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Abstract

This note presents a relational formalization of Hennessy-Milner properties in Schröder categories.

Keywords : process algebra, Schröder category, Dedekind category, relational calculus.

1 Introduction

The modal logic for labeled transition systems due to Hennessey and Milner[2] (Hennessey–Milner logic, or HM logic for short) specifies behaviors of processes with logical formulae. This logic presents modality as occurrence of actions in transition systems and actions are classified into necessity and possibility. Labeled transition systems are used for operational semantics for parallel programs and concurrent processes which provide a notion of equivalence 'observational equivalence'. Between such logical systems and semantic structure equivalent correspondences are shown in [2]. It is called Henessey–Milner properties (HM properties, for short) [1]. This means that HM logic is necessary and sufficient to describe behaviors of concurrent processes. Moreover choice of logical operations determines semantic structures with respect to observational equivalence and etc. However, investigating the proof of HM properties in [2], one would notice that their argument is assumed in logics of 2–valued Boolean algebra.

Dedekind category is a general theory of relational calculus (relation algebra). It consists of category of relations and lattice structures of homsets. In this paper we present an alternative proof of HM properties in Boolean Dedekind categories. According to our study it turns out that HM properties are valid also in generally Boolean valued modal logics, for instance, in a powerset of two point set. For the sake of this generalization one may guarantees that behaviors of concurrent processes are characterized by logical formulae which is valid in values between true and false.

HM properties are classified into three levels with respect to semantics of labeled transition systems; observational level , semi–observational level and experimental level. They correspond to HM logic with negation and conjunction, with conjunction and only with modality, respectively. The contents of this paper follow this classification. In section 2 Dedekind categories are introduced. From section **??** to section **??** we verify HM property in

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terms of observational equivalence. Especially in section ?? binomial equivalence relations are introduced as logical equivalence relations over a set of processes. These relations denote equality of sets of valid formulae in HM logic. Readers may find out that methods of proof are algebraically unified in Dedekind categories. In section ?? and ?? semi–observational level and experimental level properties are proved respectively.

2 Dedekind Categories

In this section we recall the fundamentals on relation categories, which we will call Dedekind categories.

Throughout this note, a morphism α from an object X into an object Y in a Dedekind category (which will be defined below) will be denoted by a half arrow $\alpha : X \rightharpoondown Y$, and the composite of a morphism $\alpha : X \rightharpoondown Y$ followed by a morphism $\beta : Y \rightharpoondown Z$ will be written as $\alpha\beta : X \rightharpoondown Z$. Also we will denote the identity morphism on X as id_X.

Definition 1
A Dedekind (Schröder) category $\mathcal{D}$ is a category satisfying the following:
D1. [Distributive Lattice/Boolean Algebra] For all pairs of objects X and Y the hom-set $\mathcal{D}(X,Y)$ consisting of all morphisms of X into Y is a distributive lattice with the least morphism 0_{XY} and the greatest morphism ∇_{XY}. Its lattice structure will be denoted by

$$\mathcal{D}(X,Y) = (\mathcal{D}(X,Y), \sqsubseteq, \sqcup, \sqcap, 0_{XY}, \nabla_{XY}).$$

D2. [Converse] There is given a converse operation $^\sharp : \mathcal{D}(X,Y) \to \mathcal{D}(Y,X)$. That is, for all morphisms $\alpha, \alpha' : X \rightharpoondown Y$, $\beta : Y \rightharpoondown Z$, the following involutive laws hold:
(a) $(\alpha\beta)^\sharp = \beta^\sharp\alpha^\sharp$, (b) $(\alpha^\sharp)^\sharp = \alpha$, (c) If $\alpha \sqsubseteq \alpha'$, then $\alpha^\sharp \sqsubseteq \alpha'^\sharp$
for all morphisms $\alpha, \alpha' : X \rightharpoondown Y$ and $\beta : Y \rightharpoondown Z$.
D3. [Dedekind Formula] For all morphisms $\alpha : X \rightharpoondown Y$, $\beta : Y \rightharpoondown Z$ and $\gamma : X \rightharpoondown Z$ the Dedekind formula $\alpha\beta \sqcap \gamma \sqsubseteq \alpha(\beta \sqcap \alpha^\sharp\gamma)$ holds.
D4. [Residues] For all morphisms $\beta : Y \rightharpoondown Z$ and $\gamma : X \rightharpoondown Z$ the residue (or division, weakest precondition) $\gamma \div \beta : X \rightharpoondown Y$ is a morphism such that $\alpha\beta \sqsubseteq \gamma$ if and only if $\alpha \sqsubseteq \gamma \div \beta$ for all morphisms $\alpha : X \rightharpoondown Y$.

An object I in a Dedekind category $\mathcal{D}$ is called a unit if $0_{II} \neq \mathrm{id}_I = \nabla_{II}$. ($\nabla_{XI}\nabla_{IX} = \nabla_{XX}$)

A morphism $f : X \rightharpoondown Y$ such that $f^\sharp f \sqsubseteq \mathrm{id}_Y$ (*univalent*) is called a *partial function* and may be introduced as $f : X \to Y$. A partial function $f : X \to Y$ such that $\mathrm{id}_X \sqsubseteq ff^\sharp$ (*total*) is called a *function*. In what follows a word *relations* is a synonym of morphisms in a Dedekind category.

3 Observational Equivalences

Let $\mathcal{D}$ be a boolean Dedekind category and A a set of labels. An (A-labelled transition) system over $\mathcal{D}$ consists of an object X of $\mathcal{D}$ and an A-indexed set of transition relations $\delta_a : X \rightharpoondown X$ ($a \in A$), which will be denoted by $(X, \delta_a : X \rightharpoondown X; a \in A)$.

Definition 2
For each relation $\theta : X \rightharpoonup X$ define relations $\theta^{\bullet} : X \rightharpoonup X$ and $\theta^{+} : X \rightharpoonup X$ by

$$\theta^{\bullet} = \bigsqcap_{a \in A} (\delta_a \theta \div \delta_a) \text{ and } \theta^{+} = \theta^{\bullet} \sqcap \theta^{\sharp \bullet \sharp},$$

respectively.

Proposition 1

1. *If θ is a reflexive trasitive relation on X, then so is $\theta^{\bullet}$,*
2. *If θ is an equivalence relation on X, then so is θ^{+}.*

Two sequences of relations $\mu_0, \mu_1, \cdots, \mu_n, \cdots : X \rightharpoonup X$ and $\theta_0, \theta_1, \cdots, \theta_n, \cdots : X \rightharpoonup X$ will be defined as follows:

$$\mu_0 = \theta_0 = \nabla_{XX}, \qquad \mu_{n+1} = {\mu_n}^{\bullet} \quad \text{and} \quad \theta_{n+1} = {\theta_n}^{+}.$$

It is easy to see from the last proposition that all relations μ_n are reflexive transitive, and all relations θ_n are equivalence relations. Set $\mu_\infty = \bigsqcap_{n \geq 0} \mu_n$ and $\theta_\infty = \bigsqcap_{n \geq 0} \theta_n$.

