.4 *Let* $\nu : \Psi \to \Phi$ *be a strict natural transformation and* (B, b) *a* Ψ*-coalgebra. Then a subset* H *of* B *is a subcoalgebra of* (B, b) *if and only if* H *is a subcoalgebra of a* Φ*-coalgebra* $(B, \nu_B b)$.

Proof. Let $i : H \to B$ be the inclusion and consider a diagram

$$\begin{array}{ccc}
H & \xrightarrow{i} & B \\
 & & \downarrow{\scriptstyle b} \\
\Psi(H) & \xrightarrow{\Psi(i)} & \Psi(B) \\
{\scriptstyle \nu_H}\downarrow & & \downarrow{\scriptstyle \nu_B} \\
\Phi(H) & \xrightarrow[\Phi(i)]{} & \Phi(B),
\end{array}$$

in which the square is a pullback by the strictness of ν. Then it is trivial that a function $h : H \to \Psi H$ with $bi = \Psi(i)h$ bijectively corresponds to a function $h' : H \to \Phi(H)$ with $\nu_B bi = \Phi(i)h'$. This completes the proof. □

As a direct result from the above lemma we have the following

Corollary 6.5 *Let* $\Phi, \Psi : \mathbf{Set} \to \mathbf{Set}$ *be endofunctors preserving intersections and* $\nu : \Psi \to \Phi$ *a strict natural transformation.*

(a) *If* (B, b) *is a* Ψ*-coalgebra, then* $[x]_{(B,b)} = [x]_{(B,\nu_B b)}$ *for all* $x \in B$.

(b) *If all trees of Φ-coalgebras are M-bounded for a set M, then so are those of Ψ-coalgebras.* $\square$

By Theorem **6.1** and Theorem **5.4** we have the following

Example 6.6 *All categories of coalgebras for the following endofunctors have final coalgebra.*

(a) *The finite powerset functor* $\wp_{\text{fin}} : \mathbf{Set} \to \mathbf{Set}$.

(b) *The Kleene functor* $X^* : \mathbf{Set} \to \mathbf{Set}$.

(c) *A polynomial functor* $\Phi(X) = A_0 + A_1 \times X + \cdots + A_n \times X^n + \cdots : \mathbf{Set} \to \mathbf{Set}$ *(where* $A_0, A_1, \cdots$ *are fixed sets).*

(d) *A functor* $\wp_M(A \times X) : \mathbf{Set} \to \mathbf{Set}$.

(e) *A functor* $(A \times X)^* : \mathbf{Set} \to \mathbf{Set}$

Proof. (a) Let ω denote the set of all natural numbers. A natural inclusion $\wp_{\text{fin}}(X) \to \wp_\omega(X)$ is a strict natural transformation. (b) A natural transformation $X^* \to \wp_\omega(X)$ assigning $\{\sigma_1, \sigma_2, \cdots, \sigma_k\} \in \wp_\omega(X)$ to $\sigma_1\sigma_2\cdots\sigma_k \in X^*$ is strict. (c) A natural transformation $\Phi(X) \to \wp_\omega(X)$ assigning $\{\sigma_1, \sigma_2, \cdots, \sigma_k\} \in \wp_\omega(X)$ to $(a, \sigma_1\sigma_2\cdots\sigma_k) \in A_k \times X^k$ $(k \geq 0)$ is strict. (d) A natural transformation $\wp_M(A \times X) \to \wp_M(X)$ induced by the projection $A \times X \to X$ is strict. (e) A natural transformation $(A \times X)^* \to X^*$ assigning $\{\sigma_1, \sigma_2, \cdots, \sigma_k\} \in \wp_\omega(X)$ to $(a_1, \sigma_1)(a_2, \sigma_2)\cdots(a_k, \sigma_k) \in (A \times X)^*$ is strict. $\square$

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Appendix

In this section we will review a known result by Barr [4], which asserts that any cocomplete and co-well-powered category with a set of generators has a final object, in order to show how it is related to our result in this note. The Barr's result is an instance of an even more general statement: the special adjoint functor theorem [6] due to Peter Freyd.

We assume that a small set is a set belonging to a fixed universe $\mathcal{U}$ satisfying the standard ZFC axioms for set theory, and that a category has small hom-sets, in other words, the set of all arrows from an object into another object is small. A category is called small if the set of its objects is small, and cocomplete if all functors from a small category into it have colimits. It is a basic fact ([6, Theorem V 2.1]) that a category is cocomplete if and only if it has coequalizers of all parallel pairs of arrows and it has coproducts indexed by all small sets. A category $\mathcal{C}$ is called co-well-powered if all objects have the small set of quotient objects, that is, for all objects X the set of all equivalence classes of epimorphisms with a domain X is a small set. Throughout the rest of the note we assume a set means a small set.

Definition 1 *Let $\mathcal{D}$ be a category.*

(a) *An object F of $\mathcal{D}$ is called final if for every object $X \in \mathcal{D}$ there is a unique arrow $X \to F$.*

(b) *An object W of $\mathcal{D}$ is called weak final if for every object $X \in \mathcal{D}$ there is at least one arrow $X \to W$.*

(c) *A set $\mathcal{S}$ of objects of $\mathcal{D}$ is called a co-solution set if for every object $X \in \mathcal{D}$ there is an object $S \in \mathcal{S}$ and an arrow $X \to S$.*

The following theorem suggests how to construct a final object from a co-solution set in cocomplete categories.