Proposition 2
If each transition relation $\delta_a : X \rightharpoonup X$ is a finite sum of partial functions, then equalities $(\mu_\infty)^{\bullet} = \mu_\infty$ and $(\theta_\infty)^{+} = \theta_\infty$ hold.

4 Binomial Equivalence Relations

Let $\mathcal{D}$ be a boolean Dedekind category.

Definition 3
For every relation $\rho : I \rightharpoonup X$ define relations $\zeta_\rho : X \rightharpoonup X$ and $\xi_\rho : X \rightharpoonup X$ by

$$\zeta_\rho = \rho^{\sharp} \div \rho^{\sharp} \text{ and } \xi_\rho = \zeta_\rho \sqcap \zeta_\rho^{\sharp},$$

respectively.

Lemma 1
$\zeta_\rho : X \rightharpoonup X$ is a reflexive transitive relation and $\xi_\rho : X \rightharpoonup X$ is an equivalence relation.

Definition 4
A sequence $\mathcal{L}_0, \mathcal{L}_1, \cdots, \mathcal{L}_n, \cdots$ of sets of formulas will be defined as follows:

1. $\mathcal{L}_0 = \{\mathrm{true}\}$,
2. If $F \in \mathcal{L}_n$, then $F \in \mathcal{L}_{n+1}$,
   If $F \in \mathcal{L}_n$ and $a \in A$, then $a.F \in \mathcal{L}_{n+1}$,
   If $F \in \mathcal{L}_{n+1}$, then $\neg F \in \mathcal{L}_{n+1}$,
   If $F_1, F_2 \in \mathcal{L}_{n+1}$, then $F_1 \wedge F_2 \in \mathcal{L}_{n+1}$.

Definition 5
A sequence $\mathcal{M}_0, \mathcal{M}_1, \cdots, \mathcal{M}_n, \cdots$ of sets of formulas will be defined as follows:

1. $\mathcal{M}_0 = \{\mathrm{true}\}$,

2. If $F \in \mathcal{M}_n$, then $F \in \mathcal{M}_{n+1}$,
   If $F \in \mathcal{M}_n$ and $a \in A$, then $a.F \in \mathcal{M}_{n+1}$,
   If $F_1, F_2 \in \mathcal{M}_{n+1}$, then $F_1 \wedge F_2 \in \mathcal{M}_{n+1}$.

Definition 6
For each formula F in $\mathcal{L}$ or $\mathcal{M}$ its interpretation $[\![F]\!] : I \rightharpoonup X$ will be defined as follows:

$$[\![\mathrm{true}]\!] = \nabla_{IX}, [\![a.F]\!] = [\![F]\!]\delta_a^\sharp, [\![\neg F]\!] = [\![F]\!]^-, \text{ and } [\![F_1 \wedge F_2]\!] = [\![F_1]\!] \sqcap [\![F_2]\!].$$

For each formula $F \in \mathcal{L}$ we then have a binomial equivalence relation $\xi_{[\![F]\!]} : X \rightharpoonup X$.

Definition 7
For each natural number n a reflexive transitive relation $\zeta_n : X \rightharpoonup X$ and an equivalence relation $\xi_n : X \rightharpoonup X$ will be defined as follows:

$$\zeta_n = \bigsqcap_{F \in \mathcal{M}_n} \zeta_{[\![F]\!]} \quad \text{and} \quad \xi_n = \bigsqcap_{F \in \mathcal{L}_n} \xi_{[\![F]\!]}.$$

5 Hennessy-Milner Properties

Proposition 3
$\mu_n \sqsubseteq \zeta_n$ and $\theta_n \sqsubseteq \xi_n$ hold for each natural number n.

Theorem 1
If each transition relation δ_a is a finite sum of partial functions, then inequalities $\zeta_n \sqsubseteq \mu_n$ and $\xi_n \sqsubseteq \theta_n$ hold for each natural number n.

6 Experimental Equivalences

Definition 8
Define a relation $\varepsilon_n : X \rightharpoonup X$ by

$$\varepsilon_n = \bigsqcap_{w \in A^{[n]}} \{(\delta_w \nabla_{XI} \div \delta_w \nabla_{XI}) \sqcap (\delta_w \nabla_{XI} \div \delta_w \nabla_{XI})^\sharp\},$$

$$\varepsilon_n = \bigsqcap_{w \in A^{[n]}} \zeta_{\delta_w \nabla_{XI}},$$

where $A^{[n]}$ is the set of all strings with length $|w| \leq n$.

It is trivial that ε_n is an equvalence relation on X such that $\varepsilon_{n+1} \sqsubseteq \varepsilon_n$.

Definition 9
A sequence $\mathcal{N}_0, \mathcal{N}_1, \cdots, \mathcal{N}_n, \cdots$ of sets of formulas will be defined as follows:

1. $\mathcal{N}_0 = \{\mathrm{true}\}$,

2. If $F \in \mathcal{N}_n$, then $F \in \mathcal{N}_{n+1}$,
   If $F \in \mathcal{N}_n$ and $a \in A$, then $a.F \in \mathcal{N}_{n+1}$,

Proposition 4
For each natural number n an equality $\varepsilon_n = \bigsqcap_{F \in \mathcal{N}_n} \xi_{[\![F]\!]}$ holds.

7 Conclusion

We presented an alternative proof of HM properties in Boolean Dedekind categories, so that HM properties hold in more than two Boolean valued modal logics. This seems to be important for concurrency theory. Because, considering behavior of communicating concurrent processes one cannot determine their logical semantics with either true or false. It is worth studying logical semantics of concurrency in general Boolean algebras and lattice–valued logical semantics.

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Appendix

Proof of 1. (a) First the reflexive law $id_X \sqsubseteq \theta^\bullet$ follows from $id_X \sqsubseteq \delta_a\theta \div \delta_a$ by $id_X\delta_a \sqsubseteq \theta\delta_a$. The transitive law $\theta^\bullet\theta^\bullet \sqsubseteq \theta^\bullet$ can be seen from $(\delta_a\theta \div \delta_a)(\delta_a\theta \div \delta_a)\delta_a \sqsubseteq (\delta_a\theta \div \delta_a)\delta_a\theta \sqsubseteq \delta_a\theta\theta \sqsubseteq \delta_a\theta$.
(b) First note that the symmetric law $\theta^\sharp = \theta$ implies $\theta^+ = \theta^\bullet \sqcap \theta^{\bullet\sharp}$. The reflexive law $id_X \sqsubseteq \theta^+$ follows from the reflexive law $id_X \sqsubseteq \theta^\bullet$ of $\theta^\bullet$. The symmetric law $\theta^{+\sharp} = \theta^+$ is trivial by $\theta^+ = \theta^\bullet \sqcap \theta^{\bullet\sharp}$. The transitive law $\theta^+\theta^+ \sqsubseteq \theta^+$ can be seen from $\theta^+\theta^+ \sqsubseteq \theta^\bullet\theta^\bullet \sqcap \theta^{\bullet\sharp}\theta^{\bullet\sharp} \sqsubseteq \theta^+$ using the transitive law of $\theta^\bullet$.

PROPOSITION 5
Since the hom set $\mathcal{D}(X, Y)$ of a Dedekind category $\mathcal{D}$ is a complete Heyting algebra, a distributive law

$$\alpha \sqcap (\bigsqcup_{j\in J} \alpha_j) = \bigsqcup_{j\in J}(\alpha \sqcap \alpha_j)$$

holds for all $\alpha, \alpha_j : X \rightharpoonup Y$ ($j \in J$), and so an extended distributive law also holds:

$$\bigsqcap_{i=1}^{k}(\bigsqcup_{j\in J} \alpha_{ij}) = \bigsqcup_{j_1\in J,\cdots,j_k\in J}(\bigsqcap_{i=1}^{k} \alpha_{ij_i})$$

for all $\alpha_{ij} : X \rightharpoonup Y$ ($i = 1, \cdots, k$ and $j \in J$).

PROPOSITION 6
Assume that every hom set of a Dedekind category $\mathcal{D}$ satisfies the distributive law in the last proposition. Let $\beta_j : Y \rightharpoonup Z$ be a decreasing sequence of relations, that is, $\beta_0 \sqsupseteq \beta_1 \sqsupseteq \cdots \sqsupseteq \beta_j \sqsupseteq \beta_{j+1} \sqsupseteq \cdots$. Then an equality $\alpha(\bigsqcap_{j\geq 0} \beta_j) = \bigsqcap_{j\geq 0} \alpha\beta_j$ holds, if α is one of the following:

1. *a finite sum of partial functions,*

2. *a disjoint sum of finite sums of partial functions.*

Proof. (a) Fisrt of all a semi-distributive law $\alpha(\sqcap_{j\geq 0}\beta_j)\sqsubseteq\sqcap_{j\geq 0}\alpha\beta_j$ is trivial. Assume that $\alpha=\sqcup_{i=1}^{k}f_i$ for partial functions f_i. Then we have

$$\begin{aligned}\alpha(\sqcap_{j\geq 0}\beta_j) &= (\sqcup_{i=1}^{k}f_i)(\sqcap_{j\geq 0}\beta_j)\\ &= \sqcup_{i=1}^{k}\{f_i(\sqcap_{j\geq 0}\beta_j)\}\\ &= \sqcup_{i=1}^{k}(\sqcap_{j\geq 0}f_i\beta_j)\\ &= \sqcap_{j_1\geq 0,\cdots,j_k\geq 0}(\sqcup_{i=1}^{k}f_i\beta_{j_i})\end{aligned}$$

and

$$\begin{aligned}\sqcap_{j\geq 0}\alpha\beta_j &= \sqcap_{j\geq 0}(\sqcup_{i=1}^{k}f_i)\beta_j\\ &= \sqcap_{j\geq 0}(\sqcup_{i=1}^{k}f_i\beta_j).\end{aligned}$$

Thus we have to see that $\sqcap_{j\geq 0}(\sqcup_{i=1}^{k}f_i\beta_j)\sqsubseteq(\sqcup_{i=1}^{k}f_i\beta_{j_i})$ for all $j_1\geq 0,\cdots,j_k\geq 0$. Set $j_0=\max\{j_1,\cdots,j_k\}$. Then $\beta_{j_0}\sqsubseteq\beta_{j_i}$ for $i=1,\cdots,k$, and hence

$$\sqcap_{j\geq 0}(\bigsqcup_{i=1}^{k}f_i\beta_j)\sqsubseteq\bigsqcup_{i=1}^{k}f_i\beta_{j_0}\sqsubseteq\bigsqcup_{i=1}^{k}f_i\beta_{j_i},$$

which completes the proof.
(b) (First we assume (N,I) distributive law $\sqcap_{j\geq 0}(\sqcup_{i\in I}\alpha_{ij})=\sqcup_{c:N\to I}(\sqcap_{j\geq 0}\alpha_{c(j)j})$.) Assume that $\alpha=\sqcup_{i\in I}\alpha_i$, where α_i is a finite sum of partial functions such that $\alpha_i^\sharp\alpha_{i'}=0_{YY}$ if $i\neq i'$. Then we have

$$\begin{aligned}\sqcap_{j\geq 0}\alpha\beta_j &= \sqcap_{j\geq 0}(\sqcup_{i\in I}\alpha_i)\beta_j\\ &= \sqcap_{j\geq 0}(\sqcup_{i\in I}\alpha_i\beta_j)\\ &= \sqcup_{c:N\to I}(\sqcap_{j\geq 0}\alpha_{c(j)}\beta_j)\\ &= \sqcup_{i\in I}(\sqcap_{j\geq 0}\alpha_i\beta_j)\\ &= \sqcup_{i\in I}[\alpha_i(\sqcap_{j\geq 0}\beta_j)]\\ &= (\sqcup_{i\in I}\alpha_i)(\sqcap_{j\geq 0}\beta_j)\\ &= \alpha(\sqcap_{j\geq 0}\beta_j).\end{aligned}$$

Note that if $c(j)\neq c(j')$, then $\sqcap_{j\geq 0}\alpha_{c(j)}\beta_j=0_{XY}$, since

$$\alpha_{c(j)}\beta_j\sqcap\alpha_{c(j')}\beta_{j'}\sqsubseteq\alpha_{c(j)}[\beta_j\sqcap\alpha_{c(j)}^\sharp\alpha_{c(j')}\beta_{j'}]=0_{XY}.$$

Proof of **2**.(continuity)
(a) It follows from $\mu_\infty^\bullet=\sqcap_{a\in A}\{\delta_a(\sqcap_{n\geq 0}\mu_n)\div\delta_a\}=\sqcap_{a\in A}\{(\sqcap_{n\geq 0}\delta_a\mu_n)\div\delta_a\}=\sqcap_{a\in A}\sqcap_{n\geq 0}(\delta_a\mu_n\div\delta_a)=\sqcap_{n\geq 0}\sqcap_{a\in A}(\delta_a\mu_n\div\delta_a)=\sqcap_{n\geq 0}\mu_n^\bullet=\sqcap_{n\geq 0}\mu_{n+1}=\mu_\infty$.
(b) By the same way as in (a) it follows that $\theta_\infty^\bullet=\sqcap_{n\geq 0}\theta_n^\bullet$, that is, $(\sqcap_{n\geq 0}\theta_n)^\bullet=\sqcap_{a\in A}\{\delta_a(\sqcap_{n\geq 0}\theta_n)\div\delta_a\}=\sqcap_{a\in A}\{(\sqcap_{n\geq 0}\delta_a\theta_n)\div\delta_a\}=\sqcap_{a\in A}\sqcap_{n\geq 0}(\delta_a\theta_n\div\delta_a)=\sqcap_{n\geq 0}\sqcap_{a\in A}(\delta_a\theta_n\div\delta_a)=\sqcap_{n\geq 0}\theta_n^\bullet$. But by noticing that θ and θ_n are equivalence relations, we have $\theta_\infty^+=\theta_\infty^\bullet\sqcap\theta_\infty^{\bullet\sharp}=(\sqcap_{n\geq 0}\theta_n^\bullet)\sqcap(\sqcap_{n\geq 0}\theta_n^\bullet)^\sharp=\sqcap_{n\geq 0}(\theta_n^\bullet\sqcap\theta_n^{\bullet\sharp})=\sqcap_{n\geq 0}\theta_n^+=\sqcap_{n\geq 0}\theta_{n+1}=\theta_\infty$.