Theorem 2 (Existence of a final object [6, Theorem V **6.1**])

(a) *A cocomplete category $\mathcal{D}$ has a co-solution set if and only if it has a weak final object.*

(b) *A cocomplete category $\mathcal{D}$ has a weak final object if and only if it has a final object.*

Proof. (a) The existence of this co-solution set is necessary. If $\mathcal{D}$ has a weak final object W, then a singleton set $\{W\}$ realizes the co-solution set, since there is always an arrow $X \to W$. Conversely, assume a co-solution set $\mathcal{S}$ of $\mathcal{D}$. Since $\mathcal{D}$ is cocomplete, it contains a coproduct object $W = \coprod_{S \in \mathcal{S}} S$ of the given set $\mathcal{S}$. For each object $X \in \mathcal{D}$, there is at least one arrow $X \to W$, for example, a composite $X \to S \to W$, where the second arrow is an injection of the coproduct. Hence W is weak final.
(b) The necessity is trivial, because a final object is weak final by definition. Assume W is a weak final object. By hypothesis, the totality $\mathcal{D}(W, W)$ of endomorphisms of W is a set and $\mathcal{D}$ is cocomplete, so we can construct the coequalizer $e : W \to F$ of the set of all the

endomorphisms of W. For each $X \in \mathcal{D}$, there is at least one arrow $X \to F$ by $X \to W \to F$. Suppose there were two, $f, g : X \to F$, and take their coequalizer e_1 as the figure below:

$$\begin{tikzcd} X \arrow[r, "f,g"] & F \arrow[r, "e_1"] & R \arrow[d, "s"] \\ & W \arrow[u, "e"] \arrow[r, "se_1e"'] & W. \end{tikzcd}$$

By the hypothesis of W, there is an arrow $s : R \to W$, so the composite se_1e is, like id_W, an endomorphism of W. But e was defined as the coequalizer of all endomorphisms of W, so

$$ese_1e = e\mathrm{id}_W = \mathrm{id}_F e.$$

Now e is a coequalizer, hence is epic; canceling e on the right gives $ese_1 = \mathrm{id}_F$. Therefore

$$f = \mathrm{id}_F f = ese_1 f = ese_1 g = \mathrm{id}_F g = g,$$

since $e_1 f = e_1 g$. This conlusion means that F is final in $\mathcal{D}$. □

Definition 3 *Let $\mathcal{D}$ be a category. A set $\mathcal{G}$ of objects of $\mathcal{D}$ is called a generating set if for any two different arrows $f, g : X \to Y$ of $\mathcal{D}$ there is an object $G \in \mathcal{G}$ and an arrow $t : G \to X$ such that $ft \neq gt$.* □

Let $\mathcal{G}$ be a set of objects of a category $\mathcal{D}$. Then for every object X of $\in \mathcal{D}$ the class

$$\mathcal{D}(\mathcal{G}, X) = \cup_{G \in \mathcal{G}} \mathcal{D}(G, X)$$

is a set. (That is, $\mathcal{D}(\mathcal{G}, X)$ is the set of all arrows from an object in $\mathcal{G}$ into the given object X.) For an arrow $t \in \mathcal{D}(\mathcal{G}, X)$ its domain will be denoted by G_t, namely, $t : G_t \to X$. It is logically trivial that $\mathcal{G}$ is a generating set if and only if any two arrows $f, g : X \to Y$ of $\mathcal{D}$ are identical if $ft = gt$ for each arrow $t : G_t \to X$ in $\mathcal{D}(\mathcal{G}, X)$.

Lemma 4 *Let $\mathcal{D}$ be a cocomplete category and $\mathcal{G}$ a generating set of $\mathcal{D}$. Then for every object X of $\in \mathcal{D}$*

(a) *There exists an epimorphism from a coproduct $G_X = \coprod_{t \in \mathcal{D}(\mathcal{G}, X)} G_t$ onto an object X,*

(b) *There exists an arrow from X into a quotient of a coproduct $G_* = \coprod_{G \in \mathcal{G}} G$.*

Proof. (a) Define an arrow $s : G_X \to X$ by a unique arrow such that $sj_t = t$ for each $t \in \mathcal{D}(\mathcal{G}, X)$, where $j_t : G_t \to G_X$ is an injection of the coproduct G_X. We will show that s is epic. Let $f, g : X \to Y$ be two arrows such that $fs = gs$. Then for each $t \in \mathcal{D}(\mathcal{G}, X)$ it follows that $ft = fsj_t = gsj_t = gt$. Hence the definition of generating sets claims $f = g$.
(b) By the result of (a) there is an epimorphism $s : G_X \to X$. On the other hand, there is a unique morphism $r : G_X \to G_*$ such that $rj_t = k_{G_t}$ for each $t \in \mathcal{D}(\mathcal{G}, X)$, where $k_G : G \to G_*$ for $G \in \mathcal{G}$ denotes an injection of the coproduct G_*. Then construct a pushout of s and r:

$$\begin{tikzcd} G_X \arrow[r, "s"] \arrow[d, "r"'] & X \arrow[d, "r'"] \\ G_* \arrow[r, "s'"'] & Q. \end{tikzcd}$$

It is a basic fact that if s is an epimorphism, then so is s'. Therefore there is an arrow r' from X into a quotient Q of G_*. □

Theorem 5 (Existence of a final object [6])
If a category $\mathcal{D}$ is cocomplete and co-well-powered and has a generating set $\mathcal{G}$, then

(a) *The set of all quotient objects of a coproduct $G_* = \coprod_{G \in \mathcal{G}} G$ is a co-solution set of $\mathcal{D}$,*

(b) *$\mathcal{D}$ has a final object.*

Proof. (a) First note that the totality of all quotients of G_* is a set by the co-well-poweredness of $\mathcal{D}$. For each object X of $\mathcal{D}$ there exists an arrow from X into a quotient of G_*, by the virtue of Lemma **4**(b). This means that the set of all quotients of G_* is a co-solution set. (b) It immediately follows from (a) and Theorem **2**. □