Proof of **1** (a) First the reflexive law $id_X\sqsubseteq\zeta_\rho$ follows from $id_X\sqsubseteq\rho^\sharp\div\rho^\sharp$ by $id_X\rho^\sharp\sqsubseteq\rho^\sharp$. The transitive law $\zeta_\rho\zeta_\rho\sqsubseteq\zeta_\rho$ can be easily seen from $(\rho^\sharp\div\rho^\sharp)(\rho^\sharp\div\rho^\sharp)\rho^\sharp\sqsubseteq(\rho^\sharp\div\rho^\sharp)\rho^\sharp\sqsubseteq\rho^\sharp$ by using $(\gamma\div\beta)\beta\sqsubseteq\gamma$. (b) First the reflexive law $id_X\sqsubseteq\xi_\rho$ follows from $id_X\sqsubseteq\zeta_\rho\sqcap\zeta_\rho^\sharp$ by $id_X\sqsubseteq\zeta_\rho$. The symmetric law ${\xi_\rho}^\sharp=\xi_\rho$ is trivial by definition. The transitive law $\xi_\rho\xi_\rho\sqsubseteq\xi_\rho$ can be seen from $(\zeta_\rho\sqcap\zeta_\rho^\sharp)(\zeta_\rho\sqcap\zeta_\rho^\sharp)\sqsubseteq\zeta_\rho\zeta_\rho\sqcap\zeta_\rho^\sharp\zeta_\rho^\sharp\sqsubseteq\zeta_\rho\sqcap\zeta_\rho^\sharp$.

Lemma 2

1. $\zeta_{\rho_1} \sqcap \zeta_{\rho_2} \sqsubseteq \zeta_{\rho_1 \sqcap \rho_2}$ *and* $\xi_{\rho_1} \sqcap \xi_{\rho_2} \sqsubseteq \xi_{\rho_1 \sqcap \rho_2}$,
2. $\delta\zeta_\rho \div \delta \sqsubseteq \zeta_{\rho\delta^\sharp}$ *for a relation* $\delta : X \rightharpoondown X$,
3. $\xi_\rho = \rho^\sharp\rho \sqcup \rho^{-\sharp}\rho^-$ *and* $\xi_{\rho^-} = \xi_\rho$.

Proof. (a) The former follows from $(\rho_1^\sharp \div \rho_1^\sharp) \sqcap (\rho_2^\sharp \div \rho_2^\sharp) \sqsubseteq \{\rho_1^\sharp \div (\rho_1 \sqcap \rho_2)^\sharp\} \sqcap \{\rho_2^\sharp \div (\rho_1 \sqcap \rho_2)^\sharp\} = (\rho_1^\sharp \sqcap \rho_2^\sharp) \div (\rho_1 \sqcap \rho_2)^\sharp$ by ???? and the latter is a simple corollary of the former. (b) $\delta(\rho^\sharp \div \rho^\sharp) \div \delta \sqsubseteq (\delta\rho^\sharp \div \rho^\sharp) \div \delta = \delta\rho^\sharp \div \delta\rho^\sharp$, since $\alpha(\beta \div \gamma) \sqsubseteq \alpha\beta \div \gamma$. (c) As $\xi_\rho = (\rho^{\sharp-}\rho \sqcup \rho^\sharp\rho^-)^-$ by ????, the result simply follows from checking a fact that $\rho^\sharp\rho \sqcup \rho^{-\sharp}\rho^-$ is the complement of $\rho^{\sharp-}\rho \sqcup \rho^\sharp\rho^-$.

Proof of 3. The statements will be proved by sturctural induction on formulas.
(a) Assume that $\mu_n \sqsubseteq \zeta_{[\![F]\!]}$ for all $F \in \mathcal{M}_n$. We will show that $\mu_{n+1} \sqsubseteq \zeta_{[\![G]\!]}$ for all $G \in \mathcal{M}_{n+1}$. (i) If $G \in \mathcal{M}_n$, then $\mu_{n+1} \sqsubseteq \mu_n \sqsubseteq \zeta_{[\![G]\!]}$. (ii) Assume that $G = a.F$ for $F \in \mathcal{M}_n$ and $a \in A$. Then from the induction hypothsis $\mu_n \sqsubseteq \zeta_{[\![F]\!]}$ we have $\mu_{n+1} \sqsubseteq \delta_a\mu_n \div \delta_a \sqsubseteq \delta_a\zeta_{[\![F]\!]} \div \delta_a \sqsubseteq \zeta_{[\![a.F]\!]} = \zeta_{[\![G]\!]}$. (iii) If $G = G_1 \wedge G_2$ for $G_1, G_2 \in \mathcal{M}_{n+1}$, then $\mu_{n+1} \sqsubseteq \zeta_{[\![G_1]\!]} \sqcap \zeta_{[\![G_2]\!]} \sqsubseteq \zeta_{[\![G_1 \wedge G_2]\!]} = \zeta_{[\![G]\!]}$. (b) It suffices to see that $\theta_{n+1} \sqsubseteq \xi_{n+1}$ can be derived from the induction hypothesis $\theta_n \sqsubseteq \xi_n$. Assume that $\theta_n \sqsubseteq \xi_{[\![F]\!]}$ for all $F \in \mathcal{L}_n$. We will show that $\theta_{n+1} \sqsubseteq \xi_{[\![G]\!]}$ for all $G \in \mathcal{L}_{n+1}$ by the structural induction. (i) If $G \in \mathcal{L}_n$, then $\theta_{n+1} \sqsubseteq \theta_n \sqsubseteq \xi_{[\![G]\!]}$. (ii) Assume that $G = a.F$ for $F \in \mathcal{L}_n$ and $a \in A$. Then from $\theta_n \sqsubseteq \xi_{[\![F]\!]} \sqsubseteq \zeta_{[\![F]\!]}$ we have $\delta_a\theta_n \div \delta_a \sqsubseteq \delta_a\zeta_{[\![F]\!]} \div \delta_a \sqsubseteq \zeta_{[\![a.F]\!]}$ and hence

$$\theta_{n+1} \sqsubseteq (\delta_a\theta_n \div \delta_a) \sqcap (\delta_a\theta_n \div \delta_a)^\sharp \sqsubseteq \zeta_{[\![a.F]\!]} \sqcap \zeta^\sharp_{[\![a.F]\!]} = \xi_{[\![G]\!]}.$$

(iii) If $G = \neg G'$ for $G' \in \mathcal{L}_{n+1}$, then $\theta_{n+1} \sqsubseteq \xi_{[\![G']\!]} = \xi_{[\![\neg G']\!]} = \xi_{[\![G]\!]}$. (iv) If $G = G_1 \wedge G_2$ for $G_1, G_2 \in \mathcal{L}_{n+1}$ such that $\theta_{n+1} \sqsubseteq \xi_{[\![G_1]\!]}$ and $\theta_{n+1} \sqsubseteq \xi_{[\![G_2]\!]}$, then $\theta_{n+1} \sqsubseteq \xi_{[\![G_1]\!]} \sqcap \xi_{[\![G_2]\!]} \sqsubseteq \xi_{[\![G_1 \wedge G_2]\!]}$.

Lemma 3
Let $\rho, \rho_1, \cdots, \rho_k, \sigma_1, \cdots, \sigma_k : I \rightharpoondown X$, *and* $\delta : X \rightharpoondown X$ *be relations in* $\mathcal{D}$. *Then*

1. *If* $\sigma_i \sqsubseteq \rho_j$ *for* $i \neq j$, *then* $\bigsqcap_{i=1}^k (\sigma_i \sqcup \rho_i) = (\bigsqcup_{i=1}^k \sigma_i) \sqcup (\bigsqcap_{i=1}^k \rho_i)$.
2. *If* $\mathcal{R}$ *is a finite set of relations* $\rho : I \rightharpoondown X$ *such that* $\bigsqcup_{\rho \in \mathcal{R}} \rho = \nabla_{IX}$ *and* $\rho \sqcap \rho' = 0_{IX}$ *for* $\rho \neq \rho' \in \mathcal{R}$, *then* $\bigsqcap_{\rho \in \mathcal{R}} (\delta\rho^\sharp \div \rho^\sharp) = \bigsqcup_{\rho \in \mathcal{R}} \delta\rho^\sharp\rho$.

Proof. (a) Since $\sigma_1 \sqsubseteq \rho_2$ and $\sigma_2 \sqsubseteq \rho_1$, it is trivial that

$$(\rho_1 \sqcup \sigma_1) \sqcap (\rho_2 \sqcup \sigma_2) = (\rho_1 \sqcap \rho_2) \sqcup \sigma_1 \sqcup \sigma_2.$$

We now remark that $\sigma_3 \sqsubseteq \rho_1 \sqcap \rho_2$ and $\sigma_1 \sqcup \sigma_2 \sqsubseteq \rho_3$, and then also the following is clear:

$$\bigsqcap_{i=1}^{3} (\rho_i \sqcup \sigma_i) = (\bigsqcap_{i=1}^{3} \rho_i) \sqcup (\bigsqcup_{i=1}^{3} \sigma_i)$$

(b) First note that $\bigsqcup_{\rho \in \mathcal{R}} \rho = \nabla_{IX}$, and when $\rho \neq \rho'$ for $\rho, \rho' \in \mathcal{R}$, we have

$$\delta\rho^\sharp\rho \sqcap \nabla_{XI}\rho' \sqsubseteq \nabla_{XI}\rho \sqcap \nabla_{XI}\rho' = \nabla_{XI}(\rho \sqcap \rho') = 0_{XX}.$$

Therefore from (b) we have

$$\begin{aligned}
\textstyle\bigsqcap_{\rho\in\mathcal{R}}(\delta\rho^{\sharp}\div\rho^{\sharp}) &= \textstyle\bigsqcap_{\rho\in\mathcal{R}}\{(\nabla_{XI}\rho)^{-}\sqcup\delta\rho^{\sharp}\rho\} \\
&= \{\nabla_{XI}(\textstyle\bigsqcup_{\rho\in\mathcal{R}}\rho)\}^{-}\sqcup(\textstyle\bigsqcup_{\rho\in\mathcal{R}}\delta\rho^{\sharp}\rho) \\
&= \textstyle\bigsqcup_{\rho\in\mathcal{R}}\delta\rho^{\sharp}\rho.
\end{aligned}$$

LEMMA 4
If a transition relation δ_a is a finite sum of partial functions, then

1. $\bigsqcap_{F\in\mathcal{M}_n}(\delta_a[\![F]\!]^{\sharp}\div[\![F]\!]^{\sharp})\sqsubseteq\delta_a\zeta_n$,

2. $\bigsqcap_{F\in\mathcal{L}_n}(\delta_a[\![F]\!]^{\sharp}\div[\![F]\!]^{\sharp})\sqsubseteq\delta_a\xi_n$.

Proof. By the assumtion we can set $\delta_a=\bigsqcup_{i=1}^{k}f_i$, where f_i is a partial function for $i=1,\cdots,k$. Then $\delta_a\zeta_n=(\bigsqcup_{i=1}^{k}f_i)\zeta_n=\bigsqcup_{i=1}^{k}f_i\zeta_n=\bigsqcup_{i=1}^{k}\{f_i(\bigsqcap_{F\in\mathcal{M}_n}\zeta_{[\![F]\!]})\}=\bigsqcup_{i=1}^{k}(\bigsqcap_{F\in\mathcal{M}_n}f_i\zeta_{[\![F]\!]})=\bigsqcap_{F_1,\cdots,F_k\in\mathcal{M}_n}\bigsqcup_{i=1}^{k}f_i\zeta_{[\![F_i]\!]}$ and also $\delta_a\xi_n=\bigsqcap_{F_1,\cdots,F_k\in\mathcal{L}_n}\bigsqcup_{i=1}^{k}f_i\xi_{[\![F_i]\!]}$.
(a) It suffices to prove $\bigsqcap_{F\in\mathcal{M}_n}(\delta_a[\![F]\!]^{\sharp}\div[\![F]\!]^{\sharp})\sqsubseteq\bigsqcup_{i=1}^{k}f_i\zeta_{[\![F_i]\!]}$ for all formulas $F_1,\cdots,F_k\in\mathcal{M}_n$. Now we will prove $\bigsqcap_{i=1}^{k}(f_i\zeta_{[\![F_i]\!]})^{-}\sqsubseteq\bigsqcup_{F\in\mathcal{M}_n}(\delta_a[\![F]\!]^{\sharp}\div[\![F]\!]^{\sharp})^{-}$. Since

$$\prod_{i=1}^{k}(f_i\zeta_{[\![F_i]\!]})^{-}=\prod_{i=1}^{k}\{(f_i\nabla_{XX})^{-}\sqcup f_i(\zeta_{[\![F_i]\!]})^{-}\}=\prod_{i=1}^{k}\{(f_i\nabla_{XX})^{-}\sqcup f_i[\![F_i]\!]^{\sharp-}[\![F_i]\!]\}$$

and

$$(\delta_a[\![F]\!]^{\sharp}\div[\![F]\!]^{\sharp})^{-}=(\delta_a[\![F]\!]^{\sharp})^{-}[\![F]\!]=\prod_{i=1}^{k}(f_i[\![F]\!]^{\sharp})^{-}[\![F]\!]=\prod_{i=1}^{k}\{(f_i\nabla_{XI})^{-}\sqcup f_i[\![F]\!]^{\sharp-}\}[\![F]\!],$$

the right hand side of the last equality is the supremum (sum) of terms(relations)

$$\{\prod_{p=1}^{s}(f_{i_p}\nabla_{XX})^{-}\}\sqcap(\prod_{q=1}^{t}f_{j_q}[\![F_{j_q}]\!]^{\sharp-}[\![F_{j_q}]\!]),$$

where $\{i_1,\cdots,i_s\}$ and $\{j_1,\cdots,j_t\}$ is a partition of a set $\{1,\cdots,k\}$.
(i) $\bigsqcap_{i=1}^{k}(f_i\nabla_{XX})^{-}=(\bigsqcup_{i=1}^{k}f_i\nabla_{XX})^{-}=(\delta_a\nabla_{XX})^{-}=(\delta_a\nabla_{XI})^{-}\nabla_{IX}=(\delta_a[\![true]\!]^{\sharp})^{-}[\![true]\!]$, where $true\in\mathcal{M}_n$.
(ii) $\bigsqcap_{i=1}^{k}f_i[\![F_i]\!]^{\sharp-}[\![F_i]\!]=(\bigsqcap_{i=1}^{k}f_i[\![F_i]\!]^{\sharp-})(\bigsqcap_{i=1}^{k}[\![F_i]\!])\sqsubseteq\bigsqcap_{i=1}^{k}\{(f_i\nabla_{XI})^{-}\sqcup f_i[\![F]\!]^{\sharp-}\}[\![F]\!]$, where $F=F_1\wedge\cdots\wedge F_k$.
(iii)

$$\begin{aligned}
\{\textstyle\bigsqcap_{p=1}^{s}(f_{i_p}\nabla_{XX})^{-}\}\sqcap(\textstyle\bigsqcap_{q=1}^{t}f_{j_q}[\![F_{j_q}]\!]^{\sharp-}[\![F_{j_q}]\!]) &= \{\textstyle\bigsqcap_{p=1}^{s}(f_{i_p}\nabla_{XX})^{-}\}\sqcap(\textstyle\bigsqcap_{q=1}^{t}f_{j_q}[\![F_{j_q}]\!]^{\sharp-})[\![F]\!] \\
&\sqsubseteq \{\textstyle\bigsqcap_{p=1}^{s}(f_{i_p}\nabla_{XX})^{-}\}\sqcap(\textstyle\bigsqcap_{q=1}^{t}f_{j_q}[\![F]\!]^{\sharp-})[\![F]\!] \\
&\sqsubseteq [\{\textstyle\bigsqcap_{p=1}^{s}(f_{i_p}\nabla_{XX})^{-}\}[\![F]\!]^{\sharp}\sqcap(\textstyle\bigsqcap_{q=1}^{t}f_{j_q}[\![F]\!]^{\sharp-})][\![F]\!] \\
&\sqsubseteq [\{\textstyle\bigsqcap_{p=1}^{s}(f_{i_p}\nabla_{XX})^{-}\}\nabla_{XI}\sqcap(\textstyle\bigsqcap_{q=1}^{t}f_{j_q}[\![F]\!]^{\sharp-})][\![F]\!] \\
&= [\{\textstyle\bigsqcap_{p=1}^{s}(f_{i_p}\nabla_{XX}\nabla_{XI})^{-}\}\sqcap(\textstyle\bigsqcap_{q=1}^{t}f_{j_q}[\![F]\!]^{\sharp-})][\![F]\!] \\
&= [\{\textstyle\bigsqcap_{p=1}^{s}(f_{i_p}\nabla_{XI})^{-}\}\sqcap(\textstyle\bigsqcap_{q=1}^{t}f_{j_q}[\![F]\!]^{\sharp-})][\![F]\!] \\
&\sqsubseteq [\textstyle\bigsqcap_{i=1}^{k}\{(f_i\nabla_{XI})^{-}\sqcup f_i[\![F]\!]^{\sharp-}\}][\![F]\!]
\end{aligned}$$

where $F=F_{j_1}\wedge\cdots\wedge F_{j_t}$.

(b) Recall that $\delta_a\xi_n = \bigsqcap_{F_1,\cdots,F_k\in\mathcal{L}_n} \bigsqcup_{i=1}^k f_i\xi_{[\![F_i]\!]}$. Hence it suffices to see that $\bigsqcap_{F\in\mathcal{L}_n}(\delta_a[\![F]\!]^\sharp \div [\![F]\!]^\sharp) \sqsubseteq \bigsqcup_{i=1}^k f_i\xi_{[\![F_i]\!]}$ for all formulas $F_1,\cdots,F_k \in \mathcal{L}_n$. Now consider the set $\mathcal{R}$ of all formulas $F_1{}' \wedge \cdots \wedge F_k{}'$ such that $F_i{}' = F_i$ or $F_i{}' = \neg F_i$ for $i = 1,\cdots,k$. That is,

$$\mathcal{R} = \{\neg^{\varepsilon_1}F_1 \wedge \cdots \wedge \neg^{\varepsilon_1}F_k \mid \varepsilon_i = 0 \text{ or } 1 (i = 1,\cdots,k)\}.$$

Since $\mathcal{R}$ is a subset of $\mathcal{L}_n$, it is enough to show that $\bigsqcap_{F\in\mathcal{R}}(\delta_a[\![F]\!]^\sharp \div [\![F]\!]^\sharp) \sqsubseteq \bigsqcup_{i=1}^k f_i\xi_{[\![F_i]\!]}$ for all formulas $F_1,\cdots,F_k \in \mathcal{L}_n$. It si trivial that $[\![F]\!]^\sharp[\![F]\!] \sqsubseteq [\![F_i]\!]^\sharp[\![F_i]\!] \sqcup [\![\neg F_i]\!]^\sharp[\![\neg F_i]\!] = \xi_{[\![F_i]\!]}$ for all $F \in \mathcal{R}$. Therefore we have $\bigsqcup_{F\in\mathcal{R}}[\![F]\!]^\sharp[\![F]\!] \sqsubseteq \xi_{[\![F_i]\!]}$ and so

$$\begin{aligned}
\bigsqcap_{F\in\mathcal{R}}(\delta_a[\![F]\!]^\sharp \div [\![F]\!]^\sharp) &= \bigsqcup_{F\in\mathcal{R}}\delta_a[\![F]\!]^\sharp[\![F]\!] && \{\text{ Lemma 5.7(b). }\} \\
&= \delta_a(\bigsqcup_{F\in\mathcal{R}}[\![F]\!]^\sharp[\![F]\!]) \\
&= (\bigsqcup_{i=1}^k f_i)(\bigsqcup_{F\in\mathcal{R}}[\![F]\!]^\sharp[\![F]\!]) \\
&= \bigsqcup_{i=1}^k f_i(\bigsqcup_{F\in\mathcal{R}}[\![F]\!]^\sharp[\![F]\!]) \\
&\sqsubseteq \bigsqcup_{i=1}^k f_i\xi_{[\![F_i]\!]}.
\end{aligned}$$

Proof of 1. (a) We will see that $\zeta_{n+1} \sqsubseteq \mu_{n+1}$ can be derived from the induction hypothesis $\zeta_n \sqsubseteq \mu_n$. Assume that $\zeta_n \sqsubseteq \mu_n$. If $\zeta_{n+1} \sqsubseteq \zeta_n{}^\bullet$, then $\zeta_{n+1} \sqsubseteq \zeta_n{}^\bullet \sqsubseteq \mu_n{}^\bullet = \mu_{n+1}$. So it suffices to prove $\zeta_{n+1} \sqsubseteq \zeta_n{}^\bullet$ and so $\zeta_{n+1} \sqsubseteq \delta_a\zeta_n \div \delta_a$ for all $a \in A$. But this fact follows from the following:

$$\begin{aligned}
\zeta_{n+1} &\sqsubseteq \bigsqcap_{F\in\mathcal{M}_n}\zeta_{[\![a.F]\!]} \\
&= \bigsqcap_{F\in\mathcal{M}_n}(\delta_a[\![F]\!]^\sharp \div \delta_a[\![F]\!]^\sharp) \\
&= \{\bigsqcap_{F\in\mathcal{M}_n}(\delta_a[\![F]\!]^\sharp \div [\![F]\!]^\sharp)\} \div \delta_a \\
&\sqsubseteq \delta_a\zeta_n \div \delta_a && \{\text{ Lemma 5.8(a). }\}
\end{aligned}$$

(b) We will see that $\xi_{n+1} \sqsubseteq \theta_{n+1}$ can be derived from the induction hypothesis $\xi_n \sqsubseteq \theta_n$. Assume that $\xi_n \sqsubseteq \theta_n$. If $\xi_{n+1} \sqsubseteq \xi_n{}^+$, then $\xi_{n+1} \sqsubseteq \xi_n{}^+ \sqsubseteq \theta_n{}^+ = \theta_{n+1}$. So it suffices to prove $\xi_{n+1} \sqsubseteq \xi_n{}^+$. But note that $\xi_{n+1} \sqsubseteq \xi_n{}^+$ iff $\xi_{n+1} \sqsubseteq \delta_a\xi_n \div \delta_a$ for all $a \in A$. But this fact follows from the following:

$$\begin{aligned}
\xi_{n+1} &\sqsubseteq \bigsqcap_{F\in\mathcal{L}_n}\zeta_{[\![a.F]\!]} \\
&= \bigsqcap_{F\in\mathcal{L}_n}(\delta_a[\![F]\!]^\sharp \div \delta_a[\![F]\!]^\sharp) \\
&= \{\bigsqcap_{F\in\mathcal{L}_n}(\delta_a[\![F]\!]^\sharp \div [\![F]\!]^\sharp)\} \div \delta_a \\
&\sqsubseteq \delta_a\xi_n \div \delta_a. && \{\text{ Lemma 5.8(b). }\}
\end{aligned}$$

PROPOSITION 7
The universal morphism $\nabla_{XI} : X \rightharpoondown I$ is a function.

Proof. (totality) $\nabla_{XI}\nabla_{XI}^\sharp = \nabla_{XI}\nabla_{IX} = \nabla_{XX} \sqsupseteq id_X$ by the assumption.
(unvalency) $\nabla_{XI}^\sharp\nabla_{XI} = \nabla_{IX}\nabla_{XI} \sqsubseteq \nabla_{II} = id_I$.

PROPOSITION 8
For each $a \in A$ the equality $\delta_a\nabla_{XI} \div \delta_a\nabla_{XI} = \delta\nabla_{XX} \div \delta_a$ holds.

Proof. As ∇_{XI} is a function, we have $\delta_a\nabla_{XI} \div \nabla_{XI} = \delta_a\nabla_{XI}\nabla_{XI}^\sharp$.

$$\begin{aligned}
\delta_a\nabla_{XI} \div \delta_a\nabla_{XI} &= (\delta_a\nabla_{XI} \div \nabla_{XI}) \div \delta_a \\
&= (\delta_a\nabla_{XI}\nabla_{XI}^\sharp) \div \delta_a \\
&= \delta_a\nabla_{XX} \div \delta_a.
\end{aligned}$$

Let $\alpha : X \rightharpoondown Y$, $\beta, \beta' : Y \rightharpoondown Z$, $\gamma, \gamma' : X \rightharpoondown Z$ and $\delta : Y' \rightharpoondown Y$ be relations and W an object.

1. $0_{WX}\alpha = 0_{WY}$ and $\alpha 0_{YW} = 0_{XW}$,
2. $\alpha(\bigsqcup_{j\in J}\beta_j)\gamma = \bigsqcup_{j\in J}\alpha\beta_j\gamma$,
3. If $\alpha \sqsubseteq \alpha'$ and $\beta \sqsubseteq \beta'$, then $\alpha\beta \sqsubseteq \alpha'\beta'$,
4. If $\alpha^\sharp\alpha \sqsubseteq id_Y$ and $J \neq \emptyset$, then $\alpha(\bigsqcap_{j\in J}\beta_j) = \bigsqcap_{j\in J}\alpha\beta_j$,
5. $0^\sharp_{XY} = 0_{YX}$ and $\nabla^\sharp_{XY} = \nabla_{YX}$,
6. $(\bigsqcup_{j\in J}\beta_j)^\sharp = \bigsqcup_{j\in J}\beta_j^\sharp$ and $(\bigsqcap_{j\in J}\beta_j)^\sharp = \bigsqcap_{j\in J}\beta_j^\sharp$,
7. $\alpha^{-\sharp} = \alpha^{\sharp -}$,
8. $(\gamma \div \beta)\beta \sqsubseteq \gamma$ and $\gamma \div \delta\beta = (\gamma \div \beta) \div \delta$,
9. $(\bigsqcap_{i\in I}\gamma_i) \div \beta = \bigsqcap_{i\in I}(\gamma_i \div \beta)$ and $\gamma_i \div (\bigsqcup_{i\in I}\beta_i) = \bigsqcap_{i\in I}(\gamma \div \beta_i)$,
10. If $\beta' \sqsubseteq \beta$ and $\gamma \sqsubseteq \gamma'$, then $\gamma \div \beta \sqsubseteq \gamma' \div \beta'$,
11. $\gamma \div \beta = (\gamma^-\beta^\sharp)^-$,
12. If $\alpha^\sharp\alpha \sqsubseteq id_Y$, then $(\alpha\beta)^- = (\alpha\nabla_{YZ})^- \sqcup \alpha\beta^-$,
13. If $\beta^\sharp\beta \sqsubseteq id_Z$, then $\gamma \div \beta = (\nabla_{XZ}\beta^\sharp)^- \sqcup \gamma\beta^\sharp$,
14. If $\beta^\sharp\beta \sqsubseteq id_Z$, then $(\bigsqcup_{i\in I}\gamma_i) \div \beta = \bigsqcup_{i\in I}(\gamma_i \div \beta)$,
15. If $\rho : I \rightharpoonup X$ and $\rho_i : I \rightharpoonup X$, then $\rho^\sharp \div (\bigsqcap_{i\in I}\rho_i^\sharp) = \bigsqcup_{i\in I}(\rho^\sharp \div \rho_i^\sharp)$.
16. $\bigsqcap_{i=1}^k \tau_i\rho_i = (\bigsqcap_{i=1}^k \tau_i)(\bigsqcap_{i=1}^k \rho_i)$.
17. If I is a strict unit, then $(\alpha\nabla_{YI})^-\nabla_{IW} = (\alpha\nabla_{YW})^-$.

Proof. (a) It directly follows from a fact that $0_{WX}\alpha \sqsubseteq 0_{WY} \iff 0_{WX} \sqsubseteq 0_{WY} \div \alpha$, in which the latter condition is true because the zero relation is the smallest relation.
(b) It follows from

$$\begin{aligned}(\textstyle\bigsqcup_{j\in J}\beta_j)\gamma \sqsubseteq \eta &\iff \textstyle\bigsqcup_{j\in J}\beta_j \sqsubseteq \eta \div \gamma\\ &\iff \forall j\in J : \beta_j \sqsubseteq \eta \div \gamma\\ &\iff \forall j\in J : \beta_j\gamma \sqsubseteq \eta\\ &\iff \textstyle\bigsqcup_{j\in J}\beta_j\gamma \sqsubseteq \eta.\end{aligned}$$

(c)

$$\begin{aligned}\alpha\beta \sqcap \{(\alpha\nabla_{YZ})^- \sqcup \alpha\beta^-\} &= \{\alpha\beta \sqcap (\alpha\nabla_{YZ})^-\} \sqcup (\alpha\beta \sqcap \alpha\beta^-)\\ &\sqsubseteq \{\alpha\nabla_{YZ} \sqcap (\alpha\nabla_{YZ})^-\} \sqcup \alpha(\beta \sqcap \beta^-)\\ &= 0_{XZ}\end{aligned}$$

and

$$\begin{aligned}\alpha\beta \sqcup \{(\alpha\nabla_{YZ})^- \sqcup \alpha\beta^-\} &= (\alpha\nabla_{YZ})^- \sqcup \alpha(\beta \sqcup \beta^-)\\ &= (\alpha\nabla_{YZ})^- \sqcup \alpha\nabla_{YZ}\\ &= \nabla_{XZ}.\end{aligned}$$

Hence the result is clear.

(d) Recall that $\gamma \div \beta = (\gamma^- \beta^\sharp)^-$. then it suffices to see that $(\nabla_{XZ}\beta^\sharp)^- \sqcup \gamma\beta^\sharp$ is the complement of $\gamma^-\beta^\sharp$. Recall that β is a partial function.

$$\begin{aligned} \gamma^-\beta^\sharp \sqcap \{(\nabla_{XZ}\beta^\sharp)^- \sqcup \gamma\beta^\sharp\} &= \{\gamma^-\beta^\sharp \sqcap (\nabla_{XZ}\beta^\sharp)^-\} \sqcup (\gamma^-\beta^\sharp \sqcap \gamma\beta^\sharp) \\ &\sqsubseteq \{\nabla_{XZ}\beta^\sharp \sqcap (\nabla_{XZ}\beta^\sharp)^-\} \sqcup (\gamma^- \sqcap \gamma)\beta^\sharp \\ &= 0_{XX} \end{aligned}$$

and

$$\begin{aligned} \gamma^-\beta^\sharp \sqcup \{(\nabla_{XZ}\beta^\sharp)^- \sqcup \gamma\beta^\sharp\} &= (\nabla_{XZ}\beta^\sharp)^- \sqcup (\gamma^- \sqcup \gamma)\beta^\sharp \\ &= (\nabla_{XZ}\beta^\sharp)^- \sqcup \nabla_{XZ}\beta^\sharp \\ &= \nabla_{XY}. \end{aligned}$$

Hence the result is clear.

(e) It is trivial that the semi-distributive law $\alpha(\bigsqcap_{j\in J}\beta_j) \sqsubseteq \bigsqcap_{j\in J}\alpha\beta_j$ holds by the monotonicy of the compoition. As $J \neq \emptyset$ there is some $j_0 \in J$. Hence we have

$$\begin{aligned} \textstyle\bigsqcap_{j\in J}\alpha\beta_j &= \alpha\beta_{j_0} \sqcap (\textstyle\bigsqcap_{j\neq j_0}\alpha\beta_j) \\ &\sqsubseteq \alpha\{\beta_{j_0} \sqcap \alpha^\sharp(\textstyle\bigsqcap_{j\neq j_0}\alpha\beta_j)\} && \{\text{ Dedekind Formula }\} \\ &\sqsubseteq \alpha\{\beta_{j_0} \sqcap (\textstyle\bigsqcap_{j\neq j_0}\alpha^\sharp\alpha\beta_j)\} && \{\text{ The semi-distributivity }\} \\ &\sqsubseteq \alpha\{\beta_{j_0} \sqcap (\textstyle\bigsqcap_{j\neq j_0}\beta_j)\} && \{\ \alpha^\sharp\alpha \sqsubseteq id_Y\ \} \\ &\sqsubseteq \alpha(\textstyle\bigsqcap_{j\in J}\beta_j), \end{aligned}$$

which completes the proof. (f) It is enough to see that $\tau_1\rho_1 \sqcap \tau_2\rho_2 = (\tau_1 \sqcap \tau_2)(\rho_1 \sqcap \rho_2)$. But $(\tau_1 \sqcap \tau_2)(\rho_1 \sqcap \rho_2) \sqsubseteq \tau_1\rho_1 \sqcap \tau_2\rho_2 \sqsubseteq (\tau_1 \sqcap \tau_2\rho_2\rho_1^\sharp)\rho_1 \sqcap (\tau_1\rho_1\rho_2^\sharp \sqcap \tau_2)\rho_2 \sqsubseteq (\tau_1 \sqcap \tau_2)\rho_1 \sqcap (\tau_1 \sqcap \tau_2)\rho_2 = (\tau_1 \sqcap \tau_2)(\rho_1 \sqcap \rho_2)$, since $\tau_1 \sqcap \tau_2$ is a partial function. (g)

$$\begin{aligned} (\alpha\nabla_{YI})^-\nabla_{IW} \sqcap \alpha\nabla_{YW} &\sqsubseteq \{(\alpha\nabla_{YI})^- \sqcap \alpha\nabla_{YW}\nabla_{IW}^\sharp\}\nabla_{IW} \\ &\sqsubseteq \{(\alpha\nabla_{YI})^- \sqcap \alpha\nabla_{YI}^\sharp\}\nabla_{IW} \\ &= 0_{XI}\nabla_{IZ} \\ &= 0_{XZ} \end{aligned}$$

and

$$\begin{aligned} (\alpha\nabla_{YI})^-\nabla_{IW} \sqcup \alpha\nabla_{YW} &= (\alpha\nabla_{YI})^-\nabla_{IW} \sqcup \alpha\nabla_{YI}\nabla_{IW} \\ &= \{(\alpha\nabla_{YI})^- \sqcup \alpha\nabla_{YI}\}\nabla_{IW} \\ &- \nabla_{XI}\nabla_{IW} \\ &= \nabla_{IW}. \end{aligned}$$